# An Abstract Model for Branching and its Application to Mixed Integer Programming

Pierre Le Bodic<sup>\*</sup> and George Nemhauser Georgia Institute of Technology

August 23, 2016

#### Abstract

The selection of branching variables is a key component of branch-andbound algorithms for solving Mixed-Integer Programming (MIP) problems since the quality of the selection procedure is likely to have a significant effect on the size of the enumeration tree. State-of-the-art procedures base the selection of variables on their "LP gains", which is the dual bound improvement obtained after branching on a variable. There are various ways of selecting variables depending on their LP gains. However, all methods are evaluated empirically. In this paper we present a theoretical model for the selection of branching variables. It is based upon an abstraction of MIPs to a simpler setting in which it is possible to analytically evaluate the dual bound improvement of choosing a given variable. We then discuss how the analytical results can be used to choose branching variables for MIPs, and we give experimental results that demonstrate the effectiveness of the method on MIPLIB 2010 "tree" instances where we achieve a 5% geometric average time and node improvement over the default rule of SCIP, a state-of-the-art MIP solver.

## 1 Introduction

Branch & Bound (B&B) [11] is currently the most successful and widely used algorithm to solve general Mixed Integer Programs (MIPs). B&B searches the solution space by recursively splitting it, which is traditionally represented by a tree, where the root node is associated with the entire solution space and where sibling nodes represent a partition of the solution space of their parent node. At each node, the subspace is encoded by a MIP, and its Linear Program (LP) relaxation is solved to provide a dual bound. If the LP is infeasible, or if the LP bound is no better than the primal bound, the node is pruned (the primal bound is the value of the best feasible solution found so far). Otherwise, the subspace at that node is partitioned and the two corresponding children nodes

<sup>\*</sup>Now at the Faculty of Information Technology of Monash University.

are recursively explored. The part of the B&B algorithm that decides how to partition the solution space is referred to as the *branching rule*. Typically, the branching rule selects one variable among the *candidate variables*, i.e. those that have a fractional value in the LP solution of the current node, but are required to be integer in the original MIP. Formally, suppose that at the current node the value of an integer variable x in the node's LP solution is  $x_{lp} \notin \mathbb{Z}$ . Branching on x would result in two children, each encoding a solution subspace, one in which x is upper-bounded by  $\lfloor x_{lp} \rfloor$ , the other in which x is lower-bounded by  $\lceil x_{lp} \rceil$ . The PhD thesis [1] provides a practical point of view of the latest advances in MIP solving and branching in particular. For a general overview of MIP solving and B&B, see [13], [5] and [6].

This research is motivated by the desire to understand the fundamentals of state-of-the-art MIP branching rules (many of which are justified experimentally). The main contribution of this paper is the introduction of theoretical decision problems to study the branching component of B&B. Based on an analysis of these models, we introduce new *scoring functions*, and prove their efficiency on both simulated experiments and MIP instances. The paper is organized as follows. After defining the abstract B&B model in Section 2, we study the simplest of our problems in Section 3. More complex problems are introduced and analyzed in Sections 4 and 5, respectively. Section 6 introduces two new scoring functions based on the theory developed in previous sections. Experimental results are presented in Section 7 and conclusions are given in Section 8.

# 2 Abstract Branch & Bound model

In this section we model an abstract version of B&B through a series of definitions. We start by defining what we consider to be a *variable*.

**Definition 1** . A variable x is encoded by a pair of non-negative (left and right) integer gains  $(l_x, r_x)$  with  $1 \le l_x \le r_x$ .

The  $l_x$  and  $r_x$  gains model the dual bound changes that occur when branching on the variable x. We suppose that these two gains are *known* and *fixed*. (Note that when solving a MIP using B&B, these gains are not fixed for a variable.)

We now define a Branch & Bound tree.

**Definition 2**. Given a set of variables, a Branch & Bound tree is a binary tree (in the graph sense) with one variable x assigned to each inner node i.

We then say that variable x is branched on at node i. Note that, in Definition 2, the root node is considered as an inner node. Also note that, in our abstract model, the gains  $(l_x, r_x)$  of a variable x do not change with the depth of the node where the variable is branched on, nor do they depend on the dual bound at this node.

These two definitions naturally lead to the definition of *dual gap closed* at a node i in a B&B tree. Note that throughout the article, the gap will always refer to the *absolute* gap (as opposed to the relative gap), i.e. the absolute difference between the primal bound and the dual bound.

**Definition 3**. The dual gap  $g_i$  closed at a node i, referred to as  $g_i$ , is given by the recursive formula

$$g_i = \begin{cases} 0 & \text{if } i \text{ is the root node} \\ g_h + l_{x(h)} & \text{if } i \text{ is the left child of node } h \\ g_h + r_{x(h)} & \text{if } i \text{ is the right child of node } h, \end{cases}$$

where x(h) is the variable branched on at node h.

For instance, if at a node i, the dual bound is  $g_i = 10$ , and variable (2,5) is branched on, the dual bound closed at the left and right children of i are 12 and 15, respectively.

The gap *closed* by a B&B tree is defined as follows:

**Definition 4**. A tree closes a gap G if for all leaves  $i, g_i \ge G$  holds.

Throughout, the *size* of a tree refers to the number of nodes of the tree. Using these definitions, the following sections define problems that model the B&B algorithm in increasing complexity.

# 3 The Single Variable Branching problem

The first problem we define using the abstract model set up in Section 2 is the SINGLE VARIABLE BRANCHING (SVB) problem, which is a tree-size measurement problem.

Problem SINGLE VARIABLE BRANCHING

**Input:** one variable encoded by  $(l,r) \in \mathbb{Z}_{>0}^2$ ,  $G \in \mathbb{Z}_{>0}$ ,  $k \in \mathbb{Z}_{>0}$ , such that  $l \leq r \leq G$ .

**Question:** Is the size of the Branch & Bound tree that closes the gap G, repeatedly using variable (l, r), at most k?

In the example given in Figure 1, a B&B tree that closes a gap of G = 6 is built using only variables with gains (2,5). Observe that any tree closing a gap G contains the *unique* tree that closes G in a minimum number of nodes. As we will see, the analysis of SVB will prove more complex and insightful than it may seem at first glance.

### 3.1 Motivation: state-of-the-art scoring functions and example

A state-of-the-art branching rule for MIP solving (as implemented in SCIP [2]) is a hybrid of mainly two methods: strong branching and pseudo-cost branching. Given a candidate variable x, strong branching computes the LP gains of

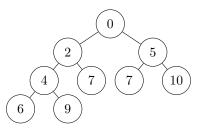


Figure 1: B&B tree with 9 nodes that closes a gap of 6 with variable gains (2, 5). The gap closed at each node is indicated by the value at the center of each node.

the would-be children of the current node if x is branched on. The candidate variable which provide the best LP bounds is then branched on. Pseudo-cost branching complements strong branching; it keeps track of the LP gains of branching variables for which the children have already been processed, and, given sufficient historical data, estimates the gains that would be computed by strong branching. Strong branching is computationally expensive, as it requires solving many LPs, while pseudo-cost branching only requires a few arithmetic operations at each node. However, pseudo-cost branching is an estimate, and it requires initialization. The state-of-the-art branching rule essentially consists of using strong branching early in the tree, i.e. at the root node and a few subsequent levels, while pseudo-cost branching is called at lower levels. In practice, there are many refinements to these methods. We direct the reader to Tobias Achterberg's PhD thesis [1] and references therein for an in-depth review of branching rules.

At each node, strong branching (or pseudo-cost branching) provides the (estimated) LP gains  $(l_x, r_x)$  resulting from branching up and down for each fractional variable x. Each candidate variable x is then scored according to its gains  $(l_x, r_x)$ , and the he highest scoring variable is selected for branching. The state-of-the-art scoring function used for this purpose is:

 $\max(\epsilon, l_x) \cdot \max(\epsilon, r_x),$ 

where  $\epsilon > 0$  is chosen close to 0 (e.g.  $\epsilon = 10^{-6}$  in [1]) to break the ties if all variables satisfy  $\min(l_x, r_x) = 0$ , but plays no role otherwise. We simply refer to this scoring function as the *product* function. Prior to the introduction of the **product** function, the *linear* (or convex) function was the standard:

$$(1-\mu)\cdot l_x+\mu\cdot r_x,$$

where  $\mu$  is a parameter in [0,1] ( $\mu = \frac{1}{6}$  in [1]). In [1], Achterberg reports that switching from the linear function to the product function yields an improvement of 14% in computing time, and 34% in number of nodes. The rationale behind the linear function with  $\mu = \frac{1}{6}$  is that it should be preferable to improve the dual bound of both children by a little rather than only one of

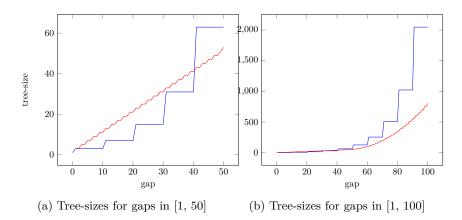


Figure 2: Plot of the sizes of the trees built with variables (10, 10) (in blue) and (2, 49) (in red).

them. The **product** function prefers equal left and right gains, i.e. it is more likely to produce a balanced tree. However, there are types of instances for which the **linear** function performs better, so the **product** function does not systematically outperform the **linear** one.

To the best of our knowledge, both the linear and the product function have been established experimentally, and no theoretical evidence supports the use of one over the other, or over any other possible function. We now give a simple example suggesting that more complex functions are required to score variables. If the product or the linear functions are good scoring functions for solving MIPs, then they should also perform well on simple models such as SVB. Consider variables (10, 10) and (2, 50): both the product and the linear function (with default linear coefficient  $\mu = \frac{1}{6}$ ) assign these variables an equal score. Observe that variable (2, 49) receives a smaller score than (2, 50) (and therefore than variable (10, 10)). We now consider the tree-sizes of two SVB instances, one that has variable (2, 49) as input, the other having variable (10, 10).

Figure 2 gives two SVB tree-size plots. For gaps no larger than 40, variable (10, 10) requires fewer nodes than (2, 49) to close the gap, but larger gaps are closed faster with (2, 49). For a gap of 1000, using only (10, 10) produces a tree 323 *million* times larger than using only (2, 49), yet both scoring functions assign a higher score to variable (10, 10). It is clear from this example that the state-of-the-art scoring functions are imperfect.

#### 3.2 A polynomial-time algorithm

There exists a simple recurrence relation that solves SVB. For a given gap G, t(G) is defined as the size of the SVB tree that closes G:

$$t(G) = \begin{cases} 1 & \text{if } G \le 0\\ 1 + t(G - l) + t(G - r) & \text{otherwise.} \end{cases}$$
(1)

Hence, for a given k, the answer to SVB is YES if and only if  $t(G) \leq k$ . Unfortunately, computing t(G) requires O(G) operations. Therefore the running time of this algorithm is pseudo-polynomial (and thus exponential) in the encoding length of the input. Moreover, observe that we do not necessarily have  $G = O(\log(k))$  for YES instances of SVB: if the given variables are all (1, G), then k = 2G + 1 is the tree-size. However, there exists a closed-form formula that can be evaluated in polynomial time:

**Theorem 1** . SVB is in P. Furthermore, a formula for the size of the  $B \mathscr{C} B$  tree is

$$t(G) = 1 + 2 \cdot \sum_{k_r=1}^{\left\lceil \frac{G}{r} \right\rceil} {\binom{k_r + \left\lceil \frac{G - (k_r - 1) \cdot r}{l} \right\rceil - 1}{k_r}}.$$
 (2)

Proof. The size of the input is  $O(\log(G) + \log(k))$ . Let T be the minimal tree that closes the gap G, and let  $d = \lceil \frac{G}{r} \rceil$  be the minimum depth of its leaf nodes. If  $k < 2^{d+1} - 1$ , the answer to the decision problem SVB is necessarily NO, therefore we can assume the opposite throughout the rest of proof. It follows that  $d = O(\log(k))$ . Starting from the root of T, we turn right at most d times before finding a leaf. For a given path from the root to a leaf, let  $k_l$  (resp.  $k_r$ ) denote the number of times we turn left (resp. right). Observe that a pair  $(k_l, k_r)$  does not uniquely identify a leaf. Furthermore, since the number of leaves is in O(k), it is impossible to iterate over all of them in a time polynomial in the size of the input. However, since all leaves satisfy  $k_r \leq d$ , it is possible to iterate over all the values of  $k_r$ . From now on we suppose that  $k_r \leq d$  is fixed, and restrict our attention to those leaves that are reached by turning right exactly  $k_r$  times. The inequality

$$k_l \cdot l + k_r \cdot r < G + r$$

must hold for each leaf, since the gap closed at each leaf cannot exceed G by r or more, as otherwise the gap would already be closed at the leaf's parent. We therefore have the bound  $k_l \leq \left\lceil \frac{G-r(k_r-1)}{l} \right\rceil - 1$ , and the depth of a leaf is bounded by  $k_r + \left\lceil \frac{G-r(k_r-1)}{l} \right\rceil - 1$ . Furthermore, observe that any  $k_r$  elements chosen in the set  $\{1, \ldots, k_r + \left\lceil \frac{G-r(k_r-1)}{l} \right\rceil - 1\}$  uniquely determine a path to a leaf, and that each leaf can be encoded using this scheme. This bijection thus ensures that the number of such leaves is  $\binom{k_r + \left\lceil \frac{G-r(k_r-1)}{l} \right\rceil - 1}{k_r}$ . The total number of leaves is therefore

$$\sum_{k_r=0}^{\left\lceil \frac{G}{r} \right\rceil} \binom{k_r + \left\lceil \frac{G - (k_r - 1) \cdot r}{l} \right\rceil - 1}{k_r}.$$
(3)

Formula (3) can be computed in  $O(d^2) = O(\log^2(k))$  operations, which is polynomial in the size of the input. Observe how the formula iterates over the possible  $k_r$ 's but avoids iterating over the range of  $k_l$ 's, as this may require O(k) steps. Since each inner node has exactly two children, the theorem is proven.

Evaluating formula (2) becomes impractical as the gap G becomes large. We present below an asymptotic study of the tree-size that can be used in practice when formula (2) is too expensive to compute.

#### 3.3 Asymptotic study

Analyzing algorithms through a related recurrence relation similar to (1) (and its mathematical properties, such as the asymptotic behavior) is a standard technique (see [7, Section 4.5]). In this section and throughout the article we however systematically provide self-sufficient explanations and analyzes, as our readership may not be familiar with algorithm analysis techniques.

As the gap to close G becomes large, we prove that the tree-size grows essentially linearly, i.e. there exists a fixed ratio determining the growth of the tree. Perhaps the most popular example of this phenomenon is the Fibonacci sequence, for which the golden ratio  $\frac{1+\sqrt{5}}{2}$  corresponds to the asymptotic growth rate between two consecutive numbers. In fact, the Fibonacci sequence is given by the recursion formula (1) with variable (1,2) up to the additive constant 1, which does not intervene in asymptotic results. This section therefore provides convergence results on sequences generalizing the Fibonacci sequence. Unfortunately, even though using the same notion for the ratio as in the Fibonacci sequence would be the most intuitive, the limit of  $\frac{t(G+1)}{t(G)}$  when G tends to infinity may not exist in our case. Indeed, consider variable (2, 2), and the limit of these two subsequences:

$$\lim_{\substack{G \to \infty \\ G \text{ odd}}} \frac{t(G+1)}{t(G)} = 1 \qquad \qquad \lim_{\substack{G \to \infty \\ G \text{ even}}} \frac{t(G+1)}{t(G)} = 2.$$

In this example, the ratio  $\frac{t(G+1)}{t(G)}$  does not converge as G tends to infinity. However, there exists another definition of the ratio, that can be shown to converge. We use the following definition:

**Definition 5**. The ratio of a variable is

$$\varphi = \lim_{G \to \infty} \sqrt[l]{\frac{t(G+l)}{t(G)}}.$$

With this definition, variable (2, 2) has ratio  $\varphi = \sqrt{2}$ . This means that, asymptotically, the number of nodes doubles for every two additional units of gap to close.

Since the ratio  $\varphi$  indicates the growth rate of a tree in the SVB setting, a variable with a small ratio requires a smaller SVB tree than one with a bigger ratio, given a large enough gap G. The ratios of the two variables taken as example in Section 3.1, namely (10, 10) and (2, 49), are approximately 1.071 and 1.049, respectively. This explains why variable (2, 49) produces smaller trees than (10, 10) for large enough gaps. Throughout this section we show that the limit defining the ratio  $\varphi$  in Definition 5 always exists and how it can be computed. We first prove Proposition 2, which gives an equivalent way of defining  $\varphi$ .

**Proposition 2**. If the limit  $\varphi$  exists, then we also have

$$\varphi = \lim_{G \to \infty} \sqrt[r]{\frac{t(G+r)}{t(G)}}.$$

Proof.

$$\begin{split} \varphi^{lr} &= \left(\lim_{G \to \infty} \frac{t(G+l)}{t(G)}\right)^r \\ &= \lim_{G \to \infty} \frac{t(G+l)}{t(G)} \cdot \frac{t(G+l)}{t(G)} \cdot \dots \cdot \frac{t(G+l)}{t(G)} \\ &= \lim_{G \to \infty} \frac{t(G+lr)}{t(G+l(r-1))} \cdot \frac{t(G+l(r-1))}{t(G+l(r-2))} \cdot \dots \cdot \frac{t(G+l)}{t(G)} \\ &= \lim_{G \to \infty} \frac{t(G+lr)}{t(G)} \\ &= \lim_{G \to \infty} \frac{t(G+lr)}{t(G+(l-1)r))} \cdot \frac{t(G+(l-1)r)}{t(G+(l-2)r)} \cdot \dots \cdot \frac{t(G+r)}{t(G)} \\ &= \left(\lim_{G \to \infty} \frac{t(G+r)}{t(G)}\right)^l. \end{split}$$

**Proposition 3**. If the limit  $\varphi$  exists, then  $\sqrt[r]{2} \leq \varphi \leq \sqrt[l]{2}$ .

*Proof.* Let T (resp.  $T^l$ ,  $T^r$ ) be the tree that closes the gap G (resp. G + l, G + r), and observe that the tree T is a sub-tree of  $T^l$  and  $T^r$  (all three have the same root). First, each leaf of T is an inner node of  $T^r$ . This means that the number of nodes at least asymptotically doubles, thus 2 is a lower bound on  $\varphi^r$ . Second, each leaf of T is either a leaf or the parent of two leaves in  $T^l$ , hence 2 is an upper bound on  $\varphi^l$ .

The limit  $\varphi$  clearly exists in the case l = r, and the bounds in the above property provide the exact value. We now prove that the limit  $\varphi$  exists in general: **Theorem 4**. When G tends to infinity, both sequences  $\sqrt[l]{\frac{t(G+l)}{t(G)}}$  and  $\sqrt[r]{\frac{t(G+r)}{t(G)}}$  converge to  $\varphi$ , which is the unique root greater than 1 of the equation  $p(x) = x^r - x^{r-l} - 1 = 0$ .

Proof. See Appendix A.1 and Theorem 6.

**Corollary 5** . A numerical approximation of  $\varphi^r$  is given by the fixed-point iteration

$$f(x) = 1 + \frac{1}{x^{\frac{l}{r}} - 1}$$

with starting point x = 2.

Proof. See Appendix A.1.

Having established Theorem 4, the definitions of  $\varphi$  and p should now make more sense, as they can be shown to originate more directly from the definition of the sequence:

$$\begin{split} t(G) &= 1 + t(G-l) + t(G-r) \\ \Rightarrow &\frac{t(G)}{t(G-r)} = \frac{1}{t(G-r)} + \frac{t(G-l)}{t(G-r)} + 1 \\ \Rightarrow &\lim_{G \to \infty} \frac{t(G)}{t(G-r)} = \lim_{G \to \infty} \left( \frac{1}{t(G-r)} + \frac{t(G-l)}{t(G)} \frac{t(G)}{t(G-r)} + 1 \right) \\ \Rightarrow &\varphi^r = \varphi^{r-l} + 1. \end{split}$$

The polynomial  $p: x \to x^r - x^{r-l} - 1$  is the so-called *characteristic polynomial* of the recurrence sequence defining t. The Abel–Ruffini theorem [8, p. 264] states that there is no general closed-form formula for roots of polynomials with degrees five or higher. Since we are dealing with a particular trinomial, a formula still might exist, however we have not been able to determine one. Throughout the paper we therefore resort to numerical methods to determine  $\varphi$  (see Section 6.2). The characteristic polynomial for trees with arbitrary degrees has been discussed at length in a SAT setting in [10], where the ratio  $\varphi$  is referred to as the  $\tau$ -value.

Examples of variables and their respective ratios are given in Table 1. In the case where the sum of the variable gains is fixed (Table 1a), choosing l close to r minimizes the ratio of a variable (l, r). However, if the product is fixed (Table 1b), then the opposite is preferable. Figure 3 gives a plot of the ratio as a function of the left and right gains.

Once the value of  $\varphi$  is determined, it is then easy to circumvent the  $O(\log^2(k))$  complexity of the closed-form Formula (2), by evaluating t(F) for some value F much smaller than G, and approximating t(G) using the formula

$$t(G) \approx \varphi^{G-F} t(F).$$

We now establish some useful properties of p.

Variable	(6, 10)	(5, 11)	(4, 12)	(3, 13)	(2, 14)	(1, 15)				
$\varphi$	1.0926	1.0955	1.1002	1.107	1.1204	1.1468				
(a) Variables that satisfy $l + r = 16$										
Variable	(6, 10)	(5, 12)	(4, 15)	(3, 20)	(2, 30)	(1, 60)				
$\frac{\text{Variable}}{\varphi}$	,	(5, 12) 1.0907	· · · /		,	$\frac{(1,60)}{1.0515}$				

Table 1: Ratios for some variables, truncated to 5 significant digits.

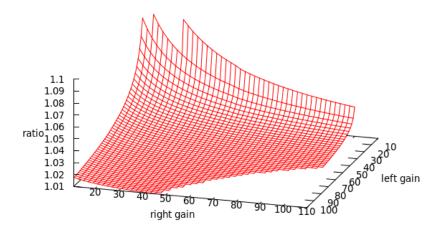


Figure 3: Variable ratio for left and right gains in [10, 100].

**Theorem 6**. The characteristic polynomial  $p: x \to x^r - x^{r-l} - 1$  satisfies the following properties.

- p is monotonically increasing in  $[1,\infty)$
- in  $[1,\infty)$ , p has a single real root  $\varphi > 1$

*Proof.* Consider the sign of  $p'(x) = rx^{r-1} - (r-l)x^{r-l-1}$  when  $x \in R^+$ .

$$p'(x) > 0 \Leftrightarrow x^{r-l-1}(rx^l - r + l) > 0$$
$$\Leftrightarrow rx^l - r + l > 0$$
$$\Leftrightarrow x > \sqrt[l]{1 - \frac{l}{r}}$$
$$\Leftrightarrow x \ge 1.$$

We thus know that p is increasing over  $[1, \infty)$ . Furthermore, p(1) = -1 and  $\lim_{x\to\infty} p(x) = \infty$ , and p is continuous, therefore there exists a single root  $\varphi$  in  $(1, \infty)$ .

This analysis is corroborated by the fact that the tree-size increases when incrementing the gap by l or r, thus we should have  $\varphi > 1$ .

# 4 The Multiple Variable Branching problem

We consider the MULTIPLE VARIABLE BRANCHING problem (MVB), which naturally extends the SVB problem, defined as follows:

Problem Multiple Variable Branching

**Input:** n variables encoded by  $(l_i, r_i), i \in \{1, ..., n\}$ , an integer G > 0, an integer k > 0.

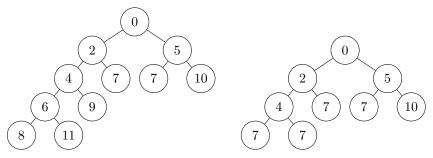
**Question:** Is there a Branch & Bound tree with at most k nodes that closes the gap G, using each variable as many times as needed?

Figure 4 illustrates the difference between MVB and SVB, given two variables (2, 5) and (3, 3), and G = 7. In the MVB tree, variable (2, 5) is branched on at all (inner) nodes except the bottom left one, where variable (3, 3) is branched on.

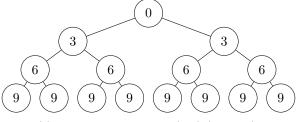
There is a recursive equation for MVB, similar to (1) for SVB. For a given gap G, t(G) is defined as the minimum size of a MVB tree that closes G:

$$t(G) = \begin{cases} 1 & \text{if } G \le 0\\ 1 + \min_{1 \le i \le n} \left( t(G - l_i) + t(G - r_i) \right) & \text{otherwise.} \end{cases}$$
(4)

The time complexity of this algorithm is O(nG), which is pseudo-polynomial in the size of the input. Unlike SVB, we do not know whether MVB can be solved in polynomial time. For a given G, we say that variable i is branched on at the corresponding node if  $i \in \arg\min_{1 \le i \le n} (t(G - l_i) + t(G - r_i))$ .



(a) SVB tree for variable (2,5) (size=11). (b) Minimum-size MVB tree (size=9).



(c) SVB tree for variable (3,3) (size=15).

Figure 4: For a gap G = 7, SVB trees using variable (2,5) or (3,3), and minimum-size MVB tree using both.

#### 4.1 Motivating example (continued)

Following the example introduced in Section 3.1 for the SVB problem, consider now the plot of the MVB tree-sizes in Figure 5 (the SVB plots are the same as in Figure 2).

As anticipated, for a given gap, the MVB tree-size is smaller than each of the SVB tree-sizes. However, the difference in tree-sizes with (2, 49) does not seem to increase drastically. Indeed, for a gap of 1000, the SVB tree-size of (2, 49) is only 1.798 times larger than the MVB tree-size (and this ratio is approximately constant for larger gaps). In fact, for this instance, we can verify experimentally that variable (2, 49) is branched on at every node for which the gap to close is at least 31. This phenomenon is the subject of the next section.

#### 4.2 Asymptotic study

Let  $\varphi$  (resp.  $\varphi_i$ ) now denote the ratio associated with the recursive equation (4) of MVB (resp., for SVB, equation (1) for variable *i*). Furthermore, for a variable *i*, let  $p_i$  be its characteristic polynomial, for i = 1, ..., n. Formally, we define the ratio for a MVB problem to be

$$\varphi = \lim_{G \to \infty} \sqrt[z]{\frac{t(G+z)}{t(G)}}$$

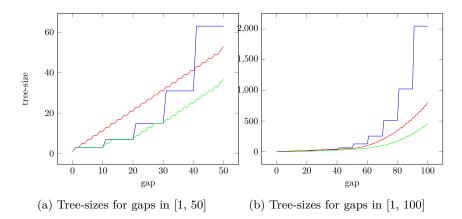


Figure 5: Plot of the sizes of the trees built with variables (10, 10) (in blue) and (2, 49) (in red), and both (in green).

where z is the least common multiple of all  $l_i$  and  $r_i$ .

**Theorem 7**.  $\varphi = \min_i \varphi_i$ 

Proof. See Appendix A.2.

Essentially, this means that for a large G, the tree-size grows at the growth rate of the *best* variable for SVB. Therefore we also have

$$t(G) \approx \varphi^{G-F} t(F)$$

for large gaps F and G, with  $G \ge F$ . Furthermore, we propose the following conjecture.

**Conjecture 1**. For each instance of MVB, there exist a gap H such that for all gaps greater than H, variable  $i = \arg \min_j \varphi_j$  is always branched on at the root node.

Proving this conjecture (and proving that H is polynomially bounded by the size of the input) may help design a polynomial-time algorithm for MVB. Figure 6 provides the value of H for instances of MVB with two variables, (10, 10) and (2, r), where r is the value on the horizontal axis. The value of H is maximized for r = 29, which also minimizes  $|\varphi_{(10,10)} - \varphi_{(2,r)}|$ .

## 5 The General Variable Branching problem

The GENERAL VARIABLE BRANCHING problem (GVB) is defined as follows:

**Problem** GENERAL VARIABLE BRANCHING Input: n variables encoded by  $(l_i, r_i), i = 1, ..., n$ , an integer G > 0, an integer k > 0, and a vector of multiplicities  $m \in \mathbb{Z}_{>0}^n$ .

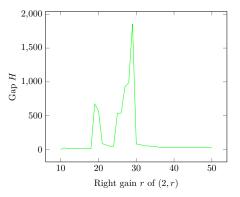


Figure 6: Value of the gap H for the MVB problem with variable (10, 10) and (2, r), for varying values of r.

**Question:** Is there a Branch & Bound tree with at most k nodes that closes the gap G, branching on each variable i at most  $m_i$  times on each path from the root to a leaf?

Observe that GVB comprises the input of problem MVB, and additionally stipulates that each variable may not be used more than a given number of times on each path from the root to a leaf. Consequently, GVB corresponds to MVB whenever the multiplicity of each input variable is large enough, therefore GVB generalizes MVB.

When solving a MIP instance using a B&B search, the minimization version of GVB models the problem of choosing a variable to branch on under the following simplifying hypotheses:

- The up and down gains are known for each variable, and are invariant.
- If the problem is feasible, the optimal value is known.

The first hypothesis is not satisfied in practice. However, as the B&B tree grows, pseudo-costs become better approximations for the up and down gains. Likewise, the second hypothesis becomes true at some point during the B&B search. So, while both hypotheses are not initially satisfied, they become more accurate as the tree grows. As we will see, these hypotheses are not sufficient to render the GVB problem theoretically tractable. A recursion for GVB similar to (4) for MVB is

$$t(G,m) = \begin{cases} 1 & \text{if } G \le 0\\ 1 + \min_{1 \le i \le n, m_i > 0} (t(G - l_i, m - v_i) + t(G - r_i, m - v_i)) & \text{otherwise} \end{cases}$$
(5)

where m and  $v_i$ , are both vectors of size n; m is the vector of multiplicities, and  $v_i$  is the indicator vector of the set  $\{i\}$  (i.e. the  $i^{\text{th}}$  element of  $v_i$  is 1, the n-1

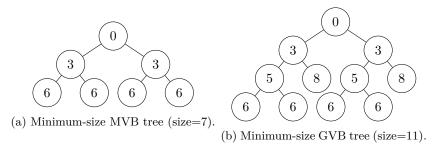


Figure 7: Minimum-size trees for MVB and GVB with input G = 6 and variables (1, 1), (2, 5), (3, 3) (in GVB, with multiplicities 1).

others are 0). This algorithm keeps track of the variables already branched on, and, in the case where all multiplicities are 1, its running time is  $O(2^n G)$ .

Figure 7 gives the minimum-size B&B tree for a GVB instance and an MVB instance that has the same variables and gap as input.

#### 5.1 Complexity of GVB

For the complexity result presented in this section, it is sufficient to consider the case where all multiplicities are one. We prove the #P-hardness of GVB by reducing it to a variant of counting knapsack solutions:

#### **Problem** #KNAPSACK

**Input:** N items with weights  $w_1, \ldots, w_N$  and W the capacity of the knapsack, a given K, all integer positive.

**Question:** Are there at least K distinct feasible solutions x to the (covering) knapsack constraint  $\sum_{i=1}^{N} w_i x_i \geq W$ , with  $x_i \in \{0, 1\}, i = 1, ..., N$ ?

Note that in our definition of the #KNAPSACK problem, we ask for *at least* K solutions. This variant of the counting problem can solve the exact counting problem by e.g. dichotomy in O(N) calls, hence it is #P-hard.

#### **Theorem 8**. GVB is (weakly) #P-hard

*Proof.* We build an instance of the GVB problem that embeds an instance of the #KNAPSACK problem. Set n = N + 1, and for each  $i \in \{1, \ldots, n-1\}$ , create a variable with gains  $(C, C + w_i)$ , where  $C = \sum_{j \in \{1, \ldots, N\}} w_j$ ; furthermore, create the  $n^{th}$  variable (C, C). Set the target gap G = (n - 1)C + W and the number of allowed nodes  $k = 2^{n+1} - 1 - 2K$ . Finally, set all multiplicities to one. Suppose, without loss of generality, that the knapsack instance is feasible, and thus  $C \geq W$ . We prove that the answer to this GVB instance is YES if and only if the answer to the instance of #KNAPSACK is YES.

First, observe that a minimum-size tree is obtained by branching on the variable with the biggest right gains, i.e. corresponding to the biggest items (w.r.t.  $w_i$ ), and finally by branching on the dummy variable with gains (C, C).

Consider a tree T that closes the gap G. Observe that each leaf of T has at least depth n-1, and at most n. In fact, there is a one-to-one mapping between (not necessarily feasible) solutions of the knapsack instance and nodes at depth n-1. Furthermore, there is a one-to-one mapping between *feasible* solutions of the knapsack instance and *leaves* at depth n-1. Therefore, since an infeasible knapsack instance would yield a perfect tree, with  $2^{n+1}-1$  nodes, each feasible solution decreases the number of nodes by 2 exactly. If there are at least K feasible solutions to #KNAPSACK, there is a tree with at most  $2^{n+1} - 1 - 2K$  nodes for their corresponding GVB instance, and vice-versa.

In this proof we use the fact that the domination relation, formally defined in Section 6, is a total order on the set of variables, and thus which variables to branch on to obtain a minimum-size tree is trivial. As a result, we actually prove Theorem 8 for a special case of the GVB problem, where at a given level, a single variable is branched on at every node. Similar to SVB, this special case is therefore a tree measurement problem.

Theorem 8 in particular implies, under the widely believed conjecture that the polynomial hierarchy PH [15] is proper to the second level, that GVB is neither in NP nor in co-NP. This is because  $PH \subseteq P^{\#P}$  [16].

# 6 Using the abstract model for scoring branching candidates

#### 6.1 Applicability of our results to rational numbers

The results we present in Section 3 through 5 suppose integer gains and gap. This does not hold in general when solving MIP instances, where numbers are encoded as rationals. It is however sufficient to notice that in SVB, MVB or GVB, if some data is rational, there exists a scaling factor  $q \in Q_+$ , such that, if the gap and all gains are multiplied by q, all data become integer. Indeed, in the abstract setting described in Section 2, which holds for SVB, MVB or GVB, suppose a B&B tree closes a gap G, and all gains are multiplied by q, then the same tree closes a gap  $q \cdot G$ , since the gap closed at each node is the sum of all gains along the path from the root to the leaf.

Furthermore, given a variable with rational gains  $(l, r) \in \mathbb{Q}^2_+$ , and a scaling factor  $q \in \mathbb{Q}_+$ , such that the scaled variable has gains  $(q \cdot l, q \cdot r) \in Z^2_+$ , then the ratio  $\varphi$  of the scaled variable can be computed as described in Section 3, and we have

$$\varphi^{qr} - \varphi^{q(r-l)} - 1 = 0 \Leftrightarrow (\varphi^q)^r - (\varphi^q)^{(r-l)} - 1 = 0,$$

therefore the ratio of variable (l, r) is given by  $\varphi^q$ .

Finally, the proof of Theorem 1 (for the closed-form formula) does not require data to be integer, or even rational, and Corollary 5 (for the fixed-point equation) inherently works on rational numbers. We implement both formulas in our code without any scaling.

#### 6.2 The ratio scoring function

Recall from Section 3.1 that a scoring function combines the left and right LP gains to score candidate variables. The variable with the highest score is then branched on. The linear and product functions are two scoring functions that were presented in Section 3.1. In this section, we introduce the *ratio* scoring function, based on Definition 5 for the SVB problem. Given two variables i and j and their respective ratios  $\varphi_i$  and  $\varphi_j$ , it selects the variable with the smallest ratio.

To help analyze the behavior of the **ratio** scoring function, we formalize the concept of domination for two variables.

**Definition 6**. Given two variables  $(l_1, r_1)$  and  $(l_2, r_2)$ , we say that  $(l_1, r_1)$  dominates  $(l_2, r_2)$  if and only if  $l_1 \ge l_2$  and  $r_1 \ge r_2$ , with at least one of the inequalities strict.

**Proposition 9**. Suppose variable  $(l_1, r_1)$  dominates variable  $(l_2, r_2)$ . Then,

- 1. Both the product and the linear function assign a better score to  $(l_1, r_1)$  than to  $(l_2, r_2)$ .
- 2. In the MVB or GVB problem setting, if both  $(l_1, r_1)$  and  $(l_2, r_2)$  are branched on on a common path, branching on  $(l_1, r_1)$  before branching on  $(l_2, r_2)$  yields a tree of size no larger than the converse.

*Proof.* Part 1 is obvious. Part 2 can be proven by observing that if in a B&B tree, variable  $(l_2, r_2)$  is branched on before  $(l_1, r_1)$ , then the tree where the variables are switched closes a gap at least as large.

In other words, the linear and the product scoring functions prefer nondominated variables, as dominated variables necessarily yield larger trees. We prove that the ratio scoring function also exhibits this property:

**Theorem 10**. Suppose variable  $(l_1, r_1)$  dominates variable  $(l_2, r_2)$ , then  $\varphi_1 < \varphi_2$ .

*Proof.* Starting with the definition of the characteristic polynomial of variable j, we establish:

$$0 = p_2(\varphi_2) = \varphi_2^{r_2} - \varphi_2^{r_2 - l_2} - 1 = \varphi_2^{r_2}(1 - \varphi_2^{-l_2}) - 1$$
  
<  $\varphi_2^{r_1}(1 - \varphi_2^{-l_1}) - 1 = p_1(\varphi_2).$ 

Therefore, by Theorem 6,  $\varphi_2 > \varphi_1$ .

As a consequence, the ratio of a dominated variable does not have to be computed. Moreover, to perform comparisons between ratios, it is not necessary to compute both of them.

**Proposition 11**. Suppose that the ratio  $\varphi_1$  of variable  $(l_1, r_1)$  has been computed, then, given a variable  $(l_2, r_2)$ ,  $\varphi_2 < \varphi_1$  if and only if  $p_2(\varphi_1) > 0$ .

*Proof.* Direct using Theorem 6.

Hence a strategy to reduce the number of ratio computations is to compute  $\varphi$  for a variable that is believed to be a good candidate for branching (e.g. the best according to the **product** function), and use Proposition 11 to test all other candidates, computing the ratio of a variable only if it is proven to be smaller than the current best one.

The scoring rule that we implement uses the strategy described in Algorithm 1. The computation of the ratio at Steps 2 and 5 is done numerically. We have implemented and tested five different methods, namely the fixed-point iteration method described in Corollary 5, a simple bisection method, Newton and Laguerre's method [14, Chapter 9]), and a direct method computing (5) using the closed-form formula described in Theorem 1 for a large enough gap. Iterative methods are initialized with starting point  $\sqrt[7]{2}$ , which is a proven lower bound (see Proposition 3). Experiments have shown that if  $\frac{r}{l} \leq 100$ , Laguerre's method was the most effective, otherwise the fixed-point iteration method should be used. Following this rule, computing the ratio of a variable takes around 20 milliseconds of CPU time on a modern computer.

Algorithm 1 Implementation of the ratio scoring function.

1: Filter out the branching candidates with dominated gains.  $\triangleright$  Theorem 10

2: Compute the ratio  $\varphi^*$  of the best variable according to the **product** function.

3: for all remaining branching candidates i do 4: if  $p_i(\varphi^*) < 0$  then  $\triangleright$  Proposition 11 5: Compute the ratio  $\varphi_i$  and set  $\varphi^* = \varphi_i$ . 6: end if 7: end for 8: return the variable with ratio  $\varphi^*$ .

Observe that if the ratio and one of the left (resp. right) gain are fixed, then the right (resp. left) gain can be computed analytically:

**Proposition 12**. Given a variable (l, r) and its ratio  $\varphi$ , then

$$= -\frac{\log(1-\varphi^{-l})}{\log\varphi} \qquad \qquad l = -\frac{\log(1-\varphi^{-r})}{\log\varphi}$$

*Proof.* Direct using the definition of the polynomial p.

r

Figure 8 has been generated using this property. The leftmost point (corresponding to variable (100, 100)) serves as a reference, and each curve is such that its points have the same score according to one scoring function: the red curve corresponds to the **linear** function with default  $\mu$ , blue to the **product**, and green to the **ratio**. Unsurprisingly, the red curve is a line and the blue curve is a parabola. Note how the **product** function and the **ratio** function match very closely when l and r are close to each other. This essentially means

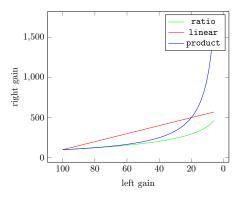


Figure 8: Right gains depending on the left gain such that the score is constant.

that if all variables have roughly equal left and right gains, the product and ratio functions behave similarly. If this is not the case, then the ratio function prefers variables with unequal left and right gains compared to the product function.

#### 6.3 The svts scoring function

Ranking the variables by ratio only makes sense for large enough gaps. Consider again the example given in Section 3.1, and suppose that variables (10, 10) and (2,49) are the only candidates for branching. The ratio function would score variable (2, 49) higher, independent of the gap. However, in the SVB setting, if the gap is no larger than 40 then (10, 10) is the better variable. We therefore define a more refined scoring function that takes the gap into account. Given the gap at a given node, this scoring rule ranks the variables according to their SVB tree-sizes. We therefore call this scoring function the svts function, for "Single Variable Tree-Size". Algorithm 2 describes the implementation of the svts function. We introduce a parameter D, which determines whether an approximation of the tree-size should be used, depending on the smallest depth of a leaf in the SVB B&B tree, i.e.  $\lceil \frac{G}{r_i} \rceil$ , given a variable  $(l_i, r_i)$  and a gap G. Parameter D effectively bounds the number of terms to add in the closedform formula (2) to compute the SVB tree-size. If  $\left\lceil \frac{G}{r_i} \right\rceil > D$  (i.e. the gap G is such that the formula (2) would require computing more than D terms), then the exact tree-size is computed for a smaller gap, namely  $r_i D$ , and an approximation of the tree-size t(G) is computed approximately using the ratio. In our implementation we have set D = 100. Deactivating Step 5 (i.e. not filtering out dominated variables) increases the average running time of a MIP solver implementing Algorithm 2 as a scoring function in B&B by 4% (in the setup of Section 7.2.1). Deactivating Step 11 (i.e. not restricting the evaluation of formula (2) to "small" trees) increases the average running time by 11%.

Note that for a large enough gap, the SVB function is equivalent to the ratio function, i.e. the rankings of variables are equal.

Algorithm 2 Implementation of the svts scoring function.

1: Compute the absolute gap G at the current node. 2: if  $G = \infty$  then  $\triangleright$  e.g. no primal solution is known. return the variable selected by Algorithm 1.  $\triangleright$  Theorem 7 3: 4: end if 5: Filter out the branching candidates with dominated gains. 6: for all remaining branching candidates i with gains  $(l_i, r_i)$  do Let  $d = \left\lceil \frac{G}{r_i} \right\rceil$ .  $\triangleright$  The minimum depth of the SVB tree 7:if  $d = \infty$  then 8: Set the tree-size  $t_i(G) = \infty$ . 9: 10:else Let  $\tilde{G} = r_i * \min(d, D)$ .  $\triangleright D$  is a parameter 11: Compute the SVB tree-size  $t_i(\tilde{G})$  using formula (2). 12:if  $\tilde{G} < G$  then 13: Compute the ratio  $\varphi_i$ . 14:Compute the SVB tree-size  $t_i(G) \approx \varphi_i^{G-\tilde{G}} t_i(\tilde{G})$ .  $\triangleright$  see (3.3) 15:end if 16:end if 17:18: **end for** 19: if all variables *i* have  $t_i(G) = \infty$  then return the variable selected by Algorithm 1. 20: 21: else **return** the variable *i* with minimum  $t_i(G)$ . 22: 23: end if

We next evaluate the **ratio** and SVB scoring function in simulations as well as MIP benchmarks.

# 7 Experimental results

Numerous MIP solving algorithms are developed and parametrized by a trialand-error loop of experiments on MIP benchmarks. The major drawback of this procedure is evidently the computational burden it involves. Another significant flaw is that the instances used to develop the algorithm are often the ones used to benchmark it. This may lead to *overfitting*, which means that the algorithm will perform much better on these instances than on general ones. The magnitude of this problem increases with the number of parameters of a method. In an effort to mitigate these issues, we carry out simulations on a large set of random instances in Section 7.1. The results we obtain on these simulated instances confirm the improvements we achieve on standard MIP benchmarks in Section 7.2.

#### 7.1 Numerical simulations

In order to evaluate the performance of the linear, product and ratio scoring functions detailed in Section 6, we run simulations in which these scoring functions are used to select variables on MVB and GVB instances. The GVB problem models the B&B algorithm more closely than the MVB problem, but in practice it is only possible to obtain exact tree-sizes for the latter problem. We do not run simulations on the svts function, as it requires computing tree-sizes for all variables and each possible gap.

We evaluate the scoring functions on synthetically generated instances. The variable gains are generated in one of four different ways:

- **Balanced** [B] Both left and right gains are integers uniformly drawn in the interval [1, 1000] (if necessary, gains are switched to ensure that the left gain is less than or equal to the right gain).
- Unbalanced [U] The left (resp. right) gain is an integer uniformly drawn in the interval [1, 500] (resp. [501, 1000]).
- Very Unbalanced [V] The left (resp. right) gain is an integer uniformly drawn in the interval [1, 250] (resp. [251, 1000]).
- Extremely Unbalanced [X] The left (resp. right) gain is an integer uniformly drawn in the interval [1, 100] (resp. [101, 1000]).

We will refer to these different ways of generating variables as different *data types*.

#### 7.1.1 Simulation results for the MVB problem

Recall that in the MVB problem, one variable can be used an arbitrary number of times. In this section, two types of methods are evaluated. The first type correspond to *scoring functions* that can be or are actually implemented in MIP solvers to choose which variable to branch on (e.g. the **product** function). The second type are *problem solving algorithms*, which use knowledge of the MVB problem, and cannot directly be implemented in MIP solvers (e.g. an exact recursive algorithm). The methods of the first type are the ones we are really interested in, while the methods of the second type help evaluate the first ones. We compute the tree-sizes that result from the following methods, listed by order of increasing exactness.

- linear (with different values of the parameter  $\mu$ ), product and ratio scoring functions. Since the ranking of variables obtained by scoring functions only depends on the left and right gains, the same variable is branched on at every node, which means the tree-size obtained by selecting variables using a scoring function corresponds to the tree-size of the SVB tree for the variable with the best score.
- Lower Bound (LB) for scoring functions: it is the minimum tree-size of all SVB instances, for all variables in the input of the MVB instance.
- The exact recursive function, as described in Section 4: this method produces a minimum-size tree for the instance.

Each branching strategy in this list necessarily produces trees with fewer nodes than those listed higher in the list. Note that the linear, product and ratio scoring functions are listed together, as it is *a priori* not clear whether any of the scoring function dominates the others. These are the scoring functions we are primarily interested in.

The other parameters have been chosen as follows:

- The number of variables for each instance is 100. Increasing the number of variables is possible, however the chances of producing a variable that dominates all others would increase.
- The gap is set at  $G = 10^5$ . Increasing G by an order of magnitude may cause the tree-sizes to exceed the maximum number that can be encoded by a double-precision floating-point representation ( $\approx 10^{307}$ ).
- The number of instances for each of the B, U, V and X type of variables is 100.

Table 2 provides two different performance measures for each type of data and each branching strategy. The first performance measure, "t-s", is the percentile change in geometric means of the tree-sizes compared to the (exact) minimum tree-size. For instance, the number 26.2 (at the top-left of the table) indicates that this branching strategy yielded trees with, on average, 26.2%

Data				$linear(\mu$				LB	
Data	$\mu =$	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	1	product	ratio	LD
В	t-s	26.2	8.23	6.23	3.10	$10^{17}$	2.84	2.84	2.83
D	wins	88	93	95	99	8	100	100	
U	t-s	$10^{08}$	96.79	6.07	8.43	$10^{23}$	6.60	3.35	3.14
U	wins	11	67	93	90	10	93	98	
V	t-s	$10^{20}$	57.93	24.85	282.28	$10^{23}$	29.82	5.75	5.75
V	wins	19	77	86	65	14	83	99	
X	t-s	$10^{47}$	6.07	731.44	$10^{5}$	$10^{31}$	93.86	6.07	6.07
л	wins	7	100	72	50	13	82	100	

Table 2: Simulation results on MVB.

more nodes than the minimum-size trees. For the cases where the difference is in orders of magnitude, only the order is indicated. Note that the tree-sizes obtained in the experiments vary from around  $10^{30}$  for the balanced data to up to  $10^{300}$  for extremely unbalanced data. The second performance measure, "wins", is the number of times a scoring function produces the smallest tree-size among the linear, product and ratio functions (LB is excluded as it necessarily produces smaller tree-sizes). In case of a tie, multiple scoring functions win.

First, observe that the performance (w.r.t. tree-size) of branching strategies generally decreases as the left and right gains of the variables become less balanced. For some values of  $\mu$ , the linear function performs extremely badly. More interestingly, it appears that the best value of  $\mu$  heavily depends on the balance of data. The value  $\mu = \frac{1}{2}$  is better for balanced data (B), while for extremely unbalanced data (X),  $\mu = \frac{1}{6}$  performs best, in a tie with the ratio function. For this type of data, the tie may be explained by Figure 8, where the linear and ratio functions tend to pick similar variables for very unbalanced gains. Similarly, the tie between the product function and the ratio function for balanced gains. The ratio function wins on 397 out of 400 instances, and is very close to the theoretical bound LB that these scoring functions can achieve. Note that both the ratio function and LB are within 7% of the actual minimum-tree size, even for the most unbalanced data.

Additional results with smaller gaps are given in Appendix B.

#### 7.1.2 Simulation results for the GVB problem

The experiments we carry out in this section are significantly different than those we did on MVB: indeed, in the GVB problem, each variable can only be used a given number of times on a path from the root to a leaf. As a consequence, we are not able to compute the exact minimum tree-sizes, as the state space of the recurrence equation (5) is too large. For simplicity, and to ensure that the results in this section would be different than those from Section 7.1, we have set multiplicities to 1. Note that the algorithms that compute the tree-sizes for scoring functions are also harder to solve than in the previous section, and the gaps G used in our instances are therefore smaller. The parameters of these experiments are presented below.

- As in the previous section, the number of variables for each instance is 100, and the number of instances for each of the B, U, V and X type of variables is also 100.
- The gap takes different values for each type of data. Indeed, while a gap of e.g. G = 5000 can be closed with a few hundred nodes in the balanced case, it can only be closed in half of the instances for extremely unbalanced data. We therefore adapt the gap to the data. In each case, the minimum gap tested is chosen such that the tree-sizes are at least 100. The maximum gap tested is chosen such that the depth of the trees is close to 100, i.e. the number of variables, but also such that all 100 instances can be solved. The average between the minimum and the maximum gap is also tested.

Table 3 presents the simulation results, in a format similar to Table 2. Since we do not know the minimum tree-sizes in these experiments, the reference is the product function, as it is the state of the art. As a consequence, negative changes may (and do) occur. The observation we made in the previous experiments on MVB, which is that the ratio function benefits from unbalancedness, clearly carries over. Furthermore, for a given type of data, it also profits from an increase in the gap. An approximate 7% decrease in the number of nodes occurs when these two facts are taken together. Moreover, the ratio function has, by far, the largest number of wins. Note that the linear function outperforms the ratio or the product function in some cases.

A table similar to Table 3 for the MVB problem is given in Appendix B.

Since the product function seems to perform slightly better for small gaps, and the ratio improves with larger gaps, the idea of changing the scoring function depending on the gap naturally arises. Table 4 provides the results of this hybrid function for the maximum gaps considered for each data type. The notation is the same as in Table 3 (including the fact that the product function is taken as a reference). The hybrid ratio-product function is parametrized by h, the height at which the scoring function used at a node switches from ratio to product (the height of a node is the length of the longest path to reach a leaf from this node). For example, h = 10 means that at nodes that would have a height of 10 or less with the ratio function, the product function is used instead. As a consequence, h = 0 corresponds to the ratio function, and h = 100 to the product function (since there are 100 variables, the depth of a leaf is at most 100).

First, observe that we do obtain a slight reduction in the number of nodes for any value of h (except for data U and h = 50). Second, note that the optimal value for the height parameter increases as the data becomes less balanced. Third, observe how for each data type, the relative change compared to the **product** function is unimodal, which suggests a low variability of the results and backs up our analysis.

5	Gap 5000 5000	$\begin{array}{c} \mu = \\ t\text{-s} \\ wins \\ t\text{-s} \\ \end{array}$	$     \begin{array}{r}       0 \\       1.94 \\       67 \\       21.34     \end{array} $		-0.15	$\frac{\frac{1}{2}}{0.41}$	1 235.83	product	ratio
	5000	wins t-s	67		-0.15	0.41	025 02	0.00	
	5000	t-s		77		0.41	233.03	0.00	-0.03
B 15			01.94		85	76	0	82	83
Б 16			21.34	9.37	1.18	2.36	1051.56	0.00	0.06
	5000	wins	2	3	17	14	0	41	45
1.01	25000	t-s	57.70	19.45	0.89	19.23	1742.11	0.00	-0.96
<u></u>	0000	wins	0	0	7	1	0	28	64
5	5000	t-s	50.86	10.63	-0.02	4.27	314.73	0.00	0.17
0	5000	wins	0	2	45	3	0	<b>48</b>	44
U 15	2500	t-s	86.38	30.30	0.75	19.99	1562.15	0.00	-0.74
0 1	12000	wins	0	0	11	0	0	29	60
20	0000	t-s	71.91	22.99	0.30	61.17	3874.17	0.00	-2.29
	20000	wins	0	0	2	0	0	14	84
5	5000	t-s	257.39	4.76	6.50	54.40	282.93	0.00	-0.91
J	5000	wins	0	1	0	0	0	35	65
V 7	7500	t-s	428.67	11.88	14.85	102.26	425.88	0.00	-2.29
v í	1000	wins	0	0	0	0	0	25	75
1(	0000	t-s	528.58	24.38	25.76	135.77	525.01	0.00	-5.81
	0000	wins	0	0	0	0	0	4	96
0	2000	t-s	445.42	8.29	42.80	63.76	89.95	0.00	3.15
2	2000	wins	0	3	0	0	0	76	28
X 3	3000	t-s	903.02	6.72	47.56	73.37	107.32	0.00	-1.16
A 0	5000	wins	0	0	0	0	0	43	58
1	1000	t-s	1537.76	2.47	41.26	68.27	105.88	0.00	-6.90
4	4000	wins	0	1	0	0	0	2	97

Table 3: Simulation results on GVB.

Data		(ratio)	(ratio) Hybrid ratio-product $(h)$								
(& Gap)	h =	0	10	20	30	40	50	100			
В	t-s	-0.96	-1.31	-0.49	-0.02	0.00	0.00	0.00			
(25000)	wins	6	71	28	13	12	12	12			
U	t-s	-2.29	-2.29	-3.37	-1.01	-0.05	0.01	0.00			
(20000)	wins	0	7	<b>79</b>	14	0	0	0			
V	t-s	-5.81	-5.81	-6.18	-8.38	-4.40	-0.46	0.00			
(10000)	wins	0	0	5	81	14	0	0			
Х	t-s	-6.90	-6.95	-7.38	-9.03	-10.57	-8.27	0.00			
(4000)	wins	0	0	0	14	<b>73</b>	13	0			

Table 4: Simulation results on GVB using a hybrid ratio-product function, parametrized by a height h.

#### 7.2 Experiments on MIP instances

We have modified the solver SCIP 3.1.1 [2] to use the ratio or the svts function, alternatively to the default product function. Besides the changes necessary for the implementation of these new scoring functions, no other change to SCIP has been made. In particular, the decision to use strong branching or pseudo-costs on a given variable and at a given node is unchanged. Furthermore, note that in SCIP, the selection of the branching variable does not solely rely on the product. There are a number of "soft" tie-breakers that can come into play if the product scores of multiple variables are almost equal. This is referred to as hybrid branching [3]. If at a given node, the variable that the hybrid branching rule selects is not the one that maximizes the product function, then we keep that variable and do not use the ratio or svts function. Finally, for the ratio, we use the parameter h = 10 as defined in Section 7.1.2, following the results of Table 4. This essentially means that whenever a node is believed to be "close" to the leaves, the product function is used rather than the ratio function.

#### 7.2.1 Benchmark instances

The benchmark test set comprises the instances from MIPLIB 3.0, MIPLIB 2003 [4], and MIPLIB 2010 Benchmark [9]. These instances are the state of the art in MIP benchmarking. The instances that are at the time of writing classified as "open" have been removed, which leaves 159 instances. For feasible instances, the optimal value is provided as a primal bound, and primal heuristics are disabled (for all instances). Cuts after the root node are also disabled to reduce performance variability [12]. The disconnected components presolver is disabled as it can lead to different transformed problems, and thus possibly increased variability. Furthermore, for each instance, ten different seeds are used to create random permutations of the input (using SCIP's internal procedure). A time limit of two hours is specified. In this setting, the testbed comprises 1590 instances, and the total running time that this experiment required on a single machine is 4 months.

Three different scoring functions are tested. The first one is the product function, which is the default in SCIP. The second one is the ratio function, defined in Section 6.2. The third one is the svts function defined in Section 6.3.

Table 5 gives the result for these three scoring functions, with **product** as the reference. Each line corresponds to the 10 permutations of each instance of the test set. There are three columns for each scoring function. The columns of the **product** scoring function gives absolute performance measures, while the two others give measures relative to **product**. For **product**, the first column provides the number of permutations solved, and, for the other two functions, the difference from the reference. Note that all 10 permutations of a given instance may not be solved the same number of times depending on the scoring function used. For each instance, we determine for each scoring function the number of permutations solved. Let N be the minimum and note that N can be as large as 6 and we could still have no single permutation solved by all three scoring functions. After having determined N for a given instance, we take the N best results for each scoring function and compute the arithmetic averages. These averages of time (in seconds) and nodes are directly displayed for the **product** function. In the nodes column, we use the letters k and m as a shorthand for thousands and millions. For the **ratio** and **svts** functions, we display the ratio of averages for both time and nodes, compared to the **product**. Considering the N best permutations rather than the intersection of the sets of solved permutations enables more data to be used, and reflects in the averages the fact that some scoring functions solve more permutations, giving them a fair advantage.

The instances for which at least one permutation can be solved at the root node by any setting, or for which no permutation is solved by all settings (i.e. N = 0), are not displayed. We similarly exclude instances that can be solved in less than a second by at least one setting and permutation. Note that these instances are accounted in the total number of instances solved, but not in the different average measures.

The total, geometric mean and shifted geometric mean (with shifts 10 and 100 for time and nodes, respectively) [1, p. 321] are provided at the bottom of the table. These are not computed on the averages given at each line, but on each value that is used to compute the line averages (i.e. an instance solved 10 times by all settings counts more than an instance solved fewer than 10 times by all settings).

Function ratio solves marginally more instances than the product function. Both functions ratio and svts slightly outperform product, in terms of time and nodes used. The difference in performance is especially apparent when considering the total resources used, as it mostly reflects the performance on large instances, on which the scoring functions we introduce perform better. In Section 7.2.2, we will consider a set of instances that contains instances that experimentally require large B&B trees, and on which this observation is comfirmed.

Instance		product			ratio			svts	
Instance	#	time $(s)$	nodes	#	time $(s)$	nodes	#	time (s)	nodes
30n20b8	10	521	10	+0	1.02	1.49	+0	1.01	1.34
aflow30a	10	36	2.5k	+0	0.98	0.99	+0	0.96	0.89
aflow40b	10	3483	210.9k	+0	0.68	0.65	+0	0.70	0.63
air04	10	30	7	+0	0.96	1.00	+0	1.03	1.00
app1-2	10	845	565	+0	0.90	1.01	+0	0.89	1.14
arki001	2	4862	1.3m	+0	0.73	0.74	+1	0.85	0.80
ash608gpia-3col	10	87	5	+0	1.04	1.00	+0	1.01	1.00
bell5	10	1	1.1k	+0	0.99	1.00	+0	0.99	1.00
biella1	10	217	2.8k	+0	0.98	1.06	+0	0.99	1.00
bienst2	10	288	112.6k	+0	1.05	0.98	+0	1.11	1.09
binkar10_1	10	431	135.8k	+0	1.09	1.03	+0	1.01	0.98
blend2	10	2	222	+0	0.99	1.00	+0	1.01	1.01
cap6000	10	8	1.8k	+0	1.00	1.04	+0	1.01	1.05
csched010	9	6035	721.2k	-1	0.86	0.86	+0	0.90	0.84
danoint	2	7047	1.2m	-1	1.00	0.96	-1	1.02	0.97
dcmulti	10	2	9	+0	1.00	1.00	+0	1.00	1.00
dfn-gwin-UUM	10	256	62.9k	+0	1.03	1.07	+0	1.05	0.96
Table 5: Compari	son c	of scoring	functio	on on	the ben	chmarl	k test	; set (c	ontinues

Table 5: Comparison of scoring function on the benchmark test set (continues in the next page).

# 10 10 0 10 10 10 10 10 10 10 10 10 10 1	103 134 - 129 5 5 4 4 4 4 4 29 2286 806 1 2 2 2 462 422 1910	nodes 496 4.4k - - 494 12 9 4 5 20 9 11.5k 83.6k 6.7k 7 16 1.0k 260	$\begin{array}{c} \# \\ +0 \\ +0 \\ +10 \\ +0 \\ +0 \\ +0 \\ +0 \\ +$	time (s) 0.97 0.82 - 0.98 1.00 1.00 1.01 0.99 0.99 0.99 1.26 0.97 1.00 0.99	0.99 0.84 - 1.18 1.00 1.00 1.00 1.00 1.00 1.34 0.99 0.96	# +0 +0 +0 +0 +0 +0 +0 +0 +0 +0 +0 +0 +0	time (s) 1.11 0.88 - 0.99 1.01 1.01 1.01 1.00 0.99 1.00 1.05 0.97 1.92	nc 0 1 1 1 1 1 1 0 1 1 0 0
10 0 10 10 10 10 10 10 10 10 1	134 - 129 5 5 4 4 4 4 29 2286 806 1 2 2 462 422	4.4k - 494 12 9 4 5 20 9 11.5k 83.6k 6.7k 7 16 1.0k 260	$ \begin{array}{c} +0 \\ +10 \\ +0 \\ +0 \\ +0 \\ +0 \\ +0 \\ +0 $	0.82 0.98 1.00 1.00 1.01 0.99 0.99 1.26 0.97 1.00	0.84 - 1.18 1.00 1.00 1.00 1.00 1.00 1.34 0.99 0.96	+0 +0 +0 +0 +0 +0 +0 +0 +0 +0 +0 +0 +0	0.88 - 0.99 1.01 1.01 1.01 1.00 <b>0.99</b> 1.00 1.05 0.97	0 1 1 1 1 1 0 1 1
0 10 10 10 10 10 10 10 10 10 10 10 10 10	- 129 5 4 4 4 29 2286 806 1 2 2 462 422	494 12 9 4 5 20 9 11.5k 83.6k 6.7k 7 16 1.0k 260	$ \begin{array}{c} +10 \\ +0 \\ +0 \\ +0 \\ +0 \\ +0 \\ +0 \\ +0 $	<ul> <li>0.98</li> <li>1.00</li> <li>1.01</li> <li>0.99</li> <li>0.99</li> <li>1.26</li> <li>0.97</li> <li>1.00</li> </ul>	1.18 1.00 1.00 1.00 1.00 1.00 1.34 0.99 0.96	$ \begin{array}{c} +0 \\ +0 \\ +0 \\ +0 \\ +0 \\ +0 \\ +0 \\ +0 $	- 0.99 1.01 1.01 1.01 1.00 <b>0.99</b> 1.00 1.05 0.97	1 1 1 1 0 1 1
10 10 10 10 10 10 10 10 10 10 10 10 10 1	<b>5</b> <b>4</b> 4 4 <b>29</b> 2286 806 1 <b>2</b> 2 <b>462</b> <b>422</b>	12 9 4 5 20 9 11.5k 83.6k 6.7k 7 16 1.0k 260	$ \begin{array}{c} +0 \\ +0 \\ +0 \\ +0 \\ +0 \\ +0 \\ +0 \\ +0 $	1.00 1.01 0.99 0.99 1.26 0.97 1.00	1.00 1.00 1.00 1.00 1.00 1.34 0.99 0.96	+0 +0 +0 +0 +0 +0 +0 +0 +0 +0 +0 +0	$1.01 \\ 1.01 \\ 1.00 \\ 0.99 \\ 1.00 \\ 1.05 \\ 0.97$	1 1 1 0 1 1
10 10 10 10 10 10 10 10 10 10 10 10 10 1	5 4 4 29 2286 806 1 2 2 462 422	9 4 5 20 9 11.5k 83.6k 6.7k 7 16 1.0k 260	$ \begin{array}{c} +0 \\ +0 \\ +0 \\ +0 \\ +0 \\ +0 \\ +0 \\ +0 $	<ol> <li>1.00</li> <li>1.01</li> <li>0.99</li> <li>0.99</li> <li>1.26</li> <li>0.97</li> <li>1.00</li> </ol>	1.00 1.00 1.00 1.00 1.34 0.99 0.96	+0 +0 +0 +0 +0 +0 +0 +0 +0 +0 +0	$1.01 \\ 1.01 \\ 1.00 \\ 0.99 \\ 1.00 \\ 1.05 \\ 0.97$	1 1 1 0 1 1
10 10 10 10 10 10 10 10 10 10 10 10 10	4 4 4 29 2286 806 1 2 2 462 422	4 5 20 9 11.5k 83.6k 6.7k 7 16 1.0k 260	+0 +0 +0 +0 +0 +0 +0 +0 +0 +0 +0 +0 +0 +	1.01 <b>0.99</b> <b>0.99</b> 1.26 <b>0.97</b> 1.00	<ol> <li>1.00</li> <li>1.00</li> <li>1.00</li> <li>1.34</li> <li>0.99</li> <li>0.96</li> </ol>	+0 +0 +0 +0 +0 +0 +0 +0 +0	1.01 1.00 <b>0.99</b> 1.00 1.05 0.97	1 1 0 1 1
10 10 10 10 10 10 10 10 10 10 10 10	4 4 29 2286 806 1 2 2 462 422	5 20 9 11.5k 83.6k 6.7k 7 16 1.0k 260	+0 +0 +0 +0 +0 +0 +0 +0 +0 +0	0.99 0.99 0.99 1.26 0.97 1.00	<ol> <li>1.00</li> <li>1.00</li> <li>1.34</li> <li>0.99</li> <li>0.96</li> </ol>	$+0 \\ +0 \\ +0 \\ +0 \\ +0 \\ +0$	1.00 <b>0.99</b> 1.00 1.05 0.97	1 0 1 1
10 10 10 10 10 10 10 10 10 10 10 10	4 4 29 2286 806 1 2 2 462 422	20 9 11.5k 83.6k 6.7k 7 16 1.0k 260	+0 +0 +0 +0 +0 +0 +0 +0 +0	0.99 <b>0.99</b> 1.26 <b>0.97</b> 1.00	1.00 1.00 1.34 0.99 0.96	$+0 \\ +0 \\ +0 \\ +0 \\ +0$	<b>0.99</b> 1.00 1.05 0.97	0 1 1
10 10 10 10 10 10 10 10 10 10	4 29 2286 806 1 2 2 462 422	9 11.5k 83.6k 6.7k 7 16 1.0k 260	+0+0+0+0+0+0+0	0.99 1.26 0.97 1.00	1.00 1.34 0.99 0.96	+0 +0 +0 +0	$1.00 \\ 1.05 \\ 0.97$	<b>1</b> 1
10 10 10 10 10 10 10 10 10 10	29 2286 806 1 2 2 462 422	<b>11.5k</b> 83.6k 6.7k <b>7</b> <b>16</b> 1.0k 260	$+0 \\ +0 \\ +0 \\ +0 \\ +0 \\ +0$	1.26 <b>0.97</b> 1.00	1.34 <b>0.99</b> <b>0.96</b>	$^{+0}_{+0}$	$1.05 \\ 0.97$	1
10 10 10 10 10 10 10 10 10	2286 806 1 2 462 422	83.6k 6.7k <b>7</b> <b>16</b> 1.0k 260	$+0 \\ +0 \\ +0 \\ +0 \\ +0$	<b>0.97</b> 1.00	$0.99 \\ 0.96$	+0	0.97	
10 10 10 10 10 10 10 10	806 1 2 462 422	6.7k 7 16 1.0k 260	$^{+0}_{+0}_{+0}$	1.00	0.96			0
10 10 10 10 10 10 10	1 2 462 422	<b>7</b> 16 1.0k 260	$^{+0}_{+0}$			+0		
10 10 10 10 10 10	2 2 462 422	<b>16</b> 1.0k 260	+0	0.99	1 00		1.00	0
10 10 10 10 10	2 462 422	1.0k 260		1.01	<b>1.00</b> 1.05	+0 +0 +0	<b>0.98</b> 1.01	1 1
10 10 10 10	$\begin{array}{c} 462 \\ 422 \end{array}$	260		0.92	0.83	+0 +0	1.01	1
10 10 10	422		$^{+0}_{+0}$	1.03	0.85	+0 +0	1.05	1
10 10		300	+0	1.06	1.00	+0	1.00	0
10	1810	3.3m	+0	0.87	0.88	+0	0.92	0
	200	484.2k	+0	0.83	0.83	+0	0.98	0
10	268	18.1k	+0	0.99	1.01	+0	0.98	0
10	1832	829.9k	+0	1.18	1.17	+0	1.30	1
10	30	356	+0	1.01	0.95	+0	1.05	1
10	1228	116.9k	+0	0.92	0.96	+0	1.13	1
10	2	125	+0	0.98	1.02	+0	0.98	0
10	1	7	+0	0.99	1.00	+0	1.01	1
								0
								1
								0
								1
								1
								0 0
								1
								0
								0
								ŏ
10	431						1.00	0
10	142	1.8k	+0	1.13	1.22	+0	1.13	1
10	275	12		0.95	0.98	+0	1.00	0
10	76	5.3k	+0	1.27	1.59	+0	1.12	1
10	2011	2.9k	+0	1.03	1.07	+0	1.08	1
								1
								0
								1
								1
								0
								0
								0
								1
								0
								1
								1
								0
10	123	325.8k		1.13	1.14		1.09	1
10	2	237		1.00	1.01		0.98	1
10	$^{2}$	153	+0	0.99	1.02	+0	0.99	1
0	-	-	+7	-	-	+4	-	
10	106	12.5k	+0	1.00	1.02	+0	0.98	0
10	11	4	+0	0.98	1.00	+0	1.00	1
10	6	4	+0	0.97	1.00	+0	0.98	1
10	130	507	+0	1.07	1.17	+0	1.08	1
	544		+0	0.82	0.82	+0	0.81	0
	-	-		-	-		-	0
	$\begin{array}{c} 10\\ 10\\ 10\\ 10\\ 10\\ 10\\ 10\\ 10\\ 10\\ 10\\$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	10         30         356           10         1228         116.9k           10         2         125           10         1         7           10         45         23.8k           10         5         12           10         74         886           10         1         20           1         20         1         3.6k           10         1959         6.3k           10         1959         6.3k           10         1213         46.6k           10         395         23.2k           10         4213         46.6k           10         395         23.2k           10         431         131           10         142         1.8k           10         275         12           10         76         5.3k           10         2011         2.9k           10         1463         13           10         2143         37           10         414         174           10         81         92           10         4192         18.6k	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	10         30 $356$ $+0$ $1.01$ 10 $1228$ $116.9k$ $+0$ $0.92$ 10         2 $125$ $+0$ $0.93$ 10         1         7 $+0$ $0.99$ 10         45 $23.8k$ $+0$ $1.24$ 10         5 $12$ $+0$ $0.99$ 10         74 $886$ $+0$ $0.97$ 10         1 $20$ $+0$ $1.00$ 10 $1959$ $6.3k$ $+0$ $0.90$ 10 $732$ $233$ $+0$ $0.90$ 10 $732$ $233$ $+0$ $0.99$ 10 $554$ $11$ $+0$ $1.01$ 10 $431$ $131$ $+0$ $83$ 10 $142$ $1.8k$ $+0$ $1.03$ 10 $275$ $126$ $1.022$ $1022$ 10 $76$ $5.3k$ $+0$	10         30 $356$ $+0$ 1.01 $0.95$ 10         1228         116.9k $+0$ $0.92$ $0.96$ 10         2         125 $+0$ $0.98$ $1.02$ 10         1         7 $+0$ $0.99$ $1.00$ 10         45 $23.8k$ $+0$ $1.24$ $1.28$ 10         5 $12$ $+0$ $0.99$ $1.07$ 10         74 $886$ $+0$ $0.97$ $0.89$ 10         1 $20$ $+0$ $1.00$ $1.04$ 1 $4417$ $13.6k$ $+1$ $1.30$ $1.33$ 10 $1959$ $6.3k$ $+0$ $0.99$ $0.89$ 10 $732$ $233$ $+0$ $0.99$ $0.89$ 10 $732$ $233$ $+0$ $0.99$ $0.89$ 10 $395$ $23.2k$ $+0$ $1.00$ $1.01$ 10 $431$	10       30 $356$ +0 $1.01$ $0.95$ +0         10       1228 $116.9k$ +0 $0.92$ $0.96$ +0         10       2 $125$ +0 $0.98$ $1.02$ +0         10       1       7       +0 $0.99$ $1.00$ +0         10       45 $23.8k$ +0 $1.24$ $1.28$ +0         10       5 $12$ +0 $0.99$ $1.07$ +0         10       74 $886$ +0 $0.97$ $0.89$ +0         10       120       +0 $1.00$ $1.04$ +0         1       4417 $13.6k$ +1 $1.30$ $1.33$ +1         10 $1959$ $6.3k$ +0 $0.99$ $0.89$ +0         10 $732$ $233$ +0 $0.99$ $0.89$ +0         10 $123$ $46.6k$ +0 $1.04$ $0.99$ +0         10 $486$ $69.9k$ +0 $1.00$ $1.01$ +0      <	$\begin{array}{cccccccccccccccccccccccccccccccccccc$

in the next page).

Instance		product			ratio		svts			
Instance	#	time $(s)$	nodes	#	time $(s)$	nodes	#	time $(s)$	nodes	
rmatr100-p10	10	109	799	+0	1.03	1.00	+0	1.02	0.93	
rmatr100-p5	10	140	387	+0	0.96	0.99	+0	0.95	0.99	
rmine6	3	3540	380.6k	+1	1.10	1.05	-2	2.00	2.39	
rocII-4-11	10	419	4.7k	+0	2.43	5.91	+0	1.27	1.80	
rococoC10-001000	1	4677	323.9k	+1	0.83	0.81	+1	0.44	0.50	
roll3000	9	3781	429.3k	-2	1.15	1.18	-3	0.84	0.83	
rout	10	111	44.6k	+0	0.73	0.73	+0	0.94	0.94	
satellites1-25	10	1590	3.4k	+0	1.03	0.98	+0	1.02	1.01	
set1ch	10	2	5	+0	1.00	0.92	+0	1.00	0.92	
sp98ic	10	4736	276.2k	+0	0.49	0.42	+0	0.70	0.58	
sp98ir	10	71	1.6k	+0	0.97	1.16	+0	0.98	1.15	
stein27	10	2	4.2k	+0	0.98	0.98	+0	1.00	0.99	
stein45	10	34	51.5k	+0	0.92	0.98	+0	0.97	0.99	
tanglegram1	7	1749	289	-1	1.13	1.06	-2	0.90	0.58	
tanglegram2	10	10	4	+0	0.98	1.00	+0	1.00	1.00	
timtab1	10	1184	774.5k	+0	1.04	1.04	+0	1.01	1.02	
tr12-30	10	3644	780.2k	+0	1.02	1.08	+0	1.00	1.02	
unitcal_7	9	3415	32.0k	+1	0.76	0.75	+0	0.91	0.87	
vpm2	10	2	180	+0	1.00	1.05	+0	0.98	1.01	
zib54-UUE	3	4914	413.0k	+0	1.20	1.18	+0	1.09	1.13	
Total	1225	705680	121.4m	+24	0.92	0.97	+1	0.94	0.97	
Geo. mean		98	1.2k		0.98	1.00		0.99	0.98	
Sh. geo. mean		148	2.8k		0.98	1.00		0.99	0.98	

Table 5: Comparison of scoring function on the benchmark test set.

#### 7.2.2 Instances with large B&B trees

MIPLIB 2010 also has a so-called tree set with 52 instances, which "contains instances that (empirically) lead to large enumeration trees" [9]. These instances have been selected because the scoring functions **ratio** and **svts** defined in Section 6 have specifically been designed to solve instances with large B&B trees. We have tested our scoring functions on these instances in the same setup as the experiments of Section 7.2.1, but we gave a time limit of 12 hours to allow for a more significant number of instances to be solved. Including permutations, the testbed comprises 520 instances, and required the equivalent of 19 months of running time on a single machine.

Table 6 provides the result for the tree test set with the same notation as Table 5 in Section 7.2.1. Both functions ratio and svts largely outperform product, both in terms of time and nodes, and solve slightly more instances. Note that some instances in this test set do not require larger trees than those in the benchmark test set (e.g. glass4, ns894788 and pg). This is probably due to the fact that MIP solvers have made significant progress for these instances since this test set was designed. A more extensive report of the numerical experiments is provided in the appendices, Section C.

Instance		product				ratio		svts			
	Instance	#	time (s)	nodes	#	time $(s)$	nodes	#	time (s)	nodes	_
	csched007	10				0.78			0.82	0.76	
_	le 6: Compar	10	1440	112.4k	+0	1.21	1.03	+0	1.14	1.06	_
Tab	le 6: Compar	ison	of scori	ng funct	ion (	on the t	ree tes	t set	(continu	ues in	the
next	z page).										

Instance		produc	t		ratio		svts			
mstance	#	time $(s)$	nodes	#	time (s)	nodes	#	time (s)	nodes	
glass4	10	28	11.5k	+0	1.37	1.34	+0	1.03	1.03	
gmu-35-40	0	-	-	+2	-	-	+1	-	-	
k16x240	10	23835	23.2m	+0	0.88	0.89	+0	0.86	0.85	
neos-1616732	10	9182	2.5m	+0	0.85	0.94	+0	0.82	0.84	
neos-942830	10	9184	1.5m	+0	0.95	0.95	+0	1.00	0.94	
neos15	1	34352	12.3m	+0	0.91	0.94	+0	0.87	0.91	
neos858960	10	3997	2.8m	+0	0.92	0.93	+0	0.92	0.93	
noswot	10	556	679.8k	+0	1.62	1.57	+0	1.44	1.36	
ns1766074	10	5747	925.7k	+0	0.95	0.98	+0	0.98	0.98	
pg	10	<b>24</b>	203	+0	1.01	0.97	+0	1.01	1.10	
ran14x18	10	23039	18.4m	+0	0.83	0.82	+0	0.87	0.81	
reblock166	4	16858	1.6m	+2	0.98	1.00	+2	0.79	0.95	
timtab1	10	1142	774.5k	+0	1.05	1.04	+0	1.02	1.02	
umts	10	8726	1.5m	+0	0.69	0.71	+0	0.64	0.65	
wachplan	10	15334	324.2k	+0	1.02	1.01	+0	1.01	1.01	
Total	155	1434131	602.1m	+4	0.87	0.87	+3	0.87	0.84	
Geo. mean		2879	618.6k		0.97	0.96		0.95	0.95	
Sh. geo. mean		3021	636.4k		0.97	0.96		0.95	0.95	

Table 6: Comparison of scoring function on the tree test set.

# 8 Conclusions

We developed one of the first models for the branching component of the Branch & Bound algorithm, over fifty years after its introduction. We proved that these models are relevant by theoretically establishing new scoring functions that are efficient for MIP solving. Numerous questions naturally arise regarding these models.

One possible line of investigation relates to the computational complexity of the decision problems we have defined. For instance, can MVB be solved in polynomial time (even for two variables)? Is there an approximation algorithm for the minimization version of GVB?

Scoring functions are the only components of B&B that we numerically analyze through the theory we develop in this paper. Are there other elements of B&B that can be studied via the current models? There certainly exist decision problems that can model the B&B algorithm more accurately than GVB (e.g. if the gains of a variable are not fixed). Would the analysis of these models deepen our understanding of B&B, and lead to additional MIP solving improvements?

## Acknowledgments

The authors would like to thank Eduardo Uchoa for pointing out reference [10] and Graham Farr for helping with the discussion at the end of Section 5.1. This research was funded by AFOSR grant FA9550-12-1-0151 of the Air Force Office of Scientific Research and the National Science Foundation Grant CCF-1415460 to the Georgia Institute of Technology.

# References

- T. Achterberg. Constraint Integer Programming. PhD thesis, Technische Universität Berlin, 2007.
- [2] T. Achterberg. SCIP: Solving constraint integer programs. Mathematical Programming Computation, 1(1):1–41, July 2009.
- [3] T. Achterberg and T. Berthold. Hybrid branching. In WJ. Hoeve and J. N. Hooker, editors, Integration of AI and OR Techniques in Constraint Programming for Combinatorial Optimization Problems, volume 5547 of Lecture Notes in Computer Science, pages 309–311. Springer, 2009.
- [4] T. Achterberg, T. Koch, and A. Martin. MIPLIB 2003. Operations Research Letters, 34(4):361–372, 2006.
- T. Achterberg and R. Wunderling. Mixed Integer Programming: Analyzing 12 Years of Progress, pages 449–481. Springer Berlin Heidelberg, Berlin, Heidelberg, 2013.
- [6] M. Conforti, G. Cornuéjols, and G. Zambelli. Integer programming, volume 271 of Graduate Texts in Mathematics. Springer, 2014.
- [7] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein. Introduction to Algorithms, Third Edition. The MIT Press, 3rd edition, 2009.
- [8] N. Jacobson. Basic Algebra I: Second Edition. Dover, 2009.
- [9] T. Koch, T. Achterberg, E. Andersen, O. Bastert, T. Berthold, R. E. Bixby, E. Danna, G. Gamrath, A. M. Gleixner, S. Heinz, A. Lodi, H. Mittelmann, T. Ralphs, D. Salvagnin, D. E. Steffy, and K. Wolter. MIPLIB 2010. *Mathematical Programming Computation*, 3(2):103–163, 2011.
- [10] O. Kullmann. Fundaments of branching heuristics. In Armin Biere, Marijn Heule, Hans van Maaren, and Toby Walsh, editors, *Handbook of Satisfiability*, volume 185 of *Frontiers in Artificial Intelligence and Applications*, pages 205–244. IOS Press, 2009.
- [11] A. H. Land and A. G. Doig. An automatic method of solving discrete programming problems. *Econometrica*, 28(3):pp. 497–520, 1960.
- [12] A. Lodi and A. Tramontani. Performance variability in mixed-integer programming. In *TutORials in Operations Research: Theory Driven by Influential Applications*, chapter 2, pages 1–12. 2013.
- [13] G. L. Nemhauser and L. A. Wolsey. Integer and Combinatorial Optimization. Wiley-Interscience, New York, NY, USA, 1988.
- [14] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery. Numerical Recipes 3rd Edition: The Art of Scientific Computing. Cambridge University Press, New York, NY, USA, 3 edition, 2007.

- [15] Larry J. Stockmeyer. The polynomial-time hierarchy. Theoretical Computer Science, 3(1):1 – 22, 1976.
- [16] Seinosuke Toda. PP is as hard as the polynomial-time hierarchy. SIAM Journal of Computing, 20(5):865–877, October 1991.

# Appendices

# A Proofs

## A.1 Proof of Theorem 4 and Corollary 5

**Theorem**. When G tends to infinity, both sequences  $\sqrt[l]{\frac{t(G+l)}{t(G)}}$  and  $\sqrt[r]{\frac{t(G+r)}{t(G)}}$  converge to  $\varphi$ , which is the unique root greater than 1 of the equation  $p(x) = x^r - x^{r-l} - 1 = 0$ .

*Proof.* Proposition 3 proves the case l = r, therefore we suppose throughout the proof that r > l. First, we define the notation

$$\underline{L} = \liminf_{G \to \infty} \frac{t(G+l)}{t(G)} \qquad \qquad \overline{L} = \limsup_{G \to \infty} \frac{t(G+l)}{t(G)}$$
$$\underline{R} = \liminf_{G \to \infty} \frac{t(G+r)}{t(G)} \qquad \qquad \overline{R} = \limsup_{G \to \infty} \frac{t(G+r)}{t(G)}.$$

Using the recursive definition (1) of t between the first and the second lines, we can easily establish:

$$\underline{L} = \left(\limsup_{G \to \infty} \frac{t(G)}{t(G+l)}\right)^{-1} = \left(\limsup_{G \to \infty} \frac{t(G-l)}{t(G)}\right)^{-1}$$
$$= \left(\limsup_{G \to \infty} \frac{t(G) - t(G-r)}{t(G)}\right)^{-1} = \left(1 - \limsup_{G \to \infty} \frac{t(G-r)}{t(G)}\right)^{-1}$$
$$= \left(1 - \limsup_{G \to \infty} \frac{t(G)}{t(G+r)}\right)^{-1} = \left(1 - \left(\liminf_{G \to \infty} \frac{t(G+r)}{t(G)}\right)^{-1}\right)^{-1}$$
$$= (1 - \underline{R}^{-1})^{-1} = 1 + \frac{1}{\underline{R} - 1}. \quad \text{(note that } \underline{R} \ge 2.)$$

Similarly,  $\bar{L} = 1 + \frac{1}{\bar{R}-1}$ . Next, we obtain

$$\liminf_{G \to \infty} \frac{t(G+lr)}{t(G)} = \liminf_{G \to \infty} \frac{t(G+lr)}{t(G+(l-1)r)} \frac{t(G+(l-1)r)}{t(G+(l-2)r)} \cdots \frac{t(G+r)}{t(G)}$$
$$\geq \liminf_{G \to \infty} \frac{t(G+lr)}{t(G+(l-1)r)} \liminf_{G \to \infty} \frac{t(G+(l-1)r)}{t(G+(l-2)r)} \cdots \liminf_{G \to \infty} \frac{t(G+r)}{t(G)}$$
$$= \left(\liminf_{G \to \infty} \frac{t(G+r)}{t(G)}\right)^l = \underline{R}^l.$$

Also, we have

$$\limsup_{G \to \infty} \frac{t(G+lr)}{t(G)} = \limsup_{G \to \infty} \frac{t(G+lr)}{t(G+l(r-1))} \dots \frac{t(G+l)}{t(G)}$$
$$\leq \limsup_{G \to \infty} \frac{t(G+lr)}{t(G+l(r-1))} \dots \limsup_{G \to \infty} \frac{t(G+l)}{t(G)}$$
$$= \left(\limsup_{G \to \infty} \frac{t(G+l)}{t(G)}\right)^r = \bar{L}^r.$$

Together, these yield

$$\bar{L}^r \geq \limsup_{G \to \infty} \frac{t(G+lr)}{t(G)} \geq \liminf_{G \to \infty} \frac{t(G+lr)}{t(G)} \geq \underline{R}^l.$$

Following the same steps, we can show  $\bar{R}^l \geq \underline{L}^r$ . Starting from this inequality, we prove

$$\bar{R}^{\frac{l}{r}} \geq \underline{L} = 1 + \frac{1}{\underline{R} - 1} \Rightarrow \underline{R} \geq 1 + \frac{1}{\bar{R}^{\frac{l}{r}} - 1}.$$

Likewise, we can show that the inequality  $\bar{R} \leq 1 + \frac{1}{\underline{R}^{\frac{l}{r}} - 1}$  holds.

We now introduce two monotonic sequences  $\alpha_n$  and  $\omega_n$  that respectively bound <u>R</u> from below and bound  $\overline{R}$  from above, and we prove that they converge to the same limit. For all non-negative n, let  $\alpha_n$  and  $\omega_n$  be defined as follows:

$$\alpha_n = \begin{cases} 2 & \text{if } n = 0\\ f(\omega_{n-1}) & \text{otherwise} \end{cases} \qquad \omega_n = f(\alpha_n)$$

where  $f(x) = 1 + \frac{1}{x^{\frac{l}{r}} - 1}$  for all  $x \in (1, \infty)$ . We first prove by induction that for all non-negative integer n,  $\alpha_n$  and  $\omega_n$  satisfy  $\alpha_n \leq \underline{R} \leq \bar{R} \leq \omega_n$ . Proposition 3 ensures that  $\alpha_0 = 2$  is a lower bound on  $\underline{R}$ . Suppose that for a given nonnegative n, the inequality  $\alpha_n \leq \underline{R}$  holds. We prove that  $\bar{R} \leq \omega_n$  holds too:

$$\begin{split} \alpha_n &\leq \underline{R} \Rightarrow 1 + \frac{1}{\alpha_n^{\frac{l}{r}} - 1} \geq 1 + \frac{1}{\underline{R}^{\frac{l}{r}} - 1} \geq \bar{R} \\ &\Rightarrow f(\alpha_n) \geq \bar{R} \\ &\Rightarrow \omega_n \geq \bar{R}. \end{split}$$

The same reasoning also proves that for all non-negative integers n,  $\omega_n \geq \overline{R}$  implies  $\alpha_{n+1} \leq \underline{R}$ . The sequence  $\alpha_n$  thus bounds  $\underline{R}$  from below, and  $\omega_n$  bounds  $\overline{R}$  from above.

We now prove that  $\alpha_n$  is monotonically increasing. Consider the inequality:

$$\begin{aligned} \alpha_1 > \alpha_0 \Leftrightarrow f(f(2)) > 2 \Leftrightarrow \frac{1}{f(2)^{\frac{l}{r}} - 1} > 1 \Leftrightarrow f(2) < 2^{\frac{r}{l}} \\ \Leftrightarrow 1 + \frac{1}{2^{\frac{l}{r}} - 1} < 2^{\frac{r}{l}} \Leftrightarrow 2^{\frac{l}{r}} < 2^{\frac{r}{l}} (2^{\frac{l}{r}} - 1) \Leftrightarrow r > l. \end{aligned}$$

Suppose now that for a given non-negative  $n, \alpha_n > \alpha_{n-1}$ . This implies that  $\omega_n < \omega_{n-1}$ :

$$\omega_n - \omega_{n-1} = \frac{1}{\alpha_n^{\frac{l}{r}} - 1} - \frac{1}{\alpha_{n-1}^{\frac{l}{r}} - 1} < 0 \Leftrightarrow \alpha_n^{\frac{l}{r}} > \alpha_{n-1}^{\frac{l}{r}}.$$

In turn, for that given n,  $\omega_n < \omega_{n-1}$  implies  $\alpha_{n+1} > \alpha_n$ . The sequence  $\alpha_n$  and  $\omega_n$  are thus increasing and decreasing, respectively. Since they are bounded, each of them converges to one of the solutions of the fixed-point equation x = f(x). We now prove that there is a unique solution to that equation:

$$\begin{aligned} x &= f(x) \Leftrightarrow x = 1 + \frac{1}{x^{\frac{l}{r}} - 1} \Leftrightarrow (x^{\frac{l}{r}} - 1)(x - 1) = 1 \\ &\Leftrightarrow x^{\frac{l+r}{r}} - x^{\frac{l}{r}} - x = 0 \\ &\Leftrightarrow p(X) = X^r - X^{r-l} - 1 = 0 \end{aligned}$$

where  $X = x^{\frac{1}{r}}$ . We establish in Theorem 6 that the polynomial p has a unique root in  $(1, \infty)$ , hence the fixed-point equation x = f(x) also has a unique solution. Consequently, it is necessary that both sequences  $\alpha_n$  and  $\omega_n$  converge to this unique fixed point, and thus  $\underline{R} = \overline{R}$ . Furthermore, the sequence  $\sqrt[r]{\frac{t(G+r)}{t(G)}}$  converges to the root  $\varphi > 1$  of the polynomial p.

converges to the root  $\varphi > 1$  of the polynomial p. Since we have established that  $\underline{L} = 1 + \frac{1}{\underline{R}-1}$  and  $\overline{L} = 1 + \frac{1}{\overline{R}-1}$ , it follows that  $\underline{L} = \overline{L}$ . Since  $p(\varphi) = 0$ ,  $\varphi$  equivalently satisfies  $\varphi^l = 1 + \frac{1}{\varphi^r - 1}$ , therefore  $\underline{L} = \overline{L} = \varphi^l$ .

**Corollary** . A numerical approximation of  $\varphi^r$  is given by the fixed-point iteration

$$f(x) = 1 + \frac{1}{x^{\frac{l}{r}} - 1}.$$

with the starting point x = 2.

*Proof.* Recall the definition of the sequences  $\alpha_n$  and  $\omega_n$  as given in the proof of Theorem 4 (notice that function f has the same definition). The sequence  $f_n$  generated by the fixed-point equation is  $(2, f(2), f(f(2)), f(f(f(2))), \ldots)$ , which is equal to  $(\alpha_0, \omega_0, \alpha_1, \omega_1, \ldots)$ . Formally, the sequence  $f_n$  generated by the fixed-point equation satisfies

$$f_n = \begin{cases} \alpha_{\frac{n}{2}} & \text{if } n \text{ is even} \\ \omega_{\frac{n+1}{2}} & \text{if } n \text{ is odd.} \end{cases}$$

In the proof of Theorem 4, we prove that both  $\alpha_n$  and  $\omega_n$  converge to  $\varphi^r$  when n tends to infinity, therefore  $f_n$  also converges to  $\varphi^r$ .

# A.2 Proof of Theorem 7

Recall that z is the least common multiple of all  $l_i$  and  $r_i$ .

**Theorem .**  $\varphi = \min_i \varphi_i$ 

*Proof.* Let  $\alpha$  be such that for all positive integers  $x \leq z$ ,

$$\alpha^x \le \liminf_{G \to \infty} \frac{t(G+x)}{t(G)} \le \limsup_{G \to \infty} \frac{t(G+x)}{t(G)}.$$
(6)

A possible value is  $\alpha = 1$ . Similar to the proof of Theorem 4, we use the notation

$$\underline{Z} = \liminf_{G \to \infty} \frac{t(G+z)}{t(G)} \qquad \qquad \overline{Z} = \limsup_{G \to \infty} \frac{t(G+z)}{t(G)}$$

For all variables i, we have:

$$\begin{split} \bar{Z} &= \limsup_{G \to \infty} \left( \frac{t(G+z)}{t(G+z-l_i)} \frac{t(G+z-l_i)}{t(G+z-2l_i)} \dots \frac{t(G+z-(\frac{z}{l_i}-1)l_i)}{t(G)} \right) \\ &\leq \left(\limsup_{G \to \infty} \frac{t(G+l_i)}{t(G)}\right)^{\frac{z}{l_i}} \\ &= \left(\limsup_{G \to \infty} \frac{1+\min_j(t(G+l_i-l_j)+t(G+l_i-r_j))}{t(G)}\right)^{\frac{z}{l_i}} \\ &\leq \left(1+\limsup_{G \to \infty} \frac{t(G+l_i-r_i)}{t(G)}\right)^{\frac{z}{l_i}} \\ &\leq \left(1+\left(\liminf_{G \to \infty} \frac{t(G+l_i-r_i)}{t(G+l_i-r_i)}\right)^{-1}\right)^{\frac{z}{l_i}} \\ &\leq \left(1+\left(\liminf_{G \to \infty} \frac{t(G+r_i-l_i)}{t(G)}\right)^{-1}\right)^{\frac{z}{l_i}} \\ &\leq \left(1+\alpha^{l_i-r_i}\right)^{\frac{z}{l_i}} \end{split}$$

where the last line follows from the lower bound on  $\alpha^{-x}$  from (6).

Suppose there exists a variable i such that  $\alpha > \varphi_i,$  then we establish, using Theorem 6:

$$\begin{split} \alpha > \varphi_i \Rightarrow p_i(\alpha) > 0 \\ \Rightarrow \alpha^{r_i} - \alpha^{r_i - l_i} - 1 > 0 \\ \Rightarrow 1 + \alpha^{l_i - r_i} < \alpha^{l_i} \\ \Rightarrow (1 + \alpha^{l_i - r_i})^{\frac{z}{l_i}} < (\alpha^{l_i})^{\frac{z}{l_i}} \\ \Rightarrow \bar{Z} < \alpha^z. \end{split}$$

This contradicts expression (6), hence for all variables  $i, \alpha \leq \varphi_i$ . Suppose that there exists a variable i such that  $\varphi_i^z < \overline{Z}$ , then

$$\begin{aligned} \alpha^{z} &\leq \varphi_{i}^{z} < \bar{Z} \leq \left(1 + \alpha^{l_{i} - r_{i}}\right)^{\frac{z}{l_{i}}} \Rightarrow \alpha^{l_{i}} < 1 + \alpha^{l_{i} - r_{i}} \\ &\Rightarrow \alpha^{r_{i} - l_{i}} p_{i}(\alpha) < 0 \\ &\Rightarrow \alpha < \varphi_{i}, \end{aligned}$$

which is a contradiction, hence for all variables  $i,\,\bar{Z}\leq \varphi_i^z.$  In addition, for each variable i,

$$\begin{split} \underline{Z} &\geq \left(\liminf_{G \to \infty} \frac{t(G+r_i)}{t(G)}\right)^{\frac{z}{r_i}} \\ &\geq \left(\liminf_{G \to \infty} \frac{1 + \min_j(t(G+r_i-l_j) + t(G+r_i-r_j))}{t(G)}\right)^{\frac{z}{r_i}} \\ &\geq \left(\min_j \liminf_{G \to \infty} \frac{t(G+r_i-l_j) + t(G+r_i-r_j)}{t(G)}\right)^{\frac{z}{r_i}}. \end{split}$$

Hence there must exist a variable j such that

 $\alpha$ 

$$\underline{Z} \ge \left(1 + \liminf_{G \to \infty} \frac{t(G + r_j - l_j)}{t(G)}\right)^{\frac{z}{r_j}}$$
$$\ge \left(1 + \alpha^{r_j - l_j}\right)^{\frac{z}{r_j}}.$$

Suppose  $\alpha < \varphi_j$ , then, using Theorem 6 in the first line:

$$<\varphi_{j} \Rightarrow p_{j}(\alpha) < 0$$
  
$$\Rightarrow 1 + \alpha^{r_{j}-l_{j}} > \alpha^{r_{j}}$$
  
$$\Rightarrow (1 + \alpha^{r_{j}-l_{j}})^{\frac{z}{r_{j}}} > (\alpha^{r_{j}})^{\frac{z}{r_{j}}}$$
  
$$\Rightarrow \underline{Z} > \alpha^{z}.$$

Since  $\varphi_j^z > \alpha^z$  implies  $\underline{Z} > \alpha^z$ , then  $\underline{Z} \ge \varphi_j^z$  must be true. This can be shortly proven by writing

$$(\varphi_j^z > \alpha^z \Rightarrow \underline{Z} > \alpha^z) \Rightarrow \neg(\varphi_j^z > \alpha^z \land \underline{Z} \le \alpha^z) \Rightarrow \neg(\varphi_j^z > \underline{Z}) \Rightarrow \varphi_j^z \le \underline{Z}.$$

Since  $\varphi_i^z \ge \bar{Z}$  holds for all variables *i*, then variable *j* satisfies  $\varphi_j^z \le \underline{Z} \le \bar{Z} \le \varphi_i^z$  for all variables *i*. We can finally conclude that

$$\lim_{G \to \infty} \sqrt[z]{\frac{t(G+z)}{t(G)}} = \min_{i} \varphi_i^z.$$

Dete	G			1	$linear(\mu$	)				LB
Data	Gap	$\mu =$	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	1	product	ratio	LB
	5000	t-s	0.13	0.15	0.15	0.46	517.40	0.16	0.16	0.11
	5000	wins	99	98	98	96	21	97	97	
В	15000	t-s	6.02	3.49	3.13	2.80	$10^{04}$	2.76	2.76	2.62
Б	15000	wins	89	94	96	98	8	99	99	
	25000	t-s	7.78	3.68	3.18	2.72	$10^{05}$	2.71	2.71	2.71
	25000	wins	88	93	95	99	8	100	100	
	5000	t-s	98.02	2.47	0.96	0.90	1353.05	0.96	0.87	0.73
	5000	wins	16	85	95	89	12	96	93	
U	12500	t-s	544.24	10.22	2.76	3.59	$10^{04}$	2.82	2.79	2.50
U	12500	wins	13	74	99	84	10	97	92	
	20000	t-s	1860.53	16.64	3.58	4.15	$10^{06}$	3.80	3.17	3.09
	20000	wins	13	70	92	87	11	90	95	
	5000	t-s	694.14	7.43	7.49	15.47	1587.97	6.52	5.80	5.49
	5000	wins	20	82	81	61	13	87	92	
V	7500	t-s	2176.92	7.79	7.81	18.20	5596.92	6.75	5.89	5.69
v	1300	wins	19	81	83	64	14	86	94	
	10000	t-s	6456.22	9.48	8.17	22.52	$10^{04}$	7.61	5.81	5.72
	10000	wins	19	76	86	66	15	83	97	
	2000	t-s	626.55	5.64	14.14	36.89	527.77	5.08	5.64	4.79
	2000	wins	11	85	64	45	12	91	85	
х	3000	t-s	1943.09	6.21	16.60	48.57	1198.56	6.76	6.21	5.67
Л	3000	wins	9	88	68	47	12	88	88	
	4000	t-s	5797.74	6.03	18.73	60.70	2449.16	7.30	6.03	5.68
	4000	wins	9	93	67	47	12	89	93	

Table 7: Simulation results on MVB presented as Table 2.

## **B** Additional numerical simulations

We give an additional set of simulations on the MVB model, with the same target gaps as in the simulations on GVB, so that one can compare both experimental setups easily. For the same experiment, we present two different tables, Table 7 and 8, where the results are presented as in Table 2 and 3, respectively. The two differences between Table 7 and 8 are thus the presence (or absence) of the last column LB, and the reference used for relative performance (for Table 7 it is the minimum tree-size, and for Table 8 it is the tree-size produced by product).

These results show that ratio performs generally better than product and linear, and that this phenomenon becomes more significant as the gap to close increases. If we compare Table 8 to Table 3, it appears that ratio is even more at an advantage on GVB than on MVB. One reason may be that, for a given instance, the best variable for ratio and product may be the same, or not very different, and in the MVB experiments this best variable would be branched on at every node. However, in GVB, this variable would be branched on only at the root node, and the subsequent best variables chosen by ratio and product are likely to differ. Indeed, observe that there are many more ties in the MVB experiments than in the GVB experiments.

Data	Can			]	$linear(\mu$	)			
Data	Gap	$\mu =$	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	1	product	ratio
	5000	t-s	-0.03	-0.02	-0.02	0.30	516.40	0.00	0.00
	5000	wins	99	98	98	96	21	97	97
В	15000	t-s	3.18	0.71	0.36	0.04	$10^{04}$	0.00	0.00
D	15000	wins	89	94	96	98	8	99	99
	25000	t-s	4.94	0.94	0.46	0.01	$10^{05}$	0.00	0.00
	23000	wins	88	93	95	99	8	100	100
	5000	t-s	96.14	1.50	0.00	-0.06	1339.26	0.00	-0.09
	5000	wins	16	85	95	89	12	96	93
U	12500	t-s	526.58	7.20	-0.06	0.75	$10^{04}$	0.00	-0.03
U	12000	wins	13	74	99	84	10	97	92
	20000	t-s	1788.75	12.37	-0.21	0.34	$10^{06}$	0.00	-0.61
	20000	wins	13	70	92	87	11	90	95
	5000	t-s	645.51	0.85	0.90	8.40	1484.61	0.00	-0.68
	5000	wins	20	82	81	61	13	87	92
V	7500	t-s	2032.98	0.97	0.99	10.73	5236.79	0.00	-0.81
v	1300	wins	19	81	83	64	14	86	94
	10000	t-s	5992.84	1.74	0.52	13.86	$10^{04}$	0.00	-1.67
	10000	wins	19	76	86	66	15	83	97
	2000	t-s	591.45	0.54	8.62	30.28	497.44	0.00	0.54
	2000	wins	11	85	64	45	12	91	85
Х	3000	t-s	1813.72	-0.52	9.21	39.16	1116.33	0.00	-0.52
$\Lambda$		wins	9	88	68	47	12	88	88
	4000	t-s	5396.50	-1.18	10.65	49.77	2275.74	0.00	-1.18
	4000	wins	9	93	67	47	12	89	93

Table 8: Simulation results on MVB presented as Table 3.

## C Detailed results for the experiments on MIP instances

We present in Section C.1 and C.2 the detailed results for the benchmark and tree test sets, as described in Section 7.2.1 and 7.2.2, respectively. There are three tables in each section, one for each of the scoring function tested, namely **product**, **ratio**, and **svts**. In each table, each line corresponds to all 10 permutations for each instance of the test set. The second column indicates the number of permuted instances solved. We provide statistics in terms of time, number of nodes and LP iterations. For each of these three measures, we give, from left to right in the table, the minimum, average over solved instances (and average over all 10 instances), and maximum. In the node and LP iterations columns, we use the letters k, m and b as a shorthand for thousands, millions and billions. Total and averages are provided at the end of each table. Note that it would not be fair to compare the measures over solved instances in this setting. Indeed, the set of solved instances differ depending on the scoring function considered. Function **ratio** generally solves more hard instances, thus the averages on solved instances have higher values.

## C.1 Benchmark test set results

Tables 9, 10 and 11 give the results on the benchmark test set for the scoring functions product, ratio, and svts, respectively.

Instance	#solved	1	time (s)			nodes		l	LP iterations	
Instance	#solved	min	avg	$\max$	min	avg	$\max$	min	avg	max
10teams	10	1	1 (1)	1	1	1 (1)	1	1.0k	1.2k (1.2k)	1.6k
30n20b8	10	483	521(521)	544	7	10 (10)	15	31.7k	37.5k(37.5k)	48.6k
alc1s1	0	7200	- (7200)	7200	308.8k	- (433.7k)	577.7k	11.9m	- (13.0m)	14.6m
acc-tight5	10	0	Ò (0)	0	0	0 (0)	0	0	0 (0)	0
aflow30a	10	33	36 (36)	42	1.8k	2.5k(2.5k)	4.2k	34.5k	45.1k (45.1k)	64.9k
aflow40b	10	2592	3483(3483)	4952	142.9k	210.9k (210.9k)	289.1k	2.9m	4.4m (4.4m)	$6.0 \mathrm{m}$
air04	10	28	30 (30)	40	6	7 (7)	8	11.6k	13.0k (13.0k)	14.6k
app1-2	10	603	845 (845)	1362	29	565(565)	2.0k	8.5k	28.3k (28.3k)	84.0k
arki001	2	3738	4862 (6732)	7200	1.0m	1.3m(1.7m)	1.9m	5.3m	6.8m (11.5m)	$17.0 \mathrm{m}$
ash608gpia-3col	10	80	87 (87)	111	5	5 (5)	9	8.0k	10.3k (10.3k)	19.1k
atlanta-ip	0	7200	- (7200)	7200	4.1k	- (6.8k)	9.4k	1.9m	- (2.4m)	$2.9\mathrm{m}$
bab5	0	7200	- (7200)	7200	10.8k	-(16.2k)	21.9k	721.2k	- (946.2k)	1.3m
beasleyC3	0	7200	- (7200)	7200	790.1k	- (1.1m)	1.5m	28.5m	- (32.8m)	36.6m
bell5	10	1	1 (1)	1	1.1k	$1.1\dot{k}$ (1.1 $\dot{k}$ )	1.1k	1.3k	1.4k (1.4k)	1.6k
biella1	10	163	217(217)	286	2.0k	2.8k(2.8k)	5.3k	266.8k	355.8k (355.8k)	518.3k
bienst2	10	208	288 (288)	414	82.9k	112.6k (112.6k)	167.1k	2.1m	2.9m (2.9m)	$3.9\mathrm{m}$
binkar10 1	10	268	431 (431)	592	85.4k	135.8k (135.8k)	196.2k	633.3k	1.0m(1.0m)	1.4m
blend2	10	1	2(2)	2	166	222 (222)	305	780	960 (960) <sup>´</sup>	1.2k
bley_xl1	10	1155	2182(2182)	4539	1	1 (1)	1	2.6k	14.4k (14.4k)	97.8k
bnatt350	10	0	0 (0)	0	0	0(0)	0	0	0 (0)	0
cap6000	10	8	8 (8)	8	1.7k	1.8k(1.8k)	1.8k	2.6k	2.7k(2.7k)	2.7k
core2536-691	10	9	11 (11)	13	1	1 (1)	1	9.1k	11.5k (11.5k)	15.5k
cov1075	0	7200	- (7200)	7200	1.1m	- (1.2m)	1.3m	37.5m	- (40.3m)	$44.0 \mathrm{m}$
csched010	9	5048	6151(6256)	7200	561.5k	738.0k (752.7k)	884.3k	28.9m	37.7m (38.4m)	45.0m
danoint	2	7047	7073 (7175)	7200	1.2m	1.2m(1.2m)	1.3m	51.7m	51.9m (53.8m)	$57.0 \mathrm{m}$
dcmulti	10	2	2(2)	3	6	9 (9)	10	1.5k	1.6k (1.6k)	1.7k
dfn-gwin-UUM	10	239	256(256)	291	62.9k	62.9k ( $62.9k$ )	62.9k	1.0m	1.0m(1.0m)	$1.0 \mathrm{m}$
disctom	10	0	1 (1)	2	0	1 (1)	1	0	358 (358)	1.1k
dsbmip	10	1	1 (1)	1	1	1(1)	1	974	1.2k(1.2k)	1.3k
egout	10	0	0 (0)	0	1	1(1)	1	37	39 (39)	42
eil33-2	10	93	103 (103)	123	442	496 (496)	598	24.3k	26.8k (26.8k)	31.0k
eilB101	10	101	134 (134)	204	2.7k	4.4k(4.4k)	8.6k	235.0k	318.5k (318.5k)	480.9k
enigma	10	0	0 (0)	0	0	0 (0)	0	0	0 (0)	0
enlight13	0	7200	-(7200)	7200	4.1m	- (4.6m)	$4.9 \mathrm{m}$	18.7m	- (20.1m)	21.2m
enlight14	0	7200	- (7200)	7200	4.7m	- (5.1m)	5.4m	20.6m	- (22.1m)	22.9m
ex9	10	2	2(2)	3	0	Ò (0)	0	0	0 (0)	0
fast0507	10	116	129(129)	140	412	494 (494)	534	48.1k	54.1k (54.1k)	58.7k
fiber	10	4	5 (5)	5	1	12(12)	51	735	1.1k (1.1k)	1.6k
fixnet6	10	5	5 (5)	5	7	9 (9)	12	1.4k	1.7k (1.7k)	2.2k
flugpl	10	0	0(0)	0	220	221(221)	221	188	188 (188)	188
Table 9: Detailed	results f	or the j	product scorin	ng funct	tion on	the benchmark	c test se	t(contin	nues in the nex	t page).

Instance	#solved		time (s)			nodes			LP iterations	
	" sorred	min	avg	max	min	avg	max	min	avg	max
gen	10	0	0 (0)	0	1	1 (1)	1	139	143 (143)	146
gesa2	10	3	4 (4)	4	2	3 (3)	3	1.2k	1.3k (1.3k)	1.4k
gesa2-o	10	4	4 (4)	4	2	4 (4)	5	1.1k	1.4k(1.4k)	1.5k
gesa2_o	10	4	4(4)	4	4	5(5)	5	1.4k	1.6k(1.6k)	1.7k
gesa3	10	4	4 (4)	4	18	20(20)	22	1.6k	1.7k(1.7k)	1.7k
gesa3_o	10	4	4 (4)	4	7	9 (9)	11	1.4k	1.9k (1.9k)	2.3k
glass4	10	15	29 (29)	45	5.4k	11.5k (11.5k)	18.9k	36.3k	75.2k (75.2k)	130.2k
gmu-35-40	0	7200	- (7200)	7200	$8.1 \mathrm{m}$	- (10.1m)	12.4m	24.9m	- (28.2m)	$33.0\mathrm{m}$
gt2	10	0	0(0)	0	1	1 (1)	1	91	91 (91)	91
harp2	0	7200	-(7200)	7200	4.5m	- (5.0m)	5.9m	5.0m	- (5.7m)	$7.0 \mathrm{m}$
iis-100-0-cov	10	2194	$228\hat{6}$ (2286)	2461	80.4k	83.6k (83.6k)	86.9k	3.3m	3.4m(3.4m)	3.5m
iis-bupa-cov	0	7200	- (7200)	7200	132.0k	- (134.2k)	137.1k	7.8m	- (8.0m)	8.2m
iis-pima-cov	10	737	806 (806)	926	6.1k	6.7k(6.7k)	7.6k	375.4k	411.6k (411.6k)	469.6k
$\hat{\rm khb}05250$	10	1	1 (1)	1	2	7 (7)	9	323	554 (554)	837
l152lav	10	1	2(2)	2	15	16(16)	17	1.5k	1.8k(1.8k)	2.0k
lectsched-4-obj	10	1	1(1)	1	0	0 (0)	0	0	0 (0)	0
lseu	10	2	2(2)	2	971	1.0k(1.0k)	1.1k	3.5k	3.6k(3.6k)	3.7k
m100n500k4r1	10	0	0 (0)	0	1	1 (1)	1	166	203 (203)	260
macrophage	0	7200	-(7200)	7200	955.7k	- (1.1m)	1.2m	17.1m	- (19.0m)	$22.1 \mathrm{m}$
manna81	10	1	1(1)	1	1	1(1)	1	3.1k	3.2k ( $3.2k$ )	3.2k
map18	10	391	462 (462)	578	225	260(260)	289	84.6k	91.1k(91.1k)	98.9k
map20	10	367	422(422)	480	277	300 (300)	329	83.5k	89.9k (89.9k)	95.7k
markshare1	0	7200	-(7200)	7200	22.2m	- (23.1m)	23.7m	45.9m	- (48.0m)	49.3m
markshare2	0	7200	- (7200)	7200	26.4m	- (26.8m)	27.2m	63.7m	- (64.8m)	65.7m
mas74	10	1729	1810 (1810)	1982	3.2m	3.3m(3.3m)	3.3m	12.5m	12.9m (12.9m)	$12.9 \mathrm{m}$
mas76	10	197	200 (200)	203	484.2k	484.2k (484.2k)	484.2k	1.6m	1.6 m (1.6 m)	1.6m
mcsched	10	209	268(268)	369	12.7k	18.1k (18.1k)	25.9k	469.7k	674.1k(674.1k)	$1.0 \mathrm{m}$
mik-250-1-100-1	10	1732	1832 (1832)	1995	787.0k	829.9k (829.9k)	904.1k	2.5m	2.7 m (2.7 m)	$2.9\mathrm{m}$
mine-166-5	10	28	30 (30)	33	186	356(356)	722	3.9k	4.8k (4.8k)	5.7k
mine-90-10	10	338	1228(1228)	2701	37.9k	116.9k (116.9k)	208.2k	236.6k	681.9k (681.9k)	$1.1 \mathrm{m}$
misc03	10	1	2 (2)	2	59	125 (125)	239	1.9k	2.8k (2.8k)	3.9k
misc06	10	1	1 (1)	2	3	7 (7)	10	1.3k	1.8k (1.8k)	2.3k
misc07	10	36	45 (45)	62	17.9k	23.8k (23.8k)	29.6k	140.6k	180.7k (180.7k)	223.9k
mitre	10	45	48 (48)	54	1	1 (1)	1	1.3k	1.4k (1.4k)	1.4k
mod008	10	5	5 (5)	5	11	12(12)	12	544	594(594)	607
mod010	10	1	1(1)	1	2	3 (3)	4	905	1.0k(1.0k)	1.3k
mod011	10	61	74 (74)	88	601	886 (886)	1.5k	49.6k	67.1k(67.1k)	96.1k
modglob	10	1	1(1)'	2	10	20 (20)	27	675	768 (768)	1.0k
momentum1	0	7200	- (7200)	7200	18.6k	-(21.1k)	26.2k	892.2k	- (1.5m)	$1.9\mathrm{m}$
momentum2	1	4417	4417(6922)	7200	13.6k	13.6k (22.9k)	27.8k	426.1k	426.1k (733.2k)	859.9k
blo 0. Dotailad	rogulta f	or the	product scorir	or fund	ion on	the henchmonly	toot an	tloontin	maa in tha nóm	t magaa)

Instance	#solved		time $(s)$			nodes			LP iterations	
		min	avg	max	min	avg	max	min	avg	max
msc98-ip	10	934	1959 (1959)	4088	2.3k	6.3k(6.3k)	16.3k	783.4k	1.9m~(1.9m)	4.3m
mspp16	0	7200	- (7200)	7200	0	- (0)	0	0	- (0)	0
mzzv11	10	672	732(732)	832	12	233(233)	776	48.6k	60.0k ( $60.0k$ )	76.0k
mzzv42z	10	493	554(554)	620	2	11 (11)	20	32.9k	40.5k ( $40.5k$ )	46.5k
n3div36	0	7200	- (7200)	7200	96.8k	- (147.7k)	173.9k	985.7k	- (1.2m)	1.5m
n3seq24	0	7200	- (7200)	7200	82.4k	- (116.2k)	190.0k	1.3m	- (2.1m)	3.3m
n4-3	10	831	1213(1213)	1846	30.8k	46.6k (46.6k)	70.9k	1.3m	1.7m(1.7m)	2.5m
neos-1109824	10	234	395 (395)	632	10.1k	23.2k (23.2k)	40.7k	103.8k	207.0k (207.0k)	353.5k
neos-1337307	0	7200	- (7200)	7200	231.6k	- (268.3k)	307.0k	3.5m	- (3.7m)	$4.0 \mathrm{m}$
neos-1396125	10	236	486 (486)	884	32.1k	69.9k ( $69.9k$ )	129.8k	911.0k	1.8m(1.8m)	3.4m
neos-1601936	10	7	7 (7)	8	1	1 (1)	1	10.4k	11.0k (11.0k)	11.4k
neos-476283	10	314	431 (431)	584	67	131 (131)	201	8.0k	11.3k (11.3k)	16.5k
neos-686190	10	97	142(142)	173	1.6k	1.8k(1.8k)	2.0k	34.2k	37.2k (37.2k)	39.9k
neos-849702	10	0	0 (0)	0	0	0 (0)	0	0	0 (0)	0
neos-916792	0	7200	- (7200)	7200	197.5k	- (239.2k)	280.2k	2.3m	- (2.8m)	3.3m
neos-934278	0	7200	- (7200)	7200	7.8k	- (11.3k)	18.1k	5.6m	- (6.8m)	$7.8 \mathrm{m}$
neos13	10	246	275 (275)	294	12	12 (12)	12	3.0k	3.0k (3.0k)	3.0k
neos18	10	58	76 (76)	119	3.6k	5.3k (5.3k)	8.1k	118.6k	177.5k (177.5k)	296.9k
net12	10	1279	2011 (2011)	2973	2.1k	2.9k (2.9k)	3.7k	1.1m	2.0 m (2.0 m)	3.3m
netdiversion	10	1032	1463(1463)	2132	5	13 (13)	57	30.5k	46.3k (46.3k)	86.3k
newdano	5	4722	5939(6569)	7200	1.1m	1.3m (1.5m)	1.9m	54.7m	69.5m (73.1m)	85.7m
noswot	10	283	573 (573)	1136	367.0k	679.8k (679.8k)	1.0m	1.2m	2.1 m (2.1 m)	3.2m
ns1208400	10	225	414 (414)	626	1	174 (174)	533	142.1k	286.5k (286.5k)	433.0k
ns1688347	10	59	81 (81)	105	6	92 (92)	438	5.8k	23.8k (23.8k)	38.2k
ns1758913	6	1651	3272 (4843)	7200	1	1 (1)	1	29.3k	44.1k (50.4k)	74.6k
ns1766074	10	5324	5740 (5740)	6120	895.3k	925.7k (925.7k)	949.9k	3.4m	3.7 m (3.7 m)	3.8m
ns1830653	10	402	492 (492)	726	15.9k	18.6k (18.6k)	24.6k	1.1m	1.3m (1.3m)	1.9m
nsrand-ipx	3	2411	5004 (6541)	7200	248.9k	584.7k (903.1k)	1.1m	1.6m	3.6m (5.4m)	6.8m
nw04	10	1979	2719(2719)	3271	7	10 (10)	12	1.3k	1.6k (1.6k)	1.9k
opm2-z7-s2	10	679	858 (858)	1106	707	977 (977)	1.4k	43.2k	64.4k (64.4k)	86.8k
opt1217	5	2	2(3601)	7200	1	1 (1.7m)	4.4m	747	798 (9.3m)	29.5m
p0033	10	0	0 (0)	0	1	1 (1)	1	57	57 (57)	57
p0201	10	2	3(3)	3	4	10(10)	39	1.1k	1.4k (1.4k)	2.8k
p0282	10	1	2(2)	2	2	3(3)	3	459	649 (649)	890
p0548	10	1	$\frac{-}{1}(\frac{-}{1})$	1	1	1(1)	1	331	331(331)	331
p2756	10	4	4(4)	4	10	10(10)	10	538	547(547)	555
pg5_34	10	1257	1397(1397)	1824	116.0k	116.0k (116.0k)	116.0k	2.2m	2.2m (2.2m)	2.2m
pigeon-10	0	7200	-(7200)	7200	929.9k	-(1.1m)	1.1m	4.2m	- (6.5m)	9.1m
pigeon-10 pk1	10	1200	123(123)	126	325.8k	325.8k (325.8k)	325.8k	1.9m	1.9m (1.9m)	1.9m
pp08a	10	2	2(2)	3	107	237 (237)	350	2.0k	3.3k (3.3k)	5.1k
la 0. Datailad	rogulta f	$\int_{0}^{1} \frac{1}{t} h o$	product scori	ng funct	tion on	the benchmark	r tost sc	t (conti	nies in the nev	t name

Instance	#solved		time (s)			nodes			LP iterations	
mstance	#301764	min	avg	max	min	avg	max	min	avg	max
pp08aCUTS	10	2	2 (2)	3	75	153 (153)	245	1.8k	2.5k (2.5k)	3.8k
protfold	0	7200	-(7200)	7200	15.6k	- (30.4k)	48.7k	15.0m	- (21.9m)	26.5m
pw-myciel4	0	7200	- (7200)	7200	482.7k	- (561.1k)	624.2k	19.4m	- (21.9m)	24.7m
qiu	10	78	$10\dot{6}$ (10 $\dot{6}$ )	125	9.6k	12.5k (12.5k)	14.3k	304.7k	410.6k (410.6k)	495.1k
qnet1	10	9	11 (11)	12	3	4 (4)	6	3.7k	4.0k (4.0k)	4.5k
qnet1 o	10	5	6 (6)	6	3	4(4)	6	2.5k	2.8k(2.8k)	3.3k
rail507	10	114	130 (130)	156	450	507 (507)	540	51.6k	54.6k (54.6k)	56.3k
ran16x16	10	491	544(544)	606	322.5k	349.3k (349.3k)	397.2k	2.6m	2.9m (2.9m)	3.2m
rd-rplusc-21	0	7200	-(7200)	7200	42.8k	- (70.9k)	174.8k	450.1k	- (729.0k)	1.6m
reblock67	10	203	250 (250)	312	41.6k	47.9k (47.9k)	57.5k	459.6k	504.7k (504.7k)	618.0k
rentacar	10	5	6 (6)	7	2	2 (2)	2	2.8k	3.2k (3.2k)	3.8k
rgn	10	1	1 (1)	1	1	1(1)	2	327	392 (392)	556
rmatr100-p10	10	106	109 (109)	116	789	799 (799)	805	64.1k	64.2k ( $64.2k$ )	64.3k
rmatr100-p5	10	128	140(140)	160	387	387 (387)	387	67.5k	67.6k (67.6k)	67.6k
rmine6	3	3540	4297 (6329)	7200	380.6k	461.5k (780.9k)	985.0k	2.6m	3.1 m (5.6 m)	7.0m
rocII-4-11	10	352	419 (419)	522	3.4k	4.7k (4.7k)	7.0k	71.6k	84.6k (84.6k)	110.5k
rococoC10-001000	1	4677	4677 (6947)	7200	37.1k	323.9k (309.5k)	562.8k	13.2m	13.2m (22.6m)	32.2m
roll3000	9	1543	4724 (4971)	7200	150.3k	529.1k (551.7k)	804.1k	2.3m	6.9m (7.3m)	10.8m
rout	10	57	111 (111)	186	18.8k	44.6k (44.6k)	68.9k	263.8k	614.3k (614.3k)	930.2k
satellites1-25	10	925	1590 (1590)	2011	1.1k	3.4k (3.4k)	5.7k	149.4k	765.6k (765.6k)	1.5m
set1ch	10	2	2 (2)	3	3	5 (5)	7	987	1.1k (1.1k)	1.5k
seymour	0	7200	- (7200)	7200	101.9k	- (111.6k)	120.6k	7.3m	- (7.9m)	8.4m
sp97ar	ő	7200	- (7200)	7200	118.5k	-(155.3k)	179.1k	4.0m	- (4.8m)	5.4m
sp98ic	10	2283	4736 (4736)	6527	123.0k	276.2k (276.2k)	389.0k	2.0m	4.3m (4.3m)	6.3m
sp98ir	10	57	71 (71)	85	1.2k	1.6k (1.6k)	1.9k	30.6k	35.6k (35.6k)	42.2k
stein27	10	2	2(2)	2	3.7k	4.2k (4.2k)	4.5k	13.7k	14.4k (14.4k)	14.9k
stein45	10	32	34(34)	38	48.6k	51.5k (51.5k)	54.3k	260.0k	267.4k (267.4k)	272.8k
swath	0	7200	- (7200)	7200	107.7k	- (147.6k)	194.0k	1.4m	- (1.8m)	2.4m
tanglegram1	7	833	2464(3885)	7200	21	745 (1.6k)	3.8k	118.3k	596.8k (848.4k)	1.7m
tanglegram2	10	7	10(10)	17	3	4 (4)	7	6.0k	6.8k (6.8k)	9.1k
timtab1	10	767	1184(1184)	1512	532.7k	774.5k (774.5k)	$1.0 \mathrm{m}$	4.4m	6.0 m (6.0 m)	8.0m
timtab2	0	7200	- (7200)	7200	3.0m	- (3.3m)	3.6m	34.6m	- (36.6m)	39.1m
tr12-30	10	3006	3644 (3644)	4240	685.0k	780.2k (780.2k)	900.4k	3.3m	3.7m(3.7m)	4.1m
triptim1	10	178	203 (203)	232	1	1 (1)	1	30.7k	33.8k (33.8k)	35.9k
unitcal 7	9	1576	3415(3794)	7200	11.8k	32.0k (35.9k)	71.3k	204.8k	499.0k (547.7k)	1.2m
vpm1	10	0	0 (0)	0	11.0K	1(1)	1	59	499.0K (041.1K) 60 (60)	60
vpm2	10	2	$ \begin{array}{c} 0 & (0) \\ 2 & (2) \end{array} $	2	151	180(180)	207	1.6k	1.8k (1.8k)	2.1k
vpm2 vpphard	10	7200	-(7200)	$\frac{2}{7200}$	814	-(8.3k)	207 21.9k	1.0K	-(1.5m)	1.7m
zib54-UUE	3	3911	-(7200) 4914 (6514)	7200 7200	378.8k	-(0.3K) 413.0k (592.5k)	21.9k 734.1k	1.5m 17.1m	-(1.5m) 20.7m (27.9m)	32.9m
			product scorin	1200	. 310.0K	+1.0.0K (092.0K)	134.1K	L 11.111	20.111 (21.911)	52.911

Instance	#solved		time (s)			nodes			LP iterations	
Instance	#sorveu	min	avg	$\max$	min	avg	max	min	avg	max
Total	1225		809064 (3437062)			128.2m(1.0b)			1.7b (7.6b)	
Arit. mean			660(2162)			104.7k (660.0k)			1.4m~(4.8m)	
Sh. geo. mean			87 (251)			1.3k (4.4k)			33.4k (111.4k)	

Table 9: Detailed results for the product scoring function on the benchmark test set.

Instance	#solved		time (s)			nodes			LP iterations	
Instance	#solved	min	avg	$\max$	min	avg	$\max$	min	avg	max
10teams	10	0	1 (1)	1	1	1 (1)	1	1.0k	1.2k (1.2k)	1.6k
30n20b8	10	485	529(529)	571	7	15(15)	34	32.6k	37.1k(37.1k)	42.9k
alc1s1	0	7200	- (7200)	7200	277.0k	- (430.3k)	545.7k	10.7m	- (13.4m)	$15.1 \mathrm{m}$
acc-tight5	10	0	0(0)	0	0	0 (0)	0	0	0 (0)	0
aflow30a	10	34	35 (35)	39	1.7k	2.5k(2.5k)	4.4k	34.2k	43.9k (43.9k)	61.4k
aflow40b	10	1596	2372(2372)	2974	93.8k	137.2k (137.2k)	171.5k	2.0m	2.9m(2.9m)	3.6m
air04	10	27	29 (29)	34	6	7 (7)	8	11.6k	13.0k (13.0k)	14.6k
app1-2	10	619	758 (758)	986	29	572(572)	2.0k	8.4k	23.6k (23.6k)	65.9k
arki001	2	3386	3566(6473)	7200	954.3k	971.4k (1.7m)	$2.1\mathrm{m}$	5.1m	5.1 m (10.2 m)	15.8m
ash608gpia-3col	10	80	91 (91)	108	5	5 (5)	9	8.0k	10.3k (10.3k)	19.1k
atlanta-ip	0	7200	- (7200)	7200	5.8k	- (7.3k)	9.2k	1.8m	- (2.4m)	$2.7\mathrm{m}$
bab5	0	7200	- (7200)	7200	15.3k	- (20.6k)	36.9k	710.2k	- (1.0m)	1.5m
beasleyC3	0	7200	- (7200)	7200	1.0m	- (1.2m)	1.5m	21.9m	- (27.0m)	34.5m
bell5	10	1	1 (1)	1	1.1k	1.1k (1.1k)	1.1k	1.3k	1.4k (1.4k)	1.6k
biella1	10	158	214(214)	269	1.9k	3.0k(3.0k)	4.9k	250.9k	341.5k (341.5k)	434.8k
bienst2	10	237	301 (301)	391	86.0k	110.2k (110.2k)	161.3k	2.2m	2.9m(2.9m)	$3.9\mathrm{m}$
binkar10_1	10	290	470 (470)	797	90.5k	140.5k (140.5k)	204.9k	675.9k	1.0m(1.0m)	1.5m
blend2	10	1	2(2)	2	183	221 (221)	313	820	975(975)	1.1k
bley_xl1	10	1124	2010(2010)	3012	1	1 (1)	1	2.6k	14.4k (14.4k)	97.8k
bnatt350	10	0	0 (0)	0	0	0(0)	0	0	0 (0)	0
cap6000	10	8	8 (8)	8	1.8k	1.8k(1.8k)	1.9k	2.6k	2.7k(2.7k)	2.8k
core2536-691	10	10	11 (11)	12	1	1 (1)	1	9.1k	11.5k (11.5k)	15.5k
cov1075	0	7200	- (7200)	7200	1.1m	- (1.2m)	1.4m	37.8m	- (41.0m)	44.4m
csched010	8	4327	5206(5605)	7200	503.2k	616.8k (673.5k)	947.6k	25.9m	32.0m (35.1m)	48.9m
danoint	1	7061	7061 (7186)	7200	1.1m	1.2m(1.2m)	1.2m	50.1m	50.1m (53.8m)	55.8m
dcmulti	10	2	2(2)	3	6	9 (9)	10	1.5k	1.6k (1.6k)	1.7k
dfn-gwin-UUM	10	250	265(265)	295	67.3k	67.3k (67.3k)	67.3k	1.0m	1.0m(1.0m)	$1.0\mathrm{m}$
disctom	10	0	1 (1)	2	0	1 (1)	1	, 0	358 (358)	1.1k
Table 10: Detail	ed results	s for th	e ratio scorin	ig funct	tion on	the benchmark	test set	(contin	ues in the next	page).

Instance	#solved	I	time (s)		I	nodes			LP iterations	
mstance	#solveu	min	avg	$\max$	min	avg	max	min	avg	max
dsbmip	10	1	1 (1)	1	1	1 (1)	1	974	1.2k (1.2k)	1.3k
egout	10	0	0 (0)	0	1	1(1)	1	37	39 (39)	42
eil33-2	10	89	100 (100)	125	438	492 (492)	518	22.5k	24.3k (24.3k)	26.3k
eilB101	10	89	110(110)	126	2.5k	3.7k(3.7k)	5.6k	173.7k	241.1k (241.1k)	314.8k
enigma	10	0	0 (0)	0	0	0 (0)	0	0	0 (0)	0
enlight13	10	2225	2414(2414)	2630	924.0k	924.0k (924.0k)	924.0k	4.2m	4.2m (4.2m)	4.2m
enlight14	0	7200	- (7200)	7200	3.8m	- (4.2m)	4.5m	18.1m	- (19.2m)	20.2m
ex9	10	2	2(2)	3	0	0(0)	0	0	0 (0)	0
fast0507	10	118	127(127)	140	462	585 (585)	644	52.2k	59.6k (59.6k)	66.7k
fiber	10	4	5(5)	5	1	12(12)	51	735	1.1k (1.1k)	1.6k
fixnet6	10	5	5(5)	5	7	9(9)	12	1.4k	1.7k (1.7k)	2.2k
flugpl	10	0	0 (0)	0	229	229(229)	229	187	187 (187)	187
gen	10	0	0(0)	0	1	1(1)	1	139	143(143)	146
gesa2	10	4	4(4)	4	2	3(3)	3	1.2k	1.3k (1.3k)	140 1.4k
gesa2-o	10	4	$\frac{4}{4}(4)$	4	2	4(4)	5	1.1k	1.4k (1.4k)	1.5k
gesa2_o	10	4	4(4)	5	4	5(5)	5	1.4k	1.6k (1.6k)	1.7k
gesa3	10	4	$\frac{4}{4}$ (4)	4	18	20(20)	22	1.4k	1.7k (1.7k)	1.7k
gesa3 o	10	4	4(4) 4(4)	4 5	7	9(9)	11	1.0k 1.4k	1.9k (1.9k)	2.3k
glass4	10	17	36(36)	80	5.5k	15.4k (15.4k)	35.9k	41.5k	102.8k (102.8k)	242.0k
gmu-35-40	0	7200	- (7200)	7200	8.9m	-(10.6m)	11.8m	24.2m	-(29.9m)	242.0k 37.1m
		1								
gt2	10	0	0 (0)     (7200)	0	1	1(1)	1	91	91(91)	91
harp2	0	7200	- (7200)	7200	5.1m	-(5.8m)	6.5m	5.6m	-(6.8m)	8.2m
iis-100-0-cov	10	2119	2217(2217)	2318	79.0k	82.9k (82.9k)	86.9k	3.2m	3.4m(3.4m)	3.6m
iis-bupa-cov	0	7200	- (7200)	7200	129.9k	- (133.5k)	137.8k	7.8m	- (8.0m)	8.2m
iis-pima-cov	10	770	804 (804)	863	6.1k	6.4k (6.4k)	6.9k	375.9k	397.2k (397.2k)	422.0k
khb05250	10	1	1(1)	1	2	7 (7)	9	323	554 (554)	837
l152lav	10	1	2(2)	2	15	17(17)	19	1.5k	1.8k(1.8k)	2.0k
lectsched-4-obj	10	1	1(1)	1	0	0 (0)	0	0	0 (0)	0
lseu	10	1	1(1)	2	827	857 (857)	910	3.1k	3.2k ( $3.2k$ )	3.2k
m100n500k4r1	10	0	0 (0)	0	1	1 (1)	1	166	203(203)	260
macrophage	0	7200	- (7200)	7200	961.8k	- (1.1m)	1.2m	14.1m	- (17.8m)	21.8m
manna81	10	1	1(1)	1	1	1(1)	1	3.1k	3.2k ( $3.2k$ )	3.2k
map18	10	406	477 (477)	552	219	258 (258)	289	83.6k	90.9k (90.9k)	98.9k
map20	10	384	445 (445)	630	279	300 (300)	329	83.9k	89.7k (89.7k)	95.5k
markshare1	0	7200	- (7200)	7200	22.6m	- (23.5m)	23.9m	46.1m	- (48.0m)	48.8m
markshare2	0	7200	- (7200)	7200	23.9m	- (24.8m)	25.6m	57.2m	- (59.4m)	61.5m
mas74	10	1518	1566 (1566)	1672	2.9m	2.9m(2.9m)	$2.9\mathrm{m}$	11.3m	11.3m (11.3m)	11.3m
mas76	10	160	165(165)	178	403.2k	403.2k (403.2k)	403.2k	1.3m	1.3 m (1.3 m)	1.3m
mcsched	10	174	266(266)	409	10.2k	18.3k (18.3k)	29.5k	378.0k	657.4k (657.4k)	$1.0\mathrm{m}$
mik-250-1-100-1	10	2025	2166(2166)	2290	901.6k	971.8k (971.8k)	$1.1 \mathrm{m}$	3.0m	3.3 m (3.3 m)	3.5m
Table 10: Detail	ed result	s for th	e ratio scórin	g func	tion on	the benchmark	test set	(continu	ues in the next	page).
				0				`		- 0 /

Instance	#solved		time (s)		I	nodes			LP iterations	
mstance	#solved	min	avg	$\max$	min	avg	max	min	avg	max
mine-166-5	10	28	31 (31)	33	186	340 (340)	572	3.9k	4.7k (4.7k)	5.7k
mine-90-10	10	327	1129(1129)	2565	32.6k	112.7k (112.7k)	203.9k	209.1k	643.3k (643.3k)	1.1m
misc03	10	1	2 (2)	2	77	128 (128)	225	2.2k	2.9k (2.9k)	3.9k
misc06	10	1	$\frac{1}{1}(1)$	2	3	7 (7)	10	1.3k	1.8k (1.8k)	2.3k
misc07	10	45	56 (56)	69	24.2k	30.6k (30.6k)	35.5k	177.9k	236.3k (236.3k)	273.8k
mitre	10	44	49(49)	58	1	1 (1)	1	1.3k	1.4k (1.4k)	1.4k
mod008	10	5	5(5)	5	12	13(13)	15	580	601 (601)	607
mod010	10	1	1(1)	1	2	3(3)	4	905	1.0k (1.0k)	1.3k
mod011	10	61	72(72)	91	549	784 (784)	997	47.6k	62.4k ( $62.4k$ )	75.8k
modglob	10	1	1 (1)	2	10	21(21)	27	694	769 (769)	1.0k
momentum1	0	7200	- (7200)	7200	20.2k	-(23.1k)	26.8k	1.2m	-(1.6m)	2.1m
momentum2	2	5734	6457(7051)	7200	16.0k	19.8k (24.7k)	20.6k	469.3k	516.5k (723.6k)	1.1m
msc98-ip	10	969	1760(1760)	4134	2.8k	6.3k (6.3k)	18.1k	743.4k	1.6m (1.6m)	4.1m
mspp16	0	7200	- (7200)	7200	0	-(0)	0	0	- (0)	4.1III 0
mspp10 mzzv11	10	631	724(724)	778	12	206 (206)	966	48.6k	58.3k (58.3k)	76.3k
mzzv42z	10	497	557 (557)	628	2	12(12)	20	32.9k	40.4k (40.4k)	46.5k
n3div36	0	7200	- (7200)	7200	99.5k	-(144.6k)	181.8k	777.6k	-(1.2m)	1.6m
n3seq24	0	7200	- (7200)	7200 7200	77.7k	-(144.0k) -(110.4k)	181.8K 184.5k	1.3m	-(1.211) -(1.9m)	$2.4 \mathrm{m}$
n3seq24 n4-3	10	868	1260(1260)	$1200 \\ 1780$	30.4k	46.0k (46.0k)	64.3k	1.3m 1.2m	1.8m (1.8m)	2.4m 2.3m
neos-1109824	10	131	205(205)	292	4.7k	8.5k (8.5k)	04.5ĸ 15.0k	54.2k	86.0k (86.0k)	2.5m 141.0k
neos-1337307	0	7200	-(7200)	7200	233.7k	-(265.2k)	305.7k	3.1m	-(3.6m)	4.4m
neos-1396125	10	276	484 (484)	911	38.4k	71.0k (71.0k)	141.8k	1.1m	1.7m(1.7m)	3.1m
neos-1601936	10	7	7 (7)	8	1	1(1)	1	10.4k	11.0k (11.0k)	11.4k
neos-476283	10	318	380(380)	569	75	129(129)	201	8.0k	11.3k (11.3k)	16.4k
neos-686190	10	101	161(161)	189	1.9k	2.2k (2.2k)	2.6k	36.1k	41.2k (41.2k)	45.1k
neos-849702	10	0	0 (0)	0	0	0 (0)	0	0	0(0)	0
neos-916792	0	7200	- (7200)	7200	142.1k	- (180.5k)	206.6k	1.7m	- (2.2m)	2.5m
neos-934278	0	7200	- (7200)	7200	7.3k	- (9.5k)	13.9k	5.9m	- (6.6m)	7.8m
neos13	10	241	262(262)	298	10	12(12)	12	3.0k	3.0k(3.0k)	3.0k
neos18	10	74	97(97)	120	6.1k	8.4k(8.4k)	11.6k	166.7k	228.9k ( $228.9k$ )	299.4k
net12	10	1608	2077 (2077)	2447	2.1k	3.1k(3.1k)	4.0k	1.5m	2.1 m (2.1 m)	2.6m
netdiversion	10	1060	1413(1413)	2078	5	12(12)	49	30.0k	50.1k ( $50.1k$ )	138.1k
newdano	4	4309	5757 (6623)	7200	1.1m	1.2m~(1.5m)	$2.0\mathrm{m}$	52.7m	66.2m (72.2m)	86.7m
noswot	10	327	$880 \ (880)$	3601	436.9k	1.1m(1.1m)	4.3m	1.4m	2.8m(2.8m)	8.6m
ns1208400	10	315	456(456)	650	1	210(210)	702	142.1k	337.6k (337.6k)	646.9k
ns1688347	10	58	82(82)	115	6	105(105)	322	5.8k	24.3k (24.3k)	44.4k
ns1758913	7	2224	4119(5043)	7200	1	1 (1)	1	29.3k	50.0k (55.8k)	88.0k
ns1766074	10	5197	5611(5611)	6066	895.4k	907.9k (907.9k)	942.4k	3.5m	3.6m(3.6m)	3.8m
ns1830653	10	367	484 (484)	791	14.9k	19.0k (19.0k)	33.8k	933.0k	1.3 m (1.3 m)	2.3m
nsrand-ipx	10	808	3768 (3768)	7097	59.9k	464.5k (464.5k)	830.8k	508.6k	3.1 m (3.1 m)	6.0 m
able 10: Detail	ed results	s for t	he <b>ratiò</b> scórin	g funct	tion on	the benchmark	test set	(continu	ues in the next	page).
				~						- 0 /

Instance	#solved		time (s)			nodes			LP iterations	
Instance	#solved	min	avg	$\max$	min	avg	max	$\min$	avg	max
nw04	10	1989	2283 (2283)	3039	7	10 (10)	12	1.3k	1.6k (1.6k)	1.9k
opm2-z7-s2	10	705	883 (883)	1092	537	976 (976)	1.3k	35.3k	68.7k (68.7k)	102.0k
opt1217	5	2	2(3601)	7200	1	1 (1.6m)	3.6m	747	798 (8.7m)	22.6m
p0033	10	0	0 (0)	0	1	1 (1)	1	57	57 (57)	57
p0201	10	2	3 (3)	3	4	10 (10)	33	1.1k	1.4k (1.4k)	2.5k
p0282	10	1	1(1)	$\tilde{2}$	2	3 (3)	3	459	649 (649)	890
p0548	10	1	1(1)	1	1	1(1)	1	331	331 (331)	331
p2756	10	4	4(4)	4	10	10 (10)	10	538	547 (547)	555
pg5 34	10	1243	1304 (1304)	1365	107.6k	107.6k (107.6k)	107.6k	2.0m	2.0m(2.0m)	$2.0\mathrm{m}$
pigeon-10	0	7200	- (7200)	7200	832.7k	- (1.0m)	1.2m	4.3m	- (5.8m)	$7.1 \mathrm{m}$
pk1	10	137	139 (139)	143	369.8k	369.8k (369.8k)	369.8k	$2.1 \mathrm{m}$	2.1 m (2.1 m)	$2.1 \mathrm{m}$
pp08a	10	2	2(2)	3	111	240 (240)	402	2.1k	3.3k (3.3k)	5.7k
pp08aCUTS	10	2	2(2)	3	75	155(155)	255	1.8k	2.5k(2.5k)	3.7k
protfold	0	7200	-(7200)	7200	21.6k	- (35.7k)	43.9k	15.7m	- (20.8m)	25.3m
pw-myciel4	7	3923	5559 (6051)	7200	447.2k	583.7k (607.0k)	780.5k	10.8m	15.5m (16.9m)	21.2m
qiu	10	79	106 (106)	134	10.2k	12.7 k (12.7k)	15.7k	315.4k	416.9k (416.9k)	522.3k
qnet1	10	9	10 (10)	12	3	4 (4)	6	3.7k	4.0k (4.0k)	4.5k
qnet1 o	10	5	5(5)'	6	3	4(4)	6	2.5k	2.8k(2.8k)	3.3k
rail507	10	113	140(140)	170	446	596 (596)	678	49.1k	60.4k(60.4k)	67.9k
ran16x16	10	405	444 (444)	522	245.1k	286.9k (286.9k)	332.6k	$2.1 \mathrm{m}$	2.4 m (2.4 m)	$2.9\mathrm{m}$
rd-rplusc-21	1	4462	4462 (6926)	7200	38.4k	135.5k(69.2k)	135.5k	248.3k	584.3k (750.0k)	$2.0\mathrm{m}$
reblock67	10	174	245(245)'	304	42.7k	48.7k (48.7k)	55.4k	451.0k	517.6k (517.6k)	571.4k
rentacar	10	5	6 (6)	6	2	2(2)	2	2.8k	3.2k(3.2k)	3.8k
rgn	10	1	1(1)	1	1	1(1)	2	327	392 (392)	556
rmatr100-p10	10	106	112(112)	124	789	799 (799)	805	64.1k	64.2k ( $64.2k$ )	64.3k
rmatr100-p5	10	128	134(134)	141	383	383 (383)	383	66.8k	66.8k(66.8k)	66.8k
rmine6	4	3898	4981 (6312)	7200	398.0k	519.3k (731.9k)	944.1k	2.5m	3.5 m (5.1 m)	6.8m
rocII-4-11	10	631	1019(1019)	1566	13.1k	27.6k (27.6k)	50.1k	162.4k	303.4k (303.4k)	523.0k
rococoC10-001000	2	3864	4825(6725)	7200	121.8k	354.5k (343.0k)	531.1k	10.2m	12.1 m (23.3 m)	37.9m
roll3000	7	2335	4709(5456)	7200	255.7k	558.7k (639.5k)	868.2k	3.6m	6.8 m (8.0 m)	12.5m
rout	10	30	81 (81)	160	7.9k	32.6k (32.6k)	66.1k	167.6k	484.9k (484.9k)	948.0k
satellites1-25	10	915	1636(1636)	1928	1.1k	3.4k (3.4k)	7.2k	124.0k	855.6k (855.6k)	1.6m
set1ch	10	2	2(2)	3	3	5 (5)	7	987	1.1k (1.1k)	1.5k
seymour	0	7200	-(7200)	7200	107.9k	- (112.7k)	130.3k	7.4m	- (8.0m)	8.6m
sp97ar	0	7200	- (7200)	7200	130.1k	- (153.3k)	174.3k	4.5m	- (4.9m)	5.2m
sp98ic	10	1287	2309 (2309)	3184	61.1k	117.2k (117.2k)	168.3k	941.4k	1.9 m (1.9 m)	2.8m
sp98ir	10	60	69 (69)	75	1.5k	1.8k (1.8k)	2.1k	32.2k	41.8k (41.8k)	46.7k
stein27	10	2	2(2)'	2	3.8k	4.1k (4.1k)	4.4k	13.4k	14.2k (14.2k)	14.7k
stein45	10	31	32(32)	34	48.3k	50.3k(50.3k)	53.0k	254.3k	262.4k ( $262.4k$ )	271.7k
swath	0	7200	- (7200)	7200	104.6k	- (160.7k)	214.4k	1.5m	- (2.0m)	2.6m
able 10. Detail	ed results	s for th	ie ratio scorin	or funct	tion on	the benchmark	tost sot	(contini	los in the next	naga)

Instance	#solved	time (s)			nodes			LP iterations			
Instance	#301764	min	avg	max	min	avg	max	min	avg	max	
tanglegram1	6	695	2543 (4406)	7200	21	573 (1.2k)	3.1k	118.3k	500.2k (864.5k)	1.6m	
tanglegram2	10	7	10 (10)	17	3	4 (4)	7	6.0k	6.8k(6.8k)	9.1k	
timtab1	10	856	1235(1235)	1607	592.7k	803.6k (803.6k)	993.3k	4.6m	6.3m(6.3m)	$7.9\mathrm{m}$	
timtab2	0	7200	- (7200)	7200	3.2m	- (3.5m)	3.8m	31.6m	- (34.8m)	39.2m	
tr12-30	10	3100	3707(3707)	4222	714.9k	846.1k (846.1k)	969.6k	3.3m	4.0m~(4.0m)	4.6m	
triptim1	10	169	188 (188)	232	1	1 (1)	1	30.7k	33.8k (33.8k)	35.9k	
unitcal_7	10	1465	2796(2796)	4669	10.3k	27.4k (27.4k)	58.0k	195.9k	455.2k (455.2k)	987.2k	
vpm1	10	0	0(0)	0	1	1 (1)	1	59	60(60)	60	
vpm2	10	2	2(2)	2	179	190(190)	205	1.7k	1.8k(1.8k)	2.0k	
vpphard	0	7200	- (7200)	7200	838	- (3.5k)	7.2k	1.2m	- (1.6m)	$1.7\mathrm{m}$	
zib54-UUE	3	5441	5899(6810)	7200	437.3k	487.7k (608.2k)	745.6k	23.7m	25.3m (29.3m)	32.4m	
Total	1249		849097 (3304297)			138.5m (996.1m)			1.6b (7.1b)		
Arit. mean			680 (2078)			110.9k (626.5k)			1.3m~(4.5m)		
Sh. geo. mean			92(245)			1.4k (4.4k)			36.2k (108.8k)		

Table 10: Detailed results for the **ratio** scoring function on the benchmark test set.

48

Instance	#solved		time (s)			nodes			LP iterations	
mstance	#solved	$\min$	avg	$\max$	min	avg	$\max$	min	avg	max
10teams	10	1	1 (1)	1	1	1 (1)	1	1.0k	1.2k (1.2k)	1.6k
30n20b8	10	479	526(526)	605	7	14(14)	36	32.6k	37.5k (37.5k)	41.7k
alc1s1	0	7200	- (7200)	7200	308.8k	- (420.7k)	510.0k	11.5m	- (13.8m)	15.4m
acc-tight5	10	0	0 (0)	0	0	0 (0)	0	0	0 (0)	0
aflow30a	10	31	35(35)	38	1.7k	2.2k (2.2k)	3.3k	32.9k	41.2k (41.2k)	52.5k
aflow40b	10	1533	2437(2437)	3328	79.0k	133.3k (133.3k)	212.9k	2.0m	3.2m(3.2m)	$4.9\mathrm{m}$
air04	10	28	31(31)	35	6	7 (7)	8	11.6k	13.0k (13.0k)	14.6k
app1-2	10	601	754 (754)	1061	29	645(645)	3.1k	8.8k	27.5k (27.5k)	105.4k
arki001	3	2992	5105(6571)	7200	747.9k	1.3m (1.5m)	1.9m	4.4m	6.3m(10.4m)	15.5m
ash608gpia-3col	10	79	88 (88)	103	5	5 (5)	9	8.0k	10.3k (10.3k)	19.1k
atlanta-ip	0	7200	- (7200)	7200	4.2k	- (7.0k)	8.5k	1.6m	- (2.3m)	$2.8\mathrm{m}$
bab5	0	7200	- (7200)	7200	11.6k	- (17.6k)	24.6k	658.3k	- (990.6k)	1.3m
beasleyC3	0	7200	- (7200)	7200	763.6k	- (966.1k)	1.3m	25.9m	- (29.9m)	33.1m
bell5	10	1	1 (1)	1	1.1k	1.1k (1.1k)	1.1k	1.3k	1.4k (1.4k)	1.6k
biella1	10	150	215(215)	281	1.5k	2.8k(2.8k)	4.6k	230.7k	354.3k (354.3k)	492.2k
bienst2	10	232	320 (320)	436	84.8k	123.1k (123.1k)	174.1k	2.2m	3.3m (3.3m)	4.5m
binkar10_1	10	248	437 (437)	729	76.9k	133.1k (133.1k)	216.8k	576.7k	1.0m(1.0m)	1.7m
Table 11: Detail	ed results	s for th	ie svts scoring	g functi	on on t	he benchmark	test set	(continu	ues in the next	page).

Instance	#solved		time $(s)$			nodes			LP iterations	
		min	avg	max	min	avg	max	min	avg	max
blend2	10	1	2 (2)	2	166	225 (225)	346	796	993 (993)	1.3k
bley_xl1	10	1102	1916 (1916)	2738	1	1 (1)	1	2.6k	14.4k (14.4k)	97.8k
bnatt350	10	0	0 (0)	0	0	0(0)	0	0	0(0)	0
cap6000	10	8	8 (8)	9	1.8k	1.9k(1.9k)	1.9k	2.6k	2.7k(2.7k)	2.8k
core2536-691	10	10	11 (11)	13	1	1 (1)	1	9.1k	11.5k (11.5k)	15.5k
cov1075	0	7200	- (7200)	7200	$1.1 \mathrm{m}$	- (1.2m)	1.3m	36.5m	- (40.5m)	44.1m
csched010	9	4332	5550(5715)	7200	483.4k	616.0k (632.5k)	780.7k	27.1m	34.7m (35.7m)	$44.1 \mathrm{m}$
danoint	1	7182	7182(7198)	7200	$1.1 \mathrm{m}$	1.2m (1.2m)	1.2m	50.4m	50.4m(54.1m)	$58.0 \mathrm{m}$
dcmulti	10	2	2(2)	2	6	9 (9)	10	1.5k	1.6k(1.6k)	1.7k
dfn-gwin-UUM	10	245	269(269)	304	60.2k	60.2k ( $60.2k$ )	60.2k	999.8k	999.8k (999.8k)	999.8k
disctom	10	0	1 (1)	2	0	1 (1)	1	0	358(358)	1.1k
dsbmip	10	1	1(1)	1	1	1(1)	1	974	1.2k (1.2k)	1.3k
egout	10	0	0(0)	0	1	1(1)	1	37	39 (39)	42
eil33-2	10	98	114 (114)	142	444	482 (482)	532	23.7k	26.4k (26.4k)	28.9k
eilB101	10	96	118 (118)	135	2.0k	3.4k(3.4k)	5.7k	203.7k	284.9k (284.9k)	360.5k
enigma	10	0	0 (0)	0	0	0 (0)	0	0	0 (0)	0
enlight13	0	7200	- (7200)	7200	3.8m	- (4.0m)	4.3m	18.7m	- (19.9m)	20.7m
enlight14	0	7200	- (7200)	7200	$4.0 \mathrm{m}$	- (4.3m)	4.6m	18.8m	- (19.6m)	20.6m
ex9	10	2	2(2)	3	0	0 (0)	0	0	0(0)	0
fast0507	10	113	128(128)	143	504	602(602)	760	56.5k	62.5k ( $62.5k$ )	69.3k
fiber	10	4	5(5)	5	1	13 (13)	57	735	1.1k(1.1k)	1.7k
fixnet6	10	5	5 (5)	5	7	10 (10)	13	1.4k	1.7k(1.7k)	2.2k
flugpl	10	0	0(0)	0	240	240(240)	240	197	197(197)	197
gen	10	0	0(0)	0	1	1 (1)	1	139	143(143)	146
gesa2	10	4	4 (4)	4	2	3(3)	3	1.2k	1.3k(1.3k)	1.4k
gesa2-o	10	4	4 (4)	4	2	4 (4)	5	1.1k	1.4k(1.4k)	1.5k
gesa2_o	10	4	4 (4)	4	4	5(5)	5	1.4k	1.6k(1.6k)	1.7k
gesa3	10	4	4 (4)	4	16	20(20)	20	1.6k	1.7k(1.7k)	1.7k
gesa3_o	10	4	4 (4)	5	7	9 (9)	11	1.4k	1.9k(1.9k)	2.3k
glass4	10	15	30 (30)	53	5.5k	11.8k (11.8k)	22.5k	37.6k	74.4k (74.4k)	147.2k
gmu-35-40	0	7200	- (7200)	7200	9.4m	- (10.6m)	12.3m	25.3m	- (31.3m)	34.3m
$_{ m gt2}$	10	0	0(0)	0	1	1 (1)	1	91	91 (91)	91
harp2	0	7200	- (7200)	7200	4.6m	- (5.3m)	$6.0\mathrm{m}$	5.4m	- (6.0m)	6.6m
iis-100-0-cov	10	2081	2229(2229)	2377	77.9k	83.1k (83.1k)	87.4k	3.2m	3.4m(3.4m)	3.5m
iis-bupa-cov	0	7200	- (7200)	7200	128.4k	- (132.7k)	137.1k	7.8m	- (7.9m)	$8.1 \mathrm{m}$
iis-pima-cov	10	722	802(802)	916	5.9k	6.4k (6.4k)	7.1k	373.6k	399.9k (399.9k)	436.0k
khb05250	10	1	1 (1)	1	2	7 (7)	9	323	554 (554)	837
l152lav	10	1	2(2)	2	15	16(16)	17	1.5k	1.8k(1.8k)	2.0k
lectsched-4-obj	10	1	1(1)	1	0	0 (0)	0	0	0 (0)	0
lseu	10	2	2(2)	2	1.1k	1.1k (1.1k)	1.2k	3.7k	3.9k(3.9k)	4.0k
Table 11. Detail	ed result	s for th	ie svts scoring	functi	on on t	he benchmark	test set	(continu	ues in the next	page).

Instance	#solved		time (s)			nodes			LP iterations	
mstance	#sorved	min	avg	max	min	avg	max	min	avg	max
m100n500k4r1	10	0	0 (0)	0	1	1 (1)	1	166	203 (203)	260
macrophage	0	7200	-(7200)	7200	907.9k	- (1.1m)	1.2m	15.4m	- (17.3m)	$19.0 \mathrm{m}$
manna81	10	1	1 (1)	1	1	1 (1)	1	3.1k	3.2k(3.2k)	3.2k
map18	10	411	484 (484)	595	223	259(259)	287	84.5k	92.3k (92.3k)	103.0k
map20	10	379	424 (424)	511	265	295(295)	327	84.5k	90.1k (90.1k)	95.9k
markshare1	0	7200	- (7200)	7200	22.6m	- (23.1m)	23.4m	$46.1 \mathrm{m}$	- (47.2m)	47.9m
markshare2	0	7200	- (7200)	7200	23.7m	- (24.2m)	24.6m	56.7m	- (58.1m)	58.9m
mas74	10	1599	1665(1665)	1835	$3.0\mathrm{m}$	3.0m(3.0m)	$3.0\mathrm{m}$	11.6m	11.6m (11.6m)	11.6m
mas76	10	193	196(196)	205	475.5k	475.5k (475.5k)	475.5k	1.6m	1.6m(1.6m)	1.6m
mcsched	10	182	261(261)	348	11.5k	17.3k (17.3k)	25.3k	414.7k	642.2k (642.2k)	961.7k
mik-250-1-100-1	10	1887	2374(2374)	2948	870.0k	1.1m (1.1m)	1.4m	$2.7\mathrm{m}$	3.4m(3.4m)	$4.1\mathrm{m}$
mine-166-5	10	29	32(32)	37	237	402 (402)	801	4.0k	4.9k ( $4.9k$ )	5.9k
mine-90-10	10	386	1387(1387)	2572	36.4k	138.0k (138.0k)	303.1k	241.2k	773.0k (773.0k)	1.4m
misc03	10	2	2 (2)	2	63	121 (121)	207	2.0k	2.7k(2.7k)	3.7k
misc06	10	1	1(1)	2	3	7 (7)	10	1.3k	1.8k (1.8k)	2.3k
misc07	10	40	48 (48)	55	16.9k	23.5k (23.5k)	27.9k	129.5k	186.8k (186.8k)	227.8k
mitre	10	43	49 (49)	61	1	1 (1)	1	1.3k	1.4k(1.4k)	1.4k
mod008	10	5	5(5)	5	11	12(12)	12	544	594(594)	607
mod010	10	1	1 (1)	1	2	3 (3)	4	905	1.0k(1.0k)	1.3k
mod011	10	63	72(72)	83	611	852 (852)	1.3k	51.1k	67.0k (67.0k)	90.2k
modglob	10	1	1 (1)	2	10	20 (20)	27	675	767 (767)	1.0k
momentum1	0	7200	-(7200)	7200	13.9k	- (22.6k)	26.6k	$1.1 \mathrm{m}$	- (1.5m)	$1.9\mathrm{m}$
momentum2	2	5089	5299(6820)	7200	15.4k	15.4k (22.5k)	25.9k	469.9k	650.1k (783.4k)	$1.1\mathrm{m}$
msc98-ip	10	677	1503(1503)	2348	2.2k	4.8k (4.8k)	12.1k	542.1k	1.3m (1.3m)	2.3m
mspp16	0	7200	- (7200)	7200	0	- (0)	0	0	- (0)	0
mzzv11	10	666	730 (730)	809	12	213(213)	991	48.5k	58.3k (58.3k)	78.4k
mzzv42z	10	502	535 (535)	600	2	12(12)	20	32.9k	40.6k (40.6k)	46.5k
n3div36	0	7200	- (7200)	7200	105.3k	- (149.2k)	180.9k	877.1k	- (1.3m)	$1.7\mathrm{m}$
n3seq24	0	7200	- (7200)	7200	79.3k	- (109.1k)	135.4k	$1.1\mathrm{m}$	- (1.9m)	$2.8 \mathrm{m}$
n4-3	10	811	$124\hat{6}$ (1246)	1934	28.8k	44.1k (44.1k)	65.6k	1.2m	1.7 m (1.7 m)	2.4m
neos-1109824	10	176	367 (367)	619	7.9k	19.7k (19.7k)	34.6k	92.4k	178.8k (178.8k)	303.9k
neos-1337307	0	7200	- (7200)	7200	239.3k	- (258.8k)	280.5k	3.6m	- (3.8m)	$4.0 \mathrm{m}$
neos-1396125	10	312	396 (396)	597	33.5k	56.9k (56.9k)	103.0k	999.3k	1.5m(1.5m)	$2.0 \mathrm{m}$
neos-1601936	10	7	7 (7)	8	1	1 (1)	1	10.4k	11.0k(11.0k)	11.4k
neos-476283	10	357	432 (432)	580	91	123(123)	168	8.0k	11.3k (11.3k)	16.3k
neos-686190	10	117	161(161)	261	1.5k	2.1k(2.1k)	5.0k	35.5k	42.8k(42.8k)	79.9k
neos-849702	10	0	0 (0)	0	0	0 (0)	0	0	0 (0)	0
neos-916792	0	7200	- (7200)	7200	172.2k	- (203.4k)	229.1k	$2.0\mathrm{m}$	-(2.4m)	$2.8 \mathrm{m}$
neos-934278	0	7200	- (7200)	7200	6.1k	- (9.3k)	13.0k	5.5m	- (6.5m)	$7.9\mathrm{m}$
neos13	10	255	$27\hat{6}$ (27 $\hat{6}$ )	309	10	12(12)	12	3.0k	$3.0\dot{k}$ (3.0k)	3.0k
Table 11: Detail	ed result		e svts`scóring	g functi		he benchmark			ues in the next	page).

	Instance			time (s)			nodes			LP iterations	
	motanee	#solved	min	avg	max	min	avg	max	$\min$	avg	max
	neos18	10	64	86 (86)	145	4.1k	6.1k (6.1k)	8.8k	128.7k	196.2k (196.2k)	281.8k
	net12	10	1061	2178(2178)	2975	1.8k	3.1k (3.1k)	4.7k	1.0m	2.2m (2.2m)	3.2m
	netdiversion	10	896	1395(1395)	2319	5	14(14)	64	30.5k	53.7k (53.7k)	153.2k
	newdano	5	4277	5616(6408)	7200	1.1m	1.4m (1.4m)	1.8m	52.3m	67.7m ( $67.8m$ )	81.2m
	noswot	10	332	817 (817)	2766	413.9k	922.0k (922.0k)	2.8m	1.4m	2.7 m (2.7 m)	7.2m
	ns1208400	10	288	466 (466)	653	1	232 (232)	830	142.1k	353.5k (353.5k)	676.5k
	ns1688347	10	58	84 (84)	116	6	85 (85)	328	5.8k	23.5k (23.5k)	36.8k
	ns1758913	6	2203	3830 (5178)	7200	1	1(1)	1	29.3k	44.1k (53.6k)	75.1k
	ns1766074	10	5159	5666 (5666)	6568	895.4k	907.9k (907.9k)	942.4k	3.5m	3.6m(3.6m)	3.8m
	ns1830653	10	409	448 (448)	473	15.0k	17.1k (17.1k)	19.5k	1.0m	1.2m (1.2m)	1.2m
	nsrand-ipx	5	805	3972(5586)	7200	67.5k	497.6k (764.0k)	1.3m	615.8k	3.0 m (4.5 m)	7.3m
	nw04	10	2024	2529(2529)	3112	7	10 (10)	12	1.3k	1.6k (1.6k)	1.9k
	opm2-z7-s2	10	590	787 (787)	1075	533	768 (768)	1.0k	36.0k	53.4k (53.4k)	61.5k
	opt1217	5	2	3 (3601)	7200	1	1 (1.6m)	$3.7 \mathrm{m}$	747	798 (8.2m)	$22.7 \mathrm{m}$
	p0033	10	ō	0 (0)	0	1	1 (1)	1	57	57 (57)	57
	p0201	10	2	3 (3)	4	4	10 (10)	33	1.1k	1.5k(1.5k)	2.5k
	p0282	10	1	2(2)	2	2	3 (3)	3	459	649 (649)	890
	p0548	10	1	1(1)	1	1	1(1)	1	331	331 (331)	331
-	p2756	10	4	4(4)	5	10	10 (10)	10	538	547 (547)	555
51	pg5_34	10	1243	1343(1343)	1652	109.5k	109.5k(109.5k)	109.5k	$2.1 \mathrm{m}$	2.1 m (2.1 m)	$2.1 \mathrm{m}$
	pigeon-10	0	7200	- (7200)	7200	854.8k	- (995.6k)	1.2m	3.7m	- (5.6m)	$7.1 \mathrm{m}$
	pk1	10	129	134(134)	142	344.3k	344.3k (344.3k)	344.3k	$2.0 \mathrm{m}$	2.0 m (2.0 m)	$2.0 \mathrm{m}$
	pp08a	10	2	2(2)	3	117	240 (240)	400	2.0k	3.4k (3.4k)	5.8k
	pp08aCUTS	10	2	2(2)	3	89	161(161)	283	1.8k	2.5k(2.5k)	3.8k
	protfold	0	7200	-(7200)	7200	18.9k	- (25.0k)	30.6k	15.3m	-(20.8m)	24.4m
	pw-myciel4	4	5756	$661\dot{7}$ ( $69\dot{6}7$ )	7200	480.0k	637.1k (669.7k)	809.1k	15.7m	18.0 m (20.0m)	$23.0\mathrm{m}$
	qiu	10	82	105(105)	120	10.0k	12.3k (12.3k)	13.7k	319.1k	406.7k (406.7k)	482.1k
	qnet1	10	9	10(10)	12	3	4 (4)	6	3.7k	4.0k(4.0k)	4.5k
	qnet1_o	10	5	5(5)	6	3	4 (4)	6	2.5k	2.8k (2.8k)	3.3k
	rail507	10	119	141 (141)	184	568	601 (601)	636	59.8k	62.8k ( $62.8k$ )	66.3k
	ran16x16	10	399	442(442)	504	247.1k	273.3k (273.3k)	308.5k	2.3m	2.5m(2.5m)	$2.9\mathrm{m}$
	rd-rplusc-21	0	7200	- (7200)	7200	48.9k	- (83.1k)	172.9k	320.6k	- (949.1k)	2.3m
	reblock67	10	162	232(232)	276	38.6k	45.1k (45.1k)	52.0k	394.4k	484.5k (484.5k)	553.7k
	rentacar	10	5	6(6)	6	2	2(2)	2	2.8k	3.2k ( $3.2k$ )	3.8k
	rgn	10	1	1(1)	1	1	1(1)	2	327	392(392)	556
	rmatr100-p10	10	106	111(111)	126	739	745(745)	755	62.2k	62.5k ( $62.5k$ )	63.0k
	rmatr100-p5	10	129	133(133)	139	381	382 (382)	383	67.6k	67.6k (67.6k)	67.6k
	rmine6	1	7074	7074(7187)	7200	818.0k	909.7k (894.0k)	942.4k	5.6m	5.6m(6.2m)	6.8m
	rocII-4-11	10	413	534(534)	807	5.1k	8.4k ( $8.4k$ )	15.2k	83.9k	119.6k (119.6k)	194.5k
	rococoC10-001000	2	2069	4622 (6684)	c <sup>7200</sup>	80.4k	360.8k (296.9k)	559.6k	6.1m	10.4m (23.8m)	36.5m
	Table 11: Detail	ed result	s for the	e svts scoring	; tuncti	on on t	he benchmark	test set	(contini	ies in the next	page).

Instance	#solved		time (s)			nodes			LP iterations	
mstance	#solved	min	avg	$\max$	min	avg	max	min	avg	$\max$
roll3000	6	1499	3186 (4791)	7200	157.0k	356.4k (570.1k)	931.1k	2.4m	4.6m (7.6m)	13.8m
rout	10	38	105(105)	164	10.7k	41.7k (41.7k)	67.1k	206.5k	607.4k ( $607.4k$ )	891.0k
satellites1-25	10	951	1619(1619)	2244	1.1k	3.5k(3.5k)	6.6k	180.8k	802.0k ( $802.0k$ )	1.8m
set1ch	10	2	2(2)	3	3	5(5)	7	987	1.1k(1.1k)	1.5k
seymour	0	7200	- (7200)	7200	106.0k	- (111.0k)	119.3k	7.5m	- (7.9m)	8.5m
sp97ar	0	7200	- (7200)	7200	122.9k	- (147.3k)	183.7k	$4.1 \mathrm{m}$	- (4.8m)	5.4m
sp98ic	10	1614	3332(3332)	4676	76.0k	159.7k (159.7k)	222.7k	1.4m	2.9m(2.9m)	$4.1 \mathrm{m}$
sp98ir	10	57	70 (70)	77	1.3k	1.8k(1.8k)	2.3k	30.6k	41.5k (41.5k)	48.8k
stein27	10	2	2(2)	2	3.7k	4.2k (4.2k)	4.5k	13.5k	14.3k (14.3k)	15.0k
stein45	10	32	33 (33)	37	49.0k	51.2k (51.2k)	55.7k	255.5k	265.5k (265.5k)	281.1k
swath	0	7200	- (7200)	7200	105.3k	- (149.3k)	193.5k	1.5m	- (1.9m)	2.5m
tanglegram1	5	827	1579(4390)	7200	21	168(876)	1.9k	118.3k	282.1k (899.9k)	$1.7\mathrm{m}$
tanglegram2	10	7	10 (10)	16	3	4 (4)	7	6.0k	6.9k(6.9k)	9.1k
timtab1	10	778	1193(1193)	1579	495.8k	790.9k (790.9k)	1.0m	$4.1 \mathrm{m}$	6.3m(6.3m)	8.5m
timtab2	0	7200	- (7200)	7200	3.0m	- (3.3m)	3.6m	30.1m	- (34.9m)	38.3m
tr12-30	10	2856	3629(3629)	4583	634.3k	792.2k (792.2k)	977.1k	3.2m	4.0m~(4.0m)	$4.7 \mathrm{m}$
triptim1	10	160	185(185)	224	1	1 (1)	1	30.7k	33.8k (33.8k)	35.9k
unitcal_7	9	1521	3101(3511)	7200	12.3k	27.8k (34.3k)	92.8k	216.6k	487.5k (558.4k)	1.2m
vpm1	10	0	0 (0)	0	1	1 (1)	1	59	60(60)	60
vpm2	10	2	2(2)	2	149	182(182)	195	1.6k	1.8k(1.8k)	1.9k
vpphard	0	7200	- (7200)	7200	687	- (6.5k)	13.3k	1.3m	- (1.5m)	1.8m
zib54-UUE	3	4985	5380(6654)	7200	447.1k	466.3k (589.3k)	708.8k	22.0m	23.4m (29.1m)	32.8m
Total	1226		778074 (3398874)			126.9m(1.0b)			1.6b~(7.3b)	
Arit. mean			635 (2138)			$103.5k \ (635.3k)$			1.3m~(4.6m)	
Sh. geo. mean			86(249)			1.3k (4.4k)			33.5k (110.8k)	

Table 11: Detailed results for the svts scoring function on the benchmark test set.

## C.2 Tree test set results

Tables 12, 13 and 14 give the results on the tree test set for the scoring functions product, ratio, and svts, respectively.

Instance	#solved		time (s)			nodes			LP iterations	
Instance	#solved	min	avg	max	min	avg	max	min	avg	max
blp-ar98	0	43200	- (43200)	43200	1.5m	- (2.0m)	2.3m	13.5m	- (17.0m)	20.2m
csched007	10	26734	31000 (31000)	39186	4.8m	5.7m (5.7m)	$7.0 \mathrm{m}$	101.7m	124.5m (124.5m)	152.4m
csched008	10	750	1440 (1440)	3065	58.2k	112.4k (112.4k)	221.9k	7.4m	15.4m (15.4m)	33.7m
enlight15	0	43200	- (43200)	43200	20.1m	- (21.9m)	22.8m	101.4m	- (111.2m)	116.3m
enlight16	0	43200	- (43200)	43200	23.2m	- (24.1m)	25.3m	99.6m	- (103.6m)	$109.1 \mathrm{m}$
glass4	10	15	28(28)	45	5.4k	11.5k (11.5k)	18.9k	36.3k	75.2k (75.2k)	130.2k
gmu-35-40	0	43200	- (43200)	43200	64.2m	- (70.4m)	78.4m	157.1m	- (172.4m)	186.6m
gmu-35-50	0	43200	- (43200)	43200	51.1m	- (58.6m)	72.5m	165.5m	- (213.7m)	246.4m
go19	0	43200	- (43200)	43200	9.9m	- (10.7m)	11.3m	333.2m	- (355.0m)	377.6m
k16x240	10	19787	23835 (23835)	29016	18.3m	23.2m (23.2m)	27.7m	74.4m	90.7m (90.7m)	112.9m
leo1	0	43200	- (43200)	43200	3.6m	- (3.9m)	4.4m	44.8m	- (49.7m)	52.8m
markshare 5 0	0	43200	- (43200)	43200	121.2m	- (123.8m)	$126.0 \mathrm{m}$	217.9m	- (222.6m)	226.5m
maxgasflow	0	43200	- (43200)	43200	1.4m	- (1.8m)	2.3m	8.3m	- (12.8m)	18.8m
mc11	0	43200	- (43200)	43200	2.4m	- (3.2m)	$3.9\mathrm{m}$	42.1m	- (49.9m)	60.4m
momentum1	0	43200	- (43200)	43200	143.2k	- (166.0k)	188.9k	4.1m	- (4.7m)	5.8m
neos-1426635	0	43200	- (43200)	43200	35.4m	- (40.6m)	44.1m	244.4m	- (263.1m)	283.0m
neos-1426662	0	43200	- (43200)	43200	18.6m	- (21.2m)	23.4m	125.3m	- (137.8m)	154.4m
neos-1436709	Ő	43200	- (43200)	43200	14.3m	-(17.1m)	18.6m	181.6m	-(214.3m)	275.9m
neos-1440460	Ő	43200	- (43200)	43200	20.0m	-(21.1m)	22.5m	275.8m	- (298.6m)	340.2m
neos-1442119	Ő	43200	- (43200)	43200	16.8m	- (18.7m)	20.4m	186.4m	-(207.1m)	227.1m
neos-1442657	Ő	43200	- (43200)	43200	15.4m	-(17.7m)	20.0m	216.4m	-(249.7m)	297.3m
neos-1616732	10	8341	9182 (9182)	9915	2.3m	2.5m (2.5m)	2.6m	31.6m	34.0m (34.0m)	36.1m
neos-1620770	0	43200	- (43200)	43200	4.8m	- (6.2m)	7.7m	70.3m	- (94.7m)	123.4m
neos-820146	Ő	43200	- (43200)	43200	8.8m	-(10.1m)	11.1m	341.1m	- (367.0m)	386.1m
neos-820157	Ő	43200	- (43200)	43200	7.0m	- (7.5m)	8.1m	290.2m	-(320.5m)	347.5m
neos-826650	Ő	43200	- (43200)	43200	1.5m	-(2.4m)	3.5m	137.8m	-(198.7m)	230.6m
neos-826841	Ő	43200	- (43200)	43200	3.8m	-(4.3m)	$4.7 \mathrm{m}$	127.2m	- (141.8m)	160.5m
neos-847302	Ő	43200	- (43200)	43200	2.6m	- (3.0m)	3.4m	407.3m	-(422.1m)	434.1m
neos-911880	Ő	43200	- (43200)	43200	18.9m	- (24.3m)	30.6m	65.2m	- (103.5m)	142.6m
neos-942830	10	1831	9184 (9184)	17919	237.4k	1.5m (1.5m)	3.2m	11.1m	72.4m (72.4m)	139.5m
neos15	1	34352	34352(42315)	43200	12.3m	12.3 m (16.5 m)	17.7m	106.9m	106.9 m (134.1 m)	148.0m
neos16	0	43200	- (43200)	43200	7.3m	- (8.6m)	9.7m	63.3m	- (77.8m)	89.9m
neos858960	10	3687	3997 (3997)	4338	2.7m	2.8m(2.8m)	3.0m	51.3m	54.0m (54.0m)	56.9m
nobel-eu-DBE	0	43200	- (43200)	43200	7.3m	- (8.7m)	10.6m	48.2m	-(56.2m)	65.1m
noswot	10	272	556(556)	868	367.0k	679.8k (679.8k)	1.0m	1.2m	2.1 m (2.1 m)	3.2m
ns1456591	0	43200	- (43200)	43200	297.4k	- (513.6k)	892.9k	11.8m	- (23.1m)	45.7m
ns1766074	10	5495	5747(5747)	6173	895.3k	925.7k (925.7k)	949.9k	3.4m	3.7m(3.7m)	3.8m
ns2081729	0	43200	- (43200)	43200	8.8m	-(18.3m)	31.1m	34.6m	-(53.1m)	75.7m
ns894788	10	40200	4 (4)	40200	1	1(1)	1	303	387 (387)	487
p2m2p1m1p0n100	0	43200	- (43200)	43200	197.4m	-(198.4m)	199.4m	103.3m	-(103.8m)	104.4m
									in the next page	

Table 12: Detailed results for the **product** scoring function on the tree test set(continues in the next page).

Instance	#solved		time (s)			nodes			LP iterations	
mstance	#solved	min	avg	max	min	avg	max	min	avg	max
p80x400b	0	43200	- (43200)	43200	24.2m	- (27.3m)	30.3m	80.4m	- (86.5m)	95.9m
pg	10	23	24(24)	24	203	203(203)	203	14.3k	14.3k (14.3k)	14.3k
pigeon-10	0	43200	- (43200)	43200	3.7m	- (4.4m)	5.2m	13.8m	- (24.9m)	39.8m
pigeon-11	0	43200	- (43200)	43200	3.0m	- (3.9m)	$4.7 \mathrm{m}$	12.6m	- (17.0m)	24.9m
pigeon-12	0	43200	- (43200)	43200	3.1m	- (3.7m)	$4.0 \mathrm{m}$	12.3m	- (23.1m)	38.6m
ran14x18	10	16956	23039 (23039)	31792	13.1m	18.4m (18.4m)	25.3m	102.2m	139.1m (139.1m)	189.5m
ran14x18-disj-8	0	17544	- (25432)	38252	5.7m	- (7.8m)	12.5m	80.9m	- (114.5m)	177.4m
reblock166	4	14474	16858 (32663)	43200	1.4m	1.6m(3.5m)	7.3m	5.3m	6.5m (13.4m)	26.4m
timtab1	10	754	1142 (1142)	1536	532.7k	774.5k (774.5k)	$1.0 \mathrm{m}$	4.4m	6.0m(6.0m)	$8.0\mathrm{m}$
umts	10	1544	8726 (8726)	24776	219.7k	1.5m(1.5m)	3.8m	2.6m	15.3m (15.3m)	49.8m
wachplan	10	12679	15334 (15334)	18784	269.8k	324.2k (324.2k)	398.4k	107.9m	131.1m (131.1m)	$162.9 \mathrm{m}$
wnq-n100-mw99-14	0	43200	- (43200)	43200	39.2k	- (42.4k)	45.8k	3.8m	- (4.0m)	4.2m
Total	155		1434174 (17024498)			602.1m (8.7b)			7.0b (57.5b)	
Arit. mean			9253 (32739)			3.9m (16.8m)			45.3m (110.6m)	
Sh. geo. mean			2136 (17459)			361.9k (3.2m)			5.7m (37.3m)	
	Table	19. Det	ailed results for t	he pro	duct see	pring function	n the t	roo tost	set	

Table 12: Detailed results for the **product** scoring function on the tree test set.

55

Instance	#solved		time (s)			nodes			LP iterations	
Instance	#solved	min	avg	max	min	avg	$\max$	min	avg	max
blp-ar98	0	43200	- (43200)	43200	1.6m	- (1.9m)	2.3m	14.9m	- (17.2m)	19.1m
csched007	10	20695	24218 (24218)	28186	$3.7\mathrm{m}$	4.2m(4.2m)	4.8m	90.5m	101.9m (101.9m)	123.7m
csched008	10	824	1737 (1737)	2992	63.6k	115.6k (115.6k)	179.2k	8.6m	19.7m (19.7m)	35.7m
enlight15	0	43200	- (43200)	43200	25.6m	- (26.9m)	$28.0 \mathrm{m}$	106.2m	- (111.8m)	117.5m
enlight16	0	43200	- (43200)	43200	23.8m	- (25.1m)	26.7m	106.6m	- (113.0m)	120.5m
glass4	10	15	38 (38)	101	5.5k	15.4k (15.4k)	35.9k	41.5k	102.8k (102.8k)	242.0k
gmu-35-40	2	7147	15954(37751)	43200	$11.1 \mathrm{m}$	23.6m (61.0m)	75.4m	26.0m	54.0m (158.9m)	217.1 m
gmu-35-50	0	43200	- (43200)	43200	$50.0 \mathrm{m}$	- (62.7m)	73.9m	169.3m	- (200.3m)	228.2m
go19	0	43200	- (43200)	43200	9.3m	- (9.8m)	10.6m	342.5m	- (355.9m)	377.2m
k16x240	10	15703	20899 (20899)	28244	16.2m	20.6m (20.6m)	$28.1 \mathrm{m}$	58.4m	86.2m (86.2m)	$118.0 \mathrm{m}$
leo1	0	43200	- (43200)	43200	3.3m	- (3.8m)	4.3m	48.1m	- (51.2m)	54.5m
markshare_5_0	0	43200	- (43200)	43200	126.2m	- (133.5m)	137.7m	231.0m	- (244.3m)	252.1m
maxgasflow	0	43200	- (43200)	43200	1.4m	- (1.6m)	1.9m	7.8m	- (15.0m)	21.4m
mc11	0	43200	- (43200)	43200	2.6m	- (3.2m)	$4.0 \mathrm{m}$	29.6m	- (39.3m)	50.2m
momentum1	0	43200	- (43200)	43200	125.9k	- (162.4k)	187.3k	3.8m	- (4.5m)	$5.1 \mathrm{m}$
mas-1426635		43200	- (43200)	43200	38.3m	- (41.1m)	44.6m	245.9m	-(266.6m)	$292.7\mathrm{m}$

Table 13: Detailed results for the ratio scoring function on the tree test set(continues in the next page).

Instance	#solved		time (s)			nodes			LP iterations	
mstance	#301764	min	avg	max	min	avg	max	min	avg	max
neos-1426662	0	43200	- (43200)	43200	20.1m	- (21.7m)	23.3m	126.7m	- (136.2m)	149.1n
neos-1436709	0	43200	- (43200)	43200	11.9m	- (18.3m)	$21.1 \mathrm{m}$	154.4m	- (203.6m)	376.6n
neos-1440460	0	43200	- (43200)	43200	$19.7 \mathrm{m}$	- (21.0m)	23.6m	259.9m	- (290.8m)	397.0n
neos-1442119	0	43200	- (43200)	43200	$18.0 \mathrm{m}$	- (19.2m)	20.7m	188.3m	- (203.6m)	230.9n
neos-1442657	0	43200	- (43200)	43200	$16.1 \mathrm{m}$	- (18.3m)	19.6m	210.2m	- (251.1m)	321.3n
neos-1616732	10	7160	7773 (7773)	8407	$2.1\mathrm{m}$	2.3m(2.3m)	2.5m	27.3m	29.6m (29.6m)	31.3m
neos-1620770	0	43200	- (43200)	43200	5.3m	- (6.3m)	7.3m	76.6m	- (94.7m)	116.6n
neos-820146	0	43200	- (43200)	43200	8.6m	- (10.3m)	12.0m	334.8m	- (373.5m)	431.3n
neos-820157	0	43200	- (43200)	43200	$6.9 \mathrm{m}$	- (7.7m)	9.4m	288.2m	- (317.3m)	348.6n
neos-826650	0	43200	- (43200)	43200	1.8m	- (2.2m)	$3.1\mathrm{m}$	151.4m	- (206.6m)	235.5n
neos-826841	0	43200	- (43200)	43200	$4.0 \mathrm{m}$	- (4.5m)	$5.1 \mathrm{m}$	114.0m	- (137.8m)	159.1n
neos-847302	0	43200	- (43200)	43200	$2.8 \mathrm{m}$	- (3.1m)	3.3m	383.0m	- (414.0m)	435.1n
neos-911880	0	43200	- (43200)	43200	$23.0\mathrm{m}$	- (27.7m)	$33.0\mathrm{m}$	86.9m	- (118.9m)	165.8n
neos-942830	10	1634	8689 (8689)	13644	228.5k	1.4m(1.4m)	$2.7\mathrm{m}$	11.2m	67.8m (67.8m)	93.8m
neos15	1	31393	31393(42019)	43200	11.6m	11.6m(16.6m)	$17.9 \mathrm{m}$	95.9m	95.9m (135.6m)	149.3n
neos16	0	43200	- (43200)	43200	$7.0\mathrm{m}$	- (8.9m)	11.5m	60.9m	- (77.6m)	97.4m
neos858960	10	3542	3676 (3676)	3886	2.5m	2.6m(2.6m)	2.8m	$46.9 \mathrm{m}$	48.7m (48.7m)	51.5m
nobel-eu-DBE	0	43200	- (43200)	43200	6.4m	- (7.6m)	8.7m	56.7m	- (64.7m)	70.8m
noswot	10	344	900 (900)	3519	436.9k	1.1m(1.1m)	4.3m	1.4m	2.8m(2.8m)	8.6m
ns1456591	0	43200	- (43200)	43200	194.3k	- (472.5k)	716.5k	13.7m	- (31.3m)	$49.7 \mathrm{m}$
ns1766074	10	5038	5463(5463)	5760	895.4k	907.9k (907.9k)	942.4k	3.5m	3.6m (3.6m)	3.8m
ns2081729	0	43200	- (43200)	43200	$10.4 \mathrm{m}$	- (22.2m)	35.9m	33.5m	- (63.9m)	97.9m
ns894788	10	4	4 (4)	5	1	1 (1)	1	303	387 (387)	487
p2m2p1m1p0n100	0	43200	- (43200)	43200	196.7m	- (199.1m)	204.2m	102.9m	- (104.2m)	106.9n
p80x400b	0	43200	- (43200)	43200	$25.0 \mathrm{m}$	- (27.2m)	28.8m	$75.0 \mathrm{m}$	- (84.6m)	94.9m
pg	10	23	24 (24)	26	197	197 (197)	197	14.5k	14.5k (14.5k)	14.5k
pigeon-10	0	43200	- (43200)	43200	4.0m	- (4.7m)	5.2m	18.2m	- (25.2m)	38.1m
pigeon-11	õ	43200	- (43200)	43200	3.1m	-(4.2m)	5.5m	13.1m	-(18.3m)	27.8m
pigeon-12	0	43200	- (43200)	43200	3.2m	- (3.9m)	4.6m	12.5m	- (19.7m)	36.0m
ran14x18	10	13454	19209 (19209)	26510	10.5m	15.0m (15.0m)	$20.7 \mathrm{m}$	81.0m	113.2m (113.2m)	152.9n
ran14x18-disj-8	0	11086	- (16055)	21427	3.6m	- (5.0m)	6.3m	49.6m	- (74.0m)	94.1m
reblock166	6	11729	19725 (29115)	43200	1.5m	2.0 m (2.7 m)	6.1m	5.7m	8.5m (12.0m)	24.9m
timtab1	10	855	1200 (1200)	1547	592.7k	803.6k (803.6k)	993.3k	4.6m	6.3m (6.3m)	7.9m
umts	10	1296	6009 (6009)	13658	186.5k	1.1 m (1.1 m)	2.4m	2.5m	10.6 m (10.6 m)	27.2m
wachplan	10	10904	15674 (15674)	24922	247.9k	327.1k (327.1k)	458.9k	92.2m	133.2m (133.2m)	217.3n
wnq-n100-mw99-14	0	43200	- (43200)	43200	31.4k	-(35.5k)	37.6k	3.3m	- (3.5m)	3.7m
Total	159		1336785 (16660531)			575.3m (8.8b)			6.5b (56.6b)	
Arit. mean			8407 (32039)			3.6m (17.0m)			40.8m (108.9m)	
Sh. geo. mean			2191 (17044)			376.9k (3.1m)			5.8m (36.9m)	

Table 13: Detailed results for the ratio scoring function on the tree test set(continues in the next page).

Instance	#solved		time (s)			nodes			LP iterations	
Instance	#solved	min	avg	max	min	avg	max	min	avg	max
	1									

Instance	#solved		time (s)			nodes			LP iterations	
Instance	#solved	min	avg	max	min	avg	max	min	avg	max
blp-ar98	0	43200	- (43200)	43200	1.4m	- (1.7m)	1.9m	13.9m	- (17.4m)	20.3m
csched007	10	18507	25355 ( $25355$ )	37366	3.0m	4.3m(4.3m)	5.9m	77.9m	105.6m (105.6m)	143.6m
csched008	10	555	1641 (1641)	4820	46.1k	118.6k (118.6k)	267.0k	5.2m	18.3m (18.3m)	59.2m
enlight15	0	43200	- (43200)	43200	23.1m	- (24.8m)	25.9m	103.0m	- (111.6m)	117.4m
enlight16	0	43200	- (43200)	43200	21.8m	- (25.2m)	26.3m	97.5m	- (113.6m)	$119.0 \mathrm{m}$
glass4	10	17	29 (29)	51	5.5k	11.8k (11.8k)	22.5k	37.6k	74.4k (74.4k)	147.2k
gmu-35-40	1	39750	39750(42855)	43200	59.9m	59.9m (71.2m)	79.2m	132.3m	132.3m (190.1m)	224.3m
gmu-35-50	0	43200	- (43200)	43200	49.0m	- (61.6m)	75.1m	178.0m	- (206.2m)	249.4m
go19	0	43200	- (43200)	43200	9.6m	- (10.3m)	11.0m	327.1m	- (352.8m)	370.0m
k16x240	10	16692	20462 (20462)	24643	15.3m	19.7m (19.7m)	$24.0 \mathrm{m}$	63.6m	82.9m (82.9m)	107.4m
leo1	0	43200	- (43200)	43200	3.1m	- (3.4m)	$3.9\mathrm{m}$	50.2m	- (54.8m)	$59.0 \mathrm{m}$
markshare_5_0	0	43200	- (43200)	43200	$125.0 \mathrm{m}$	- (128.0m)	$130.1 \mathrm{m}$	228.9m	- (234.2m)	$238.1 \mathrm{m}$
maxgasflow	0	43200	- (43200)	43200	1.3m	- (1.7m)	$2.1\mathrm{m}$	8.2m	- (16.2m)	20.7m
mc11	0	43200	- (43200)	43200	2.3m	- (3.1m)	3.5m	35.8m	- (47.8m)	57.0m
momentum1	0	43200	- (43200)	43200	128.9k	- (176.4k)	197.3k	4.4m	- (5.1m)	5.8m
neos-1426635	0	43200	- (43200)	43200	36.5m	- (40.8m)	44.2m	208.8m	- (249.2m)	276.9m
neos-1426662	0	43200	- (43200)	43200	17.7m	- (20.8m)	22.9m	119.7m	- (138.4m)	153.3m
neos-1436709	0	43200	- (43200)	43200	16.5m	- (18.8m)	20.8m	163.8m	- (184.7m)	212.8m
neos-1440460	0	43200	- (43200)	43200	18.4m	- (20.7m)	21.9m	246.9m	- (282.5m)	332.5m
neos-1442119	0	43200	- (43200)	43200	18.6m	- (19.5m)	20.4m	192.1m	- (200.8m)	$215.1 \mathrm{m}$
neos-1442657	0	43200	- (43200)	43200	15.4m	- (18.1m)	19.5m	211.3m	- (243.0m)	$275.0 \mathrm{m}$
neos-1616732	10	6605	7487 (7487)	8361	1.8m	2.1 m (2.1 m)	2.3m	25.8m	28.4m (28.4m)	31.3m
neos-1620770	0	43200	- (43200)	43200	4.2m	- (5.9m)	$6.9\mathrm{m}$	64.9m	- (90.7m)	$112.0 \mathrm{m}$
neos-820146	0	43200	- (43200)	43200	9.3m	- (10.2m)	10.9m	355.7m	- (373.3m)	393.6m
neos-820157	0	43200	- (43200)	43200	7.1m	- (7.7m)	8.6m	297.0m	- (319.2m)	346.5m
neos-826650	0	43200	- (43200)	43200	1.8m	- (2.3m)	3.3m	162.4m	- (206.9m)	234.4m
neos-826841	0	43200	- (43200)	43200	3.6m	- (4.3m)	$5.0 \mathrm{m}$	116.9m	- (142.3m)	171.9m
neos-847302	0	43200	- (43200)	43200	2.6m	- (2.9m)	3.4m	396.7m	- (410.4m)	424.2m
neos-911880	0	43200	- (43200)	43200	15.4m	- (26.3m)	31.9m	68.0m	- (104.3m)	136.6m
neos-942830	10	2538	9145 (9145)	18941	375.5k	1.4m(1.4m)	$3.0\mathrm{m}$	18.7m	73.7m (73.7m)	156.3m
neos15	1	29801	29801 (41860)	43200	11.2m	11.2m(16.2m)	17.8m	97.9m	97.9m (139.3m)	151.4m
neos16	0	43200	- (43200)	43200	8.1m	- (8.9m)	10.3m	72.2m	- (79.6m)	89.7m
Table 14	Detailed	results	for the syts sco	oring fun	iction or	the tree test	set(cont	inues in	the next nage)	

Table 13: Detailed results for the **ratio** scoring function on the tree test set.

Table 14: Detailed results for the svts scoring function on the tree test set(continues in the next page).

57

Instance	#solved		time (s)			nodes			LP iterations	
Instance	#sorved	min	avg	max	min	avg	max	min	avg	max
neos858960	10	3359	3665(3665)	3784	2.5m	2.6m (2.6m)	2.8m	46.9m	48.7m (48.7m)	51.5n
nobel-eu-DBE	0	43200	- (43200)	43200	7.5m	- (8.2m)	8.9m	41.5m	- (58.2m)	69.4n
noswot	10	331	799 (799)	2625	413.9k	922.0k (922.0k)	2.8m	1.4m	2.7m(2.7m)	7.2m
ns1456591	0	43200	- (43200)	43200	178.7k	- (463.2k)	696.1k	14.8m	- (31.5m)	48.1n
ns1766074	10	5349	5607(5607)	6443	895.4k	907.9k (907.9k)	942.4k	3.5m	3.6m(3.6m)	3.8m
ns2081729	0	43200	- (43200)	43200	11.0m	- (22.8m)	32.6m	38.5m	- (67.6m)	97.2n
ns894788	10	4	4 (4)	5	1	1 (1)	1	303	387(387)	487
p2m2p1m1p0n100	0	43200	- (43200)	43200	196.3m	- (197.8m)	199.5m	102.7m	- (103.5m)	104.41
p80x400b	0	43200	- (43200)	43200	24.2m	- (26.4m)	29.2m	82.3m	- (89.2m)	96.4n
pg	10	23	24 (24)	25	223	223(223)	223	14.7k	14.7k (14.7k)	14.7
pigeon-10	0	43200	- (43200)	43200	3.4m	- (4.4m)	5.0m	15.5m	- (24.3m)	41.6n
pigeon-11	0	43200	- (43200)	43200	3.0m	- (4.2m)	5.9m	12.6m	- (18.3m)	28.4n
pigeon-12	0	43200	- (43200)	43200	3.1m	- (3.9m)	4.4m	11.5m	- (20.4m)	40.1n
ran14x18	10	13372	19987 (19987)	26400	10.1m	14.8m(14.8m)	19.8m	85.9m	122.7m (122.7m)	158.61
ran14x18-disj-8	0	14834	- (20026)	23964	4.4m	- (5.9m)	7.2m	70.4m	- (94.7m)	118.81
reblock166	6	10968	17803 (27962)	43200	1.2m	1.9m~(2.8m)	7.8m	5.0m	8.1m(12.0m)	26.4n
timtab1	10	739	1168(1168)	1685	495.8k	790.9k (790.9k)	1.0m	4.1m	6.3m(6.3m)	8.5m
umts	10	1536	5581 (5581)	9636	258.5k	999.4k (999.4k)	1.5m	2.6m	8.9m (8.9m)	17.9n
wachplan	10	11060	15542 (15542)	25357	247.9k	327.1k (327.1k)	458.9k	92.2m	133.2m (133.2m)	217.3i
wnq-n100-mw99-14	0	43200	- (43200)	43200	29.9k	- (33.5k)	35.9k	3.1m	- (3.4m)	3.7m
Total	158		1341327 (16747991)			572.3m (8.8b)			6.6b (56.7b)	
Arit. mean			8489 (32208)			3.6m~(16.9m)			42.0 m (109.1 m)	
Sh. geo. mean			2140(17096)			364.2k (3.1m)			5.7m (37.2m)	

Table 14: Detailed results for the svts scoring function on the tree test set.