On the Douglas–Rachford algorithm

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Abstract

The Douglas–Rachford algorithm is a very popular splitting technique for finding a zero of the sum of two maximally monotone operators. However, the behaviour of the algorithm remains mysterious in the general inconsistent case, i.e., when the sum problem has no zeros. More than a decade ago, however, it was shown that in the (possibly inconsistent) convex feasibility setting, the shadow sequence remains bounded and it is weak cluster points solve a best approximation problem.

In this paper, we advance the understanding of the inconsistent case significantly by providing a complete proof of the full weak convergence in the convex feasibility setting. In fact, a more general sufficient condition for the weak convergence in the general case is presented. Several examples illustrate the results.

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1 Introduction

In this paper we assume that

X is a real Hilbert space,

with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. A classical problem in optimization is to find a minimizer of the sum of two proper convex lower semicontinuous functions. This problem can be modelled as

find
$$x \in X$$
 such that $0 \in (A + B)x$, (1)

where *A* and *B* are maximally monotone operators on *X*, namely the subdifferential operators of the functions under consideration. For detailed discussions on problem (1) and the connection to optimization problems we refer the reader to [8], [16], [18], [20], [22], [32], [33], [31], [36], [37], and the references therein.

Due to its general convergence results, the Douglas–Rachford algorithm has become a very popular splitting technique to solve the sum problem (1) provided that the solution set is nonempty. The algorithm was first introduced in [23] to numerically solve certain types of heat equations. Let $x \in X$, let $T = T_{(A,B)}$ be the Douglas–Rachford operator associated with the ordered pair (A, B) (see (5)) and let J_A be the resolvent of A (see Fact 2.3). In their masterpiece [27], Lions and Mercier extended the algorithm to be able to find a zero of the sum of two, not necessarily linear and possibly multivalued, maximally monotone operators. They proved that the "governing sequence" $(T^n x)_{n \in \mathbb{N}}$ converges weakly to a fixed point of T, and that if A + B is maximally monotone, then the weak cluster points of the "shadow sequence" $(J_A T^n x)_{n \in \mathbb{N}}$ are solutions of (1). In [34], Svaiter provided a proof of the weak convergence of the shadow sequence, regardless of A + B.

Nonetheless, very little is known about the behaviour of the algorithm in the inconsistent setting, i.e., when the set of zeros of the sum is empty. In [9] (see Remark 5.6), the authors showed that for the case when *A* and *B* are normal cone operators of two nonempty closed convex subsets of *X*, and $P_{\overline{ran}(Id-T)} \in ran(Id-T)$ (see Fact 5.1), then the shadow sequence $(J_A T^n x)_{n \in \mathbb{N}}$ is bounded and its weak cluster points solve a certain best approximation problem.

In this paper we derive some new and useful identities for the Douglas–Rachford operator. The main contribution of the paper is generalizing the results in [9] by proving the full weak convergence of the shadow sequence in the convex feasibility setting (see Theorem 5.5). While the general case case remains open (see Example 5.7 and Remark 5.8), we provide some sufficient conditions for the convergence of the shadow sequence in some special cases (see Theorem 5.4).

As a by product of our analysis we present a new proof for the result in [34] concerning the weak convergence of the shadow sequence (see Theorem 6.2). Our proof is in the spirit of the techniques used in [27].

The notation used in the paper is standard and follows largely [8].

Useful identities for the Douglas-Rachford operator 2

We start with two elementary identities which are easily verified directly.

Lemma 2.1. Let $(a, b, z) \in X^3$. Then the following hold:

- (i) $\langle z, z a + b \rangle = ||z a + b||^2 + \langle a, z a \rangle + \langle b, 2a z b \rangle$.
- (i) $(z, z) = ||a b||^2 + \langle a, z a \rangle + \langle b, 2a z b \rangle.$ (ii) $||z||^2 = ||z a + b||^2 + ||b a||^2 + 2\langle a, z a \rangle + 2\langle b, 2a z b \rangle.$

Lemma 2.2. Let $(a, b, x, y, a^*, b^*, u, v) \in X^8$. Then

$$\langle (a,b) - (x,y), (a^*,b^*) - (u,v) \rangle = \langle a-b,a^* \rangle + \langle x,u \rangle - \langle x,a^* \rangle - \langle a-b,u \rangle + \langle b,a^*+b^* \rangle + \langle y,v \rangle - \langle y,b^* \rangle - \langle b,u+v \rangle.$$
(2)

Unless stated otherwise, we assume from now on that

 $A : X \rightrightarrows X$ and $B : X \rightrightarrows X$ are maximally monotone operators.

The following result concerning the resolvent $J_A := (\mathrm{Id} + A)^{-1}$ and the reflected resolvent $R_A := 2J_A - \text{Id is well known; see, e.g., } [8, \text{Corollary } 23.10(i)\&(ii)].$

Fact 2.3. $J_A : X \to X$ is firmly nonexpansive and $R_A : X \to X$ is nonexpansive.

Let us recall the well-known inverse resolvent identity (see, [31, Lemma 12.14]):

$$J_A + J_{A^{-1}} = \mathrm{Id},\tag{3}$$

and the following useful description of the graph of A.

Fact 2.4 (Minty parametrization). (See [29].) $M: X \to \operatorname{gra} A : x \mapsto (J_A x, J_{A^{-1}} x)$ is a continuous bijection, with continuous inverse M^{-1} : gra $A \to X : (x, u) \to x + u$; consequently,

gra
$$A = M(X) = \{ (J_A x, x - J_A x) \mid x \in X \}.$$
 (4)

Definition 2.5. The Douglas–Rachford splitting operator associated with (A, B) is

$$T := T_{(A,B)} = \frac{1}{2}(\mathrm{Id} + R_B R_A) = \mathrm{Id} - J_A + J_B R_A.$$
 (5)

We will simply use *T* instead of $T_{(A,B)}$ provided there is no cause for confusion.

The following result will be useful.

Lemma 2.6. Let $x \in X$. Then the following hold:

(i) $x - Tx = J_A x - J_B R_A x = J_{A^{-1}} x + J_{B^{-1}} R_A x.$ (ii) $(J_A x, J_B R_A x, J_{A^{-1}} x, J_{B^{-1}} R_A x)$ lies in gra $(A \times B)$.

Proof. (i): The first identity is a direct consequence of (5). In view of (3) $J_A x - J_B R_A x - J_{A^{-1}} x - J_{B^{-1}} R_A x = J_A x - (x - J_A x) - (J_B + J_{B^{-1}}) R_A x = R_A x - R_A x = 0$, which proves the second identity. (ii): Use (12) and Fact 2.4 applied to $A \times B$ at $(x, R_A x) \in X \times X$.

The next theorem is a direct consequence of the key identities presented in Lemma 2.1.

Theorem 2.7. Let $x \in X$ and let $y \in X$. Then the following hold:

(i)
$$\langle Tx - Ty, x - y \rangle = ||Tx - Ty||^2 + \langle J_A x - J_A y, J_{A^{-1}} x - J_{A^{-1}} y \rangle$$

 $+ \langle J_B R_A x - J_B R_A y, J_{B^{-1}} R_A x - J_{B^{-1}} R_A y \rangle.$
(ii) $\langle (Id - T)x - (Id - T)y, x - y \rangle = ||(Id - T)x - (Id - T)y||^2 + \langle J_A x - J_A y, J_{A^{-1}} x - J_{A^{-1}} y \rangle$
 $+ \langle J_B R_A x - J_B R_A y, J_{B^{-1}} R_A x - J_{B^{-1}} R_A y \rangle.$
(iii) $||x - y||^2 = ||Tx - Ty||^2 + ||(Id - T)x - (Id - T)y||^2 + 2\langle J_A x - J_A y, J_{A^{-1}} x - J_{A^{-1}} y \rangle$
 $+ 2\langle J_B R_A x - J_B R_A y, J_{B^{-1}} R_A x - J_{B^{-1}} R_A y \rangle.$
(iv) $||J_A x - J_A y||^2 + ||J_{A^{-1}} x - J_{A^{-1}} y||^2 - ||J_A T x - J_A Ty||^2 - ||J_{A^{-1}} T x - J_{A^{-1}} Ty||^2$
 $= ||(Id - T)x - (Id - T)y||^2 + 2\langle J_A T x - J_A Ty, J_{A^{-1}} T x - J_{A^{-1}} Ty \rangle$
 $+ 2\langle J_B R_A x - J_B R_A y, J_{B^{-1}} R_A x - J_{B^{-1}} R_A y \rangle.$
(v) $||J_A T x - J_A Ty||^2 + ||J_{A^{-1}} T x - J_{A^{-1}} Ty||^2 \le ||J_A x - J_A y||^2 + ||J_{A^{-1}} x - J_{A^{-1}} y||^2.$

Proof. (i)–(iii): Use Lemma 2.1–(iii) respectively, with z = x - y, $a = J_A x - J_A y$ and $b = J_B R_A x - J_B R_A y$, (4) and (5). (iv): It follows from the (3) that

$$\|x - y\|^2 = \|J_A x - J_A y + J_{A^{-1}} x - J_{A^{-1}} y\|^2$$
(6a)

$$= \|J_A x - J_A y\|^2 + \|J_{A^{-1}} x - J_{A^{-1}} y\|^2 + 2\langle J_A x - J_A y, J_{A^{-1}} x - J_{A^{-1}} y\rangle.$$
(6b)

Applying (6) to (Tx, Ty) instead of (x, y) yields

 $||Tx - Ty||^{2} = ||J_{A}Tx - J_{A}Ty||^{2} + ||J_{A^{-1}}Tx - J_{A^{-1}}Ty||^{2}$

$$+2\langle J_{A}Tx - J_{A}Ty, J_{A^{-1}}Tx - J_{A^{-1}}Ty \rangle.$$
(7)

Now combine (6), (7) and (iii) to obtain (iv). (v): In view of (4), the monotonicity of *A* and *B* implies $\langle J_A Tx - J_A Ty, J_{A^{-1}} Tx - J_{A^{-1}} Ty \rangle \ge 0$ and $\langle J_B R_A x - J_B R_A y, J_{B^{-1}} R_A x - J_{B^{-1}} R_A y \rangle \ge 0$. Now use (iv).

3 The Douglas–Rachford operator, Attouch–Théra duality and solution sets

The Attouch–Théra dual pair (see [1]) of (A, B) is $(A, B)^* := (A^{-1}, B^{-\otimes})$, where

$$B^{\odot} := (-\operatorname{Id}) \circ B \circ (-\operatorname{Id}) \quad \text{and} \quad B^{-\odot} := (B^{-1})^{\odot} = (B^{\odot})^{-1}.$$
(8)

We use

$$Z := Z_{(A,B)} = (A+B)^{-1}(0) \quad \text{and} \quad K := K_{(A,B)} = (A^{-1} + B^{-\Im})^{-1}(0) \quad (9)$$

to denote the primal and dual solutions, respectively (see, e.g., [7]).

Let us record some useful properties of $T_{(A,B)}$.

Fact 3.1. *The following hold:*

- (i) (Lions and Mercier). $T_{(A,B)}$ is firmly nonexpansive.
- (ii) (Eckstein). $T_{(A,B)} = T_{(A^{-1},B^{-1})}$.
- (iii) (Combettes). $Z = J_A(\text{Fix }T)$.
- (iv) $K = J_{A^{-1}}(\text{Fix } T)$.

Proof. (i): See, e.g., [27, Lemma 1], [24, Corollary 4.2.1 on page 139], [25, Corollary 4.1], or Theorem 2.7(i) (ii): See e.g., [24, Lemma 3.6 on page 133] or [7, Proposition 2.16]). (iii): See [21, Lemma 2.6(iii)]. (iv): See [7, Corollary 4.9].

The following notion, coined by Iusem [28], is very useful. We say that $C : X \rightrightarrows X$ is *paramonotone* if it is monotone and we have the implication

$$\begin{cases} (x,u) \in \operatorname{gra} C\\ (y,v) \in \operatorname{gra} C\\ \langle x-y,u-v \rangle = 0 \end{cases} \Rightarrow \{ (x,v), (y,u) \} \subseteq \operatorname{gra} C.$$
 (10)

Example 3.2. Let $f : X \to]-\infty, +\infty]$ be proper, convex and lower semicontinuous. Then ∂f is paramonotone by [28, Proposition 2.2] (or by [8, Example 22.3(*i*)]).

We now recall that the so-called "extended solution set" (see [26, Section 2.1] and also [7, Section 3]) is defined by

$$\mathcal{S} := \mathcal{S}_{(A,B)} := \{ (z,k) \in X \times X \mid -k \in Bz, k \in Az \} \subseteq Z \times K.$$
(11)

Fact 3.3. *Recalling Fact 2.4, we have the following:*

- (i) $S = M(\text{Fix } T) = \{ (J_A \times J_{A^{-1}})(y, y) \mid y \in \text{Fix } T \}.$
- (ii) Fix $T = M^{-1}(S) = \{z + k \mid (z, k) \in S\}.$
- (iii) (Eckstein and Svaiter). S is closed and convex.

If A and B are paramonotone, then we additionally have:

- (iv) $S = Z \times K$.
- (v) Fix T = Z + K.

Proof. (i)&(ii): This is [7, Theorem 4.5]. (iii): See [26, Lemma 2]. Alternatively, since Fix *T* is closed, and *M* and M^{-1} are continuous, we deduce the closedness from (i). The convexity was proved in [7, Corollary 3.7]. (iv) & (v): See [7, Corollary 5.5(ii)&(iii)].

Working in $X \times X$, we recall that (see, e.g., [8, Proposition 23.16])

$$A \times B$$
 is maximally monotone and $J_{A \times B} = J_A \times J_B$. (12)

Corollary 3.4. *Let* $x \in X$ *and let* $(z, k) \in S$ *. Then*

$$\|(J_A T x, J_{A^{-1}} T x) - (z, k)\|^2 = \|J_A T x - z\|^2 + \|J_{A^{-1}} T x - k\|^2$$
(13a)

$$= \|J_A I x - z\|^2 + \|J_{A^{-1}} I x - k\|^2$$
(13a)
$$\leq \|J_A x - z\|^2 + \|J_{A^{-1}} x - k\|^2$$
(13b)

$$= \|(J_A x, J_{A^{-1}} x) - (z, k)\|^2.$$
(13c)

Proof. It follows from [7, Theorem 4.5] that $z + k \in \text{Fix } T$, $J_A(z + k) = z$ and $J_{A^{-1}}(z + k) = k$. Now combine with Theorem 2.7(v) with *y* replaced by z + k.

We recall, as consequence of [8, Corollary 22.19] and Example 3.2, that when $X = \mathbb{R}$, the operators *A* and *B* are paramonotone. In view of Fact 3.3(iv), we then have $S = Z \times K$.

Lemma 3.5. Suppose that $X = \mathbb{R}$. Let $x \in X$ and let $(z, k) \in Z \times K$. Then the following hold:

(i) $||J_A T x - z||^2 \le ||J_A x - z||^2$. (ii) $||J_{A^{-1}} T x - k||^2 \le ||J_{A^{-1}} x - k||^2$.

Proof. (i): Set

$$q(x,z) := \|J_A T x - z\|^2 - \|J_A x - z\|^2.$$
(14)

If $x \in \text{Fix } T$ we get q(x, z) = 0. Suppose that $x \in \mathbb{R} \setminus \text{Fix } T$. Since *T* is firmly nonexpansive we have that Id -T is firmly nonexpansive (see [8, Proposition 4.2]), hence monotone by [8, Example 20.27]. Therefore $(\forall x \in \mathbb{R} \setminus \text{Fix } T)(\forall f \in \text{Fix } T)$ we have

$$(x - Tx)(x - f) = ((\mathrm{Id} - T)x - (\mathrm{Id} - T)f)(x - f) > 0.$$
(15)

Notice that (14) can be rewritten as

$$q(x,z) = (J_A T x - J_A x)((J_A T x - z) + (J_A x - z)).$$
(16)

We argue by cases.

Case 1: x < Tx. It follows from (15) that

$$(\forall f \in \operatorname{Fix} T) \ x < f. \tag{17}$$

On the one hand, since J_A is firmly nonexpansive, we have J_A is monotone and therefore $J_ATx - J_Ax \ge 0$. On the other hand, it follows from Fact 3.1(iii) that $(\exists f \in \text{Fix } T)$ such that $z = J_A f = J_A T f$. Using (17) and the fact that J_A is monotone we conclude that $J_Ax - z = J_Ax - J_Af \le 0$. Moreover, since J_A and T are firmly nonexpansive operators on \mathbb{R} , we have $J_A \circ T$ is firmly nonexpansive hence monotone and therefore (17) implies that $J_ATx - z = J_ATx - J_ATf \le 0$. Combining with (16) we conclude that (i) holds.

Case 2: x > Tx. The proof follows similar to *Case 1*.

(ii) Apply the results of (i) to A^{-1} and use (3).

In view of (11) one might conjecture that Corollary 3.4 holds when we replace S by $Z \times K$. The following example gives a negative answer to this conjecture. It also illustrates that when $X \neq \mathbb{R}$, the conclusion of Lemma 3.5 could fail.

Example 3.6. Suppose that $X = \mathbb{R}^2$, that A is the normal cone operator of \mathbb{R}^2_+ , and that $B: X \to X: (x_1, x_2) \mapsto (-x_2, x_1)$ is the rotator by $\pi/2$. Then Fix $T = \mathbb{R}_+ \cdot (1, -1)$, $Z = \mathbb{R}_+ \times \{0\}$ and $K = \{0\} \times \mathbb{R}_+$. Moreover, $(\exists x \in \mathbb{R}^2) \ (\exists (z,k) \in Z \times K)$ such that $\|(J_A Tx, J_{A^{-1}}Tx) - (z,k)\|^2 - \|(J_A x, J_{A^{-1}}x) - (z,k)\|^2 > 0$ and $\|J_A Tx - z\|^2 - \|J_A x - z\|^2 = \frac{5}{4}a^2 > 0$.

Proof. Let $(x_1, x_2) \in \mathbb{R}^2$. Using [6, Proposition 2.10] we have

$$J_B(x_1, x_2) = (\frac{1}{2}(x_1 + x_2), \frac{1}{2}(-x_1 + x_2))$$
 and $R_B(x_1, x_2) = (x_2, -x_1) = -B(x_1, x_2).$ (18)

Hence, $R_B^{-1} = (-B)^{-1} = B$. Using (5) we conclude that $(x_1, x_2) \in \text{Fix } T \Leftrightarrow (x_1, x_2) \in \text{Fix } R_B R_A$. Hence

Fix
$$T = \{(x_1, x_2) \in \mathbb{R}^2 \mid (x_1, x_2) = R_B R_A(x_1, x_2)\}$$
 (19a)

$$= \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid R_B^{-1}(x_1, x_2) = 2J_A(x_1, x_2) - (x_1, x_2) \right\}$$
(19b)

$$= \{ (x_1, x_2) \in \mathbb{R}^2 \mid B(x_1, x_2) + (x_1, x_2) = 2P_{\mathbb{R}^2_+}(x_1, x_2) \}$$
(19c)

$$= \{ (x_1, x_2) \in \mathbb{R}^2 \mid (x_1 - x_2, x_1 + x_2) = 2P_{\mathbb{R}^2_+}(x_1, x_2) \}.$$
(19d)

We argue by cases.

Case 1: $x_1 \ge 0$ and $x_2 \ge 0$. Then $(x_1, x_2) \in \text{Fix } T \Leftrightarrow (x_1 - x_2, x_1 + x_2) = 2P_{\mathbb{R}^2_+}(x_1, x_2) = (2x_1, 2x_2) \Leftrightarrow x_1 = -x_2$ and $x_1 = x_2 \Leftrightarrow x_1 = x_2 = 0$.

Case 2: $x_1 < 0$ and $x_2 < 0$. Then $(x_1, x_2) \in \text{Fix } T \Leftrightarrow (x_1 - x_2, x_1 + x_2) = 2P_{\mathbb{R}^2_+}(x_1, x_2) = (0, 0) \Leftrightarrow x_1 = x_2$ and $x_1 = -x_2 \Leftrightarrow x_1 = x_2 = 0$, which contradicts that $x_1 < 0$ and $x_2 < 0$.

Case 3: $x_1 \ge 0$ and $x_2 < 0$. Then $(x_1, x_2) \in \text{Fix } T \Leftrightarrow (x_1 - x_2, x_1 + x_2) = 2P_{\mathbb{R}^2_+}(x_1, x_2) = (2x_1, 0) \Leftrightarrow x_1 = -x_2$.

Case 4: $x_1 < 0$ and $x_2 \ge 0$. Then $(x_1, x_2) \in \text{Fix } T \Leftrightarrow (x_1 - x_2, x_1 + x_2) = 2P_{\mathbb{R}^2_+}(x_1, x_2) = (0, 2x_2) \Leftrightarrow x_1 = x_2$, which never occurs since $x_1 < 0$ and $x_2 \ge 0$. Altogether we conclude that Fix $T = R_+ \cdot (1, -1)$, as claimed.

Using Fact 3.1(iii)&(iv) we have $Z = J_A(\text{Fix } T) = \mathbb{R}_+ \times \{0\}$, and $K = J_{A^{-1}}(\text{Fix } T) = (\text{Id} - J_A)(\text{Fix } T) = \{0\} \times \mathbb{R}_-$.

Now let a > 0, let x = (a,0), set $z := (2a,0) \in Z$ and set $k := (0,-a) \in K$. Notice that $Tx = x - P_{\mathbb{R}^2_+} + \frac{1}{2}(\mathrm{Id} - B)x = (a,0) - (a,0) + \frac{1}{2}((a,0) - (0,a)) = (\frac{1}{2}a, -\frac{1}{2}a)$. Hence, $J_A x = P_{\mathbb{R}^2_+}(a,0) = (a,0), J_{A^{-1}}x = P_{\mathbb{R}^2_-}(a,0) = (0,0), J_A Tx = P_{\mathbb{R}^2_+}(\frac{1}{2}a, -\frac{1}{2}a) = (\frac{1}{2}a, 0)$, and $J_{A^{-1}}x = P_{\mathbb{R}^2}(\frac{1}{2}a, -\frac{1}{2}a) = (0, -\frac{1}{2}a)$. Therefore

$$\begin{aligned} &\|(J_A Tx, J_{A^{-1}} Tx) - (z, k)\|^2 - \|(J_A x, J_{A^{-1}} x) - (z, k)\|^2 \\ &= \|J_A Tx - z\|^2 + \|J_{A^{-1}} Tx - k\|^2 - \|J_A x - z\|^2 - \|J_{A^{-1}} x - k\|^2 \\ &= \|(\frac{1}{2}a, 0) - (2a, 0)\|^2 + \|(0, -\frac{1}{2}a) - (0, -a)\|^2 - \|(a, 0) - (2a, 0)\|^2 - \|(0, 0) - (0, -a)\|^2 \\ &= \frac{9}{4}a^2 + \frac{1}{4}a^2 - a^2 - a^2 = \frac{1}{2}a^2 > 0. \end{aligned}$$

Similarly one can verify that $||J_A T x - z||^2 - ||J_A x - z||^2 = \frac{5}{4}a^2 > 0.$

4 Linear relations

In this section, we assume that¹

 $A: X \rightrightarrows X$ and $B: X \rightrightarrows X$ are maximally monotone linear relations;

equivalently, by [12, Theorem 2.1(xviii)], that

$$J_A$$
 and J_B are linear operators from X to X. (20)

This additional assumption leads to stronger conclusions.

Lemma 4.1. Id $-T = J_A - 2J_BJ_A + J_B$.

Proof. Let $x \in X$. Then indeed $x - Tx = J_A x - J_B R_A x = J_A x - J_B (2J_A x - x) = J_A x - 2J_B J_A x + J_B x$.

Lemma 4.2. Suppose that U is a linear subspace of X and that $A = P_U$. Then A is maximally monotone,

$$J_A = J_{P_U} = \frac{1}{2} (\mathrm{Id} + P_{U^{\perp}}), \quad and \quad R_A = P_{U^{\perp}} = \mathrm{Id} - A.$$
 (21)

Proof. Let $(x, y) \in X \times X$. Then

$$y = J_A x \Leftrightarrow x = y + P_U y. \tag{22}$$

Now assume $y = J_A x$. Since P_U is linear, (22) implies that $P_{U^{\perp}} x = P_{U^{\perp}} y$. Moreover, $y = x - P_U y = \frac{1}{2}(x + x - 2P_U y) = \frac{1}{2}(x + y + P_U y - 2P_U y) = \frac{1}{2}(x + y - P_U y) = \frac{1}{2}(x + P_U^{\perp} y) = \frac{1}{2}(x + P_U^{\perp} x)$. Next, $R_A = 2J_A - \text{Id} = (\text{Id} + P_U^{\perp}) - \text{Id} = P_U^{\perp}$.

We say that a linear relation *A* is *skew* (see, e.g., [15]) if $(\forall (a, a^*) \in \text{gra } A) \langle a, a^* \rangle = 0$.

Lemma 4.3. Suppose that $A : X \to X$ and $B : X \to X$ are both skew, and $A^2 = B^2 = -$ Id. Then Id $-T = \frac{1}{2}($ Id -BA).

Proof. It follows from [6, Proposition 2.10] that $R_A = A$ and $R_B = B$. Therefore (5) implies that $Id - T = \frac{1}{2}(Id - R_BR_A) = \frac{1}{2}(Id - BA)$.

Example 4.4. Suppose that A and B are skew. Let $x \in X$ and let $y \in X$. Then the following hold:

- (i) $\langle Tx Ty, x y \rangle = ||Tx Ty||^2$.
- (ii) $\langle (\mathrm{Id} T)x (\mathrm{Id} T)y, x y \rangle = \| (\mathrm{Id} T)x (\mathrm{Id} T)y \|^2$.

¹*A* : $X \rightrightarrows X$ is a *linear relation* if gra *A* is a linear subspace of $X \times X$.

(iii)
$$||x - y||^2 = ||Tx - Ty||^2 + ||(Id - T)x - (Id - T)y||^2$$
.
(iv) $||J_Ax - J_Ay||^2 + ||J_{A^{-1}}x - J_{A^{-1}}y||^2 - ||J_ATx - J_ATy||^2 - ||J_{A^{-1}}Tx - J_{A^{-1}}Ty||^2$
 $= ||(Id - T)x - (Id - T)y||^2$.
(v) $||x||^2 = ||Tx||^2 + ||x - Tx||^2$.
(vi) $\langle Tx, x - Tx \rangle = 0$.

Proof. (i)–(iv): Apply Theorem 2.7, and use (4) as well as the skewness of *A* and *B*. (v): Apply (iii) with y = 0. (vi): We have $2\langle Tx, x - Tx \rangle = ||x||^2 - ||Tx||^2 - ||x - Tx||^2$. Now apply (v).

Suppose that *U* is a closed affine subspace of *X*. One can easily verify that

$$(\forall x \in X)(\forall y \in X) \quad \langle P_U x - P_U y, (\mathrm{Id} - P_U) x - (\mathrm{Id} - P_U) y \rangle = 0.$$
(23)

Example 4.5. Suppose that U and V are closed affine subspaces of X such that $U \cap V \neq \emptyset$, that $A = N_U$, and that $B = N_V$. Let $x \in X$, and let $(z,k) \in Z \times K$). Then

$$\|(P_U x, (\mathrm{Id} - P_U)x) - (z, k)\|^2 - \|(P_U T x, (\mathrm{Id} - P_U)Tx) - (z, k)\|^2$$
(24a)

$$= \|x - (z+k)\|^{2} - \|Tx - (z+k)\|^{2}$$
(24b)

$$= \|x - Tx\|^2$$
(24c)

$$= \|P_{U}x - P_{V}x\|^{2}.$$
 (24d)

Proof. As subdifferential operators, *A* and *B* are paramonotone (by Example 3.2). It follows from Fact 3.3(v) and [7, Theorem 4.5] that

$$z + k \in \text{Fix } T, \quad P_U(z + k) = z \text{ and } (\text{Id} - P_U)(z + k) = k.$$
 (25)

Hence, in view of (23) we have

$$\|(P_U x, (\mathrm{Id} - P_U)x) - (z, k)\|^2$$
(26a)

$$= \|P_{U}x - z\|^{2} + \|(\mathrm{Id} - P_{U})x - k\|^{2}$$
(26b)

$$= \|P_{U}x - P_{U}(z+k)\|^{2} + \|(\mathrm{Id} - P_{U})x - (\mathrm{Id} - P_{U})(z+k)\|^{2}$$
(26c)

$$+2\langle P_{U}x - P_{U}(z+k), (\mathrm{Id} - P_{U})x - (\mathrm{Id} - P_{U})(z+k)\rangle$$
(26d)

$$= \|P_{U}x - P_{U}(z+k) + (\mathrm{Id} - P_{U})x - (\mathrm{Id} - P_{U})(z+k)\|^{2}$$
(26e)

$$= \|x - (z+k)\|^2.$$
(26f)

Applying (26) with x replaced by Tx yields

$$\|(P_U Tx, (\mathrm{Id} - P_U)Tx) - (z, k)\|^2 = \|Tx - (z+k)\|^2.$$
(27)

Combining (26) and (27) yields (24b). It follows from (23) and Theorem 2.7(iii) applied with (*A*, *B*, *y*) replaced by (N_U , N_V , z + k) that $||x - (z + k)||^2 - ||Tx - T(z + k)||^2 = ||x - Tx - ((z + k) - T(z + k))||^2$, which in view of (25), proves (24c).

Now we turn to (24d). Let $w \in U \cap V$. Then U = w + par U and V = w + par V. Suppose momentarily that w = 0. In this case, par U = U and par V = V. Using [5, Proposition 3.4(i)], we have

$$T = T_{(U,V)} = P_V P_U + P_{V^{\perp}} P_{U^{\perp}}.$$
(28)

Therefore

$$x - Tx = P_{U}x + P_{U^{\perp}}x - P_{V}P_{U}x - P_{V^{\perp}}P_{U^{\perp}}x = (\mathrm{Id} - P_{V})P_{U}x + (\mathrm{Id} - P_{V^{\perp}})P_{U^{\perp}}x$$
(29a)
= $P_{V^{\perp}}P_{U}x + P_{V}P_{U^{\perp}}x.$ (29b)

Using (29b) we have

$$\|x - Tx\|^{2} = \|P_{V^{\perp}}P_{U}x + P_{V}P_{U^{\perp}}x\|^{2} = \|P_{U}x - P_{V}P_{U}x + P_{V}x - P_{V}P_{U}x\|^{2}$$
(30a)

$$= \|P_U x - 2P_V P_U x + P_V x\|^2$$
(30b)

$$= \|P_{U}x\|^{2} + \|P_{V}x\|^{2} + 4\|P_{V}P_{U}x\|^{2}$$
(30c)

$$+2\langle P_{U}x, P_{V}x\rangle - 4\langle P_{U}x, P_{V}P_{U}x\rangle - 4\langle P_{V}x, P_{V}P_{U}x\rangle$$
(30d)

$$= \|P_{U}x\|^{2} + \|P_{V}x\|^{2} - 2\langle P_{U}x, P_{V}x\rangle = \|P_{U}x - P_{V}x\|^{2}.$$
 (30e)

Now, if $w \neq 0$, by [11, Proposition 5.3] we have $Tx = T_{(\text{par } U, \text{par } V)}(x - w) + w$. Therefore, (30) yields $||x - Tx||^2 = ||(x - w) - T_{(\text{par } U, \text{par } V)}(x - w)||^2 = ||P_{\text{par } U}(x - w) - P_{\text{par } V}(x - w)||^2 = ||W + P_{\text{par } U}(x - w) - (w + P_{\text{par } V}(x - w))||^2 = ||P_U x - P_V x||^2$, where the last equality follows from [8, Proposition 3.17].

5 Main results

In this section we consider the case when the set *Z* is possibly empty.

We recall the following important fact.

Fact 5.1 (Infimal displacement vector). (See, e.g., [2],[19] and [30].) Let $T : X \to X$ be nonexpansive. Then $\overline{ran}(Id - T)$ is convex; consequently, the infimal displacement vector

$$v := P_{\overline{\operatorname{ran}}(\operatorname{Id} - T)}.$$
(31)

is the unique and well-defined element in $\overline{\operatorname{ran}}(\operatorname{Id} - T)$ such that $||v|| = \inf_{x \in X} ||x - Tx||$.

Following [6], the *normal problem* associated with the ordered pair (A, B) is to²

find
$$x \in X$$
 such that $0 \in {}_{v}Ax + B_{v}x = Ax - v + B(x - v)$. (32)

We shall use

$$Z_v := Z_{(vA,B_v)}$$
 and $K_v := K_{((vA)^{-1},(B_v)^{-1})}$ (33)

to denote the *primal normal* and *dual normal solutions*, respectively. It follows from [6, Proposition 3.3] that

$$Z_v \neq \emptyset \Leftrightarrow v \in \operatorname{ran}(\operatorname{Id} - T).$$
(34)

Corollary 5.2. Let $x \in X$ and let $y \in X$. Then the following hold:

$$\sum_{n=0}^{\infty} \| (\mathrm{Id} - T)T^n x - (\mathrm{Id} - T)T^n y \|^2 < +\infty,$$
 (35a)

$$\sum_{n=0}^{\infty} \underbrace{\langle J_A T^n x - J_A T^n y, J_{A^{-1}} T^n x - J_{A^{-1}} T^n y \rangle}_{\geq 0} < +\infty,$$
(35b)

$$\sum_{n=0}^{\infty} \underbrace{\langle J_B R_A T^n x - J_B R_A T^n y, J_{B^{-1}} R_A T^n x - J_{B^{-1}} R_A T^n y \rangle}_{\geq 0} < +\infty.$$
(35c)

Consequently,

$$(\mathrm{Id} - T)T^{n}x - (\mathrm{Id} - T)T^{n}y \to 0, \qquad (36a)$$

$$\langle J_A T^n x - J_A T^n y, J_{A^{-1}} T^n x - J_{A^{-1}} T^n y \rangle \to 0,$$
 (36b)

$$\langle J_B R_A T^n x - J_B R_A T^n y, J_{B^{-1}} R_A T^n x - J_{B^{-1}} R_A T^n y \rangle \to 0.$$
 (36c)

Proof. Let $n \in \mathbb{N}$. Applying (4), to the points $T^n x$ and $T^n y$, we learn that $\{(J_A T^n x, J_{A^{-1}} T^n x), (J_A T^n y, J_{A^{-1}} T^n y)\} \subseteq \text{gra } A$, hence, by monotonicity of A we have $\langle J_A T^n x - J_A T^n y, J_{A^{-1}} T^n x - J_{A^{-1}} T^n y \rangle \ge 0$. Similarly $\langle J_B R_A T^n x - J_B R_A T^n y, J_{B^{-1}} R_A T^n x - J_{B^{-1}} R_A T^n y \rangle \ge 0$. Now (35) and (36) follow from Theorem 2.7(iii) by telescoping.

The next result on Fejér monotone sequences is of critical importance in our analysis. (When $(u_n)_{n \in \mathbb{N}} = (x_n)_{n \in \mathbb{N}}$ one obtains a well-known result; see, e.g., [8, Theorem 5.5].)

Lemma 5.3 (new Fejér monotonicity principle). Suppose that *E* is a nonempty closed convex subset of *X*, that $(x_n)_{n \in \mathbb{N}}$ is a sequence in *X* that is Fejér monotone with respect to *E*, *i.e.*,

$$(\forall e \in E)(\forall n \in \mathbb{N}) \quad ||x_{n+1} - e|| \le ||x_n - e||,$$
(37)

²Let $w \in X$ be fixed. For the operator A, the *inner and outer shifts* associated with A are defined by $A_w: X \rightrightarrows X: x \mapsto A(x-w)$ and $_wA: X \rightrightarrows X: x \mapsto Ax - w$. Note that A_w and $_wA$ are maximally monotone.

that $(u_n)_{n \in \mathbb{N}}$ is a bounded sequence in X such that its weak cluster points lie in E, and that

$$(\forall e \in E) \ \langle u_n - e, u_n - x_n \rangle \to 0.$$
 (38)

Then $(u_n)_{n \in \mathbb{N}}$ *converges weakly to some point in E*.

Proof. It follows from (38) that

$$(\forall (e_1, e_2) \in E \times E) \quad \langle e_2 - e_1, u_n - x_n \rangle = \langle u_n - e_1, u_n - x_n \rangle - \langle u_n - e_2, u_n - x_n \rangle \to 0.$$
(39)

Now obtain four subsequences $(x_{k_n})_{n \in \mathbb{N}}$, $(x_{l_n})_{n \in \mathbb{N}}$, $(u_{k_n})_{n \in \mathbb{N}}$ and $(u_{l_n})_{n \in \mathbb{N}}$ such that $x_{k_n} \rightharpoonup x_1$, $x_{l_n} \rightharpoonup x_2$, $u_{k_n} \rightharpoonup e_1$ and $u_{l_n} \rightharpoonup e_2$. Taking the limit in (39) along these subsequences we have $\langle e_2 - e_1, e_1 - x_1 \rangle = 0 = \langle e_2 - e_1, e_2 - x_2 \rangle$, hence

$$||e_2 - e_1||^2 = \langle e_2 - e_1, x_2 - x_1 \rangle.$$
 (40)

Since $\{e_1, e_2\} \subseteq E$, we conclude, in view of [3, Theorem 6.2.2(ii)] or [10, Lemma 2.2], that $\langle e_2 - e_1, x_2 - x_1 \rangle = 0$. By (39), $e_1 = e_2$.

We are now ready for our main result.

Theorem 5.4 (shadow convergence). Suppose that $x \in X$, that the sequence $(J_A T^n x)_{n \in \mathbb{N}}$ is bounded and its weak cluster points lie in Z_v , that $Z_v \subseteq \text{Fix}(v+T)$ and that $(\forall n \in \mathbb{N})$ $(\forall y \in \text{Fix}(v+T)) J_A T^n y = y$. Then the "shadow" sequence $(J_A T^n x)_{n \in \mathbb{N}}$ converges weakly to some point in Z_v .

Proof. Let $y \in Fix(v + T)$. Using (36b) and [13, Proposition 2.4(iv)] we have

$$\langle J_A T^n x - y, T^n x + nv - J_A T^n x \rangle = \langle J_A T^n x - y, T^n x - J_A T^n x - (y - nv - y) \rangle$$
(41a)

$$= \langle J_A T^n x - J_A T^n y, (\mathrm{Id} - J_A) T^n x - (\mathrm{Id} - J_A) T^n y \rangle \quad (41b)$$

$$\rightarrow 0.$$
 (41c)

Note that [13, Proposition 2.4(vi)] implies that $(T^n x + nv)_{n \in \mathbb{N}}$ is Fejér monotone with respect to Fix(v + T) and consequently with respect to Z_v . Now apply Lemma 5.3 with E replaced by Z_v , $(u_n)_{n \in \mathbb{N}}$ replaced by $(J_A T^n x)_{n \in \mathbb{N}}$, and $(x_n)_{n \in \mathbb{N}}$ replaced by $(T^n x + nv)_{n \in \mathbb{N}}$.

As a powerful application of Theorem 5.4, we obtain the following striking strengthening of a previous result on normal cone operators.

Theorem 5.5. Suppose that U and V are nonempty closed convex subsets of X, that $A = N_U$, that $B = N_V$, that $v = P_{\overline{ran}(Id - T)}$ and that $U \cap (v + V) \neq \emptyset$. Let $x \in X$. Then $(P_U T^n x)_{n \in \mathbb{N}}$ converges weakly to some point in $Z_v = U \cap (v + V)$.

Proof. It follows from [9, Theorem 3.13(iii)(b)] that $(P_U T^n x)_{n \in \mathbb{N}}$ is bounded and its weak cluster points lie in $U \cap (v + V)$. Moreover [9, Theorem 3.5] implies that $Z_v = U \cap (v + V) \subseteq U \cap (v + V) + N_{\overline{U-V}}(v) \subseteq \operatorname{Fix}(v + T)$. Finally, [9, Lemma 3.12 & Proposition 2.4(ii)] imply that $(\forall y \in \operatorname{Fix}(v + T))(\forall n \in \mathbb{N}) P_U T^n y = P_U(y - nv) = y$, hence all the assumptions of Theorem 5.4 are satisfied and the result follows.

Remark 5.6. Suppose that $v \in \operatorname{ran}(\operatorname{Id} - T)$. More than a decade ago, it was shown in [9] that when $A = N_U$ and $B = N_V$, where U and V are nonempty closed convex subsets of X, that $(P_U T^n x)_{n \in \mathbb{N}}$ is bounded and its weak cluster points lie in $U \cap (v + V)$. Theorem 5.5 yields the much stronger result that $(P_U T^n x)_{n \in \mathbb{N}}$ converges weakly to a point in $U \cap (v + V)$.

Here is another instance of Theorem 5.4.

Example 5.7. Suppose that U is a closed linear subspace of X, that $b \in U^{\perp} \setminus \{0\}$, that $A = N_U$ and that $B = \text{Id} + N_{(-b+U)}$. Then $Z = \emptyset$, $v = b \in \text{ran}(\text{Id} - T)$, $Z_v = \{0\}$ and $K_v = U^{\perp}$. Moreover, $(\forall x \in X) \ (\forall n \in \mathbb{N}) \ P_U T^n x = \frac{1}{2^n} P_U x \to 0 \text{ and } \|P_{U^{\perp}} T^n x\| \to \infty$.

Proof. By the Brezis-Haraux theorem (see [17, Theorems 3 & 4] or [8, Theorem 24.20]) we have $X = \operatorname{int} X \subseteq \operatorname{int} \operatorname{ran} B = \operatorname{int}(\operatorname{ran} \operatorname{Id} + \operatorname{ran} N_{(-b+U)}) \subseteq X$, hence $\operatorname{ran} B = X$. Using [14, Corollary 5.3(ii)] we have $\overline{\operatorname{ran}}(\operatorname{Id} - T) = \overline{(\operatorname{dom} A - \operatorname{dom} B)} \cap \overline{(\operatorname{ran} A + \operatorname{ran} B)} = (U + b - U) \cap (U^{\perp} + X) = b + U$. Consequently, using [6, Definition 3.6] and [8, Proposition 3.17] we have

$$v = P_{\overline{\operatorname{ran}}(\operatorname{Id} - T)} 0 = P_{b+U} 0 = b + P_U(-b) = b \in U^{\perp} \smallsetminus \{0\}.$$
(42)

Note that dom $_vA = \text{dom } A = U$ and dom $B_v = v + \text{dom } B = b - b + U = U$, hence $\text{dom}(_vA + B_v) = U \cap U = U$. Let $x \in U$. Using (42) we have

$$x \in Z_v \Leftrightarrow 0 \in N_U x - b + x - b + N_{-b+U}(x-b) = N_U x - b + x - b + N_U x$$
(43a)

$$\Leftrightarrow 0 \in U^{\perp} - b + x - b + U^{\perp} = x + U^{\perp} \Leftrightarrow x \in U^{\perp},$$
(43b)

hence $Z_v = \{0\}$, as claimed. As subdifferentials, both *A* and *B* are paramonotone, and so are the translated operators $_vA$ and B_v . Since $Z_v = \{0\}$, in view of [7, Remark 5.4] and (42) we learn that

$$K_v = (N_U 0 - b) \cap (0 - b + N_{-b+U} (0 - b)) = (U^{\perp} - b) \cap (-b + U^{\perp}) = U^{\perp}.$$
 (44)

Next we claim that

$$(\forall x \in X) \quad P_U T x = \frac{1}{2} P_U x. \tag{45}$$

Indeed, note that $J_B = (\mathrm{Id} + B)^{-1} = (2 \mathrm{Id} + N_{-b+U})^{-1} = (2 \mathrm{Id} + 2N_{-b+U})^{-1} = (\mathrm{Id} + N_{-b+U})^{-1} \circ (\frac{1}{2} \mathrm{Id}) = P_{-b+U} \circ (\frac{1}{2} \mathrm{Id}) = -b + \frac{1}{2} P_U$, where the last identity follows from [8, Proposition 3.17)] and (42). Now, using that $P_U R_U = P_U$ and (42) we have

³It follows from [8, Corollary 3.20] that P_U is linear, hence, $P_U R_U = P_U (2P_U - \text{Id}) = 2P_U - P_U = P_U$.

 $P_UT = P_U(P_{U^{\perp}} + J_BR_U) = P_UJ_BR_U = P_U(-b + \frac{1}{2}P_U)R_U = P_U(-b + \frac{1}{2}P_U) = \frac{1}{2}P_U$. To show that $(\forall x \in X) \ (\forall n \in \mathbb{N}) \ P_UT^n x = \frac{1}{2^n}P_Ux$, we use induction. Let $x \in X$. Clearly, when n = 0, the base case holds. Now suppose that for some $n \in \mathbb{N}$, we have, for every $x \in X$, $P_UT^n x = \frac{1}{2^n}P_Ux$. Now applying the inductive hypothesis with x replaced by Tx, and using (45), we have $P_UT^{n+1}x = P_UT^n(Tx) = \frac{1}{2^n}P_UTx = \frac{1}{2^n}P_U(\frac{1}{2}P_Ux) = \frac{1}{2^{n+1}}P_Ux$, as claimed. Finally, using (42) and [30, Corollary 6(a)] we have $||T^nx|| \to +\infty$, hence

$$\|P_{U^{\perp}}T^{n}x\|^{2} = \|T^{n}x\|^{2} - \|P_{U}T^{n}x\|^{2} = \|T^{n}x\|^{2} - \frac{1}{4^{n}}\|P_{U}x\|^{2} \to +\infty.$$
(46)

In fact, as we shall now see, the shadow sequence may be unbounded in the general case, even when one of the operators is a normal cone operator.

Remark 5.8. (shadows in the presence of normal solutions)

- (i) Example 5.7 illustrates that even when normal solutions exists, the shadows need not converge. Indeed, we have $K_v = U^{\perp} \neq \emptyset$ but the dual shadows satisfy $\|P_{U^{\perp}}T^n x\| \to +\infty$.
- (ii) Suppose that A and B are as defined in Example 5.7. Set $\widetilde{A} = A^{-1}$, $\widetilde{B} = B^{-\otimes}$ and $\widetilde{Z} = Z_{(\widetilde{A},\widetilde{B})}$. By [7, Proposition 2.4(v)] $\widetilde{Z} \neq \emptyset \Leftrightarrow K_{(\widetilde{A},\widetilde{B})} = Z_{(A,B)} \neq \emptyset$, hence $\widetilde{Z} = \emptyset$. Moreover, [6, Remarks 3.13 & 3.5] imply that $v = b \in \operatorname{ran}(\operatorname{Id} T)$ and $\widetilde{Z}_v = U^{\perp} + b = U^{\perp} \neq \emptyset$. However, in the light of (i) the primal shadows satisfy $\|J_{\widetilde{A}}T^nx\| = \|J_{A^{-1}}T^nx\| = \|P_{U^{\perp}}T^nx\| \to +\infty$.
- (iii) Concerning Theorem 5.4, it would be interesting to find other conditions sufficient for weak convergence of the shadow sequence or to even characterize this behaviour.

6 A proof of the Lions-Mercier-Svaiter theorem

In this section, we work under the assumptions that

$$Z \neq \varnothing$$
 and Fix $T \neq \varnothing$. (47)

Parts of the following two results are implicit in [34]; however, our proofs are different.

Proposition 6.1. *Let* $x \in X$ *. Then the following hold:*

- (i) $T^n x T^{n+1} x = J_A T^n x J_B R_A T^n x = J_{A^{-1}} T^n x + J_{B^{-1}} R_A T^n x \to 0.$
- (ii) The sequence $(J_A T^n x, J_B R_A T^n x, J_{A^{-1}} T^n x, J_{B^{-1}} R_A T^n x)_{n \in \mathbb{N}}$ is bounded and lies in $\operatorname{gra}(A \times B)$.

Suppose that (a, b, a^*, b^*) is a weak cluster point of $(J_A T^n x, J_B R_A T^n x, J_{A^{-1}} T^n x, J_{B^{-1}} R_A T^n x)_{n \in \mathbb{N}}$. Then:

- (iii) $a b = a^* + b^* = 0$.
- (iv) $\langle a, a^* \rangle + \langle b, b^* \rangle = 0.$
- (v) $(a, a^*) \in \operatorname{gra} A$ and $(b, b^*) \in \operatorname{gra} B$.
- (vi) For every $x \in X$, the sequence $(J_A T^n x, J_{A^{-1}} T^n x)_{n \in \mathbb{N}}$ is bounded and its weak cluster points *lie in* S.

Proof. (i): Apply Lemma 2.6(i) with *x* replaced by $T^n x$. The claim of the strong limit follows from combining Fact 3.1(i) and [2, Corollary 2.3] or [8, Theorem 5.14(ii)].

(ii): The boundedness of the sequence follows from the weak convergence of $(T^n x)_{n \in \mathbb{N}}$ (see, e.g.,[8, Theorem 5.14(iii)]) and the nonexpansiveness of the resolvents and reflected resolvents of monotone operators (see, e.g., [8, Corollary 23.10(i) and (ii)]). Now apply Lemma 2.6(ii) with *x* replaced by $T^n x$. (iii): This follows from taking the weak limit along the subsequences in (i). (iv): In view of (iii) we have $\langle a, a^* \rangle + \langle b, b^* \rangle = \langle a, a^* + b^* \rangle =$ $\langle a, 0 \rangle = 0$. (v): Let $((x, y), (u, v)) \in \text{gra}(A \times B)$ and set

$$a_n := J_A T^n x, a_n^* := J_{A^{-1}} T^n x, b_n := J_B R_A T^n x, b_n^* := J_{B^{-1}} R_A T^n x.$$
(48)

Applying Lemma 2.2 with (a, b, a^*, b^*) replaced by (a_n, b_n, a_n^*, b_n^*) yields

$$\langle (a_n, b_n) - (x, y), (a_n^*, b_n^*) - (u, v) \rangle = \langle a_n - b_n, a_n^* \rangle + \langle x, u \rangle - \langle x, a_n^* \rangle - \langle a_n - b_n, u \rangle + \langle b_n, a_n^* + b_n^* \rangle + \langle y, v \rangle - \langle y, b_n^* \rangle - \langle b_n, u + v \rangle.$$
(49)

By (12), $A \times B$ is monotone. In view of (48), (49) and Proposition 6.1(ii), we deduce that

$$\begin{aligned} \langle a_n - b_n, a_n^* \rangle + \langle x, u \rangle - \langle x, a_n^* \rangle - \langle a_n - b_n, u \rangle \\ + \langle b_n, a_n^* + b_n^* \rangle + \langle y, v \rangle - \langle y, b_n^* \rangle - \langle b_n, u + v \rangle \ge 0. \end{aligned}$$
 (50)

Taking the limit in (50) along a subsequence and using (48), Proposition 6.1(i), (iii) and (iv) yield

$$0 \leq \langle x, u \rangle - \langle x, a^* \rangle + \langle y, v \rangle - \langle y, b^* \rangle - \langle b, u + v \rangle$$

= $\langle x, u \rangle - \langle x, a^* \rangle + \langle y, v \rangle - \langle y, b^* \rangle - \langle a, u \rangle - \langle b, v \rangle + \langle a, a^* \rangle + \langle b, b^* \rangle$
= $\langle a - x, a^* - u \rangle + \langle b - y, b^* - v \rangle = \langle (a, b) - (x, y), (a^*, b^*) - (u, v) \rangle.$ (51)

By maximality of $A \times B$ (see (12)) we deduce that $((a, b), (a^*, b^*)) \in \operatorname{gra}(A \times B)$. *B*). Therefore, $(a, a^*) \in \operatorname{gra} A$ and $(b, b^*) \in \operatorname{gra} B$. (vi): The boundedness of the sequence follows from (ii). Now let (a, b, a^*, b^*) be a weak cluster point of $((J_A T^n x, J_B R_A T^n x, J_{A^{-1}} T^n x, J_{B^{-1}} R_A T^n x))_{n \in \mathbb{N}}$. By (v) we know that $(a, a^*) \in \operatorname{gra} A$ and $(b, b^*) = (a, b^*) \in \operatorname{gra} B$, which in view of (iv) implies $a^* \in Aa$ and $-a^* = b^* \in Bb = Ba$, hence $(a, a^*) \in S$, as claimed (see (11)).

Theorem 6.2. Let $x \in X$ and let $(z, k) \in S$. Then the following hold:

(i) For every $n \in \mathbb{N}$,

$$(J_A T^{n+1} x, J_{A^{-1}} T^{n+1} x) - (z, k) \|^2 = \|J_A T^{n+1} x - z\|^2 + \|J_{A^{-1}} T^{n+1} x - k\|^2$$
(52a)

$$\leq \|J_A T^n x - z\|^2 + \|J_{A^{-1}} T^n x - k\|^2$$
(52b)

$$= \|(J_A T^n x, J_{A^{-1}} T^n x) - (z, k)\|^2.$$
 (52c)

(ii) The sequence $(J_A T^n x, J_{A^{-1}} T^n x)_{n \in \mathbb{N}}$ is Fejér monotone with respect to S. (iii) The sequence $(J_A T^n x, J_{A^{-1}} T^n x)_{n \in \mathbb{N}}$ converges weakly to some point in S.

Proof. (i): Apply Corollary 3.4 with *x* replaced by $T^n x$. (ii): This follows directly from (i). (iii): Combine Proposition 6.1(vi), (ii), Fact 3.3(iii) and [8, Theorem 5.5].

Corollary 6.3. (Lions–Mercier–Svaiter). $(J_A T^n x)_{n \in \mathbb{N}}$ converges weakly to some point in Z.

Proof. This follows from Theorem 6.2(iii); see also Lions and Mercier's [27, Theorem 1] and Svaiter's [34, Theorem 1].

Remark 6.4 (brief history). The Douglas–Rachford algorithm has its roots in the 1956 paper [23] as a method for solving a system of linear equations. Lions and Mercier, in their brilliant seminal work [27] from 1979, presented a broad and powerful generalization to its current form. (See [11] and [21] for details on this connection.) They showed that $(T^n x)_{n \in \mathbb{N}}$ converges weakly to a point in Fix T and that the bounded shadow sequence $(J_A T^n x)_{n \in \mathbb{N}}$ has all its weak cluster points in Z provided that A + B was maximally monotone. (Note that resolvents are not weakly continuous in general; see, e.g., [35] or [8, Example 4.12].) Building on [4] and [26], Svaiter provided a beautiful complete answer in 2011 (see [34]) demonstrating that A + B does not have to be maximally monotone and that the shadow sequence $(J_A T^n x)_{n \in \mathbb{N}}$ in fact does converge weakly to a point in Z. (He used Theorem 6.2; however, his proof differs from ours which is more in the style of the original paper by Lions and Mercier [27].) Nonetheless, when $Z = \emptyset$, the complete understanding of $(J_A T^n x)_{n \in \mathbb{N}}$ remains open — to the best of our knowledge, Theorem 5.4 is currently the most powerful result available.

In our final result we show that when $X = \mathbb{R}$, the Fejér monotonicity of the sequence $(J_A T^n x, J_{A^{-1}} T^n x)_{n \in \mathbb{N}}$ with respect to *S* can be decoupled to yield Fejér monotonicity of $(J_A T^n x)_{n \in \mathbb{N}}$ and $(J_{A^{-1}} T^n x)_{n \in \mathbb{N}}$ with respect to *Z* and *K*, respectively.

Lemma 6.5. Suppose that $X = \mathbb{R}$. Let $x \in X$ and let $(z,k) \in Z \times K$. Then the following hold:

- (i) The sequence $(J_A T^n x)_{n \in \mathbb{N}}$ is Fejér monotone with respect to Z.
- (ii) The sequence $(J_{A^{-1}}T^nx)_{n\in\mathbb{N}}$ is Fejér monotone with respect to K.

Proof. Apply Lemma 3.5 with *x* replaced by $T^n x$.

We point out that the conclusion of Lemma 6.5 does not hold when dim $X \ge 2$, see [5, Section 5 & Figure 1].

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