### SOCP Reformulation for the Generalized Trust Region Subproblem via a Canonical Form of Two Symmetric Matrices

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**Abstract** We investigate in this paper the generalized trust region subproblem (GTRS) of minimizing a general quadratic objective function subject to a general quadratic inequality constraint. By applying a simultaneous block diagonalization approach, we obtain a congruent canonical form for the symmetric matrices in both the objective and constraint functions. By exploiting the block separability of the canonical form, we show that all GTRSs with an optimal value bounded from below are second order cone programming (SOCP) representable. Our result generalizes the recent work of Ben-Tal and Hertog (Math. Program. 143(1-2):1-29, 2014), which establishes the SOCP representability of the GTRS under the assumption of the simultaneous diagonalizability of the two matrices in the objective and constraint functions. Compared with the state-of-the-art approach to reformulate the GTRS as a semi-definite programming problem, our SOCP reformulation delivers a much faster solution algorithm. We further extend our method to two variants of the GTRS in which the inequality constraint is replaced by either an equality constraint or an interval constraint. Our methods also enable us to obtain simplified versions of the classical S-lemma, the S-lemma with equality, and the S-lemma with interval bounds.

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#### 1 Introduction

We consider in this paper the following generalized trust region subproblem (GTRS):

(P) min 
$$f(x) = \frac{1}{2}x^T Dx + e^T x$$
  
s.t.  $h(x) = \frac{1}{2}x^T Ax + b^T x + c \le 0$ ,

where A and D are  $n \times n$  symmetric matrices but not necessarily positive semi-definite,  $b, e \in \Re^n$  and  $c \in \Re$ .

The GTRS has been widely investigated in the optimization literature and includes the classical trust region subproblem as its special case where the constraint reduces to a ball constraint, i.e.,  $x^T x \leq 1$ . The classical trust region subproblem is fundamental in trust region methods for nonlinear optimization problems, see [9, 13, 19]. Other prominent applications of the classical trust region subproblem can be found in regularization and robust optimization [2], etc. Rendl and Wolkowicz [12] first solve the classical trust region subproblem via a semi-definite programming (SDP) reformulation. The past two decades have witnessed numerous methods developed for solving the GTRS under various assumptions, see, for example, [3,4,9,11,14,19]. Sturm and Zhang further reveal that problem (P) admits an SDP reformulation. Still, the relatively large computational complexity of SDP algorithms prevents them from scaling to large-scale problems. Most fast algorithms [9,11] for the GTRS are developed under a regular condition that there exists a  $\lambda \in \Re$  such that  $D + \lambda A \succ 0$ . together with some other mild conditions. Recently, Ben-Tal and Hertog [2] show that if the two matrices in both the objective and constraint functions are simultaneously diagonalizable, the GTRS can be then transformed into an equivalent second order cone programming (SOCP) problem formulation. Simultaneous diagonalizability is actually a more general condition than the regular condition [4]. Conditions for simultaneous diagonalizability and corresponding algorithms are investigated recently in [7]. Compared with the SDP representation of the GTRS, the SOCP representation delivers a much faster solution algorithm, which is critical for solving large-scale problems in practice. This recognition motivates the investigation in our study.

We advance the state-of-the-art methods for solving the GTRS in this paper. More specifically, we prove that all GTRSs with an optimal value bounded from below are SOCP representable. To obtain the SOCP representation, we invoke and extend the congruent canonical form in Uhlig [15]. We first transform the two matrices into their canonical form of block diagonal matrices via congruence, and then make use of the block separability of the matrices in the canonical form. In particular, we derive necessary conditions from the canonical form for the GTRS to be bounded from below, and then show that the problem can further be transformed to an SOCP reformulation under such necessary conditions. Moreover, the attainableness of the optimal value is determined by the associated coefficients in the linear terms in both the objective and constraint functions. Note that the transformation into separable quadratic forms (the canonical form of block diagonal matrices) can be done off-line (independent of the algorithm) and the time complexity of the transformation is almost negligible when compared with that of the SDP reformulation. In fact, our method using simultaneous block diagonalization is a generalization of the simultaneous diagonalizability in [2].

We also extend our approach to two variants of problem (P) where the constraint is replaced by either an equality constraint,

(EP) min 
$$f(x) = \frac{1}{2}x^T Dx + e^T x$$
  
s.t.  $h(x) = \frac{1}{2}x^T Ax + b^T x + c = 0$ ,

or an interval constraint,

(IP) min 
$$f(x) = \frac{1}{2}x^T Dx + e^T x$$
  
s.t.  $c_1 \le h(x) = \frac{1}{2}x^T Ax + b^T x \le c_2$ 

Moré [9] presents a method for problem (EP) by using the saddle point optimality condition under some mild assumptions. Xia et al. [17] transform the problem (EP) to an SDP reformulation by using the S-lemma with equality under the conditions that  $A \neq 0$  and h(x) can take both positive and negative values. One application of problem (EP) is time of arrival problem [6]. Stern and Wolkowicz [13] propose a method for problem (IP) under b = 0 and the regular condition. By assuming b = 0 and the simultaneous diagonalizability of A and D, Ben-Tal and Teboulle [3] derive the hidden convexity of problem (IP) and thus transform the problem to an SOCP reformulation. Ye and Zhang [18] further show that problem (IP) admits an SDP reformulation if both the primal and dual Slater conditions are satisfied. Recently, strong duality conditions of (IP) are studied in Pong and Wolkowicz [11] and a fast method is provided under the regular condition. Wang and Xia [16] further simplify the conditions in [11] and develop the S-lemma with interval bounds to solve (IP). Ben-Tal and Hertog [2] further show that (IP) can be solved as an SOCP when A and D are simultaneously diagonalizable without the assumption of b = 0. Note that (IP) includes the equality constrained problem (EP) as a special case when setting  $c_1 = c_2$ . On the other hand, we will discuss in latter sections that solution methods for (EP) can also be used to solve (IP). Essentially, we will show that some slightly modified versions of our previous results for problem (P) hold true for the equality constrained problem (EP) and the interval constrained problem (IP).

To summarize, we derive necessary conditions for problem (P) and its variants with equality constraint or interval constraint to be bounded from below and further transform the problems to their SOCP reformulations by exploiting the block separability of the canonical form. Besides, we also derive the conditions for the attainableness of the problems. We emphasize that our methods are applicable for general situations without specific assumptions such as the regular condition. We would also like to indicate that our methods avoid involvement of linear matrix inequalities (LMI) as LMIs are usually hard to handle for large-scale problems. As by-products of our research, we further obtain simplified versions of the classical S-lemma, the S-lemma with equality and the S-lemma with interval bounds.

We organize our paper as follows. In Section 2, we introduce and extend a canonical form for the two matrices in both the objective and constraint functions by a real congruent transformation. After identifying all the situations in which the optimal value is bounded from below, we show that all remaining bounded cases of problem (P) can be transformed into an SOCP reformulation. In Section 3, we extend our methods to problems (EP) and (IP). Finally, we conclude our paper in Section 4.

**Notations:** Throughout this paper,  $I_m$  represents the  $m \times m$  identical matrix.  $\overline{1}$  denotes the all one vector  $(1, \ldots, 1)^T$ . The notation  $\Re^n$  represents the *n* dimensional vector space. For symmetric matrices *A* and *B*,  $A \succeq B$  denotes that matrix A - B is positive semi-definite. We denote the *Moore–Penrose pseudoinverse* by  $A^+$ . We use  $\operatorname{sign}(x)$  to denote the sign of a real number *x*, i.e.,  $\operatorname{sign}(x) = 1$ , if  $x \ge 0$ , otherwise  $\operatorname{sign}(x) = -1$ . And we use dim *A* to denote the dimension of a square matrix *A*. And  $A_{k:l,k:l}$  denotes the submatrix of matrix *A* by selecting the rows  $k, k + 1, \ldots, l$ , and the columns  $k, k + 1, \ldots, l$ . We also denote by  $\operatorname{diag}(A_1, \ldots, A_k)$  the block diagonal matrix

$$\left( \begin{array}{ccc} A_1 & 0 \\ & \cdot & \\ & \cdot & \\ 0 & A_k \end{array} \right).$$

We denote by E the anti-diagonal matrix

$$\begin{pmatrix}
0 & 1 \\
& \cdot \\
& \cdot \\
& \cdot \\
& 1 & 0
\end{pmatrix},$$
(1)

and by F the lower striped matrix

$$\begin{pmatrix} & 0 \\ & 0 & 1 \\ & \dots & \\ & 0 & 1 \\ & 0 & 1 \end{pmatrix}.$$
 (2)

We use  $J(\lambda, m)$  to denote an  $m \times m$  Jordan block

$$\begin{pmatrix} \lambda \ e \\ & \ddots \\ & \ddots \\ & \lambda \ e \\ & & \lambda \end{pmatrix}.$$

If the eigenvalue is a real number, i.e.,  $\lambda \in \Re$ , then e = 1 for  $m \ge 2$ , while  $J = (\lambda)$  for m = 1. If the eigenvalues form a complex pair, i.e.,  $a \pm bi$ , then  $\lambda = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ , with a and  $b \in \Re$ , and  $b \neq 0$ , and  $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  for  $m \ge 4$ , while  $J = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$  for m = 2.

Let  $v(\mathbf{P})$  denote the optimal value of problem (P). For an optimization problem  $\min\{f(x) \mid x \in X\}$  with a nonempty feasible set X, we define its optimal value as  $v = \inf\{f(x) \mid x \in X\}$ . For any  $\epsilon > 0$ , we call a solution  $\overline{x} \in X$  an  $\epsilon$  optimal solution, if  $f(\overline{x}) - v \leq \epsilon$ . Note that we reuse the notations h(x) and f(x) in problems (P), (EP) and (IP) and the simplified versions of the S-lemma with different constraints to keep the notations simple and help readers follow the main theme of the paper with ease.

#### 2 SOCP reformulation for GTRS

In this section, we first use congruent transformation to transform any two symmetric matrices into a canonical form of two block diagonal matrices. Then we explore different cases of the canonical form with respect to the boundedness of the optimal value and its attainability and transform problem (P) with its optimal value bounded from below to an equivalent SOCP problem.

2.1 Canonical form of two symmetric matrices

We invoke the following lemmas from Uhlig [15] to obtain a canonical form of any two real symmetric matrices.

**Lemma 1** (Theorem 1 in [15]) Let A and D be two  $n \times n$  real symmetric matrices. Suppose A is nonsingular. Let  $A^{-1}D$  have a Jordan normal form  $diag(J_1, \ldots, J_k)$ , where  $J_1, \ldots, J_k$  are Jordan blocks either with real eigenvalues or with complex eigenvalues. Then there exists an  $n \times n$  real congruent matrix S such that

$$S^T A S = \operatorname{diag}(\varepsilon_1 E_1, \dots, \varepsilon_k E_k)$$

and

$$S^T DS = \operatorname{diag}(\varepsilon_1 E_1 J_1, \dots, \varepsilon_k E_k J_k),$$

where  $\varepsilon_i = \pm 1$  and  $E_i$  is defined in (1). Furthermore, the signs of  $\varepsilon_i$ , i = 1, ..., k, are uniquely (up to permutations) associated with the Jordan blocks,  $J_i$ , i = 1, ..., k. In particular,  $\varepsilon_i = 1$  if  $J_i$  has a pair of complex eigenvalues.

**Lemma 2** (Theorem 2 in [15]) Let A and D be two singular real symmetric matrices and assume that there exists a  $\mu \in \Re$  such that  $C = A + \mu D$  is nonsingular. Let

$$J = \text{diag}(J(\lambda_1, n_1), \dots, J(\lambda_k, n_k), J(0, n_{k+1}), \dots, J(0, n_p), J(1/\mu, n_{p+1}), \dots, J(1/\mu, n_m))$$

be the Jordan normal form of  $C^{-1}D$ . Then there exists an  $n \times n$  real congruent matrix S such that

$$S^{T}AS = \text{diag}(\tau_{1}E_{1}, \dots, \tau_{k}E_{k}, \tau_{k+1}E_{k+1}, \dots, \tau_{p}E_{p}, \tau_{p+1}F_{p+1}, \dots, \tau_{m}F_{m})$$
(3)

and

$$S^{T}DS = \text{diag}(\tau_{1}E_{1}J(\kappa_{1}, n_{1}), \dots, \tau_{k}E_{k}J(\kappa_{k}, n_{k}), \tau_{k+1}F_{k+1}, \dots, \tau_{p}F_{p},$$
  
$$\tau_{p+1}E_{p+1}, \dots, \tau_{m}E_{m}),$$
(4)

where  $E_i$  and  $F_i$  are defined in (1) and (2), respectively,  $\tau_i = \pm 1$ ,  $i = 1, \ldots, m$ , dim  $E_i = \dim F_i = n_i$ ,  $i = k + 1, \ldots, m$ , and  $\kappa_i = \lambda_i/(1 - \mu\lambda_i)$ ,  $i = 1, \ldots, k$ . The signs of  $\tau_i$  are uniquely (up to permutations) determined by the associated Jordan blocks  $J(\kappa_i, n_i)$ ,  $E_i$  or  $F_i$ . In particular,  $\tau_i = 1$  if  $J(\lambda_i, n_i)$  has a pair of complex eigenvalues,  $i = 1, \ldots, k$ . Furthermore,  $p - k \ge 1$  and  $m - p \ge 1$ .

Next we generalize the results in the previous two lemmas to general situations where we do not assume the existence of the nonsingular matrix pencil for two symmetric matrices.

**Theorem 1** For any two  $n \times n$  real symmetric matrices A and D, there exists an  $n \times n$  real invertible matrix S such that

$$S^{T}AS = \operatorname{diag}(\tau_{1}E_{1}, \dots, \tau_{k}E_{k}, \tau_{k+1}E_{k+1}, \dots, \tau_{p}E_{p}, \tau_{p+1}F_{p+1}, \dots, \tau_{m}F_{m}, 0, \dots, 0)$$
(5)

and

$$S^{T}DS = \text{diag}(\tau_{1}E_{1}J(\kappa_{1},n_{1}),\dots,\tau_{k}E_{k}J(\kappa_{k},n_{k}),\tau_{k+1}F_{k+1},\dots,\tau_{p}F_{p},$$
  
$$\tau_{p+1}E_{p+1},\dots,\tau_{m}E_{m},0\dots,0)$$
(6)

where dim  $E_i$  = dim  $F_i$  =  $n_i$ , i = k + 1, ..., m, and  $\tau_i = \pm 1$ , i = 1, ..., m. The signs of  $\tau_i$  are uniquely (up to permutations) determined by the associated Jordan blocks  $J(\kappa_i, n_i)$ ,  $E_i$  or  $F_i$ . The values of  $\kappa_i$  are uniquely (up to permutations) determined by the associated Jordan blocks  $J(\kappa_i, n_i)$ . *Proof* Given Lemmas 1 and 2, we only need to consider in the proof the case where A and D are both singular and there does not exist a  $\mu \in \Re$  such that  $A + \mu D$  is nonsingular.

We can always find a congruent matrix  $Q_1$  such that  $\overline{A} \triangleq Q_1^T A Q_1 = \text{diag}(A_1, 0, \dots, 0)$ , where  $A_1$  is a  $q \times q$  diagonal matrix and q = rank(A). Denote

$$\bar{D} \triangleq Q_1^T D Q_1 = \begin{pmatrix} D_1 & D_2 \\ D_2^T & D_3 \end{pmatrix},$$

where  $D_1$  is a  $q \times q$  matrix. We can always find a congruent matrix S such that  $S^T D_3 S = \text{diag}(D_6, 0, \dots, 0)$ , where  $D_6$  is a nonsingular  $s \times s$  diagonal matrix. Let  $Q_2 \triangleq \text{diag}(I_q, S)$ . Then  $\hat{A} \triangleq Q_2^T \bar{A} Q_2 = \bar{A}$ , and

$$\hat{D} \triangleq Q_2^T \bar{D} Q_2 = \begin{pmatrix} D_1 \ D_4 \ D_5 \\ D_4^T \ D_6 \ 0 \\ D_5^T \ 0 \ 0 \end{pmatrix}.$$

Let

$$Q_3 \triangleq \begin{pmatrix} I_q & 0 & 0 \\ -D_6^{-1} D_4^T & I_s & 0 \\ 0 & 0 & I_{n-q-s} \end{pmatrix}.$$

Then,

$$\tilde{D} \triangleq Q_3^T \hat{D} Q_3 = \begin{pmatrix} D_1 - D_4 D_6^{-1} D_4^T & 0 & D_5 \\ 0 & D_6 & 0 \\ D_5^T & 0 & 0 \end{pmatrix}$$

and  $\tilde{A} \triangleq Q_3^T \hat{A} Q_3 = \hat{A} = \bar{A}$ . We can always choose a  $\mu \in \Re$  such that the first q columns of  $\tilde{D} + \mu \tilde{A}$  are linearly independent. For example, we can choose  $\mu = \max_{i=1,\dots,q} \sum_{j=1}^{q} |b_{ij}|/|a_{ii}| + 1$ , where  $b_{ij}$  is the element in the *i*th row and the *j*th column of  $D_1 - D_4 D_6^{-1} D_4^T$  and  $a_{ii}$  is the *i*th diagonal element of  $A_1$ . Then  $\mu A_1 + D_1 - D_4 D_6^{-1} D_4^T$  is nonsingular.

If the columns in  $D_5$  are linearly independent, then  $D + \mu A$  is nonsingular and thus  $D + \mu A$  is nonsingular, which contradicts our assumption of no nonsingular matrix pencil. Thus the columns in  $D_5$  are linearly dependent. We can always find a congruent matrix  $Q_4$  such that  $\check{A} = Q_4^T \tilde{A} Q_4 = \hat{A} = \bar{A}$ , and

$$\check{D} \triangleq Q_4^T \tilde{D} Q_4 = \begin{pmatrix} D_1 - D_4 D_6^{-1} D_4 & 0 & D_5' & 0 \\ 0 & D_6 & 0 & 0 \\ D_5'^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where  $D'_5$  is of full column rank. Let

$$A' \triangleq \begin{pmatrix} A_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, D' \triangleq \begin{pmatrix} D_1 - D_4 D_6^{-1} D_4^T & 0 & D_5' \\ 0 & D_6 & 0 \\ D_5'^T & 0 & 0 \end{pmatrix}.$$

Then there exists a  $\mu$  such that  $D' + \mu A'$  is nonsingular. From Lemma 2 we know that A' and D' can be congruent to the canonical form in (3) and (4). So A and D can be congruent to the canonical form in (5) and (6).

**Remark 1** If A and D are both singular and there does not exist an  $\mu \in \Re$  such that  $A + \mu D$  is nonsingular, then the number of the common 0 terms in the lower right part of (5) and (6) is equal to  $n - \operatorname{rank}(A' + \mu D')$ .

From Lemma 1, Lemma 2 and Theorem 1, we know that (5) and (6) represent a canonical form for any two real symmetric matrices A and D via congruence. Without loss of generality, we assume from now on that matrices A and D in problem (P) satisfy:

 $\begin{aligned} A &= \operatorname{diag}(A_1, A_2, \dots, A_s) \\ &= \operatorname{diag}(\tau_1 E_1, \dots, \tau_k E_k, \tau_{k+1} E_{k+1}, \dots, \tau_p E_p, \tau_{p+1} F_{p+1}, \dots, \tau_m F_m, 0, \dots, 0), \\ D &= \operatorname{diag}(D_1, D_2, \dots, D_s) \\ &= \operatorname{diag}(\tau_1 E_1 J(\kappa_1, n_1), \dots, \tau_k E_k J(\kappa_k, n_k), \tau_{k+1} F_{k+1}, \dots, \tau_p F_p, \tau_{p+1} E_{p+1}, \dots, \tau_m E_m, 0, \dots, 0). \end{aligned}$ 

Note that we have four kinds of block pairs  $(A_i, D_i)$ :  $(\tau_i E_i, \tau_i E_i J(\kappa_i, n_i))$ ,  $(\tau_i E_i, \tau_i F_i)$ ,  $(\tau_i F_i, \tau_i E_i)$  and (0, 0). In fact, the second kind of block pairs is a special case of the first kind with  $\kappa_i = 0$  due to  $E_i J(0, n_i) = F_i$ . We call the first two kinds of block pairs type A block pairs, the third kind of block pairs type B block pairs and the last one type C block pairs.

#### 2.2 SOCP reformulation from canonical form

Without loss of generality, we make the following assumptions.

**Assumption 1** i) There is at least one feasible solution in problem (P); ii) The following three conditions do not hold true at the same time:  $A \succeq 0$ ,  $b \in \text{Range}(A)$  and  $c = \frac{1}{2}b^T A^+ b$ .

Note that problem (P) is infeasible if and only if  $A \succeq 0$ ,  $b \in \text{Range}(A)$  and  $c = \frac{1}{2}b^TA^+b+k$  for some k > 0, which leads to  $h(x) = \frac{1}{2}(x+A^+b)^TA(x+A^+b)+k > 0$ . If  $A \succeq 0$ ,  $b \in \text{Range}(A)$  and  $c = \frac{1}{2}b^TA^+b$ , then  $h(x) = \frac{1}{2}x^TAx + b^Tx + c = \frac{1}{2}(x+A^+b)^TA(x+A^+b) \ge 0$ . Thus, the inequality constraint in problem (P) becomes an equality constraint which means all the feasible solutions are in the boundary. Actually, Assumption 2.1 is equivalent to Slation condition, i.e., there exists an  $\overline{x}$  Such that  $h(\overline{x}) < 0$ .

Moreover, when the three conditions in ii) hold together, problem (P) reduces to an unconstrained quadratic problem: Decompose A as  $A = L^T L$ , where  $L \in \Re^{r \times n}$  with r being the rank of A. Then, the constraint becomes  $(x + A^+b)^T L^T L(x + A^+b) = 0$  and thus  $L(x + A^+b) = 0$ . Rewrite  $x = -A^+b + Vy$ , where  $V \in \Re^{n \times (n-r)}$  is a matrix basis of the null space of L and  $y \in \Re^{n-r}$ . The problem then reduces to an unconstrained quadratic optimization problem.

Let us recall the S-lemma [10], which states the equivalence of the following two statements under Slation condition:

 $(\mathbf{S}_1) \ (\forall x \in \Re^n) \ h(x) \le 0 \Rightarrow f(x) \ge 0.$ 

(S<sub>2</sub>)  $\exists \mu \ge 0$  such that  $f(x) + \mu h(x) \ge 0, \forall x \in \Re^n$ .

In general, f(x) can be represented as  $f(x) = \frac{1}{2}x^T Dx + e^T x + v$  with an additional constant v in (S<sub>1</sub>) and (S<sub>2</sub>), and we use this representation of f(x) in the following of this section when discussing about the S-lemma and its variants. The connection between problem (P) and the S-Lemma is illustrated in [17] by

$$v(P) = \inf_{x \in \Re^n} \{ f(x) \mid h(x) \le 0 \}$$
  
= 
$$\sup_{\eta \in \Re} \{ \eta : \{ x \in \Re^n \mid f(x) < \eta, h(x) \le 0 \} = \emptyset \}$$
  
= 
$$\sup_{\eta \in \Re} \{ \eta \mid \exists \mu \ge 0 \text{ such that } f(x) - \eta + \mu h(x) \ge 0, \ \forall x \in \Re^n \}$$
  
= 
$$\sup_{\eta \in \Re, \mu \ge 0} \{ \eta \mid \left( \begin{array}{c} D + \mu A & e + \mu b \\ e^T + \mu b^T \ 2(v + \mu c - \eta) \end{array} \right) \succeq 0 \}.$$
(7)

By invoking the S-lemma, we have the following theorems.

**Theorem 2** Consider the case where a type A block pair  $(\tau_i E_i, \tau_i E_i J(\kappa_i, n_i))$ exists in problem (P). If the size of the associated Jordan block  $J(\kappa_i, n_i)$  is greater than 2 and the associated eigenvalue of the Jordan block is real, then the objective value of (P) is unbounded from below, i.e.,  $v(P) = -\infty$ .

*Proof* If the size of the associated Jordan block  $J(\kappa_i, n_i)$  is greater than 2, then  $\tau_i(E_i J(\kappa_i, n_i) + \mu E_i)$  takes the following form

$$\tau_i(E_i J(\kappa_i, n_i) + \mu E_i) = \tau_i \begin{pmatrix} \kappa_i + \mu \\ \kappa_i + \mu & 1 \\ \dots & \dots \\ \kappa_i + \mu & 1 \end{pmatrix}$$

Since the  $(n_i - 1) \times (n_i - 1)$  principal minor

$$\tau_i \begin{pmatrix} \kappa_i + \mu \ 1 \\ \cdots & \cdots \\ \kappa_i + \mu \ 1 \\ 1 \end{pmatrix}$$

is non-positive semi-definite when its size  $n_i - 1$  is greater than or equal to 2,  $D_i + \mu A_i = \tau_i (E_i J(\kappa_i, n_i) + \mu E_i)$  cannot be positive semi-definite. Thus, there is no  $\mu$  such that  $D + \mu A = \text{diag}(D_1 + \mu A_1, \dots, D_m + \mu A_m, 0, \dots, 0) \succeq 0$ . So the problem in (7) is infeasible and by the S-lemma we have  $v(\mathbf{P}) = -\infty$ .  $\Box$ 

Using similar proofs, we have the following theorems.

**Theorem 3** Consider the case where a type A block pair  $(\tau_i E_i, \tau_i E_i J(\kappa_i, n_i))$ exists in problem (P). If the eigenvalues of the associated Jordan block  $J(\kappa_i, n_i)$ form a complex pair, then the objective value of problem (P) is unbounded from below, i.e.,  $v(P) = -\infty$ . *Proof* If the eigenvalues of the associated Jordan block  $J(\kappa_i, n_i)$  form a complex pair, then there does not exist a  $\mu \in \Re$  such that

$$\tau_i(E_i J(\kappa_i, n_i) + \mu E_i) = \tau_i \begin{pmatrix} b_i & a_i + \mu \\ a_i + \mu & -b_i \\ \dots & \dots \\ b_i & a_i + \mu \\ a_i + \mu & -b_i \end{pmatrix} \succeq 0,$$

because the  $4 \times 4$  principal minor (if  $n_i = 4k$  for some positive integer k),

$$\tau_i \begin{pmatrix} b_i & a_i + \mu \\ a_i + \mu & -b_i \end{pmatrix} \\ b_i & a_i + \mu \\ a_i + \mu & -b_i \end{pmatrix}$$

or the  $2 \times 2$  principal minor (if  $n_i = 4k + 2$  for some positive integer k),

$$\tau_i \begin{pmatrix} b_i & a_i + \mu \\ a_i + \mu & -b_i \end{pmatrix}$$

is non-positive semi-definite. So the problem in (7) is infeasible and by the S-lemma we get  $v(P) = -\infty$ .

**Theorem 4** Consider the case where a type B block pair  $(\tau_i F_i, \tau_i E_i)$  exists in problem (P). If dim  $F_i \ge 2$ , then problem (P) is unbounded from below, i.e.,  $v(P) = -\infty$ .

*Proof* If the size of the associated Jordan block  $J(\kappa_i, n_i)$  is larger than or equal to 2, then there does not exist a  $\mu \in \Re$  such that

$$\tau_i(E_i + \mu F_i) = \tau_i \begin{pmatrix} 1 \\ 1 \\ \mu \\ \dots \\ 1 \\ \mu \\ 1 \\ \mu \end{pmatrix} \succeq 0.$$

So the problem in (7) is infeasible and by the S-lemma we have  $v(\mathbf{P}) = -\infty$ .

**Remark 2** Assumption 1 is necessary in Theorem 4. Otherwise the S-lemma does not hold and we have the following counter example:  $\min f(x) = x_1x_2$  subject to  $h(x) = x_2^2 \leq 0$ . The problem has a size 2 block pair  $(F_{2\times 2}, E_{2\times 2})$  but a finite optimal value of  $\min_{h(x) \leq 0} f(x) = 0$ .

So if problem (P) has a finite optimal solution, then any type B block pairs are of size 1 and any type A block pairs are of a size less than or equal to 2 and the eigenvalues in the associated Jordan blocks are real. Now let us consider

a type A block pair with size 2, and, without loss of generality, let it be the first block  $(A_1, D_1) = (\tau_1 E_1, \tau_1 E_1 J_1(\lambda, 2))$  with

$$E_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ E_1 J_1(\lambda, 2) = \begin{pmatrix} 0 & \lambda \\ \lambda & 1 \end{pmatrix}$$

Denote  $D = \text{diag}(D_1, D_J)$ ,  $A = \text{diag}(A_1, A_J)$ ,  $I = \{1, 2\}$ ,  $J = \{3, \dots, n\}$ ,  $e_I = (e_1, e_2)^T$ ,  $e_J = (e_3, \dots, e_n)^T$ ,  $b_I = (b_1, b_2)^T$ ,  $b_J = (b_3, \dots, b_n)^T$ ,  $z = (x_1, x_2)^T$ , and  $y = (x_3, \dots, x_n)^T$ . We can then represent problem (P) as follows,

(RP) min 
$$\frac{1}{2}y^T D_J y + e_J^T y + \frac{1}{2}z^T D_1 z + e_I^T z$$
  
s.t.  $\frac{1}{2}y^T A_J y + b_J^T y + \frac{1}{2}z^T A_1 z + b_I^T z + c \le 0$ 

The term in the constraint associated with  $(A_1, D_1)$  is

$$\frac{1}{2}z^{T}A_{1}z + b_{I}^{T}z = \frac{1}{2}\tau_{1}z^{T}E_{1}z + b_{I}^{T}z = \tau_{1}z_{1}z_{2} + b_{1}z_{1} + b_{2}z_{2},$$
(8)

and the term in the objective function associated with  $(A_1, D_1)$  is

$$\frac{1}{2}z^{T}D_{1}z + e_{I}^{T}z = \frac{1}{2}\tau_{1}z^{T}E_{1}J_{1}(\lambda,2)z + e_{I}^{T}z = \tau_{1}\lambda z_{1}z_{2} + \frac{1}{2}\tau_{1}z_{2}^{2} + e_{1}z_{1} + e_{2}z_{2}.$$
 (9)

Without loss of generality, we further assume  $b_1 = b_2 = 0$ . Since otherwise when letting  $z'_1 = z_1 + \tau_1 b_2$  and  $z'_2 = z_2 + \tau_1 b_1$ , the constraint function will become  $\frac{1}{2}y^T A_J y + b^T_J y + \frac{1}{2}{z'}^T A_1 z' + c'$ , where  $c' = c - \tau_1 b_1 b_2$ , and the objective function will become  $\frac{1}{2}y^T D_J y + e^T_J y + \frac{1}{2}{z'}^T D_1 z' + e'^T_I z' + d_0$ , where  $e'_1 = e_1 - \lambda b_1$ ,  $e'_2 = e_2 - b_1 - \lambda b_2$ , and  $d_0 = -e_1 \tau_1 b_2 - e_2 \tau_1 b_1 + \tau_1 \lambda b_1 b_2 + \frac{1}{2} \tau_1 b_1^2$ . Note that  $\tau_1 = \pm 1$  according to Theorem 1.

From now on, we assume that the coefficients in b corresponding to any  $2 \times 2$  type A Jordan block pair are 0.

**Theorem 5** Consider the case where there exists a type A block pair  $(\tau_1 E_1, \tau_1 E_1 J_1(\lambda, 2))$  in problem (P) and the eigenvalue of the associated Jordan block  $J_1(\lambda, 2)$  is real. Assume there is a feasible solution  $\bar{x} = (\bar{z}^T, \bar{y}^T)^T$  and let  $\pi = \tau_1 \bar{z}_1 \bar{z}_2$ . Let  $\rho = \inf\{ (9) \mid (8) = \tau_1 z_1 z_2 \leq \pi \}$ . We have the following three cases:

- 1. When  $\tau_1 = 1$ . If  $(\lambda \le 0, e_1 = 0, e_2 \ne 0)$  or  $(\lambda = 0, e_1 = 0, e_2 = 0, \pi \ge 0)$  or  $(\lambda < 0, e_1 = 0, e_2 = 0, \pi = 0)$ , then  $\rho = \lambda \pi \frac{1}{2}e_2^2$  and the infimum is attainable;
- 2. When  $\tau_1 = 1$ . If  $(\lambda = 0, e_1 = 0, e_2 = 0, \pi < 0)$  or  $(\lambda < 0, e_1 = 0, e_2 = 0, \pi \neq 0)$ , then  $\rho = \lambda \pi \frac{1}{2}e_2^2$  and the infimum is unattainable;
- 3. Otherwise,  $\rho = -\infty$  and thus problem (P) is unbounded from below.

*Proof* We consider the problem in the following cases:

- When  $\tau_1 = 1$ , (8) becomes  $z_1 z_2 \leq \pi$  and (9) becomes  $\lambda z_1 z_2 + \frac{1}{2} z_2^2 + e_1 z_1 + e_2 z_2$ . We then have the following cases corresponding to the values of  $\lambda$ ,  $e_1$ ,  $e_2$  and  $\pi$ .
  - When  $\lambda > 0$ , set  $z_1 = -\frac{M}{\lambda} M$  and  $z_2 = M$ , where  $M \in \Re$  is chosen such that  $-(1 + \frac{1}{\lambda})M^2 \leq \pi$ . Then  $z_1 z_2 \leq \pi$  and  $\rho = -(\lambda + \frac{1}{2})M^2 - (\frac{e_1}{\lambda} + e_1 - e_2)M \rightarrow -\infty$  when  $M \rightarrow \infty$ . (case 3)
  - When  $\lambda = 0$ , we have the following subcases:
    - When  $e_1 \neq 0$ , set  $z_1 = -\operatorname{sign}(e_1)M$  and  $z_2 = -\operatorname{sign}(e_1)\frac{\pi}{M}$ , where  $M \in \Re$  and  $z_1 z_2 = \pi$ . Then  $\rho = -|e_1|M + C_1 + C_2 \cdot \frac{1}{M} + C_3 \cdot \frac{1}{M^2} \to -\infty$  when  $M \to +\infty$ , where  $C_i, i = 1, 2, 3$ , are the reduced constants. (case 3)
    - When  $e_1 = 0$ , (9) becomes  $\frac{1}{2}z_2^2 + e_2z_2 = \frac{1}{2}(z_2 + e_2)^2 \frac{1}{2}e_2^2 \ge -\frac{1}{2}e_2^2$  $\Rightarrow \rho \ge -\frac{1}{2}e_2^2$ .
      - If  $e_2 \neq 0$ , set  $z_1 = -\frac{\pi}{e_2}$  and  $z_2 = -e_2$ , then  $z_1 z_2 = \pi$  and  $(9) = -\frac{1}{2}e_2^2$ . Thus  $\rho = -\frac{1}{2}e_2^2$  and the infimum is attainable. (case 1)
      - If  $e_2 = 0$  and  $\pi \ge 0$ , we can set  $z_1$  at any real value and  $z_2 = 0$  such that  $z_1 z_2 \le \pi$  and thus  $\rho = -\frac{1}{2}e_2^2$  and the infimum is attainable. (case 1)
      - If otherwise  $e_2 = 0$  and  $\pi < 0$ , we cannot set  $z_2 = -e_2 = 0$ , which contradicts the constraint  $z_1 z_2 \le \pi < 0$ . So the infimum is unattainable. But we can set  $z_1 = M\pi$  and  $z_2 = \frac{1}{M}$   $(M \in \Re)$ such that  $z_1 z_2 = \pi$  and  $(9) = \frac{1}{2M^2} \to 0$  when  $M \to \infty$ . Thus  $\rho = \lim_{M \to \infty} \frac{1}{2M^2} - \frac{1}{2}e_2^2 = 0$  but the infimum is unattainable. (case 2)
  - When  $\lambda < 0$ , we have the following subcases:
    - When  $e_1 \neq 0$ , set  $z_1 = -\text{sign}(e_1)M$  and  $z_2 = -\frac{\text{sign}(e_1)\pi}{M}$   $(M \in \Re)$ such that  $z_1 z_2 = \pi$  and then  $(9) = -|e_1|M + C_1 + C_2 \cdot \frac{1}{M} + C_3 \cdot \frac{1}{M^2} \rightarrow -\infty$  when  $M \rightarrow \infty$ , where  $C_i$ , i = 1, 2, 3 are the reduced constants. Thus  $\rho = -\infty$ . (case 3)
    - When  $e_1 = 0$ ,  $(9) = \lambda z_1 z_2 + \frac{1}{2} z_2^2 + e_2 z_2 \ge \lambda \pi + \frac{1}{2} (z_2 + e_2)^2 \frac{1}{2} e_2^2 \ge \lambda \pi \frac{1}{2} e_2^2$ . Next we show that  $\rho = \lambda \pi \frac{1}{2} e_2^2$ . We first note that, to achieve  $\lambda z_1 z_2 = \lambda \pi$  in the above inequality, we need to set  $z_1 z_2 = \pi$ .
      - · If  $e_2 \neq 0$ , set  $z_1 = -\frac{\pi}{e_2}$  and  $z_2 = -e_2$ , such that  $z_1 z_2 = \pi$  and then  $(9) = \lambda \pi - \frac{1}{2} e_2^2$ . (case 1)
      - If  $e_2 = 0$  and  $\pi \neq 0$ , we cannot set  $z_2 = -e_2 = 0$ , which contradicts the constraint  $z_1 z_2 = \pi \neq 0$ . So the infimum is unattainable. But we can set  $z_1 = M\pi$  and  $z_2 = \frac{1}{M}$   $(M \in \Re)$ such that  $z_1 z_2 = \pi$  and  $(9) = \lambda \pi + \frac{1}{2M^2} \to \lambda \pi$  when  $M \to \infty$ . So  $\rho = \lambda \pi - \frac{1}{2}e_2^2 = \lambda \pi$  and the infimum is unattainable. (case 2)
      - · If  $e_2 = 0$  and  $\pi = 0$ , we can set  $z_1$  at any real value and  $z_2 = 0$ and thus attain the infimum  $\rho = \lambda \pi - \frac{1}{2}e_2^2 = \lambda \pi$ . (case 1)

- When  $\tau_1 = -1$ , set  $z_1 = -\frac{\pi}{M}$  and  $z_2 = M$   $(M \in \Re)$  such that  $\tau_1 z_1 z_2 = -z_1 z_2 = \pi$  and  $(9) = -\frac{1}{2}M^2 + C_1 \cdot M + C_2 \cdot \frac{1}{M} + C_3 \to -\infty$  when  $M \to \infty$ , where  $C_i$ , i = 1, 2, 3, are the reduced constants. Thus  $\rho = -\infty$ . (case 3)

Since  $\inf\{ (9) \mid (8) = \tau_1 z_1 z_2 \leq \pi \}$  is a subproblem of (RP), if there is a feasible solution  $(\bar{z}^T, \bar{y}^T)^T$  for (RP) with  $\bar{z}_1 \bar{z}_2 = \pi$ , and  $\rho = -\infty$ , then  $v(P) = v(RP) = -\infty$ .

**Remark 3** 1. Case 2 in the above theorem is the only case where problem (P) is bounded from below but its infimum is unattainable.

- 2. If two matrices A and D are simultaneously diagonalizable via congruence, which covers most conditions discussed in the existing literature ([2,4,9]), then the optimal value is attainable when problem (P) is bounded from below.
- 3. All the bounded cases require  $e_1 = 0$  in the linear terms in the objective function associated with  $2 \times 2$  Jordan blocks.

The following theorem is a direct result of Theorems 2–5.

**Theorem 6** If problem (P) has an optimal value bounded from below, then:

- 1. dim  $E_i \leq 2$ , i = 1, ..., p, dim  $E_i = 1$ , i = p + 1, ..., m, and there is no complex eigenvalue pair in  $J(\kappa_i, n_i)$ ;
- 2. If for some index i, dim  $E_i = 2$ , then the *i*th block satisfies the conditions in either case 1 or case 2 in Theorem 5.

Note that the conditions in items 1 and 2 of Theorem 6 are necessary for problem (P) to be bounded from below and we assume these conditions hold in the following discussion of this section. Rearrange the block pairs with single elements to the upper left part of the diagonal in the canonical form and express A and D in the following forms,

$$A = \operatorname{diag}(\alpha_1, \dots, \alpha_l, E_1, \dots, E_{\frac{n-l}{2}}), \tag{10}$$

$$D = \operatorname{diag}(\delta_1, \dots, \delta_l, E_1 J(\zeta_1, 2), \dots, E_{\frac{n-l}{2}} J(\zeta_{\frac{n-l}{2}}, 2)),$$
(11)

where l is the number of block pairs with size 1 in the canonical form of (5) and (6). In the following, we assume A and D are in the form of (10) and (11) and further assume  $b_i = 0$ , i = l + 1, ..., n, as we discussed after (9).

Moreover, from item 3 in Remark 3, we have  $e_{l+2j-1} = 0$ ,  $j = 1, \ldots, \frac{n-l}{2}$ . Then problem (P) can be reduced to the following form:

(P<sub>1</sub>) min 
$$f(x) = \sum_{i=1}^{l} (\delta_i x_i^2 + e_i x_i) + \sum_{j=1,\dots,\frac{n-l}{2}} (\zeta_j x_{l+2j-1} x_{l+2j} + \frac{1}{2} x_{l+2j}^2 + e_{l+2j} x_{l+2j})$$
  
s.t.  $h(x) = \sum_{i=1}^{l} (\alpha_i x_i^2 + b_i x_i) + \sum_{j=1,\dots,\frac{n-l}{2}} (x_{l+2j-1} x_{l+2j}) + c \le 0.$ 

**Theorem 7** Assume that items 1 and 2 in Theorem 6 are satisfied, then  $v(P) = v(P_1) = v(P_2)$ , where  $(P_2)$  is the following SOCP problem:

(P<sub>2</sub>) min 
$$\sum_{i=1}^{l} (\delta_i y_i + e_i x_i) + \sum_{j=1}^{\frac{n-l}{2}} \zeta_j z_j + c_0$$
  
s.t.  $\sum_{i=1}^{l} (\alpha_i y_i + b_i x_i) + \sum_{j=1}^{\frac{n-l}{2}} z_j + c \le 0,$   
 $\frac{1}{2} x_i^2 - y_i \le 0, \ \forall i = 1, 2, \dots, l,$   
 $x, y \in \Re^l, \ z \in \Re^{\frac{n-l}{2}},$ 

where  $c_0 = -\sum_{j=1,\dots,\frac{n-l}{2}} \frac{1}{2} e_{l+2j}^2$ .

More specifically, if  $(P_2)$  admits an optimal solution, then there exists an optimal solution  $(\bar{x}, \bar{y}, \bar{z})$  to  $(P_2)$  with  $\frac{1}{2}\bar{x}_i^2 = \bar{y}_i$ , i = 1, 2, ..., l. Moreover, we can find an optimal solution (or an  $\epsilon$  optimal solution)  $\tilde{x}$  to  $(P_1)$  with

$$\tilde{x}_{i} = \bar{x}_{i}, i = 1, \dots, l, 
\tilde{x}_{l+2j} = \begin{cases}
1/M & \text{if } \begin{cases} \zeta_{j} = 0, e_{l+2j} = 0, \bar{z}_{j} < 0, \\ \text{or } \zeta_{j} < 0, e_{l+2j} = 0, \bar{z}_{j} \neq 0, \\
-e_{l+2j} & \text{otherwise}, \\
\tilde{x}_{l+2j-1} = \frac{\bar{z}_{j}}{\bar{x}_{l+2j}}, \quad j = 1, \dots, \frac{n-l}{2}.
\end{cases}$$
(12)

Particularly, if  $(P_1)$  is bounded from below, then the optimal value of  $(P_1)$  is unattainable if and only if  $\zeta_j = 0$ ,  $e_{l+2j} = 0$ ,  $\bar{z}_j < 0$  or  $\zeta_j < 0$ ,  $e_{l+2j} = 0$ ,  $\bar{z}_j \neq 0$ . In this case, for any  $\epsilon > 0$ , there exists an  $\epsilon$  optimal solution  $\tilde{x}$  such that  $f(\tilde{x}) - v(P_1) < \epsilon$  with a sufficiently large M > 0.

Proof Because of Theorem 6, (P<sub>1</sub>) is equivalent to (P). And the main differences between (P<sub>1</sub>) and (P<sub>2</sub>) are the terms associated to the 2 × 2 Jordan blocks. Let us consider how to simplify the terms associated with the 2 × 2 Jordan blocks. According to Assumption 1, (P<sub>1</sub>) is feasible. For any feasible solution  $\hat{x}$  of (P<sub>1</sub>), we let  $\pi_j = \hat{x}_{l+2j-1}\hat{x}_{l+2j}$ . Now let us concentrate on problem  $\inf\{\zeta_j x_{l+2j-1} x_{l+2j} + \frac{1}{2}x_{l+2j}^2 + e_{l+2j}x_{l+2j} | x_{l+2j-1}x_{l+2j} = \pi_j\} = \inf\{\zeta_j \pi_j + \frac{1}{2}x_{l+2j}^2 + e_{l+2j}x_{l+2j} = \pi_j\} = \inf\{\zeta_j \pi_j + \frac{1}{2}(x_{l+2j}^2 + e_{l+2j})^2 - \frac{1}{2}e_{l+2j}^2 | x_{l+2j-1}x_{l+2j} = \pi_j\}$ . Thus setting  $x_{l+2j-1} = -e_{l+2j}$  (if  $e_{l+2j} = 0$ , set  $x_{l+2j} = \frac{1}{M}$  as in the proof of Theorem 5) and  $x_{l+2j-1} = \frac{\pi_{j+2j}}{x_{l+2j}}$ , the objective function  $(\zeta_j x_{l+2j-1}x_{l+2j} + \frac{1}{2}x_{l+2j}^2 + e_{l+2j}x_{l+2j})$  has an infimum  $\zeta_j \pi_j - \frac{1}{2}e_{l+2j}^2$  under the constraint  $x_{l+2j-1}x_{l+2j} = \pi_j$ , which is linear with the cross term  $x_{l+2j-1}x_{l+2j} = \pi_j$ .

Using such a separability, we denote

$$z_j = x_{l+2j-1}x_{l+2j}$$
 and  $c_0 = -\sum_{j=1,\dots,\frac{n-l}{2}} \frac{1}{2}e_{l+2j}^2$ ,

and have the following problem which has the same objective value with  $(P_1)$ :

(P<sub>3</sub>) min 
$$\sum_{i=1}^{l} (\delta_i x_i^2 + e_i x_i) + \sum_{j=1}^{\frac{n-l}{2}} \zeta_j z_j + c_0$$
  
s.t.  $\sum_{i=1}^{l} (\alpha_i x_i^2 + b_i^T x_i) + \sum_{j=1}^{\frac{n-l}{2}} z_j + c \le 0.$ 

Moreover, if there is an optimal solution  $(\bar{x}, \bar{z})$  of  $(\mathbf{P}_3)$ , we can also find an optimal solution (or an  $\epsilon$  optimal solution)  $\tilde{x}$  of  $(\mathbf{P}_1)$  in the form of (12). In this case, the optimal value of  $(\mathbf{P}_1)$  is unattainable if and only if  $\zeta_j = 0$ ,  $e_{l+2j} = 0$ ,  $\bar{z}_j < 0$  or  $\zeta_j < 0$ ,  $e_{l+2j} = 0$ ,  $\bar{z}_j \neq 0$  from Theorem 5. Furthermore, for any  $\epsilon > 0$ , if we set  $M \ge \sqrt{\frac{1}{2\epsilon}}$ , then  $f(\tilde{x}) - v(\mathbf{P}_1) = \frac{1}{2M^2} \le \epsilon$ .

Introducing  $y_i = \frac{1}{2}x_i^2$ , i = 1, 2, ..., l, (P<sub>3</sub>) is then equivalent to the following (P<sub>4</sub>):

(P<sub>4</sub>) min 
$$\sum_{i=1}^{l} (\delta_i y_i + e_i x_i) + \sum_{j=1}^{\frac{n-l}{2}} \zeta_j z_j + c_0$$
  
s.t.  $\sum_{i=1}^{l} (\alpha_i y_i + b_i x_i) + \sum_{j=1}^{\frac{n-l}{2}} z_j + c \le 0$   
 $\frac{1}{2} x_i^2 - y_i = 0, \ \forall i = 1, 2, \dots, l,$   
 $x, y \in \Re^l, \ z \in \Re^{\frac{n-l}{2}}.$ 

We next prove the equivalence of  $(P_4)$  and  $(P_2)$  in two parts:

- 1. If  $(P_2)$  is unbounded from below, then  $(P_4)$  is unbounded from below.
- 2. If (P<sub>2</sub>) has an optimal solution  $(x^*, y^*, z^*)$ , then we can always find a solution  $(\bar{x}, \bar{y}, \bar{z})$  with  $\bar{y}_i = \frac{1}{2}\bar{x}_i^2$ , i = 1, ..., l and  $\bar{z} = z^*$ , which is optimal not only to (P<sub>2</sub>) but also to (P<sub>4</sub>).

The first part is proved in the following Lemma 3. Now let us prove part 2. Note that if  $(P_2)$  is bounded from below, then there must exist an optimal solution  $(x^*, y^*, z^*)$  since Slation condition is satisfied. Denote

$$J := \{i : \frac{1}{2}(x_i^*)^2 < y_i^*, \ i = 1, \dots, l\}.$$

If  $J = \emptyset$ , then  $(x^*, y^*, z^*)$  is also an optimal solution of  $(P_4)$ . If  $J \neq \emptyset$ , by Theorem 3 in [2], we can transform the optimal solution  $(x^*, y^*, z^*)$  of  $(P_2)$  to an optimal solution  $(\bar{x}, \bar{y}, z^*)$  of  $(P_2)$  with  $\bar{y}_i = \frac{1}{2}\bar{x}_i^2$ ,  $i = 1, \ldots, l$ , and  $(\bar{x}, \bar{y}, z^*)$ is also a feasible solution of  $(P_4)$ , since  $\bar{y}_i = \frac{1}{2}\bar{x}_i^2$ ,  $i = 1, \ldots, l$ . So  $v(P_2) \ge v(P_4)$ . But  $(P_2)$  is a relaxation of  $(P_4)$ , so  $v(P_2) \le v(P_4)$ . Thus  $v(P_2) = v(P_4)$  and  $(\bar{x}, \bar{y}, z^*)$  is optimal to  $(P_4)$ . **Lemma 3** If  $(P_2)$  is unbounded from below, then  $(P_4)$  is unbounded from below.

*Proof* We only need to prove that  $(P_3)$  (since  $(P_4)$  is equivalent to  $(P_3)$ ) is bounded from below implies that  $(P_2)$  is bounded from below.

In this proof, we only consider the cases with no z term in (P<sub>2</sub>) and (P<sub>3</sub>), since z only appears in linear terms in both the objective and constraint functions., which can be regarded as a special case of the x variable (i.e., the coefficients before  $z_j^2$  are  $0, j = 1, \ldots, \frac{n-l}{2}$ ).

Denote the Lagrangian function of (P<sub>3</sub>) as  $L(x,\nu) = f(x) + \nu h(x)$ , and the dual function as  $\theta(\nu) = \min_{x} L(x,\nu)$ , where  $\nu \ge 0$ . If (P<sub>3</sub>) is bounded from below, from the S-lemma (as Slation condition holds here), we know there is no duality gap between the primal problem (P<sub>3</sub>) and its Lagrangian dual problem of  $\max_{\nu\ge 0} \theta(\nu)$ , i.e., there exists  $(\bar{x},\bar{\nu})$  such that  $\min_{h(x)\le 0} f(x) = f(\bar{x}) =$  $\theta(\bar{\nu}) = \max_{\nu\ge 0} \theta(\nu)$ . So  $(\bar{x},\bar{\nu})$  is a saddle point of the Lagrangian function  $L(x,\nu)$ . Then  $f(\bar{x}) = \min_{x\in\Re^n} L(x,\bar{\nu})$ ,  $h(\bar{x}) \le 0$ ,  $\bar{\nu} \ge 0$ ,  $\bar{\nu}h(\bar{x}) = 0$ . From  $\min_{x\in\Re^n} L(x,\bar{\nu}) =$  $\min_{x\in\Re^n} \sum_{i=1}^l (\frac{1}{2}(\delta_i + \bar{\nu}\alpha_i)x_i^2 + (e_i + \bar{\nu}b_i)x_i) + c_0 + \bar{\nu}c = f(\bar{x})$ , we get  $\delta_i + \bar{\nu}\alpha_i \ge 0$ and  $(\delta_i + \bar{\nu}\alpha_i)\bar{x}_i + (e_i + \bar{\nu}b_i) = 0$  and if, in addition,  $\delta_i + \bar{\nu}\alpha_i = 0$ , we have  $e_i + \bar{\nu}b_i = 0$ ,  $i = 1, \ldots, l$ . So  $(\bar{x}, \bar{\nu})$  satisfies the KKT conditions of (P<sub>3</sub>), i.e.,  $(\delta_i + \bar{\nu}\alpha_i)\bar{x}_i + e_i + \bar{\nu}b_i = 0$ ,  $i = 1, \ldots, l$ ,  $\bar{\nu} \ge 0$ ,  $\bar{\nu}h(\bar{x}) = 0$ .

Next we can construct a KKT point of (P<sub>2</sub>) from the saddle point  $(\bar{x}, \bar{\nu})$ . Denote  $\bar{y}_i = \frac{1}{2}\bar{x}_i^2$  and  $\bar{\mu}_i = \delta_i + \bar{\nu}\alpha_i \ge 0$ , i = 1, ..., l. Then  $(\bar{x}, \bar{y}, \bar{\mu}, \bar{\nu})$  satisfies the KKT condition of (P<sub>2</sub>):  $\delta_i + \bar{\nu}\alpha_i - \bar{\mu}_i = 0$ ,  $e_i + \bar{\nu}b_i + \bar{\mu}_i\bar{x}_i = 0$ ,  $\bar{\mu}_i(\frac{1}{2}\bar{x}_i^2 - \bar{y}_i) = 0$ , i = 1, ..., l,  $\bar{\nu}(\sum_{i=1}^{l} (\alpha_i \bar{y}_i + b_i \bar{x}_i) + c) = 0$ . Thus  $(\bar{x}, \bar{y})$  is a global optimal solution of (P<sub>2</sub>) because of the convexity of (P<sub>2</sub>). So we conclude that (P<sub>2</sub>) is bounded from below.

**Example 1** Consider the following problem:

$$\min -x_1 x_2 + 0.5 x_2^2 - x_3^2 + x_4^2 + 2x_2 - x_4$$
  
s.t.  $x_1 x_2 + x_3^2 + 0.75 x_4^2 \le 1.25$ ,

where the related matrices can be expressed as

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1.5 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

 $e = (0, 2, 0, -1)^T$ , b = 0 and c = -1.25. Note that A and D are not simultaneously diagonalizable but in the canonical form (5) and (6). According to

Theorem 7, we get the following equivalent SOCP reformulation,

$$\min z_1 - 2y_3 + 2y_4 - x_4 - 2 \\ \text{s.t.} \quad z_1 + 2y_3 + 1.5y_4 \le 1.25 \\ \frac{1}{2}x_3^2 - y_3 \le 0 \\ \frac{1}{2}x_4^2 - y_4 \le 0.$$

Solving the above SOCP problem yields the optimal solution  $z_1^* = -12.9773$ ,  $x_3^* = 0, x_4^* = 0.2857, y_3^* = 7.0830$  and  $y_4^* = 0.0408$ . Note  $\frac{1}{2}(x_3^*)^2 - y_3^* = -7.0830 < 0$ . Using the transformation method in Theorem 3 in [2], we obtain a new solution  $(\bar{x}, \bar{y}, \bar{z})$ , with  $\bar{x}_3 = \pm \sqrt{2y_3^*} = \pm 3.7638$ ,  $\bar{z}_1 = z_1^*$ ,  $\bar{x}_3 = x_3^*, \bar{y}_3 = y_3^*$  and  $\bar{y}_4 = y_4^*$ . By applying Theorem 7, we get  $\tilde{x}_2 = -2$ ,  $\tilde{x}_1 = \frac{\bar{z}_1}{\bar{x}_2} = 6.4886, \tilde{x}_3 = \bar{x}_3$  and  $\tilde{x}_4 = \bar{x}_4$ . So we obtain an optimal solution  $\tilde{x} = (6.4886, -2, \pm 3.7638, 0.2857)^T$  to the origin problem, with an optimal value -3.3929.

#### 2.3 Dual problem of SOCP reformulation

The following theorem shows that the dual problem of  $(P_2)$  is a simple concave maximization problem with a single variable.

**Theorem 8** Under Assumption 2.1, the objective values of  $(P_2)$  and the following Lagrangian dual problem are equal:

(D<sub>1</sub>) 
$$\max_{\nu \ge 0} \{ \rho(\nu) = c\nu + c_0 + \sum_{i=1}^{l} h_i(\nu) + g(\nu) \}$$

where

$$h_i(\nu) = \begin{cases} -\frac{(\nu b_i + e_i)^2}{2(\nu \alpha_i + \delta_i)} & \text{if } \nu \alpha_i + \delta_i > 0, \\ 0 & \text{if } \nu \alpha_i + \delta_i = 0 \text{ and } e_i + \nu b_i = 0, \\ -\infty & \text{otherwise}, \end{cases}$$
(13)

 $i = 1, \ldots, l, and$ 

$$g(\nu) = \begin{cases} 0 & \text{if } \zeta_j + \nu = 0, \ \forall j = 1, \dots, \frac{n-l}{2}, \\ -\infty & \text{otherwise.} \end{cases}$$
(14)

*Proof* The Lagrangian function of  $(P_2)$  is:

$$\begin{split} L(x,y,z;\mu,\nu) &= \delta^T y + e^T x + \zeta^T z + c_0 + \nu (\alpha^T y + b^T x + \bar{1}^T z + c) \\ &+ \sum_{i=1}^l \mu_i (\frac{1}{2} x_i^2 - y_i) \\ &= \sum_{i=1}^l y_i (\delta_i + \nu \alpha_i - \mu_i) + \sum_{i=1}^l x_i (e_i + \frac{1}{2} \mu_i x_i + \nu b_i) \\ &+ \sum_{j=1}^{\frac{n-l}{2}} (\zeta_j + \nu) z_j + cv + c_0, \end{split}$$

where  $\mu \geq 0$  and  $\nu \geq 0$ .

The dual objective function is then given as

$$m(\mu,\nu) = \min_{x,y,z} L(x,y,z;\mu,\nu)$$

If  $m(\mu, \nu)$  is bounded from below, then the coefficients of  $y_i$  and  $z_i$  should be equal to 0. As the objective function is separable, we can get an analytical form of  $m(\mu, \nu)$ :

$$\max_{\mu \ge 0, \nu \ge 0} m(\mu, \nu) = \max_{\nu \ge 0} (c\nu + c_0 + \sum_{i=1}^{l} h_i(\nu) + g(\nu)),$$

where  $h_i(\nu)$  and  $g(\nu)$  are given in (13) and (14), respectively.

**Remark 4** If variable z exists in (P<sub>2</sub>), i.e., (P<sub>2</sub>) has some  $2 \times 2$  block pairs, then either all  $\zeta_i$ ,  $i = 1, \ldots, \frac{n-l}{2}$ , in (P<sub>2</sub>), are the same (there exists a  $\nu \ge 0$ such that  $\nu + \zeta_i = 0$ ,  $i = 1, \ldots, \frac{n-l}{2}$ , and  $v(D_1) = \rho(\nu)$ ), or, otherwise, there does not exist such a  $\nu$  and  $v(P_2) = v(D_1) = -\infty$ . The first case requires that all  $\zeta_i$  are equal, which is equivalent to that all  $2 \times 2$  Jordan blocks have the same eigenvalue, and we can directly calculate the optimal value by the dual problem (D<sub>1</sub>) using  $\nu = \zeta_i$ .  $\forall i$ . In fact, we characterize all the possible situations of the GTRS: If the GTRS is bounded from below, then there only exist  $1 \times 1$  or  $2 \times 2$  Jordan blocks in the canonical form and in addition all  $2 \times 2$  Jordan blocks have the same eigenvalue. While the results in Ben-Tal and Hertog [2] are restricted to simultaneously diagonalizable cases, our results identify all the cases where the GTRS is bounded from below, though only one situation (i.e., there exist  $2 \times 2$  Jordan blocks in the canonical form and all  $2 \times 2$  Jordan blocks have the same eigenvalue. While the results in Ben-Tal and Hertog [2] are restricted to simultaneously diagonalizable cases, our results identify all the cases where the GTRS is bounded from below, though only one situation (i.e., there exist  $2 \times 2$  Jordan blocks in the canonical form and all  $2 \times 2$  Jordan blocks have the same eigenvalue is proved to be possible bounded from below.

If the variable z does not exist in  $(P_2)$ ,  $(D_1)$  can be formulated as an SOCP:

$$\begin{aligned} (\mathbf{D}'_1) & \min c\nu + c_0 + \sum_{i=1}^n w_i \\ \text{s.t.} & (\nu\alpha_i + \delta_i + \frac{1}{2}w_i)^2 \ge (\nu\alpha_i + \delta_i - \frac{1}{2}w_i)^2 + (\nu b_i + e_i)^2, \ \forall i \\ & \nu\alpha_i + \delta_i \ge 0, \ \forall i \\ & \nu \ge 0. \end{aligned}$$

#### 2.4 A simplified S-lemma

Denote  $h(x) = \frac{1}{2}x^T Ax + b^T x + c$  and  $f(x) = \frac{1}{2}x^T Dx + e^T x + v$  in the following of this section when discussing about the S-lemma and its variants. First recall the classical S-lemma in Section 2.2 and note that LMI are involved in the statement (S<sub>2</sub>) from (7).

By applying the canonical form and the SOCP reformulation  $(P_2)$  of problem (P), we can obtain a simplified S-lemma, which shows the equivalence of the following statements  $(S_1)$  and  $(\tilde{S}_2)$  when Slation condition holds:

$$\begin{array}{l} (\mathbf{S}_{1}) \; (\forall x \in \Re^{n}) \; h(x) \leq 0 \Rightarrow f(x) \geq 0. \\ (\tilde{\mathbf{S}}_{2}) \; \exists x, y \in \Re^{l}, \; z \in \Re^{\frac{n-l}{2}} : \begin{cases} \delta^{T} y + e^{T} x + \zeta^{T} z + c_{0} + v \geq 0 \\ \alpha^{T} y + b^{T} x + \bar{1}^{T} z + c \leq 0 \\ \frac{1}{2} x_{i}^{2} - y_{i} \leq 0, \; \forall \; i = 1, \dots, l. \end{cases}$$

When solving the SOCP problem in  $(\tilde{S}_2)$ , we avoid LMI, which are hard to handle for large-scale problems. Moreover, our simplified version of S-lemma includes the simplified version of S-lemma in Ben-Tal and Hertog [2] as a special case.

#### 2.5 Algorithms for computing the canonical form

According to the proof of Theorem 1 in Uhlig [15], we present a method in the appendices to get the canonical form for two arbitrary  $n \times n$  symmetric matrices under the following conditions:

- 1. A is nonsingular;
- 2. The Jordan normal form of  $A^{-1}D$  has only  $1 \times 1$  and  $2 \times 2$  Jordan blocks with real eigenvalues and all the  $2 \times 2$  Jordan blocks have the same eigenvalue.

We assume that the first condition holds because from Lemma 2 and Theorem 1, calculation for the canonical form of two arbitrary matrices can be reduced to the situation that at least one of the matrices is nonsingular. We further assume the second condition because otherwise  $v(P) = -\infty$  (according to Remark 4).

In general, numerical computation for Jordan normal form is sometimes unstable, see Chapter 7 in [5] and [8]. Large Jordan blocks are difficult to handle due to the unstableness in calculating Jordan blocks, see [5,8]. On the other hand, problem (P) itself also has some problematic cases, i.e., a small perturbation of the matrices will cause a significant change of the optimal solution, see [11], which, we believe, are strongly related to the unstableness of the Jordan decomposition methods. Our analysis above sheds some light on the relationship between the unstable cases of the Jordan normal form and the problematic cases of problem (P). Fortunately, as we only need to calculate the Jordan normal form for real eigenvalues and the blocks are of small sizes of  $1 \times 1$  and  $2 \times 2$ . Furthermore, if there are multiple  $2 \times 2$  blocks, they must have the same eigenvalue. Thus, problem (P) is of a special structure in our implementation. The steps in the algorithms in appendices show that if the  $2 \times 2$  blocks are of the same eigenvalues and the number of the  $2 \times 2$  blocks are small, which covers all non-problematic cases of problem (P), the method to calculate the canonical form will be very fast, i.e., with time complexity  $O(n^3)$ . In the literature, there are  $O(n^3)$  Jordan decomposition methods, see [1,8].

# **3** Extension to equality constrained and interval bounded variants of GTRS

This section extends the usage of the canonical form and the SOCP reformulation to the equality constrained problem (EP) and the interval bounded problem (IP).

3.1 GTRS with equality constraint

We first make an assumption similarly to Assumption 1 to avoid some trivial cases.

**Assumption 2** i) There is at least one feasible solution in (EP); ii) The following conditions do not hold true at the same time:  $A \succeq 0$ ,  $b \in \text{Range}(A)$  and  $c = \frac{1}{2}b^T A^+ b$ ; and iii)  $A \neq 0$ .

This assumption is a "two-side" Slater condition (Assumption 1 in [17]) plus the condition  $A \neq 0$ . In fact, if A = 0, we can transform the constraint  $b^T x + c = 0$  to another quadratic equality constraint which satisfies all the three conditions in ii), i.e.,  $(b^T x + c)^2 = 0$ . So problem (EP) can be transformed to an unconstrained quadratic optimization problem using the null space representation of  $L(x + A^+b) = 0$ , by decomposing A as  $A = L^T L$ , when ii) is violated, or using the null space representation of  $b^T x + c = 0$  when iii) is violated.

With the same notations as in Section 2, Theorems 2, 3 and 4 still hold here, which can be proved in a similar way by the S-lemma with equality [17]. However, Theorem 5 needs some modifications. In the following of this section, we still use (8) and (9) to denote the associated terms in both the constraint and objective functions, i.e.,  $(8) = \frac{1}{2}\tau_1 z^T E_I z = \tau_1 z_1 z_2 = \pi$  and  $(9) = \tau_1 \lambda z_1 z_2 + \frac{1}{2}\tau_1 z_2^2 + e_1 z_1 + e_2 z_2.$ 

**Theorem 9** Consider the case where there exists a type A block pair  $(\tau_1 E_1, \tau_1 E_1 J_1(\lambda, 2))$  in problem (P) and the eigenvalue of the associated Jordan block  $J_1(\lambda, 2)$  is real. Assume there is a feasible solution  $\bar{x} = (\bar{z}^T, \bar{y}^T)^T$  and let  $\pi = \tau_1 \bar{z}_1 \bar{z}_2$ . Let  $\rho = \inf\{ (9) \mid (8) = \tau_1 z_1 z_2 = \pi \}$ . We have the following three cases:

- 1. When  $\tau_1 = 1$ . If  $(e_1 = 0, e_2 \neq 0)$  or  $(e_1 = 0, e_2 = 0, \pi = 0)$ , then  $\rho = \lambda \pi \frac{1}{2}e_2^2$  and the infimum is attainable;
- 2. When  $\tau_1 = 1$ . If  $e_1 = 0$ ,  $e_2 = 0$ ,  $\pi \neq 0$ , then  $\rho = \lambda \pi \frac{1}{2}e_2^2$  and the infimum is unattainable:
- 3. Otherwise,  $\rho = -\infty$  and thus (EP) is unbounded from below.

*Proof* The proof is similar to that of Theorem 5.

**Theorem 10** If the optimal value of problem (EP) is bounded from below, then:

- 1. dim  $E_i \leq 2, i = 1, ..., p, dim E_i = 1, i = p + 1, ..., m, and there is no$ complex eigenvalue pair in  $J(\kappa_i, n_i)$ ;
- 2. If for some index i, dim  $E_i = 2$ , then the ith block satisfies case 1 or case 2 in Theorem 9.

Note that the conditions in items 1 and 2 of Theorem 10 are necessary for problem (EP) to be bounded from below and we assume that the conditions hold in the following of this section. In the same way as the method in solving problem (P), we can then assume that A and D have the form in (10) and (11),  $b_j = 0$ , for j = l + 1, ..., n, and  $e_{l+2j-1} = 0$ ,  $j = 1, ..., \frac{n-l}{2}$ .

Similarly to Theorem 7, using the S-lemma with equality [17] under Assumption 2, we have the following theorem.

**Theorem 11** Assume that items 1 and 2 in Theorem 10 are satisfied, then problem (EP) has the same optimal value with the following SOCP reformu*lation:* 

(EP<sub>1</sub>) min 
$$\sum_{i=1}^{l} (\delta_i y_i + e_i x_i) + \sum_{j=1}^{\frac{n-l}{2}} \zeta_j z_j + c_0$$
  
s.t.  $\sum_{i=1}^{l} (\alpha_i y_i + b_i x_i) + \sum_{j=1}^{\frac{n-l}{2}} z_j + c = 0$   
 $\frac{1}{2} x_i^2 - y_i \le 0, \ \forall i = 1, 2, \dots, l,$   
 $x, y \in \Re^l, \ z \in \Re^{\frac{n-l}{2}},$ 

where  $c_0 = -\sum_{j=1,...,\frac{n-l}{2}} \frac{1}{2} e_{l+2j}^2$ . More specifically, if  $(\text{EP}_1)$  admits an optimal solution, then there exists an optimal solution  $(\bar{x}, \bar{y}, \bar{z})$  to  $(\text{EP}_1)$  with  $\frac{1}{2}\bar{x}_i^2 = \bar{y}_i$ ,  $i = 1, 2, \dots, l$ . Moreover, we can find an optimal solution (or an  $\epsilon$  optimal solution)  $\tilde{x}$  to (EP) with

$$\begin{aligned} \tilde{x}_i &= \bar{x}_i, \ i = 1, \dots, l, \\ \tilde{x}_{l+2j} &= \begin{cases} 1/M & \text{if } e_{l+2j} = 0, \ \bar{z}_j \neq 0, \\ -e_{l+2j} & \text{otherwise}, \\ \tilde{x}_{l+2j-1} &= \frac{\bar{z}_j}{\bar{x}_{l+2j}}, \ j = 1, \dots, \frac{n-l}{2}. \end{cases}$$

Particularly, if (EP) is bounded from below, the optimal value of (EP) is unattainable if and only if  $e_{l+2j} = 0$  and  $\bar{z}_j \neq 0$ . In this case, for any  $\epsilon > 0$ , there exists an  $\epsilon$  optimal solution  $\tilde{x}$  such that  $f(\tilde{x}) - v(\text{EP}) < \epsilon$  with a sufficient large M > 0.

Furthermore, if  $\zeta_i \neq \zeta_j$  for some  $i \neq j$ , where  $i, j \in \{1, \dots, \frac{n-l}{2}\}$ , problem (EP) is unbounded from below.

The dual problem of  $(EP_1)$  has the same form with  $(D_1)$ , except that  $\nu > 0$ is replaced by  $\nu \in \Re$ . From Theorem 11, we can also conclude that all the  $2 \times 2$ Jordan blocks have the same eigenvalue, i.e.,  $\zeta_i = \zeta_j$  for  $1 \le i, j \le \frac{n-l}{2}$ , if (EP<sub>1</sub>) is bounded from below.

**Theorem 12** Under Assumption 2, the objective values of  $(EP_1)$  and the following Lagrangian dual problem are equal:

(D<sub>1</sub>) 
$$\max_{\nu \in \Re} (c\nu + c_0 + \sum_{i=1}^{l} h_i(\nu) + g(\nu)),$$

where

$$h_i(\nu) = \begin{cases} -\frac{(\nu b_i + e_i)^2}{2(\nu \alpha_i + \delta_i)} & \text{if } \nu \alpha_i + \delta_i > 0, \\ 0 & \text{if } \nu \alpha_i + \delta_i = 0 \text{ and } e_i + \nu b_i = 0, \\ -\infty & \text{otherwise}, \end{cases}$$

i = 1, ..., l, and

$$g(\nu) = \begin{cases} 0 & \text{if } \zeta_j + \nu = 0, \ \forall j = 1, \dots, \frac{n-l}{2}, \\ -\infty & \text{otherwise.} \end{cases}$$

Assumption 2 is necessary in the above theorem, since otherwise  $(EP_1)$  may not be equivalent to (EP). In fact,  $v(EP_1) = -\infty$  and  $v(EP) > -\infty$  hold true for the following case stated in Theorem 3 in [17]: D has exactly one negative eigenvalue,  $A = 0, b \neq 0$  and

$$\begin{pmatrix} V^T D V & V^T (Dx_0 + e) \\ (x_0^T D + e^T) V & f(x_0) \end{pmatrix} \succeq 0,$$

where  $x_0 = -\frac{e}{2b^T b}b$ ,  $V \in \Re^{n \times (n-1)}$  is the matrix basis of the null space of the  $1 \times n$  matrix  $b^T$ . For example, consider the problem  $\min\{f(x) \mid h(x) = 0\}$ , where  $f(x) = 2x_1^2 - x_2^2$  and  $h(x) = x_1 - x_2$ . In this case  $v(\text{EP}_1) = -\infty$  but v(EP) = 0.

Now let us denote  $h(x) = \frac{1}{2}x^T Ax + b^T x + c$  and  $f(x) = \frac{1}{2}x^T Dx + e^T x + v$  in the following of this section. Recently, Xia et al. [17] give the conditions when the S-lemma with equality holds, with or without Assumption 1 in [17] (the "two-side" Slater condition). The S-lemma with equality asks if the following two statements are equivalent:

(E<sub>1</sub>)  $(\forall x \in \Re^n) h(x) = 0 \Rightarrow f(x) \ge 0.$ 

(E<sub>2</sub>)  $\exists \mu \in \Re$  such that  $f(x) + \mu h(x) \ge 0, \forall x \in \Re^n$ .

Theorem 3 in [17] states that under the "two-side" Slater condition, (E<sub>1</sub>) is equivalent to (E<sub>2</sub>) except for the following case: D has exactly one negative eigenvalue,  $A = 0, b \neq 0$  and

$$\begin{pmatrix} V^T D V & V^T (Dx_0 + e) \\ (x_0^T D + e^T) V & f(x_0) \end{pmatrix} \succeq 0,$$

where  $x_0 = -\frac{c}{2b^T h} b$  and  $V \in \Re^{n \times (n-1)}$  is the matrix basis of the null space of the  $1 \times n$  matrix  $b^T$ . In fact, the above case violates condition iii) in Assumption 2. So under our assumption, this case is out of consideration and the S-lemma with equality always holds in our investigation.

Under Assumption 2, by applying the canonical form and the SOCP reformulation of (EP), we obtain a simplified S-lemma with equality, which shows the following two statements are equivalent:

$$\begin{array}{l} (\mathbf{E}_1) \ (\forall x \in \Re^n) \ h(x) = 0 \Rightarrow f(x) \geq 0. \\ (\tilde{\mathbf{E}}_2) \ \exists x, y \in \Re^l, \ z \in \Re^{\frac{n-l}{2}} : \begin{cases} \delta^T y + e^T x + \zeta^T z + c_0 + v \geq 0 \\ \alpha^T y + b^T x + \bar{1}^T z + c = 0 \\ \frac{1}{2} x_i^2 - y_i \leq 0, \ \forall i = 1, \dots, l. \end{cases}$$

The equivalence between  $(E_1)$  and  $(\tilde{E}_2)$  can be directly derived from Theorem 11. One advantage of our simplified S-lemma with equality is its feature of LMI free, thus being more tractable for large-scale problems, while the S-lemma in [17] is not.

#### 3.2 GTRS with interval constraint

Similarly to the equality constrained case, we make the following assumption.

Assumption 3 i) There is at least one feasible solution in (IP); ii) The following conditions do not hold true at the same time:  $A \succeq 0, b \in \text{Range}(A)$ and  $c_i = \frac{1}{2}b^T A^+ b$  for i = 1 or 2; and iii)  $A \neq 0$ .

Theorem 10 holds in this case and thus we can still assume, without loss of generality, A and D have the form in (10) and (11),  $b_j = 0$ , for  $j = l+1, \ldots, n$ , and  $e_{l+2j-1} = 0$ ,  $j = 1, \ldots, \frac{n-l}{2}$ .

Theorem 13 Assume that the conditions in items 1 and 2 in Theorem 10 are satisfied, problem (IP) has the same optimal value with the following SOCP problem:

(IP<sub>1</sub>) min 
$$\sum_{i=1}^{l} (\delta_i y_i + e_i x_i) + \sum_{j=1}^{\frac{n-l}{2}} \zeta_j z_j + c_0$$
  
s.t.  $\bar{h}(x, y, z) = \sum_{i=1}^{l} (\alpha_i y_i + b_i x_i) + \sum_{j=1}^{\frac{n-l}{2}} z_j \ge c_1$   
 $\bar{h}(x, y, z) = \sum_{i=1}^{l} (\alpha_i y_i + b_i x_i) + \sum_{j=1}^{\frac{n-l}{2}} z_j \le c_2$   
 $\frac{1}{2} x_i^2 - y_i \le 0, \ \forall i = 1, 2, \dots, l,$   
 $x, y \in \Re^l, \ z \in \Re^{\frac{n-l}{2}},$ 

 $c_0 = -\sum_{j=1,\dots,\frac{n-l}{2}} \frac{1}{2} e_{l+2j}^2.$ More specifically, if (IP<sub>1</sub>) admits an optimal solution, then there exists an optimal solution  $(\bar{x}, \bar{y}, \bar{z})$  to (IP<sub>1</sub>) with  $\frac{1}{2}\bar{x}_i^2 = \bar{y}_i$ , i = 1, 2, ..., l. Moreover, we can find an optimal solution (or an  $\epsilon$  optimal solution) to (IP) with

$$\begin{aligned} \tilde{x}_i &= \bar{x}_i, \ i = 1, \dots, l, \\ \tilde{x}_{l+2j} &= \begin{cases} 1/M & \text{if } e_{l+2j} = 0, \ \bar{z}_j \neq 0, \\ -e_{l+2j} & \text{otherwise}, \end{cases} \\ \tilde{x}_{l+2j-1} &= \frac{\bar{z}_j}{\bar{x}_{l+2j}}, \ j = 1, \dots, \frac{n-l}{2}. \end{aligned}$$

Particularly, if (IP) is bounded from below, the optimal value of (IP) is unattainable if and only if  $e_{l+2j} = 0$  and  $\bar{z}_j \neq 0$ . In this case, for any  $\epsilon > 0$ , there exists an  $\epsilon$  optimal solution  $\tilde{x}$  such that  $f(\tilde{x}) - v(IP) < \epsilon$  with a sufficient large M > 0.

Furthermore, if  $\zeta_i \neq \zeta_j$  for some  $i \neq j$ , where  $i, j \in \{1, \ldots, \frac{n-l}{2}\}$ , problem (IP) is unbounded from below.

*Proof* (IP) is equivalent to the following (IP<sub>2</sub>):

(IP<sub>2</sub>) min 
$$\sum_{i=1}^{l} (\delta_i y_i + e_i x_i) + \sum_{j=1}^{\frac{n-l}{2}} \zeta_j z_j + c_0$$
  
s.t.  $-\sum_{i=1}^{l} (\alpha_i y_i + b_i x_i) - \sum_{j=1}^{\frac{n-l}{2}} z_j + c_1 \le 0$   
 $\sum_{i=1}^{l} (\alpha_i y_i + b_i x_i) + \sum_{j=1}^{\frac{n-l}{2}} z_j - c_2 \le 0$   
 $\frac{1}{2} x_i^2 - y_i = 0, \ \forall i = 1, 2, \dots, l,$   
 $x, y \in \Re^l, \ z \in \Re^{\frac{n-l}{2}}.$ 

So we only need to prove the equivalence between  $(IP_1)$  and  $(IP_2)$ .

By the S-lemma with interval bounds [16], similarly to Lemma 3, we know that if  $v(IP_1)$  is unbounded from below, then  $v(IP_2)$  is unbounded from below.

Now we consider the case where  $v(\text{IP}_1)$  is bounded from below. Then there exists a global minimum  $(x^*, y^*, z^*)$  for  $(\text{IP}_1)$ . The Fritz-John conditions of  $(\text{IP}_1)$  are stated as following: There exist  $\nu_0 \ge 0$ ,  $\nu_1 \ge 0$ ,  $\nu_2 \ge 0$ ,  $\mu_i \ge 0$ ,  $i = 1, \ldots, l$ , not all of which are zero, such that

$$\nu_0 \delta_i - (\nu_1 - \nu_2) \alpha_i - \mu_i = 0, \ \forall i = 1, \dots, l, \nu_0 e_i - (\nu_1 - \nu_2) b_i + \mu_i x_i^* = 0, \ \forall i = 1, \dots, l, \nu_0 \zeta_j - (\nu_1 - \nu_2) = 0, \ \forall j = 1, \dots, \frac{n-l}{2}.$$

We assume that  $\alpha_i$  and  $b_i$  are not both zero for  $i = 1, \ldots, l$ , otherwise  $x_i$  is a free variable only appearing in the objective function and then (IP<sub>1</sub>) is either unbounded from below or can be reduced to a new problem without variable  $x_i$ . Moreover, we cannot take equality in both sides of the quadratic constraint, so there must exist at least one strict inequality. Then, from the last equation

in Fritz-John conditions and the complementary slack conditions, we conclude  $\nu_1(\bar{h}(x^*, y^*, z^*) - c_1) = 0$ ,  $\nu_2(\bar{h}(x^*, y^*, z^*) - c_2) = 0$ , and one of  $\nu_1$  and  $\nu_2$  must be 0. Then from the first equation in Fritz-John conditions we know that if there exists some index i such that  $\frac{1}{2}(x_i^*)^2 - y_i^* < 0$ , together with the complementary slack conditions  $\mu_i(\frac{1}{2}(x_i^*)^2 - y_i^*) = 0$ , we conclude  $\mu_i = 0$  and  $\nu_0 > 0$  (otherwise  $(\nu_1 - \nu_2)\alpha_i = (\nu_1 - \nu_2)b_i = 0 \Rightarrow \nu_1 - \nu_2 = 0 \Rightarrow \nu_1 = \nu_2 = 0$  and thus  $\mu_i = 0$  for all i, which contradicts the fact that  $\nu_0, \nu_1, \nu_2, \mu_i, i = 1 \cdots, l$  are not all zero). So the Fritz-John conditions is reduced to the KKT conditions. Because one of  $\nu_1$  and  $\nu_2$  is 0, Assumption 6 in [2] holds. Then, by applying Theorem 7 in [2], we can get another optimal solution  $(\bar{x}, \bar{y}, \bar{z})$  to (IP<sub>1</sub>) with  $\frac{1}{2}\bar{x}_i^2 = \bar{y}_i, i = 1, 2, \ldots, l$ , and  $\bar{z} = z^*$ , which is also optimal to (IP<sub>2</sub>). The remaining of the proof is similar to that of Theorem 7.

Similarly, the dual problem of  $(IP_1)$  is a simple problem with two variables.

**Theorem 14** Under Assumption 2, the objective values of  $(IP_1)$  and the following Lagrangian dual problem are equal:

(ID<sub>1</sub>) 
$$\max_{\nu \ge 0} (\nu_1 c_1 - \nu_2 c_2 + c_0 + \sum_{i=1}^l h_i(\nu) + g(\nu)),$$

where

$$h_{i}(\nu) = \begin{cases} -\frac{(-\nu_{1}b_{i}+\nu_{2}b_{i}+e_{i})^{2}}{2(-\nu_{1}\alpha_{i}+\nu_{2}\alpha_{i}+\delta_{i})} & \text{if } -\nu_{1}\alpha_{i}+\nu_{2}\alpha_{i}+\delta_{i} > 0, \\ 0 & \text{if } -\nu_{1}\alpha_{i}+\nu_{2}\alpha_{i}+\delta_{i} = 0 \text{ and } e_{i}-\nu_{1}b_{i}+\nu_{2}b_{i} = 0, \\ -\infty & \text{otherwise}, \end{cases}$$
(15)

 $i = 1, \ldots, l, and$ 

$$g(\nu) = \begin{cases} 0 & \text{if } \zeta_j - \nu_1 + \nu_2 = 0, \text{ for all } j = 1, \dots, \frac{n-l}{2}, \\ -\infty & \text{otherwise.} \end{cases}$$
(16)

*Proof* The Lagrangian of  $(IP_1)$  is:

$$\begin{split} L(x,y,z;\mu,\nu) &= \delta^T y + e^T x + \zeta^T z + c_0 + \nu_1 (-\alpha^T y - b^T x - \bar{1}^T z + c_1) \\ &+ \nu_2 (\alpha^T y + b^T x + \bar{1}^T z - c_2) + \sum_{i=1}^l \mu_i (\frac{1}{2} x_i^2 - y_i) \\ &= \sum_{i=1}^l y_i (\delta_i + \nu_2 \alpha_i - \nu_1 \alpha_i - \mu_i) + \sum_{i=1}^l x_i (e_i + \frac{1}{2} \mu_i x_i + \nu_2 b_i) \\ &- \nu_1 b_i) + \sum_{j=1}^{\frac{n-l}{2}} (\zeta_j + \nu_2 - \nu_1) z_j + c_1 \nu_1 - c_2 \nu_2 + c_0. \end{split}$$

The dual Lagrangian function is then read as

$$m(\mu,\nu) = \min_{x,y,z} L(x,y,z;\mu,\nu)$$

As the objective function is separable, we can get an analytical form of  $m(\mu, \nu)$ :

$$\max_{\mu \ge 0, \nu \ge 0} m(\mu, \nu) = \max_{\nu \ge 0} (c_1 \nu_1 - c_2 \nu_2 + c_0 + \sum_{i=1}^{l} h_i(\nu) + g(\nu)),$$

where  $h_i(\nu)$  and  $g(\nu)$  are given in (15) and (16), respectively.

Since one of the two Lagrangian multipliers  $\nu_1$  and  $\nu_2$  must be zero, we can separate problem (ID<sub>1</sub>) into two problems with a single variable  $\nu_1$  or  $\nu_2$  by setting either  $\nu_1 = 0$  or  $\nu_2 = 0$ . Then problem (ID<sub>1</sub>) is reduced to a problem with the same form as problem (D<sub>1</sub>). Besides, problem (ID<sub>1</sub>) is also equivalent to an SOCP problem with two variables similar as problem (D<sub>1</sub>).

**Remark 5** Actually, problem (IP) must have an optimal solution on the boundary, except for the case where  $D \succeq 0$ ,  $e \in Range(D)$  and  $x = -D^+e$  is in the interior of the interval constraint. This is because, if the optimal solution  $x^*$  is not on the boundary, then  $x^*$  must be a local minimum and (IP) has only one local minimum under the conditions that D is semi-definite positive,  $e \in Range(D)$  and the local minimum is  $x = -D^+e$ . So we can first verify whether the conditions  $D \succeq 0$  and  $e \in Range(D)$  are satisfied and then check whether  $x = -D^+e$  is in the interior of the constraint. Otherwise, the optimal solution must be on the boundary. Then we can separate the problem into two equality constrained problems with an equality constraint  $h(x) = c_1$ or  $h(x) = c_2$ , and solve them with the methods for the equality constrained case. And the solution with the smaller optimal value of the above two equality constrained problems is the optimal solution of problem (IP).

We now denote  $h(x) = \frac{1}{2}x^T A x + b^T x$  and  $f(x) = \frac{1}{2}x^T D x + e^T x + v$ . Similarly to the simplified S-lemma with equality, we can simplify the S-lemma with interval bounds in [16] under Assumption 3 to the following form: (I<sub>1</sub>)  $\Leftrightarrow$  ( $\tilde{I}_2$ ).

$$\begin{array}{l} (\mathbf{I}_{1}) \; (\forall x \in \Re^{n}) \; c_{1} \leq h(x) \leq c_{2} \Rightarrow f(x) \geq 0. \\ (\tilde{\mathbf{I}}_{2}) \; \exists x, y \in \Re^{l}, \; z \in \Re^{\frac{n-l}{2}} : \begin{cases} \delta^{T}y + e^{T}x + \zeta^{T}z + c_{0} + v \geq 0, \\ c_{1} \leq \alpha^{T}y + b^{T}x + \bar{1}^{T}z \leq c_{2}, \\ \frac{1}{2}x_{i}^{2} - y_{i} \leq 0, \; \forall i = 1, \dots, l. \end{cases}$$

#### 4 Conclusions

In this study, we have successfully developed an SOCP reformulation for the GTRS, which is in fact a quadratic programming over a single nonconvex quadratic constraint. Particularly, we have derived the SOCP reformulation

under the condition that the GTRS is bounded from below, via a canonical congruent form of the two matrices in both the objective and constraint functions. While Ben-Tal and Hertog investigate in [2] the simultaneous diagonalizability of the two matrices, we explore the simultaneous block diagonalizability, which applies to arbitrary two matrices. More specifically, we introduce and extend the canonical form for two real matrices in [15] to find a block diagonal form for two matrices in both the objective and constraint functions. Exploiting the separability of the block diagonal form of the two matrices, we show that problem (P) is SOCP representable if it is bounded from below. We also establish the attainableness of the problem from the canonical form without any additional calculation. Moreover, as our SOCP reformulation is LMI free, it can be solved much faster than the SDP reformulation for the GTRS. We further extend our methods to solve variants of problem (P) where the inequality constraint is replaced by either an equality constraint or an interval constraint. Moreover, we obtain simplified versions of the S-lemma under the three kinds of constraints.

One of our future research is to consider variants of the GTRS with additional linear inequality constraints or two general quadratic constraints.

## Appendices

In the appendices, we present an algorithm to compute the canonical form if all the eigenvalues of the Jordan matrix of  $A^{-1}D$  (A is assumed invertible) are real and there only exist  $1 \times 1$  and  $2 \times 2$  Jordan blocks, where all the  $2 \times 2$  Jordan blocks have the same eigenvalue. Without loss of generality, we assume the non-singularity of matrix A for simplicity of the analysis.

**Definition 1** A single element a,  $a \ 2 \times 1 \ matrix \begin{pmatrix} 0 \\ a \end{pmatrix}$ ,  $a \ 1 \times 2 \ matrix (0, a)$ or  $a \ 2 \times 2 \ matrix \begin{pmatrix} 0 & a \\ a & b \end{pmatrix}$  is called a lower striped matrix, where  $a, b \in \Re$ .

**Definition 2** Let  $J(\lambda, n_1)$ ,  $J(\lambda, n_2)$ , ...,  $J(\lambda, n_l)$  denote all the Jordan blocks associated with the same eigenvalue  $\lambda$  of a real matrix A. Then  $C(\lambda) =$ diag $(J(\lambda, n_1), \ldots, J(\lambda, n_l))$ , where dim $J_i \geq$  dim $J_{i+1}$  for  $i = 1, \ldots, l-1$ , is called the full chain of Jordan blocks or full Jordan chain of length l associated with  $\lambda$ .

We present the following two algorithms to calculate the canonical form. Algorithm 1 is the main algorithm to compute the canonical form. Algorithm 2 is a subroutine to calculate the canonical form for the full Jordan chain associated with  $\lambda$ .

Let us look into Algorithm 1 first. Line 2 is to find the Jordan normal form of  $A^{-1}D$ , i.e.,  $J = V^{-1}A^{-1}DV$ , where V is some invertible matrix. Lines 3–5 are just to avoid some unbounded cases of problem (P). Line 6

updates A and D:  $A = V^T A V = \text{diag}(A_1, A_2, \dots, A_m)$  and  $D = V^T D V = \text{diag}(D_1, D_2, \dots, D_m)$ . According to the proof of Theorem 1 in [15], A and D are now both block diagonal matrices and the blocks  $A_i$  and  $D_i$  have the same dimension with  $C(\lambda_i)$ , (in fact  $A_i^{-1}D_i = C(\lambda_i)$ ), where  $C(\lambda_i)$  is the full Jordan chain associated with  $\lambda_i$ , and has the form  $C(\lambda_i) = \text{diag}(J_1(\lambda_i), J_2(\lambda_i), \dots, J_{l_i}(\lambda_i))$ , where

$$J_1(\lambda_i) = \ldots = J_k(\lambda_i) = \begin{pmatrix} \lambda_i & 1 \\ 0 & \lambda_i \end{pmatrix}$$
 and  $J_{k+1}(\lambda_i) = \ldots = J_{l_i}(\lambda_i) = \lambda_i$ .

Note that from the assumption there is at most one Jordan chain that has  $2 \times 2$ Jordan blocks and we further assume that this Jordan chain is associated with the eigenvalue  $\lambda_i$ . According to the Lemma 2 in [15],  $A_i$  is a block matrix in the following form,

$$\begin{pmatrix} A_{11}^i \cdots A_{1l}^i \\ \vdots & \vdots \\ A_{l1}^i \cdots A_{ll}^i \end{pmatrix},$$

where  $A_{pq}^i$  is a dim  $J_p(\lambda_i) \times \dim J_q(\lambda_i)$  lower striped matrix,  $p, q = 1, \ldots, l$  (we use  $l = l_i$  here and in the following for simplicity).

 $\label{eq:algorithm1} Algorithm \ 1 \ {\rm Calculate \ the \ canonical \ form \ for \ two \ real \ symmetric \ matrices}$ 

- **Input:** A and D are  $n \times n$  real symmetric matrices and A is nonsingular
- **Output:**  $\tilde{A}$  and  $\tilde{D}$  in the canonical form and the congruent matrix U and the final matrices  $\tilde{A} = U^T A U$  and  $\tilde{D} = U^T D U$
- 1: function CANONICAL-TRANSFORMATION(A, D)
- 2: Calculate the Jordan normal form for  $A^{-1}D$ , find a nonsingular matrix V such that  $J = V^{-1}A^{-1}DV = \text{diag}(C(\lambda_1), \ldots, C(\lambda_m))$ , where  $C(\lambda_i)$  is the full Jordan chain associated with the eigenvalue  $\lambda_i$ , denote  $l_i$  the number of blocks on the diagonal of  $C(\lambda_i)$  and  $k_i$  the number of  $2 \times 2$  blocks.<sup>1</sup>
- 3: if J has complex eigenvalues or Jordan blocks with size greater than 2 or more than one full Jordan chain has  $2 \times 2$  Jordan blocks **then**
- 4: return (in this case problem (P) is unbounded from below)
- 5: **end if**
- 6:  $A = V^T A V = \operatorname{diag}(A_1, A_2, \cdots, A_m) \text{ and } D = V^T D V = \operatorname{diag}(D_1, D_2, \cdots, D_m)$
- 7: **for**  $i = 1; i \le m; i + do$
- 8:  $(\check{A}_i, \check{D}_i, U_i) = \text{JBF}(A_i, D_i, C(\lambda_i), k_i, l_i)$ 9: end for
- 10:  $\check{A} = \operatorname{diag}(\check{A}_1, \dots, \check{A}_m), \, \check{D} = \operatorname{diag}(\check{D}_1, \dots, \check{D}_m), \, U = \operatorname{diag}(U_1, \dots, U_m)$ 11: end function

On Lines 7–10 of Algorithm 1, we switch to Algorithm 2 for the Jordan chain  $C(\lambda_i)$  and then on Line 10 we use diagonal matrix  $U = \text{diag}(U_1, \ldots, U_m)$  as a congruent matrix to compute the canonical form of A and B.

Next we describe Lines 2–12 in Algorithm 2. Assuming the first k Jordan blocks in  $C(\lambda_i)$  are of size  $2 \times 2$  (we use  $k = k_i$  here and in the following for simplicity), we next make a transformation to make  $A_{11}^i$  nonsingular if  $A_{11}^i$  is

Algorithm 2 Calculate the canonical form for a Jordan chain

- **Input:** A and D are  $l \times l$  real symmetric matrices and A is nonsingular, J is the Jordan matrix of  $A^{-1}D$  and has only  $1 \times 1$  and  $2 \times 2$  Jordan blocks with a real eigenvalue, where the first k blocks are  $2 \times 2$  blocks (k could be equal to 0) and the last l k blocks are  $1 \times 1$  blocks
- **Output:**  $\check{A}$  and  $\check{D}$  in the canonical form and the congruent matrix U such that  $\check{A} = U^T A U$ and  $\check{D} = U^T D U$

1: function  $(\check{A}, \check{D}, U) = \text{JBF}(A, D, J, k, l)$ 2:  $U = I_{l+k}$ 3: for  $j = 1; j \le k; j + +$  do  $X_j = I_{l+k}$ if the (j, j)th block of  $A, A_{jj}$  is singular then 4: 5: 6: 7:else $X_j = (18)$ 8: end if 9:  $Y_j = (19), \ U = UX_j Y_j$ 10: end for 11:  $\bar{A} = U^T A U$ 12:13:Find the associated congruent matrix Q of the spectral decomposition of lower right (l $(k-k) \times (l-k)$  submatrix  $\bar{A}_{(k+1):l,(k+1):l}$  $\bar{Q} = \operatorname{diag}(I_{2k}, Q), \, \tilde{A} = \bar{Q}^T \bar{A} \bar{Q}$ 14: for  $j = 1; j \le k; j + do$   $P_j = \begin{pmatrix} \frac{1}{\sqrt{|a_{j1}|}} & -\frac{a_{j2}}{2a_{j1}\sqrt{|a_{j1}|}} \\ 0 & \frac{1}{\sqrt{|a_{j1}|}} \end{pmatrix}$ (Note the *j*th block of  $\tilde{A}$  is  $\begin{pmatrix} 0 & a_{j1} \\ a_{j1} & a_{j2} \end{pmatrix}$ ) 1516:end for  $P_{k+1} = \text{diag}(\frac{1}{\sqrt{|a_{k+1}|}}, \cdots, \frac{1}{\sqrt{|a_l|}})$  (where  $a_i$  is the (2k+i)th element in the diagonal 17:18: $\dot{P} = \operatorname{diag}(P_1, \dots, P_k, P_{k+1}), \ \check{A} = P^T \tilde{A} P, \check{D} = \check{A} J, \ U = U \bar{Q} P$ 19: 20: end function

singular. If  $A_{11}^i$  is singular, we show that after a suitable permutation to  $A_i$ , the new  $A_{11}^i$  will be nonsingular. If there exists some s,  $2 \le s \le k$  such that  $A_{ss}^i$  is nonsingular, let

$$X_{i}^{1} = \operatorname{diag}\left(\begin{pmatrix} 0 & 0 & \cdots & 0 & I_{2} \\ 0 & I_{2} & & & \\ \vdots & \ddots & & \\ 0 & & I_{2} \\ I_{2} & & & 0 \end{pmatrix}_{2s \times 2s}, I_{2}, \dots, I_{2}, I_{l-k}\right)$$

then  $(X_i^1)^T A_i X_i^1$  has a nonsingular (1, 1) block. Otherwise all  $A_{ss}^i$  are singular,  $s = 1, \ldots, k$ . Since  $A_i$  is nonsingular and  $A_{11}^i, A_{12}^i, \ldots, A_{1l}^i$  are low striped matrices, there must be at least one nonsingular matrix among  $A_{12}^i, \ldots, A_{1k}^i$ (otherwise the first row of  $A_i$  is 0, which contradicts the nonsingularity of  $A_i$ ).

<sup>&</sup>lt;sup>1</sup> In the algorithm, we arrange all the  $2 \times 2$  blocks to the upper left part of  $C(\lambda_i)$  and all  $1 \times 1$  blocks to the lower right part when calculating the Jordan normal form.

Suppose that  $A^i_{1s}$  is nonsingular for  $2 \leq s \leq k$  and let

$$X_i^1 = I_{l+k} + \begin{pmatrix} 0_{2\times 2} & 0_{2\times(2s-2)} & -I_2 & 0_{2\times(l+k-2s)} \\ 0_{(2s-2)\times 2} & \ddots & 0_{(2s-2)\times 2} \\ I_2 & 0_{2\times(2s-2)} & 0_{2\times 2} \\ 0_{(l+k-2s)\times 2} & & & \end{pmatrix}.$$

Then

$$(X_i^1)^T A_i X_i^1 = \begin{pmatrix} A_{11}^i + A_{1s}^i + (A_{1s}^i)^T + A_{ss}^i * \\ * & * \end{pmatrix}$$

with nonsingular (1,1) block. Now, as  $A_i = (X_i^1)^T A_i X_i^1$  has nonsingular (1,1) block, we can transform  $A_i$  next to a simpler matrix with only 0 blocks in (1,t) and (t,1) positions via congruence, for all t = 2, ..., l. Let

$$Y_i^1 = I_{l+k} + \begin{pmatrix} 0 - (A_{11}^i)^{-1} A_{12}^i \cdots - (A_{11}^i)^{-1} A_{1l}^i \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then we get

$$A_i^1 = (Y_i^1)^T A_i Y_i^1 = \begin{pmatrix} A_{11}^{i,1} & 0 & \cdots & 0 \\ 0 & A_{22}^{i,1} & \cdots & A_{2l}^{i,1} \\ 0 & \vdots & \vdots \\ 0 & A_{l2}^{i,1} & \cdots & A_{ll}^{i,1} \end{pmatrix}.$$

Next, we conduct similar operations to the sub-matrix,

$$\begin{pmatrix} A_{22}^{i,1} \cdots A_{2l}^{i,1} \\ \vdots & \vdots \\ A_{l2}^{i,1} \cdots A_{ll}^{i,1} \end{pmatrix}$$

and then conduct similar operations again iteratively for k-2 times with the associated congruent matrices in the *j*th iteration. Let us be more specific using similar notations as in the first iteration. If some  $(s_j, s_j)$  block is nonsingular,  $j+1 \leq s_j \leq k$ , we introduce

$$X_{i}^{j} = \operatorname{diag}(\underbrace{I_{2}, \dots, I_{2}}_{j-1}, K, I_{2}, \dots, I_{2}, I_{l-k}),$$
(17)

where

$$K = \begin{pmatrix} 0 & 0 & \cdots & 0 & I_2 \\ 0 & I_2 & & & \\ \vdots & \ddots & & & \\ 0 & & I_2 & & \\ I_2 & & & 0 \end{pmatrix}_{2s_j \times 2s_j} .$$

Otherwise, we denote

$$X_{i}^{j} = I_{l+k} + \begin{pmatrix} 0_{(2j-2)\times(2j-2)} & 0 & 0 & 0 & 0\\ 0 & 0_{2\times2} & 0_{2\times(2s_{j}-2)} & -I_{2} & 0_{2\times m} \\ 0 & 0_{(2s_{j}-2)\times2} & \ddots & 0_{(2s_{j}-2)\times2} & 0\\ 0 & I_{2} & 0_{2\times(2s_{j}-2)} & 0_{2\times2} & 0\\ 0 & 0_{m\times2} & 0 & 0 & 0 \end{pmatrix}, (18)$$

where  $m = l + k - 2s_j - 2j$ , and

$$Y_{i}^{j} = I_{l+k} + \begin{pmatrix} 0_{(2j-2)\times(2j-2)} & 0_{2\times2} - (A_{22}^{i})^{-1}A_{23}^{i} \cdots - (A_{22}^{i})^{-1}A_{2l}^{i} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$
(19)

Then we get  $\overline{A}_i = A_i^k = (X_i^1 Y_i^1 \dots X_i^k Y_i^k)^T A_i (X_i^1 Y_i^1 \dots X_i^k Y_i^k) =$ 

$$\begin{pmatrix} A_{11}^{i,k} & & & \\ & \ddots & & & \\ & & A_{kk}^{i,k} & & \\ & & & A_{k+1,k+1}^{i,k} \dots & A_{k+1,l}^{i,k} \\ & & & \vdots & & \\ & & & & A_{l,k+1}^{i,k} \dots & A_{l,l}^{i,k} \end{pmatrix},$$

where  $A_{jj}^{i,k}$  is a 2 × 2 low striped matrix for j = 1, ..., k, and  $A_{pq}^{i,k}$  is a single number for p, q > k+1. (We rewrite in the following  $A_{jj}^k = A_{jj}^{i,k}$  for simplicity.) Lines 13 - 14 of Algorithm 2 state that: using spectral decomposition to

the following submatrix,

$$\bar{A}_{(k+1):l,(k+1):l} = \begin{pmatrix} A_{k+1,k+1}^k \dots A_{k+1,l}^k \\ \vdots & \vdots \\ A_{l,k+1}^k \dots & A_{l,l}^k \end{pmatrix},$$

we finally get  $Q_i^T \bar{A}_{(k+1):l,(k+1):l} Q_i = \text{diag}(a_{k+1}, \ldots, a_l)$ . Denote

$$W_i = X_i^1 Y_i^1 \dots X_i^k Y_i^k$$
 and  $\bar{Q}_i = \operatorname{diag}(I_{2k}, Q_i),$ 

and we get

$$\tilde{A}_i = \bar{Q}_i^T \bar{A}_i \bar{Q}_i = \operatorname{diag}(A_{11}^k, \dots, A_{kk}^k, a_{k+1}, \dots, a_l).$$

Since both  $W_i$  and  $\bar{Q}_i$  commute with  $C(\lambda_i)$  according to [15],

$$\begin{split} \tilde{D}_i &= (W_i \bar{Q}_i)^T D_i W_i \bar{Q}_i \\ &= (W_i \bar{Q}_i)^T A_i C(\lambda_i) W_i \bar{Q}_i \\ &= (W_i \bar{Q}_i)^T A_i W_i \bar{Q}_i C(\lambda_i) \\ &= \tilde{A}_i C(\lambda_i) \\ &= \text{diag}(D_{11}^k, \dots, D_{kk}^k, d_{k+1}, \dots, d_l) \end{split}$$

has the same block diagonal form with  $\tilde{A}$ .

The remaining of Algorithm 2 considers the transformation of the following block to the canonical form,

$$A_{jj}^k = \begin{pmatrix} 0 & a_{j1} \\ a_{j1} & a_{j2} \end{pmatrix}.$$

Introducing the following congruent matrix

$$P_i^j = \begin{pmatrix} \frac{1}{\sqrt{|a_{j1}|}} & -\frac{a_{j2}}{2a_{j1}\sqrt{|a_{j1}|}} \\ 0 & \frac{1}{\sqrt{|a_{j1}|}} \end{pmatrix},$$

we get

$$(P_i^j)^T A_{jj}^k P_i^j = \epsilon_i^j E_i^j$$

where  $E_i^j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ,  $\epsilon_i^j = 1$  if  $a_{j1} > 0$  or otherwise  $\epsilon_i^j = -1$ . Then from  $(P_i^j)^T A_{jj}^k P_i^j = \epsilon_i^j E_i^j$ , we know

$$(P_{i}^{j})^{T} D_{jj}^{k} P_{i}^{j} = (P_{i}^{j})^{T} A_{jj}^{k} P_{i}^{j} (P_{i}^{j})^{-1} J_{j}(\lambda_{i}) P_{i}^{j}$$
$$= (P_{i}^{j})^{T} A_{jj}^{k} P_{i}^{j} J_{j}(\lambda_{i}) (P_{i}^{j})^{-1} P_{i}^{j}$$
$$= \epsilon_{i}^{j} E_{i}^{j} J_{j}(\lambda_{i}),$$

where the second last equality is due to that  $P_i^j$  commutes with  $J_j(\lambda_i)$ , see Lemma 2 in [15]. Thus the *j*th block of  $(A_i, D_i)$  is in the canonical form. For the last l - k elements in  $\tilde{D}_i$ , we have

$$\tilde{A}_{(k+1):l,(k+1):l} = \operatorname{diag}(a_{k+1},\ldots,a_l)$$

Let

$$P_i^{k+1} = \operatorname{diag}(\frac{1}{\sqrt{|a_{k+1}|}}, \dots, \frac{1}{\sqrt{|a_l|}}),$$

then

$$(P_i^{k+1})^T \tilde{A}_{(k+1):l,(k+1):l} P_i^{k+1} = \operatorname{diag}(\epsilon_i^{k+1}, \dots, \epsilon_i^l)$$

Denote  $P_i = \text{diag}(P_i^1, \dots, P_i^k, P_i^{k+1})$ , then

$$\check{A}_i = P_i^T \tilde{A}_i P_i = \operatorname{diag}(\epsilon_i^1 E_i^1, \dots, \epsilon_i^k E_i^k, \epsilon_i^{k+1}, \dots, \epsilon_i^l)$$

is already in the canonical form (5). Besides,

$$\check{D}_i = P_i^T \tilde{D}_i P_i = P_i^T \tilde{A}_i C(\lambda_i) P_i = P_i^T \tilde{A}_i P_i C(\lambda_i) = \check{A}_i C(\lambda_i)$$

satisfies (6) (the second last equality is due to that the two block diagonal matrices  $C(\lambda_i)$  and  $P_i$  commute).

Note in the above procedure, if there are too many  $2 \times 2$  blocks in the same Jordan chain, the time cost is very large. But this is really rare in practice since it is actually rare that any two eigenvalues of  $A^{-1}D$  are the same and much rarer if two  $2 \times 2$  blocks have the same eigenvalue, in which case at least 4 eigenvalues are the same.

For other Jordan chains, we only have  $1 \times 1$  Jordan blocks and thus it can be easily transformed to the canonical form by Algorithm 2 on Lines 13–14 and 18–19 and the time required is very little when compared with the Jordan chains that has  $2 \times 2$  blocks.

#### References

- Beelen, T., van Dooren, P.: Computational aspects of the Jordan canonical form. In: Cox, M.G., Hammarling, S.J. (eds.) Reliable Numerical Computation, pp. 57–72. Oxford University Press. Oxford (1990)
- Ben-Tal, A., den Hertog, D.: Hidden conic quadratic representation of some nonconvex quadratic optimization problems. Math. Program. 143(1-2), 1–29 (2014)
- Ben-Tal, A., Teboulle, M.: Hidden convexity in some nonconvex quadratically constrained quadratic programming. Math. Program. 72(1), 51–63 (1996)
- Feng, J.-M., Lin, G.-X., Sheu, R.-L., Xia, Y.: Duality and solutions for quadratic programming over single non-homogeneous quadratic constraint. J. Global Optim. 54(2), 275–293 (2012)
- 5. Golub, G.H., van Loan, C.F.: Matrix computations, 3nd edn. Johns Hopkins University Press, Baltimore (1996)
- Hmam, H.: Quadratic optimization with one quadratic equality constraint. Warfare and Radar Division DSTO Defence Science and Technology Organisation, Australia, Report DSTO-TR-2416 (2010)
- Jiang, R., Li, D.: Simultaneous diagonalization of matrices and its application in quadratically constrained quadratic programming. Arxiv, preprint arXiv: 1507.05703 (2015)
- 8. Kågström, B., Ruhe, A.: An algorithm for numerical computation of the jordan normal form of a complex matrix. ACM Trans. Math. Software. **6**(3), 398–419 (1980)
- Moré, J.J.: Generalizations of the trust region problem. Optim. Methods Softw. 2(3-4), 189–209 (1993)
- 10. Pólik, I., Terlaky, T.: A survey of the s-lemma. SIAM Rev. 49(3), 371-418 (2007)
- Pong, T.K., Wolkowicz, H.: The generalized trust region subproblem. Comput. Optim. Appl. 58(2), 273–322 (2014)
- 12. Rendl, F., Wolkowicz, H.: A semidefinite framework for trust region subproblems with applications to large scale minimization. Math. Program. **77**(1), 273–299 (1997)
- Stern, R., Wolkowicz, H.: Indefinite trust region subproblems and nonsymmetric eigenvalue perturbations. SIAM J. Optim. 5(2), 286–313 (1995)
- Sturm, J.F., Zhang, S.: On cones of nonnegative quadratic functions. Math. Oper. Res. 28(2), 246–267 (2003)
- Uhlig, F.: A canonical form for a pair of real symmetric matrices that generate a nonsingular pencil. Linear Algebra Appl. 14(3), 189–209 (1976)
- Wang, S., Xia, Y.: Strong duality for generalized trust region subproblem: S-lemma with interval bounds. Optim. Lett. 9, 1063-C1073, (2015)

- 17. Xia, Y., Wang, S., Sheu, R.L.: S-lemma with equality and its applications. Math.
- Program., Ser. A., to appear 2015, doi:10.1007/s10107-015-0907-0
  Ye, Y., Zhang, S.: New results on quadratic minimization. SIAM J. Optim. 14(1), 245-267 (2003)
  Yuan, Y.X.: On a subproblem of trust region algorithms for constrained optimization.
- Math. Program. 47(1-3), 53-63 (1990)