TENSOR EIGENVALUE COMPLEMENTARITY PROBLEMS

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ABSTRACT. This paper studies tensor eigenvalue complementarity problems. Basic properties of standard and complementarity tensor eigenvalues are discussed. We formulate tensor eigenvalue complementarity problems as constrained polynomial optimization. When one tensor is strictly copositive, the complementarity eigenvalues can be computed by solving polynomial optimization with normalization by strict copositivity. When no tensor is strictly copositive, we formulate the tensor eigenvalue complementarity problem equivalently as polynomial optimization by a randomization process. The complementarity eigenvalues can be computed sequentially. The formulated polynomial optimization can be solved by Lasserre's hierarchy of semidefinite relaxations. We show that it has finite convergence for generic tensors. Numerical experiments are presented to show the efficiency of proposed methods.

1. Introduction

Let \mathbb{R} be the real field, \mathbb{R}^n be the space of all real *n*-dimensional vectors, and $\mathbb{R}^{n \times n}$ be the space of all real *n*-by-*n* matrices. Denote by \mathbb{R}^n_+ the nonnegative orthant, i.e., the set of vectors in \mathbb{R}^n whose entries are all nonnegative.

The classical matrix eigenvalue complementarity problem (MEiCP) is that: for given two matrices $A, B \in \mathbb{R}^{n \times n}$, we want to find a number $\lambda \in \mathbb{R}$ and a nonzero vector $x \in \mathbb{R}^n$ such that

$$(1.1) 0 \le x \perp (\lambda Bx - Ax) \ge 0.$$

In the above, $a \perp b$ means that the two vectors a, b are perpendicular to each other. For (λ, x) satisfying (1.1), λ is called a complementary eigenvalue of (A, B) and x is called the associated complementary eigenvector. MEiCPs have wide applications, such as static equilibrium states of mechanical systems with unilateral friction [34], the dynamic analysis of structural mechanical systems [23, 24] and the contact problem in mechanics [25]. The MEiCP (1.1) has at least one solution if $x^T B x \neq 0$ for all $x \in \mathbb{R}^n_+ \setminus \{0\}$ (cf. [16, 36]). When A and B are symmetric, the problem (1.1) can be reduced to finding a stationary point of the quotient $x^T A x / x^T B x$ over the standard simplex. For such cases, nonlinear optimization methods can be applied to solve MEiCPs (cf. [15, 40]). When A, B are not symmetric, other approaches were proposed for solving MEiCPs, such as the branch-and-bound technique [14, 16], the scaling-and-projection and the power iteration [35, 36], semismooth Newton-type

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methods [1, 2]. Most existing methods aim at computing one of the complementarity eigenvalues. The matrix complementarity problem is NP-hard [14].

Eigenvalues were recently studied for tensors [9, 13, 21, 38]. For an integer m > 0, an m-th order n-dimensional tensor \mathcal{A} is a multi-array indexed as

$$\mathcal{A} := (\mathcal{A}_{i_1,\dots,i_m})_{1 \leq i_1,\dots,i_m \leq n}.$$

Let $T^m(\mathbb{R}^n)$ be the space of all such real tensors. For $x := (x_1, \dots, x_n) \in \mathbb{R}^n$, denote by $\mathcal{A}x^{m-1}$ the vector in \mathbb{R}^n such that, for each $i = 1, 2, \dots, n$,

(1.2)
$$(\mathcal{A}x^{m-1})_i = \sum_{i_2,\dots,i_m=1}^n \mathcal{A}_{i,i_2,\dots,i_m} x_{i_2} \cdots x_{i_m}.$$

Denote by Ax^m the homogeneous polynomial

$$Ax^{m} = \sum_{i_{1}, i_{2}, \dots, i_{m}=1}^{n} A_{i_{1}, i_{2}, \dots, i_{m}} x_{i_{1}} x_{i_{2}} \cdots x_{i_{m}}.$$

Clearly, $Ax^m = \sum_{j=1}^n x_j (Ax^{m-1})_j$. Lim [21] and Qi [38] introduced the notion of tensor eigenvalues. Generalized eigenvalues can be defined similarly for tensors [9]. For two nonzero tensors $A, B \in T^m(\mathbb{R}^n)$, if a pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ satisfies the equation

$$\lambda \mathcal{B}x^{m-1} - \mathcal{A}x^{m-1} = 0,$$

then λ is called a \mathcal{B} -eigenvalue of \mathcal{A} and x is the associated \mathcal{B} -eigenvector. Such (λ, x) is called a \mathcal{B} -eigenpair. Recently, Cui, Dai and Nie [7] studied \mathcal{B} -eigenvalues of symmetric tensors. They proposed a semidefinite relaxation approach for computing all real \mathcal{B} -eigenvalues sequentially, from the largest to the smallest. Each eigenvalue can be computed by solving a finite hierarchy of semidefinite relaxations. This approach was originally used for computing the hierarchy of local minimums for polynomial optimization [33].

Recently, Ling et al. [22] introduced the tensor eigenvalue complementarity problem (TEiCP): for two given tensors $\mathcal{A}, \mathcal{B} \in \mathcal{T}^m(\mathbb{R}^n)$, we want to find a number $\lambda \in \mathbb{R}$ and a nonzero vector $x \in \mathbb{R}^n$ such that

$$(1.4) 0 \le x \perp (\lambda \mathcal{B}x^{m-1} - \mathcal{A}x^{m-1}) \ge 0.$$

For such a pair (λ, x) , λ is called a complementary eigenvalue of $(\mathcal{A}, \mathcal{B})$ and x is called the associated complementary eigenvector. For convenience, the complementary eigenvalues and eigenvectors are respectively called C-eigenvalues and C-eigenvectors. The above (λ, x) is called a C-eigenpair. Clearly, when m = 2, the TEiCP is reduced to the classical matrix eigenvalue complementarity problem. TEiCPs have wide applications such as higher-order Markov chains [26], magnetic resonance imaging [39]. We refer to [5, 22] for more applications of TEiCPs.

In the existing references (cf. [22]), C-eigenvalues defined as in (1.4) are also called *Pareto-eigenvalues*. Indeed, Ling et al. [22] considered more general tensor eigenvalue complementarity problems, where the conditions $x \geq 0$ and $\lambda \mathcal{B} x^{m-1} - \mathcal{A} x^{m-1} \geq 0$ are replaced by

$$x \in K$$
, $\lambda \mathcal{B} x^{m-1} - \mathcal{A} x^{m-1} \in K^*$.

Here, K is a closed convex cone and K^* is the dual cone. In [22], it was shown that the TEiCP has at least one solution, under the assumption that $\mathcal{B}x^m \neq 0$ for all $x \in \mathbb{R}^n_+ \setminus \{0\}$. They also gave an upper bound for the number of C-eigenvalues, for

nonsingular tensor pairs (A, B) (see §3.1 for the definition). Moreover, a scaling-and-projection algorithm was given for solving TEiCPs. Recently, Chen et al. [5] have further new work on TEiCPs. When the tensors are symmetric, they reformulated the problem as nonlinear optimization and then proposed a shifted projected power method. Chen and Qi [4] reformulated the TEiCP as a system of nonlinear equations and proposed a damped semi-smooth Newton method for solving it. Some properties of Pareto-eigenvalues are further studied in [42]. Generally, the tensor eigenvalue complementarity problem is difficult to solve. It is also NP-hard, since the TEiCP includes the MEiCP as a special case.

Contributions In this paper, we study how to solve TEiCPs. Our aim is to compute all C-eigenvalues, if there are finitely many ones. We formulate TEiCPs equivalently as polynomial optimization problems, and then solve them by Lasserre type semidefinite relaxations. Throughout the paper, a property is said to be generically true in a tensor space if it holds in an open dense subset of that space, in the Zariski topology. For such a property, a tensor in that open dense set is called a generic tensor.

First, we study properties of generalized eigenvalues of tensor pairs. For non-singular tensor pairs, it is known that the number of eigenvalues is finite (cf. [9]). For generic tensors, we show a further new result: for each eigenvalue, there is a unique eigenvector, up to scaling. Thus, the number of normalized eigenvectors is also finite. Similarly, for generic tensors, we can also show that the number of C-eigenvalues and C-eigenvectors (up to scaling) are finite. These results are given in Section 3.

Second, we show how to solve tensor eigenvalue complementarity problems when the tensor \mathcal{B} is strictly copositive (i.e., $\mathcal{B}x^m > 0$ for all $x \in \mathbb{R}^n_+ \setminus \{0\}$). For such cases, the complementarity eigenvectors can be normalized such that $\mathcal{B}x^m = 1$. Then, we formulate the problem as constrained polynomial optimization. The complementarity eigenvalues can be computed sequentially, from the smallest to the biggest. Each of them can be solved by a sequence of semidefinite relaxations. We prove that such sequence has finite convergence for generic tensors, subject to that \mathcal{B} is strictly copositive. This will be shown in Section 4.

Third, we study how to solve tensor eigenvalue complementarity problems when \mathcal{B} is not not copositive. For such tensors, a C-eigenvector x may not be normalized as $\mathcal{B}x^m=1$. Thus, we formulate TEiCPs as polynomial optimization in a different way. By a randomization process, the complementarity eigenvectors are classified in two cases. For each case, the TEiCP is equivalently formulated as a polynomial optimization problem. The C-eigenvectors can be computed in order, by choosing a randomly chosen objective. Each of them can be computed by a sequence of semidefinite relaxations. For generic tensors, we show that it converges in finitely many steps. The results are shown in Section 5.

In Section 6, we present numerical experiments for solving tensor eigenvalue complementarity problems. Some preliminaries in polynomial optimization and moment problems are given in Section 2.

2. Preliminaries

Notation The symbol \mathbb{N} (resp., \mathbb{R} , \mathbb{C}) denotes the set of nonnegative integers (resp., real, complex numbers). For integer n > 0, [n] denotes the set $\{1, \ldots, n\}$.

For two vectors $a, b \in \mathbb{R}^n$, $a \circ b$ denotes the Hadamard product of a and b, i.e., the product is defined componentwise. For $x = (x_1, \dots, x_n)$ and $\alpha = (\alpha_1, \dots, \alpha_n)$, denote the monomial power

$$x^{\alpha} := x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

The symbol $[x]_d$ denotes the following vector of monomials

$$[x]_d^T = [1 \quad x_1 \quad \cdots \quad x_n \quad x_1^2 \quad x_1 x_2 \quad \cdots \quad x_1^d \quad x_1^{d-1} x_2 \quad \cdots \quad x_n^d],$$

The symbol $\mathbb{R}[x] := \mathbb{R}[x_1, \dots, x_n]$ denotes the ring of polynomials in $x := (x_1, \dots, x_n)$ and with real coefficients. The ring $\mathbb{C}[x] := \mathbb{C}[x_1, \dots, x_n]$ is similarly defined over the complex field. The deg(p) denotes the degree of a polynomial p. The cardinality of a set S is denoted as |S|. For $t \in \mathbb{R}$, [t] (resp., [t]) denotes the smallest integer not smaller (resp., the largest integer not bigger) than t. For a matrix A, A^T denotes its transpose. For a symmetric matrix $X, X \succeq 0$ (resp., $X \succ 0$) means X is positive semidefinite (resp., positive definite). For a vector u, ||u|| denotes its standard Euclidean norm. The e_i denotes the standard i-th unit vector in \mathbb{N}^n .

2.1. **Polynomial optimization.** In this section, we review some basics in polynomial optimization. We refer to [17, 18, 20] for more details.

An ideal I in $\mathbb{R}[x]$ is a subset of $\mathbb{R}[x]$ such that $I \cdot \mathbb{R}[x] \subseteq I$ and $I + I \subseteq I$. For a tuple $h = (h_1, \dots, h_m)$ in $\mathbb{R}[x]$, denote the ideal

$$I(h) := h_1 \cdot \mathbb{R}[x] + \dots + h_m \cdot \mathbb{R}[x].$$

The k-th truncation of the ideal I(h), denoted as $I_k(h)$, is the set

$$(2.1) h_1 \cdot \mathbb{R}[x]_{k-\deg(h_1)} + \dots + h_m \cdot \mathbb{R}[x]_{k-\deg(h_m)}.$$

In the above, $\mathbb{R}[x]_t$ is the set of polynomials in $\mathbb{R}[x]$ with degrees at most t. Clearly, $I(h) = \bigcup_{k \in \mathbb{N}} I_k(h)$.

A polynomial ψ is said to be a sum of squares (SOS) if $\psi = q_1^2 + \cdots + q_k^2$ for some $q_1, \ldots, q_k \in \mathbb{R}[x]$. The set of all SOS polynomials in x is denoted as $\Sigma[x]$. For a degree m, denote the truncation

$$\Sigma[x]_m := \Sigma[x] \cap \mathbb{R}[x]_m$$
.

For a tuple $g = (g_1, \ldots, g_t)$, its quadratic module is the set

$$Q(q) := \Sigma[x] + q_1 \cdot \Sigma[x] + \dots + q_t \cdot \Sigma[x].$$

The k-th truncation of Q(g) is the set

$$(2.2) Q_k(g) := \sum [x]_{2k} + g_1 \cdot \sum [x]_{d_1} + \dots + g_t \cdot \sum [x]_{d_t}$$

where each $d_i = 2k - \deg(g_i)$. Note that $Q(g) = \bigcup_{k \in \mathbb{N}} Q_k(g)$.

The set I(h) + Q(g) is said to be archimedean if there exists N > 0 such that $N - ||x||^2 \in I(h) + Q(g)$. For the tuples h, g as above, denote

(2.3)
$$E(h) := \{x \in \mathbb{R}^n \mid h(x) = 0\}, \quad S(g) := \{x \in \mathbb{R}^n \mid g(x) \ge 0\}.$$

Clearly, if I(h) + Q(g) is archimedean, then the set $E(h) \cap S(g)$ is compact. On the other hand, if $E(h) \cap S(g)$ is compact, then I(h) + Q(g) can be forced to be archimedean by adding the polynomial $M - ||x||^2$ to the tuple g, for sufficiently large M.

If $f \in I(h) + Q(g)$, then $f \geq 0$ on the set $E(h) \cap S(g)$. Conversely, if f > 0 on $E(h) \cap S(g)$ and I(h) + Q(g) is archimedean, then $f \in I(h) + Q(g)$. This is called *Putinar's Positivstellensatz* (cf. [37]) in the literature. Interestingly, when f

is only nonnegative on $E(h) \cap S(g)$, we also have $f \in I(h) + Q(g)$, if some standard optimality conditions hold (cf. [30]).

2.2. Moment and localizing matrices. For $\alpha = (\alpha_1, \dots, \alpha_n)$, denote $|\alpha| := \alpha_1 + \dots + \alpha_n$ and

$$\mathbb{N}_d^n := \{ \alpha \in \mathbb{N}^n : |\alpha| \le d \}.$$

Let $\mathbb{R}^{\mathbb{N}_d^n}$ be the space of real vectors indexed by $\alpha \in \mathbb{N}_d^n$. A vector in $\mathbb{R}^{\mathbb{N}_d^n}$ is called a truncated multi-sequence (tms) of degree d. For $y \in \mathbb{R}^{\mathbb{N}_d^n}$, define the operation

(2.4)
$$\left\langle \sum_{\alpha \in \mathbb{N}_d^n} p_{\alpha} x_1^{\alpha_1} \cdots x_n^{\alpha_n}, y \right\rangle := \sum_{\alpha \in \mathbb{N}_d^n} p_{\alpha} y_{\alpha}.$$

(In the above, each p_{α} is a coefficient.) We say that y admits a representing measure supported in a set T if there exists a Borel measure μ such that its support, denoted as $\text{supp}(\mu)$, is contained in T and

$$y_{\alpha} = \int_{T} x^{\alpha} \mathrm{d}\mu \quad \forall \alpha \in \mathbb{N}_{d}^{n}.$$

For a polynomial $q \in \mathbb{R}[x]_{2k}$, the k-th localizing matrix of q, generated by a tms $y \in \mathbb{R}^{\mathbb{N}^n_{2k}}$, is the symmetric matrix $L_q^{(k)}(y)$ satisfying

$$vec(p_1)^T \Big(L_q^{(k)}(y)\Big) vec(p_2) = \langle qp_1p_2, y\rangle,$$

for all $p_1, p_2 \in \mathbb{R}[x]$ with $\deg(p_1), \deg(p_2) \leq k - \lceil \deg(q)/2 \rceil$. In the above, $vec(p_i)$ denotes the coefficient vector of the polynomial p_i . When q = 1 (the constant one polynomial), $L_q^{(k)}(y)$ becomes a *moment matrix* and is denoted as

$$(2.5) M_k(y) := L_1^{(k)}(y).$$

When $q = (q_1, \ldots, q_r)$ is a tuple of r polynomials, then we denote

(2.6)
$$L_q^{(k)}(y) := \left(L_{q_1}^{(k)}(y), \dots, L_{q_r}^{(k)}(y)\right).$$

We refer to [8, 10, 32] for localizing and moment matrices.

Let $h = (h_1, \ldots, h_m)$ and $g = (g_1, \ldots, g_t)$ be two polynomial tuples. In applications, people are often interested in whether or not a tms $y \in \mathbb{R}^{\mathbb{N}^n_{2k}}$ admits a representing measure whose support is contained in $E(h) \cap S(g)$, as in (2.3). For this to be true, a necessary condition (cf. [8, 10]) is that

(2.7)
$$M_k(y) \succeq 0, \quad L_g^{(k)}(y) \succeq 0, \quad L_h^{(k)}(y) = 0.$$

However, the above is typically not sufficient. Let

$$d_0 = \max\{1, \lceil \deg(h)/2 \rceil, \lceil \deg(g)/2 \rceil\}.$$

If y satisfies (2.7) and the rank condition

(2.8)
$$\operatorname{rank} M_{k-d_0}(y) = \operatorname{rank} M_k(y),$$

then y admits a measure supported in $E(h) \cap S(g)$ (cf. [8]). In such case, y admits a unique finitely atomic measure on $E(h) \cap S(g)$. For convenience, we just call that y is flat with respect to h = 0 and $g \ge 0$ if (2.7) and (2.8) are both satisfied.

For $t \leq d$ and $w \in \mathbb{R}^{\mathbb{N}_d^n}$, denote the truncation of w:

$$(2.9) w|_t = (w_\alpha)_{\alpha \in \mathbb{N}^n}.$$

For two tms' $y \in \mathbb{R}^{\mathbb{N}^n_{2k}}$ and $z \in \mathbb{R}^{\mathbb{N}^n_{2l}}$ with k < l, we say that y is a truncation of z (equivalently, z is an extension of y), if $y = z|_{2k}$. For such case, y is called a flat truncation of z if y is flat, and z is a flat extension of y if z is flat. Flat extensions and flat truncations are very useful in solving polynomial optimization and truncated moment problems (cf. [28, 31, 32]).

3. Properties of tensor eigenvalues

This section studies some properties of standard eigenvalues and complementarity eigenvalues, for generic tensor pairs.

3.1. **Tensor eigenvalues and eigenvectors.** For two given tensors $\mathcal{A}, \mathcal{B} \in \mathrm{T}^m(\mathbb{C}^n)$, a number $\lambda \in \mathbb{C}$ is called a generalized eigenvalue of the pair $(\mathcal{A}, \mathcal{B})$ if there exists a vector $x \in \mathbb{C}^n \setminus \{0\}$ such that

$$\mathcal{A}x^{m-1} - \lambda \mathcal{B}x^{m-1} = 0.$$

If so, such x is called a generalized eigenvector, associated with λ , of the pair $(\mathcal{A}, \mathcal{B})$. We refer to Ding and Wei [9] for generalized tensor eigenvalues. For convenience, we just call that the above λ (resps., x) is an eigenvalue (resp., eigenvector) of $(\mathcal{A}, \mathcal{B})$, and (λ, x) is called an eigenpair.

Tensor eigenvalues are closely related to the notion of *resultant*, denoted as Res, for tuples of homogeneous polynomials. For a tuple $f = (f_1, \ldots, f_n)$ of n homogeneous polynomials in $x := (x_1, \ldots, x_n)$, its resultant is the polynomial Res(f), in the coefficients of f, such that Res(f) = 0 if and only if the homogeneous equation

$$f_1(x) = \cdots = f_n(x) = 0$$

has a nonzero solution in \mathbb{C}^n . The Res(f) is an irreducible polynomial, and is homogeneous in the coefficients of each f_i . We refer to Cox, Little and O'Shea [6] for resultants. For a tensor $\mathcal{F} \in T^m(\mathbb{C}^n)$, $\mathcal{F}x^{m-1}$ is a tuple of n homogeneous polynomials of degree m-1. For convenience, denote the resultant:

$$(3.2) R(\mathcal{F}) := Res(\mathcal{F}x^{m-1}).$$

Clearly, λ is an eigenvalue of $(\mathcal{A}, \mathcal{B})$ if and only if

$$R(\mathcal{A} - \lambda \mathcal{B}) = 0.$$

Note that $R(A - \lambda B)$ is a polynomial in λ and its degree is $n(m-1)^{n-1}$. As in [9], (A, B) is called a nonsingular tensor pair if the equation

$$Ax^{m-1} = Bx^{m-1} = 0$$

has the only zero solution. Clearly, if $R(\mathcal{B}) \neq 0$ then $(\mathcal{A}, \mathcal{B})$ is nonsingular.

Theorem 3.1. Let $A, B \in T^m(\mathbb{C}^n)$ and $D := n(m-1)^{n-1}$.

- (i) ([9, Theorem 2.1]) If $R(\mathcal{B}) \neq 0$, then $(\mathcal{A}, \mathcal{B})$ has D eigenvalues, counting multiplicities.
- (ii) If \mathcal{A}, \mathcal{B} are generic tensors in $T^m(\mathbb{C}^n)$, then $(\mathcal{A}, \mathcal{B})$ has D distinct eigenvalues. Moreover, for each eigenvalue, there is a unique eigenvector, up to scaling.

Proof. (i) This item can be found in Theorem 2.1 of Ding and Wei [9]. If $R(\mathcal{B}) \neq 0$, then $(\mathcal{A}, \mathcal{B})$ is a nonsingular tensor pair.

(ii) The resultant $R(\mathcal{F})$ is an irreducible polynomial in the entries of \mathcal{F} . The hypersurface

$$\mathcal{H} = \{ \mathcal{F} \in \mathbf{T}^m(\mathbb{C}^n) : R(\mathcal{F}) = 0 \}$$

is irreducible in the space $T^m(\mathbb{C}^n)$. Its minimum degree defining polynomial is $R(\mathcal{F})$, with the degree D. The hypersurface \mathscr{H} is smooth, except a subset $\mathscr{E} \subseteq \mathscr{H}$ whose dimension is smaller than that of \mathscr{H} . For generic \mathcal{A}, \mathcal{B} , the line

$$\mathscr{L} = \{ \mathcal{A} - \lambda \mathcal{B} : \lambda \in \mathbb{C} \}$$

does not intersect the set \mathscr{E} . That is, \mathscr{L} intersects \mathscr{H} only at smooth points of \mathscr{H} (i.e., the intersection is transversal). This implies that for all λ satisfying

$$\phi(\lambda) := R(\mathcal{A} - \lambda \mathcal{B}) = 0,$$

we have $\phi'(\lambda) \neq 0$. The roots of ϕ are all simple. Therefore, $(\mathcal{A}, \mathcal{B})$ has D distinct eigenvalues, when \mathcal{A}, \mathcal{B} are generic tensors in $T^m(\mathbb{C}^n)$.

Let X be the determinantal projective variety

$$X = \{ x \in \mathbb{P}^{n-1} : \operatorname{rank} \left[\mathcal{A} x^{m-1} \quad \mathcal{B} x^{m-1} \right] < 2 \}.$$

(The \mathbb{P}^{n-1} is the projective space of equivalent classes of vectors in \mathbb{C}^n .) Clearly, if $R(\mathcal{B}) \neq 0$, then (λ, x) is an eigenpair of $(\mathcal{A}, \mathcal{B})$ if and only if $x \in X$. When \mathcal{A}, \mathcal{B} are generic, we have $R(\mathcal{B}) \neq 0$, and the set X is zero-dimensional (i.e., X is a finite set), and its cardinality is equal to the number D. This can be implied by Propositions A.5, A.6 of [27].

When \mathcal{A}, \mathcal{B} are generic tensors, $(\mathcal{A}, \mathcal{B})$ has D distinct eigenvalues. For each eigenvalue, there is at least one eigenvector. This implies that there is a unique eigenvector up to scaling.

3.2. Combinatorial eigenvalues and eigenvectors. First, we give the definition of combinatorial eigenvalues for tensor pairs. Recall the Hadamard product \circ as in $\S 2$.

Definition 3.2. Let $\mathcal{A}, \mathcal{B} \in T^m(\mathbb{C}^n)$ be tensors. If there exist a number $\lambda \in \mathbb{C}$ and a vector $x \in \mathbb{C}^n \setminus \{0\}$ such that

$$(3.3) x \circ (\mathcal{A}x^{m-1} - \lambda \mathcal{B}x^{m-1}) = 0,$$

then λ (resp., x) is called a combinatorial eigenvalue (resp., combinatorial eigenvector) of the pair (A, B). Such (λ, x) is called a combinatorial eigenpair.

For convenience of writing, the combinatorial eigenvalues (resp., eigenvectors, eigenpairs) defined in (3.3) are called CB-eigenvalues (resp., CB-eigenvectors, CB-eigenpairs). In particular, C-eigenvalues (resp., C-eigenvectors, C-eigenpairs) as in (1.4) are also CB-eigenvalues (resp., CB-eigenvectors, CB-eigenpairs).

For a subset $J = \{i_1, \ldots, i_k\} \subseteq [n]$, denote $x_J = (x_{i_1}, \ldots, x_{i_k})$. For a tensor $\mathcal{F} \in \mathrm{T}^m(\mathbb{C}^n)$, let \mathcal{F}_J be the principal sub-tensor of \mathcal{F} corresponding to the set J, i.e., \mathcal{F}_J is a tensor in $\mathrm{T}^m(\mathbb{C}^k)$ indexed by (j_1, \ldots, j_m) such that

$$(\mathcal{F}_J)_{j_1,\ldots,j_m} = \mathcal{F}_{j_1,\ldots,j_m}, \quad j_1,\ldots,j_m \in J.$$

Similar to $\mathcal{F}x^{m-1}$, $\mathcal{F}_J(x_J)^{m-1}$ is defined to be the k-dimensional vector, indexed by $j \in J$ such that

(3.4)
$$(\mathcal{F}_J(x_J)^{m-1})_j = \sum_{i_2,\dots,i_m \in J} \mathcal{F}_{j,i_2,\dots,i_m} x_{i_2} \cdots x_{i_m}.$$

Like (3.2), let $R_J(\mathcal{F})$ be the resultant of the homogeneous tuple $\mathcal{F}_J(x_J)^{m-1}$

$$(3.5) R_J(\mathcal{F}) := Res(\mathcal{F}_J(x_J)^{m-1}).$$

When (A, B) is a nonsingular tensor pair, Ling et al. [22, Theorem 4.1] gave an upper bound for the number of C-eigenvalues. We give a similar result for CB-eigenvalues. Furthermore, we also give upper bound for the number of CB-eigenvectors (up to scaling), for generic tensors A, B. Thus, the number of C-eigenvectors (up to scaling) can also be bounded.

Theorem 3.3. Let $\mathcal{A}, \mathcal{B} \in T^m(\mathbb{C}^n)$.

- (i) If $R_J(\mathcal{B}) \neq 0$ for each $\emptyset \neq J \subseteq [n]$, then $(\mathcal{A}, \mathcal{B})$ has at most nm^{n-1} CB-eigenvalues.
- (ii) If \mathcal{A}, \mathcal{B} are generic tensors in $T^m(\mathbb{C}^n)$, then, for each CB-eigenvalue, there is a unique CB-eigenvector (up to scaling).

Proof. (i) This can be done by following the approach in the proof of Theorem 4.1 of [22]. Suppose λ is a CB-eigenvalue, with the CB-eigenvector $u \neq 0$ such that

$$u \circ (\mathcal{A}u^{m-1} - \lambda \mathcal{B}u^{m-1}) = 0.$$

Let $J = \{j : u_j \neq 0\}$, a nonempty set. Then, the above implies that

$$\mathcal{A}_J(u_J)^{m-1} - \lambda \mathcal{B}_J(u_J)^{m-1} = 0.$$

So, λ is an eigenvalue of the sub-tensor pair $(\mathcal{A}_J, \mathcal{B}_J)$. By Theorem 3.1(i), $(\mathcal{A}_J, \mathcal{B}_J)$ has at most $|J|(m-1)^{|J|-1}$ eigenvalues. By enumerating all possibilities of J, the number of CB-eigenvalues of $(\mathcal{A}, \mathcal{B})$ is at most the number

$$\sum_{|J|=1}^{n} \binom{n}{|J|} |J| (m-1)^{|J|-1} = nm^{n-1}.$$

(ii) When \mathcal{A}, \mathcal{B} are generic in the space $T^m(\mathbb{C}^n)$, for each $\emptyset \neq I \subseteq [n]$, the subpair $(\mathcal{A}_I, \mathcal{B}_I)$ is also generic in $T^m(\mathbb{C}^{|I|})$. Hence, $(\mathcal{A}_I, \mathcal{B}_I)$ has a unique eigenvector (up to scaling) for each eigenvalue, by Theorem 3.1(ii). For each CB-eigenpair (λ, u) of $(\mathcal{A}, \mathcal{B})$, we showed in the item (i) that λ is an eigenvalue of the sub-tensor pair $(\mathcal{A}_I, \mathcal{B}_I)$ with the eigenvector u_I , with the index set $J = \{j : u_i \neq 0\}$.

Suppose v is another CB-eigenvector associated to λ . Let $I = \{j : v_j \neq 0\}$. Clearly, λ is also an eigenvalue of the sub-tensor pair (A_I, B_I) . We show that I = J. Define the set

$$V = \{ \mathcal{C} \in \mathbf{T}^m(\mathbb{C}^n) : R_I(\mathcal{C}_I) = R_I(\mathcal{C}_I) = 0 \}.$$

The polynomial $R_I(\mathcal{C}_I)$ is irreducible in the entries of the subtensor \mathcal{C}_I . The same is true for $R_J(\mathcal{C}_J)$. When $I \neq J$, the dimension of the set V is at most dim $(\mathbb{T}^m(\mathbb{C}^n))$ – 2. When \mathcal{A}, \mathcal{B} are generic tensors, the line in the space $\mathbb{T}^m(\mathbb{C}^n)$

$$\mathcal{L} = \{ \mathcal{A} - \lambda \mathcal{B} : \lambda \in \mathbb{C} \}$$

does not intersect V. Therefore, if $I \neq J$, then λ cannot be a common eigenvalue of the two different sub-tensor pairs $(\mathcal{A}_I, \mathcal{B}_I)$ and $(\mathcal{A}_J, \mathcal{B}_J)$. Hence, I = J and u_J, v_J are both eigenvectors of $(\mathcal{A}_J, \mathcal{B}_J)$. By Theorem 3.1(ii), u is a scaling of v.

4. TEICPS WITH STRICT COPOSITIVITY

In this section, we discuss how to compute C-eigenvalues of a tensor pair $(\mathcal{A}, \mathcal{B})$ when \mathcal{B} is strictly copositive. Note that $\mathcal{B} \in \mathrm{T}^m(\mathbb{R}^n)$ is said to be copositive (resp., strictly copositive) if $\mathcal{B}x^m \geq 0$ (resp., $\mathcal{B}x^m > 0$) for all $x \in \mathbb{R}^n_+ \setminus \{0\}$. Recall that (λ, x) is a C-eigenpair of $(\mathcal{A}, \mathcal{B})$ if x is a nonzero vector and

$$0 \le x \perp (\lambda \mathcal{B}x^{m-1} - \mathcal{A}x^{m-1}) \ge 0.$$

Any positive scaling of such x is also a C-eigenvector. When \mathcal{B} is strictly copositive, we can always scale x such that $\mathcal{B}x^m = 1$. Under this normalization, the C-eigenpair (λ, x) satisfies

$$0 = x^{T} (\lambda \mathcal{B} x^{m-1} - \mathcal{A} x^{m-1}) = \lambda \mathcal{B} x^{m} - \mathcal{A} x^{m} = \lambda - \mathcal{A} x^{m}.$$

So, we get $\lambda = \mathcal{A}x^m$. The C-eigenvalues of $(\mathcal{A}, \mathcal{B})$ can be found by solving the polynomial system

(4.1)
$$\begin{cases} \mathcal{B}x^m = 1, \ x \circ ((\mathcal{A}x^m)\mathcal{B}x^{m-1} - \mathcal{A}x^{m-1}) = 0, \\ x \ge 0, \ (\mathcal{A}x^m)\mathcal{B}x^{m-1} - \mathcal{A}x^{m-1} \ge 0, \end{cases}$$

where o denotes the Hadamard product of two vectors. If we define

$$a(x) := x \circ \mathcal{A}x^{m-1}, \quad b(x) := x \circ \mathcal{B}x^{m-1}.$$

Then, it clearly holds that

$$x \circ (\lambda \mathcal{B}x^{m-1} - \mathcal{A}x^{m-1}) = \lambda b(x) - a(x).$$

The polynomial system (4.1) can be rewritten as

(4.2)
$$\begin{cases} \mathcal{B}x^{m} = 1, \ (\mathcal{A}x^{m})b(x) - a(x) = 0, \\ x \ge 0, \ (\mathcal{A}x^{m})\mathcal{B}x^{m-1} - \mathcal{A}x^{m-1} \ge 0. \end{cases}$$

When \mathcal{B} is strictly copositive, the solution set of (4.2) is compact, because $\{x \in \mathbb{R}^n : \mathcal{B}x^m = 1, x \geq 0\}$ is compact. The tensor pair $(\mathcal{A}, \mathcal{B})$ has at least one C-eigenvalue when \mathcal{B} (or $-\mathcal{B}$) is strictly copositive (cf. [22, Theorem 2.1]). Moreover, under some generic conditions on \mathcal{B} , $(\mathcal{A}, \mathcal{B})$ has finitely many C-eigenvalues (cf. Theorem 3.3). They can be ordered monotonically as

$$(4.3) \lambda_1 < \lambda_2 < \dots < \lambda_N.$$

For convenience, denote the polynomial tuples

(4.4)
$$\begin{cases} f_0 = \mathcal{A}x^m, \\ p = \left(\mathcal{B}x^m - 1, (\mathcal{A}x^m)b(x) - a(x)\right), \\ q = \left(x, (\mathcal{A}x^m)\mathcal{B}x^{m-1} - \mathcal{A}x^{m-1}\right). \end{cases}$$

4.1. The first C-eigenvalue. The first eigenvalue λ_1 equals the optimal value of the optimization problem

(4.5)
$$\begin{cases} \lambda_1 = \min & f_0(x) \\ \text{s.t.} & p(x) = 0, \ q(x) \ge 0. \end{cases}$$

We apply Lasserre type semidefinite relaxations [17] to solve (4.5). For the orders k = m, m + 1, ..., the k-th Lasserre relaxation is

(4.6)
$$\begin{cases} \nu_{1,k} := & \min \quad \langle f_0, y \rangle \\ & \text{s.t.} \quad \langle 1, y \rangle = 1, L_p^{(k)}(y) = 0, \\ & M_k(y) \succeq 0, L_q^{(k)}(y) \succeq 0, y \in \mathbb{R}^{\mathbb{N}_{2k}^n}. \end{cases}$$

In the above, $\langle 1, y \rangle = 1$ means that the first entry of y is one, and the matrices $M_k(y), L_p^{(k)}(y), L_q^{(k)}(y)$ are defined as in (2.5)-(2.6). Its dual problem is

(4.7)
$$\begin{cases} \tilde{\nu}_{1,k} := \max_{j \in \mathcal{I}} \gamma \\ \text{s.t.} \quad f_0 - \gamma \in I_{2k}(p) + Q_k(q). \end{cases}$$

Suppose $y^{1,k}$ is an optimizer of (4.6). If, for some $t \in [m, k]$, the truncation $\hat{y} = y^{1,k}|_{2t}$ (see (2.9)) satisfies

(4.8)
$$\operatorname{rank} M_{t-m}(\hat{y}) = \operatorname{rank} M_t(\hat{y}),$$

then $\nu_{1,k} = \lambda_1$ and we can get rank $M_t(\hat{y})$ global optimizers of (4.5) (cf. [28]).

4.2. The second and other eigenvalues. We discuss how to compute λ_i for i = 2, ..., N. Suppose λ_{i-1} is already computed. We need to determine the next C-eigenvalue λ_i . Consider the optimization problem

(4.9)
$$\begin{cases} \min & f_0(x) \\ \text{s.t.} & p(x) = 0, \ q(x) \ge 0, \ f_0(x) - \lambda_{i-1} - \delta \ge 0. \end{cases}$$

The optimal value of (4.9) is equal to λ_i if

$$(4.10) 0 < \delta < \lambda_i - \lambda_{i-1}.$$

Similarly, Lasserre type semidefinite relaxations can be applied to solve (4.9). For the orders $k=m,m+1,\ldots$, the k-th Lasserre relaxation is

(4.11)
$$\begin{cases} \nu_{i,k} := & \min \quad \langle f_0, z \rangle \\ & \text{s.t.} \quad \langle 1, z \rangle = 1, L_p^{(k)}(z) = 0, M_k(z) \succeq 0, \\ & L_q^{(k)}(z) \succeq 0, L_{f_0 - \lambda_{i-1} - \delta}^{(k)}(z) \succeq 0, z \in \mathbb{R}^{\mathbb{N}_{2k}^n}. \end{cases}$$

The dual problem of (4.11) is

(4.12)
$$\begin{cases} \tilde{\nu}_{i,k} := \max_{j \in \mathcal{I}} \gamma \\ \text{s.t.} \quad f_0 - \gamma \in I_{2k}(p) + Q_k(q, f_0 - \lambda_{i-1} - \delta). \end{cases}$$

Suppose $y^{i,k}$ is an optimizer of (4.11). If a truncation $\hat{y} = y^{i,k}|_{2t}$ satisfies (4.8) for some $t \in [m, k]$, then $\nu_{i,k} = \lambda_i$ and we can get optimizers of (4.9) (cf. [28]).

In practice, the existence of λ_i is usually not known in advance. Even if it exists, its value is typically not available. So, we need to determine the value of δ satisfying (4.10). Consider the polynomial optimization problem:

(4.13)
$$\begin{cases} \tau := & \max \quad f_0(x) \\ & \text{s.t.} \quad p(x) = 0, \ q(x) \ge 0, \ f_0(x) \le \lambda_{i-1} + \delta. \end{cases}$$

Its optimal value τ can be computed by Lasserre relaxations like (4.11)-(4.12). As in Proposition 4.5, δ satisfies (4.10) if and only if $\tau = \lambda_{i-1}$. When $\tau = \lambda_{i-1}$, λ_i does not exist if and only if (4.11) is infeasible for some k.

4.3. An algorithm for computing C-eigenvalues. Assume that the tensor \mathcal{B} is strictly copositive. So, the C-eigenvectors can be normalized as $\mathcal{B}x^m=1$. We propose an algorithm to compute the C-eigenvalues sequentially, from the smallest one λ_1 to the biggest one λ_N . Since \mathcal{B} is strictly copositive, λ_1 always exists [22]. We assume there are finitely many C-eigenvalues.

First, we compute λ_1 by solving semidefinite relaxations (4.6)-(4.7). After getting λ_1 , we solve (4.11)-(4.12) for λ_2 . If λ_2 does not exist, then λ_1 is the biggest C-eigenvalue and we stop; otherwise, we continue to determine λ_3 . Repeating this procedure, we can get all the C-eigenvalues of $(\mathcal{A}, \mathcal{B})$.

Algorithm 4.1. For two tensors $\mathcal{A}, \mathcal{B} \in \mathrm{T}^m(\mathbb{R}^n)$ with \mathcal{B} strictly copositive, compute a set Λ of all C-eigenvalues and a set U of C-eigenvectors, for the pair $(\mathcal{A}, \mathcal{B})$. Let $U := \emptyset$, $\Lambda := \emptyset$, i := 1, k := m.

- Step 1. Solve (4.6) with the order k for an optimizer $y^{1,k}$.
- Step 2. If (4.8) is satisfied for some $t \in [m, k]$, then update $U := U \cup S$, with S a set of optimizers of (4.5); let $\lambda_1 = \nu_{1,k}$, $\Lambda := \{\lambda_1\}$, i := i + 1 and go to Step 3. If such t does not exist, let k := k + 1 and go to Step 1.
- Step 3. Let $\delta = 0.05$, and compute the optimal value τ of (4.13). If $\tau > \lambda_{i-1}$, let $\delta := \delta/2$ and compute τ again. Repeat this, until we get $\tau = \lambda_{i-1}$. Let k := m.
- Step 4. Solve (4.11) with the order k. If it is infeasible, then (4.2) has no further C-eigenvalues, and stop. Otherwise, compute an optimizer $y^{i,k}$ for (4.11).
- Step 5. If (4.8) is satisfied for some $t \in [m, k]$, then update $U := U \cup S$ where S is a set of optimizers of (4.9); let $\lambda_i = \nu_{i,k}$, $\Lambda := \Lambda \cup \{\lambda_i\}$, i := i + 1 and go to Step 3. If such t does not exist, let k := k + 1 and go to Step 4.

The semidefinite relaxation (4.6) can be solved by the software GloptiPoly 3 [12] and SeDuMi [41]. When (4.8) holds, it can be shown that $\lambda_{i,k} = \lambda_i$, and we can get a set of optimizers of (4.5), (4.9). Such optimizers are the associated eigenvectors for the C-eigenvalue λ_i . In Steps 2 and 5, the method in Henrion and Lasserre [11] can be used to compute the set S.

4.4. **Properties of relaxations.** First, we discuss when Algorithm 4.1 has finite convergence. For the polynomial tuple p, denote the sets

$$(4.14) V_{\mathbb{C}}(p) := \{ u \in \mathbb{C}^n \mid p(u) = 0 \}, \quad V_{\mathbb{R}}(p) := V_{\mathbb{C}}(p) \cap \mathbb{R}^n.$$

Theorem 4.2. Let $A, B \in T^m(\mathbb{R}^n)$. Suppose B is strictly copositive. Then, we have the properties:

(i) The smallest C-eigenvalue λ_1 of $(\mathcal{A}, \mathcal{B})$ always exists. Moreover, if the set $V_{\mathbb{R}}(p)$ is finite, then for all k sufficiently large,

$$\nu_{1,k} = \tilde{\nu}_{1,k} = \lambda_1$$

and the condition (4.8) must be satisfied.

(ii) For $i \geq 2$, suppose λ_i exists and $0 < \delta < \lambda_i - \lambda_{i-1}$. If the set $V_{\mathbb{R}}(p)$ is finite, then for all k sufficiently large,

$$\nu_{i,k} = \tilde{\nu}_{i,k} = \lambda_i$$

and the condition (4.8) must be satisfied.

- *Proof.* (i) Since \mathcal{B} is strictly copositive, $(\mathcal{A}, \mathcal{B})$ has at least one C-eigenvalue (cf. [22, Theorem 2.1]). So, λ_1 always exists. If $V_{\mathbb{R}}(p)$ is finite, the equation p(x) = 0 has finitely many real solutions. Thus, when the relaxation order k is sufficiently large, we must have $\nu_{1,k} = \tilde{\nu}_{1,k} = \lambda_1$ and the flat truncation condition (4.8) must be satisfied. This can be implied by Proposition 4.6 of [19] and Theorem 1.1 of [29].
- (ii) If $0 < \delta < \lambda_i \lambda_{i-1}$ holds, the optimal value of (4.9) is equal to λ_i . When $V_{\mathbb{R}}(p)$ is finite, the equation p(x) = 0 has finitely many real solutions. The conclusion can be implied by Proposition 4.6 of [19] and Theorem 1.1 of [29].

Remark 4.3. In Theorem 4.2, if $V_{\mathbb{R}}(p)$ is not a finite set, $\nu_{1,k}$ and $\tilde{\nu}_{1,k}$ may not have finite convergence to λ_1 , but the asymptotic convergence can be established. When \mathcal{B} is strictly copositive, the set $\{x \in \mathbb{R}^n : \mathcal{B}x^m = 1, x \geq 0\}$ is compact, say,

contained in the ball $\{x \in \mathbb{R}^n : M - x^T x \ge 0\}$, where M > 0 is a sufficiently large number. If we add $M - x^T x$ to the polynomial tuple q, then $\nu_{1,k}$ and $\tilde{\nu}_{1,k}$ have asymptotic convergence to λ_1 . This is because such Q(q) is archimedean, and the asymptotic convergence can be implied by the results in [17].

However, interestingly, the set $V_{\mathbb{R}}(p)$ is finite for generic tensors \mathcal{A}, \mathcal{B} .

Proposition 4.4. Let p be as in (4.4). If A, B are generic tensors, then $V_{\mathbb{C}}(p)$ and $V_{\mathbb{R}}(p)$ are finite sets.

Proof. The equation p(x) = 0 implies that

$$\mathcal{B}x^{m} = 1$$
, $a(x) - (\mathcal{A}x^{m})b(x) = x \circ (\mathcal{A}x^{m-1} - (\mathcal{A}x^{m})\mathcal{B}x^{m-1}) = 0$.

So, x must be a nonzero vector. Let $J = \{j : x_j \neq 0\}$. Then we get

$$\mathcal{A}_J(x_J)^{m-1} - (\mathcal{A}x^m)\mathcal{B}_J(x_J)^{m-1} = 0.$$

Hence, x_J is an eigenvector of the sub-tensor pair $(\mathcal{A}_J, \mathcal{B}_J)$. When \mathcal{A}, \mathcal{B} are generic, such x must be finitely many, by Theorem 3.3(ii). The conclusion holds over the complex field. So, $V_{\mathbb{C}}(p)$, as well as $V_{\mathbb{R}}(p)$, is finite, for generic \mathcal{A}, \mathcal{B} .

The existence of λ_i and the relation (4.10) can be checked as follows.

Proposition 4.5. Let $A, B \in T^m(\mathbb{R}^n)$. Suppose B is strictly copositive. Let Λ be the set of all C-eigenvalues of (A, B). Assume Λ is finite. Let λ_i be the i-th smallest C-eigenvalue of (A, B), and λ_{max} be the maximum of them. For all $i \geq 2$ and all $\delta > 0$, we have the following properties:

- (i) If (4.11) is infeasible for some k, then $\Lambda \cap [\lambda_{i-1} + \delta, \infty) = \emptyset$.
- (ii) If $\Lambda \cap [\lambda_{i-1} + \delta, \infty) = \emptyset$ and $V_{\mathbb{R}}(p)$ is finite, then (4.11) must be infeasible for some k.
- (iii) If $\tau = \lambda_{i-1}$ and λ_i exists, then δ satisfies (4.10).
- (iv) If $\tau = \lambda_{i-1}$ and (4.11) is infeasible for some k, then λ_i does not exist.

Proof. Since \mathcal{B} is strictly copositive, every C-eigenvector x can be scaled such that $\mathcal{B}x^m=1$.

- (i) Note that, for every eigenpair (λ, u) of $(\mathcal{A}, \mathcal{B})$ with $\lambda \geq \lambda_{i-1} + \delta$, the tms $[u]_{2k}$ (see the notation in §2) is always feasible for (4.11). If (4.11) is infeasible for some k, then $(\mathcal{A}, \mathcal{B})$ clearly has no C-eigenvalues $\geq \lambda_{i-1} + \delta$.
- (ii) Suppose (A, \mathcal{B}) has no C-eigenvalues $\geq \lambda_{i-1} + \delta$ and $V_{\mathbb{R}}(p)$ is finite. The feasible set of (4.9) is empty. By the Positivstellensatz (cf. [3, Theorem 4.4.2]), we have

$$-2 = \phi + \psi$$
, $\phi \in I(p)$, $\psi \in Pr(q, f_0 - \lambda_{i-1} - \delta)$,

where $Pr(q, f_0 - \lambda_{i-1} - \delta)$ denotes the preodering generated by the tuple $(q, f_0 - \lambda_{i-1} - \delta)$. (We refer to [3] for preorderings.) Since $V_{\mathbb{R}}(p)$ is finite, the ideal I(p) is archimedean. (This is because $-\|p\|^2$ belongs to I(p) and the set $\{x \in \mathbb{R}^n : -\|p\|^2 \ge 0\}$ is compact.) So, $I(p) + Q(q, f_0 - \lambda_{i-1} - \delta)$ is also archimedean. Note that $1 + \psi$ is strictly positive on $\{x \in \mathbb{R}^n : p = 0, q \ge 0, f_0 - \lambda_{i-1} - \delta \ge 0\}$. By Putinar's Positivstellensatz (cf. [37]), $1 + \psi \in I(p) + Q(q, f_0 - \lambda_{i-1} - \delta)$. Thus, we get

$$-1 = \phi + \sigma$$
, $\phi \in I_{2k}(p)$, $\sigma \in Q_k(q, f_0 - \lambda_{i-1} - \delta)$

where $\sigma = 1 + \psi$ and k is sufficiently large. This implies that (4.12) has an improving direction and it is unbounded from the above. By weak duality, the relaxation (4.11) must be infeasible, for k big enough.

(iii) If $\tau = \lambda_{i-1}$, then the maximum C-eigenvalue, which is less than or equal to $\lambda_{i-1} + \delta$, is still λ_{i-1} . So, if λ_i exists, we must have $\lambda_i > \lambda_{i-1} + \delta$, i.e., (4.10) is satisfied.

(iv) When (4.11) is infeasible for some k, $(\mathcal{A}, \mathcal{B})$ has no C-eigenvalues $\geq \lambda_{i-1} + \delta$. So, if $\tau = \lambda_{i-1}$, λ_{i-1} is the maximum C-eigenvalue, and λ_i does not exist. \square

5. Solving general TeiCPs

In this section, we discuss how to compute complementarity eigenvalues of $(\mathcal{A}, \mathcal{B})$ for generic tensors $\mathcal{A}, \mathcal{B} \in \mathrm{T}^m(\mathbb{R}^n)$. Recall that λ is a C-eigenvalue of $(\mathcal{A}, \mathcal{B})$ if there exists a nonzero vector $x \in \mathbb{R}^n$ such that

$$0 \le x \perp (\lambda \mathcal{B}x^{m-1} - \mathcal{A}x^{m-1}) \ge 0.$$

5.1. Polynomial optimization reformulations. As in §4, we still denote

$$a(x) := x \circ \mathcal{A}x^{m-1}, \quad b(x) := x \circ \mathcal{B}x^{m-1}.$$

If we normalize x to have unit length, then (λ, x) is a C-eigenpair of $(\mathcal{A}, \mathcal{B})$ if and only if it is a solution of the polynomial system

(5.1)
$$\begin{cases} x^T x = 1, \ \lambda b(x) - a(x) = 0, \\ x \ge 0, \ \lambda \mathcal{B} x^{m-1} - \mathcal{A} x^{m-1} \ge 0. \end{cases}$$

When $b(x) \neq 0$, the equation $a(x) - \lambda b(x) = 0$ holds if and only if

$$rank \begin{bmatrix} a(x) & b(x) \end{bmatrix} \le 1,$$

which is equivalent to that

$$(5.2) a(x)_i b(x)_j - b(x)_i a(x)_j = 0 (1 \le i < j \le n).$$

Suppose (5.1) has finitely many real solutions. For a generic vector $\xi \in \mathbb{R}^n$, we have $\xi^T b(x) \neq 0$ for all x satisfying (5.1) and

(5.3)
$$\lambda = \frac{\xi^T a(x)}{\xi^T b(x)}.$$

The C-eigenvalues of $(\mathcal{A}, \mathcal{B})$ can be computed in two cases.

Case I: $\xi^T b(x) > 0$. In this case, the system (5.1) is equivalent to

(5.4)
$$\begin{cases} x^T x = 1, a(x)_i b(x)_j - b(x)_i a(x)_j = 0 \ (1 \le i < j \le n), \\ x \ge 0, \ \xi^T b(x) \ge 0, \ (\xi^T a(x) \mathcal{B} x^{m-1} - \xi^T b(x) \mathcal{A} x^{m-1}) \ge 0. \end{cases}$$

Note that (5.4) does not use λ directly. For generic $(\mathcal{A}, \mathcal{B})$, (5.4) has finitely many solutions. Once a solution x is found, the C-eigenvalue λ can be computed by (5.3). The system (5.4) can be solved as a polynomial optimization problem. Generate a random polynomial $f(x) \in \mathbb{R}[x]_{2m}$. Consider the optimization problem

(5.5)
$$\begin{cases} \min & f(x) \\ \text{s.t.} & h(x) = 0, \ g(x) \ge 0, \end{cases}$$

where the polynomial tuples h, g are given as

(5.6)
$$\begin{cases} h(x) = \left(x^T x - 1, \left(a(x)_i b(x)_j - b(x)_i a(x)_j\right)_{1 \le i < j \le n}\right), \\ g(x) = \left(x, \xi^T b(x), \xi^T a(x) \mathcal{B} x^{m-1} - \xi^T b(x) \mathcal{A} x^{m-1}\right). \end{cases}$$

Clearly, x satisfies (5.4) if and only if x is feasible for (5.5).

Case II: $\xi^T b(x) < 0$. In this case, the system (5.1) is equivalent to

(5.7)
$$\begin{cases} x^T x = 1, a(x)_i b(x)_j - b(x)_i a(x)_j = 0 \ (1 \le i < j \le n), \\ x \ge 0, -\xi^T b(x) \ge 0, \xi^T b(x) \mathcal{A} x^{m-1} - \xi^T a(x) \mathcal{B} x^{m-1} \ge 0. \end{cases}$$

Like (5.4), the system (5.7) does not use λ directly. Once a point x satisfying (5.7) is obtained, the C-eigenvalue λ can be obtained by (5.3). Clearly, x satisfies (5.7) if and only if it is feasible for the optimization problem

(5.8)
$$\begin{cases} \min & f(x) \\ \text{s.t.} & h(x) = 0, \ \tilde{g}(x) \ge 0, \end{cases}$$

where h is the same as in (5.6) while the tuple \tilde{g} is given as

(5.9)
$$\tilde{g}(x) = \left(x, -\xi^T b(x), \xi^T b(x) \mathcal{A} x^{m-1} - \xi^T a(x) \mathcal{B} x^{m-1}\right).$$

The feasible sets of (5.5) and (5.8) are compact, since they are contained in the unit sphere. However, they are possibly empty.

The C-eigenpairs (λ, x) satisfying (5.1) can be found by computing feasible points of the optimization problems (5.5) and (5.8). When the number of C-eigenvectors (normalized to have unit lengths) is finite, we can compute all the feasible points of (5.5) and (5.8). In the following subsections, we show how to do this.

5.2. Compute C-eigenvectors. Assume that there are finitely many C-eigenvectors (normalized to have unit lengths) for the tensor pair (A, B). We propose an algorithm for computing all of them.

5.2.1. C-eigenpairs for case I.

We discuss how to compute the C-eigenvectors satisfying (5.4). Assume the feasible set of (5.5) is nonempty and finite. When it is generically chosen, f achieves different values at different feasible points of (5.5), say, they are monotonically ordered as

$$(5.10) f_1^{(1)} < f_2^{(1)} < \dots < f_{N_1}^{(1)}.$$

We aim to compute the C-eigenvectors, along with the values $f_i^{(1)}$, in the order $i = 1, ..., N_1$. Choose a number ℓ_i such that

$$(5.11) f_{i-1}^{(1)} < \ell_i < f_i^{(1)}.$$

(For the case i=1, $f_0^{(1)}$ can be chosen to be any value smaller than $f_1^{(1)}$.) Note that $f_i^{(1)}$ is equal to the optimal value of

(5.12)
$$\begin{cases} \min & f(x) \\ \text{s.t.} & h(x) = 0, \ g(x) \ge 0, \ f(x) - \ell_i \ge 0. \end{cases}$$

We apply Lasserre type semidefinite relaxations to solve (5.12). For the orders $k = m, m + 1, \ldots$, the k-th Lasserre relaxation is

(5.13)
$$\begin{cases} \mu_{1,k} := & \min \quad \langle f, y \rangle \\ & \text{s.t.} \quad \langle 1, y \rangle = 1, L_h^{(k)}(y) = 0, M_k(y) \succeq 0, \\ & L_g^{(k)}(y) \succeq 0, L_{f-\ell_i}^{(k)}(y) \succeq 0, y \in \mathbb{R}^{\mathbb{N}_{2k}^n}. \end{cases}$$

(See §2.2 for the notation in the above.) The dual problem of (5.13) is

(5.14)
$$\begin{cases} \tilde{\mu}_{1,k} := \max_{j \in I_{2k}(h)} \gamma \\ \text{s.t.} \quad f - \gamma \in I_{2k}(h) + Q_k(g, f - \ell_i), \end{cases}$$

where $I_{2k}(h)$ and $Q_k(g, f - \ell_i)$ are defined as in (2.1)-(2.2). By weak duality, it can be shown that (cf. [17])

(5.15)
$$\tilde{\mu}_{1,k} \leq \mu_{1,k} \leq f_i^{(1)}, \quad \forall k \geq m.$$

Moreover, both $\{\mu_{1,k}\}$ and $\{\tilde{\mu}_{1,k}\}$ are monotonically increasing.

When (5.4) has a solution, the semidefinite relaxation (5.13) is always feasible. Suppose $y^{i,k}$ is an optimizer of (5.13). If for some $t \in [m,k]$, the truncation $\hat{y} := y^{i,k}|_{2t}$ satisfies the rank condition

(5.16)
$$\operatorname{rank} M_{t-m}(\hat{y}) = \operatorname{rank} M_t(\hat{y}),$$

then one can show that $\mu_{1,k} = \tilde{\mu}_{1,k} = f_i^{(1)}$ and we can get rank $M_t(\hat{y})$ optimizers of (5.12) (cf. [28]). The method in [11] can be applied to compute the minimizers of (5.12). Interestingly, we will show that the rank condition (5.16) must be satisfied, for generic tensors \mathcal{A}, \mathcal{B} (cf. Theorem 5.2).

5.2.2. C-eigenpairs for case II.

Now we show how to find the C-eigenvectors satisfying (5.7). The computation is similar to the case I. Assume the feasible set of (5.8) is nonempty and finite. Order its objective values monotonically as

(5.17)
$$f_1^{(2)} < f_2^{(2)} < \dots < f_{N_2}^{(2)}.$$

We compute the C-eigenvectors and the value $f_i^{(2)}$ in the order $i = 1, ..., N_2$. Choose a number $\tilde{\ell}_i$ such that

$$(5.18) f_{i-1}^{(2)} < \tilde{\ell}_i < f_i^{(2)}.$$

(For i = 1, choose $f_0^{(2)}$ to be any value smaller than $f_1^{(2)}$.) Note that $f_i^{(2)}$ is equal to the minimum value of

$$\begin{cases}
\min \quad f(x) \\
\text{s.t.} \quad h(x) = 0, \ \tilde{g}(x) \ge 0, \ f(x) - \tilde{\ell}_i \ge 0.
\end{cases}$$

For an order $k \geq m$, the k-th Lasserre relaxation (cf. [17]) for solving (5.19) is

(5.20)
$$\begin{cases} \mu_{2,k} := & \min \quad \langle f, z \rangle \\ & \text{s.t.} \quad \langle 1, z \rangle = 1, L_h^{(k)}(z) = 0, M_k(z) \succeq 0, \\ & L_{\tilde{g}}^{(k)}(z) \succeq 0, L_{f-\tilde{\ell}_i}^{(k)}(z) \succeq 0, z \in \mathbb{R}^{\mathbb{N}_{2k}^n}. \end{cases}$$

Its dual optimization problem is

(5.21)
$$\begin{cases} \tilde{\mu}_{2,k} := \max \quad \gamma \\ \text{s.t.} \quad f - \gamma \in I_{2k}(h) + Q_k(\tilde{g}, f - \tilde{\ell}_i). \end{cases}$$

Suppose $z^{i,k}$ is an optimizer of (5.20). If for some $t \in [m,k]$, the truncation $\hat{z} := z^{i,k}|_{2t}$ satisfies the rank condition

(5.22)
$$\operatorname{rank} M_{t-m}(\hat{z}) = \operatorname{rank} M_t(\hat{z}),$$

then $\mu_{2,k} = \tilde{\mu}_{2,k} = f_i^{(2)}$ and we can get rank $M_t(\hat{z})$ optimizers of (5.19) (cf. [28]). We will show that the condition (5.22) must be satisfied for generic tensors \mathcal{A}, \mathcal{B} (cf. Theorem 5.2).

5.2.3. An algorithm for computing C-eigenpairs.

In practice, the f, ℓ_i, ℓ_i need to be chosen properly. We propose to choose f in the form as

(5.23)
$$f = [x]_m^T (R^T R) [x]_m,$$

where R is a random square matrix. For f as in (5.23), we almost always have

$$f_1^{(1)} > 0, \quad f_1^{(2)} > 0.$$

Thus, we can choose

(5.24)
$$f_0^{(1)} = f_0^{(2)} = -1, \quad \ell_1 = \tilde{\ell}_1 = 0.$$

In the computation of $f_i^{(1)}, f_i^{(2)}$, suppose the values of $f_{i-1}^{(1)}, f_{i-1}^{(2)}$ are already computed. In practice, for $\delta > 0$ small enough, we can choose

$$\ell_i = f_{i-1}^{(1)} + \delta, \qquad \tilde{\ell}_i = f_{i-1}^{(2)} + \delta,$$

to satisfy (5.11) and (5.18). Such value of δ can be determined by solving the following maximization problems:

(5.25)
$$\begin{cases} \theta_1 = \max & f(x) \\ & \text{s.t.} & h(x) = 0, g(x) \ge 0, f(x) \le f_{i-1}^{(1)} + \delta, \end{cases}$$
(5.26)
$$\begin{cases} \theta_2 = \max & f(x) \\ & \text{s.t.} & h(x) = 0, \tilde{g}(x) \ge 0, f(x) \le f_{i-1}^{(2)} + \delta. \end{cases}$$

(5.26)
$$\begin{cases} \theta_2 = \max & f(x) \\ \text{s.t.} & h(x) = 0, \ \tilde{g}(x) \ge 0, \ f(x) \le f_{i-1}^{(2)} + \delta. \end{cases}$$

Their optimal values can be computed by Lasserre type semidefinite relaxations. When h(x) = 0 has finitely many real solutions, we must have $\theta_1 = f_{i-1}^{(1)}$ and $\theta_2 = f_{i-1}^{(2)}$, for $\delta > 0$ sufficiently small. For such case, the relations (5.11) and (5.18) will be satisfied. This is justified by Lemma 5.6.

Note that f achieves only finitely many values in the feasible sets of (5.5), (5.8), when $(\mathcal{A}, \mathcal{B})$ has finitely many normalized C-eigenvectors.

Algorithm 5.1. For two given tensors $\mathcal{A}, \mathcal{B} \in \mathrm{T}^m(\mathbb{R}^n)$, compute a set Λ of Ceigenvalues, and a set U of C-eigenvectors, for the pair $(\mathcal{A}, \mathcal{B})$.

- Step 0. Choose f as in (5.23), with R a random square matrix. Choose a random vector $\xi \in \mathbb{R}^n$. Let $U = \emptyset$, i = 1, k = m, $\ell_1 = 0$, $\tilde{\ell}_1 = 0$.
- Step 1. Solve (5.13) for the order k. If it is infeasible, then (5.4) has no further C-eigenvectors (except those in U); let k = m, i = 1 and go to Step 4. Otherwise, compute an optimizer $y^{i,k}$ for (5.13).
- Step 2. If (5.16) is satisfied for some $t \in [m, k]$, then update $U := U \cup S$, where S is a set of optimizers of (5.12); let i := i + 1 and go to Step 3. If such t does not exist, let k := k + 1 and go to Step 1.
- Step 3. Let $\delta = 0.05$, and compute the optimal value θ_1 of (5.25). If $\theta_1 > f_{i-1}^{(1)}$, let $\delta := \delta/2$ and compute θ_1 again. Repeat this process, until $\theta_1 = f_{i-1}^{(1)}$ is met. Let $\ell_i := f_{i-1}^{(1)} + \delta$, k = m, then go to Step 1.
- Step 4. Solve (5.20) for the order k. If it is infeasible, then (5.7) has no further C-eigenvectors (except those in U) and go to Step 7. Otherwise, compute an optimizer $z^{i,k}$ for it.
- Step 5. Check whether or not (5.22) is satisfied for some $t \in [m, k]$. If yes, update $U := U \cup S$, where S is a set of optimizers of (5.19); let i := i + 1 and go to Step 6. If no, let k := k + 1 and go to Step 4.

Step 6. Let $\delta = 0.05$, and compute the optimal value θ_2 of (5.26). If $\theta_2 > f_{i-1}^{(2)}$, let $\delta := \delta/2$ and compute θ_2 again. Repeat this process, until we get $\theta_2 = f_{i-1}^{(2)}$. Let $\tilde{\ell}_i = f_{i-1}^{(2)} + \delta$, k = m, and go to Step 4. Step 7. Let $\Lambda := \{\xi^T a(u)/\xi^T b(u): u \in U\}$.

The Lasserre type semidefinite relaxations (5.13) and (5.20) can be solved by the software GloptiPoly 3 [12] and SeDuMi [41]. In Step 2 and Step 5, the method in Henrion and Lasserre [11] can be used to compute optimizers of (5.12). The same is true for (5.19) and its Lasserre relaxation (5.20).

5.3. Properties of the relaxations. First, we prove that Algorithm 5.1 converges in finitely many steps for generic tensors \mathcal{A}, \mathcal{B} . Let T_1, T_2 be the feasible sets of (5.12) and (5.19), respectively. Let $V_{\mathbb{R}}(h)$ be defined as in (4.14).

Theorem 5.2. Let $\mathcal{A}, \mathcal{B} \in T^m(\mathbb{R}^n)$ be two tensors. Let h, g, \tilde{g} be the polynomial tuples as in (5.6), (5.9), constructed from \mathcal{A}, \mathcal{B} and a vector $\xi \in \mathbb{R}^n$. Then, for all ℓ_i, ℓ_i satisfying (5.11) and (5.18), we have the following properties:

- (i) The relaxation (5.13) is infeasible for some order k if and only if the feasible set T_1 of (5.12) is empty.
- (ii) Suppose $T_1 \neq \emptyset$. If $V_{\mathbb{R}}(h)$ is a finite set, then for k sufficiently large, the rank condition (5.16) must be satisfied and

$$\mu_{1,k} = \tilde{\mu}_{1,k} = f_i^{(1)}.$$

- (iii) The relaxation (5.20) is infeasible for some order k if and only if the feasible set T_2 of (5.19) is empty.
- (iv) Suppose $T_2 \neq \emptyset$. If $V_{\mathbb{R}}(h)$ is a finite set, then for k sufficiently large, the rank condition (5.22) must be satisfied and

$$\mu_{2,k} = \tilde{\mu}_{2,k} = f_i^{(2)}.$$

Proof. (i) "only if" direction: If the relaxation (5.13) is infeasible for some order k, then the feasible set of (5.12) must be empty. This is because, if otherwise (5.12)has a feasible point, say, u, then the tms $[u]_{2k}$ (see the notation in §2) generated by u must be feasible for (5.13).

"if" direction: Since $T_1 = \emptyset$, by the Positivstellensatz (cf. [3, Theorem 4.4.2]), we have

$$-2 = \phi + \psi, \quad \phi \in I(h), \ \psi \in Pr(g, f - \ell_i).$$

Here, $Pr(g, f - \ell_i)$ is the preordering of the tuple $(g, f - \ell_i)$. (We refer to [3] for preorderings.) Note that the sum $1 + \psi$ is strictly positive on $\{x \in \mathbb{R}^n : h = 1\}$ $0, g \ge 0, f - \ell_i \ge 0$. The ideal I(h) is archimedean, because $1 - ||x||^2 \in I(h)$. So, $I(h) + Q(g, f - \ell_i)$ is also archimedean. By Putinar's Positivstellensatz, $1 + \psi \in$ $I(h) + Q(g, f - \ell_i)$. This implies that

$$-1 = \phi + \sigma$$
, $\phi \in I_{2k}(h)$, $\sigma \in Q_k(g, f - \ell_i)$,

where $\sigma = 1 + \psi$ and k is sufficiently large. So, the dual optimization problem (5.14) has an improving direction and it is unbounded from the above. By weak duality, the optimization (5.13) must be infeasible.

(ii) When the set $V_{\mathbb{R}}(h)$ is finite, the Lasserre's hierarchy (5.13)-(5.14) must have finite convergence, and the condition (5.16) must be satisfied, when k is sufficiently large. This can be implied by Theorem 1.1 of [29] and Proposition 4.6 of [19].

(iii)-(iv): These two items can be proved exactly in the same way as for (i)-(ii). The proof is omitted here, for cleanness of the paper. \Box

Remark 5.3. In Theorem 5.2(ii), (iv), if $V_{\mathbb{R}}(h)$ is not finite, then we can only get the asymptotic convergence

$$\lim_{k \to \infty} \mu_{1,k} = \lim_{k \to \infty} \tilde{\mu}_{1,k} = f_i^{(1)}, \qquad \lim_{k \to \infty} \mu_{2,k} = \lim_{k \to \infty} \tilde{\mu}_{2,k} = f_i^{(2)}.$$

This is because $V_{\mathbb{R}}(h)$ is contained in the unit sphere $\{x \in \mathbb{R}^n : x^T x = 1\}$ and the ideal I(h) is archimedean. The asymptotic convergence can be implied by [17]. However, the set $V_{\mathbb{R}}(h)$ is finite for generic tensors \mathcal{A}, \mathcal{B} , as shown below.

Proposition 5.4. Let h be as in (5.6). If A, B are generic tensors, then $V_{\mathbb{C}}(h)$ and $V_{\mathbb{R}}(h)$ are finite sets.

Proof. By the construction of h as in (5.6), h(x) = 0 if and only if

(5.27)
$$x^T x - 1 = 0$$
, rank $[a(x) \ b(x)] \le 1$.

Let $J = \{j : x_j \neq 0\}$. We claim that $b(x) \neq 0$. Suppose otherwise b(x) = 0, then

$$\mathcal{B}_J(x_J)^{m-1} = 0.$$

(See §3.2 for the notation \mathcal{B}_J .) Since x_J is a nonzero vector, we get $R_J(\mathcal{B}) = 0$. This is impossible, when \mathcal{B} is a generic tensor. Thus, in (5.27), $b(x) \neq 0$ and there exists λ such that

$$a(x) - \lambda b(x) = 0.$$

Thus, we get that

$$x \circ (\mathcal{A}x^{m-1} - \lambda \mathcal{B}x^{m-1}) = a(x) - \lambda b(x) = 0.$$

This implies that x is a C-eigenvector, associated to λ . By Theorem 3.3, there are finitely many normalized C-eigenvectors, when \mathcal{A}, \mathcal{B} are generic. Therefore, h(x) = 0 has finitely many complex solutions, for generic \mathcal{A}, \mathcal{B} . So, both $V_{\mathbb{C}}(h)$ and $V_{\mathbb{R}}(h)$ are finite.

Proposition 5.5. Let T_1 (resp., T_2) be the feasible set of (5.12) (resp., (5.19)). For all $\xi \in \mathbb{R}^n$, we have the properties:

- (i) If $T_1 = \emptyset$, then there is no C-eigenvector x satisfying (5.4) and $f(x) \ge \ell_i$.
- (ii) If $T_2 = \emptyset$, then there is no C-eigenvector x satisfying (5.7) and $f(x) \geq \tilde{\ell}_i$.
- (iii) For the case i=1, if $T_1=\emptyset$ then the set (5.4) is empty; if $T_2=\emptyset$ then the set (5.7) is empty. Thus, if $T_1=T_2=\emptyset$, then the pair $(\mathcal{A},\mathcal{B})$ has no C-eigenpairs.

Proof. For every C-eigenpair (λ, x) , it holds that $a(x) - \lambda b(x) = 0$, so

$$\xi^T a(x) - \lambda \xi^T b(x) = 0.$$

If $\xi^T b(x) > 0$, x satisfies (5.4). If $\xi^T b(x) < 0$, x satisfies (5.7). If $\xi^T b(x) = 0$, then $\xi^T a(x) = 0$ and x satisfies both (5.4) and (5.7).

- (i) Every C-eigenvector x satisfying (5.4) and $f(x) \ge \ell_i$ belongs to the set T_1 . So, if $T_1 = \emptyset$, then no C-eigenvector x satisfies (5.4) and $f(x) \ge \ell_i$.
 - (ii) The proof is same as for the item (i).
- (iii) For the case i=1, the set T_1 is same as (5.4), and T_2 is same as (5.7), because $\ell_1 \leq f_1^{(1)}$ and $\tilde{\ell}_1 \leq f_1^{(2)}$. So, if $T_1 = \emptyset$, then (5.4) is empty; if $T_2 = \emptyset$, then (5.7) is empty. If $T_1 = T_2 = \emptyset$, then (5.4) and (5.7) are both empty, i.e., $(\mathcal{A}, \mathcal{B})$ has no C-eigenpairs.

Lemma 5.6. Assume that $V_{\mathbb{R}}(h)$ is a finite set. Let θ_1, θ_2 be as in (5.25), (5.26). Then, for $\delta > 0$, $\ell_i = f_{i-1}^{(1)} + \delta$ satisfies (5.11) if and only if $\theta_1 = f_{i-1}^{(1)}$, and $\tilde{\ell}_i = f_{i-1}^{(2)} + \delta$ satisfies (5.18) if and only if $\theta_2 = f_{i-1}^{(2)}$.

Proof. Since $V_{\mathbb{R}}(h)$ is a finite set, (5.5) has finitely many objective values on its feasible set, and they can be ordered as in (5.10). The optimal value θ_1 of (5.25) is the maximum objective value of (5.5) that is less than or equal to $f_{i-1}^{(1)} + \delta$. Then, (5.11) is satisfied if and only if $\theta_1 = f_{i-1}^{(1)}$. The proof is same for the case of $\tilde{\ell}_i$. \square

6. Numerical Experiments

In this section, we present numerical experiments for solving tensor eigenvalue complementarity problems. The Lasserre type semidefinite relaxations are solved by the software GloptiPoly 3 [12] and SeDuMi [41]. The experiments are implemented on a laptop with an Intel Core i5-2520M CPU (2.50GHz) and 8GB of RAM, using Matlab R2014b. We display 4 decimal digits for numerical numbers.

We use \mathcal{I} to denote the identity tensor (i.e., $\mathcal{I}_{i_1\cdots i_m}=1$ if $i_1=\cdots=i_m$, and $\mathcal{I}_{i_1\cdots i_m}=0$ otherwise). When \mathcal{B} is strictly copositive. Algorithm 4.1 is applied to solve the TEiCP; otherwise, Algorithm 5.1 is used.

Example 6.1. (i) ([22, Example 5.1]). Consider the tensors $\mathcal{A}, \mathcal{B} \in T^4(\mathbb{R}^2)$ given as

```
 \begin{array}{l} \mathcal{A}(:,:,1,1) = \left( \begin{array}{ccc} 0.8147 & 0.5164 \\ 0.5164 & 0.9134 \\ \end{array} \right), & \mathcal{A}(:,:,1,2) = \left( \begin{array}{ccc} 0.4218 & 0.8540 \\ 0.8540 & 0.9595 \\ \end{array} \right), \\ \mathcal{A}(:,:,2,1) = \left( \begin{array}{ccc} 0.4218 & 0.8540 \\ 0.8540 & 0.9595 \\ \end{array} \right), & \mathcal{A}(:,:,2,2) = \left( \begin{array}{ccc} 0.4218 & 0.8540 \\ 0.8540 & 0.9595 \\ \end{array} \right), \\ \mathcal{B}(:,:,1,1) = \left( \begin{array}{ccc} 1.6324 & 1.1880 \\ 1.1880 & 1.5469 \\ 1.4824 & 1.9340 \\ \end{array} \right), & \mathcal{B}(:,:,1,2) = \left( \begin{array}{ccc} 1.6557 & 1.4424 \\ 1.4424 & 1.9340 \\ \end{array} \right), \\ \mathcal{B}(:,:,2,1) = \left( \begin{array}{ccc} 1.6557 & 1.4424 \\ 1.4424 & 1.9340 \\ \end{array} \right), & \mathcal{B}(:,:,2,2) = \left( \begin{array}{ccc} 1.6557 & 1.4386 \\ 1.4386 & 1.0318 \\ \end{array} \right). \end{array}
```

The tensor \mathcal{B} is strictly copositive, beause all its entries are positive. By Algorithm 4.1, we get three C-eigenpairs (λ_i, u_i) :

$$\begin{array}{ll} \lambda_1 = 0.4678, & u_1 = (0.8328, 0.0585), \\ \lambda_2 = 0.4848, & u_2 = (0.2577, 0.6538), \\ \lambda_3 = 0.4991, & u_3 = (0.8847, 0.0000). \end{array}$$

The computation takes about 2 seconds.

(ii) ([22, Example 5.2]). Consider the tensors $\mathcal{A}, \mathcal{B} \in T^4(\mathbb{R}^3)$ given as:

$$\mathcal{B}(:,:,2,3) = \left(\begin{array}{ccc} 0.4632 & 0.2043 & 0.2823 \\ 0.2043 & 0.7282 & 0.7400 \\ 0.2823 & 0.7400 & 0.9369 \end{array}\right), \\ \mathcal{B}(:,:,3,1) = \left(\begin{array}{ccc} 0.7585 & 0.6433 & 0.2306 \\ 0.6433 & 0.8986 & 0.3427 \\ 0.2306 & 0.3427 & 0.5390 \end{array}\right)$$

$$\mathcal{B}(:,:,3,2) = \left(\begin{array}{ccc} 0.4632 & 0.2043 & 0.2823 \\ 0.2043 & 0.7282 & 0.7400 \\ 0.2823 & 0.7400 & 0.9369 \end{array} \right), \ \mathcal{B}(:,:,3,3) = \left(\begin{array}{ccc} 0.8200 & 0.5914 & 0.4983 \\ 0.5914 & 0.0762 & 0.2854 \\ 0.4983 & 0.2854 & 0.1266 \end{array} \right)$$

The tensor \mathcal{B} is also strictly copositive, beause all its entries are positive. By Algorithm 4.1, we get three C-eigenpairs (λ_i, u_i) :

```
\begin{array}{ll} \lambda_1 = 1.5520, & u_1 = (0.2201, 0.1572, 0.8680), \\ \lambda_2 = 2.3562, & u_2 = (0.0000, 0.0312, 1.5404), \\ \lambda_3 = 2.7583, & u_3 = (0.0000, 0.0000, 1.6765). \end{array}
```

The computation takes about 15 seconds.

Example 6.2. ([5, §5]). Consider the tensors $\mathcal{A}, \mathcal{B} \in T^6(\mathbb{R}^4)$ with $\mathcal{B} = \mathcal{I}$ (the identity tensor) and \mathcal{A} listed as in Table 1. Note that \mathcal{A} is a symmetric tensor, i.e., $\mathcal{A}_{i_1i_2i_3i_4i_5i_6} = \mathcal{A}_{j_1j_2j_3j_4j_5j_6}$ whenever $(i_1, i_2, i_3, i_4, i_5, i_6)$ is a permutation of $(j_1, j_2, j_3, j_4, j_5, j_6)$. So, only its upper triangular entries are listed. The tensor \mathcal{B}

TABLE 1. The symmetric tensor $A \in T^6(\mathbb{R}^4)$ in Example 6.2.

```
A_{111111} = 0.5000,
                         A_{111112} = -0.2369,
                                                 A_{111113} = 0.1953,
                                                                          A_{111114} = -0.2691,
A_{111122} = 0.0835,
                         A_{111123} = -0.2016,
                                                  A_{111124} = -0.0441,
                                                                          A_{111133} = 0.0567,
A_{111134} = -0.2784,
                         A_{111144} = 0.2321,
                                                 A_{111222} = -0.1250,
                                                                          A_{111223} = 0.0333,
A_{111224} = 0.0235,
                        A_{111233} = 0.0093,
                                                 A_{111234} = -0.0304,
                                                                          A_{111244} = -0.0167,
                         A_{111334} = -0.0385,
A_{111333} = 0.1028,
                                                 A_{111344} = 0.0068,
                                                                          A_{111444} = 0.1627,
A_{112222} = -0.1002,
                         A_{112223} = 0.0733,
                                                 A_{112224} = 0.0607,
                                                                          A_{112233} = -0.1125,
A_{112234} = 0.0096,
                         A_{112244} = -0.0810,
                                                  A_{112333} = -0.0299,
                                                                          A_{112334} = 0.0153,
A_{112344} = 0.0572,
                                                 A_{113333} = 0.1927,
                                                                          A_{113334} = -0.1024,
                         A_{112444} = 0.0251,
A_{113344} = -0.0885,
                        A_{113444} = 0.0289,
                                                 A_{114444} = -0.0668,
                                                                          A_{122222} = -0.2707,
A_{122223} = -0.1066,
                         A_{122224} = -0.1592,
                                                 A_{122233} = 0.0805,
                                                                          A_{122234} = -0.0540,
                         A_{122333} = -0.0048,
                                                 A_{122334} = -0.0118,
A_{122244} = -0.0434,
                                                                          A_{122344} = 0.0196,
A_{122444} = -0.0585,
                         A_{123333} = -0.0442,
                                                 A_{123334} = -0.0618,
                                                                          A_{123344} = 0.0318,
                                                  A_{133333} = 0.1291,
A_{123444} = 0.0332,
                         A_{124444} = -0.2490,
                                                                          A_{133334} = 0.0704,
A_{133344} = -0.0032,
                                                  A_{134444} = 0.0232,
                         A_{133444} = 0.0270,
                                                                           A_{144444} = -0.3403,
A_{222222} = -0.6637,
                         A_{222223} = 0.2191,
                                                 A_{222224} = 0.3280,
                                                                          A_{222233} = 0.1834,
A_{222234} = 0.0627,
                         A_{222244} = 0.0860,
                                                 A_{222333} = 0.1590,
                                                                          A_{222334} = -0.0217,
A_{222344} = 0.1198,
                                                 A_{223333} = 0.0549,
                         \mathcal{A}_{222444}\,=\,\text{-}0.1674,
                                                                          A_{223334} = -0.0868,
                         \mathcal{A}_{223444} \ = \ 0.0101,
                                                 \mathcal{A}_{224444}\,=\,\text{-}0.0307,
                                                                          A_{233333} = -0.3553,
A_{223344} = 0.0043,
A_{233334} = 0.0207,
                         A_{233344} = 0.1544,
                                                  A_{233444} = -0.1707,
                                                                           A_{234444} = -0.3557,
A_{244444} = -0.1706,
                                                 A_{333334} = -0.3628,
                         A_{333333} = 0.7354,
                                                                          A_{333344} = -0.2650,
                                                                          A_{444444} = 0.6136.
A_{333444} = -0.0479,
                         A_{334444} = -0.0084,
                                                 A_{344444} = -0.0559,
```

is strictly copositive. We apply Algorithm 4.1 and get fifteen C-eigenpairs (λ_i, u_i) :

```
u_1 = (0.7814, 0.7331, 0.7630, 0.8654),
         -12.7096,
                        u_2 = (0.7414, 0.8448, 0.1123, 0.8819),
          -9.3276,
                        u_3 = (0.0000, 0.5798, 0.8395, 0.9214),
          -6.9921,
          -4.8469,
                        u_4 = (0.7907, 0.0000, 0.8629, 0.8365),
\lambda_5 =
          -3.1530,
                        u_5 = (0.1704, 0.0000, 0.9300, 0.8406),
                        u_6 = (0.0000, 0.8032, 0.0000, 0.9492),
          -0.9797,
                       u_7 = (0.4471, 0.0000, 0.0186, 0.9987),
          -0.0933,
\lambda_8 = \lambda_9 =
            0.3394,
                       u_8 = (1.0000, 0.0000, 0.0000, 0.1880),
            0.6136,
                       u_9 = (0.0000, 0.0000, 0.0000, 1.0000),
\lambda_{10} =
            0.9215,
                      u_{10} = (0.5942, 0.5831, 0.9856, 0.0000),
\lambda_{11} =
            1.7772,
                      u_{11} = (0.9431, 0.0000, 0.0000, 0.8165),
\lambda_{12} =
                      u_{12} = (0.0000, 0.9338, 0.8342, 0.0887),
            3.0313,
            3.1009,
                      u_{13} = (0.0000, 0.9239, 0.8504, 0.0000),
\lambda_{14} =
                      u_{14} = (0.0000, 0.9619, 0.7672, 0.4016),
            4.5057, u_{15} = (0.8754, 0.0000, 0.9051, 0.0000).
```

The computation takes about 16083 seconds.

Example 6.3. ([4, §5]) Consider the tensors $\mathcal{A}, \mathcal{B} \in T^6(\mathbb{R}^4)$ with $\mathcal{B} = \mathcal{I}$ and \mathcal{A} listed as in Table 2. The tensor \mathcal{A} is symmetric, so only the upper triangular entries are listed. The tensor \mathcal{B} is copositive. We apply Algorithm 4.1 and get only one

TABLE 2. The symmetric tensor $\mathcal{A} \in T^6(\mathbb{R}^4)$ in Example 6.3.

$A_{111111} = 0.1197,$	$A_{111112} = 0.4859,$	$A_{111113} = 0.4236,$	$\mathcal{A}_{111114} = 0.1775,$
$A_{111122} = 0.4639,$	$A_{111123} = 0.4951,$	$A_{111124} = 0.5322,$	$A_{111133} = 0.4219,$
$A_{111134} = 0.4606,$	$A_{111144} = 0.4646,$	$A_{111222} = 0.4969,$	$A_{111223} = 0.4649,$
$A_{111224} = 0.5312,$	$A_{111233} = 0.5253,$	$A_{111234} = 0.4635,$	$A_{111244} = 0.4978,$
$A_{111333} = 0.5562,$	$A_{111334} = 0.5183,$	$A_{111344} = 0.4450,$	$A_{111444} = 0.4754,$
$A_{112222} = 0.4992,$	$A_{112223} = 0.5420,$	$A_{112224} = 0.4924,$	$A_{112233} = 0.5090,$
$A_{112234} = 0.4844,$	$A_{112244} = 0.5513,$	$A_{112333} = 0.5040,$	$A_{112334} = 0.4611,$
$A_{112344} = 0.4937,$	$A_{112444} = 0.5355,$	$A_{113333} = 0.4982,$	$A_{113334} = 0.4985,$
$A_{113344} = 0.4756,$	$A_{113444} = 0.4265,$	$A_{114444} = 0.5217,$	$A_{122222} = 0.2944,$
$A_{122223} = 0.5123,$	$A_{122224} = 0.4794,$	$A_{122233} = 0.5046,$	$A_{122234} = 0.4557,$
$A_{122244} = 0.5332,$	$A_{122333} = 0.5161,$	$A_{122334} = 0.5236,$	$A_{122344} = 0.5435,$
$A_{122444} = 0.5576,$	$A_{1233333} = 0.5685,$	$A_{123334} = 0.5077,$	$A_{123344} = 0.5138,$
$A_{123444} = 0.5402,$	$A_{124444} = 0.4774,$	$A_{133333} = 0.6778,$	$A_{133334} = 0.4831,$
$A_{133344} = 0.5030,$	$A_{133444} = 0.4865,$	$A_{134444} = 0.4761,$	$A_{144444} = 0.3676,$
$A_{222222} = 0.1375,$	$A_{222223} = 0.5707,$	$A_{222224} = 0.5440,$	$A_{222233} = 0.5135,$
$A_{222234} = 0.5770,$	$A_{222244} = 0.6087,$	$A_{222333} = 0.5075,$	$A_{222334} = 0.4935,$
$A_{222344} = 0.5687,$	$A_{222444} = 0.5046,$	$A_{223333} = 0.5226,$	$A_{223334} = 0.4652,$
$A_{223344} = 0.5289,$	$A_{223444} = 0.4810,$	$A_{224444} = 0.5310,$	$A_{233333} = 0.6187,$
$A_{233334} = 0.5811,$	$A_{233344} = 0.4811,$	$\mathcal{A}_{233444} = 0.4883,$	$A_{234444} = 0.4911,$
$\mathcal{A}_{244444} = 0.4452,$	$A_{333333} = 0.1076,$	$A_{333334} = 0.6543,$	$A_{333344} = 0.4257,$
$A_{333444} = 0.5786,$	$A_{334444} = 0.5956,$	$A_{344444} = 0.4503,$	$A_{444444} = 0.3840.$

C-eigenpair:

$$\lambda_1 = 515.4181, \quad u_1 = (0.7909, 0.7957, 0.7941, 0.7941).$$

The computation takes about 140 seconds.

Example 6.4. Consider the tensors $\mathcal{A}, \mathcal{B} \in T^3(\mathbb{R}^n)$ given as:

$$A_{ijk} = \frac{(-1)^j}{i} + \frac{(-1)^k}{j} + \frac{(-1)^i}{k}, \ \mathcal{B} = \mathcal{I}.$$

By Algorithm 4.1, for n = 3, we get seven C-eigenpairs (λ_i, u_i) :

$$\begin{array}{lll} \lambda_1 = & -8.7329, & u_1 = (0.8432, 0.2568, 0.7266), \\ \lambda_2 = & -8.1633, & u_2 = (0.8529, 0.0000, 0.7241), \\ \lambda_3 = & -3.1458, & u_3 = (0.9982, 0.1768, 0.0000), \\ \lambda_4 = & -3.0000, & u_4 = (1.0000, 0.0000, 0.0000), \\ \lambda_5 = & -1.2863, & u_5 = (0.0000, 0.3171, 0.9893), \\ \lambda_6 = & -1.0000, & u_6 = (0.0000, 0.0000, 1.0000), \\ \lambda_7 = & 2.1458, & u_7 = (0.3491, 0.9856, 0.0000). \end{array}$$

When n = 4, we get seven C-eigenpairs (λ_i, u_i) :

$$\begin{array}{lll} \lambda_1 = & -8.3411, & u_1 = (0.8498, 0.0000, 0.7253, 0.1674), \\ \lambda_2 = & -8.1633, & u_2 = (0.8529, 0.0000, 0.7241, 0.0000), \\ \lambda_3 = & -3.0413, & u_3 = (0.9996, 0.0000, 0.0000, 0.1043), \\ \lambda_4 = & -3.0000, & u_4 = (1.0000, 0.0000, 0.0000, 0.0000), \\ \lambda_5 = & -1.0971, & u_5 = (0.0000, 0.0000, 0.9960, 0.2284), \\ \lambda_6 = & -1.0000, & u_6 = (0.0000, 0.0000, 1.0000, 0.0000), \\ \lambda_7 = & 6.6817, & u_7 = (0.4382, 0.7963, 0.0000, 0.7434). \end{array}$$

For n = 3, the computation takes about 21 seconds; for n = 4, it takes about 138 seconds. When n = 5, thirteen C-eigenvalues are obtained. The computer is out of memory for computing the resting C-eigenvalues.

Example 6.5. Consider the tensors $\mathcal{A}, \mathcal{B} \in T^5(\mathbb{R}^n)$ such that

$$\mathcal{A}_{i_1\cdots i_5} = \left(\sum_{j=1}^{5} (-1)^{j+1} \exp(i_j)\right)^{-1}, \ \mathcal{B} = \mathcal{I}.$$

By Algorithm 4.1, for n = 3, we get only one C-eigenpair:

$$\lambda_1 = 2.4335, \quad u_1 = (0.7526, 0.6080, 0.9245).$$

When n = 4, we get only one C-eigenpair:

$$\lambda_1 = 5.4419, \quad u_1 = (0.7391, 0.6412, 0.7719, 0.8313).$$

When n = 5, we get only one C-eigenpair:

$$\lambda_1 = 8.8555, \quad u_1 = (0.7347, 0.6513, 0.7212, 0.7404, 0.7585).$$

For n = 3, the computation takes about 7 seconds; for n = 4, it takes about 44 seconds; for n = 5, it takes about 2662 seconds.

In the following examples, the tensor \mathcal{B} is not strictly copositive. So, Algorithm 5.1 is applied.

Example 6.6. Consider the tensors $\mathcal{A}, \mathcal{B} \in T^3(\mathbb{R}^n)$ given as:

$$\mathcal{A}_{ijk} = \tan(i - \frac{j}{2} + \frac{k}{3}), \ \mathcal{B}_{ijk} = \frac{(-1)^j}{i} + \frac{(-1)^k}{j} + \frac{(-1)^i}{k}.$$

By Algorithm 5.1, for n=3, we get two C-eigenpairs (λ_i, u_i) :

$$\lambda_1 = -4.0192, \quad u_1 = (0.5171, 0.8559, 0.0000),$$

 $\lambda_2 = -0.3669, \quad u_2 = (1.0000, 0.0000, 0.0000).$

When n = 4, we get two C-eigenpairs (λ_i, u_i) :

$$\lambda_1 = -0.8408, \quad u_1 = (0.7095, 0.4519, 0.0000, 0.5407),$$

 $\lambda_2 = -0.2332, \quad u_2 = (0.9962, 0.0000, 0.0000, 0.0874).$

When n = 5, we get six C-eigenpairs (λ_i, u_i) :

$$\begin{array}{lll} \lambda_1 = & -13.3912, & u_1 = (0.0000, 0.0000, 0.0000, 0.3370, 0.9415), \\ \lambda_2 = & -4.1204, & u_2 = (0.0000, 0.0398, 0.0000, 0.0470, 0.9981), \\ \lambda_3 = & -0.8408, & u_3 = (0.7095, 0.4519, 0.0000, 0.5407, 0.0000), \\ \lambda_4 = & -0.8216, & u_4 = (0.7004, 0.4548, 0.0000, 0.5501, 0.0068), \\ \lambda_5 = & -0.4376, & u_5 = (0.6150, 0.1435, 0.4245, 0.3803, 0.5257), \\ \lambda_6 = & -0.2332, & u_6 = (0.9962, 0.0000, 0.0000, 0.0874, 0.0000). \end{array}$$

For n = 3, the computation takes about 2 seconds; for n = 4, it takes about 9 seconds; for n = 5, it takes about 3003 seconds.

Example 6.7. Consider the tensors $\mathcal{A}, \mathcal{B} \in T^4(\mathbb{R}^n)$ such that

$$\mathcal{A}_{i_1 i_2 i_3 i_4} = \frac{1}{10} (i_1 + 2i_2 + 3i_3 + 4i_4 - \sqrt{i_1^2 + i_2^2 + i_3^2 + i_4^2}),$$

$$\mathcal{B}_{i_1 i_2 i_3 i_4} = \arctan(i_1 i_2 i_3 i_4).$$

We apply Algorithm 5.1 to compute the C-eigenpairs. When n=3, we get three C-eigenpairs (λ_i, u_i) :

$$\begin{array}{lll} \lambda_1 = & 0.8706, & u_1 = (1.0000, 0.0000, 0.0000), \\ \lambda_2 = & 0.9780, & u_2 = (0.6209, 0.0000, 0.7839), \\ \lambda_3 = & 1.3163, & u_3 = (0.0000, 0.0000, 1.0000). \end{array}$$

When n = 4, we also get three C-eigenpairs (λ_i, u_i) :

$$\lambda_1 = 0.8706, \quad u_1 = (1.0000, 0.0000, 0.0000, 0.0000),$$

 $\lambda_2 = 1.0698, \quad u_2 = (0.7850, 0.0000, 0.0000, 0.6195),$
 $\lambda_3 = 1.7455, \quad u_3 = (0.0000, 0.0000, 0.0000, 1.0000).$

When n = 5, we also get three C-eigenpairs (λ_i, u_i) :

$$\lambda_1 = 0.8706, \quad u_1 = (1.0000, 0.0000, 0.0000, 0.0000, 0.0000),$$
 $\lambda_2 = 1.1536, \quad u_2 = (0.8527, 0.0000, 0.0000, 0.0000, 0.5224),$
 $\lambda_3 = 2.1787, \quad u_3 = (0.0000, 0.0000, 0.0000, 0.0000, 1.0000).$

For n=3, the computation takes about 6 seconds; for n=4, it takes about 35 seconds; for n=5, it takes about 716 seconds.

Example 6.8. Consider the tensors $\mathcal{A}, \mathcal{B} \in \mathrm{T}^4(\mathbb{R}^n)$ such that

$$\mathcal{A}_{i_1 i_2 i_3 i_4} = (1 + i_1 + 2i_2 + 3i_3 + 4i_4)^{-1}, \ \mathcal{B}_{i_1 i_2 i_3 i_4} = \tan(i_1) + \dots + \tan(i_4).$$

Algorithm 5.1 is applied. When n=3,4,5, the relaxations (5.13) and (5.20) for the order k=4 are both infeasible, so there are no C-eigenvalues. For n=3, the computation takes about 2 seconds; for n=4, it takes about 8 seconds; for n=5, it takes about 43 seconds.

Example 6.9. Consider two randomly generated tensors $\mathcal{A}, \mathcal{B} \in \mathbb{T}^3(\mathbb{R}^5)$:

$$A(:,:,1) = \left(\begin{array}{ccccc} 0.0195 & -0.8385 & -0.5971 & -0.9968 & 0.8617 \\ 1.2397 & 1.8190 & 0.3261 & -1.0365 & -0.6295 \\ -0.1187 & 0.2297 & 1.5407 & -1.0985 & 0.1256 \\ 1.3101 & -0.9982 & 1.1868 & 0.2386 & 2.4171 \\ 1.4264 & 2.4354 & -0.4358 & -1.4201 & 0.5474 \\ \end{array} \right),$$

$$A(:,:,2) = \left(\begin{array}{ccccc} 1.0276 & -1.0345 & -0.6651 & -0.7659 & 0.1898 \\ 0.3746 & 0.6527 & -1.1189 & 0.8586 & 0.8419 \\ 0.8412 & 0.8611 & 0.0405 & -0.3752 & -0.2802 \\ 0.0881 & -1.3853 & 1.8775 & 0.2183 & 0.1735 \\ 0.7881 & 0.5900 & 0.0509 & -0.7750 & 0.0494 \\ \end{array} \right),$$

$$A(:,:,3) = \begin{pmatrix} -0.5525 & -0.2713 & -0.3630 & 0.8350 & -0.0891 \\ 0.2359 & -1.8207 & 0.6906 & -1.7055 & -1.2772 \\ 2.0821 & -1.3487 & -0.4501 & 0.8657 & 1.4453 \\ -0.7056 & 0.2588 & -0.5409 & -0.6727 & -1.7967 \\ 0.7374 & 0.3692 & -0.8594 & 0.7489 & 0.3929 \end{pmatrix}$$

$$A(:,:,4) = \begin{pmatrix} 0.4575 & 0.3137 & 0.4335 & 0.9388 & 0.0927 \\ -0.0042 & 1.6425 & 0.8085 & -1.5722 & -0.9639 \\ -0.1085 & -0.0514 & -1.3662 & 0.3091 & -3.1922 \\ -1.2807 & -0.2399 & -1.1180 & -1.2672 & 0.2671 \\ -1.0279 & -0.9839 & -0.3586 & 0.7765 & 0.4211 \end{pmatrix}$$

$$A(:,:,5) = \begin{pmatrix} 2.0504 & 0.4528 & -1.7698 & -2.5073 & -0.1142 \\ -0.0395 & 0.3460 & -0.1017 & -1.5303 & 0.1027 \\ -0.4152 & -1.2332 & -0.1069 & -1.2440 & 1.6888 \\ -0.8989 & -0.3438 & -2.5825 & -0.4245 & -0.8625 \\ -1.6842 & -0.7582 & -1.7254 & -0.1353 & -0.0564 \end{pmatrix}$$

$$B(:,:,1) = \begin{pmatrix} 0.1278 & -1.2405 & -1.5521 & -0.3097 & 0.4371 \\ 1.0476 & -1.1941 & -0.1954 & -1.3133 & 0.3712 \\ -0.8638 & 0.4681 & 0.1090 & 0.8267 & -0.7007 \\ -0.5110 & -0.2755 & -0.8768 & -0.3897 & -0.2546 \end{pmatrix}$$

$$B(:,:,2) = \begin{pmatrix} 0.1286 & 0.5255 & 0.3809 & 0.1088 & -1.2674 \\ 0.2852 & -1.1047 & -0.8320 & 0.9058 & -2.3433 \\ -1.5964 & 0.3327 & 0.1657 & 0.2164 & 0.4927 \\ -0.9393 & -0.9674 & -0.4843 & 0.4749 & 0.4720 \\ -0.9393 & -0.9674 & -0.4843 & 0.4749 & 0.4720 \\ -0.6881 & 1.7844 & 2.0353 & 0.5464 & 0.7580 \end{pmatrix}$$

$$B(:,:,3) = \begin{pmatrix} 0.4473 & 0.8023 & 2.1941 & 1.7633 & -2.0100 \\ 0.8716 & 0.1619 & 0.0832 & 1.0375 & 1.0234 \\ -0.4001 & 1.0824 & 0.4427 & 1.6162 & 0.1706 \\ -0.2331 & 0.2375 & -0.0875 & -0.5156 & -1.0727 \\ 0.6626 & 0.1542 & 0.3014 & 1.1429 & -0.1337 \end{pmatrix}$$

$$B(:,:,4) = \begin{pmatrix} 0.4473 & 0.8023 & 2.1941 & 1.7633 & 2.0100 \\ -0.9912 & 0.3709 & 0.4380 & -0.2003 & -0.4898 \\ -0.9912 & 0.3709 & 0.4380 & -0.2003 & -0.4898 \\ -0.9912 & 0.3709 & 0.4380 & -0.2003 & -0.4898 \\ -0.99912 & 0.3709 & 0.4380 & -0.2666 & -0.4003 \\ 0.7989 & -0.5784 & -1.1768 & -1.1067 & 0.1850 \end{pmatrix}$$

$$B(:,:,5) = \begin{pmatrix} 0.3332 & 0.3925 & 1.5851 & -0.2666 & -0.4003 \\ 0.8811 & 0.4142 & -0.7639 & 0.6644 & 0.0389 \\ 0.4362 & 0.3792 & 1.5087 & -0.6220 & -0.4257 \\ 2.3515 & -0.7528 & 0.9182 & -1.3888 & 0.0862 \\ 0.8837 & -0.6053 & 2.66$$

By Algorithm 5.1, we get five C-eigenpairs (λ_i, u_i) :

$$\begin{array}{lll} \lambda_1 = & -0.3593, & u_1 = (0.1195, 0.2810, 0.9522, 0.0000, 0.0000), \\ \lambda_2 = & 0.0717, & u_2 = (0.8084, 0.0000, 0.3062, 0.4481, 0.2278), \\ \lambda_3 = & 0.2998, & u_3 = (0.0000, 0.9292, 0.3696, 0.0000, 0.0000), \\ \lambda_4 = & 0.8616, & u_4 = (0.7547, 0.0000, 0.3079, 0.3919, 0.4267), \\ \lambda_5 = & 2.1402, & u_5 = (0.7067, 0.3554, 0.3536, 0.2436, 0.4358). \end{array}$$

The computation takes about 995 seconds.

References

- [1] S. Adly and A. Seeger, A nonsmooth algorithm for cone-constrained eigenvalue problems, Comput. Optim. Appl., 49 (2011), pp. 299–318.
- [2] S. Adly and H. Rammal, A new method for solving Pareto eigenvalue complementarity problems, Comput. Optim. Appl., 55 (2013), pp. 703–731.
- [3] J. Bochnak, M. Coste and M-F. Roy, Real Algebraic Geometry, Springer, 1998.
- [4] Z. Chen and L. Qi, A semismooth Newton method for tensor eigenvalue complementarity problem, arXiv preprint arXiv:1510.08570, 2015.
- [5] Z. CHEN, Q. YANG AND L. YE, Generalized eigenvalue complementarity problem for tensors, arXiv preprint arXiv:1505.02494, 2015.
- [6] D. A. COX, J. LITTLE AND D. O'SHEA, Using Algebraic Geometry, Springer Science & Business Media, 2006.
- [7] C. Cui, Y. Dai and J. Nie, All real eigenvalues of symmetric tensors, SIAM J. Matrix Anal. Appl., 35 (2014), pp. 1582–1601.
- [8] R. Curto and L. Fialkow, Truncated K-moment problems in several variables, J. Operator Theory, 54 (2005), pp. 189–226.

- [9] W. DING AND Y. WEI, Generalized tensor eigenvalue problems, SIAM J. Matrix Anal. Appl., 36 (2015), pp. 1073-1099.
- [10] J. W. HELTON AND J. NIE, A semidefinite approach for truncated K-moment problems, Found. Comput. Math., 12 (2012), pp. 851–881.
- [11] D. HENRION AND J. LASSERRE, Detecting global optimality and extracting solutions in GloptiPoly, Positive Polynomials in Control, Lecture Notes in Control and Inform. Sci. Springer, Berlin, 312 (2005), pp. 293–310.
- [12] D. Henrion, J. Lasserre and J. Loefberg, GloptiPoly 3: moments, optimization and semidefinite programming, Optim. Methods Softw., 24 (2009), pp. 761–779.
- [13] S. Hu, Z. Huang, C. Ling and L. Qi, On determinants and eigenvalue theory of tensors, J. Symbolic Comput., 50 (2013), pp. 508-531.
- [14] J. J. JÚDICE, I. M. RIBEIRO AND H. D. SHERALI, The eigenvalue complementarity problem, Comput. Optim. Appl., 37 (2007), pp 139–156.
- [15] J. J. JÚDICE, M. RAYDAN, S. S. ROSA AND S. A. SANTOS, On the solution of the symmetric eigenvalue complementarity problem by the spectral projected gradient algorithm, Numer. Algor., 47 (2008), pp. 391–407.
- [16] J. J. JUDICE, H. D. SHERALI, I. M. RIBEIRO AND S. S. ROSA, On the asymmetric eigenvalue complementarity problem, Optim. Methods Softw., 24 (2009), pp. 549–568.
- [17] J. B. LASSERRE, Global optimization with polynomials and the problem of moments, SIAM J. Optim., 11 (2001), pp. 796–817.
- [18] J. B. LASSERRE, Moments, Positive Polynomials and Their Applications, Imperial College Press, 2009.
- [19] J. LASSERRE, M. LAURENT AND P. ROSTALSKI, Semidefinite characterization and computation of zero-dimensional real radical ideals, Found. Comput. Math., 8 (2008), pp. 607–647.
- [20] M. LAURENT, Sums of squares, moment matrices and optimization over polynomials, Emerging Applications of Algebraic Geometry, Vol. 149 of IMA Volumes in Mathematics and its Applications, M. Putinar and S. Sullivant (eds), Springer, 2009, pp. 157–270.
- [21] L. H. Lim, Singular values and eigenvalues of tensors: a variational approach, In: Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Addaptive Processing, CAMSAP05, pp. 129–132. IEEE Computer Society Press, Piscataway (2005).
- [22] C. Ling, H. He and L. Qi, On the cone eigenvalue complementarity problem for higherorder tensors, Comput. Optim. Appl., 63 (2016), pp. 143-168.
- [23] J. A. C. MARTINS, S. BARBARIN, M. RAOUS AND A. PINTO DA COSTA, Dynamic stability of finite dimensional linearly elastic systems with unilateral contact and Coulomb friction, Comput. Methods Appl. Mech. Eng., 177 (1999), pp. 289–328.
- [24] J. A. C. MARTINS AND A. PINTO DA COSTA, Stability of finite-dimensional nonlinear elastic systems with unilateral contact and friction, Int. J. Solids Struct., 37 (2000), pp. 2519–2564.
- [25] J. A. C. MARTINS AND A. PINTO DA COSTA, Bifurcations and instabilities in frictional contact problems: theoretical relations, computational methods and numerical results, In: European Congress on Computational Methods in Applied Sciences and Engineering: EC-COMAS (2004).
- [26] M. NG, L. QI AND G. ZHOU G, Finding the largest eigenvalue of a nonnegative tensor, SIAM J. Matrix Anal. Appl., 31 (2009), pp. 1090–1099.
- [27] J. NIE AND K. RANESTAD, Algebraic degree of polynomial optimization, SIAM J. Optim., 20 (2009), pp. 485–502.
- [28] J. Nie, Certifying convergence of Lasserre's hierarchy via flat truncation, Math. Program., Ser. A, 142 (2013), pp. 485–510.
- [29] J. Nie, Polynomial optimization with real varieties, SIAM J. Optim., 23 (2013), pp. 1634– 1646.
- [30] J. Nie, Optimality conditions and finite convergence of Lasserre's hierarchy, Math. Program., Ser. A, 146 (2014), pp. 97–121.
- [31] J. Nie, The A-truncated K-moment problem, Found. Comput. Math., 14 (2014), pp. 1243–1276.
- [32] J. Nie, Linear optimization with cones of moments and nonnegative polynomials, Math. Program., Ser. B, 153 (2015), pp. 247–274.
- [33] J. Nie, The hierarchy of local minimums in polynomial optimization, Math. Program., Ser. B, 151 (2015), pp. 555–583.

- [34] A. Pinto da Costa, I. N. Figueiredo, J. J. Júdice and J. A. C. Martins, A complementarity eigenproblem in the stability analysis of finite dimensional elastic systems with frictional contact, In: Ferris, M., Pang, J.S., Mangasarian, O. (eds.) Complementarity: Applications, Algorithms and Extensions, pp. 67–83. Kluwer Academic, New York (2001).
- [35] A. Pinto da Costa and A. Seeger, Numerical resolution of cone-constrained eigenvalue problems, J. Comput. Appl. Math., 28 (2009), pp. 37–61.
- [36] A. Pinto da Costa and A. Seeger, Cone-constrained eigenvalue problems: theory and algorithms, Comput. Optim. Appl., 45 (2010), pp. 25–57.
- [37] M. PUTINAR, Positive polynomials on compact semi-algebraic sets, Ind. Aniv. Math. J., 42 (1993), pp. 969–984.
- [38] L. QI, Eigenvalues of a real supersymmetric tensor, J. Symbolic Comput., 40 (2005), pp. 1302–1324.
- [39] L. QI, G. Yu And E. X. Wu, Higher order positive semidefinite diffusion tensor imaging, SIAM J. Imaging Sci., 3 (2010), pp. 416–433.
- [40] M. QUEIROZ, J. J. JÚDICE AND C. HUMES, The symmetric eigenvalue complementarity problem, Math. Comput., 73 (2004), pp. 1849–1863.
- [41] J. F. STURM, SeDuMi 1.02: A MATLAB toolbox for optimization over symmetric cones, Optim. Methods Softw., 11 & 12 (1999), pp. 625-653. http://sedumi.ie.lehigh.edu/
- [42] F. Xu and C. Ling, Some properties on Pareto-eigenvalues of higher-order tensors, Operations Research Transactions, 19 (2015), pp. 34–41.

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