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# Convex sweeping process in the framework of measure differential inclusions and evolution variational inequalities

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**Abstract** In this paper, we analyze and discuss the well-posedness of two new variants of the so-called sweeping process, introduced by Moreau in the early 70s (Moreau in *Sém Anal Convexe Montpellier*, 1971) with motivation in plasticity theory. The first new variant is concerned with the perturbation of the normal cone to the moving convex subset  $C(t)$ , supposed to have a bounded variation, by a Lipschitz mapping. Under some assumptions on the data, we show that the perturbed differential measure inclusion has one and only one right continuous solution with bounded variation. The second variant, for which a large analysis is made, concerns a first order sweeping process with velocity in the moving set  $C(t)$ . This class of problems subsumes as a particular case, the evolution variational inequalities [widely used in applied mathematics and unilateral mechanics (Duvaut and Lions in *Inequalities in mechanics and physics*. Springer, Berlin, 1976)]. Assuming that the moving subset  $C(t)$  has a continuous variation for every  $t \in [0, T]$  with  $C(0)$  bounded, we show that the problem has at least a Lipschitz continuous solution. The well-posedness of this class of sweeping process is obtained under the coercivity assumption of the involved operator. We also discuss some applications of the sweeping process to the study of vector hysteresis operators in the elastoplastic model (Krejčí in *Eur J Appl Math* 2:281–292, 1991), to

the planning procedure in mathematical economy (Henry in *J Math Anal Appl* 41:179–186, 1973 and Cornet in *J. Math. Anal. Appl.* 96:130–147, 1983), and to nonregular electrical circuits containing nonsmooth electronic devices like diodes (Acary et al. *Nonsmooth modeling and simulation for switched circuits. Lecture notes in electrical engineering.* Springer, New York 2011). The theoretical results are supported by some numerical simulations to prove the efficiency of the algorithm used in the existence proof. Our methodology is based only on tools from convex analysis. Like other papers in this collection, we show in this presentation how elegant modern convex analysis was influenced by Moreau’s seminal work.

**Keywords** Differential inclusion · Moreau’s sweeping process · Normal cones · Evolution variational inequality · Convex integral functional

**Mathematics Subject Classification** 49J53 · 34G25 · 58E35

## 1 Introduction

In the seventies Moreau introduced and thoroughly studied the *sweeping process*, which is a particular differential inclusion. As a partial viewpoint, consider a time-moving closed convex set  $C(t)$  which drags a point  $u(t)$ , so this point must stay in  $C(t)$  at every time  $t$ , and the opposite of its velocity, say  $-\frac{du}{dt}(t)$ , has to be normal to the set  $C(t)$ . To take into account the nonsmoothness of the boundary of the convex set  $C(t)$ , the law of motion is formulated as

$$\begin{cases} -\frac{du}{dt}(t) \in N(C(t); u(t)) \\ u(0) = u_0 \in C(0) \\ u(t) \in C(t) \quad \forall t, \end{cases} \quad (1.1)$$

where  $N(C(t); u(t))$  is the (outward) *normal cone* to the set  $C(t)$  at the point  $u(t)$  in the sense of Modern Convex Analysis. The following interpretation arises (see [45]) for the way how the point  $u(t)$  is “swept”: as long as the point  $u(t)$  happens to be in the interior of  $C(t)$ , the normal cone  $N(C(t); u(t))$  is reduced to zero, so  $u(t)$  does not move. When the point is “caught up with” by the boundary of  $C(t)$  it moves, subject to an inward normal direction, as if pushed by this boundary. Concrete original motivations of the *sweeping process* by Moreau are: quasi-static evolution in elastoplasticity, contact dynamics, friction dynamics, granular material (see [43,47] and the references therein). The sweeping process model is also of great interest in nonsmooth mechanics, convex optimization, mathematical economics and more recently in the modeling and simulation of switched electrical circuits [1,2,4,6,12]. Existence and uniqueness results when the convex sets  $C(t)$  are absolutely continuous or have bounded retraction are provided in [45]. Those results will be discussed in Sect. 3.1. Moreau [45] also introduced the second order sweeping process for the study of Lagrangian mechanical systems subject to frictionless unilateral constraints. For such systems the velocity may be discontinuous at the impact time. In this case, the acceleration can be defined as a measure. This kind of problems fall within the formalism of measure differential

inclusions. For the sweeping process with nonconvex sets  $C(t)$ , we refer the reader to [7, 9, 13, 16, 18–20, 26, 58–60] and the references therein.

A natural variant of the sweeping process is the differential inclusion

$$\begin{cases} -\frac{du}{dt}(t) \in N(C(t); u(t)) + F(t, u(t)) \\ u(0) = u_0 \in C(0) \\ u(t) \in C(t), \quad \forall t \in [0, T], \end{cases} \quad (1.2)$$

where  $F$  is a set-valued mapping from  $[0, T] \times H$  into weakly compact convex sets of a Hilbert space  $H$ . The particular case where all the sets  $C(t)$  are equal to a fixed convex set  $K$  is related to the Henry's mathematical model of planning procedure, see [29] and Sect. 3.3 for more details; this model has been largely developed by Cornet [21]. For the existence of solutions with time-varying convex/nonconvex sets  $C(t)$  we refer the reader to [9, 14–16, 25–28, 32–34, 37, 58]. Applications to the crowd motion modeling have been realized in [36].

From a numerical point of view, the time-integration (also known as time-stepping) schemes have been applied to find an approximation of the solution to the sweeping process. The so called “catching-up” algorithm was introduced by Moreau [40, 45, 46] to prove the existence of a solution to (1.1) and is defined by

$$u_0 \in H, \quad (u_{k+1} - u_k) \in -N(C(t_{k+1}); u_{k+1}), \quad (1.3)$$

where  $u_k$  stands for an approximation of  $u$  at the time  $t_k$ . Using the fact that  $[I + N(C; \cdot)]^{-1} = \text{proj}_C$  (the metric projection operator onto  $C$ ), one sees that (1.3) is equivalent to

$$u_0 \in H, \quad u_{k+1} = \text{proj}_{C(t_{k+1})}(u_k). \quad (1.4)$$

When the time step goes to zero, under various assumptions on the variation of  $C(t)$ , the approximation constructed via  $(u_k)_k$  in (1.4) contains a subsequence which converge weakly in  $H$  to some  $u$  satisfying (1.1) a.e. (see [34, 45]).

Another interesting mathematical formalism, called *Differential Variational Inequalities* (DVI), was introduced by Pang and Stewart [50]. It is a combination of an ordinary differential equation with a variational inequality or a complementarity constraint. A DVI consists to find trajectories  $t \mapsto x(t)$  and  $t \mapsto u(t)$  such that

$$\begin{cases} \frac{dx}{dt}(t) = f(t, x(t), u(t)) \\ \langle F(t, x(t), u(t)), v - u(t) \rangle \geq 0, \quad \forall v \in K, \quad \text{a.e. } t \in [0, T], \end{cases}$$

where  $K$  is a closed convex subset of  $H$ ,  $f$  and  $F$  are given mappings. The DVI formalism unifies several known mathematical problems such as: ordinary differential equations with discontinuous right-hand term, differential algebraic equations, dynamic complementarity problems etc . . . (see [50] for more details). The DVI formalism was proved to be powerful for the treatment of many problems in science and engineering such as: unilateral contact problems in mechanics, finance, traffic networks, electrical circuits etc . . .

The main aim of the present paper is to analyze two variants of the sweeping process and to establish existence results for them. The first new variant is concerned with the

case where the sweeping process (1.1) is perturbed by a Lipschitz mapping and where the moving convex set  $C(t)$  has a bounded variation. The mathematical formulation is then a *measure differential inclusion* (see Sect. 4).

The second variant is of the form

$$\begin{cases} A_1 \frac{du}{dt}(t) + A_0 u(t) - f(t) \in -N(C(t); \frac{du}{dt}(t)) \\ u(0) = u_0 \in H, \end{cases} \quad (1.5)$$

where  $A_0, A_1 : H \rightarrow H$  are two linear bounded, symmetric and semidefinite operators,  $f : [0, T] \rightarrow H$  is a bounded continuous mapping and  $C(t)$  is a nonempty closed and convex set of  $H$  for every  $t \in [0, T]$ . Problem (1.5) includes as a special case the following evolution variational inequality:

$$\begin{cases} \text{Find } u : [0, T] \rightarrow H, \text{ with } u(0) = u_0 \in H, \\ \text{such that } \dot{u}(t) \in C(t) \text{ a.e. } t \in [0, T] \text{ and} \\ a_0(u(t), v - \dot{u}(t)) + a_1(\dot{u}(t), v - \dot{u}(t)) \geq \langle l(t), v - \dot{u}(t) \rangle \text{ for all } v \in C(t). \end{cases} \quad (1.6)$$

Here  $a_0(\cdot, \cdot)$  and  $a_1(\cdot, \cdot)$  are real bilinear, bounded and symmetric forms,  $l \in W^{1,2}([0, T], H)$  and  $\dot{u}(t) := \frac{du}{dt}(t)$ .

The evolution variational inequalities of type (1.6) are widely used in applied mathematics, unilateral mechanics and various fields of sciences and engineering such as for instance traffic networks, energy market, transportation, elastoplasticity etc ... (see e.g. [24]).

The remainder of this manuscript is organized as follows. Section 2 is devoted to several results of convex analysis which are used throughout the paper; in particular, Rockafellar's theorem on the Legendre–Fenchel conjugate of a convex integral functional on a functional space is stated in Sect. 2.2. In Sect. 3 we review the significance of the differential measure formulation of (1.1) where  $C(t)$  has a bounded variation and state Moreau's theorem on existence and uniqueness of solution; various known variants in the literature with applications to *hysteresis phenomena*, *planning procedures* and *electrical circuits* are also briefly described. The first new variant presented above of the perturbation with a Lipschitz mapping of the sweeping process involving convex set  $C(t)$  with bounded variation is studied in great detail in Sect. 4; a theorem of existence and uniqueness is established. Section 5 is dedicated to the second variant (1.5), for which a large analysis is made and an existence theorem is provided; under the coercivity of the linear operator  $A_0$  the uniqueness is also obtained. Section 6 is devoted to some illustrative numerical simulations.

## 2 Notation and preliminaries

This section is devoted to concepts and preliminary results which will be used in the paper.

## 2.1 Subdifferential, normal cone, conjugate

Given a normed space  $X$  with topological dual  $X^*$  and a convex function  $\varphi : X \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ , the *subdifferential* of  $\varphi$  at a point  $x \in X$  with  $|\varphi(x)| < +\infty$  is defined as

$$\partial\varphi(x) := \{x^* \in X^* : \langle x^*, x' - x \rangle \leq \varphi(x') - \varphi(x), \forall x' \in X\},$$

and the *effective domain* of  $\varphi$  is  $\text{dom } \varphi := \{x' \in X : \varphi(x') < +\infty\}$ ; the function  $\varphi$  is said to be *proper* whenever  $\text{dom } \varphi \neq \emptyset$  and  $\varphi$  does not take on the value  $-\infty$ . The subdifferential is related to the *directional derivative*  $\varphi'(x; \cdot)$  in the sense that

$$\partial\varphi(x) = \{x^* \in X^* : \langle x^*, z \rangle \leq \varphi'(x; z), \forall z \in X\},$$

where

$$\varphi'(x; z) := \inf_{\tau > 0} \tau^{-1}(\varphi(x + \tau z) - \varphi(x)) = \lim_{\tau \downarrow 0} \tau^{-1}(\varphi(x + \tau z) - \varphi(x)).$$

This characterization shows, for any convex function  $\varphi_0$  Gâteaux differentiable at  $x$ , that

$$\partial\varphi_0(x) = \{D\varphi_0(x)\} \quad \text{and} \quad \partial(\varphi_0 + \varphi)(x) = D\varphi_0(x) + \partial\varphi(x). \quad (2.1)$$

Through the directional derivative and the Hahn-Banach theorem, one also sees that, whenever  $\varphi$  is finite and continuous at  $x$ , the set  $\partial\varphi(x)$  is nonempty and weakly\* compact in  $X^*$  and

$$\varphi'(x; z) = \max \{ \langle x^*, z \rangle : x^* \in \partial\varphi(x) \}. \quad (2.2)$$

Concerning the continuity, we recall (see, e.g., [51,56]) that a lower semicontinuous convex function on a Banach space is continuous on the interior of its effective domain.

Three particular convex functions arise in general in many problems involving Modern Convex Analysis (see [8,30,48,54,56,57,62]). Given a nonempty closed convex set  $C$  of  $X$ , those functions correspond to the *indicator* and *support functions*  $\iota_C$  and  $\sigma_C$  [or  $\sigma(C, \cdot)$ ] of  $C$  respectively, and to the *distance function*  $d_C$  from the set  $C$ , defined by

$$\begin{aligned} \iota_C : X &\rightarrow \mathbb{R} \cup \{+\infty\} \quad \text{with } \iota_C(x) = 0 \text{ if } x \in C \text{ and } \iota_C(x) = +\infty \text{ if } x \notin C, \\ \sigma_C : X^* &\rightarrow \mathbb{R} \cup \{+\infty\} \quad \text{with } \sigma_C(x^*) := \sup_{x \in C} \langle x^*, x \rangle, \\ d_C : X &\rightarrow \mathbb{R} \quad \text{with } d_C(x) := \inf_{y \in C} \|x - y\|. \end{aligned}$$

From the definition of  $\sigma_C$ , we see that  $\sigma_C$  coincides with the Legendre–Fenchel conjugate of  $\iota_C$ , that is,  $\sigma_C = (\iota_C)^*$  where, for the above function  $\varphi$ , its *Legendre–Fenchel conjugate* is defined as

$$\varphi^* : X^* \rightarrow \mathbb{R} \cup \{-\infty, +\infty\} \quad \text{with} \quad \varphi^*(x^*) := \sup_{x \in X} (\langle x^*, x \rangle - \varphi(x)).$$

The Legendre–Fenchel conjugate is also related to the subdifferential. Indeed, for  $\varphi(x)$  finite, one has

$$x^* \in \partial\varphi(x) \Leftrightarrow \varphi^*(x^*) + \varphi(x) = \langle x^*, x \rangle,$$

so, provided the convex function  $\varphi$  is proper and lower semicontinuous, the set-valued operator  $\partial\varphi^* : X^* \rightrightarrows X$  is the inverse of the set-valued operator  $\partial\varphi$ ; this ensures, in particular when the Banach space  $X$  is reflexive, that the set-valued operator  $\partial\varphi : X \rightrightarrows X^*$  is surjective if and only  $\text{dom } \varphi^* = X^*$ . Hence in particular

$$\begin{cases} \partial\varphi \text{ is surjective whenever } \text{dom } \varphi \text{ is bounded and} \\ \varphi \text{ is bounded from below.} \end{cases} \quad (2.3)$$

Indeed, these boundedness properties taken together clearly imply that  $\varphi^*$  is finite on  $X^*$ , so  $\varphi^*$  is continuous on  $X^*$  (by the continuity property recalled above) and hence  $\text{Dom } \partial\varphi^* = X^*$  according to (2.2), which is equivalent (by a property recalled above) to the surjectivity of  $\partial\varphi$ . (Above,  $\text{Dom } \partial\varphi^*$  denotes the effective domain of  $\partial\varphi^*$ , where  $\text{Dom } M := \{s \in S : M(s) \neq \emptyset\}$  for any set-valued mapping  $M : S \rightrightarrows Y$  between two sets  $S$  and  $Y$ ).

For  $\varphi = \iota_C$  and  $x \in C$ , it is readily seen that  $x^* \in \partial\iota_C(x)$  if and only if  $\langle x^*, x' - x \rangle \leq 0$ , for all  $x' \in C$ , so  $\partial\iota_C(x)$  is the set  $N(C; x)$  of *outward normals* of the convex set  $C$  at the point  $x \in C$ ; the latter inequality characterization also says that

$$x^* \in N(C; x) \quad \text{if and only if} \quad \sigma(C, x^*) = \langle x^*, x \rangle \quad \text{and} \quad x \in C. \quad (2.4)$$

When  $X$  is a Hilbert space  $H$ , it is also clear from the inequality characterization above that

$$y - \text{proj}_C(y) \in N(C; \text{proj}_C(y)) \quad \text{for all } y \in H, \quad (2.5)$$

where  $\text{proj}_C(y)$  denotes the nearest point of  $y$  in  $C$ , hence  $\text{proj}_C$  is the metric projection onto  $C$ . For the normed space  $X$ , it is known and not difficult to see, for  $x \in C$ , that

$$N(C; x) \cap \mathbb{B}_{X^*} = \partial d_C(x) \quad \text{and} \quad \sigma(\partial d_C(x), z) = d'_C(x; z), \quad \forall z \in X, \quad (2.6)$$

where  $\mathbb{B}_{X^*} := \{x^* \in X^* : \|x^*\| \leq 1\}$  (resp.  $\mathbb{B}_X := \{x \in X : \|x\| \leq 1\}$ ) denotes the closed unit ball of  $X^*$  (resp.  $X$ ) centered at the origin.

The fundamental concepts of subdifferential or normal cone, directional derivative and Legendre–Fenchel conjugate will be at the heart of our present paper. From the definitions it directly follows the monotonicity property of the subdifferential of the convex function  $\varphi$  (resp. normal cone of the convex set  $C$ ) (property crucial for the paper), say

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq 0 \quad \text{for all } x_i^* \in \partial\varphi(x_i) \quad (\text{resp. } x_i^* \in N(C; x_i)).$$

It is worth mentioning that the converse (which is not obvious) also holds true, that is, a lower semicontinuous function  $\psi$  on a Banach space  $X$  is convex if and only if

$\partial_0\psi$  is monotone, where  $\partial_0$  is any subdifferential with appropriate fuzzy sum rule (see [22,52]) on the Banach space  $X$ . Another deep important property of the subdifferential in Convex Analysis (established in the Hilbert setting by Moreau [39, Proposition 12.b] and in the Banach setting by Rockafellar [53, Theorem A, p. 210] concerns the maximal monotonicity[10]):

**Theorem 2.1** (Moreau in Hilbert spaces and Rockafellar in Banach spaces) *Let  $X$  be a Banach space. Then, the subdifferential of a proper lower semicontinuous convex function  $\varphi$  on  $X$  is maximal monotone in the sense that there is no monotone set-valued operator from  $X$  into  $X^*$  whose graph is larger than the graph of  $\partial\varphi$ .*

## 2.2 Normal convex integrand

Assume that the normed space  $X$  is separable and complete; a set-valued mapping  $M$  from a measurable space  $(S, \mathcal{S})$  into closed subsets of  $X$  is  $\mathcal{S}$ -measurable provided that, for each open set  $U$  of  $X$ , one has  $M^{-1}(U) \in \mathcal{S}$ . So, following Rockafellar [55], an extended real-valued function  $\varphi : S \times X \rightarrow \mathbb{R} \cup \{+\infty\}$  is called a *normal integrand* whenever  $\varphi(s, \cdot)$  is proper and lower semicontinuous for all  $s \in S$  and the (epigraphical) set-valued mapping  $s \mapsto \text{epi } \varphi(s, \cdot)$  (from  $S$  into  $X \times \mathbb{R}$ ) is  $\mathcal{S}$  measurable. As usual,  $\mathcal{B}(X)$  denotes the Borel  $\sigma$ -field of  $X$ . When, in addition  $\varphi(s, \cdot)$  is convex for all  $s \in S$ , one says that  $\varphi$  is a normal convex integrand. For a set-valued mapping  $M : S \rightrightarrows X$  with  $\text{Dom } M \neq \emptyset$ , it is readily seen that the function  $(s, x) \mapsto \iota_{M(s)}(x)$  is a normal (convex) integrand if and only if the set-valued mapping  $M$  is measurable and takes on closed (convex) values. For a normal integrand, it is known (see, for example, [17,55]) that, for any measurable mapping  $u(\cdot) : S \rightarrow X$ , the function  $s \mapsto \varphi(s, u(s))$  is measurable. Furthermore,  $(s, x^*) \mapsto \varphi^*(s, x^*)$  is a normal convex integrand, where by convenience of notation

$$\varphi^*(s, x^*) := (\varphi(s, \cdot))^*(x^*) \quad \text{for all } x^* \in X^*.$$

Suppose that the separable Banach space  $X$  is reflexive and  $\mu$  is a  $\sigma$ -finite measure on  $\mathcal{S}$ . For any  $p \in [1, +\infty]$  denote by  $L_\mu^p(S, X)$  the vector space of classes of measurable mappings  $u(\cdot) : S \rightarrow X$  such that the function  $s \mapsto \|u(s)\|$  belongs to the standard space  $L_\mu^p(S, \mathbb{R})$  and denote by  $\|u\|_p$  the norm of  $u$  in  $L_\mu^p(S, X)$  given by the norm of  $s \mapsto \|u(s)\|$  in  $L_\mu^p(S, \mathbb{R})$ , so  $\|u\|_p = \left(\int_S \|u(s)\|^p d\mu(s)\right)^{1/p}$  for  $1 \leq p < +\infty$ , and similarly for  $p = +\infty$ . For any real  $p \in [1, +\infty[$ , taking  $q$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ , the topological dual of  $L_\mu^p(S, X)$  is identified with  $L_\mu^q(S, X^*)$  through the standard pairing. When  $\mu$  is the Lebesgue measure  $\lambda$  on some interval  $I$  of  $\mathbb{R}$  and there is no risk of confusion, we will just write  $L^p(I, X)$ .

With any  $p \in [1, +\infty]$  and any normal integrand  $\varphi : S \times X \rightarrow \mathbb{R} \cup \{+\infty\}$  one can associate the integral functional  $I_\varphi : L_\mu^p(S, X) \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  with

$$I_\varphi(u) := \int_S \varphi(s, u(s)) d\mu(s) \quad \text{for all } u(\cdot) \in L_\mu^p(S, X);$$

we recall that, for a measurable function  $\psi : S \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ , the extended real  $\int_S \psi(s) d\mu(s)$  is the infimum of integrals  $\int_S \rho(s) d\mu(s)$  of integrable real-valued functions  $\rho : S \rightarrow \mathbb{R}$  such that  $\psi(s) \leq \rho(s)$  for  $\mu$ -almost every  $s \in S$  (with the standard convention  $\inf \emptyset = +\infty$ ). One of the key results concerning normal convex integrand is the following theorem due to Rockafellar (see [55,56]). For various extensions outside the reflexive setting, we refer the reader to Castaing and Valadier [17].

**Theorem 2.2** (Rockafellar theorem on conjugate integrand) *Let  $X$  be a separable reflexive Banach space and let  $p \in [1, +\infty[$ . Let  $\varphi : S \times X \rightarrow \mathbb{R} \cup \{+\infty\}$  be a normal convex integrand such that  $I_\varphi$  is finite at some point in  $L_\mu^p(S, X)$ . Then, the Legendre–Fenchel conjugate of  $I_\varphi$  coincides with  $I_{\varphi^*}$ , that is,*

$$(I_\varphi)^*(u^*) = I_{\varphi^*}(u^*) := \int_S \varphi^*(s, u^*(s)) d\mu(s) \quad \text{for all } u^*(\cdot) \in L_\mu^q(S, X^*).$$

Although the above concepts and results are recalled in the context of normed spaces for completeness of their statements, the framework of the rest of the paper is that of a Hilbert space  $H$ .

### 3 Convex sweeping process and some variants in the literature

#### 3.1 Convex sweeping process

In 1971, Moreau [41,42] introduced the “sweeping process” (in the absolutely continuous framework) as the evolution differential inclusion

$$\frac{du}{dt}(t) \in -N(C(t); u(t)) \quad \text{for a.e. } t \in C(t), \quad \text{with } u(T_0) = u_0 \in C(T_0), \quad (3.1)$$

where  $0 \leq T_0 < T < +\infty$ ; for convenience, we will write sometimes, as usual,  $\dot{u}(t)$  in place of  $\frac{du}{dt}(t)$ . In an earlier paper [40], Moreau showed how such an evolution equation arises in the theory of elastic mechanical systems submitted to nonsmooth efforts as *dry friction*; note that the velocity in such cases may present discontinuity in time. He also provided later in a 1973 paper [43] more details on applications to elasticity and other fields of mechanics.

The paper [41] is concerned with the situation where the discontinuity of the velocity is exhibited by an absolute continuity property of the state of the system. The main result of that paper [41] can be stated as follows.

**Theorem 3.1** (Moreau’s theorem for absolutely continuous convex sweeping process) *Assume that the sets  $C(t)$  of the Hilbert space  $H$  are nonempty closed convex sets for which there is a nondecreasing absolutely continuous function  $v(\cdot) : [T_0, T] \rightarrow \mathbb{R}_+ := [0, +\infty[$  such that, for each  $y \in H$ ,*

$$d(y, C(t)) \leq d(y, C(s)) + v(t) - v(s) \quad \text{for all } T_0 \leq s \leq t \leq T.$$

Then, the evolution equation (3.1) admits one and only one absolutely continuous solution.

To take into account the more general situation where there are jumps, Moreau transformed the above model into a measure differential inclusion and proved in [45] an existence result that we give in the following form.

**Theorem 3.2** (Moreau's theorem for convex sweeping process with bounded variation) *Assume that the sets  $C(t)$  of the Hilbert space  $H$  are nonempty closed convex sets for which there is a positive Radon measure  $\mu$  on  $[T_0, T]$  such that, for each  $y \in H$ ,*

$$d(y, C(t)) \leq d(y, C(s)) + \mu([s, t]), \quad \text{for all } T_0 \leq s \leq t \leq T.$$

Then, the measure differential evolution inclusion

$$\begin{cases} du \in -N(C(t); u(t)) \\ u(T_0) = u_0 \in C(T_0) \end{cases} \quad (3.2)$$

admits one and only one right continuous solution with bounded variation.

A mapping  $u(\cdot) : [T_0, T] \rightarrow H$  is a solution of the measure differential inclusion in the theorem provided that it is right continuous with bounded variation with  $u(T_0) = u_0$  and  $u(t) \in C(t)$  for all  $t \in [T_0, T]$  and the differential measure  $du$  associated with  $u$  admits the derivative measure  $\frac{du}{d\mu}$  (see the next section for the meaning) as a density relative to  $\mu$  and

$$\frac{du}{d\mu}(t) \in -N(C(t); u(t)) \quad \text{for } \mu - \text{a.e. } t \in [T_0, T].$$

In [26] it is shown that  $u(\cdot)$  is a solution if and only if the latter inclusion is fulfilled with some positive Radon measure  $\nu$  on  $[T_0, T]$  in place of  $\mu$ .

### 3.2 An elasto-plastic model and hysteresis

Many problems from thermo-plasticity, phase transition (etc) in the literature lead to variational inequalities in the form below. Consider, for example, the following elasto-plastic one (see, e.g. [31]). Let  $Z$  be a closed convex set of the  $\frac{1}{2}N(N+1)$ -dimensional vector space  $E$  of symmetric tensors  $N \times N$ . Assume that the interior of  $Z$  is nonempty, so  $\text{int } Z \neq \emptyset$  corresponds to the elasticity domain and  $\text{bdry } Z$  to the plasticity. Write the strain tensor  $\varepsilon = (\varepsilon)_{i,j}$  (depending on time  $t$ ) as  $\varepsilon := \varepsilon^e + \varepsilon^p$ , where  $\varepsilon^e$  is the elastic strain and  $\varepsilon^p$  the plastic strain. The elastic strain  $\varepsilon^e$  is related to the stress tensor  $\sigma = (\sigma)_{i,j}$  linearly, that is,  $\varepsilon^e = A^2\sigma$ , where  $A$  is a (constant) symmetric positive definite matrix. The system is then subjected to the variational inequality:

$$\langle \dot{\varepsilon}^p(t), z \rangle \leq \langle \dot{\varepsilon}^p(t), \sigma(t) \rangle, \quad \forall z \in Z : \quad \text{principle of maximal dissipation}$$

and to the region constraint  $\sigma(t) \in Z$  for all  $t \in [0, T]$ ; in this system, the tensor strain  $\varepsilon$  is supposed to be given as an absolutely continuous mapping and the initial tensor stress  $\sigma_0$  is given in  $Z$ . Observing that the above inequality can be written as

$$\left\langle -A\dot{\sigma}(t) + A^{-1}\dot{\varepsilon}(t), A\sigma(t) - Az \right\rangle \geq 0,$$

Setting

$$\zeta(t) := A\sigma(t) - A^{-1}\varepsilon(t),$$

yields to the equivalent inequality

$$\left\langle -\dot{\zeta}(t), \zeta(t) - \left( Az - A^{-1}\varepsilon(t) \right) \right\rangle \geq 0 \quad \text{for all } z \in Z. \quad (3.3)$$

By setting,

$$C(t) := -A^{-1}\varepsilon(t) + A(Z),$$

problem (3.3) is reduced to the convex sweeping process

$$\begin{cases} \dot{\zeta}(t) \in -N(C(t); \zeta(t)) \\ \zeta(0) = A\sigma_0 - A^{-1}\varepsilon(0) \in C(0). \end{cases}$$

Clearly, we have

$$d(x, C(t)) \leq d(x, C(s)) + \|A^{-1}\| \int_s^t \|\dot{\varepsilon}(r)\| dr.$$

This provides according to Theorem 3.1 above (besides to [31, Proposition 2.2]) another proof of existence and uniqueness of solution for that system.

This defines a mapping  $\Phi : W^{1,1}([0, T], E)$  assigning to each absolutely continuous mapping  $\varepsilon \in W^{1,1}([0, T], E)$  the solution  $\Phi(\varepsilon) := \sigma_\varepsilon$  of the system. This mapping  $\Phi$  enjoys two particular properties:

• Rate independence

Denoting by  $\sigma_\varepsilon$  the solution associated with  $\varepsilon$  and taking any absolutely continuous increasing bijection  $\theta : [0, T] \rightarrow [0, T]$ , for almost every  $t \in [0, T]$ , we have  $\langle -A^2\dot{\sigma}_\varepsilon(\theta(t)) + \dot{\varepsilon}(\theta(t)), \sigma_\varepsilon(\theta(t)) - z \rangle \geq 0$  hence

$$\left\langle -A^2\dot{\sigma}_\varepsilon(\theta(t))\dot{\theta}(t) + \dot{\varepsilon}(\theta(t))\dot{\theta}(t), \sigma_\varepsilon(\theta(t)) - z \right\rangle \geq 0$$

(since  $\dot{\theta}(t) \geq 0$  a.e.), from which it can be obtained, for almost every  $t \in [0, T]$ ,

$$\begin{cases} \left\langle -A^2\frac{d}{dt}(\sigma_\varepsilon \circ \theta)(t) + \frac{d}{dt}(\varepsilon \circ \theta)(t), (\sigma_\varepsilon \circ \theta)(t) - z \right\rangle \geq 0 \\ (\sigma_\varepsilon \circ \theta)(0) = \sigma_0 \in C(0). \end{cases}$$

The uniqueness property guarantees that  $\sigma_\varepsilon \circ \theta$  is the solution associated with  $\varepsilon \circ \theta$ , otherwise stated,  $\Phi(\varepsilon \circ \theta) = \Phi(\varepsilon) \circ \theta$ . The latter equality is known in the literature as the *rate independence property* (see, e.g., [11,31,61]).

• Causality

For each  $\tau \in [0, T]$  and  $\varepsilon \in W^{1,1}([0, T], E)$ , denoting by  $\sigma_\varepsilon$  the solution on  $[0, T]$  of the system above, the restriction of  $\sigma|_{[0,\tau]}$  to  $[0, \tau]$  coincides with the solution on  $[0, \tau]$  of the system associated with  $\varepsilon|_{[0,\tau]}$  according to the same uniqueness property above. Then, for  $\varepsilon_i \in W^{1,1}([0, T], E)$  ( $i = 1, 2$ ), we have

$$\varepsilon_1|_{[0,\tau]} = \varepsilon_2|_{[0,\tau]} \Rightarrow \sigma_{\varepsilon_1}(t) = \sigma_{\varepsilon_2}(t) \quad \forall t \in [0, \tau],$$

otherwise stated

$$\varepsilon_1|_{[0,\tau]} = \varepsilon_2|_{[0,\tau]} \Rightarrow \Phi(\varepsilon_1)(t) = \Phi(\varepsilon_2)(t) \quad \forall t \in [0, \tau];$$

this is generally called the *causality property* (see, e.g., [31]).

Both rate independence and causality properties translate that  $\Phi$  is an *hysteresis operator* according to [11,31,61] where those properties are brought to light with various physical examples with hysteresis phenomena.

For several other models, we refer the reader to [11]. Of course, by Theorem 3.1 the mathematical features and properties above still hold in the context of a Hilbert space  $H$  with any closed convex set  $Z$  (without any condition on its interior) and any coercive bijective bounded symmetric linear operator  $A : H \rightarrow H$ .

### 3.3 Planning procedure

In mathematical economy, Henry [29] introduced, as mathematical model for the planning procedure, the differential inclusion

$$\dot{x}(t) \in \text{proj}_{T_K(x(t))} F(x(t)) \quad \text{a.e. } t \in [0, T], \quad x(0) = x_0 \in K,$$

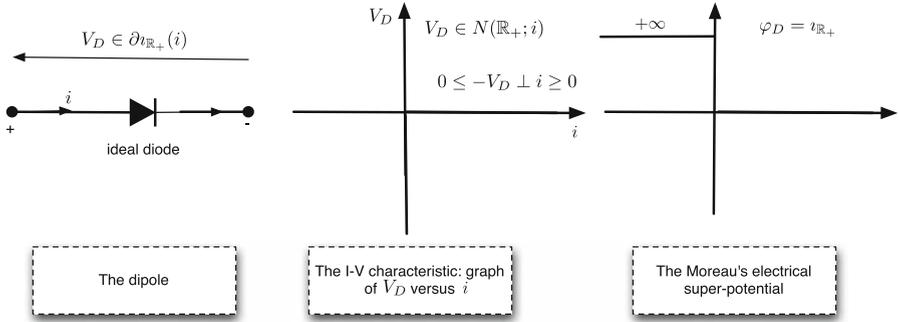
where  $K$  is a closed convex set of  $\mathbb{R}^N$ ,  $F : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$  is an upper semi-continuous set-valued mapping with nonempty compact convex values, and  $T_K(y)$  denotes the tangent cone of  $K$  at  $y$ . This differential inclusion is known (see [21,29]) to be completely linked to the following system

$$\dot{x}(t) \in -N(x(t); K) + F(x(t)), \quad x(0) = x_0 \in K,$$

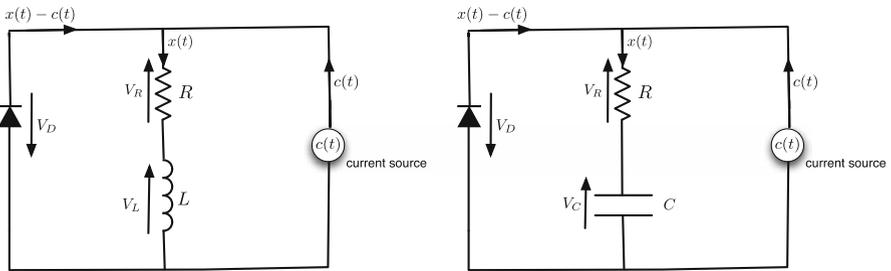
which enters in the following class of perturbed sweeping processes

$$\begin{cases} \dot{x}(t) \in -N(C(t); x(t)) + F(x(t)) \quad \text{a.e. } t \in [0, T] \\ x(0) = x_0 \in C(0), \end{cases}$$

where  $C(t)$  is, as in Sect. 3.1, a closed convex set moving in an absolutely continuous way. Existence results for such perturbed sweeping process are established in finite



**Fig. 1** Ideal diode model



**Fig. 2** Resistor–inductor–diode model (RLD, *left*) and resistor–capacitor–diode model (RCD, *right*)

dimensions in [14,37,58], and in [9,37] for the Hilbert setting under compactness growth conditions for the set-valued mapping  $F$ . Under compactness growth assumptions on  $F$ , existence of solution has been proved in [26] when the set  $C(t)$  moves with a bounded variation, and also in [37] when the set  $C(t)$  has a bounded retraction and  $F$  is weakly-norm upper semicontinuous.

### 3.4 Non-regular electrical circuits

The aim of this section is to illustrate the sweeping process in the theory of non-regular electrical circuits. Electrical devices like diodes are described in terms of Ampere–Volt characteristic (I, V) which is (possibly) a multifunction expressing the difference of potential  $V_D$  across the device as a function of current  $i$  going through the device. The diode is a device that constitutes a rectifier which permits the easy flow of charges in one direction but restrains the flow in the opposite direction. Figure 1 illustrates the ampere–volt characteristic of an ideal diode model.

Let us consider the left circuit depicted in Fig. 2 involving a load resistance  $R > 0$ , an inductor  $L > 0$ , a diode (assumed to be ideal) and a current source  $c(t)$ . Using Kirchhoff's laws, we have

$$V_R + V_L + V_D = 0.$$

Hence,

$$L\dot{x}(t) + Rx(t) \in -N(\mathbb{R}_+; x(t) - c(t)). \quad (3.4)$$

We have

$$\begin{aligned} N(\mathbb{R}_+; x(t) - c(t)) &= \partial_{I_{\mathbb{R}_+}}(x(t) - c(t)) = \partial_{I_{[c(t), +\infty[}}(x(t)) \\ &= N(C(t); x(t)), \text{ with } C(t) = [c(t), +\infty[. \end{aligned}$$

Therefore, the inclusion (3.4) is equivalent to

$$L\dot{x}(t) + Rx(t) \in -N(C(t); x(t)), \text{ with } C(t) = [c(t), +\infty[, \quad (3.5)$$

which is of the form (1.2).

The right circuit depicted in Fig. 2 involves a load resistance  $R \geq 0$ , a capacitor  $C > 0$ , a diode (assumed to be ideal) and a current source  $c(t)$ . Using Kirchhoff's laws, we have

$$V_R + V_C = -V_D.$$

Therefore,

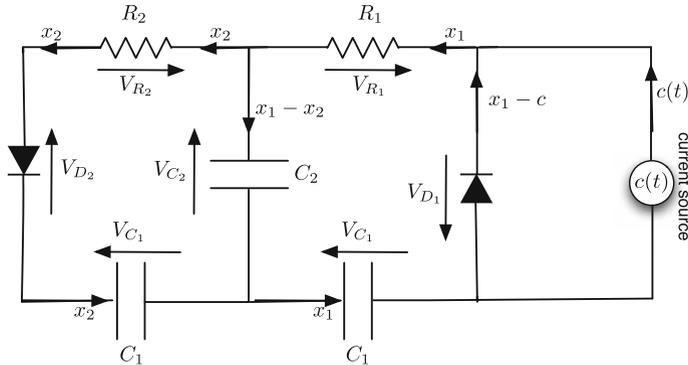
$$Rx(t) + \frac{1}{C} \int x(t)dt \in -N(\mathbb{R}_+; x(t) - c(t)). \quad (3.6)$$

If the charge on the capacitor is  $q$  and the current flowing in the circuit is  $x$ , then  $x(t) = \frac{dq}{dt}(t) = \dot{q}(t)$ . Hence (3.6) is equivalent to

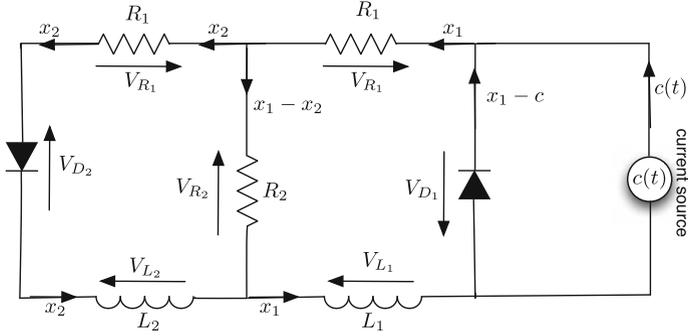
$$R\dot{q}(t) + \frac{1}{C}q(t) \in -N(C(t); \dot{q}(t)), \quad (3.7)$$

which is of the form (1.5).

Let us consider now the electrical system shown in Fig. 3 that is composed of two resistors  $R_1 \geq 0$ ,  $R_2 \geq 0$  with voltage/current laws  $V_{R_k} = R_k x_k$  ( $k = 1, 2$ ), three



**Fig. 3** Electrical circuit with resistors, capacitors and ideal diodes (RCD)



**Fig. 4** Electrical circuit with resistors, inductors and ideal diodes (RLD)

capacitors  $C_1 > 0$ ,  $C_2 > 0$  with voltage/current laws  $V_{C_k} = \frac{1}{C_k} \int x_k(t) dt$ ,  $k = 1, 2$  and two ideal diodes with characteristics  $0 \leq -V_{D_k} \perp i_k \geq 0$ . Using Kirchoff's laws, we have

$$\begin{cases} V_{R_1} + V_{C_1} + V_{C_2} = -V_{D_1} \in -N(\mathbb{R}_+; x_1 - c) \\ V_{R_2} + V_{C_1} - V_{C_2} = -V_{D_2} \in -N(\mathbb{R}_+; x_2). \end{cases}$$

Therefore the dynamics of this circuit is given by

$$\underbrace{\begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}}_{A_1} \underbrace{\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix}}_{\dot{q}} + \underbrace{\begin{pmatrix} \frac{1}{C_1} + \frac{1}{C_2} & -\frac{1}{C_2} \\ -\frac{1}{C_2} & \frac{1}{C_1} + \frac{1}{C_2} \end{pmatrix}}_{A_0} \underbrace{\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}}_q \in -N(C(t); \dot{q}(t)), \quad (3.8)$$

with  $C(t) = [c(t), +\infty[ \times ]0, +\infty[$  and  $\dot{q}_i(t) = x_i(t)$ ,  $i = 1, 2$ . We observe that if  $R_1 > 0$  and  $R_2 > 0$ , then (3.8) is equivalent to

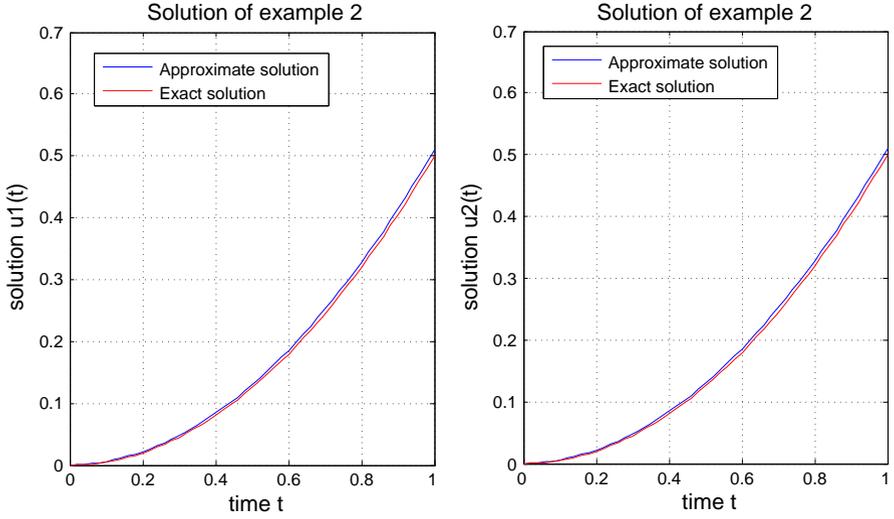
$$\dot{q}(t) = \text{proj}_{C(t)} \left( -A_1^{-1} A_0 q(t) \right),$$

which is an ordinary differential equation (see Remark 1, Sect. 5 for more detail). If  $R_1 = 0$  or  $R_2 = 0$ , then the operator  $[A_1 + N(C(t); \cdot)]^{-1}$  may be set-valued (see Example 1) and (3.8) is of the form (1.5). The same analysis holds for the dynamics (3.7) while  $R > 0$ .

In the same way, we can show that the dynamical behavior of the circuit depicted in Fig. 4 is given by the following sweeping process

$$\underbrace{\begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}}_{A_1} \underbrace{\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}}_{\dot{x}} + \underbrace{\begin{pmatrix} R_1 + R_2 & -R_2 \\ -R_2 & R_1 + R_2 \end{pmatrix}}_{A_0} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_x \in -N(C(t); x(t)), \quad (3.9)$$

with  $C(t) = [c(t), +\infty[ \times ]0, +\infty[$ . It is clear that (3.9) is of the form of the perturbed sweeping process (1.2).



**Fig. 5** Comparison of the exact and the approximate solution of Example 2

Some other circuits containing Zener diodes, transistors, rectifier–stabilizer circuits, DC–DC Buck and Boost converters can be analyzed in the same way [1–6]. The usage of tools from Modern Convex Analysis (and particularly the notion of Moreau’s convex superpotential) in electronics for the study of electrical circuits is fairly recent. It is a quite promising topic of research which may help engineers for the simulation of complicated electrical circuits. Due to the lack of smoothness in some circuits, most used softwares like Simulation Program with Integrated Circuits Emphasis (SPICE) can not simulate non-regular circuits without approximation of  $i$ – $v$  characteristic of the involved nonlinear electrical devices (Fig. 5).

In the next sections we study and prove existence of solutions of two *new variants* of Moreau’s sweeping process.

#### 4 Lipschitz single-valued perturbation variant of BV sweeping process

In this section we are concerned with the *differential inclusion*

$$\begin{cases} du \in -N(C(t); u(t)) + f(t, u(t)), \\ u(0) = u_0 \in C(0) \end{cases} \quad (4.1)$$

where  $f : I \times H \rightarrow H$  is a Carathéodory mapping and where the variation of  $C(t)$  is expressed by a given positive Radon measure  $\mu$  on  $I$  as in the line of Theorem 3.2. The case of a set-valued mapping  $F : I \times H \rightrightarrows H$  (in place of  $f$ ) has been studied in [15, 16] in the finite dimensional setting and in [26] under the assumption  $F(t, x) \subset \beta(t)(1 + \|x\|)K$  where  $K$  is a fixed normed compact subset of  $H$ . Our aim here is to study in the Hilbert setting the new variant where  $f$  satisfies a Lipschitz condition and no compactness condition is assumed.

Before defining the concept of solution of the measure differential inclusion (4.1), some preliminaries are necessary. Throughout the rest of this section, all the measures on a compact interval  $I = [T_0, T]$  of  $\mathbb{R}$  will be Radon measures.

We start this section by recalling some results from vector measures. For two positive Radon measures  $\nu$  and  $\hat{\nu}$  on  $I$  and for  $I(t, r) := I \cap [t - r, t + r]$ , it is known (see, e.g., [35, Theorem 2.12]) that the limit

$$\frac{d\hat{\nu}}{d\nu}(t) := \lim_{r \downarrow 0} \frac{\hat{\nu}(I(t, r))}{\nu(I(t, r))}$$

(with the convention  $\frac{0}{0} = 0$ ) exists and is finite for  $\nu$ -almost every  $t \in I$  and it defines a Borel function of  $t$ , called the *derivative* of  $\hat{\nu}$  with respect to  $\nu$ . Furthermore, the measure  $\hat{\nu}$  is absolutely continuous with respect to  $\nu$  if and only if  $\frac{d\hat{\nu}}{d\nu}(\cdot)$  is a density of  $\hat{\nu}$  relative to  $\nu$ , or otherwise stated, if and only if the equality  $\hat{\nu} = \frac{d\hat{\nu}}{d\nu}(\cdot)\nu$  holds true. Under such an absolute continuity assumption, a mapping  $u(\cdot) : I \rightarrow H$  is  $\hat{\nu}$ -integrable on  $I$  if and only if the mapping  $t \mapsto u(t)\frac{d\hat{\nu}}{d\nu}(t)$  is  $\nu$ -integrable on  $I$ ; furthermore, in that case,

$$\int_I u(t) d\hat{\nu}(t) = \int_I u(t) \frac{d\hat{\nu}}{d\nu}(t) d\nu(t). \quad (4.2)$$

When  $\nu$  and  $\hat{\nu}$  are each one absolutely continuous with respect to the other, we will say that they are *absolutely continuously equivalent*.

Now suppose that the mapping  $u(\cdot) : I \rightarrow H$  has *bounded variation* and denote by  $du$  the differential measure associated with  $u$  (see [23,44]); if in addition,  $u(\cdot)$  is right continuous, then

$$u(t) = u(s) + \int_{]s,t]} du \quad \text{for all } s, t \in I \text{ with } s \leq t.$$

Conversely, if there exists some mapping  $\hat{u}(\cdot) \in L^1_\nu(I, H)$  such that  $u(t) = u(T_0) + \int_{]T_0,t]} \hat{u} d\nu$  for all  $t \in I$ , then  $u(\cdot)$  is of bounded variation and right continuous and  $du = \hat{u} d\nu$ ; so  $\hat{u}(\cdot)$  is a *density* of the vector measure  $du$  relative to  $\nu$ . Then, putting  $I^-(t, r) := [t - r, t]$  and  $I^+(t, r) := [t, t + r]$ , according to Moreau and Valadier [49], for  $\nu$ -almost every  $t \in I$ , the following limits exist in  $H$  and

$$\hat{u}(t) = \frac{du}{d\nu}(t) := \lim_{r \downarrow 0} \frac{du(I(t, r))}{d\nu(I(t, r))} = \lim_{r \downarrow 0} \frac{du(I^-(t, r))}{d\nu(I^-(t, r))} = \lim_{r \downarrow 0} \frac{du(I^+(t, r))}{d\nu(I^+(t, r))}. \quad (4.3)$$

In particular, the last equality ensures that

$$\frac{du}{d\nu}(t) = \frac{du(\{t\})}{d\nu(\{t\})} \quad \text{and} \quad \frac{d\lambda}{d\nu}(t) = 0, \quad \text{whenever } \nu(\{t\}) > 0. \quad (4.4)$$

Above and in the rest of the paper  $\lambda$  denotes the Lebesgue measure.

**Definition 1** A mapping  $u : I \rightarrow H$  is a solution of the measure differential inclusion (4.1) if:

- (i)  $u(\cdot)$  is of bounded variation, right continuous, and satisfies  $u(T_0) = u_0$  and  $u(t) \in C(t)$  for all  $t \in I$ ;
- (ii) there exists a positive Radon measure  $\nu$  absolutely continuously equivalent to  $\mu + \lambda$  and with respect to which the differential measure  $du$  of  $u(\cdot)$  is absolutely continuous with  $\frac{du}{d\nu}(\cdot)$  as an  $L^1_\nu(I, H)$ -density and

$$\frac{du}{d\nu}(t) + f(t, u(t)) \frac{d\lambda}{d\nu}(t) \in -N(C(t); u(t)) \quad \nu - \text{a.e. } t \in I.$$

The following proposition concerning a particular chain rule for differential measures will be needed. Its statement is a consequence of a more general result from Moreau [44].

**Proposition 1** *Let  $H$  be a Hilbert space,  $\nu$  be a positive Radon measure on the closed bounded interval  $I$ , and  $u(\cdot) : I \rightarrow H$  be a right continuous with bounded variation mapping such that the differential measure  $du$  has a density  $\frac{du}{d\nu}$  relative to  $\nu$ . Then, the function  $\Phi : I \rightarrow \mathbb{R}$  with  $\Phi(t) := \|u(t)\|^2$  is a right continuous with bounded variation function whose differential measure  $d\Phi$  satisfies, in the sense of ordering of real measures,*

$$d\Phi \leq 2 \left\langle u(\cdot), \frac{du}{d\nu}(\cdot) \right\rangle d\nu.$$

The next result is a substitute of Grownwall's lemma relative to Radon measures. We refer, for example, to [37] for its statement (see also [38]).

**Lemma 1** *Let  $\nu$  be a positive Radon measure on  $[T_0, T]$  and let  $g(\cdot) \in L^1_\nu([T_0, T], \mathbb{R}_+)$ . Assume that, for a fixed real number  $\theta \geq 0$ , one has, for all  $t \in ]T_0, T]$ ,*

$$0 \leq g(t)\nu(\{t\}) \leq \theta < 1.$$

*Let  $\varphi \in L^\infty_\nu([T_0, T], \mathbb{R}_+)$  and let some fixed real number  $\alpha \geq 0$  satisfying, for all  $t \in [T_0, T]$ ,*

$$\varphi(t) \leq \alpha + \int_{]T_0, t]} g(s)\varphi(s) d\nu(s).$$

*Then, for all  $t \in [T_0, T]$ ,*

$$\varphi(t) \leq \alpha \exp \left\{ \frac{1}{1 - \theta} \int_{]T_0, t]} g(s)\varphi(s) d\nu(s) \right\}.$$

We establish now a stability property of the subdifferential of the distance function from a continuous moving set.

**Proposition 2** *Let  $E$  be a metric space,  $C : E \rightrightarrows H$  be a set-valued mapping with nonempty closed convex sets of a normed space  $X$ , and let  $t_0 \in \text{cl } Q$  with  $Q \subset E$ . Assume that there exists a non-negative real-valued function  $\eta : Q \rightarrow \mathbb{R}$  with  $\lim_{Q \ni t \rightarrow t_0} \eta(t) = 0$  such that, for all  $t \in Q$ ,*

$$d(x, C(t)) \leq d(x, C(t_0)) + \eta(t) \quad \text{for all } x \in X.$$

*Let  $(t_n)_n$  be a sequence in  $Q$  tending to  $t_0$  and let  $(x_n)_n$  be a sequence in  $H$  converging to some  $x \in C(t_0)$  with  $x_n \in C(t_n)$  for all  $n$ . Then, for all  $z \in X$ ,*

$$\limsup_{n \rightarrow \infty} d'_{C(t_n)}(x_n; z) \leq d'_{C(t_0)}(x; z).$$

*Proof* Let  $(t_n)_n$  and  $(x_n)_n$  be as in the statement. Fix any  $z \in X$ . Then, for each real  $\tau > 0$ , we have, for all  $n$ ,

$$\begin{aligned} d'_{C(t_n)}(x_n; z) &\leq \tau^{-1}[d_{C(t_n)}(x_n + \tau z) - d_{C(t_n)}(x_n)] = \tau^{-1}d_{C(t_n)}(x_n + \tau z) \\ &\leq \tau^{-1}[d_{C(t_0)}(x + \tau z) + \|x_n - x\| + \eta(t_n)], \end{aligned}$$

which gives

$$\limsup_{n \rightarrow \infty} d'_{C(t_n)}(x; z) \leq \tau^{-1}d_{C(t_0)}(x + \tau z) = \tau^{-1}[d_{C(t_0)}(x + \tau z) - d_{C(t_0)}(x)].$$

This justifies the desired inequality

$$\limsup_{n \rightarrow \infty} d'_{C(t_n)}(x; z) \leq d'_{C(t_0)}(x; z).$$

□

We can now prove, using some ideas from [26], the theorem concerning the above measure differential inclusion. The case of prox-regular sets  $C(t)$  will be treated elsewhere.

**Theorem 4.1** *Let  $H$  be a Hilbert space and  $C(\cdot) : [T_0, T] \rightrightarrows H$  be a set-valued mapping from  $[T_0, T]$  into the nonempty closed convex subsets of  $H$  for which there exists a positive Radon measure  $\mu$  on  $I := [T_0, T]$  such that for all  $s, t \in I$  with  $s \leq t$*

$$|d(y, C(t)) - d(y, C(s))| \leq \mu([s, t]) \quad \text{for all } y \in H.$$

*Let  $f : I \times H \rightarrow H$  be a mapping such that*

(i) there exists a non-negative function  $\beta(\cdot) \in L^1_\lambda(I, \mathbb{R})$  such that

$$\|f(t, x)\| \leq \beta(t)(1 + \|x\|) \quad \text{for all } x \in \bigcup_{t \in I} C(t);$$

(ii) for each real  $r > 0$ , the functions  $(f(\cdot, x))_{x \in r\mathbb{B}_H}$  are equicontinuous and there exists some non-negative function  $L_r(\cdot) \in L^1_\lambda(I, \mathbb{R})$  such that

$$\|f(t, x) - f(t, y)\| \leq L_r(t)\|x - y\| \quad \text{for all } t \in I, x, y \in r\mathbb{B}_H.$$

Then, for each  $u_0 \in C(T_0)$ , the following perturbed sweeping process

$$\begin{cases} -du \in N(C(t); u(t)) + f(t, u(t)) \\ u(T_0) = u_0 \end{cases}$$

has one and only one right continuous with bounded variation solution.

*Proof* I- First, let us suppose that

$$\int_{T_0}^T (\beta(s) + 1) d\lambda(s) \leq 1/4, \quad (4.5)$$

and let us construct a sequence of appropriate right continuous with bounded variation mappings.

Put

$$l := 2(\mu([T_0, T]) + \|u_0\| + 1),$$

and consider on  $I$  the positive Radon measure

$$\nu := \mu + (l + 1)(\beta(\cdot) + 1)\lambda. \quad (4.6)$$

As in Moreau [45] and Castaing and Monteiro Marques [16], consider the function  $v(\cdot) : I \rightarrow \mathbb{R}$  defined by

$$v(t) := \nu([T_0, t])$$

and set

$$V := v(T) = \nu([T_0, T]).$$

The function  $v(\cdot)$  is increasing and right continuous with  $v(T_0) = 0$ . Let  $(\varepsilon_n)_n$  be a sequence of positive real numbers with  $\varepsilon_n \downarrow 0$ . For each  $n \in \mathbb{N}$ , let  $0 = V_0^n < V_1^n < \dots < V_{q_n}^n = V$  be a partition such that

$$V_{j+1}^n - V_j^n \leq \varepsilon_n \forall j = 0, \dots, q_n - 1, \quad \text{and} \quad \left\{ V_0^n, \dots, V_{q_n}^n \right\} \subset \left\{ V_0^{n+1}, \dots, V_{q_{n+1}}^{n+1} \right\}. \quad (4.7)$$

Put  $V_{1+q_n}^n := V + \varepsilon_n$ . For each  $n \in \mathbb{N}$ , consider the partition of  $I$  associated with the subsets

$$J_j^n := v^{-1} \left( [V_j^n, V_{j+1}^n[ \right), \quad j = 0, 1, \dots, q_n,$$

and note that  $(J_j^m)_{j=0}^{q_m}$  is a refinement of  $(J_j^n)_{j=0}^{q_n}$  whenever  $m \geq n$ . Since  $v(\cdot)$  is increasing and right continuous, it is easy to see that, for each  $j = 0, 1, \dots, q_n$ , the set  $J_j^n$  is either empty or an interval of the form  $[r, s[$  with  $r < s$ . Furthermore, we have  $J_{q_n}^n = \{T\}$ . This produces an integer  $p(n) \in \mathbb{N}$  and a finite sequence

$$T_0 = t_0^n < t_1^n < \dots < t_{p(n)}^n = T$$

such that, for each  $i \in \{0, \dots, p(n) - 1\}$ , there is some  $j \in \{0, \dots, q_n - 1\}$  for which  $[t_i^n, t_{i+1}^n[ = J_j^n$ . It ensues that, for any  $i \in \{0, \dots, p(n) - 1\}$ ,

$$v([t_i^n, t]) = v(t) - v(t_i^n) \leq \varepsilon_n \quad \text{for all } t \in [t_i^n, t_{i+1}^n[, \quad (4.8)$$

which entails

$$v([t_i^n, t_{i+1}^n]) \leq \varepsilon_n, \quad (4.9)$$

hence (since  $\lambda \leq v$ )

$$t_{i+1}^n - t_i^n \leq \varepsilon_n. \quad (4.10)$$

For each  $i \in \{0, \dots, p(n) - 1\}$ , put

$$\sigma_i^n := (l + 1) \int_{t_i^n}^{t_{i+1}^n} (\beta(s) + 1) d\lambda(s) \quad \text{and} \quad \eta_i^n := t_{i+1}^n - t_i^n, \quad (4.11)$$

and observe that  $\eta_i^n \rightarrow 0$  as  $n \rightarrow \infty$ . For each  $i \in \{0, \dots, p(n) - 1\}$ , choose some  $s_i^n \in [t_i^n, t_{i+1}^n[$  such that

$$\beta(s_i^n) \leq \inf_{s \in [t_i^n, t_{i+1}^n[} \beta(s) + 1, \quad (4.12)$$

and define the function  $\rho_n : I \rightarrow I$  by  $\rho_n(T) := s_{p(n)-1}^n$  and

$$\rho_n(t) := s_i^n \text{ if } t \in [t_i^n, t_{i+1}^n[ \quad (0 \leq i \leq p(n) - 1). \quad (4.13)$$

Now, put  $u_0^n := u_0$ ,  $y_0^n := f(\rho_n(t_0^n), u_0^n)$  and  $u_1^n := \text{proj}_{C(t_1^n)}(u_0^n - \eta_0^n y_0^n)$ , and define by induction  $\{u_i^n : i = 0, \dots, p(n)\}$  and  $\{y_i^n : i = 0, \dots, p(n) - 1\}$  such that

$$y_i^n := f(\rho_n(t_i^n), u_i^n) \quad \text{and} \quad u_{i+1}^n := \text{proj}_{C(t_{i+1}^n)}(u_i^n - \eta_i^n y_i^n). \quad (4.14)$$

Fix any  $i \in \{0, \dots, p(n) - 1\}$ . From (4.14) we have, by the variation assumption of  $C(\cdot)$ ,

$$\begin{aligned} \|u_{i+1}^n - u_i^n + \eta_i^n y_i^n\| &= d_{C(t_{i+1}^n)}(u_i^n - \eta_i^n y_i^n) \leq d_{C(t_i^n)}(u_i^n - \eta_i^n y_i^n) + \mu(Jt_i^n, t_{i+1}^n) \\ &\leq \mu(Jt_i^n, t_{i+1}^n) + \eta_i^n \|y_i^n\|, \end{aligned} \quad (4.15)$$

which implies

$$\|u_{i+1}^n\| \leq \|u_i^n\| + \mu(Jt_i^n, t_{i+1}^n) + 2\eta_i^n \|y_i^n\|,$$

and hence

$$\|u_{i+1}^n\| \leq \|u_0^n\| + \sum_{k=0}^i (\mu(Jt_k^n, t_{k+1}^n) + 2\eta_k^n \|y_k^n\|). \quad (4.16)$$

On the other hand, from assumption (i),

$$\|y_i^n\| \leq \beta(\rho_n(t_i^n)) (1 + \|u_i^n\|) \leq \beta(\rho_n(t_i^n)) \left(1 + \max_{0 \leq k \leq p(n)} \|u_k^n\|\right), \quad (4.17)$$

and this latter inequality combined with (4.16) yields

$$\|u_{i+1}^n\| \leq \|u_0^n\| + \sum_{k=0}^i \mu(Jt_k^n, t_{k+1}^n) + 2 \left(1 + \max_{0 \leq k \leq p(n)} \|u_k^n\|\right) \sum_{k=0}^i \eta_k^n \beta(\rho_n(t_k^n)).$$

Noting by (4.12) that

$$\sum_{k=0}^i \eta_k^n \beta(\rho_n(t_k^n)) = \sum_{k=0}^i \int_{t_k^n}^{t_{k+1}^n} \beta(\rho_n(t_k^n)) d\lambda(s) \leq \int_{T_0}^{t_{i+1}^n} (\beta(s) + 1) d\lambda(s),$$

we obtain

$$\|u_{i+1}^n\| \leq \|u_0^n\| + \mu(JT_0, T) + 2 \left(1 + \max_{0 \leq k \leq p(n)} \|u_k^n\|\right) \int_{T_0}^T (\beta(s) + 1) d\lambda(s).$$

Thanks to (4.5) it results that

$$\max_{0 \leq k \leq p(n)} \|u_k^n\| \leq \|u_0^n\| + \mu(JT_0, T) + \frac{1}{2} \left(1 + \max_{0 \leq k \leq p(n)} \|u_k^n\|\right),$$

which, according to the definition of  $l$ , gives

$$\max_{0 \leq k \leq n} \|u_k^n\| \leq 2 \left( \mu(JT_0, T) + \|u_0\| + \frac{1}{2} \right) \leq l. \quad (4.18)$$

The latter inequality, combined with (4.12), yields

$$\eta_i^n \|y_i^n\| \leq \eta_i^n \beta(\rho_n(t_i^n)) (1 + \|u_i^n\|) \leq (l+1) \int_{t_i^n}^{t_{i+1}^n} (\beta(s) + 1) d\lambda(s) = \sigma_i^n. \quad (4.19)$$

Consequently, by (4.15),

$$d_{C(t_{i+1}^n)}(u_i^n - \eta_i^n y_i^n) \leq \mu([t_i^n, t_{i+1}^n]) + \eta_i^n \|y_i^n\| \leq \mu([t_i^n, t_{i+1}^n]) + \sigma_i^n, \quad (4.20)$$

and hence by (4.6) and (4.11) [since  $u_{i+1}^n = \text{proj}_{C(t_{i+1}^n)}(u_i^n - \eta_i^n y_i^n)$ ]

$$\|u_{i+1}^n - u_i^n + \eta_i^n y_i^n\| \leq \mu([t_i^n, t_{i+1}^n]) + \sigma_i^n \leq \nu([t_i^n, t_{i+1}^n]). \quad (4.21)$$

**Step 1.** Construction of the sequence  $(u_n(\cdot))$ .

Following [16,45], define the mapping  $u_n(\cdot) : I \rightarrow H$  by  $u_n(T) := u_{p(n)}^n$  and

$$u_n(t) = u_i^n + \frac{\nu([t_i^n, t])}{\nu([t_i^n, t_{i+1}^n])} (u_{i+1}^n - u_i^n + \eta_i^n y_i^n) - (t - t_i^n) y_i^n \quad \text{if } t \in [t_i^n, t_{i+1}^n]. \quad (4.22)$$

We observe that  $u_n(\cdot)$  is well defined on  $I$  and it is right continuous with bounded variation on each interval  $[t_i^n, t_{i+1}^n]$ , so it is right continuous with bounded variation on the whole interval  $I$ . Furthermore, the definition of  $u_n(\cdot)$  can be rewritten, for any  $t \in I$ , as

$$u_n(t) = u_n(T_0) + \int_{]T_0, t[} \Pi_n(s) d\nu(s) - \int_{]T_0, t[} f(\rho_n(s), u_n(\delta_n(s))) d\lambda(s),$$

where

$$\Pi_n(t) := \sum_{i=0}^{p(n)-1} \frac{u_{i+1}^n - u_i^n + \eta_i^n y_i^n}{\nu([t_i^n, t_{i+1}^n])} \mathbf{1}_{]t_i^n, t_{i+1}^n[}(t),$$

and  $\delta_n(s) := t_i^n$  if  $t \in [t_i^n, t_{i+1}^n[$  and  $\delta_n(T) := t_{p(n)-1}^n$ . Since, by (4.6), the measure  $\lambda$  is absolutely continuous with respect to  $\nu$ , it has  $\frac{d\lambda}{d\nu}(\cdot)$  as a density in  $L_v^\infty(I, \mathbb{R}_+)$  relative to  $\nu$  and then by (4.2), for every  $t \in I$ ,

$$u_n(t) = u_n(T_0) + \int_{]T_0, t[} \left\{ \Pi_n(s) - f(\rho_n(s), u_n(\delta_n(s))) \frac{d\lambda}{d\nu}(s) \right\} d\nu(s).$$

This tells us that the vector measure  $du_n$  has the latter integrand as a density in  $L_v^\infty(I, H)$  relative to  $\nu$ , so by the first equality in (4.3)

$$\frac{du_n}{d\nu}(\cdot) \text{ is a density of } du_n \text{ with respect to } \nu, \quad (4.23)$$

and, for  $\nu$ -almost every  $t \in I$ ,

$$\frac{du_n}{d\nu}(t) + f(\rho_n(t), u_n(\delta_n(t))) \frac{d\lambda}{d\nu}(t) = \Pi_n(t) = \sum_{i=0}^{p(n)-1} \frac{u_{i+1}^n - u_i^n + \eta_i^n y_i^n}{\nu([t_i^n, t_{i+1}^n])} \mathbf{1}_{[t_i^n, t_{i+1}^n]}(t). \quad (4.24)$$

Taking (4.21) into account, it results that

$$\left\| \frac{du_n}{d\lambda}(t) + f(\rho_n(t), u_n(\delta_n(t))) \frac{d\lambda}{d\nu}(t) \right\| \leq 1 \quad \nu - \text{a.e. } t \in I. \quad (4.25)$$

On the other hand, by (4.6) again, the measure  $(l+1)(\beta(\cdot)+1)\lambda$  is absolutely continuous with respect to  $\nu$ , thus it has  $\frac{d((l+1)(\beta(\cdot)+1)\lambda)}{d\nu}$  as a density relative to  $\nu$  with, for  $\nu$ -almost every  $t \in I$ ,

$$0 \leq (l+1)(\beta(t)+1) \frac{d\lambda}{d\nu}(t) = \frac{d((l+1)(\beta(\cdot)+1)\lambda)}{d\nu} \leq 1. \quad (4.26)$$

Note also, by (4.12) and (4.18), that

$$\left\| f(\rho_n(t), u_n(\delta_n(t))) \right\| \leq (l+1)(\beta(t)+1), \quad \text{for all } t \in I,$$

which ensures, for  $\nu$ -almost every  $t \in I$ ,

$$\left\| f(\rho_n(t), u_n(\delta_n(t))) \frac{d\lambda}{d\nu}(t) \right\| \leq (l+1)(\beta(t)+1) \frac{d\lambda}{d\nu}(t) \leq 1, \quad (4.27)$$

hence

$$\left\| \frac{du_n}{d\nu}(t) \right\| \leq 2. \quad (4.28)$$

Defining the functions  $\theta_n(\cdot) : I \rightarrow I$  by  $\theta_n(T_0) := T_0$  and

$$\theta_n(t) := t_{i+1}^n \quad \text{if } t \in [t_i^n, t_{i+1}^n] \quad (0 \leq i \leq p(n)-1), \quad (4.29)$$

we also see by (2.5) that, for  $\nu$ -almost every  $t \in I$ ,

$$\frac{du_n}{d\nu}(t) + f(\rho_n(t), u_n(\delta_n(t))) \frac{d\lambda}{d\nu}(t) \in -N(C(\theta_n(t)); u_n(\theta_n(t))),$$

and hence according to (2.6) and (4.25)

$$\frac{du_n}{dv}(t) + f(\rho_n(t), u_n(\delta_n(t))) \frac{d\lambda}{dv}(t) \in -\partial d_{C(\theta_n(t))}(u_n(\theta_n(t))). \quad (4.30)$$

**Step 2.** Cauchy property of  $(u_n(\cdot))_n$ .

Consider any integers  $n, m \in \mathbb{N}$ . Since  $u_0^n = u_0 \in C(t_0^n)$  and  $u_{i+1}^n = \text{proj}_{C(t_{i+1}^n)}(u_i^n - \eta_i^n y_i^n)$ , we note by (4.22) and (4.29) that

$$u_n(\theta_n(t)) \in C(\theta_n(t)) \quad \text{for all } t \in I. \quad (4.31)$$

This allows us to write, for every  $t \in I$ ,

$$\begin{aligned} d_{C(\theta_n(t))}(u_m(t)) &= d_{C(\theta_n(t))}(u_m(t)) - d_{C(\theta_m(t))}(u_m(\theta_m(t))) \\ &\leq d_{C(\theta_n(t))}(u_m(t)) - d_{C(\theta_m(t))}(u_m(t)) + \|u_m(\theta_m(t)) - u_m(t)\|, \end{aligned}$$

and hence, according to the variation assumption on  $C(\cdot)$  and to the fact that one of the partitions  $(J_j^m)_{j=0}^{q_m}$  and  $(J_j^n)_{j=0}^{q_n}$  is a refinement of the other (depending on either  $n \leq m$  or  $m < n$ ),

$$d_{C(\theta_n(t))}(u_m(t)) \leq \max\{\mu(\lceil t, \theta_n(t) \rceil), \mu(\lceil t, \theta_m(t) \rceil)\} + \|u_m(\theta_m(t)) - u_m(t)\|, \quad (4.32)$$

so we see through (4.28) that

$$d_{C(\theta_n(t))}(u_m(t)) \leq \mu(\lceil t, \theta_n(t) \rceil) + \mu(\lceil t, \theta_m(t) \rceil) + 2v(\lceil t, \theta_m(t) \rceil). \quad (4.33)$$

Putting

$$\gamma_n(t) := \mu(\lceil t, \theta_n(t) \rceil) + v(\lceil t, \theta_n(t) \rceil), \quad (4.34)$$

we obtain

$$d_{C(\theta_n(t))}(u_m(t)) \leq \gamma_n(t) + 2\gamma_m(t). \quad (4.35)$$

Consequently, by (4.30), we deduce, for  $v$ -almost every  $t \in I$ ,

$$\begin{aligned} \left\langle \frac{du_n}{dv}(t) + f(\rho_n(t), u_n(\delta_n(t))) \frac{d\lambda}{dv}(t), u_n(\theta_n(t)) - u_m(t) \right\rangle &\leq d_{C(\theta_n(t))}(u_m(t)) \\ &\leq \gamma_n(t) + 2\gamma_m(t), \end{aligned}$$

which entails by (4.25) and (4.28)

$$\left\langle \frac{du_n}{dv}(t) + f(\rho_n(t), u_n(\delta_n(t))) \frac{d\lambda}{dv}(t), u_n(t) - u_m(t) \right\rangle \quad (4.36)$$

$$\leq \gamma_n(t) + 2\gamma_m(t) + \|u_n(t) - u_n(\theta_n(t))\| \leq \gamma_n(t) + 2\gamma_m(t) + 2v(\lceil t, \theta_n(t) \rceil). \quad (4.37)$$

On the other hand, for every  $t \in I$ , write

$$\begin{aligned} & \{f(\rho_n(t), u_n(\delta_n(t))) - f(t, u_n(t))\} \frac{d\lambda}{dv}(t) \\ &= \{f(\rho_n(t), u_n(\delta_n(t))) - f(t, u_n(\delta_n(t)))\} \frac{d\lambda}{dv}(t) \\ & \quad + \{f(t, u_n(\delta_n(t))) - f(t, u_n(t))\} \frac{d\lambda}{dv}(t), \end{aligned}$$

and observe by the equicontinuity assumption in (ii) that the first expression  $\{\cdot\} \frac{d\lambda}{dv}(t)$  in the right-hand side tends to 0 as  $n \rightarrow \infty$  since there is some real  $r > 0$  such that  $\|u_n(t)\| \leq r$ , for all  $t \in I$  and  $n \in \mathbb{N}$ , according to (4.18), (4.19) and (4.22). By the inequality

$$\|u_n(\delta_n(t)) - u_n(t)\| \frac{d\lambda}{dv}(t) \leq 2\nu(\lfloor \delta_n(t), t \rfloor) \frac{d\lambda}{dv}(t)$$

[due to (4.28)] and by (4.4), we also see that  $\|u_n(\delta_n(t)) - u_n(t)\| \rightarrow 0$  and hence the second expression  $\{\cdot\} \frac{d\lambda}{dv}(t)$  tends to 0 as  $n \rightarrow \infty$  according to the Lipschitz property of  $f(t, \cdot)$  on  $r\mathbb{B}_H$  in the assumption (ii). Consequently,

$$\{f(\rho_n(t), u_n(\delta_n(t))) - f(t, u_n(t))\} \frac{d\lambda}{dv}(t) \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (4.38)$$

and, for

$$\varphi_{n,m}(t) := \frac{d\lambda}{dv}(t) \|f(\rho_n(t), u_n(\delta_n(t))) - f(t, u_n(t))\| \|u_n(t) - u_m(t)\|,$$

we have  $\int_{\lfloor T_0, T \rfloor} \varphi_{n,m}(t) dv(t) \rightarrow 0$  as  $n, m \rightarrow \infty$  by the Lebesgue dominated convergence theorem (because  $\|u_n(t)\| \leq r$  as seen above). From this and (4.36), for any  $n, m$ , it ensures that, for  $\nu$ -almost every  $t \in I$ ,

$$\begin{aligned} & \left\langle \frac{du_n}{dv}(t) - \frac{du_m}{dv}(t), u_n(t) - u_m(t) \right\rangle \\ & \leq L_r(t) \frac{d\lambda}{dv}(t) \|u_n(t) - u_m(t)\|^2 + 4(\gamma_n(t) + \gamma_m(t)) + 2\nu(\lfloor t, \theta_n(t) \rfloor) \\ & \quad + 2\nu(\lfloor t, \theta_m(t) \rfloor) + \varphi_{n,m}(t) + \varphi_{m,n}(t), \end{aligned}$$

and, for

$$\begin{aligned} \alpha_{n,m} := & \int_{\lfloor T_0, T \rfloor} \{4(\gamma_n(t) + \gamma_m(t)) + 2\nu(\lfloor t, \theta_n(t) \rfloor) + 2\nu(\lfloor t, \theta_m(t) \rfloor) + \varphi_{n,m}(t) \\ & + \varphi_{m,n}(t)\} dv(t), \end{aligned}$$

we have  $\alpha_{n,m} \rightarrow 0$  as  $n, m \rightarrow \infty$ . On the other hand, Proposition 1 says that

$$d(\|u_n(\cdot) - u_m(\cdot)\|^2) \leq 2 \left\langle \frac{du_n}{dv}(\cdot) - \frac{du_m}{dv}(\cdot), u_n(\cdot) - u_m(\cdot) \right\rangle,$$

thus, putting  $\psi_{n,m}(t) := \|u_n(t) - u_m(t)\|^2$  and noting that  $u_n(T_0) = u_m(T_0)$ , we deduce that, for all  $t \in I$ ,

$$\psi_{n,m}(t) \leq \int_{]T_0, t]} 2L_r(s) \frac{d\lambda}{dv}(s) \psi_{n,m}(s) dv(s) + \alpha_{n,m}.$$

Noting that  $L_r(s) \frac{d\lambda}{dv}(s) \nu(\{s\}) = 0$  for all  $s$  (since  $\frac{d\lambda}{dv}(s) = 0$  if  $\nu(s) > 0$  according to (4.4)) we can apply Lemma 1 and this yields

$$\psi_{n,m}(t) \leq \alpha_{n,m} \exp \left( \int_{]T_0, t]} 2L_r(s) \frac{d\lambda}{dv}(s) dv(s) \right) = \alpha_{n,m} \exp \left( \int_{]T_0, t]} 2L_r(s) d\lambda(s) \right),$$

thus

$$\sup_{t \in I} \psi_{n,m}(t) \leq \alpha_{n,m} \exp \left( \int_{]T_0, T]} 2L_r(s) d\lambda(s) \right).$$

This ensures that the sequence  $(u_n(\cdot))_n$  satisfies the Cauchy property with respect to the norm of uniform convergence on the space of all bounded mappings from  $I$  into  $H$ . Consequently, this sequence  $(u_n(\cdot))_n$  converges uniformly on  $I$  to some mapping  $u(\cdot)$ . This also tells us that the mapping  $u(\cdot)$  does not depend on the partition  $V_0^n < \dots < V_{q_n}^n$  of  $[0, V]$  satisfying (4.7). Furthermore, by (4.28), extracting a subsequence if necessary, we may suppose that  $(\frac{du_n}{dv}(\cdot))_n$  converges weakly in  $L_v^2(I, H)$  to some mapping  $h(\cdot) \in L_v^2(I, H)$ , so, for every  $t \in I$ ,

$$\int_{]T_0, t]} \frac{du_n}{dv}(s) dv(s) \longrightarrow \int_{]T_0, t]} h(s) dv(s) \text{ weakly in } H.$$

Since  $\frac{du_n}{dv}(\cdot)$  is, by (4.23), a density of  $du_n$  relative to  $\nu$ , we also have  $u_n(t) = u_0 + \int_{]T_0, t]} \frac{du_n}{dv}(s) dv(s)$ , thus it ensues that  $u(t) = u_0 + \int_{]T_0, t]} h(s) dv(s)$ , and this tells us that  $u(\cdot)$  is right continuous with bounded variation on  $I$ , and the vector measure  $du$  has  $h(\cdot) \in L_v^2(I, H)$  as a density relative to  $\nu$  and  $\frac{du}{dv}(\cdot) = h(\cdot)$   $\nu$ -a.e. We also deduce that

$$\frac{du_n}{dv}(\cdot) \longrightarrow \frac{du}{dv}(\cdot) \text{ weakly in } L_v^2(I, H).$$

**Step 3.** Let us prove that  $u(\cdot)$  is a solution.

First, for each  $t \in I$ , noting by (4.10) that  $0 \leq \theta_n(t) - t \leq \varepsilon_n$ , and writing by (4.28)

$$\|u_n(\theta_n(t)) - u(t)\| \leq \|u_n(t) - u(t)\| + 2v(\lfloor t, \theta_n(t) \rfloor),$$

we see that, as  $n \rightarrow \infty$ ,

$$\theta_n(t) \downarrow t \quad \text{and} \quad u_n(\theta_n(t)) \rightarrow u(t).$$

Furthermore, taking into account the closedness of  $C(t)$  and observing, by the variation assumption, that

$$\begin{aligned} d_{C(t)}(u_n(\theta_n(t))) &\leq d_{C(\delta_n(t))}(u_n(\theta_n(t))) + \mu(\lfloor \delta_n(t), t \rfloor) \\ &\leq d_{C(\delta_n(t))}(u_n(\delta_n(t))) + \|u_n(\theta_n(t)) - u_n(\delta_n(t))\| + \mu(\lfloor \delta_n(t), t \rfloor) \\ &= \|u_n(\theta_n(t)) - u_n(\delta_n(t))\| + \mu(\lfloor \delta_n(t), t \rfloor), \end{aligned}$$

we also see (since  $\mu(\{t\}) = v(\{t\})$ ) that

$$u(t) \in C(t) \quad \text{for every } t \in I \text{ with } v(\{t\}) = 0.$$

We also know by construction that  $u_n(T_0) \in C(T_0)$  and  $u_n(T) \in C(T)$ , so  $u(T_0) \in C(T_0)$  and  $u(T) \in C(T)$ . Consider now any  $\bar{t} \in ]T_0, T[$  with  $v(\{\bar{t}\}) > 0$ . We may choose the partitions  $V_0^n < \dots < V_{q_n}^n$  (satisfying (4.7)) such that  $v(\bar{t}) \in \{V_1^n, \dots, V_{q_n}^n\}$ , say  $v(\bar{t}) = V_{j_n(\bar{t})+1}^n$  for some  $j_n(\bar{t}) \in \{0, \dots, q_n - 1\}$ . It follows that  $J_{j_n(\bar{t})+1}^n = v^{-1}([V_{j_n(\bar{t})+1}^n, V_{j_n(\bar{t})+2}^n])$  is of the form  $[\bar{t}, \tau[$  with  $\tau > \bar{t}$ , thus  $\bar{t} = t_{i_n(\bar{t})+1}^n$  for some  $i_n(\bar{t}) \in \{0, \dots, p(n) - 1\}$ . It results, for all  $n$ , that  $\theta_n(\bar{t}) = \bar{t}$  hence  $u_n(\bar{t}) \in C(\bar{t})$ , and this gives  $u(\bar{t}) \in C(\bar{t})$ , since  $u_n(\bar{t}) \rightarrow u(\bar{t})$  as  $n \rightarrow \infty$  according to the aforementioned independence property of  $u(\cdot)$  with respect to the partition  $V_0^n < \dots < V_{q_n}^n$ . So,

$$u(t) \in C(t) \quad \text{for all } t \in I. \tag{4.39}$$

Now let us show that

$$\frac{du}{dv}(t) + f(t, u(t)) \frac{d\lambda}{dv}(t) \in -N(C(t); u(t)) \quad v - \text{a.e. } t \in I.$$

As above, we write by (4.28)

$$\|u_n(\delta_n(t)) - u(t)\| \leq \|u_n(t) - u(t)\| + 2v(\lfloor \delta_n(t), t \rfloor),$$

which entails, for each  $t \in I$  with  $v(\{t\}) = 0$ , that  $u_n(\delta_n(t)) \rightarrow u(t)$ . Using this and the assumption (ii), for each  $t \in I$  with  $v(\{t\}) = 0$ , we can see (as for (4.38)) that

$f(\rho_n(t), u_n(\delta_n(t))) \frac{d\lambda}{d\nu}(t) \rightarrow f(t, u(t)) \frac{d\lambda}{d\nu}(t)$  as  $n \rightarrow \infty$ . Moreover, for each  $t \in I$  with  $\nu(\{t\}) > 0$ , we have  $\frac{d\lambda}{d\nu}(t) = 0$  by (4.4), so it results, for any  $t \in I$ , that

$$e_n(t) := f(\rho_n(t), u_n(\delta_n(t))) \frac{d\lambda}{d\nu}(t) \rightarrow f(t, u(t)) \frac{d\lambda}{d\nu}(t) =: e(t)$$

as  $n \rightarrow \infty$ . The Lebesgue dominated convergence theorem then yields that  $(e_n(\cdot))_n$  converges strongly to  $e(\cdot)$  in  $L^2_\nu(I, H)$ . Putting  $\zeta_n(t) := \frac{du_n}{d\nu}(t) + e_n(t)$  and  $\zeta(t) := \frac{du}{d\nu}(t) + e(t)$ , the sequence  $(\zeta_n(\cdot))_n$  converges weakly in  $L^2_\nu(I, H)$  to  $\zeta(\cdot)$  and by Mazur's lemma there exists a sequence  $(\xi_n(\cdot))_n$  converging strongly in  $L^2_\nu(I, H)$  to  $\zeta(\cdot)$  with

$$\xi_n(\cdot) \in \text{conv}\{\zeta_k : k \geq n\}, \quad \text{for all } n \in \mathbb{N}.$$

This sequence  $(\xi_n(\cdot))_n$  has a subsequence (that we do not relabel) converging  $\nu$ -almost everywhere to  $\zeta(\cdot)$ , hence, there is some Borel set  $I_0 \subset I$  with  $\nu(I \setminus I_0) = 0$  such that, for all  $t \in I_0$ ,

$$\zeta(t) \in \bigcap_n \overline{\text{conv}}\{\zeta_k(t) : k \geq n\}.$$

Fixing any  $t \in I_0$  and any  $z \in H$ , it results from (4.30) that

$$\langle -\zeta(t), z \rangle \leq \inf_{n \in \mathbb{N}} \sup_{k \geq n} \langle -\zeta_k(t), z \rangle \leq \limsup_{n \rightarrow \infty} d'_{C(\theta_n(t))}(u_n(\theta_n(t)); z),$$

and, since  $u_n(\theta_n(t)) \rightarrow u(t)$  with  $u(t) \in C(t)$  [see (4.39)], Proposition 2 implies that

$$\langle -\zeta(t), z \rangle \leq d'_{C(t)}(u(t); z).$$

The latter inequality means, for each  $t \in I_0$ , that  $-\zeta(t) \in \partial d_{C(t)}(u(t))$  hence

$$\frac{du}{d\nu}(t) + f(t, u(t)) \frac{d\lambda}{d\nu}(t) \in -N(C(t); u(t)), \quad (4.40)$$

which finishes the proof of existence of a solution in the case where  $\int_{T_0}^T (\beta(s) + 1) d\lambda(s) \leq 1/4$ .

**II.** Case where  $\int_{T_0}^T (\beta(s) + 1) d\lambda(s) > 1/4$ .

First, from (4.40), we note that the mapping  $u(\cdot)$  in the above case is also a solution with the measure  $\mu + \lambda$  in place of  $\nu$  therein, since the measure  $\mu + \lambda$  is absolutely continuous with respect to  $\nu$  and vice versa. Let  $T_0, T_1, \dots, T_p$  be a subdivision of  $[T_0, T]$  such that, for each  $i = 1, \dots, p$ ,

$$\int_{T_{i-1}}^{T_i} (\beta(s) + 1) d\lambda(s) \leq \frac{1}{4}.$$

For each  $i = 1, \dots, p$ , denote by  $\mu_i$  the Radon measure induced on  $[T_{i-1}, T_i]$  by  $\mu$  and set  $\nu_i := \mu_i + \lambda$ . Then, the part I provides a right continuous with bounded variation mapping  $u_1 : [T_0, T_1] \rightarrow H$  such that  $u_1(T_0) = u_0$ ,  $u_1(t) \in C(t)$  for all  $t \in [T_0, T_1]$ ,  $du_1$  has  $\frac{du_1}{d\nu_1}$  as a density in  $L^1_{\nu_1}([T_0, T_1], H)$  relative to  $\nu_1$ , and

$$\frac{du_1}{d\nu_1}(t) + f(t, u_1(t)) \frac{d\lambda}{d\nu_1}(t) \in -N(C(t); u_1(t)) \quad \nu_1 - \text{a.e. } t \in [T_0, T_1].$$

Similarly, there is a right continuous with bounded variation mapping  $u_2 : [T_1, T_2] \rightarrow H$  such that  $u_2(T_1) = u_1(T_1)$ ,  $u_2(t) \in C(t)$  for all  $t \in [T_1, T_2]$ ,  $du_2$  has  $\frac{du_2}{d\nu_2}$  as a density in  $L^1_{\nu_2}([T_1, T_2], H)$  relative to  $\nu_2$ , and

$$\frac{du_2}{d\nu_2}(t) + f(t, u_2(t)) \frac{d\lambda}{d\nu_2}(t) \in -N(C(t); u_2(t)) \quad \nu_2 - \text{a.e. } t \in [T_1, T_2].$$

So, by induction, we obtain a finite sequence of right continuous with bounded variation mappings  $u_i : [T_{i-1}, T_i] \rightarrow H$  ( $i = 1, \dots, p$ ) such that  $u_i(T_{i-1}) = u_{i-1}(T_{i-1})$ ,  $u_i(t) \in C(t)$  for all  $t \in [T_{i-1}, T_i]$ , the vector measure  $du_i$  has  $\frac{du_i}{d\nu_i}$  as a density in  $L^1_{\nu_i}([T_{i-1}, T_i], H)$  relative to  $\nu_i$ , and

$$\frac{du_i}{d\nu_i}(t) + f(t, u_i(t)) \frac{d\lambda}{d\nu_i}(t) \in -N(C(t); u_i(t)) \quad \nu_i - \text{a.e. } t \in [T_{i-1}, T_i].$$

Then, the mapping  $u : [T_0, T] \rightarrow H$  with  $u(t) := u_i(t)$  if  $t \in [T_{i-1}, T_i]$  ( $i = 1, \dots, p$ ) is well defined and right continuous with bounded variation, and the inclusions  $u(t) \in C(t)$ , for all  $t \in [T_0, T]$ , along with the equality  $u(T_0) = u_0$  are obviously fulfilled. On the other hand, putting

$$g(t) := \mathbf{1}_{[T_0, T_1]}(t) \frac{du_1}{d\nu_1}(t) + \sum_{i=2}^p \mathbf{1}_{[T_{i-1}, T_i]}(t) \frac{du_i}{d\nu_i}(t) \quad \text{for all } t \in [T_0, T]$$

and considering the Radon measure  $\nu_0 := \mu + \lambda$  on  $[T_0, T]$ , we easily see that  $u(t) = u(T_0) + \int_{]T_0, t]} g(s) d\nu_0(s)$  for all  $t \in [T_0, T]$ , so the vector measure  $du$  has  $g(\cdot) \in L^1_{\nu_0}([T_0, T], H)$  as a density relative to  $\nu_0$   $\frac{du}{d\nu_0}(\cdot) = g(\cdot)$   $\nu_0$ -a.e. Furthermore, for  $\nu_0$ -almost every  $t \in [T_0, T]$ ,

$$\frac{du}{d\nu_0}(t) + f(t, u(t)) \frac{d\lambda}{d\nu_0}(t) \in -N(C(t); u(t)),$$

so  $u(\cdot)$  is a solution on the whole interval  $[T_0, T]$ .

### III. Uniqueness.

Let  $u_i : [T_0, T] \rightarrow H$  ( $i = 1, 2$ ) be two solutions which are right continuous with bounded variation, so,  $u_i(T_0) = u_0$ ,  $u_i(t) \in C(t)$  for all  $t \in [T_0, T]$ , and there exist two Radon measures  $\nu_i$  on  $[T_0, T]$  ( $i = 1, 2$ ) absolutely continuously equivalent to

$\mu + \lambda$  and such that the vector measure  $du_i$  has  $\frac{du_i}{dv_i}(\cdot) \in L^1_{v_i}([T_0, T], H)$  as a density relative to  $v_i$  and

$$-\frac{du_i}{dv_i}(t) - f(t, u_i(t)) \frac{d\lambda}{dv_i}(t) \in N(C(t); u_i(t)) \quad \text{for } v_i - \text{a.e. } t \in [T_0, T].$$

With respect to the positive Radon measure  $\nu := \nu_1 + \nu_2$  absolutely continuously equivalent to  $\mu + \lambda$  hence to  $v_i$ , the measures  $du_i$  and  $\lambda$  have densities in  $L^1_\nu([T_0, T], H)$  and  $L^1_\nu([T_0, T], \mathbb{R}_+)$  respectively, and

$$\frac{du_i}{d\nu}(t) = \frac{du_i}{dv_i}(t) \frac{dv_i}{d\nu}(t) \quad \text{and} \quad \frac{d\lambda}{d\nu}(t) = \frac{d\lambda}{dv_i}(t) \frac{dv_i}{d\nu}(t)$$

with  $\frac{dv_i}{d\nu}(t) \geq 0$ . Since  $N(C(t); u(t))$  is a cone, it ensues that, for  $\nu$ -almost every  $t \in [T_0, T]$ ,

$$-\frac{du_i}{d\nu}(t) - f(t, u(t)) \frac{d\lambda}{d\nu}(t) \in N(C(t); u(t)).$$

Noting that  $u_1(\cdot)$  and  $u_2(\cdot)$  are bounded on  $[0, T]$  as mappings with bounded variation on  $[T_0, T]$ , choose a real  $r > 0$  such that  $\|u_i(t)\| \leq r$  for all  $i = 1, 2$  and  $t \in [T_0, T]$ . The monotonicity of the normal cone and the Lipschitz property of  $f(t, \cdot)$  on  $r\mathbb{B}$  yield, for  $\nu$ -almost every  $t \in [T_0, T]$ ,

$$\left\langle \frac{du_1}{d\nu}(t) - \frac{du_2}{d\nu}(t), u_1(t) - u_2(t) \right\rangle \leq L_r(t) \frac{d\lambda}{d\nu}(t) \|u_1(t) - u_2(t)\|^2,$$

thus Proposition 1 says that, for all  $t \in [T_0, T]$ ,

$$\|u_1(t) - u_2(t)\| \leq \int_{[T_0, t]} 2L_r(s) \frac{d\lambda}{d\nu}(s) \|u_1(s) - u_2(s)\|^2 d\nu(s),$$

and Lemma 1 entails, for all  $t \in [T_0, T]$ , that  $\|u_1(t) - u_2(t)\|^2 \leq 0$ , which confirms the uniqueness of solution and finishes the proof of the theorem.  $\square$

## 5 A variant with velocity in the moving set

In this section we are interested in the following variant of the sweeping process:

$$\begin{cases} A_1 \dot{u}(t) + A_0 u(t) - f(t) \in -N(C(t); \dot{u}(t)) & \text{a.e. } t \in [0, T], \\ u(0) = u_0 \in H, \end{cases} \quad (5.1)$$

where  $f : [0, T] \rightarrow H$  is a continuous mapping and  $A_0, A_1 : H \rightarrow H$  are two bounded symmetric linear operators which are semi-definite positive in the standard

sense, that is,

$$\langle A_i x, x \rangle \geq 0, \quad \text{for all } x \in H \text{ and } i = 0, 1.$$

By a solution, we mean an absolutely continuous mapping  $u(\cdot) : [0, T] \rightarrow H$  with  $u(0) = u_0$  such that the above inclusion is fulfilled for almost every  $t \in [0, T]$ . We have to be careful with such a variant. Indeed, even in the simple case where  $A_0$  and  $A_1$  are the null operators, that is, the system is reduced to

$$f(t) \in N(C(t); \dot{u}(t)) \quad \text{with } u(0) = u_0 \in H, \quad (5.2)$$

a solution does not always exist. Taking, for example,  $C(t) = [t, +\infty[$  in  $\mathbb{R}$  and  $f(t) = 1$ , no solution exists in  $[0, T]$ . However, assuming that  $C(t)$  is bounded for all  $t \in [0, T]$ , the property (2.3) tells us that the set-valued operator  $N(C(t); \cdot)$  is surjective and hence (5.2) has at least one solution  $u(\cdot)$ . Such boundedness condition of  $C(t)$  will be assumed in our analysis below.

*Remark 1* Writing the inclusion in (5.1) as

$$-A_0 u(t) + f(t) \in \partial(\varphi_{A_1} + \iota_{C(t)})(\dot{u}(t)),$$

we see that it is equivalent to

$$\dot{u}(t) \in \partial(\varphi_{A_1} + \iota_{C(t)})^*(-A_0 u(t) + f(t)).$$

By setting

$$g(t, x) := (\varphi_{A_1} + \iota_{C(t)})^*(-x), \quad \text{where } \varphi_{A_1}(x) = \frac{1}{2} \langle A_1 x, x \rangle,$$

it appears that (5.1) is equivalent to the differential evolution inclusion

$$\begin{cases} \dot{u}(t) + \partial g(t, -A_0 u(t) + f(t)) \ni 0 \\ u(0) = u_0 \in H. \end{cases}$$

We emphasize that, in the latter differential inclusion, the convex function  $g(t, \cdot)$  depends on the time  $t$ . Instead of continuing in this direction, our aim here is to show how an adaptation of Moreau's catching up algorithm leads to a constructive proof of existence of a solution to (5.1).

### 5.1 A constructive existence result

**Theorem 5.1** *Let  $H$  be a separable Hilbert space. Let  $A_0, A_1 : H \rightarrow H$  be two bounded semi-definite symmetric linear operators and let  $f : [0, T] \rightarrow H$  be a continuous mapping which is bounded by a real  $\beta \geq 0$ , that is,*

$$\|f(t)\| \leq \beta, \quad \text{for all } t \in [0, T].$$

Assume that  $C(0) \subset R\mathbb{B}_H$  and the nonempty closed sets  $C(t)$  of  $H$  have a continuous variation in the sense that there is some nondecreasing continuous function  $v(\cdot) : [0, T] \rightarrow \mathbb{R}$  with  $v(0) = 0$  and such that

$$|d(y, C(t)) - d(y, C(s))| \leq |v(t) - v(s)| \quad \text{for all } y \in H \text{ and } s, t \in [0, T].$$

Then, for any initial point  $u_0 \in H$ , the evolution variational inequality (5.1) admits at least a Lipschitz continuous solution  $u : [0, T] \rightarrow H$ .

*Proof* Consider, for each  $n \in \mathbb{N}$  the following partition of the interval  $I := [0, T]$

$$\begin{aligned} t_i^n &:= i \frac{T}{n} \quad \text{for } 0 \leq i \leq n, \\ I_i^n &:= ]t_i^n, t_{i+1}^n] \quad \text{for } 0 \leq i \leq n-1. \end{aligned} \quad (5.3)$$

By assumption  $C(0) \subset R\mathbb{B}_H$ , hence the continuity of  $v$  implies:

$$C(t) \subset \rho\mathbb{B}_H, \quad \text{for all } t \in [0, T], \quad \text{with } \rho := R + \max_{t \in [0, T]} |v(t)|. \quad (5.4)$$

Put  $u_0^n = u_0 := T/n$  and  $f_i^n := f(t_i^n)$  for all  $i = 1, \dots, n$ . Consider the continuous convex function  $\varphi_{A_k} : H \rightarrow \mathbb{R}$ ,  $k = 0, 1$ , with

$$\varphi_{A_k}(x) := \frac{1}{2} \langle A_k(x), x \rangle \quad \text{for all } x \in H,$$

and note by (2.1) that

$$\eta_n A_0 + A_1 + N(C(t_1^n); \cdot) = \partial(\eta_n \varphi_{A_0} + \varphi_{A_1} + \iota_{C(t_1^n)})$$

with  $\text{dom}(\eta_n \varphi_{A_0} + \varphi_{A_1} + \iota_{C(t_1^n)})$  bounded in  $H$  since  $C(t_1^n) \subset \rho\mathbb{B}_H$ . The function  $\eta_n \varphi_{A_0} + \varphi_{A_1} + \iota_{C(t_1^n)}$  is also bounded from below on  $\rho\mathbb{B}_H$  (containing its effective domain), since the linear operators  $A_0$  and  $A_1$  are bounded. By (2.3) the set-valued operator  $\eta_n A_0 + A_1 + N(C(t_1^n); \cdot)$  is surjective, so we can choose some  $z_1^n \in H$  such that

$$f_1^n - A_0 u_0^n \in (\eta_n A_0 + A_1 + N(C(t_1^n); \cdot))(z_1^n),$$

and clearly  $z_1^n \in C(t_1^n) \subset \rho\mathbb{B}_H$ . Put  $u_1^n = u_0^n + \eta_n z_1^n$ . Now suppose that  $u_0^n, u_1^n, \dots, u_i^n, z_1^n, z_2^n, \dots, z_i^n$  are constructed.

As above, the set-valued operator  $\eta_n A_0 + A_1 + N(C(t_{i+1}^n); \cdot)$  is surjective, so we can find  $z_{i+1}^n \in C(t_{i+1}^n)$  such that

$$f_{i+1}^n - A_0 u_i^n \in (\eta_n A_0 + A_1 + N(C(t_{i+1}^n); \cdot))(z_{i+1}^n),$$

and we set  $u_{i+1}^n := u_i^n + \eta_n z_{i+1}^n$ . We then obtain by induction finite sequences  $(u_i^n)_{i=0}^n$  and  $(z_i^n)_{i=1}^n$  such that, for all  $i = 0, \dots, n-1$ ,

$$f_{i+1}^n - A_0 u_i^n \in (\eta_n A_0 + A_1 + N(C(t_{i+1}^n); \cdot)) (z_{i+1}^n) \text{ and } u_{i+1}^n = u_i^n + \eta_n z_{i+1}^n. \quad (5.5)$$

Through the sequences  $(u_i^n)_{i=0}^n, (z_i^n)_{i=1}^n$  and  $(f_i^n)_{i=0}^n$  in  $H$ , we construct two sequences of mappings  $(u_n(\cdot))_n$  from  $[0, T]$  into  $H$  and  $(f_n(\cdot))_n$  from  $[0, T]$  into  $H$  as follows: we set  $f_n(0) := f_1^n$  and  $u_n(0) := u_0^n$ , and for each  $i = 0, \dots, n-1$ , we set

$$f_n(t) := f_{i+1}^n \text{ and } u_n(t) := u_i^n + \frac{(t - t_i^n)}{\eta_n} (u_{i+1}^n - u_i^n) \text{ for all } t \in ]t_i^n, t_{i+1}^n].$$

Clearly, the mapping  $u_n(\cdot)$  is Lipschitz continuous on  $[0, T]$ , and  $\rho$  is a Lipschitz constant of  $u_n(\cdot)$  on  $[0, T]$  since for every  $t \in ]t_i^n, t_{i+1}^n[$

$$\dot{u}_n(t) = \frac{(u_{i+1}^n - u_i^n)}{\eta_n} = z_{i+1}^n \in C(t_{i+1}^n) \subset \rho \mathbb{B}_H.$$

Furthermore, for every  $t \in [0, T]$ , one has  $u_n(t) = u_0 + \int_0^t \dot{u}_n(s) ds$ , hence  $\|u_n(t)\| \leq \|u_0\| + \rho T$ .

Using the linearity of  $A_0$  and the definition of  $u_{i+1}^n$ , we see that

$$f_{i+1}^n - A_0 u_{i+1}^n - A_1 z_{i+1}^n \in N(C(t_{i+1}^n); z_{i+1}^n).$$

So, defining the function  $\theta_n$  from  $[0, T]$  to  $[0, T]$  by  $\theta_n(0) = t_1^n$  and  $\theta_n(t) = t_{i+1}^n$  for any  $t \in ]t_i^n, t_{i+1}^n]$ , the latter inclusion becomes

$$f_n(t) - A_0 u_n(\theta_n(t)) - A_1 \dot{u}_n(t) \in N(C(\theta_n(t)); \dot{u}_n(t)) \text{ a.e. } t \in [0, T], \quad (5.6)$$

and we also note that

$$\sup_{t \in [0, T]} |\theta_n(t) - t| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Now, let us prove the convergence of the sequences  $(u_n(\cdot))_n, (\dot{u}_n(\cdot))_n$  and  $(f_n(\cdot))_n$ . We have, for all  $n$ ,

$$\begin{cases} \|u_n(t)\| \leq \|u_0\| + \rho T, & \|f_n(t)\| \leq \beta \text{ for all } t \in [0, T] \text{ and} \\ \|\dot{u}_n(t)\| \leq \rho \text{ for almost all } t \in [0, T], \end{cases}$$

so the sequence of mappings  $(u_n(\cdot))_n$  is uniformly bounded in norm and variation. Hence, (see [[38], Theorem 0.2.1]), we find a mapping  $u : [0, T] \rightarrow H$  with bounded variation and a subsequence still denoted  $(u_n(\cdot))_n$ , such that

$$u_n(t) \rightharpoonup u(t) \text{ weakly in } H \text{ for all } t \in [0, T], \quad (5.7)$$

$$u_n(\cdot) \rightharpoonup u(\cdot) \text{ in the weak-star topology of } L^\infty([0, T], H), \quad (5.8)$$

and, for some  $\zeta(\cdot) \in L^2([0, T], H)$ ,

$$\dot{u}_n(\cdot) \rightarrow \zeta(\cdot) \text{ in the weak topology of } L^2([0, T], H). \quad (5.9)$$

In particular,  $u(0) = u_0$ . The Lipschitz continuity of the mapping  $u_n$  and the weak lower semicontinuity of the norm give

$$\|u(t) - u(s)\| \leq \liminf_{n \rightarrow \infty} \|u_n(t) - u_n(s)\| \leq \rho|t - s| \text{ for all } t, s \in [0, T].$$

The mapping  $u(\cdot)$  is then Lipschitz continuous on  $[0, T]$ , and hence the derivative mapping  $\dot{u}(\cdot)$  exists almost everywhere. Fix any  $t \in [0, T]$ . For each  $z \in H$  with  $\|z\| \leq 1$ , we can write

$$\begin{aligned} |\langle z, u_n(\theta_n(t)) - u(t) \rangle| &\leq |\langle z, u_n(\theta_n(t)) - u_n(t) \rangle| + |\langle z, u_n(t) - u(t) \rangle| \\ &\leq \rho|\theta_n(t) - t| + |\langle z, u_n(t) - u(t) \rangle|, \end{aligned}$$

which entails, according to (5.7), that  $u_n(\theta_n(t)) \rightarrow u(t)$  weakly in  $H$  as  $n \rightarrow \infty$ . On the other hand,

$$\langle z, u_n(t) \rangle = \langle z, u_0 \rangle + \int_0^t \langle \mathbf{1}_{[0,t]}(s)z, \dot{u}_n(s) \rangle ds,$$

and using (5.9) and taking the limit as  $n \rightarrow \infty$  give

$$\langle z, u(t) \rangle = \langle z, u_0 \rangle + \int_0^t \langle \mathbf{1}_{[0,t]}(s)z, \zeta(s) \rangle ds = \left\langle z, u_0 + \int_0^t \zeta(s) ds \right\rangle.$$

The latter equality being true for all  $z \in H$ , we deduce that  $u(t) = u_0 + \int_0^t \zeta(s) ds$ , and this guarantees that  $\dot{u}(\cdot) = \zeta(\cdot)$  almost everywhere hence

$$\dot{u}_n(\cdot) \rightarrow \dot{u}(\cdot) \text{ weakly in } L^2([0, T], H) \quad (5.10)$$

according to (5.9) again. Furthermore, since  $f_n(t) = f(\theta_n(t))$  and  $f(\cdot)$  is continuous, we have, for every  $t \in [0, T]$ , that  $f_n(t) \rightarrow f(t)$  strongly in  $H$  as  $n \rightarrow \infty$ .

Let us prove that  $\dot{u}(t) \in C(t)$ , for almost every  $t \in [0, T]$ .

First, using the assumption on the variation of  $C(\cdot)$ , we note that

$$\dot{u}_n(t) = z_{i+1}^n \in C(\theta_n(t)) \subset C(t) + |v(\theta_n(t)) - v(t)|\mathbb{B}_H, \text{ for a.e. } t \in ]t_i^n, t_{i+1}^n],$$

so

$$\dot{u}_n(t) \in C(t) + |v(\theta_n(t)) - v(t)|\mathbb{B}_H, \text{ for a. e. } t \in [0, T]. \quad (5.11)$$

Fix  $\varepsilon > 0$  and let

$$D_\varepsilon = \left\{ \phi \in L^2([0, T]; H) : \phi(t) \in C(t) + \varepsilon \mathbb{B}_H \text{ a. e. } t \in [0, T] \right\}.$$

Then  $D_\varepsilon$  is closed and convex in  $L^2([0, T], H)$ , hence weakly closed, and  $\dot{u}_n \in D_\varepsilon$  for large  $n$ , by (5.11) since  $v(\cdot)$  is uniformly continuous on  $[0, T]$ . The weak convergence of  $\dot{u}_n$  to  $\dot{u}$  in  $L^2([0, T], H)$  implies that  $\dot{u} \in D_\varepsilon$  for all  $\varepsilon > 0$ . Since every  $C(t)$  is closed, the claim follows.

Now let us prove the inclusion in (5.1). Put

$$\zeta_n(t) := -A_0 u_n(\theta_n(t)) - A_1(\dot{u}_n(t)) + f_n(t) \quad \text{for all } t \in [0, T].$$

We then note that the inclusion (5.7) is equivalent by (2.4) to the inequality

$$\sigma(C(\theta_n(t)), \zeta_n(t)) + \langle -\zeta_n(t), \dot{u}_n(t) \rangle \leq 0, \quad \text{a.e.}$$

since  $\dot{u}_n(t) \in C(\theta_n(t))$ , and integrating on  $[0, T]$  we get

$$\int_0^T \sigma(C(\theta_n(t)), \zeta_n(t)) dt + \int_0^T \langle -\zeta_n(t), \dot{u}_n(t) \rangle dt \leq 0. \quad (5.12)$$

Furthermore, using the strong convergence of  $f_n(t)$  to  $f(t)$  for all  $t \in [0, T]$  along with the inequality  $\|f_n(t)\| \leq \beta$ , we see that  $(f_n(\cdot))_n$  converges strongly in  $L^2([0, T], H)$  as  $n \rightarrow \infty$ . This combined with the weak convergence of  $(\dot{u}_n(\cdot))_n$  to  $\dot{u}(\cdot)$  in  $L^2([0, T], H)$  ensures that

$$\int_0^T \langle f(t), \dot{u}(t) \rangle dt = \lim_{n \rightarrow \infty} \int_0^T \langle f_n(t), \dot{u}_n(t) \rangle dt. \quad (5.13)$$

On the other hand, we have  $A_0 = \nabla \varphi_0$ , for the continuous convex function  $\varphi_0(x) = \frac{1}{2} \langle A_0 x, x \rangle$ . Therefore, the absolute continuity of  $\varphi_0 \circ u$  and  $\varphi_0 \circ u_n$  gives

$$\begin{aligned} \int_0^T \langle A_0 u(t), \dot{u}(t) \rangle dt &= \int_0^T \frac{d}{dt} \varphi_0(u(t)) dt = \varphi_0(u(T)) - \varphi_0(u(0)) \\ &\leq \liminf_{n \rightarrow \infty} \left( \varphi_0(u_n(T)) - \varphi_0(u_n(0)) \right) \\ &= \liminf_{n \rightarrow \infty} \left( \int_0^T \frac{d}{dt} \varphi_0(u_n(t)) dt \right) \\ &= \liminf_{n \rightarrow \infty} \int_0^T \langle A_0 u_n(t), \dot{u}_n(t) \rangle dt, \end{aligned} \quad (5.14)$$

where the inequality is due to the weak lower semicontinuity of  $\varphi_0$  on  $H$  and to the fact that  $u_n(T) \rightarrow u(T)$  weakly in  $H$  as  $n \rightarrow \infty$ . We have also

$$\liminf_{n \rightarrow \infty} \int_0^T \langle A_0 u_n(t), \dot{u}_n(t) \rangle dt = \liminf_{n \rightarrow \infty} \int_0^T \langle A_0 u_n(\theta_n(t)), \dot{u}_n(t) \rangle dt \quad (5.15)$$

since

$$\int_0^T |\langle A_0 u_n(t) - A_0 u_n(\theta_n(t)), \dot{u}_n(t) \rangle| dt \leq \rho^2 \|A_0\| \int_0^T |t - \theta_n(t)| dt.$$

From the properties of  $A_1$ , it is easy to verify that the function  $x(\cdot) \mapsto \int_0^T \langle A_1 x(t), x(t) \rangle dt$  is convex and strongly continuous on  $L^2([0, T]; H)$  thus weakly lower semicontinuous on  $L^2([0, T], H)$ . Recalling by (5.10) that

$$\dot{u}_n(\cdot) \rightharpoonup \dot{u}(\cdot) \text{ weakly in } L^2([0, T], H) \text{ as } n \rightarrow \infty,$$

we obtain

$$\int_0^T \langle A_1 \dot{u}(t), \dot{u}(t) \rangle dt \leq \liminf_{n \rightarrow \infty} \int_0^T \langle A_1 \dot{u}_n(t), \dot{u}_n(t) \rangle dt. \quad (5.16)$$

On the other hand, by Theorem 2.2, the convex function  $x(\cdot) \mapsto \int_0^T \sigma(C(t), x(t)) dt$  is weakly lower semicontinuous on  $L^2([0, T], H)$ . Also, the mappings  $\widehat{A}_i : L^2([0, T], H) \rightarrow L^2([0, T], H)$  ( $i = 0, 1$ ), with  $(\widehat{A}_i x)(t) := A_i x(t)$  for all  $t \in [0, T]$ , satisfy  $\|\widehat{A}_i x\|_2 \leq \|A_i\| \|x\|_2$ ; then they are continuous and hence weakly  $\times$  weakly continuous from  $L^2([0, T], H)$  into itself. Since the sequences  $(u_n(\cdot))_n$  and  $(\dot{u}_n(\cdot))_n$  converge weakly in  $L^2([0, T], H)$  to  $u(\cdot)$  and  $\dot{u}(\cdot)$  respectively, it results that (recall  $\zeta_n(t) := -A_0 u_n(\theta_n(t)) - A_1(\dot{u}_n(t)) + f_n(t)$ ),

$$\int_0^T \sigma(C(t), -A_0 u(t) - A_1(\dot{u}(t)) + f(t)) dt \leq \liminf_{n \rightarrow \infty} \int_0^T \sigma(C(t), \zeta_n(t)) dt.$$

Since  $C(t) \subset C(\theta_n(t)) + |v(\theta_n(t)) - v(t)| \mathbb{B}_H$  (according to the assumption on  $C(\cdot)$ ), we also observe that

$$\begin{aligned} \int_0^T \sigma(C(t), \zeta_n(t)) dt &\leq \int_0^T \sigma(C(\theta_n(t)), \zeta_n(t)) dt + \int_0^T \|\zeta_n(t)\| |v(\theta_n(t)) - v(t)| dt \\ &\leq \int_0^T \sigma(C(\theta_n(t)), \zeta_n(t)) dt + \left( (\|u_0\| + \rho T) \|A_0\| + \rho \|A_1\| + \beta \right) \int_0^T |v(\theta_n(t)) - v(t)| dt. \end{aligned}$$

It is easily seen that  $\int_0^T |v(t) - v(\theta_n(t))| ds \rightarrow 0$  as  $n \rightarrow \infty$ , thus

$$\liminf_{n \rightarrow \infty} \int_0^T \sigma(C(\theta_n(t)), \zeta_n(t)) dt \geq \liminf_{n \rightarrow \infty} \int_0^T \sigma(C(t), \zeta_n(t)) dt.$$

We then deduce that

$$\liminf_{n \rightarrow \infty} \int_0^T \sigma(C(\theta_n(t)), \zeta_n(t)) dt \geq \int_0^T \sigma(C(t), -A_0u(t) - A_1\dot{u}(t) + f(t)) dt. \quad (5.17)$$

Using all the inequalities (5.12)–(5.17) together, it follows that

$$\int_0^T \langle A_0u(t) + A_1\dot{u}(t) - f(t), \dot{u}(t) \rangle dt + \int_0^T \sigma(C(t), -A_0u(t) - A_1\dot{u}(t) + f(t)) dt \leq 0, \quad (5.18)$$

by taking  $\liminf_{n \rightarrow \infty}$  on both sides of (5.12). On the other hand, for almost every  $t \in [0, T]$ , the inclusion  $\dot{u}(t) \in C(t)$  yields

$$\sigma(C(t), -A_0u(t) - A_1\dot{u}(t) + f(t)) \geq \langle -A_0u(t) - A_1\dot{u}(t) + f(t), \dot{u}(t) \rangle,$$

or stated differently,

$$\sigma(C(t), -A_0u(t) - A_1\dot{u}(t) + f(t)) + \langle A_0u(t) + A_1\dot{u}(t) - f(t), \dot{u}(t) \rangle \geq 0.$$

Taking the latter inequality into account, it results from (5.18) that, for almost every  $t \in [0, T]$ ,

$$\sigma(C(t), -A_0u(t) - A_1\dot{u}(t) + f(t)) + \langle A_0u(t) + A_1\dot{u}(t) - f(t), \dot{u}(t) \rangle = 0,$$

which means, according to (2.4),

$$A_0u(t) + A_1\dot{u}(t) - f(t) \in -N_{C(t)}(\dot{u}(t)), \text{ for a. e. } t \in [0, T].$$

This translates the desired inclusion (5.1) and completes the proof of the theorem.  $\square$

Next is a uniqueness result related to Theorem 5.1 when the linear operator  $A_0$  is coercive.

**Theorem 5.2** *Assume in addition to the hypotheses in Theorem 5.1, that  $A_0$  is coercive, that is, for all  $x \in H$ ,*

$$\langle A_0x, x \rangle \geq \alpha_0 \|x\|^2,$$

for some real constant  $\alpha_0 > 0$ . Then, for any initial point  $u_0 \in H$ , there exists one and only one Lipschitz continuous solution of (5.1).

*Proof* Suppose that  $(u_1, u_2)$  are two solutions for (5.1) such that  $u_1(0) = u_2(0) = u_0$ . Then, for almost every  $t \in [0, T]$ , we have

$$\langle A_0 u_i(t) + A_1 \dot{u}_i(t) - f(t), \dot{u}_i(t) - z \rangle \leq 0, \text{ for all } z \in C(t).$$

Using the fact that  $\dot{u}_i(t) \in C(t)$  a.e., we obtain, for a.e.  $t \in [0, T]$ ,

$$\begin{cases} \langle A_0 u_1(t) + A_1 \dot{u}_1(t) - f(t), \dot{u}_1(t) - \dot{u}_2(t) \rangle \leq 0, \\ \langle -A_0 u_2(t) - A_1 \dot{u}_2(t) + f(t), \dot{u}_1(t) - \dot{u}_2(t) \rangle \leq 0, \end{cases}$$

and adding both inequalities yields

$$\langle A_0 u_1(t) - A_0 u_2(t) + A_1 \dot{u}_1(t) - A_1 \dot{u}_2(t), \dot{u}_1(t) - \dot{u}_2(t) \rangle \leq 0, \text{ a.e. } t \in [0, T].$$

Since  $A_1$  is monotone, we deduce

$$\langle A_0 u_1(t) - A_0 u_2(t), \dot{u}_1(t) - \dot{u}_2(t) \rangle \leq 0, \text{ a.e. } t \in [0, T].$$

Consequently, for almost every  $t \in [0, T]$ ,

$$\frac{d}{dt} \langle A_0 u_1(t) - A_0 u_2(t), u_1(t) - u_2(t) \rangle = 2 \langle A_0 u_1(t) - A_0 u_2(t), \dot{u}_1(t) - \dot{u}_2(t) \rangle \leq 0.$$

By integrating, using the equalities  $u_1(0) = u_2(0) = u_0$  and the coerciveness of  $A_0$ , we have

$$\alpha_0 \|u_1(t) - u_2(t)\|^2 \leq \langle A_0 u_1(t) - A_0 u_2(t), u_1(t) - u_2(t) \rangle \leq 0, \text{ for all } t \in [0, T],$$

which completes the proof of Theorem 5.2.  $\square$

The next example shows that the coerciveness of  $A_0$  is essential to have the uniqueness of solution.

*Example 1* Let  $H = \mathbb{R}^2$ ,  $T = 1$ ,  $u_0 = (0, 0)$ ,  $A_0 = A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  and  $C(t) = [t, 1] \times [0, 1]$  for  $t \in [0, 1]$ . Here  $A_0$  is a symmetric and positive semi-definite matrix (not coercive). The set-valued mapping  $C(\cdot)$  is Lipschitz continuous. Consider  $f(t) = (0, 0)$  for all  $0 \leq t \leq 1$ . The first solution is  $u(t) = (\frac{1}{2}t^2, t)$  for  $0 \leq t \leq 1$ , with  $\dot{u}(t) = (t, 1) \in C(t)$  for all  $t \in [0, 1]$ , and hence satisfies (5.1), since

$$N([t, 1] \times [0, 1]; (t, 1)) = \begin{cases} ] - \infty, 0] \times [0, +\infty[ & \text{if } t \in [0, 1[, \\ \mathbb{R} \times [0, +\infty[ & \text{if } t = 1. \end{cases}$$

We define the second solution as follows  $u(t) = (\frac{1}{2}t^2, \frac{1}{2}t)$  for  $0 \leq t \leq 1$ , with  $\dot{u}(t) = (t, \frac{1}{2}) \in C(t)$  for  $t \in [0, 1]$ .  
A simple computation shows that

$$N([t, 1] \times [0, 1]; (t, \frac{1}{2})) = \begin{cases} ]-\infty, 0] \times \{0\} & \text{if } t \in [0, 1[, \\ \mathbb{R} \times \{0\} & \text{if } t = 1. \end{cases}$$

Therefore,  $u(t) = (\frac{1}{2}t^2, \frac{1}{2}t)$  satisfies (5.1).

## 5.2 Application

As a direct application of Theorem (5.2) we obtain an existence and uniqueness result for the evolution variational inequality given in (1.6).

**Theorem 5.3** *Let  $a_i(\cdot, \cdot) : H \times H \rightarrow \mathbb{R}$  be real bilinear, bounded and symmetric forms such that for all  $u \in H$ ;  $a_1(u, u) \geq 0$  and  $a_0(u, u) \geq \alpha_0 \|u\|^2$  for some positive constant  $\alpha_0 > 0$ . Let  $l \in W^{1,2}([0, T], H)$  be uniformly bounded, that is, there exists  $\beta > 0$  such that  $\|l(t)\| \leq \beta$  for all  $t \in [0, T]$ . Assume that  $C(0) \subset R\mathbb{B}_H$  and the nonempty closed sets  $C(t)$  of  $H$  have a continuous variation in the sense that there is some nondecreasing continuous function  $v(\cdot) : [0, T] \rightarrow \mathbb{R}$  with  $v(0) = 0$  and such that*

$$|d(y, C(t)) - d(y, C(s))| \leq |v(t) - v(s)| \quad \text{for all } y \in H \text{ and } s, t \in [0, T].$$

*Then, for every  $u_0 \in H$ , there exists one and only one Lipschitz continuous solution  $u : [0, T] \rightarrow H$  of (1.6).*

*Proof* For  $i = 0, 1$  we note by  $A_i$  the linear, bounded and symmetric operators associated respectively with  $a_i(\cdot, \cdot)$ , that is,  $a_i(u, v) = \langle A_i u, v \rangle$  for all  $u, v \in H$ . Since  $C$  has convex values, the evolution variational inequality of type (1.6) can be rewritten in the form

$$\begin{cases} A_1 \dot{u}(t) + A_0 u(t) - l(t) \in -N_{C(t)}(\dot{u}(t)) & \text{a.e. } t \in [0, T], \\ u(0) = u_0 \in H, \end{cases} \quad (5.19)$$

By Sobolev embedding theorem,  $W^{1,2}([0, T], H) \subset C([0, T], H)$ , we conclude that  $l$  is continuous. Thus all assumptions of Theorem 5.2 are satisfied and so the proof is complete.  $\square$

## 6 Numerical experiments

In this section, we will give some numerical simulation to illustrate the theoretical results discussed in the last sections. In order to solve numerically problem (5.1), we will use the following algorithm discussed in the proof of Theorem 5.1. Let us suppose

that the dimension of  $H$  is finite, i.e.,  $\dim_{\mathbb{R}}(H) < +\infty$ . For  $n \in \mathbb{N}$ , let

$$0 = t_0^n < t_1^n < \dots < t_i^n < \dots < t_n^n = T,$$

be a finite partition of the interval  $[0, T]$ . We denote by  $\eta_i^n = t_{i+1}^n - t_i^n$  the length of the time step.

For simplicity, we will suppose that  $\eta_i^n = \eta^n = \frac{T}{n}$ ,  $i = 0, 1, \dots, n$  which means that  $t_i^n = i \frac{T}{n}$ .

The approximation of  $f(t_i^n)$  will be denoted by  $f_i^n$ .

**Algorithm 1** Fix  $n \geq 2$  and set  $\eta_n = \frac{T}{n}$ ,  $u_0^n = u_0$  and  $f_0^n = f(t_0^n)$ .

For  $i = 0, 1, \dots, n - 1$

- Compute  $f_{i+1}^n = f(t_{i+1}^n)$
- Solve for  $z_{i+1}^n$  the following variational inequalities (see Remark 2)

$$f_{i+1}^n - A_0 u_i^n \in \left[ \eta_n A_0 + A_1 + N_{C(t_{i+1}^n)} \right] (z_{i+1}^n) \quad (6.1)$$

- Update  $u_{i+1}^n = u_i^n + \eta_n z_{i+1}^n$ .

end

*Remark 2* The discretized variational inclusion (6.1) is equivalent to

$$z_{i+1}^n \in \left[ \eta_n A_0 + A_1 + N_{C(t_{i+1}^n)} \right]^{-1} (f_{i+1}^n - A_0 u_i^n). \quad (6.2)$$

Since  $C(t_{i+1}^n)$  is bounded, convex and closed in a finite dimensional space, by the classical result of Stampacchia, the variational inequality (6.1) has a solution. If one of the matrices  $A_0$  or  $A_1$  is positive definite, then this solution is unique and the operator  $[\eta_n A_0 + A_1 + N_{C(t_{i+1}^n)}]^{-1}$  is single valued and non-expansive.

Since  $A_0$  and  $A_1$  are symmetric, (6.2) is equivalent to solve the following optimization problem

$$\min_{z \in C(t_{i+1}^n)} \frac{1}{2} \langle A_1 z, z \rangle + \eta_n \frac{1}{2} \langle A_0 z, z \rangle - \langle A_0 u_i^n - f_{i+1}^n, z \rangle. \quad (6.3)$$

We note that this optimization problem is convex since  $A_1, A_0$  are positive semidefinite,  $\eta_n > 0$  and the set  $C(t_{k+1}^n)$  is (closed and) convex.

The choice of the adequate solver for solving the optimization problem (6.3) depends on the structure of the set  $C(t)$ . If the set  $C(t)$  is polyhedral, i.e., described by linear inequalities and equalities of the form  $Ax \leq b$  and  $Cx = d$  with  $A \in \mathbb{R}^{m \times n}$ ,  $b \in \mathbb{R}^m$ ,  $C \in \mathbb{R}^{p \times n}$  and  $d \in \mathbb{R}^p$ , then it is possible to use any quadratic programming solver (e.g., quadprog in Matlab).

If the set  $C(t)$  is described by finitely many nonlinear inequalities and linear equalities

$$C(t) = \{x \in H : g_j(t, x) \leq 0, j=1, 2, \dots, m \text{ and } h_k(t, x)=0, k = 1, 2, \dots, p\},$$

then we can use any nonlinear programming solver (e.g., Sequential Quadratic Programming, interior point method or fmincon in Matlab).

*Example 2* Let  $H = \mathbb{R}^2$ ,  $T = 1$ ,  $u_0 = (0, 0)$ ,  $A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ,  $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  and  $C(t) = \{t\} \times [t, 1]$  for  $0 \leq t \leq 1$ . For  $f(t) = (0, \frac{1}{2}t^2)$ ,  $t \in [0, 1]$ , it is easy to check that the unique solution of (5.1) is given by

$$u(t) = \left( \frac{1}{2}t^2, \frac{1}{2}t^2 \right), \quad t \in [0, 1].$$

## 7 By the way of conclusion

In this paper, using tools from convex analysis, we studied the well-posedness of some variants of the sweeping process within the framework of measure differential inclusions and evolution variational inequalities. We proved that the perturbed measure differential inclusion (4.1) has a unique right continuous solution with bounded variation. Under the assumption that the moving set  $C(t)$  has a continuous variation for every  $t \in [0, T]$  with  $C(0)$  bounded and the coercivity of the linear operator  $A_0$ , we proved that the sweeping process (1.5) with velocity in the moving set has a unique Lipschitz continuous solution. There remain many issues that need answers and further investigation. For example, as a consequence of the preceding assumption on  $C$ , it results that the set  $C(t)$  is bounded for every  $t \in [0, T]$ . This assumption is essential in the proof of Theorem 5.1. It would be interesting to extend this result to the case of unbounded convex moving sets. As shown in the counter-example 1, the sweeping process problem (1.5) generated by an unbounded moving set can fail to have a solution for some  $f$ . We think that some compatibility conditions on  $f$  are needed to prove the existence of at least one solution. In some applications, the assumption of the convexity of  $C(t)$  is not satisfied, it will be also interesting to investigate the case of prox-regular sets  $C(t)$ . This is out of the scope of the current manuscript and will be the aim of a future work.

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