

The Diameters of Network-flow Polytopes satisfy the Hirsch conjecture

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Abstract. We solve a problem in the combinatorics of polyhedra motivated by the network simplex method. We show that the Hirsch conjecture holds for the diameter of the graphs of all network-flow polytopes, in particular the diameter of a network-flow polytope for a network with n nodes and m arcs is never more than $m + n - 1$. A key step to prove this is to show the same result for classical transportation polytopes.

Keywords: Combinatorial diameter, diameter of graphs of polyhedra, Hirsch conjecture, simplex method, network simplex method, network-flow polytopes, transportation polytopes.

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1 Introduction

Every convex polyhedron has an underlying graph or 1-skeleton (defined by the zero- and one-dimensional faces). The distance between two zero-dimensional faces (vertices) of the polyhedron is the length of a shortest path between them. The *combinatorial diameter* of a polyhedron is the maximum possible distance between a pair of its vertices. Motivated by the study of the worst-case performance of the simplex algorithm to solve linear optimization problems, researchers have considered the geometric challenge of deciding what is the largest possible (combinatorial) graph diameter of convex polytopes with given number of facets and dimension (see [8,17] for an overview about this problem and its implications in optimization theory).

One of the most famous conjectures associated with the diameter is the *Hirsch conjecture*, stated in 1957 by Warren M. Hirsch [7]. It claimed an upper bound of $f - d$ on the (combinatorial) diameter of any d -dimensional polyhedron with f facets. It is known to be true for several special classes of polyhedra (see [17] for a list), such as $(0, 1)$ -polytopes [20] or dual transportation polyhedra [1], and some of these results extend to lattice polytopes, e.g., [11,12]. Today, we know the Hirsch bound does not hold in general, neither for unbounded polyhedra nor for bounded polytopes [18,23]. It remains a famous open problem to prove or disprove that the diameter of polyhedra can be exponential in the dimension and the number of facets of the polyhedron.

Network-flow problems are among the simplest and oldest linear optimization problems, appearing in all introductory linear programming textbooks. It is well-known that the simplex method has a simple graph-theoretical meaning for these problems [25]. The network structure allows for a well-known translation of the simplex method steps in terms of arc additions (e.g., variables are arcs, basic feasible bases are some of the spanning trees of the network and the pivot entering/leaving variables are arcs sharing a cycle of the network). Network-flow problems show exponential pivoting behavior [27], but unlike the general simplex method, we now know versions of the (primal) network simplex method run in polynomial time [21]. The set of possible feasible flows in the network correspond to the points of a *network-flow polyhedron*. Here we restrict ourselves to study network-flow polytopes, i.e., where the solution space is bounded. These polytopes satisfy some special properties (e.g., for integral input data, they have integer vertices too (see [24]), and there are already some polynomial upper bounds on the diameter of network-flow polytopes (see [3,13,21])). However, the exact bound on the diameter of network-flow polytopes diameter has also remained an open question. In this paper we present a final solution of this question showing that all network-flow polytopes have a diameter no larger than the Hirsch bound, and that this is tight in many cases.

Theorem 1. *The diameters of all network-flow polytopes satisfy the Hirsch conjecture. There are network-flow polytopes for which the Hirsch bound is tight. As another consequence, the diameter of a network with n nodes and m arcs is no more than the linear bound $m + n - 1$.*

To prove Theorem 1 we reduce it to a special type of networks-flow problems and prove the Hirsch conjecture for the corresponding polytopes. A transportation problem models the minimum-cost of transporting goods from N_1 supply nodes to N_2 demand nodes, where each of these $n = N_1 + N_2$ (total) nodes sends, respectively receives, a specified quantity of a product (we assume demand is equal to supply and that the cost is given by the sum of the costs at each connection). This is readily represented as a network-flow problem on a bipartite network. The network has $N_1 \cdot N_2$ arcs. See [9,10,19,26] and references therein for detailed information about transportation polytopes. In prior work, the Hirsch bound was shown to hold for all $2 \times N$ - and $3 \times N$ -transportation polytopes [5]. The so-called partition polytopes [4] and the Birkhoff polytopes [2], both classes of 0,1-transportation polytopes, satisfy even much smaller bounds. Despite a lot of research (see [9] for an overview), the value of the exact diameter of transportation polytopes has remained an open problem until now. In this paper we finally solve this problem.

For general transportation polytopes, the best published bound until now was $8(N_1 + N_2 - 2)$ presented in [6], a factor of eight away from the bound claimed by the Hirsch conjecture (other improvements were presented, but are unpublished, see summary in [9]). For $N_1 \times N_2$ -transportation polytopes, the Hirsch conjecture states the diameter bound of a given transportation polytope is $N_1 + N_2 - 1 - \mu$, where μ is the number of so-called *critical pairs* of a supply and a demand node. These are the variables that are strictly positive in every feasible solution to our transportation problem (a value that is not purely combinatorial, but depends on the demands and supply values). Here we finally prove the Hirsch conjecture holds for all $N_1 \times N_2$ -transportation polytopes and their faces.

Theorem 2. *The diameter of an $N_1 \times N_2$ -transportation polytope is bounded above by $N_1 + N_2 - 1 - \mu$, where μ is the number of critical pairs of the transportation polytope. Therefore, the Hirsch conjecture is true for all $N_1 \times N_2$ -transportation polytopes. More strongly, all faces of $N_1 \times N_2$ -transportation polytopes satisfy the Hirsch conjecture.*

One can prove that any network-flow polytope is in fact isomorphic to a face of a transportation polytope. Therefore, once we prove Theorem 2 we can use it to prove Theorem 1.

The proof of Theorem 2 is an algorithm that connects any two vertices of a transportation polytope by a walk on the 1-skeleton that has length at most $N_1 + N_2 - 1 - \mu$. Since the walk produced by the algorithm stays in the minimal face containing both vertices the rest of Theorem 2 follows immediately. Note that the faces of transportation polytopes are transportation problems with some prescribed ‘missing edges’. From the proof it is clear that the diameter of any $N_1 \times N_2$ -transportation polytope is never more than $N_1 + N_2 - 1$. We remark that for each value of N_1, N_2 with $N_1 \geq 3, N_2 \geq 4$, there exist concrete $N_1 \times N_2$ -transportation polytopes that have no critical pairs and attain the bound of $N_1 + N_2 - 1$ [26]. Therefore, some transportation polytopes are in fact *Hirsch-sharp polytopes* in the sense of [14,16].

Our paper is structured as follows. In Section 2, we recall the necessary background on network flows and transportation polytopes and introduce our general notation. Using the language of networks we show that Theorem 1 follows from Theorem 2. In Section 3, we present our algorithm that constructs a walk on the skeleton of the faces of transportation polytopes adhering to the bounds in Theorem 2. Section 4 is dedicated to the correctness proof of the algorithm and the proof of Theorem 2.

2 Background and How Theorem 2 implies Theorem 1

Recall a *network* is a graph with n nodes and m directed edges (or arcs), where each node v has an integer value specified, the so called *excess* of v , and each arc has an assigned positive integer or infinite value called its *capacity*. A *feasible flow* is an assignment of non-negative real values to the arcs of the network so that for any node v the sum of values in outgoing arcs minus the sum of values in incoming arcs equals the prescribed excess of the node v and the capacities of the arcs are not surpassed.

The set of all feasible flows with given excess vector b and capacity vector c is a convex polyhedron, the *network-flow polyhedron*, which is defined by the constraints $\Phi_G x = b$, $0 \leq x \leq c$, where Φ_G denotes the node-arc incidence matrix of G (a *network matrix*). The incidence matrix Φ_G has one column per arc and one row per node. For an arc going from i to j , its corresponding column has zeros everywhere except at the i -th and j -th entries. The j -th entry, the *head* of the arrow, receives a -1 and the i -th entry, *tail* of the arrow, a 1 . The optimization problem $\min r^\top x$, $\Phi_G x = b$, $0 \leq x \leq c$ is the *min-cost flow* problem [25].

In what follows, we assume the network-flow polyhedra we consider are actually *polytopes*, i.e., they are bounded subsets of space. Boundedness can be easily checked in terms of the network, by testing for directed cycles on the network (see [15]). Note that boundedness is obvious for all networks with finite capacities in their arcs. It is worth noticing that it was much harder to disprove the Hirsch conjecture for polytopes (by Santos [23]) than for unbounded polyhedra (by Klee & Walkup [18]). Here we leave open the possibility that an unbounded network-flow polyhedron violates the Hirsch conjecture.

We continue with some background on transportation polytopes and their 1-skeleton. An $N_1 \times N_2$ -transportation problem has N_1 supply points and N_2 demand points. Each supply point holds a quantity $u_i > 0$ and each demand point needs a quantity $v_j > 0$ of a product. The vectors $u = (u_1, \dots, u_{N_1})$ and $v = (v_1, \dots, v_{N_2})$ are the *margins* for the transportation polytope. The total supply equals the total demand, so formally $\sum_{i=1}^{N_1} u_i = \sum_{j=1}^{N_2} v_j$. Let $y_{ij} \geq 0$ denote the flow from supply point i to demand point j . Then the set of *feasible flows* $y \in \mathbb{R}^{N_1 \times N_2}$ can be described as

$$\begin{aligned} \sum_{j=1}^{N_2} y_{ij} &= u_i & i &= 1, \dots, N_1, \\ \sum_{i=1}^{N_1} y_{ij} &= v_j & j &= 1, \dots, N_2, \\ y_{ij} &\geq 0 & i &= 1, \dots, N_1, \quad j = 1, \dots, N_2. \end{aligned}$$

The set of all real solutions of this system of equations and inequalities constitutes the *transportation polytope* $\text{TP}(u, v)$.

When discussing an $N_1 \times N_2$ -transportation problem, it is common practice to think of the supply and demand points as nodes in the complete bipartite graph K_{N_1, N_2} . In the following we denote supply nodes by σ and demand nodes by δ . For a feasible solution of a transportation problem, we define the *support graph* as the subgraph of K_{N_1, N_2} that contains precisely the edges of non-zero flow, i.e., $y_{ij} > 0$.

In general, the points of transportation polytopes do not have connected support graphs. However, this is the case for the 0-dimensional faces, or *vertices*, of non-degenerate transportation polytopes. An $N_1 \times N_2$ -transportation polytope is *non-degenerate* if every vertex of the polytope has exactly $N_1 + N_2 - 1$ non-negative entries. It is well-known that this is the case if and only if there are no non-empty proper subsets $I \subsetneq \{1, \dots, N_1\}$ and $J \subsetneq \{1, \dots, N_2\}$ such that $\sum_{i \in I} u_i = \sum_{j \in J} v_j$, see [26]. Note that for each degenerate $N_1 \times N_2$ -transportation polytope there is a non-degenerate $N_1 \times N_2$ -transportation polytope of the same or larger combinatorial diameter [26]. Therefore, it suffices to consider non-degenerate transportation polytopes to prove upper bounds. We exploit this in the upcoming proof.

For transportation polytopes the vertices can be characterized in terms of their support graphs: A feasible solution y is a vertex if and only if its support graph contains no cycles, that is, it is a spanning forest. Observe that a vertex is uniquely determined by (the edge set of) its support graph and the vertices of non-degenerate transportation polytopes are given by spanning trees (see for example [19]). Therefore, we refer to the support graphs of vertices of transportation polytopes, as well as to the vertices themselves, simply as *trees* and typically denote them by the capital letters O (for ‘original’ tree), F (for ‘final tree’), and C and C' (for the ‘current’ and ‘succeeding’ tree, corresponding to neighboring vertices of the transportation polytope). Here, the margins of the polytope play an important role: They define which trees appear as vertices of the polytope, and which do not.

We continue with characterizing the 1-dimensional faces of the transportation polytope in terms of the support graphs. Note that, to avoid confusion, we use the term ‘edge’ only for the edges of the underlying bipartite graphs, but not for the 1-faces of the transportation polytope.

Proposition 1 (see e.g., Lemma 4.1 in [26]). *Let C and C' be two trees that correspond to vertices of an $N_1 \times N_2$ -transportation polytope $TP(u, v)$. Then the vertices are adjacent in the 1-skeleton of $TP(u, v)$ if and only if $C \cup C'$ contains a unique cycle.*

In particular, walking from some vertex to a neighboring vertex in the polytope (taking a step on the skeleton / walking along a 1-face of the polytope) corresponds to changing the flow on the edges of K_{N_1, N_2} : Being at a vertex (spanning tree) C , we insert an arbitrary edge $\{\sigma, \delta\} \notin C$ into C . This closes a cycle of even length. We alternately increase and decrease flow on the edges of this cycle, where we increase on the edge we inserted. All edges are changed by the same amount which is the minimum existing flow among the edges that are decreased. Due to non-degeneracy (which we may assume here) this deletes exactly one edge and hence leads to a tree C' that is a neighboring vertex of the transportation polytope. Note that the flow on the edges in C is determined by the margins of the polytope, so the margins determine which edge is deleted when inserting an edge.

As mentioned before, to prove validity of the Hirsch conjecture, we have to show an upper bound of $f - d = N_1 + N_2 - 1 - \mu$ on the combinatorial diameter of $N_1 \times N_2$ transportation polytopes for μ the number of *critical pairs*. A critical pair (σ, δ) corresponds to an edge $\{\sigma, \delta\}$ that is present in the support graph of all feasible solutions. Equivalently, they are the variables satisfying $y_{ij} > 0$ for all feasible solutions y . To see that $N_1 + N_2 - 1 - \mu$ is in fact the Hirsch bound, note that the dimension of an $N_1 \times N_2$ -transportation polytope is $d = (N_1 - 1)(N_2 - 1)$ [19], and for non-degenerate transportation polytopes the number of facets is $f = N_1 \cdot N_2 - \mu$. This follows immediately from Theorem 2 in [19].

Example 1 (Critical pairs). Consider a 2×3 -transportation polytope with supply nodes σ^1, σ^2 with margins $u_1 = 5, u_2 = 3$ and demand nodes $\delta^1, \delta^2, \delta^3$ with margins $v_1 = 4, v_2 = 2, v_3 = 2$. Then the edge $\{\sigma^1, \delta^1\}$ has to be present in every feasible tree: Demand node δ^1 with demand $v_1 = 4$ cannot be connected only to the supply node σ^2 with total supply $u_2 = 3$. The Figure 1 illustrates two trees that correspond to vertices of the polytope. Nodes are labeled with the margins, edges with the flow.



Fig. 1. The edge connecting the nodes with margins 5 and 4 is present in both trees.

Note that the two trees differ in exactly one edge, and thus are neighbors as vertices in the 1-skeleton of the transportation polytope. In particular, this example shows that even though the edge corresponding to the critical pair is present in any tree, the flow on this edge might change during a walk on the skeleton of the transportation polytope. Therefore, we cannot simply disregard the edge and assume to have a transportation polytope without critical pairs. \square

Note that $N_1 + N_2 - 1 - \mu$ is precisely the maximum number of edges in which two trees O, F , within the same transportation polytope, can differ. In particular, for proving the Hirsch conjecture for $N_1 \times N_2$ -transportation polytopes it is enough to show that there is a finite sequence of steps from the original tree O to the final tree F that inserts the edges in $F \setminus O$ one after another such that no inserted edge is deleted at a later point. However, the following example, first mentioned in [6], shows that it might be necessary to first delete an edge that is contained in the final tree F , and only reinsert it at a later step.

Example 2. Consider the walk from a tree O to a tree F in Figure 2 on the skeleton of a transportation polytope. All supply nodes (bottom row) have supply 3, all demand nodes (top row) have demand 2. The edges are labeled with the current flow. The dashed edges are the edges we insert in the respective pivot:

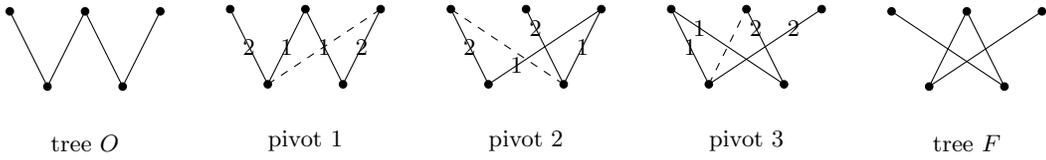


Fig. 2. A walk from tree O to tree F of length three.

Note that O and F differ by only two edges and that all edges that are not in O are in F , and vice versa. No matter which edge we insert in the first step, we have to delete an edge that is contained in F . Then we need at least two more steps, as we still have to insert two edges from F and can only insert exactly one edge in a single step. Therefore the walk above is a walk of minimum length between O and F . \square

It is important to be aware of the fundamental role that the numerical margins play for a walk on the skeleton of a transportation polytope. The margins define which particular spanning trees of K_{N_1, N_2} appear as vertices of the polytope. Let us contrast this with a similar family of polytopes. For the (graphical) matroid polytopes $P_{\text{Mat}(K_{N_1, N_2})}$ of complete bipartite graphs K_{N_1, N_2} , the vertices correspond to *all* the spanning trees of K_{N_1, N_2} . Like before, two spanning trees differing in exactly one edge correspond to neighboring vertices of the polytope $P_{\text{Mat}(K_{N_1, N_2})}$. For a walk on the skeleton of these polytopes, we can simply add the edges contained in the final tree F by exchanging them with edges in the original tree in any order. We do not care about flow values, any two vertices of $P_{\text{Mat}(K_{N_1, N_2})}$ are connected by a walk of length equal to the number of edges in which the two spanning trees differ. In such a sequence we never have to delete an edge that is contained in the target spanning tree F , thus the situation from Example 2 does not occur. Thus proving the Hirsch conjecture for graphical matroid polytopes is a purely combinatorial process, whereas the analysis for transportation polytopes requires the margins.

The following Lemma 1 allows us to derive Theorem 1 from Theorem 2.

Lemma 1. *Given a network G with n nodes and m arcs, with capacity vector c and excess vector b , there is a bipartite uncapacitated network \hat{G} with $n + m$ nodes ($N_1 = n, N_2 = m$), $M = 2m$ arcs, and excess vector \hat{b} (a linear combination of b, c) such that the integral flows in both networks are in bijection and the two corresponding polytopes are isomorphic. The network \hat{G} is obtained from G by replacing each arc by two new arcs and a new node and changing the excesses at each node as illustrated in Figure 3.*



Fig. 3. Construction for Lemma 1.

Proof. For the network G with capacity vector c , the flows are the solutions of

$$\Phi_G x = b, \quad 0 \leq x \leq c. \quad (1)$$

There is a clear bijection (a deletion of redundant dependent variables) between the solutions of system (1) and the solutions of

$$\begin{bmatrix} \Phi_G & 0 \\ I & I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}, \quad x, y \geq 0.$$

The new enlarged matrix is called the *extended network matrix*. It is clear that the polytope of the original network G is isomorphic to the polytope of the extended network matrix. This auxiliary matrix will be useful to prove our claim. From it we apply an invertible linear map to the polytope as follows.

To the network G , with its set of nodes V and its set of arcs E , we have associated the new network \widehat{G} . The set of nodes of \widehat{G} is the disjoint union of the two sets V and E and the network \widehat{G} is obtained from G by replacing each arc by two new arcs as illustrated in Figure 3: that is to each $e \in E$ is associated $f_1 = (j, e)$ and $f_2 = (i, e)$ where $e \in E$ is the common head of both of the new arcs f_1, f_2 and $i, j \in V$ are the tails. Thus \widehat{G} is a directed graph, with a set \widehat{V} of $n + m$ nodes and a set \widehat{E} of $2m$ arcs. We define a new excess vector $\widehat{b} \in \mathbb{R}^{n+m}$. The projection of \widehat{b} on \mathbb{R}^n has coordinates $\widehat{b}_i = b_i + \sum_{f \in E | \text{head}(f)=i} c(f)$. The projection of \widehat{b} on \mathbb{R}^m is the negative of the capacity vector c .

Now we describe the explicit isomorphism between the two polytopes by describing a (special) linear map between the matrix representation of the old capacitated network and the new uncapacitated bipartite network.

Let $T_G \in \mathbb{R}^{n \times m}$ be the matrix with one column per arc and one row per node defined as follows. The column corresponding to an arc has just *one* nonzero entry: the *head* of the arc receives a 1. Then $\Phi_G + T_G$ is the matrix with one column per arc and just the *tail* of the arc receives a 1. All other entries are 0. Consider the matrix transformation

$$\begin{bmatrix} I & T_G \\ 0 & -I \end{bmatrix} \begin{bmatrix} \Phi_G & 0 \\ I & I \end{bmatrix} = \begin{bmatrix} \Phi_G + T_G & T_G \\ -I & -I \end{bmatrix}.$$

The right-hand side is equal to the incidence matrix $\Phi_{\widehat{G}}$ for the new network \widehat{G} .

The m first columns correspond to new arcs f_1 , and the last m columns correspond to new arcs f_2 . Solutions of

$$\begin{bmatrix} \Phi_G & 0 \\ I & I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}$$

are solutions of the equation

$$\Phi_{\widehat{G}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} I & T_G \\ 0 & -I \end{bmatrix} \begin{bmatrix} b \\ c \end{bmatrix} = \widehat{b}.$$

Thus we obtain a linear unimodular bijection between feasible flows of the capacitated network G and feasible flows of the uncapacitated network \widehat{G} . In particular, this implies that both polytopes have the same 1-skeleton. Note that the isomorphism we provided implies that, if x_{ij} was the value of the flow on arc (i, j) of the original capacitated network, then the corresponding new flow in the uncapacitated network will give $y_{i,(i,j)} = x_{ij}$ and $y_{j,(i,j)} = c_{ij} - x_{ij}$. □

Example 3. In Figure 4 we show an example of the transformation described in Lemma 1. We also show how to interpret the resulting uncapacitated network as a transportation problem (in this case missing several edges, which are forced by conditions of type $x_{ij} = 0$). □

Proof of Theorem 1. Assume that Theorem 2 holds, that is, the Hirsch conjecture holds for all the faces of transportation polytopes. We show that this implies the Hirsch conjecture for network-flow polytopes.

First note that every uncapacitated network-flow problem (or network-flow problem with some uncapacitated arcs), whose solution space is a polytope, is identical to a *capacitated* network-flow problem: For each arc (i, j) with infinite capacity we replace the infinite capacity by a finite capacity that is *strictly* larger than the maximum value of the variable x_{ij} . This preserves the same polytope. In particular it preserves the number of inequalities that can be satisfied with equality and thus the number of facets (as the new inequalities are never met with equality). Thus we can restrict our discussion to capacitated network-flow polytopes.

The isomorphism between a capacitated network-flow problem with n nodes and m arcs and the bipartite uncapacitated network with $N_1 = n$ supply nodes and $N_2 = m$ demand nodes provided by Lemma 1 shows that the two polytopes have the same facets and dimension, as well as the same diameter. If one satisfies the Hirsch conjecture the other one does too. Now note

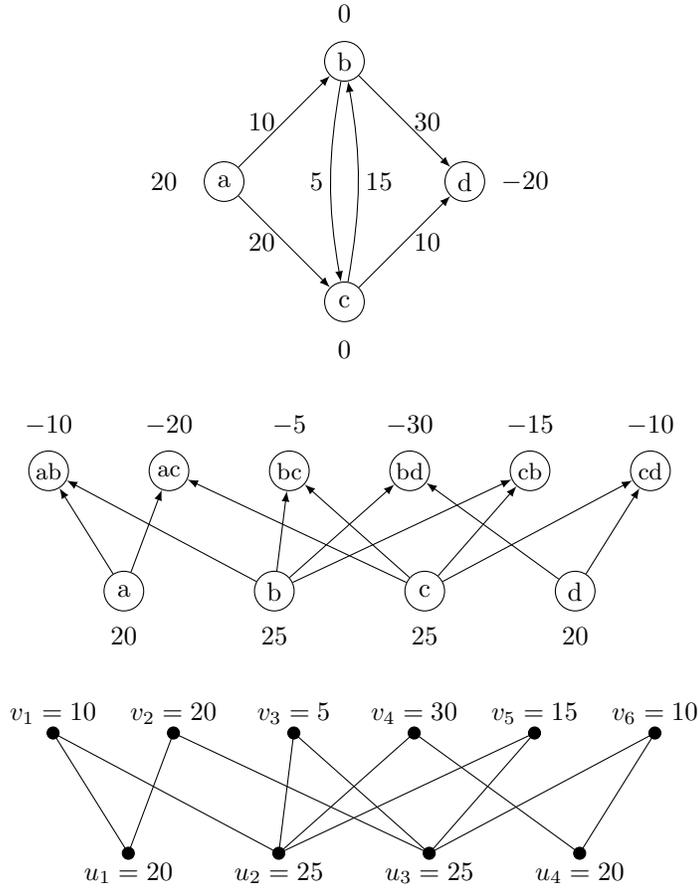


Fig. 4. From a capacitated network to an uncapacitated bipartite network to a transportation problem.

that the faces of a transportation polytope are described by a set of edges with flow fixed to zero (or, in other words, a set of edges that do not exist). Thus the bipartite uncapacitated network constructed in Lemma 1 is in fact a face of a transportation polytope. Therefore, Theorem 2 implies that the Hirsch conjecture holds for capacitated network-flow problems, too.

To see tightness of the Hirsch bound, recall there exist Hirsch-sharp transportation polytopes [26]. It is also easy to give Hirsch-sharp instances for general networks.

It remains to certify the diameter bound of $m + n - 1$ to the original network-flow polytope. The number of facets f of a capacitated network with m arcs and n nodes is bounded above by $2m$, the dimension d of the polytope is $2m - (m + n - 1)$ (because the rank of the node-arc incidence matrix is $m + n - 1$). As the diameter of the polytope satisfies the Hirsch bound of $f - d$, it also is bounded above by $m + n - 1 \geq f - d$. Of course, in many instances, the diameter is much smaller. \square

3 Algorithm to prove Theorem 2

We now present an algorithm that constructs a walk from an initial tree O to a final tree F on the 1-skeleton of a non-degenerate transportation polytope. The walk is fully specified by the corresponding concrete finite sequence of trees, starting with O , ending with F , where each tree differs from the previous one by exactly one edge. We will prove the sequence has at most $N_1 + N_2 - 1 - \mu$ steps, which proves the validity of the Hirsch conjecture for all transportation polytopes. Before we give a pseudo-code description of the algorithm, let us explain its most important features in some detail. The algorithm is based on the following important principle:

Construct a sequence of trees, starting from a tree O and ending at a tree F , by successively inserting edges contained in F and such that no previously inserted edge is ever deleted.

Note that in Example 2, we do not delete an edge that was inserted in a previous step, but it is fine to delete edges of F otherwise. To keep track of the edges that may not be deleted, we *shade* them. There will be two general situations in which we shade an edge:

- Whenever we *insert* an edge from F into the current tree, we *shade* it.
- We may *shade an edge from F that already exists* in the current tree.

Note that only the first case corresponds to a step on the skeleton of the transportation polytope from a tree C to a tree C' . When inserting an edge into a tree, we also *delete an unshaded edge*. In contrast, we keep the current tree in the second case. We will often refer to both of the above situations at the same time. We then write “(insert and) shade an edge”.

In our discussion, we will frequently refer to a tree and the set of shaded edges in the tree at the same time. To this end, we introduce the following two terms:

- We call a tree with a (not necessarily strict) subset of edges shaded a *partially shaded tree*.
- We call a tree with all edges shaded a *fully shaded tree*. As we only shade edges from F in our algorithm, a fully shaded tree is in fact the tree F .

The most important aspect of our algorithm is the order in which edges are (inserted and) shaded; recall again Example 2. Based on the edge that we insert in the current tree, the margins of the transportation polytope uniquely define which edge is deleted. The order of insertion is determined by the following labeling of the edges in F , in which every edge is *labeled + or -*.

- Choose an arbitrary demand node δ^* and, in F , consider all paths starting at δ^* .
- Label the edges on these paths alternatingly + and -, beginning with a +.

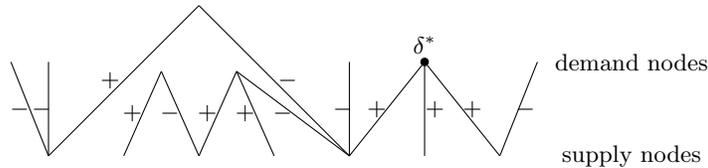


Fig. 5. A labeling of the edges in F .

Figure 5 is an example for such a labeling. Note that in particular the following properties hold: (a) each supply node is incident to exactly one +edge, (b) each demand node $\neq \delta^*$ is incident to exactly one -edge, and (c) δ^* is only incident to +edges.

These labels for edges in F will not change during our algorithm and can be preprocessed. The main part of the algorithm then begins with an original tree O and all edges unshaded. In each iteration, we (insert and) shade an edge from the final tree F . When inserting an edge, the margins tell us which edge will be deleted. We proceed like this until we reach F with all edges shaded.

We now take a closer look at a shading step (an iteration of the algorithm). Let C be a partially shaded tree. We take a supply node σ , one that satisfies a special property in C to be discussed in Section 4. We consider the edges that are incident to σ in F and that are still unshaded. We either shade one of these edges from $C \cap F$, or we insert an edge from $F \setminus C$ and then shade it. Thus in each iteration we shade an edge and obtain a succeeding tree C' with one additional edge shaded. The decision of which edge to (insert and) shade is based on the above labeling: For each supply node, its unique incident +edge will be the last edge to be shaded.

Algorithm 1 gives a description in pseudo-code of the method: Steps 1 and 2 describe the preprocessing of labeling the edges in F and the initialization of the algorithm. The main loop is stated in Step 3 and we refer to each run through the loop as an *iteration* of the algorithm. Note

that the actual walk on the skeleton greatly depends on the labeling we fix in the beginning. The chosen labeling has an impact on the if-else clauses in the main loop and also affects the subroutine (Algorithm 2). Choosing a different labeling, i.e., selecting a different demand node δ^* (Step 1 of Algorithm 1) may change the walk significantly; this might even change the number of steps (the length of the walk on the skeleton). The same holds for choosing a different initial supply node σ (Step 2 of Algorithm 1). Thus the constructed walk is not necessarily of minimum length.

Algorithm 1 Hirsch-walk in a Transportation Polytope

Input: Trees O and F corresponding to vertices of an $N_1 \times N_2$ -transportation polytope $\text{TP}(u, v)$

Output: A finite sequence S of trees corresponding to a walk from O to F on the skeleton of $\text{TP}(u, v)$. The number of steps described by S is at most $N_1 + N_2 - 1 - \mu$, where μ is the number of critical pairs of $\text{TP}(u, v)$

1. Choose an arbitrary demand node δ^* and consider all paths in F starting at δ^* . Label the edges on these paths alternatingly $+$ and $-$, beginning with a $+$.
 2. Choose an arbitrary supply node σ . All edges in O are unshaded. Start with sequence S only containing O . Set $C = O$, the current tree of the walk.
 3. **repeat**
 - **If** *there is an unshaded $-$ edge incident to σ in F* **then**
 - (insert and) shade an unshaded $-$ edge $e = \{\sigma, \delta\}$ incident to σ in F into C to obtain the succeeding tree C'
 - if** *the edge e was inserted (not only shaded)* **then**
 - append C' to the end of S ,
 - set δ' as the demand node incident to the deleted edge
 - else**
 - set $\delta' := \delta$
 - **Else**
 - (insert and) shade the unique $+$ edge $e = \{\sigma, \delta\}$ incident to σ in F into C to obtain the succeeding tree C'
 - if** *the edge e was inserted (not only shaded)* **then**
 - append C' to the end of S
 - set δ' as the demand node incident to the deleted edge
 - else**
 - set $\delta' := \delta$.
 - **If** δ^* *is only incident to shaded edges in C'* **then**
 - return** S and **stop**
 - Update σ by calling *Algorithm 2* with input δ' and C'
 - Set $C := C'$
- end repeat**
-

Algorithm 2 Find New Supply Node σ .

Input: Demand node δ' and a partially shaded tree C' (with $+/-$ -labels for the shaded edges), corresponding to a vertex of $\text{TP}(u, v)$

Output: A new supply node σ in C'

- **if** *there is an unshaded edge $\{\sigma', \delta'\}$ in C'* **then**
 - return** $\sigma = \sigma'$
 - **else**
 - set σ'' as the supply node of the unique $-$ edge $\{\sigma'', \delta'\}$ incident to δ'
 - return** $\sigma = \sigma''$
-

Let us illustrate how the algorithm works by applying it to Example 2.

Example 4 (Example 2 revisited). We show a run of Algorithm 1 for the instance from Example 2. As before, we want to connect O and F as depicted in Figure 6 with a walk on the skeleton of the transportation polytope. Figure 6 also shows a choice of δ^* and the corresponding labeling of the edges in F according to Step 1 in Algorithm 1.



Fig. 6. Initial tree O with all edges unshaded and final tree F with edge labels.

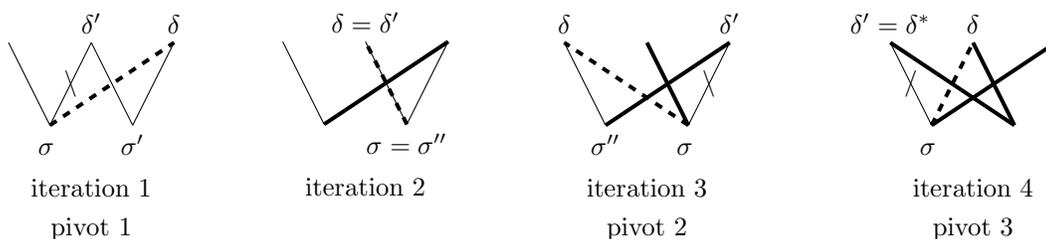


Fig. 7. The iterations of Algorithm 1 for the input from Figure 6.

The algorithm inserts and shades edges in four iterations of Algorithm 1, which are depicted in Figure 7. As we will see, iterations 1,3, and 4 correspond to actual steps on the skeleton of the underlying transportation polytope. In contrast, iteration 2 illustrates a situation where we remain at the current tree. The names of supply nodes and demand nodes in the figure correspond to the notation used in Algorithms 1 and 2. In particular, recall that in each iteration we (insert and) shade an edge incident to the carefully chosen current supply node σ . Further, recall that δ refers to the demand node of this edge. Let us follow a run of Algorithm 1 for this example in some detail. The reader should keep in mind that in all calculations we use the margins of Example 2 where supply nodes (bottom row) have supply 3 and demand nodes (top row) have demand 2.

In the first iteration, the node σ is an arbitrary supply node chosen in Step 2 of the algorithm. We begin with all edges unshaded. By inserting and shading the $-$ edge $\{\sigma, \delta\}$, we delete the edge $\{\sigma, \delta'\}$, which is an edge incident to δ' . It is important to note that the margins dictate which edge is deleted after one is inserted. The demand node δ' is the input for the first call of the subroutine to find a new supply node (Algorithm 2). There still is the unshaded edge $\{\delta', \sigma'\}$ incident to δ' in the succeeding tree, so the subroutine returns the supply node σ' and we continue with it as the new supply node σ .

In the second iteration, there is a $-$ edge $\{\sigma, \delta\}$ incident to the new σ in F that already exists in the current tree, but is still unshaded. We choose $e = \{\sigma, \delta\}$ and only shade it (no insertion). Therefore we have $\delta' = \delta$. Since there is no unshaded edge incident to δ' , the update subroutine returns the supply node σ'' of the unique $-$ edge incident to δ' as the new σ for the next iteration. (In this case, it is the same supply node as for the previous iteration.)

In the third iteration, there is only one edge $\{\sigma, \delta\}$ (a $+$ edge) left to shade incident to the new σ . We insert and shade it. Once more dictated by the margins at nodes, this forces the deletion of the unshaded edge incident to δ' . There are no unshaded edges incident to δ' and thus we continue with σ'' , the supply node of the unique $-$ edge incident to δ' . We update $\sigma := \sigma''$.

In the fourth iteration, we again insert the only unshaded edge $\{\sigma, \delta\}$ (a $+$ -edge) incident to σ . Given the margins of the instance, this forces the deletion of the unshaded edge incident to $\delta' = \delta^*$. Now δ^* is only incident to shaded edges in the new tree C' . This is the stopping criterion for Algorithm 1. We have reached the final tree F with all edges shaded. \square

4 Proof of Correctness

We now turn to the proof of correctness for Algorithm 1. First, we introduce notation and two properties for trees, respectively their nodes, that are at the core of the proof.

4.1 Well-Connectedness and the UNO and SIN properties

We distinguish two states for each node and shaded edge in the current tree.

Definition 1. *Let C be the partially shaded current tree and F the final tree.*

We say a supply node σ is well-connected if all edges that are incident to σ in F exist in the current tree C and are shaded ('nothing left to insert'). Otherwise it is open.

A demand node δ is well-connected if it is not incident to an unshaded edge in C ('nothing to delete'). Otherwise it is open.

A shaded edge incident to at least one well-connected node is a well-connected edge.

Note that to detect well-connectedness during a run of Algorithm 1, it is enough to consider the current tree and the edge labels from the final tree F . For the demand nodes, this is obvious. A supply node is well-connected if and only if it is incident to a shaded $+$ -edge. This is because the Algorithm 1 first (inserts and) shades all $-$ -edges incident to a supply node and only then the unique $+$ -edge. Further, shaded edges are never deleted.

The well-connected edges together with their incident nodes form an important structure for our proofs.

Definition 2. *A well-connected component in a partially shaded tree C is a connected subgraph induced by the well-connected edges. Further, any node not incident to a well-connected edge forms a component by itself.*

The well-connected components are exactly the connected components of the forest we would obtain deleting all edges that are not well-connected from the current tree. Note that every node and every well-connected edge belongs to a unique well-connected component. Also, every shaded $+$ -edge belongs to some well-connected component (its supply node is well-connected). However, this is not always the case for shaded $-$ -edges as its supply and demand node might both be open (edges to insert / unshaded edges); see Example 5 for a illustration of these concepts.

In what follows we will simply use “component(s)” when we talk about the well-connected components.

Definition 3. *We say a component of a partially shaded tree C satisfies property (UNO) (unique open node property) if it contains a unique open node. (UNO) holds in a partially shaded tree C if all its components satisfy (UNO).*

We will show that Algorithm 1 always preserves property (UNO) until we reach the final tree F with all edges shaded and all nodes well-connected (see Lemma 2). This property is crucial for our arguments that follow.

Example 5 (UNO). In this example, we illustrate several well-connected components and their open/well-connected nodes. The bold edges are shaded, ovals indicate the components. Recall that all shaded edges are contained in the final tree F and thus have a $+$ or $-$ label. A supply node is well-connected if and only if it is incident to a shaded $+$ -edge, while a demand node is well-connected if and only if there are no unshaded edges incident to this node.

In the left-hand component in Figure 8, the supply node σ is open as there is no shaded $+$ -edge incident to σ and therefore there is an edge left to insert. All other nodes are well-connected. In the right-hand component, the demand node δ is the unique open node as it is incident to an unshaded edge. In particular, both components satisfy (UNO).

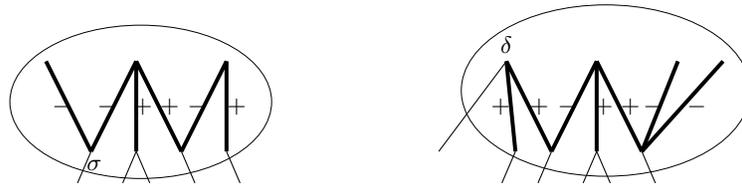


Fig. 8. Components with unique open nodes σ and δ , respectively.

Figure 9 illustrates a configuration in which two components are connected by a shaded $-$ edge $\{\sigma, \delta\}$. This edge is not well-connected because both endpoints are open: σ is not incident to a $+$ edge while δ is incident to an unshaded edge. Note that (UNO) holds in both components.

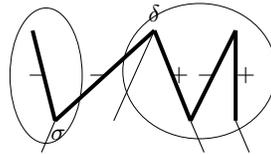


Fig. 9. Two components satisfying (UNO), connected by a shaded edge.

Finally, Figure 10 depicts two components that do not satisfy (UNO): They have two open nodes each, σ and δ , respectively δ and δ' . Observe that here all shaded edges are well-connected as they are incident to a well-connected supply or demand node.



Fig. 10. Components with two open nodes each (σ and δ / δ and δ').

□

Besides (UNO), we need a second property. It is concerned with the supply node σ that is incident to the edge we (insert and) shade. We distinguish odd and even edges with respect to the node σ in the current tree C : Every edge of the current tree C lies on a unique path starting at supply node σ . We number the edges on these paths, where the edges incident to σ are the first edges. Then an edge is *odd* with respect to σ if it has an odd number, otherwise it is *even*. In particular, these paths alternate between odd and even edges. The property (SIN) now imposes a condition on the odd edges on the paths starting at σ .

Definition 4. A supply node σ satisfies property (SIN) (*supply node insertion property*) in a partially shaded tree C if all edges that are odd with respect to σ are unshaded edges or (shaded) $-$ edges incident to well-connected demand nodes.

The odd edges on paths starting at a supply node σ are of particular interest: They are the only edges that could theoretically be deleted when inserting an edge incident to σ . Note again that the margins determine which odd edge is deleted. However, as we will see, by following Algorithms 1 and 2 the odd edges on paths starting at σ that are already shaded will not be deleted. In fact, we will prove later that Algorithm 2 returns a supply node with the (SIN) property. Proving these two statements is a key part of our upcoming proof.

Typically, we will only distinguish between odd and even edges on paths starting at a fixed supply node σ with the (SIN) property. To have a simpler wording, we will refer to these edges only as *odd* or *even* if it is clear which supply node is being considered.

Example 6 (SIN). We illustrate the property (SIN) with the tree depicted in Figure 11. Edges highlighted by wavy lines are the odd edges on paths starting at σ . These edges are either unshaded or they are shaded (bold) –edges with well-connected demand node (δ^1 and δ^2 in Figure 11 are not incident to an unshaded edge and thus well-connected). There are no conditions on the even edges on paths starting at σ .

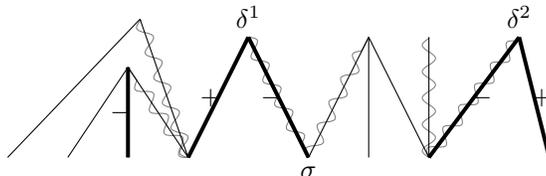


Fig. 11. A node σ with the (SIN) property.

Recall that the odd (wavy) edges are the ones that might be deleted when inserting an edge incident to σ . However, (UNO), (SIN), and our insertion strategy (first –edges, then +edge) imply that we always delete an unshaded edge. In particular, a shaded –edge will not be deleted if it is incident to a well-connected demand node. \square

Example 7 (Example 4 revisited). We illustrate the well-connected nodes and components throughout the run of the algorithm in Example 4 in Figure 7.

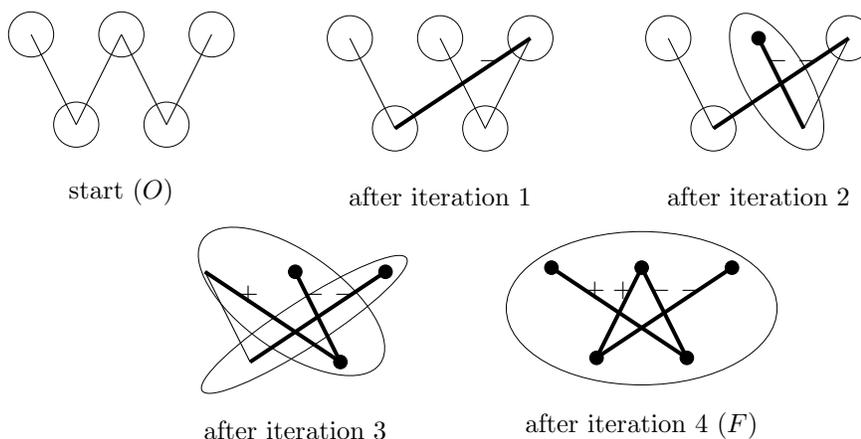


Fig. 12. Components (encircled) and well-connected nodes (filled) throughout a run of Algorithm 1.

Note that not all shadings affect the components (iteration 1). In the other iterations, smaller components are successively merged to larger ones as nodes and edges become well-connected. Well-connected components can be joined

- when a supply node becomes well-connected by (inserting and) shading the +edge incident to the node (see bottom right node in iteration 3) or
- when the last unshaded edge incident to a demand node is shaded (top center in iteration 2) or deleted (top right in iteration 3).

Then the open node of one of the components becomes the open node of the new, larger component. In particular, the trees satisfy (UNO) up to after iteration 3. \square

We close this subsection by proving two lemmas, Lemma 2 and 3, which will be useful later in the main steps of the proof.

Recall that our algorithm has to stop if C is fully shaded. Then in particular $C = F$, as only edges contained in the final tree F are shaded during the run of the algorithm. The first lemma connects this criterion to the (UNO) property.

Lemma 2. *Let C be a partially shaded tree, corresponding to a vertex of a transportation polytope $TP(u, v)$. If there is a well-connected component without an open node in C , then it is because C is fully shaded.*

Proof. Let K be a component in the current tree C with all nodes well-connected and assume C is not fully shaded already. Let V be the set of nodes of the underlying bipartite graph. Then in particular, $V(K)$ is not connected to $V - V(K)$ by a shaded edge in C , but only by an unshaded edge. Similarly, $V(K)$ and $V - V(K)$ are connected in F by at least one edge e . This edge e cannot exist in C already: Assume it does. Then e has to be unshaded. Otherwise it would be a shaded edge incident to a well-connected node in K and thus would be a well-connected edge which makes component K larger. There are no unshaded edges incident to well-connected demand nodes, so e has to be incident to a supply node in $V(K)$. This contradicts the definition of a well-connected supply node. Thus e still has to be inserted and it must be incident to a demand node of the component K .

As we explain in Section 2 it is standard practice to think of a transportation problem as a min-cost flow problem on a bipartite network and we will make use of this now. The feasible flows for given margins u, v correspond to the maximum flows in this network. Thus the difference of two feasible flows is a circulation, which is well-known to decompose into a set of cycles with flow through them [13]. Therefore, the difference $y^F - y^C$ of the feasible flows y^C, y^F of two trees C and F can be decomposed into such a set of cycles. Even more, there is such a decomposition such that flow on edges with $\{\sigma^i, \delta^j\}$ with $y_{ij}^F > y_{ij}^C$ is only increased and flow on edges with $y_{ij}^F < y_{ij}^C$ is only decreased. In particular there is a cycle \mathcal{C} that increases flow on our edge e that exists in F but not in C . Then the cycle \mathcal{C} connects $V(K)$ and $V - V(K)$ by at least one other edge $e' \neq e$. The edge e' is unshaded or does not even exist in the current tree C ; otherwise it would be well-connected.

Note that any pair of a supply node and a demand node is connected by a path with an odd number of edges. Every pair of two demand nodes is connected by a path with an even number of edges. Further, flow on the edges in the cycle \mathcal{C} is alternatingly increased and decreased. Thus, flow on the edge e' has to be increased, if it is incident to a supply node, or decreased, if it is incident to a demand node of K . But there are no such edges because for the supply nodes of K , all edges connecting to $V - V(K)$ are edges to delete (with flow to decrease) and for the demand nodes all such edges are edges to insert (flow to increase), a contradiction. Thus C is fully shaded. \square

During a run of our algorithm, we will keep the (UNO) property for all trees before the final step. Because of this, the above statement will result in a termination criterion. The next lemma is a simple but useful observation.

Lemma 3. *Let C be a partially shaded tree, corresponding to a vertex of a transportation polytope $TP(u, v)$. Assume that in C , there is a supply node satisfying (SIN). Then there is no demand node only incident to shaded +edges in C . In particular, every well-connected demand node is incident to a shaded -edge.*

Proof. Assume there is a demand node δ only incident to shaded +edges. Then, for any supply node, one of these shaded +edges is an odd edge. But this is a contradiction to having a supply node with the (SIN) property in C . \square

4.2 Proof of Theorem 2

Before starting with the actual proof, let us briefly stress its main points. The key point is to show that we always avoid deletion of a shaded edge. We do so in the following way (which precisely corresponds to Algorithms 1 and 2):

- If the current tree satisfies (UNO), then deletion of a shaded edge can be avoided if we (insert and) shade an edge incident to a supply node satisfying (SIN); first all -edges are shaded and only then the unique +edge. (this is the content of Lemma 4).

- When proceeding like this, we keep the (UNO) property throughout the whole walk from O to F . Further we can always find a new supply node with the (SIN) property for the next iteration. (this is the content of Lemma 5).

In the end, we combine our results to obtain Theorem 2.

We must also stress again that the margins of a (non-degenerate) transportation polytope determine which trees appear as vertices of a transportation polytope and which unique edge is deleted when inserting an edge into a tree. Therefore, we can avoid dealing with explicit margins in our proofs. Instead our arguments always refer to the unique edge that is deleted, which allows for a less technical proof. Note that the edge is unique because of our assumption on the polytope being non-degenerate.

In Lemma 4 we prove the key aspect for the correctness of our algorithm: The properties (UNO) and (SIN) and our shading order (first $-$ edges, then the unique $+$ edge) imply that no shaded edge is ever deleted.

Lemma 4. *Let C be a partially shaded tree, corresponding to a vertex of a transportation polytope $TP(u, v)$, and assume the following three conditions hold for C :*

1. *(UNO) holds in C .*
2. *There are no shaded $+$ edges incident to open supply nodes.*
3. *There is a supply node σ satisfying (SIN) in C .*

Choose an edge e incident to σ , as follows:

If in the current tree C there is a $-$ edge incident to σ still left to be (inserted and) shaded, then take edge e to be one such edge.

Otherwise, e is picked to be the unique $+$ edge incident to σ .

Under this rule of selection, no shaded edge is deleted when (inserting and) shading e in C to produce C' .

Proof. If we only shade an existing edge, the statement is obvious because in that operation we delete no edges. Let $e = \{\sigma, \delta\}$ be the edge we insert into the tree C . Recall that the margins of $TP(u, v)$ determine the edge $\{\sigma', \delta'\}$ that is deleted in this step. Let C' be the succeeding tree. Now assume $\{\sigma', \delta'\}$ was shaded in C . Note that by (SIN) for σ in C , $\{\sigma', \delta'\}$ was a $-$ edge with δ' well-connected in C . Let K be the component in C that contains both σ' and δ' . Then $E(K) - \{\sigma', \delta'\}$ induces two connected components, the component $K_{\delta'}$ containing δ' and $K_{\sigma'}$ containing σ' .

First observe that the open node of component K (in C) must be contained in $K_{\sigma'}$. To see this, assume all nodes in $K_{\sigma'}$ are well-connected. Then each supply node must be incident to a shaded $+$ edge and by Lemma 3 each demand node is incident to a shaded $-$ edge. Further, all these edges are contained in $K_{\sigma'}$. But then $K_{\sigma'}$ has at least $|V(K_{\sigma'})|$ many edges, a contradiction to being cycle-free as a subgraph of a tree.

Thus, all nodes in the other connected component $K_{\delta'}$ are well-connected in C and therefore also in C' . This is because no unshaded edge is inserted (in particular not incident to a demand node in $K_{\delta'}$) and in this step no edge incident to a supply node in $K_{\delta'}$ is deleted. If $\{\sigma, \delta\}$ does not connect to $K_{\delta'}$, then $K_{\delta'}$ forms a component in C' and all nodes in this component are well-connected. Lemma 2 implies $C' = F$, a contradiction to $\{\sigma', \delta'\} \in F \setminus C'$.

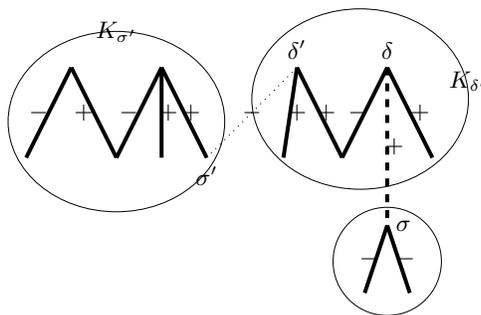


Fig. 13. The dotted edge $\{\sigma', \delta'\}$ was deleted and the bold dashed edge $\{\sigma, \delta\}$ was inserted; ovals illustrate $K_{\sigma'}$, $K_{\delta'}$, and the component with open node σ in C . In this example, the unnamed left-most supply node is the open node in $K_{\sigma'}$.

Therefore, $\{\sigma, \delta\}$ has to connect to $K_{\delta'}$, so one of σ or δ is contained in $K_{\delta'}$. All supply nodes in $K_{\delta'}$ were already well-connected in C , but σ was not. Thus δ must be the node that is contained in $K_{\delta'}$. Figure 13 depicts the situation. Observe that δ must be incident to a $-$ edge in C by Lemma 3. Further, recall that there is at most one $-$ edge incident to each demand node. Therefore, $\{\sigma, \delta\}$ is a $+$ edge. In particular, this is the unique $+$ edge incident to σ and thus σ is well-connected in C' (note that the shaded $-$ edge that we deleted was not incident to σ as then F would contain a cycle). Therefore, $\{\sigma, \delta\}$ is well-connected in C' and its insertion merged the components of σ and δ to a larger component in C' . But this component does not have an open node: We already saw that all nodes in δ 's component $K_{\delta'}$ are well-connected in C' . Further σ was the unique open node of its component, but σ is well-connected now. Again Lemma 2 implies $C' = F$, but we have $\{\sigma', \delta'\} \in F \setminus C'$.

Therefore, when inserting $\{\sigma, \delta\}$, we do not delete a shaded edge incident to a well-connected demand node. This proves the claim. \square

Note that, in particular, (UNO) and (SIN) appear as prerequisites for the application of Lemma 4. They appear as conditions 1 and 3.

Lemma 5 is the final statement we need to see the correctness of Algorithm 1. We prove that conditions 1 to 3 in Lemma 4 remain valid when following Algorithm 1. In particular, (UNO) holds in each iteration and Algorithm 2 returns a node satisfying (SIN) in the succeeding tree. Further, we prove that the termination criterion is correct. The observations from Example 7 are particularly helpful in the proof.

Lemma 5. *Let C be a partially shaded tree, corresponding to a vertex of a transportation polytope $TP(u, v)$, and assume the following three conditions hold for C :*

1. (UNO) holds in C .
2. There are no shaded $+$ edges incident to open supply nodes.
3. There is a supply node σ satisfying (SIN) in C .

Choose an edge e incident to σ , as follows:

If in the current tree C there is a $-$ edge incident to σ still left to be (inserted and) shaded, then take edge e to be one such edge.

Otherwise, e is picked to be the unique $+$ edge incident to σ .

Then, under this rule, one of the following holds:

- (i) Let δ^* be the demand node only incident to $+$ edges in F . If δ^* is well-connected in the succeeding tree C' , then C' is fully shaded, that is, it equals the final tree F and all edges are shaded.
- (ii) Otherwise, there is still an open supply node with the (SIN) property in the succeeding tree C' and we can find such a node by proceeding as in Algorithm 2. In particular, C' is not fully shaded. Further, (UNO) holds in the tree C' .

Proof. Our proof goes as follows: We will show that if C' is not fully shaded, then there is a supply node satisfying (SIN) in C' (see cases 1 and 2). But if there is a supply node with the (SIN) property, then δ^* cannot be well-connected in C by Lemma 3. Thus, if δ^* is well-connected, then C' is fully shaded. This proves (i).

It remains to prove (ii). Clearly, if δ^* (or any other node) is not well-connected, then C' cannot be fully shaded. We show the remaining statements from (ii) in cases 1 and 2. Observe that the selection of a new supply node with the (SIN) property in C' in cases 1 and 2 matches precisely the routine described in Algorithm 2.

We first summarize some fundamental observations. Let $e = \{\sigma, \delta\}$ be the edge we (insert and) shade in the tree C , yielding the succeeding tree C' . Recall that if we perform an actual insertion, then the margins determine which edge is deleted, and thus they determine the succeeding tree C' .

All nodes/edges that were well-connected in C are still well-connected in C' , as by Lemma 4 no shaded edge is deleted. At most two nodes might become well-connected in this step: The supply node σ (which is the case if and only if $\{\sigma, \delta\}$ is a +edge) and the demand node δ if we only shade an edge, respectively, the node δ' incident to the edge deleted in case we perform an insertion.

All shaded edges incident to σ in C are already well-connected by (SIN) for σ in C . In particular, they all belong to the component with open node σ . Therefore, the only edges that might become well-connected are $\{\sigma, \delta\}$ and the shaded edges incident to the demand node δ (if only shading), respectively, δ' (if insertion/deletion). If we get new well-connected edges, we join several smaller components to a larger component.

Case 1 The edge $\{\sigma, \delta\}$ already exists in C , so we only shade it.

In C , σ and δ are the unique open nodes of their components, because there is the edge $\{\sigma, \delta\}$ left to shade at σ and δ is incident to an unshaded edge $\{\sigma, \delta\}$. Every supply node σ' connected to δ by an unshaded edge or a $-$ edge in C' is open in C' , as σ' cannot be incident to a shaded +edge by the (SIN) property for σ in C (see also Figure 14).

First assume δ is open in C' . This case is depicted in the left-most picture in Figure 14. Then δ is still incident to an unshaded edge $\{\sigma', \delta\}$ and σ' is open. Further, σ' satisfies (SIN) in C' . To see this, note that for all edges, except for $\{\sigma, \delta\}$ and $\{\sigma', \delta\}$, this follows from (SIN) for σ in C (these edges have the same 'parity' with respect to both nodes σ and σ'). $\{\sigma', \delta\}$ is the only remaining odd edge for σ' . But this edge is unshaded.

For (UNO), observe that the shading affects the components only if $\{\sigma, \delta\}$ is a +edge, in which case σ becomes well-connected. But then we joined the two components with open node σ and δ , respectively, to a larger component with unique open node δ . Thus (UNO) holds in C' .

Otherwise, shading the edge $\{\sigma, \delta\}$ makes δ well-connected. If δ is only incident to shaded +edges in C' , then C' is fully shaded. The reason is that δ is well-connected, and thus the edge $\{\sigma, \delta\}$ also is well-connected. Therefore, we joined the two components with open nodes σ and δ to a single component in C' . But this larger component does not have an open node, as both σ and δ are well-connected in C' (recall that σ is incident to the shaded +edge $\{\sigma, \delta\}$). Therefore, we reached F with all edges shaded by Lemma 2.

Thus, if C' is not fully shaded, then δ is incident to its unique $-$ edge $\{\sigma'', \delta\}$ in C' and it is shaded (note that $\sigma'' = \sigma$ is possible); see center and right-hand side of Figure 14. Recall that σ'' is open in C' and observe that it satisfies (SIN) in C' . This is because the odd edge $\{\sigma'', \delta\}$ is a $-$ edge incident to a well-connected demand node. For the remaining odd edges the property follows from (SIN) for σ in C .

For (UNO), observe that $\{\sigma, \delta\}$ and $\{\sigma'', \delta\}$ became well-connected. σ , σ'' , and δ were open in C , but δ is well-connected in C' . Therefore, we merged the components with open nodes σ , σ'' , and δ , respectively, to a larger component. Further, σ'' is the unique open node of this component. For $\sigma = \sigma''$, this is obvious. Otherwise σ is well-connected in C' , as $\{\sigma, \delta\}$ has to be a +edge. This is because there is at most one $-$ edge incident to every demand node, which is $\{\sigma'', \delta\}$ in this situation. Therefore, (UNO) holds in C' . This concludes case 1.

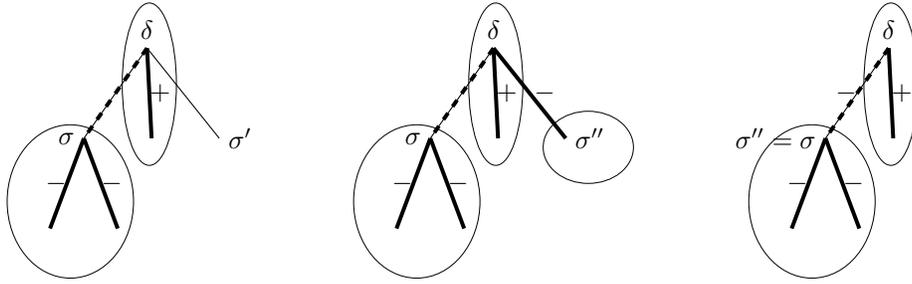


Fig. 14. Possible configurations when shading $\{\sigma, \delta\}$ (from left to right): δ open, δ well-connected and $\sigma'' \neq \sigma$, δ well-connected and $\sigma'' = \sigma$. The circles illustrate the components in C .

Case 2 We inserted and shaded $\{\sigma, \delta\}$ and deleted an unshaded edge incident to the demand node δ' (possibly with $\delta' = \delta$).

In C , σ and δ' are the unique open nodes of their components, because there is the edge $\{\sigma, \delta\}$ left to insert at σ and δ' is incident to an unshaded edge. Observe that every supply node σ' connected to δ' in C' by an unshaded edge or a $-$ edge is open in C' . To see this, recall that a supply node is open if and only if it is not incident to a shaded $+$ -edge. Thus we have to show that there is no shaded $+$ -edge incident to σ' . $\{\sigma', \delta'\}$ is no shaded $+$ -edge by assumption. Any other edge incident to σ' in C is an odd edge with respect to σ and thus cannot be a shaded $+$ -edge by (SIN) for σ in C . Thus σ' is open in C and, if $\sigma' \neq \sigma$, σ' is also open in C' .

For the case $\sigma' = \sigma$, first observe that if $\{\sigma, \delta'\} \in C'$, then this is the edge we inserted: As the step from C to C' deletes an edge incident to δ' and inserts an edge incident to σ , $\{\sigma, \delta'\}$ must be contained in the cycle that describes this step. We delete (decrease) another edge incident to δ' , so $\{\sigma, \delta'\}$ must be increased in this step. But the increased edge incident to σ is the edge we insert. Thus, if $\{\sigma, \delta'\} \in C'$, then $\{\sigma, \delta'\}$ has to be the edge we inserted and if it is a shaded $-$ -edge, then σ is open by our shading order (by condition 4: first $-$ -edges, then the unique $+$ -edge). Figure 15 depicts the situations that may arise in Case 2.

First assume δ' is open in C' . Then δ' is still incident to an unshaded edge $\{\sigma', \delta'\}$ and σ' is open by the above argument. Further, σ' satisfies (SIN) in C' : The odd edge $\{\sigma', \delta'\}$ is unshaded. The inserted edge $\{\sigma, \delta\}$ is even for σ' . For all remaining edges, it follows from (SIN) for σ in C (note that after inserting $\{\sigma, \delta\}$ and deleting an edge incident to δ' , odd (respectively even) edges for σ' in C' are odd (respectively even) with respect to σ in C ; see also Figure 15).

As δ' stays open in C' , the components are affected only if the inserted edge $\{\sigma, \delta\}$ is well-connected in C' . If $\{\sigma, \delta\}$ is a $-$ -edge, then σ is still open in C' . In case $\delta = \delta'$, δ is open by assumption. If $\delta \neq \delta'$, then δ is open in C' , as δ is incident to an unshaded edge in C by Lemma 3 and as $\delta \neq \delta'$, δ is still incident to this unshaded edge in C' . Thus, in both cases, the components are not affected if $\{\sigma, \delta\}$ is a $-$ -edge. If $\{\sigma, \delta\}$ is a $+$ -edge, the step from C to C' joins the components of σ and δ to a larger component whose unique open node is the one from δ' 's component. In particular, (UNO) holds in C' .

Now assume that the deletion of the edge incident to δ' makes δ' well-connected. If δ' is only incident to shaded $+$ -edges in C' , then C' is fully shaded: If we inserted the $+$ -edge $\{\sigma, \delta'\}$, then we joined the components with open nodes σ and δ' , respectively, to a larger component. As σ and δ' are well-connected in C' , this component in C' does not have an open node. Otherwise, all shaded $+$ -edges incident to δ' in C' are already well-connected in C . Thus the component with open node δ' in C does not change, but it does not have an open node in C' . In either case, we reached F with all edges shaded by Lemma 2.

Thus, if δ' is well-connected in C' and C' is not fully shaded, then δ' is incident to its unique $-$ -edge $\{\sigma'', \delta'\}$ in C' and the edge is shaded (note that $\sigma'' = \sigma$ is possible). Recall that σ'' is open in C' and observe that it satisfies (SIN). This is because the odd edge $\{\sigma'', \delta'\}$ is a $-$ -edge incident to a well-connected demand node. For all other odd edges the property follows just like for σ' in the case where δ' was open.

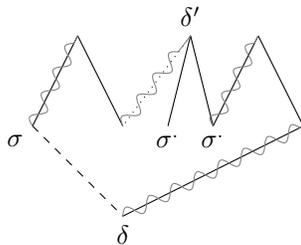


Fig. 15. Insertion of $\{\sigma, \delta\}$ deletes the dotted edge incident to δ' (wavy: odd edges for σ in C). The two σ' may refer to σ' or σ'' .

For (UNO), observe that δ' and the incident $-$ edge $\{\delta', \sigma''\}$ became well-connected in this step. Therefore, we joined the components with open nodes δ' and σ'' , respectively, to a component with open node σ'' . As before, the components containing σ and δ , respectively, are joined in case $\{\sigma, \delta\}$ is a $+$ edge. Note that $\delta = \delta'$ is possible; then σ'' is the open node of the whole component. This concludes Case 2. \square

Combining all our results, we obtain

Proof of Theorem 2. We have to show that for any two vertices of a non-degenerate transportation polytope $\text{TP}(u, v)$, Algorithm 1 finds a walk on the skeleton of $\text{TP}(u, v)$ of length at most $M + N - 1 - \mu$ connecting them, where μ is the number of critical pairs of $\text{TP}(u, v)$.

First, recall the conditions used as prerequisites for the application of both Lemma 4 and 5:

1. (UNO) holds in C .
2. There are no shaded $+$ edges incident to open supply nodes.
3. σ satisfies (SIN) in C .

Algorithm 1 starts with an original tree O in which all edges are unshaded. (UNO) is satisfied as every node forms its own component with exactly one open node (condition 1). As there are no shaded edges, condition 2 is satisfied trivially. Further, all supply nodes satisfy (SIN) (condition 3). Thus we may assume that we are at a partially shaded tree C and have a supply node σ such that all of the above conditions are satisfied.

Clearly, Algorithm 1 was designed precisely to choose an edge e to (insert and) shade following the rule indicated in Lemmas 4 and 5, namely:

If in the current tree C there is a $-$ edge incident to σ still left to be (inserted and) shaded, then take edge e to be one such edge.

Otherwise, e is picked to be the unique $+$ edge incident to σ .

Then, by Lemma 4, this step does not delete a shaded edge. Therefore, the succeeding tree C' satisfies condition 2. If δ^* is well-connected in C' , then we reached the final tree F with all edges shaded by Lemma 5 (i). Thus, our termination criterion as stated in Algorithm 1 is correct. Otherwise, by Lemma 5 (ii), the succeeding tree C' satisfies (UNO) (condition 1) and contains a supply node satisfying (SIN) (condition 3). Such a supply node σ' is found by Algorithm 2. We continue with another iteration, with C' and σ' satisfying conditions 1 to 3.

We now show that the sequence produced by Algorithm 1 has length at most $N_1 + N_2 - 1 - \mu$. As we shade every edge we insert and we never delete a shaded edge, every edge is inserted at most once. We only insert edges contained in F and there are exactly $N_1 + N_2 - 1$ such edges. However, the edges corresponding to the μ critical pairs exist in every tree. Therefore, Algorithm 1 does not perform an insertion when shading them. Thus, we have at most $N_1 + N_2 - 1 - \mu$ steps along the skeleton of the transportation polytope.

Finally, it remains to show that the faces of the polytope satisfy the Hirsch conjecture as well. To see this, note that a face of a transportation polytope is described by a set of edges that do not exist in any support graph of its vertices. Therefore, the walk constructed by Algorithm 1 stays in the face (of minimum dimension) containing the two vertices corresponding to the trees O and F , because the trees of the sequence only contain edges from $O \cup F$. Further, observe that

when restricting to a face of a transportation polytope, the dimension of the face and its number of facets both are reduced by the same number k , which is the number of edges that are never used and thus the Hirsch bound does not change.

This completes the proof of Theorem 2. The Hirsch conjecture is true for all transportation polytopes and all their faces. \square

Remark. It is worth noticing that the walk we obtain from the initial tree O to the final tree F is not always necessarily a shortest path, but there are certainly transportation polytopes for which this is the case. Moreover, the walk produced by Algorithm 1 is not necessarily monotone with respect to a linear objective function, unlike the walks used by the simplex method.

At the same time, while the Hirsch conjecture for transportation polytopes is true and guarantees a short walk between vertices, it is known that there are *long* monotone decreasing walks. More precisely, for any $1/n > \alpha > 0$, the objective function $c \cdot x = x_{1,1} + \alpha x_{1,2} + \dots + \alpha^{N-1} x_{1,N} + \alpha^N x_{2,1} + \dots + \alpha^{N^2-1} x_{N,N}$ has a monotone decreasing sequence of vertices of the $N \times N$ -Birkhoff polytope of length $cN!$ for a universal constant c (see [22]).

A computational enumeration of all $N_1 \times N_2$ -transportation polytopes, for small values of N_1 and N_2 , supports the conjecture that all integers between 1 and $N_1 + N_2 - 1$ are diameters of some $N_1 \times N_2$ transportation polytope, but we have not verified this is the case.

Finally, we remind the reader our arguments are only valid for polytopes. It is still possible that the diameter of the (bounded portion) of the 1-skeleton of an unbounded network-flow polytope is longer than the Hirsch bound and we leave this as an interesting open question.

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