Noname manuscript No.
(will be inserted by the editor)

On the Optimization of Pit Stop Strategies via Dynamic Programming

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Received: date / Accepted: date

Abstract Pit stops are a key element of racing strategy in several motor sports. Typically, these stops involve decisions such as in which laps to stop, and which type of tire, of three possible compounds, to set at each of these stops. There are several factors that increase the complexity of the task: the impact of lap times depending on the tire compound, the wear of the tires, unexpected events on the track such as safety cars and the weather, among others. This work presents a Dynamic Programming formulation that addresses the pit-stop strategy problem in order to optimize the laps in which to stop, and the tire changes that minimize the total race time. We show the relative performance of the optimal strategies for starting with tires of different compounds with different yellow-flag scenarios. Then, we extend the Dynamic Program (DP) to a Stochastic Dynamic Programming (SDP) formulation that incorporates random events such as yellow flags or rainy weather. We are able to visualize and compare these optimal pit-stop strategies obtained with these models in different scenarios. We show that the SDP solution, compared to the DP solution, tends to delay pit stops in order to benefit from a possible yellow flag. Finally, we show that the SDP outperforms the DP, especially in races in which yellow flags are likely to be waved more frequently.

Keywords Dynamic Programming · Race Strategy · Formula 1 · Pit Stop

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1 Introduction

Winning a race is the result of a combination of multiple factors, such as the driver's skill, the car and engine performance, the pit stop strategy, the tire compounds, the weather, the competitors, and several random events that may occur during the race. Consequently, racing teams invest significant effort and amounts of resources (Forbes, 2018) in each of these factors in order to obtain the best possible results in races.

Among the key factors mentioned above, in this work we look into pit stop and tire compound strategies, specifically, in which laps the car should change tires, and tires of which compounds should be used. This problem is framed under the Formula 1 setting, however, the methodology developed here can be easily extended and therefore applied to other racing federations that face similar settings. For this reason, we do not analyze refueling as part of the pit stop decisions, since refueling was banned from Formula 1 in the year 2010 (it had also been banned in the decade before 1994). But, we do consider the problem of finding the optimal fuel level with which to start the race, which is not trivial since, as we will show later in more detail, the fuel consumption might depend on the compound of the tire being used. Still, the method of solution presented here can be modified easily to incorporate other aspects such as refueling.

In Formula 1, a race, also called a *Grand Prix*, consists of a specific track on which racing cars have to complete a fixed number of laps in the shortest period of time. During each lap, each car faces the decision of either: (a) continuing on the track, or (b) doing a pit stop in order to have a change of tires[1]. The reason for changing tires is twofold: (i) tires do not last for the whole race, and (ii) there is a rule in Formula 1 that states that a car should use tires of at least two different compounds during a race[2]. There are various types of tire compounds. There is a set of tires for dry weather whose material ranges from hard to soft compounds[3]. Tires of softer compounds tend to be faster on the track compared to harder tires, especially in circuits with many turns. But, these tires of softer compounds have a shorter life cycle compared to the harder ones. Also, the racing time of a car will depend not just on the compound of the tire being used, but also on the tire's wear (due to graining, blistering, marbles, flat spots, and others). There are tires for wet weather, besides the dry weather tires, which allow better performance in the presence of rain. There are two tire compounds for rainy weather, *intermediate*, and *full wets*. The former is better suited for races with light rain, whereas the latter

[1] Note that there are other reasons why the driver would choose to make a pit stop, such as changing the front wing of the car in case it were damaged. For simplicity, we are not going to consider this type of case.

[2] This rule was set in 2016.

[3] Since 2019, there have been a total of five tire compounds for dry weather, 3 of which are available during each race.

is better suited for races with moderate to heavy rain[4]. As expected, using a wet tire compound in the absence of rain results in slower times compared to using the dry ones. Similarly, using the dry-weather compounds in rainy weather will increase lap times. Moreover, the driver will have to be especially careful during turns, since the tires will not have the appropriate grip, and, therefore, the risk of getting out of the track (which would possibly cause a retirement from the race) will increase.

Tire strategy is not a trivial matter since it presents several complexities and challenges that must be taken into account: (i) There is a trade-off between tires of softer and harder compounds: the former allow faster laps, but softer tires tend to wear down more quickly, and will therefore cause the need for more pit stops. (ii) Racing performance does not depend only on the tire compound, but also on the wear of the tires being used (see Farroni et al. (2017)). Indeed, the term *falling off the cliff* is often used to refer to the point at which tires are so deteriorated that the lap time increases significantly compared to the lap time with low or medium tire degradation (see Terms (2019) for more on terminology). (iii) Laps tend to be made more quickly as more laps are completed since the car has less fuel and therefore less weight. (iv) Weather can be a very uncertain factor in some races, since a prediction of rain is not certain enough. It is also important to predict in which lap the rain will start, and with what intensity. (v) *Yellow flags*[5] are perfect times for cars to make pit stops, since competitors on the track have a speed limit, so therefore a car in the pit lane loses less time in relative terms when the stop is made during a yellow flag.

In this work, we address the pit stop and race strategy problem taking into account all five complexities mentioned above. More specifically, we formulate a model to determine the optimal racing strategy by using dynamic programming. In order to do so, we divide the problem in different *stages*, i.e. laps. In each of these *stages* the driver has to make a *decision*, such as to make a pit stop or not, given the current *state* during that particular *stage*. Thus, in each lap information on the car is encoded in the *state*, which summarizes all previous *decisions*, such as the compound of the current tires, the tire wear, etc. We first solve a deterministic version of the problem in which there are no uncertain events, although taking (i), (ii), and (iii) into account (see the previous paragraph). The outcome of this will be the optimal action that should be taken in each lap. We then show how to use this deterministic formulation that

[4] In case of rain, when using any of the two wet tire compounds, the rule that states that tires of two different compounds should be used is no longer valid.

[5] A yellow flag indicates the period of time during the race when the cars have to slow down and no passing is allowed because an incident has taken place at some point on the track which obstructs the normal execution of the race. The two main events in Formula 1 that take place during a yellow flag are: (a) Safety Car (SC) or (b) Virtual Safety Car (VSC). In the former case, the safety car is deployed and all cars must follow with no overtaking (and therefore time differences between cars are dramatically shrunk), while in the latter (VSC) cars must decrease their speed (and so time differences between cars are maintained). For the sake of simplicity, throughout the rest of the paper we consider yellow flags to represent a VSC event.

incorporates information on *yellow flag* events in case these happen. Finally, we introduce uncertainty into the model in order to capture racing events that are not fully known beforehand, such as, for example, weather changes, and yellow flags ((iv) and (v) from the previous paragraph). The solution of this model will return the optimal strategy to be used in each lap given the current state, while taking the possible events that might happen in the future into account. In reality pit-stop decisions are also affected by the interaction between drivers; however, we have decided to neglect these from the model for tractability reasons. The disadvantages of not considering driver interactions is the omission of (a) the game theory aspect among drivers, and (b) events which might have an impact on lap times, such as overtaking and blocking.

1.1 Related Work

Motor sports is not a new field of study. For several decades, various approaches have been adopted to analyze this sport from different perspectives. Thirty years ago, Foxall and Johnston (1991) published their work on innovation, and the evolution of technology, organization, and strategy in Grand Prix motor racing, while ten years later, Jenkins and Floyd (2001) focused their efforts on analyzing the technological-development aspect of Formula 1. More recently, Choo (2015) studied the impact of the pit crew and driver performances, tire change decisions, and caution periods on race outcomes through a data-based approach, which is arguably the most successful and popular methodology in current practice (Bi, 2014). As opposed to our goal of computing optimal strategies, Choo's work focuses on using past data to evaluate and compare past decisions in order to predict the outcome for future races.

Not only is the study of these phenomena interesting for its potential applications on motor sports themselves, but it can lead to original solutions in completely different fields. For example, in medicine, Catchpole et al. (2007) were able to reduce technical errors, information handover omissions, and duration in patient handovers from surgery to intensive care using Formula 1 pit stop models, while Vergales et al. (2015) improved delivery room and admission efficiency, as well as the treatment of prematurely born infants, by applying NASCAR pit-stop models.

In terms of optimization, we find a variety of approaches, such as the study of air flow effects (Chandra et al., 2011), maximization of parameters such as power, weight, tire grip, drag, and lift (Wright and Matthews, 2001), and race-line optimization (Beltman, 2008; Xiong et al., 2010; Vesel, 2015; Jain and Morari, 2020). Using simulation, Bekker and Lotz (2009) model a Formula 1 race by simulating events such as car failures, passing maneuvers, and pit stops. A similar approach is used by both Phillips (2014) and Heilmeier et al. (2018). In these studies (Bekker and Lotz, 2009; Phillips, 2014; Heilmeier et al., 2018), the pit-stop strategies are an input of the model, and not the result of an optimization process. Thus, the best strategy can be chosen by comparing the output of all the strategies that were explored. McLaren Racing Limited

(2019) addresses the optimal fuel strategy, which consists of the fuel level at the start of the race, plus the timing and amount of fuel for refilling during the race. An illustrative and graphic representation of the pit stop strategy problem is presented in Chain Bear (2017). In a recent paper, Heilmeier et al. (2020) use neural networks (NN) calibrated on real data to predict if a pit stop should be made during the current lap, and the new tire compound to be used. By computing the optimal race strategy under a no-competition setting with a mixed-integer quadratic program (MIQP), the authors integrate these NN for all drivers in a simulation to determine the race outcome. Our work differs from their MIQP since our optimization framework does not require a linear effect of tire degradation on lap times. In addition, our optimization model is extended to a stochastic setting.

Interesting approaches can be found in other disciplines, as well: Tagliaferri et al. (2014), for example, studied yacht racing tactics by framing them as a stochastic shortest-path problem in which uncertain elements, such as wind direction, are considered. To solve this problem, the authors rely on dynamic programming, i.e., optimal policies being computed for a determined time horizon, and for every possible state of the system that is being studied. Dynamic programming has been applied in several other contexts; see Bertsekas (1995) for more details. This solving method is suitable for our study: Every lap of a Formula 1 race can be interpreted as a period in which decisions, such as making pit stops, need to be made, and in which uncertain events, such as accidents, can occur. Another important element of Tagliaferri et al., which we include in our work, is the fact that interaction between boats (in our case cars) is ignored: Even though interaction is obviously important in practice, a simpler model is more convenient in terms of computations and focusing on specific phenomena, rather than keeping a constant view of all the elements that are involved in a race, such as car design and human error.

While there are clear similarities between our work and the approach used by Tagliaferri et al., it is important to emphasize how different Formula 1 and yacht racing are. In particular, one of the main focuses of our study is tire strategy whereas in their work, Tagliaferri et al. focus on whether to tack the boat or continue on the same direction. Deciding on the optimal instant in which to replace tires, and the compound of the tires to be used could be compared better to machine-maintenance scheduling and related fields (Yang et al., 2008), in which dynamic programming can also be a valuable tool (Fallahnezhad, 2014). Clear parallels can be drawn between these models and pit stop planning since just as machine components degrade over time and need to be repaired or replaced, tires become more worn out the more laps they are used. While the degradation of a machine's components might lead to decreased product quality, or even to a complete production stop, worn tires will result in increasing lap times, and even a blowout, in the worst case, which could lead to retirement from the race. Consequently, the following trade-off must be taken into account: While a significant cost (in terms of money or time) must be incurred when stopping production in order to repair a machine (or carrying out a pit stop in order to change tires in our case), a considerable performance improvement can be expected during the succeeding periods. Not only does this mean that dynamic programming is well suited for both race-strategy optimization and machine-maintenance scheduling, but the latter field could also be a rich source of models and techniques for future studies of the former.

1.2 Organization of the Paper

The remainder of this paper is structured as follows: In Section 2 a deterministic model that optimizes pit-stop and tire-compound strategies is presented. A stochastic version of the problem that incorporates uncertainty, such as weather and yellow flags, is described in Section 3. Sections 2.2 and 3.1 contain the results of numerical computations run with the models presented. Finally, in Section 4, the main conclusions of our work are summarized.

2 Deterministic Model

Consider a race of N laps. Each lap is going to be one *stage* of the problem. At each *stage*, the driver has to decide[6] whether to continue on track, or make a pit stop to change tires. Consider $\mathcal{T} = \{1, \ldots, T\}$ to be the set of tire compounds allowed to be used during the race. Let us denote by x_n the driver *decision* at lap n such that $x_n = 0$ if there is no pit stop[7], otherwise $x_n = t$ for some $t \in \mathcal{T}$ where t indicates the compound of the new tires installed on the car.

As for the *state* variable, we will denote this by s_n where n indicates the lap. As mentioned above, the *state* variable should summarize the past information up to lap n in order to make the decision (x_n). The information in this case encompasses the following: the compound type and wear of the tires in use, and the fuel level. In addition, we must take the fact into account that the car must use at least two different tire compounds during the race. Thus, at the end of the race, the state variable should have this information. As a result, at each lap n, the state variable will reflect whether or not tires of two or more different compounds have been used so far. Then, the state variable s_n is denoted as

$$s_n = (t_n, w_n, f_n, m_n)\,. \tag{1}$$

$t_n \in \mathcal{T}$ represents the compound of the tire in use at lap n. For simplification purposes, we assume that the pits are located at the end of the lap[8]. Therefore,

[6] This and other decisions are made together by the driver and engineers of the team. In this paper, we will mention either the car, driver, or engineers, without distinction, to refer to the decision maker on the race strategy.

[7] We consider that it is not possible to do a pit stop in the last lap of the race. For simplicity, we omit the description of this constraint.

[8] In Formula 1, the pit lane is usually located in parallel to the finish line. The proposed model can be easily adapted to the real setting.

a change of tires in lap n (i.e. $x_n \neq 0$) will not affect t_n, but rather will affect t_{n+1}. Then

$$t_{n+1} = t_n \cdot \mathbb{1}_{\{x_n=0\}} + x_n \cdot \mathbb{1}_{\{x_n\neq 0\}}. \tag{2}$$

The fuel level at the beginning of lap n is denoted by f_n. The fuel consumption per lap could depend on the compound of the tires in use. Indeed, tires of softer compounds allow maneuvers such as late breaking or taking curves at higher speeds which might increase fuel consumption. Nonetheless, the softer tires could require less acceleration after corners[9]. Let c_t be the fuel consumption per lap when using a tire compound of type $t \in \mathcal{T}$. It can be seen easily that the fuel at the next lap can be expressed as the fuel level at the start at the lap minus the fuel consumption, namely

$$f_{n+1} = \max\{f_n - c_{t_n}, 0\}. \tag{3}$$

To model tire wear, $w_n \in [0, 1]$ is a parameter that reflects the wear of the tires at lap n, so that $w_n = 0$ represents a new set of tires, whereas $w_n = 1$ represents tires that are totally worn out, and therefore the car cannot continue to race. Tires degrade depending on the tire compound as well as on the fuel level. Thus, the degradation of tires at a given lap n can be modeled as a function of t_n and f_n, $\gamma(t_n, f_n)$, which is decreasing on the hardness of the tire compound, but increasing on the fuel level, since more fuel in the car implies more weight, which increases the degradation of the tires. As a result, if there is no pit stop, the tire wear of lap $n + 1$ can be expressed as $w_{n+1} = w_n + \gamma(t_n, f_n)$, whereas if there is a pit stop $w_{n+1} = 0$. Thus,

$$w_{n+1} = (w_n + \gamma(t_n, f_n)) \cdot \mathbb{1}_{\{x_n=0\}}. \tag{4}$$

Finally, with respect to the *state*, we define the variable m_n which is equal to 1 if the car has used tires of two or more different compounds up to lap n, and 0 otherwise. Then, the transition of this variable from one lap to the next can be written as[10]

$$m_{n+1} = \max\{m_n, \mathbb{1}_{\{x_n\neq 0, x_n\neq t_n\}}\}. \tag{5}$$

Since m_n acts as an indicator variable that tires of two different compounds have been used by lap n, we will call this the *two-tire-compound-indicator*. This variable will be used for the border condition. More details of this are given below. Putting Equations (2), (4), (3) together, and (5), we obtain the transition of the state variable given the state and decisions of the previous stage, namely, $s_{n+1}(s_n, x_n)$.

[9] Although we will assume that softer tires consume slightly more fuel than harder tires, this might no be the case for all motorsports.

[10] This deterministic model assumes that the weather is dry. If the weather should be rainy, there is no need to have this state variable since in this case cars are not required to use tires of two different compounds. We do not consider weather changes in the deterministic section, as this will be presented in the stochastic model in the next section.

As mentioned in Section 1, lap times are a function of the tire compound, tire wear, fuel level, and whether or not a pit stop is made. We define the *lap-time function* $\mu(t, w, f, x)$ that returns the lap time, given the tire of compound $t \in \mathcal{T}$ in use, the tire wear $w \in [0, 1]$, the fuel level f, and whether a pit stop has been made ($x \neq 0$), or not ($x = 0$). This function must be: increasing on w since the more tire wear, the higher the lap times will be; increasing on f since more fuel in the car implies more weight; and increasing on $\mathbb{1}_{\{x=0\}}$ since making a pit stop takes more time than not having one.

Given all the above, we can state the Bellman equation as:

$$V_n(s_n, x_n) = \mu\left(t_n, w_n, f_n, x_n\right) + V^*_{n+1}(s_{n+1}(s_n, x_n)), \tag{6}$$

where

$$V^*_n(s_n) = \min_{x_n \in \{0\} \cup \mathcal{T}} V_n(s_n, x_n).$$

The border condition must be such that if the car has not used tires of more than one compound during the race (i.e., $m_{N+1} = 0$), then the race time is set to infinity, namely

$$V^*_{N+1}(s_{N+1}) = \begin{cases} +\infty & \text{if } m_{N+1} = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

We solve the dynamic program from stage $N + 1$ to stage 1, obtaining the optimal decisions at each stage for each possible state. Then, the optimal strategy for starting the race should state the best tire compound and the fuel level with which to start the race. The latter can be obtained by:

$$(t_1, B) = \operatorname*{argmin}_{(t,b) \in \mathcal{T} \times [0,\infty)} V^*_1\left(s_1\right), \tag{8}$$

where $s_1 = (t, 0, b, 0)$. Note that at the start of the race, the tires are all completely new (i.e., $w_1 = 0$), and the state variable m_1 is set to 0. For those cars that are required to start with a particular tire compound type[11], t_1 will take the value of this specific tire compound, and w_1 will take the value of possible tire wear if there is any. Still, it is of great interest to analyze the optimal strategies starting with the different tire compounds in order to analyze the relative performance among them, and how these adapt to uncertain events that might occur in the race (more of this will be discussed below). This dynamic program allows us to obtain the best race strategy to be used during the race, at least from an *a-priori* stand point, i.e. if there are no random events that might occur during the race. In other words, before the race starts, we can obtain the best pit stop race strategy with respect to the compounds of the tires to use at each lap. Note that if during the race some of these elements

[11] In Formula 1, only cars that start after the 10^{th} position are allowed to choose the compound of the tires with which to start the race since the year 2016, whereas the rest of the cars are required to re-use a specific tire (with a particular compound) used in the qualifying event. The qualifying is an event that happens the day before the race, when drivers try to make the best lap times as these will decide the starting grid positions.

evolve in a different way from what we assumed at the beginning, the dynamic program can be re-run in order to adapt to the race contingencies. This can be done whenever a contingent event can be captured by the state variables defined in Equation (1). For example, if, in one lap, a driver makes a maneuver that wears the tires more than usual (more than $\gamma(t_n, f_n)$ which stands for the usual tire degrade per lap under the tire compound t_n and fuel level f_n), then in the next lap, we can update the tire wear w_{n+1} to the actual value and re-run the dynamic programming from there. One of the key elements of the dynamic program is the *lap-time function* μ, which must be properly defined when solving this formulation.

2.1 Functions

2.1.1 Lap-Tire Wear

The *lap-tire wear* function, $\gamma(t_n, f_n)$ represents the tire degradation produced on a lap n given the current tires (t_n) and the fuel level (f_n). As mentioned before, we expect that the wear will be more pronounced for softer tire compounds as well as when the car has more fuel. In order to capture these effects, we consider the function of tire wear per lap as:

$$\gamma(t_n, f_n) = d_{t_n} \cdot (1+\delta)^{f_n/F}, \tag{9}$$

where $d_{soft} > d_{medium} > d_{hard}$ and $\delta > 0$. Note that γ is a product of two factors which represent the effects of the tire compound in use, and the fuel level. More precisely, d_t is a coefficient that represents the lap tire wear when using tires of compound t. Since tires of a harder compound last longer, it is natural to assume that $d_{soft} > d_{medium} > d_{hard}$. As for the second factor of the RHS of Equation (9), $\delta > 0$ is a parameter that represents the additional tire wear as the car has more fuel, while $F > 0$ is considered to be the fuel tank capacity. Then, $\delta = 0.2$ means that if the car has a full tank of fuel, its tire wear per lap will be 20% greater than when it has almost no fuel. On the contrary, $\delta = 0$ represents the case where the tire wear is independent of the fuel level. It can be seen that the tire wear per lap is increasing on the fuel level f_n.

2.1.2 Lap-time function

The time a car takes to drive a lap at any given lap n will be denoted by the following function:

$$\begin{aligned} &\mu\left(t_n, w_n, f_n, x_n\right) \\ &= \begin{cases} \mu_0 + p_0 \cdot \mathbb{1}_{\{x_n \neq 0\}} + \alpha_{t_n} + \beta(w_n) + g \cdot f_n & \begin{array}{l}\text{if } w_n + \gamma(t_n, f_n) < 1 \\ \quad \text{and } f_n - c_{t_n} > 0,\end{array} \\ +\infty & \text{otherwise.} \end{cases} \end{aligned} \tag{10}$$

The first case of the *lap-time function* described in Equation (10) occurs when the two following things hold: (i) the wear of the tires is strictly less than one (i.e. tires are not fully worn out), and (ii) the car has a sufficient amount of fuel to finish the current lap. If either of these conditions, (i) or (ii), do not hold, then we set the lap time to be infinity. The parameter μ_0 is a *baseline* (hypothetical) lap time if the car has a new set of soft compound tires, with almost no fuel, and makes no pit stop. The parameter p_0 accounts for the extra time a pit stop takes with respect to not stopping. Then, the second term of the first case of Equation (10) captures the time spent for a pit stop.

The parameter α_t for $t \in \mathcal{T}$ corresponds to the relative lap time between racing a lap with tires of compound t, versus using the fastest tire compound (for the same level of tire wear). For example, if the tires are *soft*, *medium*, or *hard*, we could have $\alpha_{soft} = 0$, $\alpha_{medium} = 0.6$, and $\alpha_{hard} = 1.0$. This means that lapping with the medium tire compound takes 0.6 seconds more than when using the soft one, and using the tires of a hard compound take a full second more than using those of a soft compound. Then, the third term of the first case of Equation (10) reflects the lap time difference caused by the tire compound in use. The function $\beta : [0, 1] \to [0, \infty)$ takes, as argument, the tire wear which ranges from zero to one and maps it to the extra time per lap relative to using a new set of tires (wear equal to zero represents a new set of tires, whereas wear equal to one means that the tires are completely worn out). Hence, it is natural to assume that β is an increasing function of the tire wear, and $\beta(0) = 0$. For more details see Appendix B. The parameter g (in Equation (10)) represents the extra time per lap for each kilogram of fuel, and the fifth term of Equation (10) corresponds to the additional lap time caused by the current fuel level. A summarized description of the dynamic programming formulation is provided in Appendix A.

2.2 Computations

In this subsection we solve an instance with the dynamic program we introduced above[12]. Consider a race that has 52 laps, and the tires that can be used are of *soft*, *medium*, and *hard* compounds, respectively. Each of these will be indexed by the numbers 1, 2, and 3, respectively. The rest of the parameters considered are: $\mu_0 = 85$ [seconds/lap], $p_0 = 21$ [seconds/pit stop], $g = 0.03$ [seconds/kg], $c = (c_1, c_2, c_3) = (1.92, 1.87, 1.83)$ [Kg/lap], $d = (d_1, d_2, d_3) = (1/25, 1/40, 1/65)$ [1/lap], $\alpha = (\alpha_1, \alpha_2, \alpha_3) = (0, 0.6, 0.9)$ [seconds/lap].

The top left panel of Figure 1 shows the lap times for the best strategies starting with each of the three (*soft*, *medium*, and *hard*) tire compounds. Tire changes (i.e. pit stops) are marked with circles with the initial of the corresponding tire compound currently installed on the car. It can be seen clearly that softer tires are faster than their harder counterparts. Indeed, from laps 1 to 17, the continuous line is below the others (the dashed and dotted lines).

[12] The code used to carry out these and all subsequent numerical experiments is available at *https://github.com/FCarrascoHeine/F1DP*.

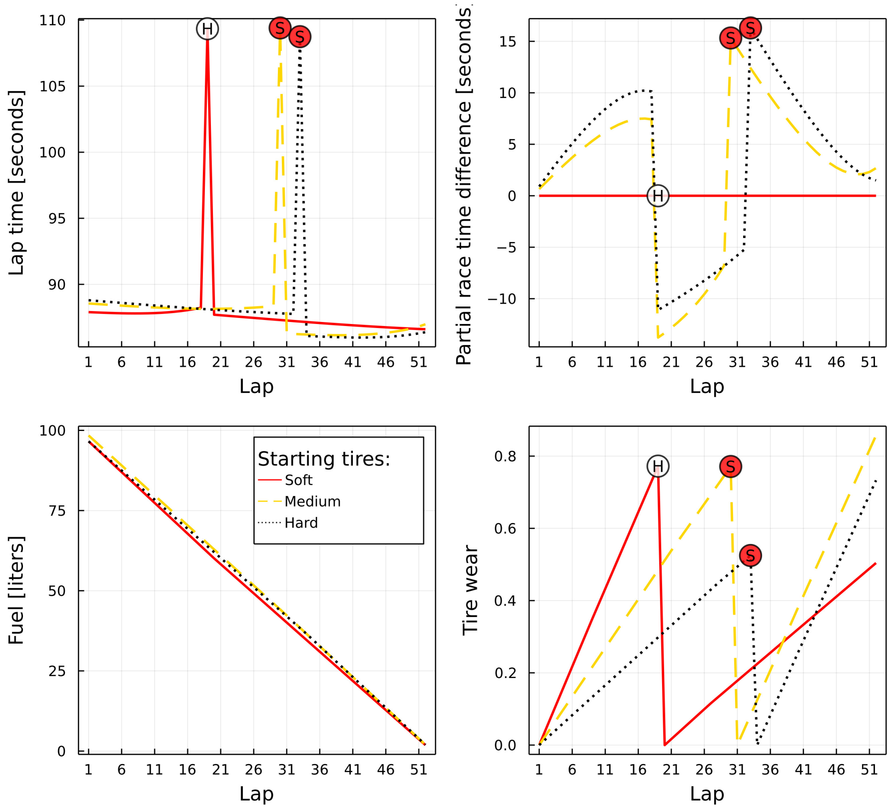

Fig. 1 Each initial tire compound's best strategy. **Top left:** Lap times. **Top right:** Partial race time difference with respect to the best partial time. **Bottom left:** Fuel level of each starting strategy. **Bottom right:** Tire wear during each lap. Filled red circles represent tire changes with the letters "S", "M", and "H" representing the soft, medium, and hard tire compounds, respectively.

Note that after lap 18 the strategy represented by the continuous line requires using tires of a hard compound, thus the lap times of this curve are slower than the other strategies once the latter change to the soft tire compound. The top right panel of Figure 1 shows the relative time difference of the race partial times for each starting tire compound strategy with respect to the strategy with the minimum race time, which in this case corresponds to starting with the soft tires. The steep increases are due to the pit stops made by the respective strategies, whereas the decreases (lap 19) are because of the pit stop made by the strategy starting with soft tires. The bottom left panel of Figure 1 shows the fuel level on each lap for each strategy. The slope of each of the curves of this plot represents the fuel consumption per lap, which is a function of the type of tire being used. The strategy starting with tires of medium compound starts the race with the largest amount of fuel. The reason for this is that this strategy uses tires of a medium and a soft compound (unlike the other two that use tires of a soft and hard tire compound). Thus, using more fuel is expected since the softer tires consume more fuel than the harder tires. As expected, all strategies end up with almost no fuel by the end of the race. Finally, the bottom right plot of Figure 1 shows the tire wear for each strategy. Note that whenever a pit stop occurs, the tire wear goes to zero. Also, the slope of each curve represents the rate of tire wear, which depends mainly on the compound of the tires being used. Tires of softer compounds have a higher slope, whereas tires of harder compounds have the lowest slope. Viewing Figure 1, in particular its top-right panel, we can conclude that the optimal, and therefore winning, strategy is to start with the soft tire compound and change to the hard one in lap 19. The best strategies starting with tires of medium and hard compounds take approximately 2.5 and 1.5 additional seconds in their respective race-times (see the top right panel of Figure 1).

2.3 Yellow Flags

Another example of an event that might happen during the race which would require re-solving the dynamic program is a *yellow flag*. These are usually caused by an incident such as an engine breakdown, or a car crash. During a yellow flag period, cars have to decrease their speed. As a result, making a pit stop during this time lapse is attractive because the rest of the cars are not racing at full speed, and so the relative time spent by stopping is less than having a pit stop during a regular lap (with no yellow flag). Although yellow flag slowdowns seem to be a convenient time to change tires, these events can not be anticipated with absolute certainty. Before solving an instance where a yellow flag occurs, we need to incorporate additional elements to the dynamic program formulation introduced in Section 2. More precisely, if, for example, there is a yellow flag during laps 21 and 22 in a race that has 52 total laps, the idea is to solve the dynamic program without considering yellow flag events, and use its optimal decisions until lap 20 (since up to this lap no yellow flag has been raised). However, at lap 21, we are aware of a yellow flag event that takes place during laps 21 and 22, and, therefore, we should solve an alternative dynamic program from lap 21, which includes the current conditions in the decisions to be made, from the current lap until the end of the race. More precisely, for each lap n, let us consider the parameter z_n which takes the value 1 if there is a yellow flag at lap n, and 0 otherwise. When there is a yellow flag, the fuel consumption is significantly less than when racing under normal conditions. Let $c_{t,a}$ be the fuel consumption per lap when using tires of compound t under a yellow flag ($a = 1$), or when there is none ($a = 0$). Naturally, $c_{t,1} < c_{t,0}$ for all $t \in \mathcal{T}$. Similarly, there is less tire wear during a yellow flag lap compared to a regular one. Let $d_{t,a}$ be the tire wear coefficient per lap of a tire of compound t, where a denotes whether or not there is a yellow flag (i.e., $a = 1$ if there is a yellow flag, and $a = 0$ if not). Then, the recursion equations for the tire wear and fuel level (given previously in

Equations (3) and (4)) are re-written as:

$$w_{n+1} = (w_n + \gamma'(t_n, f_n, z_n)) \cdot \mathbb{1}_{\{x_n=0\}} \tag{11}$$
$$f_{n+1} = \max\{f_n - c_{t_n,z_n}, 0\}, \tag{12}$$

where the *tire wear function* is now written as:

$$\gamma'(t_n, f_n, z_n) = d_{t_n,z_n} \cdot (1+\delta)^{f_n/F}, \tag{13}$$

which depends on the tire compound t_n, the fuel level f_n, and whether or not the lap is being raced during a yellow flag event, i.e., $z_n = 1$ or $z_n = 0$ respectively. Similar to the case of the *tire wear function*, the *lap-time function* has to include the fact that the car's speed is reduced during a yellow flag. We define p_1 as the additional lap time when doing a pit stop during a yellow flag.

In order to reduce notation, let us define the event $\Omega = \{w_n + \gamma'(t_n, f_n, z_n) < 1\}$ and $\{f_n - c_{t_n,z_n} > 0\}$, which is the case when the car's tires are not fully worn out, and there is still fuel left. Then, the *lap-time function* can be re-written as:

$$\begin{aligned}
&\mu'\left(t_n, w_n, f_n, x_n, z_n\right) \\
&\quad = \begin{cases} \mu_0 + p_0 \cdot \mathbb{1}_{\{x_n \neq 0\}} + \alpha_{t_n} + \beta(w_n) + g \cdot f_n & \text{if } z_n = 0 \text{ and } \Omega, \\ \mu_1 + p_1 \cdot \mathbb{1}_{\{x_n \neq 0\}} & \text{if } z_n = 1 \text{ and } \Omega, \\ +\infty & \text{otherwise,} \end{cases}
\end{aligned} \tag{14}$$

where μ_1 denotes the lap time under a yellow flag and no pit stop. The Bellman equation is defined as above in Equation (6), but using μ' instead of μ, with the same border condition shown in Equation (7). Let Z be the set of laps with yellow flags, and $DP(Z)$ be the solution of the dynamic program that considers these events. Note that $DP(\varnothing)$ is the same as the dynamic program described previously in Section 2. Following the example given above of a race with 52 laps in which yellow flags occur on laps 21 and 22 (which is not known *a priori* at the start of the race), we solve $DP(\varnothing)$ and $DP(\{21, 22\})$, and use the solution of $DP(\varnothing)$ for laps 1 to 20, and the solution of $DP(\{21, 22\})$ from laps 21 to 52. This is formalized in Algorithm 1 presented here.

Algorithm 1 Deterministic DP in presence of yellow flags

Input: Remaining laps with yellow flag $y \in \{0, 1, \ldots, L\}^N$**, Starting tire** t
Output: Decisions v_n**, lap-times** u_n**,** $\forall n \in \{1, \ldots, N\}$
Set $\left(\{x_k(s_k), \forall s_k \in \mathcal{S}_k\}_{k=1}^N, B\right) \leftarrow$ **solve** $DP(\varnothing)$**;** $s_1 \leftarrow (t, 0, B, 0)$**;** $n \leftarrow 1$
While $n \leq N$ **do**
  If $y_n > 0$ **And** $y_{n-1} \leq y_n$ **do**
    $Z \leftarrow \{i \in \{1, \ldots, N\} | y_i \geq 1, 1 \leq i - (n-1) \leq y_n\}$
    update $\{x_k(s_k), \forall s_k \in \mathcal{S}_k\}_{k=n}^N \leftarrow$ **solve** $DP(Z)$
  End
  $v_n \leftarrow x_n(s_n)$
  $u_n \leftarrow \mu'\left(t_n, w_n, f_n, x_n(s_n), y_n\right)$
  $s_{n+1} \leftarrow s_{n+1}(s_n, x_n(s_n))$
  $n \leftarrow n + 1$
End While

Algorithm 1 solves the problem for a fixed scenario of yellow flag occurrences. It receives the duration of the yellow flags at each lap, and the compound of the starting tires as input. Note that it is important to know not just if there is a yellow flag, but how long it will last. For example if $y = (0\ 2\ 1\ 2\ 1\ 0\ 0\ \dots)^T$, it means that there is a yellow flag in lap 2 that lasts for two laps, while there is also another yellow flag in lap 4 which also lasts for two laps. The output of Algorithm 1 is the decisions at each lap, and the resulting lap times. In Line 3, we obtain the optimal decisions for each state on every stage for the case in which there are no yellow flags. In Line 7 we check whether or not we are in the presence of a new yellow flag event. In the earlier example, where $y = (0\ 2\ 1\ 2\ 1\ 0\ 0\ \dots)^T$, this condition is satisfied in laps two and four. In such a case, we have new information and we, therefore, can solve the dynamic program from the current lap to the last lap while considering the yellow flag information. More precisely, the set Z contains those laps which will have yellow flags because of the current event. In the previous example, if we are on lap 2, then $Z = \{2, 3\}$; while if we are on lap 4, $Z = \{4, 5\}$. Lines 11 to 13 save the decisions, compute the lap time, and update the state. We will denote as DP the solution methods described in Algorithm 1.

We solve the same instance as described in Section 2.2 but introducing a yellow flag in some laps during the race[13]. More specifically, we illustrate the two following scenarios: (i) there is a yellow flag during laps 21 and 22, and (ii) there is a yellow flag during laps 31 and 32. Scenario (i) is depicted in Figure 2, while case (ii) is analyzed in Appendix D.

In (i), both the strategies that start with medium and hard-compound tires use the yellow flag period to make a pit stop during these laps. This is clearly beneficial as the extra time spent for the pit stop is only around 10 seconds whereas a pit stop during a regular lap (with no yellow flag) would result in a loss of more than 20 seconds. Contrarily, the starting strategy with tires of the softer compound does not make a stop since it had already stopped two laps before (in lap 19). The winning strategy in this case is the one starting with tires of the medium compound (see the top-right panel of Figure 2). In particular, once the yellow flag event occurs, this strategy is updated with respect to its original version not just on the timing of the pit stop, but also on the number of stops. Originally, in the case with no yellow flags, this strategy had planned to do a single pit stop during the race at lap 30 (see top-left panel of Figure 1). However, the yellow flag event in the middle of the race opens a chance to change tires at a low cost in terms of time. Then, a change of tires is performed in lap 21 (for the strategy starting with tires of the medium compound), before the lap in which it was supposed to take place (lap 30). Still, it is worth noticing that there is an additional pit stop considered in the race strategy (in lap 37). What is the advantage, then, of making a pit stop during the yellow flag slowdown if there is still a future pit stop to perform? The answer lies on the fact that the additional pit stop implies that the tires will

[13] We assume that the fuel consumption during a yellow flag is of 0.5 [Kg/lap] and that there is no tire wear (also during a yellow flag). In addition, we consider that the additional time lost to the pit stop in a lap with yellow flag is equal to $\mu_1 = 10$ [s].

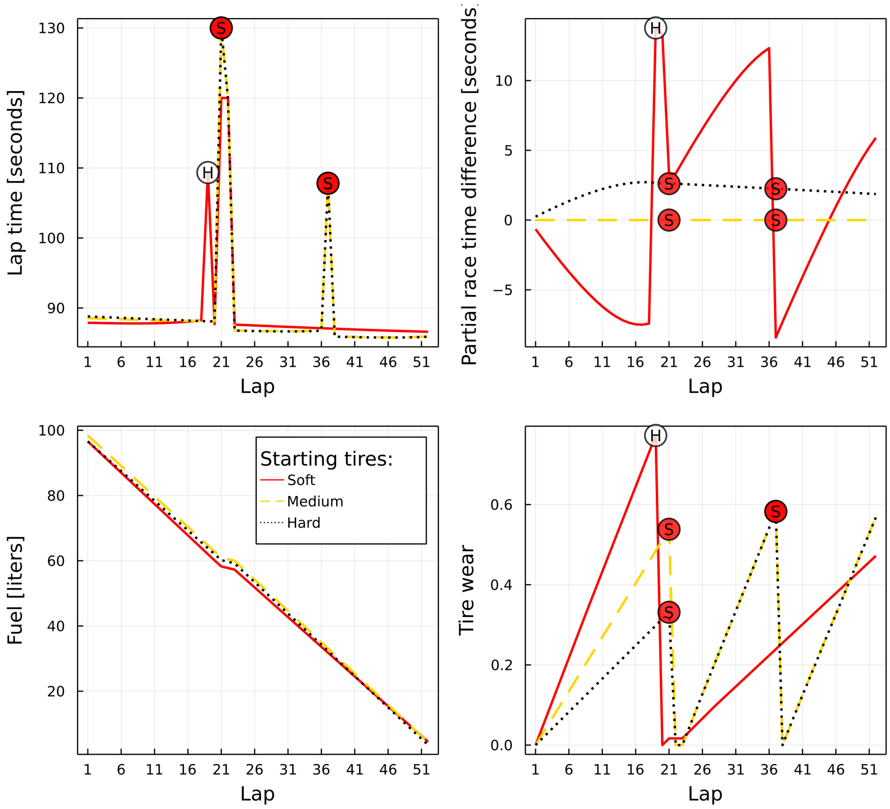

Fig. 2 Race with yellow flags in laps 21 and 22 with each initial tire compound's best strategy. **Top left:** Lap times. **Top right:** Partial race time difference with respect to the best partial time. **Bottom left:** Fuel level for each initial strategy. **Bottom right:** Tire wear for each lap. Filled circles represent tire changes, where the letters "S", "M", and "H" represent the soft, medium, and hard tire compounds, respectively.

be significantly less degraded overall compared to those of the other strategies (see bottom-right plot of Figure 2), offsetting the extra time due to a second stop.

As observed above, the occurrence of yellow flag events, and in particular their timing, has a significant impact on the pit stop strategies, and, therefore, on the total race time. In order to expand the analysis performed above, we consider a *sweep* over all possible yellow flag scenarios that last for two laps (with a single occurrence). Namely, we consider the event of having a yellow flag in laps 1 and 2, 2 and 3, 3 and 4, etc. The top panel of Figure 3 shows the total race times for each strategy starting with tires of each compound, for the different laps in which the yellow flag event takes place. For example, if the yellow flag slowdown takes place during laps 6 and 7, the starting strategies' total race times with soft, medium, and hard tire compounds are approximately of 4630, 4633, and 4632 seconds, respectively.

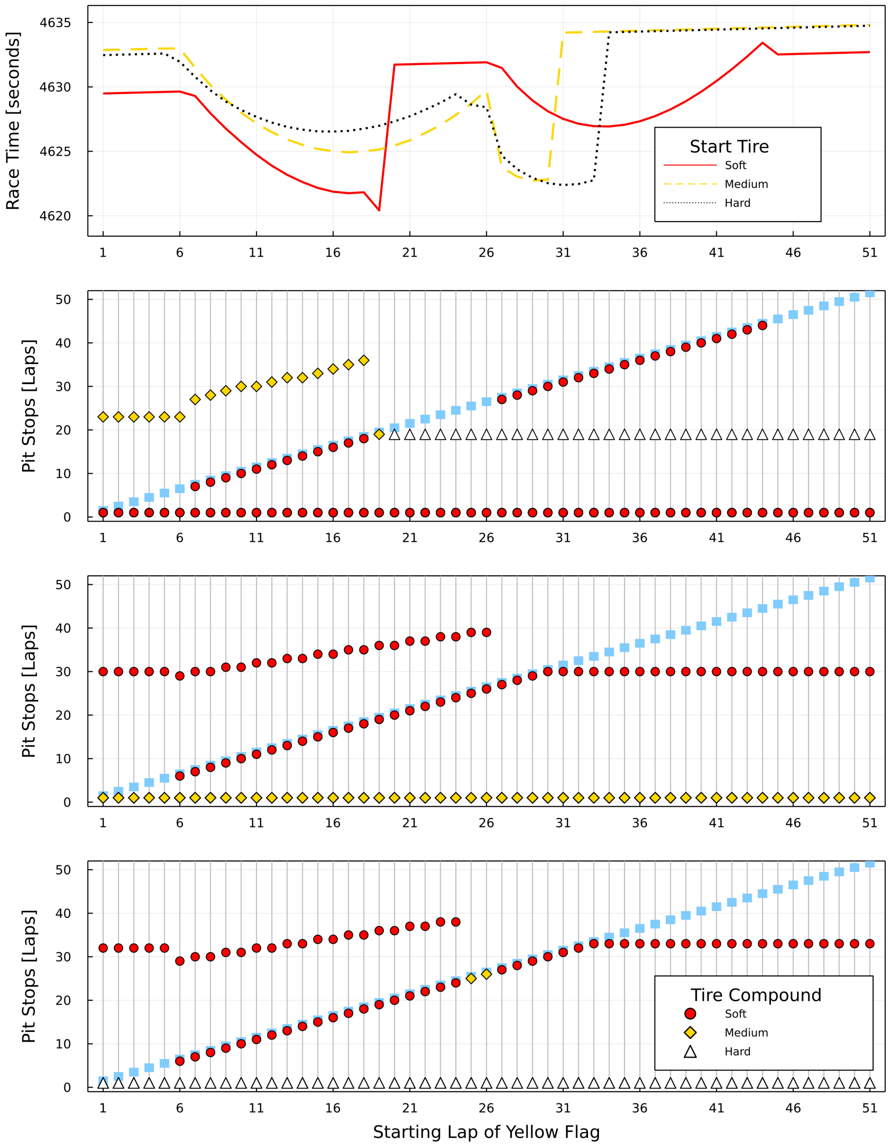

Fig. 3 Deterministic case. **First panel:** Race times starting with tires of different compounds, and yellow flag start in the x-axis. **Second, third, and fourth panels:** Pit stop starting with the soft, medium, and hard tire compounds, respectively. The blue squares in the diagonals highlight the laps in which the yellow flag is waved.

At the top panel of Figure 3 we can see that if the yellow flag event takes place before lap 20 or after lap 33, the winning strategy is the one starting with tires of the soft compound, while if the yellow flag occurs between laps 20 and 33, the winning strategy alternates between the medium and the hard tire compound. With all three strategies shown on the top panel of Figure 3, we can see that the race times follow a slightly increasing straight line during some specific lap time segments. More specifically, the race time of the strategy starting with the soft compound tires follows a straight line from laps 1 to 6, 20 to 26, and from laps 44 to 51. We see that when the yellow flag event occurs between laps 7 to 19, there is a significant reduction in the race time with respect to the "interpolated" race time values of the straight line. These race time reductions are a result of stopping during the yellow flag period, and adding a future pit stop if necessary, which is also the case for the strategies starting with the hard and medium compound tires. This can be observed on the three lower panels of Figure 3. These panels depict the particular laps at which the pit stops are made and the tires of other compounds changed in each case. The second, third, and fourth panels (from top to bottom) of Figure 3 show to the cases starting with the soft, medium, and hard compound tires respectively. We can observe from the second panel of Figure 3 that a pit stop is made during the yellow flag event whenever this takes place during laps 7 to 19. In particular, if the yellow flag event takes place between laps 7 to 18, then the dynamic program suggests using another set of soft tires and adding an additional pit stop at a future lap for changing to medium compound tires. Contrarily, if the yellow flag shows in between laps 20 to 26, it is not worth making an additional pit stop; instead, it is better to change to hard compound tires to use until the end of the race. The same analysis can be performed on the strategies starting with the medium and hard compound tires (see the bottom two panels of Figure 3).

The DP described is able to capture random events such as yellow flags in a myopic manner by updating this information as they unveil. However, the DP does not incorporate the probability of this uncertainty. In the following section we present a stochastic dynamic program that takes uncertain events into account that might happen during the race, and which are important to take into consideration in order to make the best race strategy decisions.

3 Stochastic Model

In this section, we generalize the presented model by introducing uncertain events that might have a considerable impact on the race strategy. In particular, we take into account the following two events: weather and yellow flags.

For each lap n, the state variable will capture the current information of the car (as in the deterministic model) plus: (i) the weather conditions of lap n, and (ii) whether or not lap n is raced under yellow flag. Close to the end of the lap, we need to decide whether to make a pit stop in order to change tires, or continue on the track. After this decision, at the very end of the lap, the uncertainty is unveiled with respect to the weather of the next lap, and the possible occurrence of a yellow flag, including how long it will last.

As for the weather, let us consider that $\mathcal{R}$ is the set of weather types; for example[14] $\mathcal{R} = \{\text{Dry}, \text{mild rain}, \text{heavy rain}\}$. Of course, the weather might not be constant throughout the whole race, and the changes and timing of these changes are not fully predictable. In most cases, we might have the probabilities of the weather during the following laps. We will assume that the weather of lap n depends on the weather of the previous lap, $n-1$. Let $R_n \in \mathcal{R}$ be the random variable that represents the weather during lap n. The transition probability from weather of type $i \in \mathcal{R}$ in lap $n-1$ to weather of type $j \in \mathcal{R}$ in lap n is denoted by $P_{ij}(n)$. Then, the matrix $P(n) \in \mathbb{R}^{|\mathcal{R}|\times|\mathcal{R}|}$ represents the weather transition matrix probabilities from lap $n-1$ to n, which resembles a markovian process. Also, note that this transition matrix depends on the lap, and can therefore be updated at each lap as new weather forecast information is revealed during the race.

With respect to yellow flags, on each lap there is a probability that a race event (which triggers a yellow flag) happens. In addition, the duration of a yellow flag, in terms of laps, can vary depending on the nature of the event. Let Y_n be the random variable that accounts for the number of remaining laps under yellow flag from lap n including this lap. For example, if $Y_4 = 2$, then laps 4 and 5 will take place under yellow flag; if $Y_4 = 0$ then lap 4 will be raced normally (with no yellow flag). We assume yellow flags can last at most L laps, so $Y_n \in \{0, 1, \cdots, L-1, L\}$. The probabilities of the occurrence of a yellow flag are represented by the following two cases: (i) given that lap n starts with no yellow flag, i.e. $Y_{n-1} \leq 1$, $q_{l,r}(n)$ denotes the probability that there will be a yellow flag which will last for l laps in lap[15] n given that the weather is of type $r \in \mathcal{R}$, thus $q_{l,r}(n) = \mathbb{P}(Y_n = l | Y_{n-1} \leq 1, R_n = r)$. Indeed, the likelihood of having a yellow flag event will depend on the weather of the current lap. For example, yellow flag events are more likely to occur during rainy weather rather than on a dry track. (ii), if there is an on-going yellow flag from the previous lap, $n-1$, with still more laps to go, i.e. $Y_{n-1} > 1$, then the current lap (n) will be on yellow flag slowdown, too, for one less lap. More precisely $Y_n = Y_{n-1} - 1$ if $Y_{n-1} > 1$. Note that in the first case, where $Y_{n-1} \leq 1$, the probability that a race event (causing a yellow flag) happens can depend on which lap we are in. For example, in the first laps of the race, the probability that a yellow flag event happens is higher than in other laps since the cars are packed together at the start, increasing the risk of contact among them.

The *decision* at each stage is the same as in the deterministic model, namely, whether or not to have a pit stop or not, and if so, what tire compound to change to. We denote this *decision* by $x_n \in \{0\} \cup \mathcal{T}$. Also, the set of tire compounds for wet weather will be denoted by $\mathcal{T}_w \subset \mathcal{T}$.

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[14] Of course we can consider a more refined enumeration of the possible weather states, but, for the sake of simplicity we will show only a reduced set of weather states.

[15] The l laps include lap n.

The *state* variable will have the information on the weather of the current lap, and the remaining number of laps under yellow flag, in addition to the elements considered for the deterministic model (tire compound, tire wear, fuel level, and the two-tire-compound indicator). The *state* variable, then, is denoted by the tuple

$$s_n = (t_n, w_n, f_n, m_n, r_n, y_n),$$

in which the first four components denote the same quantities as in the deterministic model, while the last two encode the information about the weather and yellow flag events. The component $r_n \in \mathcal{R}$ of the *state* variable has the weather information by lap n, while y_n denotes the number of remaining laps under yellow flag from lap n. During yellow flags events cars must reduce their speed by a considerable amount, and, as a consequence, the tire wear and the fuel consumption will differ from those when racing in a regular mode (i.e. no yellow flag). As in the deterministic setting, let $\gamma'(t, f, a)$ be the function that returns the tire wear as a function of the tire compound t, fuel-level f, and whether or not there is a yellow flag displayed ($a = 1$ if this is the case, otherwise $a = 0$; see Equation (13)). Similarly, we define the fuel consumption $c_{t,a}$ when using tire compound t and there is a yellow flag ($a = 1$), or when there is not ($a = 0$). Before presenting the state transition equation, it is important to mention that if the car uses any of the wet tire compounds, the condition of using at least two different tire compounds vanishes. Then, the transition of state variables can be expressed as:

$$\begin{align}
t_{n+1} &= t_n \cdot \mathbb{1}_{\{x_n=0\}} + x_n \cdot \mathbb{1}_{\{x_n \neq 0\}} \tag{15}\\
w_{n+1} &= (w_n + \gamma'(t_n, f_n, a)) \cdot \mathbb{1}_{\{x_n=0\}}, \quad \text{where } a = \mathbb{1}_{\{y_n \geq 1\}} \tag{16}\\
f_{n+1} &= \max\{f_n - c_{t_n,a}, 0\}, \quad \text{where } a = \mathbb{1}_{\{y_n \geq 1\}} \tag{17}\\
m_{n+1} &= \min\{m_n + \mathbb{1}_{\{x_n \neq 0, x_n \neq t_n\}} + \mathbb{1}_{\{x_n \in \mathcal{T}_w\}}, 1\} \tag{18}\\
r_{n+1} &= R_{n+1} \tag{19}\\
y_{n+1} &= Y_{n+1}. \tag{20}
\end{align}$$

Note that the *two-tire-compound-indicator* state variable (m_n) is forced to be 1 if the driver uses a wet tire compound.

The Bellman Equation in the stochastic case is given by:

$$\begin{aligned}
V_n(s_n, x_n) = \mu''\left(t_n, w_n, f_n, x_n, \mathbb{1}_{\{y_n \geq 1\}}, r_n\right)& \\
+ \mathbb{E}\left[V^*_{n+1}(s_{n+1}(s_n, x_n, R_{n+1}, Y_{n+1}))\right]&,
\end{aligned}$$

where

$$V^*_n(s_n) = \min_{x_n \in \{0\} \cup \mathcal{T}} V_n(s_n, x_n).$$

The border condition is given by:

$$V^*_{N+1}(s_{N+1}) = \begin{cases} +\infty & \text{if } m_{N+1} = 0, \\ 0 & \text{otherwise} \end{cases}$$

Moreover, μ'' is the lap time function, which can be expressed as:

$$\mu''\left(t_n, w_n, f_n, x_n, z_n, r_n\right) = \begin{cases} \mu_0 + p_0 \cdot \mathbb{1}_{\{x_n \neq 0\}} + \alpha_{t_n, r_n} + \beta(w_n) + g \cdot f_n & \text{if } z_n = 0 \text{ and } \Omega, \\ \mu_1 + p_1 \cdot \mathbb{1}_{\{x_n \neq 0\}} & \text{if } z_n = 1 \text{ and } \Omega, \\ +\infty & \text{otherwise,} \end{cases} \tag{21}$$

where α_{r_n,t_n} denotes the extra lap time of using tire compound t_n in weather of type r_n with respect to using the fastest tire compound and weather type.

As in the deterministic case, we seek to find the initial fuel level and the best tire compound with which to start the race, namely:

$$(t_1, B) = \underset{(t,b) \in \mathcal{T} \times [0,\infty)}{\operatorname{argmin}} \mathbb{E}\left[V_1^*\left(s_1\right)\right], \tag{22}$$

where $s_1 = (t, 0, B, 0, r_1, y_1)$, where $r_1 = R_1$ is the weather for the first lap, and $y_1 = Y_1$ is the number of laps with yellow flags from the first lap. We can assume that we know the state of the weather at a time before the race, R_0, and that there is no yellow flag before the start of the race; therefore $\mathbb{P}(Y_1 = l) = q_l(1)$. The resulting stochastic dynamic programming returns the best decision to be made for each state we could possibly have, under the particular probabilities on hand, $P(n)$ and $q(n)$. This stochastic deterministic program will be denoted as SDP in the rest of the paper. A summarized description of the SDP is provided in Appendix E.

3.1 Computations

We solve the SDP at the same instance described in Section 2.2, but including yellow flag events. We set the probability of yellow flags such that there is a 70% chance that at least one yellow flag event will occur during the race (see Appendix F for a more detailed description on how to compute the probability of having a yellow flag in a lap). In reality, there are Formula 1 races which have high chances of having a yellow flag, such as the Monaco or Singapore Grand Prix, whereas there are others in which this probability is less likely. For simplicity, we assume that yellow flags last for two laps. As in the deterministic setting, we evaluate the performance and decisions of the SDP for each case in which there is a yellow flag, which lasts for two laps, starting at each lap of the race. Details on the running times and hardware are provided in Appendix G.

The top panel of Figure 4 shows the race times, while the bottom three panels show the decisions for different starting tire strategies. The results are similar to the ones observed when doing the same exercise but using the solution of Algorithm 1 (i.e. the DP). However, we can see that there are some cases in which the solution of the SDP extends the use of tires, weighing the odds that a yellow flag will occur (making a pit stop if this is the case), and the extra time due to the delay of the stop. For example, when starting with the hard compound tire, the solution of the SDP indicates that if by lap 34

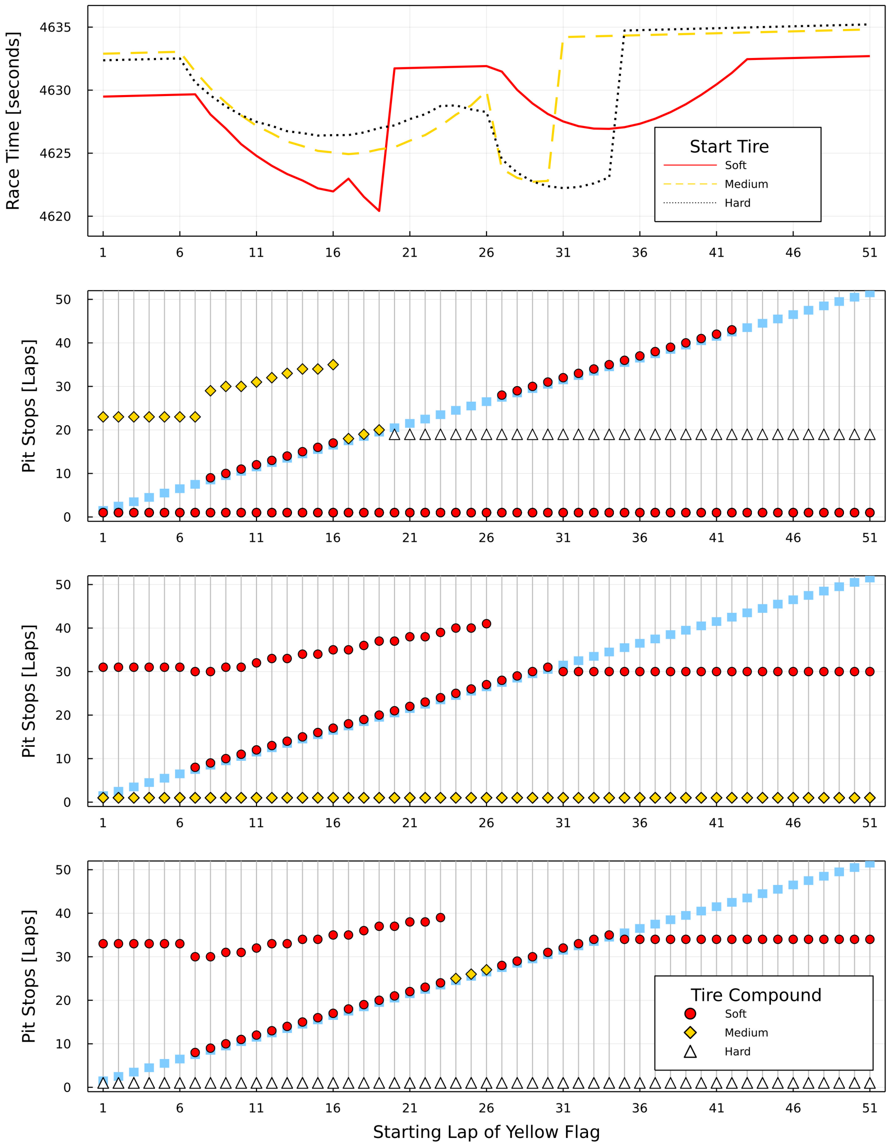

Fig. 4 Stochastic case. **First panel:** Race Times for different starting tire compounds, with the start of the yellow flag being shown in the x-axis. **Second, third, and fourth panels:** Pit stop starting with the soft, medium, and hard compound tires, respectively. The blue squares in the diagonals highlight the laps in which the yellow flag is waved.

there has been no yellow flag, then a stop should be made. However, when using the deterministic solution, this threshold becomes 33, and therefore, if the first yellow flag occurs in lap 34, the SDP solution will be leveraged from there.

We also compare the SDP solution and the DP on various sampled scenarios. In particular, once we solve the DP, we evaluate these solutions on 1000 sampled scenarios of yellow flag occurrences during the race, which are generated with the same probabilities with which we solve the SDP.

		RT diff. SDP vs DP				Freq. Min. RT			Avg. RT diff. if		
Tire	Y.F.	Avg	Sd	Min	Max	SDP	DP	Even	SDP Wins	DP Wins	Fuel diff.
	0	0.00	0.00	0.00	0.00	0	0	314			
	1	0.01	0.25	-0.94	1.23	22	72	273	-0.64	0.25	
Soft	2	-0.22	1.91	-11.00	9.82	27	57	132	-3.12	0.63	0.00
	3	-1.05	3.55	-16.29	8.56	16	18	37	-5.69	0.91	
	4	-0.37	1.17	-3.70	1.32	6	5	13	-1.93	0.52	
	0	0.00	0.00	0.00	0.00	0	0	314			
	1	0.03	0.05	0.00	0.22	0	125	242		0.08	
Medium	2	-0.12	1.53	-11.00	0.89	6	75	135	-8.01	0.29	0.00
	3	-0.37	2.02	-11.00	0.82	6	19	46	-5.72	0.43	
	4	-1.13	3.21	-11.00	1.05	4	2	18	-7.16	0.80	
	0	0.18	0.00	0.18	0.18	0	314	0		0.18	
	1	-0.11	1.57	-11.15	0.57	210	157	0	-0.50	0.43	
Hard	2	-0.34	2.11	-11.15	4.94	146	70	0	-0.74	0.49	-0.10
	3	-0.46	2.13	-11.14	3.95	53	18	0	-0.85	0.66	
	4	-1.12	3.12	-11.13	0.42	23	1	0	-1.19	0.42	

Table 1 Comparison of the SDP and DP Race Times (RT) over sampled scenarios.

The columns of Table 1 represent: (1) the starting tire compounds, (2) the number of yellow flags waved during the race, (3) - (6) the statistics of the race times (RT) of the SDP minus the RT obtained from the solution of Algorithm 1, (7) - (9) the frequency of cases that result in the lower RT, (10) - (11) the average RT difference conditioned to the cases where either SDP or DP has the lowest RT, and (12) the difference between the initial fuel load used by the stochastic and the deterministic strategies. We see in the third column of Table 1 that in almost all instances the stochastic solution results in lower race times compared to the deterministic solution. In the cases when there are no yellow flags (1^{st}, 6^{th}, and 11^{th} rows of Table 1), we observe that the deterministic solution equals or outperforms the stochastic one. This is expected since if there are no yellow flags, the assumptions made by the problem solved by the deterministic DP from lap 1 match exactly with what occurs. We observe that when starting with a soft or medium tire compound, the deterministic and stochastic solution approaches (i.e., DP and SDP respectively) start the race with the same fuel level, and their race times coincide when there are no yellow flags. In the case of having exactly one yellow flag, the DP outperforms the SDP by a narrow margin on some instances. This is because of the pit-stop postponing effect of the SDP strategy (see in the second and third panels of

Figures 3 and 4 how the second pit stops are delayed in the SDP with respect to the analogous cases of the DP). As for the cases in which there are two or more yellow flags, (also in the case when starting with soft and medium tires), the SDP wins in less scenarios than the DP (as shown by Columns 7, 8, and 9 of Table 1); nonetheless, the average race time difference is less for the SDP solution than the DP. When starting the race with the hard tire compound, the SDP starts the race with slightly less fuel than the DP due to the delaying of the first pit stop in the SDP solution. Thus, if there are no yellow flags, the DP will win by a small margin. However, in the case in which there is one or more yellow flags, the pit-stop postponing of the SDP outperforms, on average, the DP solution.

So far, we have focused mainly on a specific instance in which the chances of having a yellow flag during the race are 70%. Nonetheless, the impact of this, and that of other parameters, on the results should not be ignored. Consequently, we solve the SDP for the instance shown in Figure 4 and Table 1, but considering different values for the probability, denoted as π, of having a yellow flag (for the stochastic model) in the set $\mathcal{P} = \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.95\}$, we solve our stochastic model. With each of these values, we additionally simulate 1000 scenarios in order to test the performance of the DP and SDP. Then, for every $\pi \in \mathcal{P}$, the results obtained for the simulated scenarios allow us to compute an expected race time for both the SDP and the DP solutions.

In Figure 5, we can see the difference between expected race times of the SDP and the DP, for every $\pi \in \mathcal{P}$, depending on the type of tires that was chosen with which to start the race. A negative difference on the y-axis of Figure 5 means that the SDP results in a faster race time than the deterministic model on average. As one would expect, the higher the yellow-flag probability is, the larger the expected advantage of the stochastic model becomes. Indeed, the SDP anticipates potential yellow flags reducing the expected race time compared to the solution obtained with Algorithm 1

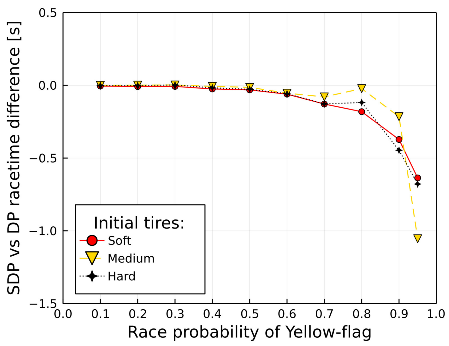

Fig. 5 SDP vs. DP Race-time differences, depending on the initial tire compound and yellow-flag probability.

4 Conclusions

We have presented two dynamic programming models to address the pit-stop strategy decision-making process in a Formula 1 race: a deterministic and a stochastic model. While both of them are used to study the impact of pit stops and tire-compound choices on race times, the stochastic formulation extends the deterministic one by including the probabilities of uncertainty, such as potential yellow-flag events, and weather events, such as rain.

Not only are we able to solve instances for both models to optimality, but we also present an algorithm that allows us to deal with the uncertainty of yellow flags by solving the deterministic dynamic programming in a non-anticipatory way. This allows us to compare the performance of both approaches in simulated scenarios. With the DP, we observed that the race performance depends not only on the starting tire compound, but also on the particular scenarios that will be revealed during the race. The DP allows us to answer questions such as “If there is a yellow flag in the current lap, is it worth making a pit stop?”, and “If yes, to tires of which compound should we change?”. The SDP, unlike the deterministic DP, tends to delay pit stops weighing the odds of a yellow flag event from which to benefit. As expected, we observed that the SDP outperforms the deterministic model when yellow flags are more likely to occur.

It is worth noticing that the models presented are not bounded to Formula 1 racing. On the contrary, these could easily be extended to other motor-racing competitions. Also, the particular functions used in this work to model tire wear, fuel consumption, or lap times, can be generalized to any other functional forms.

It is worth mentioning the limitations of the presented model. Probably the most relevant factor that is not considered is the competition with other drivers. The on-track interactions between drivers, such as blocking or overtaking, have an impact on lap times. The main challenge of including competition would be to model the game theory aspect of the problem. Another shortcoming of the model is the simplification of yellow flags which are only considered as VSC that take place between the start and end of laps. Thus, we believe there are many further research directions of the problem to be addressed in the future.

Acknowledgements The authors gratefully acknowledge financial support from CONICYT PIA/BASAL AFB180003. This research was partially supported by the supercomputing infrastructure of the NLHPC (ECM-02). The authors additionally acknowledge the support from the Research Centre for Operations Management, KU Leuven.

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Appendices

A Deterministic Dynamic Programming Model

- **Parameters and functions:**
  N: number of laps.
  $\mathcal{T}$: set of tire compounds.
  c_t: fuel usage per lap when using tires of compound $t \in \mathcal{T}$.
  $\mu(t, w, f, x)$: function that returns the lap time as a function of the tire compound $t \in \mathcal{T}$ in use, the tire wear $w \in [0, 1]$, the fuel level f, and whether a pit stop has been made ($x \in \mathcal{T}$) or not ($x = 0$).
- **Stages:**
  Lap $n \in \{1, \dots, N\}$
- **Decision (Control) Variables:**
  $x_n \in \{0\} \cup \mathcal{T}$: Tire compound chosen to be changed at the end of lap n. If there is no pit stop (and, thus, no change of tires), $x_n = 0$.
  B: Initial fuel level.
- **State Variables:**
  $s_n = (t_n, w_n, f_n, m_n)$: state at lap n.
  t_n: Tire compound used during lap n.
  w_n: Tire wear at the beginning of lap n.
  f_n: Fuel level at the beginning of lap n.
  m_n: Equals 0 if the car has used only one type of tire compound until the beginning of lap n. Otherwise $m_n = 1$ if the car has already used more than one type of tire.
- **State Transition:**
  $t_{n+1} = t_n \cdot \mathbb{1}_{\{x_n=0\}} + x_n \cdot \mathbb{1}_{\{x_n \neq 0\}}$
  $w_{n+1} = (w_n + \gamma(t_n, f_n)) \cdot \mathbb{1}_{\{x_n=0\}}$
  $f_{n+1} = \max\{f_n - c_{t_n}, 0\}$
  $m_{n+1} = \min\{m_n + \mathbb{1}_{\{x_n \neq 0, x_n \neq t_n\}}, 1\}$
- **Bellman Equation:**
  $$V_n(s_n, x_n) = \mu\left(t_n, w_n, f_n, x_n\right) + V^*_{n+1}(s_{n+1}(s_n, x_n))$$
  where
  $$V^*_n(s_n) = \min_{x_n \in \{0\} \cup \mathcal{T}} V_n(s_n, x_n)$$
- **Border Conditions:**
  $$V^*_{N+1}(s_{N+1}) = \begin{cases} +\infty & m_{N+1} = 0 \\ 0 & \text{otherwise} \end{cases}$$

B Lap time vs tire wear

The precise way we build the β function is by considering a finite set of points, i.e. *tire wear-extra time* tuples, and interpolate a cubic spline forced with null second derivatives at the extreme points of the $[0, 1]$. Figure 6 depicts an example of the β function in which the filled marker points are the input tuples, and the line corresponds to the interpolated values. According to Figure 6, if the wear of the set of tires is at 70%, then the car will take 1.2 seconds more to do a lap with respect to having a new set of tires (of the same type). Note that the function β grows slowly in the first part of the unit interval, but the slope becomes steeper as the tires become more worn (see the right side of Figure 6). In the latter case, lap times have a significant increase, also known as *falling from the cliff*. Note that the β function is quite flexible, since it allows the user to pick any arbitrary set of (*tire wear, extra time*)-tuples from which the function can be generated.

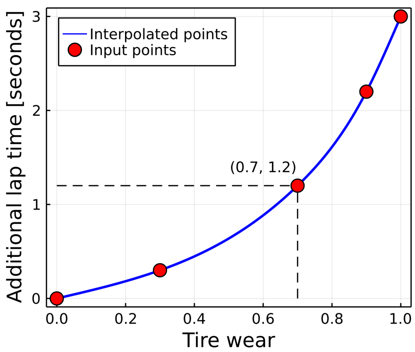

Fig. 6 Additional lap time for each level of tire wear. The filled red dots are input time-wear tuples, while the continuous blue line shows the interpolated points using a cubic spline.

C Lap Segments

A lap is depicted in Figure 7, where points “A” and “B” represent the exit from and entrance to the pits, respectively.

D Case (ii)

Figure 8 illustrates the case in which yellow flags occur in laps 31 and 32. In this case, we can see that both the strategies benefiting from this are the ones starting with soft and hard tires. However, in the former strategy, there had been a stop during lap 19 already, whereas in the latter strategy just a single stop takes place. As a result, the strategy starting with hard tires ends up being the winning strategy, (see top-right panel of Figure 8). Note that the strategy starting with the medium tires does not stop during the yellow

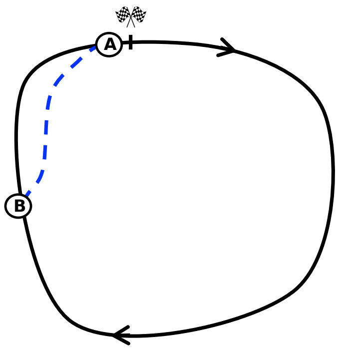

Fig. 7 Illustration of a lap, with the actual track being represented by the continuous line, while the pit lane is represented by the dashed line.

flag since it had made a pit stop just one lap earlier, changing to a soft tire compound. Thus, making a stop during the yellow flag is not convenient.

E Stochastic Dynamic Programming Model

- **Parameters and functions:**
  N: number of laps.
  $\mathcal{T}$: set of tire compounds.
  $\mathcal{T}_w$: set of wet tire compounds.
  $c_{t,a}$: fuel usage per lap when using tires of compound $t \in \mathcal{T}$, given that there is no yellow flag ($a = 0$), or if there is a yellow flag, $a = 1$.
  $\mu''(t, w, f, x, a, r)$: function that returns the lap time as a function of the tire compound $t \in \mathcal{T}$ in use, the tire wear $w \in [0, 1]$, the fuel level f, the existence of a pit stop during the lap ($x \in \mathcal{T}$), or not ($x = 0$), the presence of a yellow flag ($a = 1$ if there is one, and $a = 0$ otherwise), and under the weather condition $r \in \mathcal{R}$.
- **Stages:**
  Lap $n \in \{1, \ldots, N\}$
- **Decision (Control) Variables:**
  $x_n \in \{0\} \cup \mathcal{T}$: Tire compound chosen to change to at the end of lap n. If there is no pit stop (and, thus, no change of tires), $x_n = 0$.
  B: Initial fuel level.
- **State Variables:**
  $s_n = (t_n, w_n, f_n, m_n, r_n, y_n)$: state at lap n.
  t_n: Tire compound used during lap n.
  w_n: Tire wear at the beginning of lap n.
  f_n: Fuel level at the beginning of lap n.
  m_n: Equals 0 if the car has used only one type of tire compound until the beginning of lap n (and no rainy weather has occurred). Otherwise $m_n = 1$

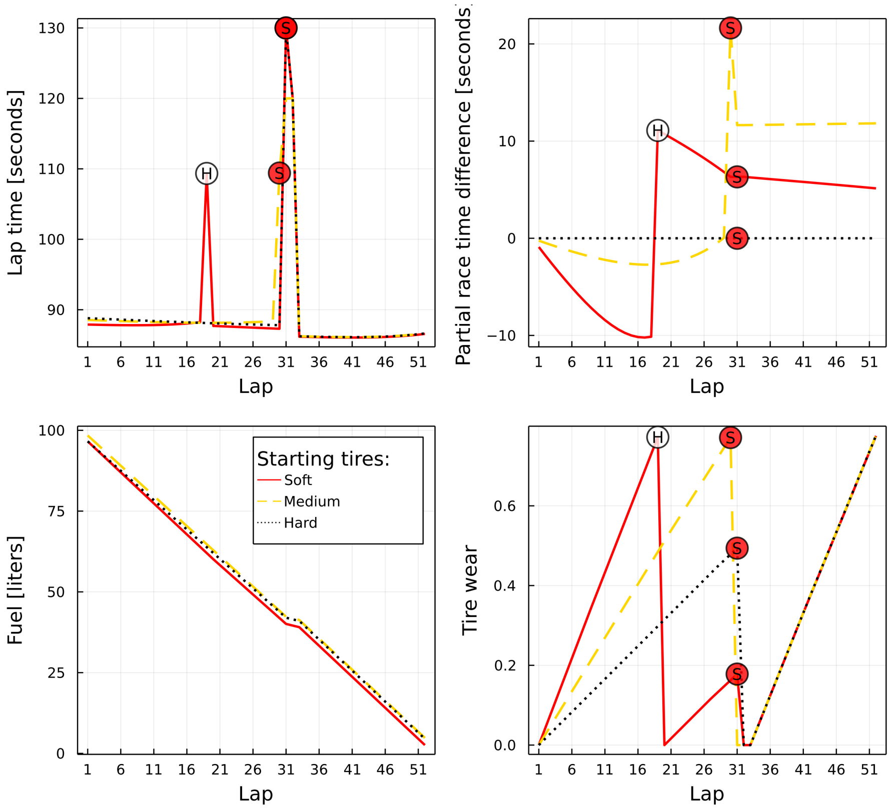

Fig. 8 Race with yellow flags in laps 31 and 32. For each initial tire compound's best strategy: **Top left** Lap times. **Top right** Partial race time difference with respect to the best partial time. **Bottom left** Fuel level for each initial strategy. **Bottom right** Tire wear for each lap. Filled circles represent tire changes where the letter "S", "M", and "H" represent the soft, medium, and hard tire compound respectively.

if the car has used more than one type of tire compound (or rainy weather has occurred).
r_n: Weather during lap n, where $r_n \in \mathcal{R}$.
y_n: Number of laps left under yellow flag. $y_n \in \{0, 1, \ldots, L\}$, where L is the maximum number of laps a yellow flag can last (due to the same event).

– **Random Variables:**
R_n: Weather during lap n, $R_n \in \mathcal{R}$. The weather transition is given by $P_{ij}(n)$ so that

$$P_{ij}(n) = \mathbb{P}(R_n = j | R_{n-1} = i)$$

where $i, j \in \mathcal{R}$.

Y_n: Number of laps with yellow flag remaining, from lap n, so that the probability of a yellow flag of length l laps is defined as

$$q_l(n) = \mathbb{P}(Y_n = l | Y_{n-1} \leq 1),$$

where $l \in \{0, 1, \ldots, L\}$, and n is the lap, and

$$Y_n = Y_{n-1} - 1$$

for the cases where $Y_{n-1} > 1$.

– **State Transition:**
$t_{n+1} = t_n \cdot \mathbb{1}_{\{x_n=0\}} + x_n \cdot \mathbb{1}_{\{x_n \neq 0\}}$
$w_{n+1} = (w_n + \gamma'(t_n, f_n, a)) \cdot \mathbb{1}_{\{x_n=0\}}$, where $a = \mathbb{1}_{\{y_n \geq 1\}}$
$f_{n+1} = \max\{f_n - c_{t_n,a}, 0\}$, where $a = \mathbb{1}_{\{y_n \geq 1\}}$
$m_{n+1} = \min\{m_n + \mathbb{1}_{\{x_n \neq 0, x_n \neq t_n\}} + \mathbb{1}_{\{x_n \in \mathcal{T}_w\}}, 1\}$, where $\mathcal{T}_w \subset \mathcal{T}$ is the set of wet tire compounds, suspending the condition of needing to use at least two tire compounds.
$r_{n+1} = R_{n+1}$
$y_{n+1} = Y_{n+1}$

– **Bellman Equation:**

$$\begin{aligned} V_n(s_n, x_n) = \mu'' &\left(t_n, w_n, f_n, x_n, \mathbb{1}_{\{y_n \geq 1\}}, r_n\right) \\ &+ \mathbb{E}\left[V^*_{n+1}(s_{n+1}(s_n, x_n, R_{n+1}, Y_{n+1}))\right], \end{aligned}$$

where

$$V^*_n(s_n) = \min_{x_n \in \{0\} \cup \mathcal{T}} V_n(s_n, x_n)$$

– **Border Conditions:**

$$V^*_{N+1}(s_{N+1}) = \begin{cases} +\infty & \text{if } m_{N+1} = 0 \\ 0 & \text{otherwise} \end{cases}$$

F Probability of Yellow Flag

Let π denote the probability that at least one yellow flag occurs during the race, and let ϕ denote the probability that a yellow flag occurs on a lap. We assume that this probability is independent of the specific lap. Then, it must hold that $1 - \pi = (1 - \phi)^N$, where N is the total number of laps. Thus, we have that $\phi = 1 - (1 - \pi)^{1/N}$.

G Running times

The algorithms were run in Julia 1.7.2, using 6-CPU of an Apple M1 Pro with 16 GB RAM. Table 2 shows running-time statistics for the DP and SDP and average number of states for each stage (lap). The number of runs for the DP are 53 since there is one run for the case with no yellow flags, and one run for each lap in which a yellow flag starts (and lasts for two laps). The runs of the SDP are the ones considering different probability of yellow flag during the race. In order to have a finite number of states, we discretize the fuel and tire wear. In particular, the instances run use a grid refinement with a width of $1/1995(\approx 0.0010)$ and $1.01 \cdot 1.92 \cdot 966/52(\approx 0.0505)$ for tire wear and fuel, respectively.

	Time [s]					
Case	Average	Std. Dev.	Maximum	Minimum	Runs	Avg. States
DP	120.72	5.94	140.81	112.3	53	2107242
SDP	635.58	12.67	657.76	616.22	10	9109140

Table 2 Running times