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A General Model for Matroids and the Greedy Algorithm

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Abstract

We present a general model for set systems to be independence families with respect to set families which determine classes of proper weight functions on a ground set. Within this model, matroids arise from a natural subclass and can be characterized by the optimality of the greedy algorithm. This model includes and extends many of the models for generalized matroid-type greedy algorithms proposed in the literature and, in particular, integral polymatroids. We discuss the relationship between these general matroids and classical matroids and provide a Dilworth embedding that allows us to represent matroids with underlying partial order structures within classical matroids. Whether a similar representation is possible for matroids on convex geometries is an open question.

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1 Introduction

A fundamental feature of combinatorial optimization is the fact that the optimality of the greedy algorithm with respect to arbitrary linear functions on finite independence systems is equivalent to the so-called Steinitz augmentation property and hence to the system giving rise to a matroid (*cf.* [7, 18, 28]). Yet, many generalized models for the greedy algorithm have been established and proved useful. Most notably among them are Edmonds' [6] integral polymatroids that apply Hoffman's [19] linear programming approach to matroids. The polymatroid model in turn was further extended by allowing certain partial orders on the ground set in question (see, *e.g.*, [12, 13, 15, 21, 23]).

On the other hand, already Dunstan, Ingleton, and Welsh [4] had introduced so-called supermatroids as a more general model for matroids. Extending distributive supermatroids, the combinatorial geometries on partially ordered sets of [9] yield a matroid type model for finite semimodular lattices. They furthermore admit an extension of the model for the greedy algorithm, which includes the polymatroid greedy algorithm (*cf.* [8]). Recently, so-called cg-matroids [17] have received attention. They generalize distributive supermatroids in that they allow a convex geometry as the underlying structure, whose lattice of closed sets is not necessarily distributive. For so-called strict cg-matroids a greedy algorithm exists and generalizes the matroid greedy algorithm (*cf.* [25]).

All those matroid generalizations have in common that they admit restrictions on the ground set while focusing on certain subclasses of linear functions that are found to be feasible for optimization by the accompanying greedy algorithms. The purpose of the present note is to provide a very general model for matroid independence systems that allow greedy optimization relative to certain classes of linear functions.

The key to the notion of a "matroid" in our approach is the link between a family $\mathcal{H}$ of subsets that includes the level sets of the linear functions to be optimized and a family $\mathcal{F}$ of feasible solutions. We first introduce our independence model and then exhibit the role the matroids play in this model with respect to greedy optimization. The striking feature of optimization under submodular constraints is Monge's [22] observation that optimal solutions have the structure of chains (see also, *e.g.*, [6, 12, 15]). We formalize this aspect in the notion of the *base chain property* of generalized matroids. The base chain property then yields a convenient framework for matroid duality, which we sketch in Section 4. We then discuss matroids that are defined with respect to closure and co-closure systems $\mathcal{H}$. Important classes of such matroids, which include matroids on convex geometries, turn out to admit descriptions in terms of their associated rank functions. We finally investigate the relationship between $\mathcal{H}$-matroids and classical matroids. We provide the construction of a Dilworth embedding for matroids on distributive lattices of closed sets, which allows us to reduce the theory of such $\mathcal{H}$-matroids to classical matroid theory (which for integral polymatroids was already pointed out in [11]).

Whether general matroids on convex geometries allow a similar embedding into classical matroids is an open problem.

2 Independence Systems

We assume throughout to be given a finite ground set E.

2.1 Constructible Families and Rank Functions

Let $\mathcal{F}$ be a non-empty family of subsets of E. $\mathcal{F}$ is called *constructible* if for all $F \in \mathcal{F}$

(C) either $F = \emptyset$ or $F \setminus e \in \mathcal{F}$ for some $e \in F$.

Note that (C) implies $\emptyset \in \mathcal{F}$. For any $F \in \mathcal{F}$, we set

$$\Gamma(F) = \{e \in E \setminus F \mid F \cup e \in \mathcal{F}\}$$

and call F a *basis* of $\mathcal{F}$ if $\Gamma(F) = \emptyset$. So the bases of $\mathcal{F}$ are exactly the (inclusion-wise) maximal members of $\mathcal{F}$. Denote by $\mathcal{B} = \mathcal{B}(\mathcal{F})$ the collection of bases.

$\mathcal{F}$ induces a *basis rank function* ρ on the collection of subsets of E *via*

$$\rho(S) = \max_{B \in \mathcal{B}} |S \cap B| = \max_{F \in \mathcal{F}} |S \cap F|.$$

Note that ρ is *normalized* (*i.e.*, $\rho(\emptyset) = 0$), and enjoys the *unit-increase property*

(UI) $\rho(S) \leq \rho(T) \leq \rho(S) + |T \setminus S|$ for all $S \subseteq T \subseteq E$.

The *restriction* of $\mathcal{F}$ to a subset $S \subseteq E$ is the family

$$\mathcal{F}(S) = \{F \in \mathcal{F} \mid F \subseteq S\}.$$

Clearly, every restriction of a constructible family is constructible. Note, however, that the basis rank function ρ_S of $\mathcal{F}(S)$ may differ from ρ. In general, one has the dominance relation

$$\rho_S(X) \leq \rho(X) \quad \text{for all } X \subseteq E.$$

2.2 $\mathcal{H}$-Independence

Let $\mathcal{H}$ be a family of subsets with $\emptyset, E \in \mathcal{H}$. Then the constructible family $\mathcal{F}$ is said to form an *independence system relative to* $\mathcal{H}$ (or an $\mathcal{H}$*-independence system*) if

(I) for all $H \in \mathcal{H}$, there exists some $F \in \mathcal{F}(H)$ such that $|F| = \rho(H)$.

In other words, $\mathcal{F}$ is an $\mathcal{H}$-independence system if and only if

$$\rho(H) = \rho_H(H) \quad \text{for all } H \in \mathcal{H}.$$

If $\mathcal{H} = 2^E$, we refer to a 2^E-independence system $\mathcal{F}$ simply as an "independence system" (or *simplicial complex*). Independence systems $\mathcal{F}$ are characterized by the fact that $F \in \mathcal{F}$ implies $F' \in \mathcal{F}$ for all subsets $F' \subseteq F$.

2.2.1 The Intersection Property

We say the $\mathcal{F}$ has the *intersection property with respect to* $\mathcal{H}$ (or the $\mathcal{H}$*-intersection property*) if

(IP) $F \cap H \in \mathcal{F}$ for all $F \in \mathcal{F}, H \in \mathcal{H}$.

It follows immediately from the definition, that (IP) implies (I), *i.e.*, every constructible family with the $\mathcal{H}$-intersection property is in particular an $\mathcal{H}$-independence system. Note that every simplicial complex has the intersection property. We illustrate the concept with an independence system of integral vectors. For a positive integer N we identify N with the set $\{1, \ldots, N\}$ in the following.

Example 2.1 *Let the integer-valued function f be defined on the collection of subsets of the set N and define*

$$\mathbb{P}(f) = \{\mathbf{x} \in \mathbb{N}^N \mid \mathbf{x}(S) = \sum_{i \in S} x_i \leq f(S), S \subseteq N\},$$

where $\mathbf{x} = (x_i \mid i = 1, \ldots, N)$. *Assuming that* f *is bounded by* $m \in \mathbb{N}$, *consider* N *pairwise disjoint sets* $K_i = \{e_{i1}, \ldots, e_{im}\}$ $(i = 1, \ldots, N)$ *and set* $E = K_1 \cup \cdots \cup K_N$. *Every* $\mathbf{x} \in \mathbb{P}(f)$ *determines a subset*

$$\mathrm{id}(x_1, \ldots, x_N) = \bigcup_{i=1}^{N} \{e_{ij} \in K_i \mid j \leq x_i\} \subseteq E.$$

The system $\mathcal{D} = \{\mathrm{id}(x_1, \ldots, x_N) \mid x_i \in \mathbb{N}, x_i \leq m\}$ *is closed under intersections and unions with corresponding vector operations:*

$$\begin{array}{rcl} \mathrm{id}(\mathbf{x}) \cup \mathrm{id}(\mathbf{y}) & \longleftrightarrow & \mathbf{x} \vee \mathbf{y} = (\ldots, \max(x_i, y_i), \ldots) \\ \mathrm{id}(\mathbf{x}) \cap \mathrm{id}(\mathbf{y}) & \longleftrightarrow & \mathbf{x} \wedge \mathbf{y} = (\ldots, \min(x_i, y_i), \ldots) \\ \mathrm{id}(\mathbf{x}) \subseteq \mathrm{id}(\mathbf{y}) & \longleftrightarrow & \mathbf{x} \leq \mathbf{y}. \end{array}$$

The family $\mathcal{F} = \{\mathrm{id}(\mathbf{x}) \mid \mathbf{x} \in \mathbb{P}(f)\}$ *has the intersection property with respect to* $\mathcal{D}$ *and gives rise to the (vector) rank function*

$$\rho(\mathbf{y}) = \max\{\mathbf{x}(N) \mid \mathbf{x} \in \mathbb{P}(f), \mathbf{x} \leq \mathbf{y}\} = \max_{F \in \mathcal{F}} |F \cap \mathrm{id}(\mathbf{y})|.$$

More generally, if $\mathcal{D}$ is the system of (order-)ideals of a partially ordered set (poset) $\mathcal{P} = (E, \leq)$, the intersection property yields generalized independence systems that have been studied in the context of (distributive) *supermatroids* (see, *e.g.*, [1, 4, 10, 27]). The *integral polymatroids* of Edmonds [6] can be understood as special distributive supermatroids (*cf.* [11] and Example 6.2 below).

Convex geometries (in the sense of [5]) generalize systems of ideals of posets. The so-called (strict) *cg-matroids* of [17] yield, in particular, independence systems with the intersection property relative to convex geometries.

The *combinatorial geometries* introduced in [9] include distributive supermatroids but do not have the intersection property in general (see Example 5.2 below).

2.2.2 $\mathcal{H}$-Matroids

An $\mathcal{H}$-independence system $\mathcal{F}$ is called an *$\mathcal{H}$-matroid* if for all $H \in \mathcal{H}$,

(M) all bases B of the restriction $\mathcal{F}(H)$ have the same cardinality $|B| = \rho(H)$.

In the case of an independence system $\mathcal{F}$ (*i.e.*, $\mathcal{H} = 2^E$), (M) yields a ("classical") *matroid*.

Remark. *In the case $\mathcal{H} = 2^E$, property* (M) *is the so-called Steinitz exchange* (*or augmentation*) *property.*

We exhibit a general submodularity property of the rank function of an $\mathcal{H}$-matroid. For large classes of families $\mathcal{H}$, this property is also sufficient to prove a normalized unit-increasing rank function to be a matroid rank function (see Section 5).

Lemma 2.1 *Let $\mathcal{M} = (E, \mathcal{F})$ be an $\mathcal{H}$-matroid with rank function ρ. Let furthermore $H_1, H_2 \in \mathcal{H}$ be such that $H_1 \subseteq H_2$. Then for all $G_1, G_2 \in \mathcal{H}$, one has*

$$\text{(S)} \quad \left.\begin{array}{l} H_1 \subseteq G_1 \cap G_2 \\ H_2 \subseteq G_1 \cup G_2 \end{array}\right\} \quad \Longrightarrow \quad \rho(H_1) + \rho(H_2) \leq \rho(G_1) + \rho(G_2).$$

Proof. Consider $B_1 \in \mathcal{F}(H_1)$ with $|B_1| = \rho(H_1)$. Because of $H_1 \subseteq H_2$ and the equicardinality property (M), B_1 is contained in some $B_2 \in \mathcal{F}(H_2)$ with $|B_2| = \rho(H_2)$. So we conclude

$$\begin{aligned} \rho(H_1) + \rho(H_2) &= |G_1 \cap G_2) \cap B_1| + |(G_1 \cup G_2) \cap B_2| \\ &= |G_1 \cap B_1| + |G_2 \cap B_2| \\ &\leq \rho(G_1) + \rho(G_2). \end{aligned}$$

$\diamond$

Hence, if $H_1, H_2 \in \mathcal{H}$ are such that $H_1 \cap H_2 \in \mathcal{H}$ and $H_1 \cup H_2 \in \mathcal{H}$, we obtain from (S) the usual submodularity inequality

$$\rho(H_1 \cap H_2) + \rho(H_1 \cup H_2) \leq \rho(H_1) + \rho(H_2).$$

3 Linear Optimization

Let $w : E \to \mathbb{R}$ be an arbitrary weight function on the ground set E. The function w extends to arbitrary subsets $X \subseteq E$ *via*

$$w(X) = \sum_{e \in X} w(e).$$

Given families $\mathcal{F}$ and $\mathcal{H}$ of subsets of E such that $\mathcal{F}$ is an $\mathcal{H}$-independence system, we are interested in the optimization problem

$$\max_{F \in \mathcal{F}} w(F). \tag{1}$$

3.1 The Greedy Algorithm

The *greedy algorithm* is the following simple-minded procedure to solve (1), where we use the notation

$$\Gamma_w(F) = \{e \in E \setminus F \mid w(e) > 0, F \cup e \in \mathcal{F}\} \quad (F \in \mathcal{F}).$$

- INITIALIZE: $F \leftarrow \emptyset$;
- ITERATE: While $\Gamma_w(F) \neq \emptyset$:

  Choose $e \in \Gamma_w(F)$ of maximal possible weight $w(e)$;
  Update $F \leftarrow (F \cup e)$;
- OUTPUT F.

We say that w is $\mathcal{H}$*-feasible* if $\mathcal{H}$ contains all level sets of w, *i.e.*,

$$W(\alpha) = \{e \in E \mid w(e) \geq \alpha\} \in \mathcal{H} \quad \text{for all } \alpha \in \mathbb{R}.$$

Theorem 3.1 *$\mathcal{M} = (E, \mathcal{F})$ is an $\mathcal{H}$-matroid if and only if the greedy algorithm is guaranteed to produce an optimal solution for* (1) *whenever w is $\mathcal{H}$-feasible.*

Proof. To see that the equicardinality property (M) necessarily holds for all bases of any $H \in \mathcal{H}$ if the greedy algorithm is optimal, consider the weight function $w = \chi_H$, where $\chi_H : E \to \{0, 1\}$ is the *characteristic function* of $H \in \mathcal{H}$ with

$$\chi_H(e) = 1 \quad \Longleftrightarrow \quad e \in H.$$

Any basis B of $\mathcal{F}(H)$ is in accordance with the greedy algorithm. So optimality implies $w(B) = |B| = \rho(H)$.

To prove sufficiency, assume that $w_1 > w_2 > \cdots > w_k$ are the distinct values of $w(e)$ $(e \in E)$, and that $w(e) > 0$ for some $e \in E$. Setting $W_i = W(w_i)$, we have

$$w = \sum_{i=1}^{k} \lambda_i \chi_{W_i}$$

where $\lambda_k = w_k$ and $\lambda_i = w_i - w_{i+1} \geq 0$ for $i = 1, \ldots, k$. Let $w_p > 0$ be the smallest among the strictly positive values of $w_1, \ldots, w_k$ and set

$$w^{(p)} = \sum_{i=1}^{p} \lambda'_i \chi_{W_i},$$

where $\lambda'_i = \lambda_i$ for $i = 1, \ldots, p-1$ and $\lambda'_p = w_p$. By definition, the greedy algorithm will select a basis B_1 of W_1 and then extend B_1 to a basis $B_2 \supseteq B_1$ of W_2 *etc.* and eventually output the basis B_p of W_p. Because $\mathcal{M}$ is an $\mathcal{H}$-matroid, we have

$$w(B_p) = \lambda'_1 \rho(W_i) + \lambda'_2 \rho(W_2) + \cdots + \lambda'_p \rho(W_p) = w^{(p)}(B_p).$$

Since no member of $\mathcal{F}$ contains more than $\rho(W_i)$ elements of W_i for all $i = 1, \ldots, p$, it is clear that the greedy solution B_p optimizes $w^{(p)}$ over $\mathcal{F}$. Because $w \leq w^{(p)}$ holds in general, B_p must be optimal for w as well.

$\diamond$

From the proof of Theorem 3.1 we can easily see the following (*cf.* [18]).

Theorem 3.2 *For an $\mathcal{H}$-matroid $(E, \mathcal{F})$ and an $\mathcal{H}$-feasible positive weight $w : E \to \mathbb{R}$, let $\hat{B}$ be an optimal solution (basis) of (1) given by the greedy algorithm, and let B be any basis of the $\mathcal{H}$-matroid. Suppose that elements of $\hat{B}$ and B are indexed as*

$$\hat{B} = \{\hat{e}_1, \ldots, \hat{e}_k\}, \qquad B = \{e_1, \ldots, e_k\}$$

such that

$$w(\hat{e}_1) \geq \cdots \geq w(\hat{e}_k), \qquad w(e_1) \geq \cdots \geq w(e_k).$$

Then we have $w(\hat{e}_i) \geq w(e_i)$ for all $i = 1, \ldots, k$.

$\diamond$

We illustrate $\mathcal{H}$-feasible functions.

Example 3.1 *Choosing $\mathcal{H} = \mathcal{D}$ in Example* 2.1*, the $\mathcal{H}$-feasible weight functions $w : E \to \mathbb{R}$ induce the so-called separable discrete concave functions (see,* e.g., [26]) *on integral vectors via*

$$\overline{w}(\mathbf{x}) = \sum_{e \in \mathbf{id}(\mathbf{x})} w(e). \tag{2}$$

Taking $\mathcal{H} = \{K_S \mid S \subseteq N\}$ (with $K_S = \bigcup_{i \in S} K_i$), $w : E \to \mathbb{R}$ is $\mathcal{H}$-feasible precisely when w is constant on each K_i, i.e., *when* (2) *induces the linear function $\overline{w}(\mathbf{x}) = w^T \mathbf{x}$ on $\mathbb{N}^N$.*

Example 3.2 *If $\mathcal{H}$ is the collection of ideals of an arbitrary poset $P = (E, \leq_P)$, then $w : E \to \mathbb{R}$ is $\mathcal{H}$-feasible if and only if for all $x, y \in E$,*

$$x \leq_P y \quad \Longrightarrow \quad w(x) \geq w(y).$$

Thus Theorem 3.1 *implies the validity of the greedy algorithm in* [8]. *If P is the trivial order, every $w : E \to \mathbb{R}$ is feasible and Theorem* 3.1 *yields the characterization of classical matroid independence systems by the greedy algorithm* [7, 18].

3.2 The Base Chain Property

We say that an (arbitrary) family $\mathcal{B}$ of subsets $B \subseteq E$ has the *base chain property* with respect to $\mathcal{H}$ if

(BC) for every chain $H_1 \subset \cdots \subset H_k$ of subsets $H_i \in \mathcal{H}$, there exists some $B \in \mathcal{B}$ such that $\rho(H_i) = |H_i \cap B|$ holds for all $i = 1, \ldots, k$.

Theorem 3.3 *Let $\mathcal{F}$ be an arbitrary constructible family with collection $\mathcal{B}$ of bases and assume that the greedy algorithm is guaranteed to produce an optimal solution for* (1) *whenever w is $\mathcal{H}$-feasible. Then $\mathcal{B}$ has the base chain property* (BC).

Proof. Let $\{H_1 \subset \ldots \subset H_k\}$ be a chain of subsets $H_i \in \mathcal{H}$ and consider the $\mathcal{H}$-feasible weight function

$$w = \sum_{i=1}^{k} \lambda_i \chi_{H_i} \quad \text{with } \lambda_1 > 0, \ldots, \lambda_k > 0.$$

Assuming w.l.o.g. $H_k = E$, note that the (optimal) greedy greedy solution yields a basis $B \in \mathcal{B}$ which does not depend on the absolute size of the weight parameters $\lambda_i > 0$. Hence the choice $\lambda_j = 1$ and $\lambda_i \approx 0$ for $i \neq j$ shows

$$|H_j \cap B| = \rho(H_j) \quad (j = 1, \ldots, k).$$

$\diamond$

Theorem 3.3 says that every $\mathcal{H}$-matroid has the base chain property. The greedy algorithm may be viewed as just a procedure to generate an appropriate basis $B \in \mathcal{F}(W_p)$ for the chain

$$W_1 \subset W_2 \subset \cdots \subset W_p.$$

Moreover, there is a certain converse to Theorem 3.3:

Any $B \in \mathcal{F}$ satisfying (BC) *with $H_i = W_i$ as in the proof of Theorem* 3.1 *is an optimal solution for problem* (1) *if the weight function w is $\mathcal{H}$-feasible and non-negative.*

Let us call the pair $\mathcal{G} = (E, \mathcal{F})$ an *$\mathcal{H}$-greedoid* if $\mathcal{F}$ is a constructible family with the equicardinality property (M). So $\mathcal{H} = 2^E$ yields exactly the *greedoids* of [20]. In the latter case, the base chain property implies that the system $\mathcal{B}$ of bases of $\mathcal{G}$ is, in fact, the system of bases of a matroid $\tilde{\mathcal{M}} = (E, \tilde{\mathcal{F}})$ where

$$\tilde{\mathcal{F}} = \{B \cap H \mid B \in \mathcal{B}, H \in \mathcal{H}\}. \tag{3}$$

In general, however, a base system $\mathcal{B}$ with the base chain property (BC) does not necessarily induce an $\mathcal{H}$-matroid *via* (3). Suppose, for example, that $\mathcal{H}$ is *graded* (or *Jordan-Dedekind*) in the sense that the length of every maximal chain in $\mathcal{H}$ equals $|E| - 1$. Then

$$\tilde{\mathcal{F}} = \{B \cap H \mid B \in \mathcal{B},\ H \in \mathcal{H}\}$$

is an $\mathcal{H}$-independence system, but not necessarily an $\mathcal{H}$-matroid. If $\mathcal{H}$ is not only Jordan-Dedekind but also closed under taking intersections, (BC) implies that $\tilde{\mathcal{M}} = (E, \tilde{\mathcal{F}})$ is an $\mathcal{H}$-matroid (see Example 5.1 in Section 5). This is a consequence of the following observation:

Corollary 3.1 *Let $\mathcal{B}$ be a family of subsets of E with the base chain property* (BC) *relative to $\mathcal{H}$. Then the associated rank function ρ is submodular* (*in the sense of condition* (S) *of Lemma* 2.1).

$\diamond$

4 Duality

The discussion of the base chain property (BC) suggests a duality framework for $\mathcal{H}$-matroids. We assume to be given a (non-empty) system $\mathcal{B}$ of equicardinal subsets $B \subseteq E$, whose members are *bases*. As before, we define an associated rank function

$$\rho(S) = \max_{B \in \mathcal{B}} |S \cap B|$$

and consider a subset family $\mathcal{H}$ with $\emptyset, E \in \mathcal{H}$. Define the *dual* of an (arbitrary) family $\mathcal{T}$ of subsets of E as the family

$$\mathcal{T}^* = \{E \setminus T \mid T \in \mathcal{T}\}.$$

Then the dual $\mathcal{B}^*$ of a base system $\mathcal{B}$ is again a base system. Clearly, $B \in \mathcal{B}$ maximizes the intersection with a set S if and only if $B^* = E \setminus B$ maximizes the intersection with $S^* = E \setminus S$. Hence we can represent the rank function ρ^* of $\mathcal{B}^*$ in terms of ρ *via*

$$\rho^*(S) = |S| - [\rho(E) - \rho(E \setminus S)] \quad \text{for all } S \subseteq E. \tag{4}$$

Moreover, we note:

Proposition 4.1 *The base system $\mathcal{B}$ satisfies* (BC) *relative to $\mathcal{H}$ if and only if its dual $\mathcal{B}^*$ satisfies* (BC) *relative to $\mathcal{H}^*$.*

$\diamond$

5 Matroids on Closure Spaces

Let $\mathcal{H}$ be as before a subset family with $\emptyset, E \in \mathcal{H}$. Recall that $\mathcal{H}$ is a *closure system* (and the pair $(E, \mathcal{H})$ a *closure space*) if $\mathcal{H}$ is intersection-closed, *i.e.*,

$$H_1 \cap H_2 \in \mathcal{H} \quad \text{for all } H_1, H_2 \in \mathcal{H}.$$

Assume henceforth that $\mathcal{H}$ is a closure system and associate with $\mathcal{H}$ the *closure operator*

$$X \;\mapsto\; \overline{X} = \bigcap\{H \in \mathcal{H} \mid X \subseteq H\} \quad (X \subseteq E).$$

X is called *closed* if $X = \overline{X}$. So $\mathcal{H}$ is precisely the collection of closed sets. Moreover, $(\mathcal{H}, \subseteq)$ is a lattice with the infimum and supremum operations

$$\begin{aligned} H_1 \wedge H_2 &= H_1 \cap H_2 \\ H_1 \vee H_2 &= \overline{H_1 \cup H_2}. \end{aligned}$$

We say that H_2 *covers* H_1 (denoted $H_1 \prec H_2$) if $H_1 \subset H_2$ holds and for all $H \in \mathcal{H}$,

$$H_1 \subset H \subseteq H_2 \quad \Longrightarrow \quad H = H_2.$$

5.1 Matroids from Rank Functions

Let $(E, \mathcal{H})$ be a closure space and consider the function $r : \mathcal{H} \to \mathbb{N}$ such that for all $H_1, H_2 \in \mathcal{H}$,

(R_0) $r(H_1) = 0 \quad$ if $H_1 = \emptyset \quad$ (normalization);

(R_1) $H_1 \prec H_2 \quad \Longrightarrow \quad r(H_1) \leq r(H_2) \leq r(H_1) + 1 \quad$ (unit-increase);

(R_2) $H_1 \prec (H_1 \vee H_2) \quad \Longrightarrow \quad r(H_1 \wedge H_2) + r(H_1 \vee H_2) \leq r(H_1) + r(H_2)$.

Associate with r the family $\mathcal{F} = \mathcal{F}(r)$ of subsets of E that can be obtained *via* the following algorithmic procedure:

- INITIALIZE: Choose some $H \in \mathcal{H}$ and a maximal chain $\mathbf{M}_H$ from $\emptyset$ to H:
$$\mathbf{M}_H = \{\emptyset = M_0 \prec M_1 \prec \ldots \prec M_k = H\};$$
- Choose a sequence $\pi = u_1 \ldots u_k$ of *representatives* $u_i \in M_i \setminus M_{i-1}$,
- OUTPUT the set $F = \{u_i \mid r(M_i) = r(M_{i-1}) + 1\}$.

It follows directly from the algorithmic definition that $\mathcal{F}$ is constructible. Moreover, in view of the unit increase property (R_1), each basis B of the restriction $\mathcal{F}(H)$ has cardinality $|B| = r(H)$. So $\mathcal{F}$ has the equicardinality property (M) of a matroid.

Theorem 5.1 *If* (R_0)–(R_2) *hold for* r, $\mathcal{F}$ *is an* $\mathcal{H}$*-independence system and* $\mathcal{M} = (E, \mathcal{F})$ *an* $\mathcal{H}$*-matroid with rank function* r.

Proof. It remains to verify that $r(H)$ equals the basis rank $\rho(H)$ of $H \in \mathcal{H}$ relative to $\mathcal{F}$. Arguing by induction on $|E|$, we may assume $H = E$ without loss of generality.

So let $B \in \mathcal{B}$ be an arbitrary independent set and assume that B arises from the chain $\mathbf{M} = \{\emptyset = M_0 \prec \ldots \prec M_k = E\}$ and the representatives $\pi = u_1, \ldots, u_k$ *via*

$$B = \{u_i \mid r(M_i) = r(M_{i-1}) + 1\}.$$

Consider the function $f : \mathcal{H} \to \mathbb{R}$ where

$$f(A) = r(A) - |A \cap B|.$$

We claim $f(A) \geq 0$. One easily checks

- $f(M_i) = 0$ for $i = 0, 1, \ldots, k$;
- $H_1 \prec (H_1 \vee H_2) \implies f(H_1 \wedge H_2) + f(H_1 \vee H_2) \leq f(H_1) + f(H_2)$.

Suppose now that the claim is false and $A \in \mathcal{H}$ a counterexample of minimal cardinality. Let j be the index such that $A \subseteq M_j$ and $A \not\subseteq M_{j-1}$ holds. Then we obtain

$$0 > f(A) \geq f(A \wedge M_{j-1}) + f(M_j) - f(M_{j-1}) = f(A \wedge M_{j-1}) \geq 0,$$

which is a contradiction. So the claim is true and $|H \cap B| \leq r(H)$ follows for all $H \in \mathcal{H}$. Hence we conclude

$$r(H) \leq \rho(H) = \max_{B \in \mathcal{F}} |H \cap B| \leq r(H) \quad \text{and thus} \quad r(H) = \rho(H).$$

$\diamond$

Example 5.1 *Let $\mathcal{C} = (E, \mathcal{H})$ be a* convex geometry*,* i.e.*, a closure space such that for all $H_1, H_2 \in \mathcal{H}$,*

$$H_1 \prec H_2 \implies |H_2| = |H_1| + 1.$$

(*Note that convex geometries are Jordan-Dedekind.*) *Then we have*

$$H_1 \prec (H_1 \vee H_2) \implies H_1 \vee H_2 = H_1 \cup H_2.$$

So Theorem 5.1 *implies that the submodularity property* (S) *of Lemma* 2.1 *is necessary and sufficient for a normalized unit-increasing function $r : \mathcal{H} \to \mathbb{N}$ to be the rank function of an $\mathcal{H}$-matroid.*

The $\mathcal{H}$-matroid $\mathcal{M} = (E, \mathcal{F})$ is a strict cg-matroid *in the sense of* [17] *when $\mathcal{F} \subseteq \mathcal{H}$ holds.*

Example 5.2 *Let $P = (E, \leq)$ be a poset with collection $\mathcal{D}$ of ideals. Then $(E, \mathcal{D})$ is in particular a convex geometry. The $\mathcal{D}$-matroids are essentially the* combinatorial geometries *of* [9] *and the strict $\mathcal{D}$-matroids are the* distributive supermatroids *of* [4].

Remark. *We point out that independence systems of type $\mathcal{F}(r)$ do* not *have the intersection property in general.*

5.2 Matroids on Co-Closure Spaces

Let $\mathcal{C} = (E, \mathcal{H})$ be a closure space. Then the dual structure $\mathcal{C}^* = (E, \mathcal{H}^*)$ is a so-called *co-closure space*. $\mathcal{H}^*$ is union-closed and $(\mathcal{H}^*, \subseteq)$ is anti-isomorphic to the lattice $(\mathcal{H}, \subseteq)$ under the supremum and infimum operations

$$\begin{aligned} S_1 \vee^* S_2 &= S_1 \cup S_2, \\ S_1 \wedge^* S_2 &= \bigcup \{S \in \mathcal{H}^* \mid S \subseteq S_1 \cap S_2\}. \end{aligned}$$

Let $r : \mathcal{H}^* \to \mathbb{N}$ be normalized and unit-increasing and consider the property

(R$_2^*$) $(S_1 \wedge^* S_2) \prec S_1 \implies r(S_1 \wedge^* S_2) + r(S_1 \vee^* S_2) \leq r(S_1) + r(S_2)$.

As before, we associate with r the family $\mathcal{F} = \mathcal{F}(r)$ of subsets of E that can be obtained *via* the following algorithmic procedure:

- INITIALIZE: Choose some $S \in \mathcal{H}^*$ and a maximal chain $\mathbf{M}_S$ from $\emptyset$ to S:
$$\mathbf{M}_S = \{\emptyset = M_0 \prec M_1 \prec \cdots \prec M_k = S\};$$
- Choose a sequence $\pi = u_1 \dots u_k$ of *representatives* $u_i \in M_i \setminus M_{i-1}$,
- OUTPUT the set $F = \{u_i \mid r(M_i) = r(M_{i-1}) + 1\}$.

Again, it is clear that the bases B of $\mathcal{F}(S)$ share the same cardinality $|B| = r(S)$. In order to establish $(E, \mathcal{M})$ as an $\mathcal{H}^*$-matroid, we need an additional assumption on $\mathcal{H}^*$.

We call the co-closure system $\mathcal{H}^*$ *locally modular* if for all $S_1, S_2 \in \mathcal{H}^*$,

(LM) $S_1 \wedge^* S_2 \prec S_1 \implies S_1 \wedge^* S_2 = S_1 \cap S_2$.

Theorem 5.2 *Assume that $\mathcal{H}^*$ is locally modular and $r : \mathcal{H}^* \to \mathbb{N}$ is a normalized unit-increasing function with property* (R$_2^*$)*. Then $\mathcal{M} = (E, \mathcal{F}(r))$ is an $\mathcal{H}^*$-matroid with rank function r.*

Proof. The key observation under the assumption of local modularity is the following. Given an arbitrary $B \in \mathcal{F}$,

$$f(S) = r(S) - |S \cap B|$$

yields a function on $\mathcal{H}^*$ with the property

$$(S_1 \wedge^* S_2) \prec S_1 \quad \Longrightarrow \quad f(S_1 \wedge^* S_2) + f(S_1 \vee^* S_2) \leq f(S_1) + f(S_2).$$

One may now argue in analogy with the proof of Theorem 5.1 and choose, if possible, an $S \in \mathcal{H}^*$ of maximal cardinality such that $f(S) < 0$. Letting j be such that $S \supseteq M_{j-1}$ but $S \not\supseteq M_j$, one arrives at a contradiction as before.

$\diamond$

5.3 Duality for Convex Geometries and Antimatroids

An *antimatroid* (*cf.* [20]) is the co-closure space $\mathcal{C}^* = (E, \mathcal{H}^*)$ associated with a convex geometry $\mathcal{C} = (E, \mathcal{H})$. Thus we have for all $S_1, S_2 \in \mathcal{H}^*$

$$S_1 \prec S_2 \quad \Longrightarrow \quad |S_2| = |S_1| + 1$$

and therefore conclude local modularity:

$$S_1 \wedge^* S_2 \prec S_1 \quad \Longrightarrow \quad S_1 \wedge^* S_2 = S_1 \cap S_2.$$

Let $r : \mathcal{H} \to \mathbb{N}$ be an arbitrary function and define $r^* : \mathcal{H}^* \to \mathbb{N}$ *via*

$$r^*(S) = |S| - [r(E) - r(E \setminus S)]. \tag{5}$$

It is straightforward to verify that r is normalized and unit-increasing if and only if r^* is normalized and unit-increasing. It follows that (5) establishes a one-to-one correspondence between the $\mathcal{H}$-matroid rank functions r and the $\mathcal{H}^*$-matroid rank functions r^* since

$$r(H) = |H| - [r^*(E) - r^*(E \setminus H)] \quad (H \in \mathcal{H}).$$

Note that this duality framework for matroids on convex geometries and matroids on antimatroids is compatible with the duality framework for general base systems of Section 4.

6 Closures and the Dilworth Completion

6.1 Closures

Let $\mathcal{C} = (E, \mathcal{H})$ be a closure space and $\mathcal{F}$ an $\mathcal{H}$-independence system with rank function ρ. For any $S \subseteq E$, we set

$$\sigma(S) = \{e \in E \mid \rho(\overline{S \cup e}) = \rho(\overline{S})\}.$$

In the case when $\sigma(S) = S$, we refer to S as *ρ-closed*. We want to know under what conditions $S \mapsto \sigma(S)$ is a closure operator. Since E is clearly ρ-closed, we thus ask when the collection Σ of ρ-closed sets is intersection-closed.

Since $X \mapsto \overline{X}$ is a closure operator, we immediately obtain

$$\overline{\sigma(S)} = \sigma(S) \in \mathcal{H} \quad \text{and thus} \quad \Sigma \subseteq \mathcal{H}.$$

Hence it suffices to study the action of σ on $\mathcal{H}$. To this end, observe for any $H \in \mathcal{H}$ the slightly more convenient representation

$$\sigma(H) = \bigvee\{G \in \mathcal{H} \mid \rho(H \vee G) = \rho(H)\}.$$

Theorem 6.1 *$S \mapsto \sigma(S)$ is a closure operator if ρ is* submodular *on $\mathcal{H}$ in the sense*

$$\rho(H_1 \wedge H_2) + \rho(H_1 \vee H_2) \leq \rho(H_1) + \rho(H_2) \quad \text{for all } H_1, H_2 \in \mathcal{H}.$$

Conversely, provided $\mathcal{H}$ is also union-closed, $S \mapsto \sigma(S)$ is a closure operator only if ρ is submodular on $\mathcal{H}$.

Proof. Let ρ be submodular and $H_1, H_2 \in \mathcal{H}$ ρ-closed . Suppose that $D = H_1 \cap H_2 = H_1 \wedge H_2$ is not ρ-closed. So there is some $G \in \mathcal{H}$ such that $\rho(D \vee G) = \rho(D)$. Assume w.l.o.g. $G \not\subseteq H_1$. Since $H_1 = \sigma(H_1)$, we have $\rho(H_1 \vee G) > \rho(H_1)$ and obtain the contradiction

$$\rho(H_1) < \rho(H_1 \vee G) \leq \rho(H_1) + \rho(D \vee G) - \rho(H_1 \wedge (D \vee G)) \leq \rho(H_1).$$

Hence we find that the collection Σ of ρ-closed sets is intersection-closed if ρ is submodular.

Conversely, assume that $\mathcal{H}$ is union closed and $H \mapsto \sigma(H)$ is a closure operator. We prove the submodularity inequality by induction on the $|H_1 \cup H_2|$. Clearly, the inequality is true if $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$. Therefore, we may assume w.l.o.g. that there exists some $H_2' \prec H_2$ with $H_2' \supseteq H_1 \cap H_2$. By induction, we thus have

$$\begin{aligned} \rho(H_1 \cap H_2) + \rho(H_1 \cup H_2') &= \rho(H_1 \cap H_2') + \rho(H_1 \cup H_2') \\ &\leq \rho(H_1) + \rho(H_2'). \end{aligned}$$

If $\rho(H_2) = \rho(H_2') + 1$, the inequality follows trivially in view of the unit-increase property of ρ. So we may assume $\rho(H_2') = \rho(H_2)$ for the remainder of the proof.

If $\rho(H_1 \vee H_2') = \rho(H_1 \vee H_2)$, there is nothing left to show. On the other hand, if $\rho(H_1 \vee H_2') < \rho(H_1 \vee H_2)$ is true, we have $H_2 \not\subseteq \sigma(H_1 \vee H_2')$ and therefore also $H_2 \not\subseteq \sigma(H_2')$, which means $\rho(H_2) > \rho(H_2')$.

$\diamond$

Example 6.1 *Let $\mathcal{H}$ be intersection- and union-closed. Then Lemma* 2.1 *implies that the rank function of every $\mathcal{H}$-matroid is submodular. Hence Theorem* 6.1 *characterizes the rank functions that arise from $\mathcal{H}$-matroids. In particular, the family $\mathcal{D}$ of ideals of any poset $P = (E, \leq)$ is union- and intersection-closed. Consequently, we find that every $\mathcal{D}$-matroid is characterized by the fact that $S \mapsto \sigma(S)$ is a closure operator.*

Remark. *The converse part in Theorem* 6.1 *may be false on general convex geometries. Assume, for example, that $\mathcal{C} = (E, \mathcal{H})$ is a convex geometry that admits sets $H_1, H_2 \in \mathcal{H}$ with $H_1 \cup H_2 \notin \mathcal{H}$. Then $r(H) = |H|$ is normalized and unit-increasing, but not submodular on $\mathcal{H}$ (because $|H_1 \vee H_2| > |H_1 \cup H_2|$). Yet, the associated operator $S \mapsto \sigma(S)$ is a closure operator with collection $\Sigma = \mathcal{H}$ of closed sets.*

6.2 The Dilworth Completion

Let $\mathcal{C} = (E, \mathcal{H})$ be a closure space and $\mathcal{M} = (E, \mathcal{F})$ an $\mathcal{H}$-matroid with rank function ρ. If ρ is submodular on $\mathcal{H}$, the system Σ of ρ-closed sets yields a lattice $(\Sigma, \subseteq)$ with infimum and supremum operations

$$\begin{aligned} S \sqcap T &= S \cap T, \\ S \sqcup T &= \sigma(S \cup T). \end{aligned}$$

Dilworth has shown that any finite lattice can be embedded into the lattice of closed sets of some classical matroid $\tilde{\mathcal{M}}$ on E (relative to the family of all subsets of E) (see [2]). We study Dilworth's construction from the point of view of independence.

We define the *Dilworth completion* $\tilde{\rho}$ of $\rho : \mathcal{H} \to \mathbb{N}$ for all subsets $X \subseteq E$ *via*

$$\tilde{\rho}(X) = \min_{H \in \mathcal{H}} \{\rho(H) + |X \setminus H|\}. \tag{6}$$

It is easy to see that $\tilde{\rho}$ is normalized and unit-increasing on the collection of all subsets of E. Moreover, $\tilde{\rho}$ extends ρ in the sense

$$\tilde{\rho}(H) = \rho(H) \quad \text{for all } H \in \mathcal{H}. \tag{7}$$

Letting

$$\tilde{\mathcal{F}} = \{X \subseteq E \mid \tilde{\rho}(X) = |X|\},$$

we observe

Lemma 6.1 $\tilde{\mathcal{F}} = \{X \subseteq E \mid |X \cap H| \leq \rho(H)$ *for all* $H \in \mathcal{H}\}$.

Proof. Note that $\tilde{\rho}(X) \leq |X|$ is always true.

If $\tilde{\rho}(X) = |X|$, then $|X \cap H| \leq \rho(H)$ follows from the definition of $\tilde{\rho}$. Conversely, if $\tilde{\rho}(X) \leq |X| - 1$ holds, there is some $H \in \mathcal{H}$ with

$$|X| - 1 \geq \tilde{\rho}(X) = \rho(H) + |X \setminus H|$$

and hence $|X \cap H| \geq \rho(H) + 1$.

$\diamond$

Lemma 6.1 shows that $\tilde{\mathcal{F}}$ is an independence system (with respect to 2^E) and contains the $\mathcal{H}$-independence system $\mathcal{F}$.

Lemma 6.2 *Assume that* ρ *is submodular on* $\mathcal{H}$. *Then one has for all* $X, Y \subseteq E$*,*

$$\tilde{\rho}(X \cap Y) + \tilde{\rho}(X \cup Y) \leq \tilde{\rho}(X) + \tilde{\rho}(Y).$$

Proof. Let $S, T \in \mathcal{H}$ be such that

$$\tilde{\rho}(X) = \rho(S) + |X \setminus S| \quad \text{and} \quad \tilde{\rho}(Y) = \rho(T) + |Y \setminus T|.$$

Then we find

$$\begin{aligned} \tilde{\rho}(X) + \tilde{\rho}(X) &= \rho(S) + \rho(T) + |X \setminus S| + |Y \setminus T| \\ &\geq \rho(S \wedge T) + |(X \cap Y) \setminus (S \wedge T)| \\ &\quad + \rho(S \vee T) + |(X \cup Y) \setminus (S \vee T)| \\ &\geq \tilde{\rho}(X \cap Y) + \tilde{\rho}(Y \cup Y). \end{aligned}$$

$\diamond$

Under the conditions of Lemma 6.2, $\tilde{\rho}$ is the rank function of a (classical) matroid $\tilde{\mathcal{M}}$, which we call the *Dilworth completion* of $\mathcal{M} = (E, \mathcal{F})$. In view of the equality (7), the greedy algorithm relative to the $\mathcal{H}$-matroid $\mathcal{M}$ may be interpreted as a special case of the greedy algorithm relative to its Dilworth completion $\tilde{\mathcal{M}}$.

Example 6.2 *With the notation of Example* 2.1 *and* $K_S = \bigcup_{i \in S} K_i$*, assume that the function* f *is submodular on the subsets of* N*. Then*

$$\tilde{f}(X) = \min_{S \subseteq N} \{f(S) + |X \setminus K_S|\} \quad (X \subseteq E)$$

is the rank function of a matroid $\tilde{\mathcal{M}}_f$ *on* E*.* $\tilde{\mathcal{M}}_f$ *is the Dilworth completion of* $\mathcal{F} = \{\mathrm{id}(x) \mid \mathbf{x} \in \mathbb{P}(f)\}$ *with respect to the family* $\mathcal{H} = \{K_S \mid S \subseteq N\}$*. The rank function* ρ_f *induced by* $\mathcal{F}$ *coincides with the rank function* $\tilde{\rho}_f$ *on the sets* $\mathrm{id}(\mathbf{x}) \in \mathcal{D}$*. So the greedy algorithm relative to* $\mathcal{F}$ *is a special case of the greedy algorithm relative to* $\tilde{\mathcal{M}}_f$*. Moreover, if* f *is monotone increasing, one has*

$$\rho_f(K_S) = \tilde{\rho}_f(K_S) = f(S) \quad \textit{for all } S \subseteq N,$$

which yields the polymatroid greedy algorithm of Edmonds [6].

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