Noname manuscript No.
(will be inserted by the editor)

An $\mathcal{O}(m \log n)$ algorithm for the weighted stable set problem in claw-free graphs with $\alpha(G) \leq 3$

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Abstract In this paper we show how to solve the *Maximum Weight Stable Set Problem* in a claw-free graph $G(V, E)$ with $\alpha(G) \leq 3$ in time $\mathcal{O}(|E| \log |V|)$. More precisely, in time $\mathcal{O}(|E|)$ we check whether $\alpha(G) \leq 3$ or produce a stable set with cardinality at least 4; moreover, if $\alpha(G) \leq 3$ we produce in time $\mathcal{O}(|E| \log |V|)$ a maximum stable set of G. This improves the bound of $\mathcal{O}(|E||V|)$ due to Faenza et alii ([2]).

Keywords claw-free graphs · stable set

1 Introduction

The *Maximum Weight Stable Set (MWSS) Problem* in a graph $G(V, E)$ with node-weight function $w : V \to \Re$ asks for a maximum weight subset of pairwise non-adjacent *nodes*. For each graph $G(V, E)$ and subset $W \subset V$ we denote by $N(W)$ (*neighborhood* of W) the set of nodes in $V \setminus W$ adjacent to some node in W. If $W = \{w\}$ we simply write $N(w)$. A *clique* is a complete subgraph of G induced by some set of nodes $K \subseteq V$. With a little abuse of notation we also regard the set K as a clique. A *claw* is a graph with four nodes w, x, y, z with w adjacent to x, y, z and x, y, z mutually non-adjacent. To highlight its structure, it is denoted as $(w : x, y, z)$. A graph G with no induced claws is said to be *claw-free* and has the property ([1]) that the symmetric difference of two stable sets induces a subgraph of G whose connected components are either (alternating) paths or (alternating) cycles. A subset $T \in V$ is *null* (*universal*) to a subset $W \subseteq V \setminus T$ if and only if $N(T) \cap W = \emptyset$ ($N(T) \cap W = W$). If $T = \{u\}$ with a little abuse of notation we say that u is *null* (*universal*) to W.

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Let $G(V,E)$ be a claw-free graph. A subset X of V is said to be *local* if there exists a node $u \in V$ such that $X \subseteq N[u]$. Observe that, by [3], a local set contains $\mathcal{O}(\sqrt{|E|})$ nodes.

Lemma 11 *Let $G(V,E)$ be a claw-free graph and $X, Y, Z, W \subseteq V$ four disjoint local sets (with W possibly empty) such that Z induces a clique in G and W is null to Z. In $\mathcal{O}(|E|)$ time we can either find a stable set $\{x,y,z\}$ with $x \in X$, $y \in Y$, $z \in Z$ or conclude that no such stable set exists. Moreover, if X is null to Y and W is non-empty, in $\mathcal{O}(|E|)$ time we can either find a stable set $\{x,y,z,w\}$ with $x \in X$, $y \in Y$, $z \in Z$, $w \in W$ or conclude that no such stable set exists.*

Proof. For any node $u \in X \cup Y$ let $h(u)$ denote the cardinality of $N(u) \cap Z$. It is easy to see that we can compute $h(u)$ for all the nodes $u \in X \cup Y$ in overall time $\mathcal{O}(|X \cup Y||Z|) = \mathcal{O}(|E|)$ (recall that X, Y, and Z are local sets, so their cardinality is $\mathcal{O}(\sqrt{|E|})$). Now let $\bar{x} \in X$ and $\bar{y} \in Y$ be any two non-adjacent nodes.

Claim (i). There exists a node $\bar{z} \in Z$ such that $\{\bar{x}, \bar{y}, \bar{z}\}$ is a stable set if and only if $h(\bar{x}) + h(\bar{y}) < |Z|$.

Proof. In fact, if $h(\bar{x}) + h(\bar{y}) < |Z|$ then the neighborhoods of nodes $\bar{x}$ and $\bar{y}$ do not cover Z, so there exists some node $\bar{z} \in Z$ which is non-adjacent to both $\bar{x}$ and $\bar{y}$. On the other hand, assume by contradiction that $h(\bar{x}) + h(\bar{y}) \geq |Z|$ and still there exists some node $\bar{z} \in Z$ which is non-adjacent to both $\bar{x}$ and $\bar{y}$. Let $Z' = Z \setminus \{\bar{z}\}$. Since we have $|N(\bar{x}) \cap Z'| + |N(\bar{y}) \cap Z'| = h(\bar{x}) + h(\bar{y}) \geq |Z'| + 1$ there exists some node $z' \in Z'$ which is adjacent to both $\bar{x}$ and $\bar{y}$. But then $(z' : \bar{x}, \bar{y}, \bar{z})$ is a claw in G, a contradiction. The claim follows.
End of Claim (i).

Now, in $\mathcal{O}(|E|)$ time, we can check if there exists some pair of nodes $x \in X$ and $y \in Y$ such that x, y are non-adjacent and $h(x) + h(y) < |Z|$. If no such pair exists, by Claim *(i)* we can conclude that no stable set $\{x, y, z\}$ with $x \in X$, $y \in Y$, $z \in Z$ exists. If, on the other hand, there exist two non-adjacent nodes $x \in X$ and $y \in Y$ satisfying $h(x) + h(y) < |Z|$ then, in $\mathcal{O}(\sqrt{|E|})$ time, we can find a node $z \in Z$ which is non-adjacent to both.

Assume now that X is null to Y. Let $\bar{w}$ be any node in W, let $\bar{X} = X \setminus N(\bar{w})$ and let $\bar{Y} = Y \setminus N(\bar{w})$. Since by assumption W is null to Z, we have that there exists a stable set $\{x, y, z, \bar{w}\}$ with $x \in X$, $y \in Y$, $z \in Z$, if and only if there exists a stable set $\{x, y, z\}$ with $x \in \bar{X}$, $y \in \bar{Y}$, $z \in Z$. Let $\bar{x} \in \bar{X}$ and $\bar{y} \in \bar{Y}$ be two nodes such that $h(\bar{x})$ and $h(\bar{y})$ are minimized. We can find such nodes in $\mathcal{O}(\sqrt{|E|})$ time and, by assumption, $\bar{x}$ and $\bar{y}$ are non-adjacent. By Claim *(i)* and the minimality of $h(\bar{x})$ and $h(\bar{y})$ there exists a stable set $\{x, y, z\}$ with $x \in \bar{X}$, $y \in \bar{Y}$, $z \in Z$ if and only if $h(\bar{x}) + h(\bar{y}) < |Z|$; moreover, if such a set exists we may assume $x \equiv \bar{x}$ and $y \equiv \bar{y}$. Hence, in $\mathcal{O}(\sqrt{|E|})$ time we can check whether there exists a stable set $\{x, y, z, \bar{w}\}$ with $x \in X$, $y \in Y$, $z \in Z$. Moreover, if the check is positive in $\mathcal{O}(\sqrt{|E|})$ time we can find a node $\bar{z} \in Z$ which is non-adjacent to $\bar{x}$, $\bar{y}$ and $\bar{w}$ so that $\{\bar{x}, \bar{y}, \bar{z}, \bar{w}\}$ is the sought-after stable set. It follows that in $\mathcal{O}(|E|)$ time we can check all the nodes in W and either find a stable set $\{x, y, z, w\}$ with $x \in X$, $y \in Y$, $z \in Z$, $w \in W$ or conclude that no such stable set exists. This concludes the proof of the lemma. ☐

Theorem 11 *Let $G(V,E)$ be a claw-free graph. In $\mathcal{O}(|E|)$ time we can construct a stable set S of G with $|S| = \min\{\alpha(G), 4\}$.*

Proof. First, observe that in $\mathcal{O}(|E|)$ time we can check whether G is a clique (in which case any singleton $S \subseteq V$ would satisfy $|S| = \alpha(G) = 1$) or construct a stable set of cardinality 2. In the first case we are done, so assume that $\{s, t\} \subseteq V$ is a stable set of cardinality 2.

We now claim that, In $\mathcal{O}(|E|)$ time, we can construct a stable set of cardinality 3 or conclude that $\alpha(G) = 2$. In fact, in $\mathcal{O}(|V|)$ time we can classify the nodes in $V \setminus \{s, t\}$ in four sets: *(i)* the set $F(s)$ of nodes adjacent to s and non-adjacent to t; *(ii)* the set $F(t)$ of nodes adjacent to t and non-adjacent to s; *(iii)* the set $W(s, t)$ of nodes adjacent both to s and to t; and *(iv)* the set SF of nodes (*super-free*) non-adjacent both to s and to t. If $SF \neq \emptyset$ then let u be any node in SF; in this case $\{s, t, u\}$ is a stable set of cardinality 3. Otherwise, in $\mathcal{O}(|E|)$ time we can check whether $F(s)$ is a clique or find a pair of non-adjacent nodes $u, v \in F(s)$. If $F(s)$ is not a clique, then $\{u, v, t\}$ is a stable set of cardinality 3. Analogously, in $\mathcal{O}(|E|)$ time we can check whether $F(t)$ is a clique or find a stable set of cardinality 3. Finally, if $SF = \emptyset$ and both $F(s)$ and $F(t)$ are cliques then, by claw-freeness, a stable set S of cardinality 3 (if any) satisfies $|S \cap F(s)| = |S \cap F(t)| = |S \cap W(s, t)| = 1$. Letting $X \equiv W(s, t)$, $Y \equiv F(s)$, $Z \equiv F(t)$ and observing that X, Y, Z are local sets, by Lemma 11 we can, in $\mathcal{O}(|E|)$ time, either conclude that $\alpha(G) = 2$ or find a stable set $\{x, y, z\}$ with $x \in X$, $y \in Y$, $z \in Z$. In the first case we are done, so assume that $T = \{s, t, u\} \subseteq V$ is a stable set of cardinality 3.

We now claim that, In $\mathcal{O}(|E|)$ time, we can construct a stable set of cardinality 4 or conclude that $\alpha(G) = 3$. In fact, in $\mathcal{O}(|V|)$ time we can classify the nodes in $V \setminus T$ in seven sets: *(i)* the set $F(s)$ of nodes adjacent to s and non-adjacent to t and to u; *(ii)* the set $F(t)$ of nodes adjacent to t and non-adjacent to s and to u; *(iii)* the set $F(u)$ of nodes adjacent to u and non-adjacent to s and to t; *(iv)* the set $W(s, t)$ of nodes adjacent both to s and to t and non-adjacent to u; *(v)* the set $W(s, u)$ of nodes adjacent both to s and to u and non-adjacent to t; *(vi)* the set $W(t, u)$ of nodes adjacent both to t and to u and non-adjacent to s; *(vii)* the set SF of nodes (*super-free*) non-adjacent to s, to t and to u. Observe that, by claw-freeness, no node can be simultaneously adjacent to s, t and u, so the above classification is complete. If $SF \neq \emptyset$ then let w be any node in SF; in this case $S = T \cup \{w\}$ is a stable set of cardinality 4. Otherwise, in $\mathcal{O}(|E|)$ time we can check whether $F(s)$ is a clique or find a pair of non-adjacent nodes $v, w \in F(s)$. If $F(s)$ is not a clique, then $\{v, w\} \cup T \setminus \{s\}$ is a stable set of cardinality 4. Analogously, in $\mathcal{O}(|E|)$ time we can check whether $F(t)$ or $F(u)$ are cliques or find a stable set of cardinality 4.

Finally, assume that SF is empty and that $F(s)$, $F(t)$, $F(u)$ are all cliques. Observe that, by claw-freeness, the symmetric difference of T and any stable set S of cardinality 4 induces a subgraph of G whose connected components are either paths or cycles where the nodes in S and T alternates. Since $|S| > |T|$, at least one component is a path P with $|P \cap S| = |P \cap T| + 1$. Since $SF = \emptyset$, the path P contains at least one node of T. If it contains a single node of T, say s, the two nodes in $P \cap S$ belong to $F(s)$, contradicting the assumption that $F(s)$ is a clique. It follows that either *(i)* P contains two nodes of T and $|P| = 5$ or *(ii)* $T \subseteq P$ and $|P| = 7$. Hence, to check whether G contains a stable set S of cardinality 4 it is sufficient to verify that there exists a path P of type *(i)* or *(ii)*. We shall prove that such check can be done in $\mathcal{O}(|E|)$ time.

Case (i).

We have three different choices for the pair of nodes in $P \cap T$. Consider, without loss of generality, $P \cap T = \{s, t\}$ and let $P = (x, s, y, t, z)$. Such a path exists if and only if there exists a stable set $\{x, y, z\}$ with $x \in F(s)$, $y \in W(s,t)$, $z \in F(t)$. Let $X \equiv F(s)$, $Y \equiv W(s,t)$, $Z \equiv F(t)$. Observe that Z is a clique and X, Y are local sets, so X, Y, Z satisfy the hypothesis of Lemma 11. Hence we can, in $\mathcal{O}(|E|)$ time, either find the stable set $\{x, y, z\}$ or conclude that there exists no such stable set. In the first case, observe that u is non-adjacent to x, y and z, so $\{x, y, z, u\}$ is the sought-after stable set of cardinality 4.

Case (ii).

We have three different choices for the order in which the three nodes s, t, u appear in the path P. Consider, without loss of generality, $P = (x, s, w, t, y, u, z)$. Such a path exists if and only if there exists a stable set $\{x, y, z, w\}$ with $x \in F(s)$, $y \in W(t,u)$, $z \in F(u)$, $w \in W(s,t)$. Let $X \equiv F(s)$, $Y \equiv W(t,u)$, $Z \equiv F(u)$, $W \equiv W(s,t)$. Observe that, by claw-freeness, X is null to Y and W is null to Z; moreover Z is a clique and X, Y, W are local sets. So X, Y, Z, W satisfy the hypothesis of Lemma 11 and we can, in $\mathcal{O}(|E|)$ time, either find the stable set $\{x, y, z, w\}$ or conclude that there exists no such stable set.

It follows that in $\mathcal{O}(|E|)$ time we can either construct a stable set of cardinality 4 or conclude that $\alpha(G) = 3$. This concludes the proof of the theorem. $\square$

Lemma 12 *Let $G(V, E)$ be a claw-free graph, $w \in \Re^V$ a weighting of V and $X, Y, Z \subseteq V$ disjoint local sets such that Z induces a clique in G. In $\mathcal{O}(|E| \log |V|)$ time we can either find a maximum-weight stable set $\{x, y, z\}$ with $x \in X$, $y \in Y$, $z \in Z$ or conclude that no such stable set exists.*

Proof. Let $z_1, z_2, \ldots, z_p$ be an ordering of the nodes in Z such that $w(z_1) \geq w(z_2) \geq \ldots \geq w(z_p)$. Let Z_i $(i = 1, \ldots, p)$ denote the set $\{z_1, \ldots, z_i\} \subseteq Z$. For any node $u \in X \cup Y$ and index $i \in \{1, \ldots, p\}$ let $h(u, i)$ denote the cardinality of $N(u) \cap Z_i$. It is easy to see that we can compute $h(u, i)$ for all the nodes $u \in X \cup Y$ and all the indices in $\{1, \ldots, p\}$ in overall time $\mathcal{O}(|X \cup Y||Z|) = \mathcal{O}(|E|)$ (recall that X, Y, and Z are local sets, so their cardinality is $\mathcal{O}(\sqrt{|E|})$). Now let $\bar{x} \in X$ and $\bar{y} \in Y$ be any two non-adjacent nodes and let i be an index in $\{1, \ldots, p\}$.

Claim (i). There exists a node $\bar{z} \in Z_i$ such that $\{\bar{x}, \bar{y}, \bar{z}\}$ is a stable set if and only if $h(\bar{x}, i) + h(\bar{y}, i) < i$.

Proof. This is a special case of Claim *(i)* in Lemma 11.
End of Claim (i).

Now, assume $h(\bar{x}, p) + h(\bar{y}, p) < p$ and let k be the smallest index in $\{1, \ldots, p\}$ such that $h(\bar{x}, k) + h(\bar{y}, k) < k$.

Claim (ii). The set $\{\bar{x}, \bar{y}, z_k\}$ is the heaviest stable set containing $\bar{x}$, $\bar{y}$ and some node in Z.

Proof. Trivial consequence of Claim *(i)* and the ordering of Z.
End of Claim (ii).

Claim (iii). If $h(\bar{x}, i) + h(\bar{y}, i) < i$ for some $i \in \{1, \ldots, p\}$ then $h(\bar{x}, j) + h(\bar{y}, j) < j$ for any $j \geq i$.

Proof. If $h(\bar{x}, i) + h(\bar{y}, i) < i$, by Claim *(i)* there exists a node $\bar{z} \in Z_i$ which is non-adjacent to both $\bar{x}$ and $\bar{y}$. If $\bar{x}$ and $\bar{y}$ had a common neighbor z' in Z_j then $(z' : \bar{x}, \bar{y}, \bar{z})$ would be a claw in G, a contradiction. It follows that $h(\bar{x}, j) + h(\bar{y}, j) = |N(\bar{x}) \cap Z_j| + |N(\bar{y}) \cap Z_j| < |Z_j| = j$ and the claim follows.
End of Claim (iii).

By Claim *(iii)* We can find k in $\lceil \log p \rceil = \mathcal{O}(\log |V|)$ constant time computations, by binary search. As a consequence, by checking all the pairs of non-adjacent nodes $x \in X$ and $y \in Y$, in $\mathcal{O}(|E| \log |V|)$ time we can either find a maximum-weight stable set $\{x, y, z\}$ with $x \in X$, $y \in Y$, $z \in Z$ or conclude that no such stable set exists. The lemma follows. □

Theorem 12 *Let $G(V, E)$ be a claw-free graph and let $w \in \Re^V$ be a weighting of V. In $\mathcal{O}(|E| \log |V|)$ time we can either conclude that $\alpha(G) \geq 4$ or construct a maximum-weight stable set S of G.*

Proof. By Theorem 11 in $\mathcal{O}(|E|)$ time we can construct a stable set S of G with $|S| = \min(\alpha(G), 4)$. If $|S| = 4$ we are done. Otherwise, $\alpha(G) \leq 3$ and, as observed in [2], $|V| = \mathcal{O}(\sqrt{|E|})$. If $|S| = \alpha(G) \leq 2$ then in $\mathcal{O}(|E|)$ time we can find a maximum-weight stable set. In fact, since S is maximal, every node in V belongs to $N[S]$, $|V| = \mathcal{O}(\sqrt{|E|})$ and the theorem follows. Hence, we can assume that $\alpha(G) = 3$ and that we have a stable set $T = \{s, t, u\}$. Moreover, since a maximum-weight stable set intersecting T can be found in $\mathcal{O}(|E|)$ time, we are left with the task of finding a maximum-weight stable set in $V \setminus T$. In $\mathcal{O}(|V|)$ time we can classify the nodes in $V \setminus T$ in six sets: *(i)* the set $F(s)$ of nodes adjacent to s and non-adjacent to t and to u; *(ii)* the set $F(t)$ of nodes adjacent to t and non-adjacent to s and to u; *(iii)* the set $F(u)$ of nodes adjacent to u and non-adjacent to s and to t; *(iv)* the set $W(s, t)$ of nodes adjacent both to s and to t and non-adjacent to u; *(v)* the set $W(s, u)$ of nodes adjacent both to s and to u and non-adjacent to t; *(vi)* the set $W(t, u)$ of nodes adjacent both to t and to u and non-adjacent to s. Observe that, by claw-freeness, no node can be simultaneously adjacent to s, t and u; moreover, since $\alpha(G) = 3$, no node can be simultaneously non-adjacent to s, t and u, so the above classification is complete. If $F(s)$ is not a clique, let v, w be two non-adjacent nodes in $F(s)$. The set $\{v, w, t, u\}$ is a stable set of cardinality 4, contradicting the assumption that $\alpha(G) = 3$. It follows that $F(s)$ and, analogously, $F(t)$ and $F(u)$ are cliques.

Observe that, by claw-freeness, the symmetric difference of T and any stable set S of cardinality 3 induces a subgraph H of G whose connected components are either paths or cycles whose nodes alternate between S and T. It follows that we can classify the stable sets non-intersecting T according to the structure of the connected components of H. Since $\alpha(G) = 3$, no connected component of H can have an odd number of nodes. We say that S is of type *(i)* if H is a path of length 6; of type *(ii)* if H is a cycle of length 6; of type *(iii)* if H contains a path of length 2. Hence, to find a maximum-weight stable set S non-intersecting T it is sufficient to construct (if it exists) a maximum-weight stable set of each one of the above three types. We now prove that this construction can be done in $\mathcal{O}(|E|)$ time.

Case (i).

If a maximum-weight stable set S of type *(i)* exists, then there exists a path P of length 6 containing S and T. We have six different choices for the order of the nodes s, t, u in P. Consider, without loss of generality, $P = (s, x, t, y, u, z)$. The set $S = \{x, y, z\}$ with $x \in W(s,t)$, $y \in W(t,u)$, $z \in F(u)$ is the sought-after maximum-weight stable set. Let $X \equiv W(s,t)$, $Y \equiv W(t,u)$, $Z \equiv F(u)$. Observe that Z is a clique and X, Y are local sets. So X, Y, Z satisfy the hypothesis of Lemma 12 and we can, in $\mathcal{O}(|E| \log |V|)$ time, either find a maximum-weight stable set $\{x, y, z\}$ with $x \in X$, $y \in Y$, $z \in Z$ or conclude that no such stable set exists.

Case (ii). If a maximum-weight stable set S of type *(ii)* exists, then there exists a cycle C of length 6 containing S and T. Let $C = (s, a, t, b, u, c)$. The set $S = \{a, b, c\}$ with $a \in W(s,t)$, $b \in W(t,u)$, $c \in W(s,u)$ is the sought-after maximum-weight stable set.

Assume first that $W(t,u)$ is a clique (we can check this in $\mathcal{O}(|E|)$ time). Let $X \equiv W(s,t)$, $Y \equiv W(s,u)$, $Z \equiv W(t,u)$. By Lemma 12 we can, in $\mathcal{O}(|E| \log |V|)$ time, either conclude that there exists no stable set of type *(ii)* or find a maximum-weight stable set of this type.

Assume now that $W(t,u)$ is not a clique and let v, v' be two non-adjacent nodes in $W(t,u)$. Let $Z_1 = W(s,u) \cap N(v)$ and $Z_2 = W(s,u) \cap N(v')$. Since u is a common neighbor to v, v' and any node in $W(s,u)$, by claw-freeness we have $W(s,u) \subseteq Z_1 \cup Z_2$. Moreover, since s is adjacent to any node in $Z_1 \cup Z_2$ and non-adjacent to v and v', again by claw-freeness we have $Z_1 \cap Z_2 = \emptyset$, so Z_1 is null to v', Z_2 is null to v and $W(s,u)$ is the disjoint union of Z_1 and Z_2. It follows that Z_1 is a clique for, otherwise, $(u : p, q, v')$ would be a claw, with p and q any two non-adjacent nodes in Z_1. Analogously, also Z_2 is a clique. Now let $X \equiv W(s,t)$, $Y \equiv W(t,u)$ and $Z \equiv Z_1$ or $Z \equiv Z_2$. By applying Lemma 12 twice we can, in $\mathcal{O}(|E| \log |V|)$ time, either conclude that there exists no stable set of type *(ii)* or find a maximum stable set $\{a, b, c\}$ with $a \in W(s,t)$, $b \in W(t,u)$, $c \in W(s,u)$.

Case (iii). If a maximum-weight stable set S of type *(iii)* exists, then there exists a path P of length 2 containing a node in S and a node in T. We have three different choices for the node in $P \cap T$. Consider, without loss of generality, $P = (s, z)$; let $Z = F(s)$. The connected components of the symmetric difference of S and T containing the nodes t and u are either *(iii-a)* two paths P_1 and P_2 of length 2; *(iii-b)* a path P_1 of length 4; or *(iii-c)* a cycle C of length 4. In the first case let $P_1 = (t, x)$, $P_2 = (u, y)$ and let $X = F(t)$, $Y = F(u)$. In the second case we have two possibilities: either t or u is an extremum of P_1. Without loss of generality, assume $P_1 = (t, x, u, y)$ and let $X = W(t,u)$, $Y = F(u)$. In either case, the set $S = \{x, y, z\}$ with $x \in X$, $y \in Y$, $z \in Z$ is the sought-after maximum-weight stable set. By applying Lemma 12 we can, in $\mathcal{O}(|E| \log |V|)$ time, either conclude that there exists no stable set of types *(iii-a)* and *(iii-b)* or find a maximum stable set $\{x, y, z\}$ with $x \in X$, $y \in Y$, $z \in Z$. In case *(iii-c)* let $C = (t, x, u, y)$. The nodes x, y belong to $W(t,u)$ and the node z to $F(s)$. Moreover, by claw-freeness, $F(s)$ is null to $W(t,u)$. Recall that $W(t,u)$ is a local sets, so its cardinality is $\mathcal{O}(\sqrt{|E|})$. It follows that the maximum-weight stable set $S = \{x, y, z\}$ can be obtained by choosing the node z having maximum weight in $F(s)$ and finding in $\mathcal{O}(|E|)$ time the pair of non-adjacent nodes $x, y \in W(t,u)$ having maximum weight. This concludes the proof of the theorem. $\square$

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