

# Tractability of Approximation for Weighted Korobov Spaces on Classical and Quantum Computers

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## Abstract

We study the approximation problem (or problem of optimal recovery in the  $L_2$ -norm) for weighted Korobov spaces with smoothness parameter  $\alpha$ . The weights  $\gamma_j$  of the Korobov spaces moderate the behavior of periodic functions with respect to successive variables. The non-negative smoothness parameter  $\alpha$  measures the decay of Fourier coefficients. For  $\alpha = 0$ , the Korobov space is the  $L_2$  space, whereas for positive  $\alpha$ , the Korobov space is a space of periodic functions with some smoothness and the approximation problem corresponds to a compact operator. The periodic functions are defined on  $[0, 1]^d$  and our main interest is when the dimension  $d$  varies and may be large. We consider algorithms using two different classes of information. The first class  $\Lambda^{\text{all}}$  consists of arbitrary linear functionals. The second class  $\Lambda^{\text{std}}$  consists of only function values and this class is more realistic in practical computations.

We want to know when the approximation problem is tractable. Tractability means that there exists an algorithm whose error is at most  $\varepsilon$  and whose information cost is bounded by a polynomial in the dimension  $d$  and in  $\varepsilon^{-1}$ . Strong tractability means that the bound does not depend on  $d$  and is polynomial in  $\varepsilon^{-1}$ . In this paper we consider the worst case, randomized and quantum settings. In each setting, the concepts of error and cost are defined differently, and therefore tractability and strong tractability depend on the setting and on the class of information.

In the worst case setting, we apply known results to prove that strong tractability and tractability in the class  $\Lambda^{\text{all}}$  are equivalent. This holds iff  $\alpha > 0$  and the sum-exponent  $s_\gamma$  of weights is finite, where  $s_\gamma = \inf \{ s > 0 : \sum_{j=1}^{\infty} \gamma_j^s < \infty \}$ .

In the worst case setting for the class  $\Lambda^{\text{std}}$  we must assume that  $\alpha > 1$  to guarantee that functionals from  $\Lambda^{\text{std}}$  are continuous. The notions of strong tractability and tractability are not equivalent. In particular, strong tractability holds iff  $\alpha > 1$  and  $\sum_{j=1}^{\infty} \gamma_j < \infty$ .

In the randomized setting, it is known that randomization does not help over the worst case setting in the class  $\Lambda^{\text{all}}$ . For the class  $\Lambda^{\text{std}}$ , we prove that strong tractability and tractability are equivalent and this holds under the same assumption as for the class  $\Lambda^{\text{all}}$  in the worst case setting, that is, iff  $\alpha > 0$  and  $s_\gamma < \infty$ .

In the quantum setting, we consider only upper bounds for the class  $\Lambda^{\text{std}}$  with  $\alpha > 1$ . We prove that  $s_\gamma < \infty$  implies strong tractability.

Hence for  $s_\gamma > 1$ , the randomized and quantum settings both break worst case intractability of approximation for the class  $\Lambda^{\text{std}}$ .

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We indicate cost bounds on algorithms with error at most  $\varepsilon$ . Let  $\mathbf{c}(d)$  denote the cost of computing  $L(f)$  for  $L \in \Lambda^{\text{all}}$  or  $L \in \Lambda^{\text{std}}$ , and let the cost of one arithmetic operation be taken as unity. The information cost bound in the worst case setting for the class  $\Lambda^{\text{all}}$  is of order  $\mathbf{c}(d) \cdot \varepsilon^{-p}$  with  $p$  being roughly equal to  $2 \max(s_\gamma, \alpha^{-1})$ . Then for the class  $\Lambda^{\text{std}}$  in the randomized setting, we obtain the total cost of order  $\mathbf{c}(d) \varepsilon^{-p-2} + d \varepsilon^{-2p-2}$ , which for small  $\varepsilon$  is roughly

$$d \varepsilon^{-2p-2}.$$

In the quantum setting, we present a quantum algorithm with error at most  $\varepsilon$  that uses about only  $d + \log \varepsilon^{-1}$  qubits and whose total cost is of order

$$(\mathbf{c}(d) + d) \varepsilon^{-1-3p/2}.$$

The speedup of the quantum setting over the randomized setting is of order

$$\frac{d}{\mathbf{c}(d) + d} \left( \frac{1}{\varepsilon} \right)^{1+p/2}.$$

Hence, we have a polynomial speedup of order  $\varepsilon^{-(1+p/2)}$ . We stress that  $p$  can be arbitrarily large, and in this case the speedup is huge.

## 1 Introduction

We study the approximation problem (or problem of optimal recovery in the  $L_2$ -norm) for periodic functions  $f : [0, 1]^d \rightarrow \mathbb{C}$  that belong to Korobov spaces. These are the most studied spaces of periodic functions. Usually, the unweighted case, in which all variables play the same role, is analyzed. As in [12, 23], in this paper we analyze a more general case of weighted Korobov spaces, in which the successive variables may have diminishing importance. We consider the unit ball of weighted Korobov spaces  $H_d$ . Hence we assume that  $\|f\|_d \leq 1$  where the norm depends on a non-negative smoothness parameter  $\alpha$  and a sequence  $\gamma = \{\gamma_j\}$  of positive weights. For  $\alpha = 0$  we have  $\|f\|_d = \|f\|_{L_2([0,1]^d)}$ , and for  $\alpha > 0$  the norm is given by

$$\|f\|_d = \left( \sum_{h \in \mathbb{Z}^d} r_\alpha(\gamma, h) |\hat{f}(h)|^2 \right)^{1/2}$$

where  $\mathbb{Z}^d = \{\dots, -1, 0, 1, \dots\}^d$ , Fourier coefficients are denoted by  $\hat{f}(h)$ , and

$$r_\alpha(\gamma, h) = \prod_{j=1}^d r_\alpha(\gamma_j, h_j) \quad \text{with} \quad r_\alpha(\gamma_j, h_j) = \begin{cases} 1 & \text{if } h_j = 0, \\ \gamma_j^{-1} |h_j|^\alpha & \text{if } h_j \neq 0, \end{cases} \quad (1)$$

The smoothness parameter  $\alpha$  measures the decay of the Fourier coefficients. It is known that the weighted Korobov space  $H_d$  consists of functions that are  $k_j$  times differentiable with respect to the  $j$ th variable if  $k_j \leq \alpha/2$ . For  $\alpha \geq 0$ , the space  $H_d$  is a Hilbert space, and for  $\alpha > 1$ , it is a Hilbert space with a reproducing kernel.

The weights  $\gamma_j$  of Korobov spaces moderate the behavior of periodic functions with respect to successive variables. For  $\|f\|_d \leq 1$  and for small  $\gamma_j$ , we have large  $r_\alpha(\gamma, h)$  with non-zero  $h_j$  and therefore the corresponding Fourier coefficient  $|\hat{f}(h)|$  must be small. In the limiting case when  $\gamma_j$  approaches zero, all Fourier coefficients  $\hat{f}(h)$  with non-zero  $h_j$  must be zero, that is, the function  $f$  does not depend on the  $j$ th variable.

We consider algorithms using different classes of information. We study the two classes  $\Lambda^{\text{all}}$  and  $\Lambda^{\text{std}}$  of information. The first one  $\Lambda^{\text{all}} = H_d^*$  consists of all continuous linear functionals, whereas the second one  $\Lambda^{\text{std}}$ , called the standard information, is more realistic in practical computations and consists only of function values, i.e., of  $L_x(f) = f(x) \forall f \in H_d$  with  $x \in [0, 1]^d$ . Such functionals are continuous only if  $\alpha > 1$ .

Our main interest is when the dimension  $d$  varies and may be large. In particular, we want to know when the approximation problem is tractable. Tractability means that there exists an algorithm whose error is at most  $\varepsilon$  and whose information cost (i.e., the number of information evaluations from  $\Lambda^{\text{all}}$  or  $\Lambda^{\text{std}}$ ) is bounded by a polynomial in the dimension  $d$  and in  $\varepsilon^{-1}$ . Strong tractability means that the bound does not depend on  $d$  and is polynomial in  $\varepsilon^{-1}$ . The exponent of strong tractability is defined roughly as the minimal non-negative  $p$  for which the bound is of order  $\varepsilon^{-p}$ .

We consider the worst case, randomized and quantum settings. Each setting has its own definition of error, information and total cost. In the worst case setting we consider only deterministic algorithms, whose error, information and total costs are defined by their worst performance. In the randomized setting we allow randomized algorithms, and their error and costs are defined on the average with respect to randomization for a worst function from the unit ball of  $H_d$ . In the quantum setting we allow quantum algorithms that run on a (hypothetical) quantum computer, with the corresponding definitions of error and costs. Clearly, the concepts of tractability and strong tractability depend on the setting and on the class of information. We are interested in checking how the setting and the class of information change conditions on tractability.

The approximation problem corresponds to the embedding operator between the weighted Korobov space  $H_d$  and the space  $L_2([0, 1]^d)$ . This operator is compact iff  $\alpha > 0$ . That is why for  $\alpha = 0$  we obtain negative results in all three settings and for the two classes of information.

In Section 3 we study the worst case setting. It is enough to consider linear algorithms

of the form

$$A_{n,d}(f) = \sum_{k=1}^n a_k L_k(f).$$

Here, the  $a_k$ 's are some elements of  $L_2([0,1]^d)$ , and the  $L_k$ 's are some continuous linear functionals from  $\Lambda^{\text{all}}$  or  $\Lambda^{\text{std}}$ . The functions  $a_k$  do not depend on  $f$ ; they form the fixed output basis of the algorithm. Necessary and sufficient conditions on tractability of approximation in the worst case setting easily follow from [12, 27, 28]. With

$$s_\gamma = \inf \left\{ s > 0 : \sum_{j=1}^{\infty} \gamma_j^s < \infty \right\},$$

we have:

1. Let  $\alpha \geq 0$ . Strong tractability and tractability of approximation in the class  $\Lambda^{\text{all}}$  are equivalent, and this holds iff  $\alpha > 0$  and the sum-exponent  $s_\gamma$  is finite. If so, the exponent of strong tractability is

$$p^*(\Lambda^{\text{all}}) = 2 \max(s_\gamma, \alpha^{-1})$$

2. Let  $\alpha > 1$ . Strong tractability of approximation in the class  $\Lambda^{\text{std}}$  holds iff

$$\sum_{j=1}^{\infty} \gamma_j < \infty.$$

If so, then  $p^*(\Lambda^{\text{all}}) \leq 2$  and the exponent of strong tractability  $p^*(\Lambda^{\text{std}})$  satisfies

$$p^*(\Lambda^{\text{std}}) \in [p^*(\Lambda^{\text{all}}), p^*(\Lambda^{\text{all}}) + 2].$$

3. Let  $\alpha > 1$ . Tractability of approximation in the class  $\Lambda^{\text{std}}$  holds iff

$$a := \limsup_{d \rightarrow \infty} \frac{\sum_{j=1}^d \gamma_j}{\ln d} < \infty.$$

In particular, we see that for the classical unweighted Korobov space, in which  $\gamma_j = 1$  for all  $j$ , the approximation problem is intractable. To break intractability we must take weights  $\gamma_j$  converging to zero with a polynomial rate, that is,  $\gamma_j = O(j^{-k})$  for some positive  $k$ . Then  $s_\gamma \leq 1/k$ .

In Section 4 we study the randomized setting. We consider randomized algorithms of the form

$$A_{n,d}(f, \omega) = \varphi_\omega(L_{1,\omega}(f), L_{2,\omega}(f), \dots, L_{n,\omega}(f)),$$

where  $\omega$  is a random element that is distributed according to a probability measure  $\varrho$ , and  $L_{k,\omega} \in \Lambda$  with  $\varphi_\omega$  being a mapping from  $\mathbb{C}^n$  into  $L_2([0, 1]^d)$ . The randomized error of an algorithm  $A_{n,d}$  is defined by taking the square root of the average value of  $\|f - A_{n,d}(f, \omega)\|_{L_2([0, 1]^d)}^2$  with respect to  $\omega$  according to a probability measure  $\varrho$ , and then by taking the worst case with respect to  $f$  from the unit ball of  $H_d$ .

It is known, see [15], that randomization does not help over the worst case setting for the class  $\Lambda^{\text{all}}$ . That is why, for the class  $\Lambda^{\text{all}}$ , tractability and strong tractability in the randomized setting are equivalent to tractability and strong tractability in the worst case setting. For the class  $\Lambda^{\text{std}}$  we prove:

1. Strong tractability and tractability of approximation are equivalent, and this holds iff  $\alpha > 0$  and  $s_\gamma < \infty$ . In this case, the exponent of strong tractability is in the interval  $[p^*(\Lambda^{\text{all}}), p^*(\Lambda^{\text{all}}) + 2]$ , where  $p^*(\Lambda^{\text{all}}) = 2 \max(s_\gamma, \alpha^{-1})$ .
2. For any  $p > p^*(\Lambda^{\text{all}})$ , we present an algorithm  $A_{n,d}$  with  $n$  of order  $\varepsilon^{-(p+2)}$  and randomized error at most  $\varepsilon$ . Let  $\mathbf{c}(d)$  be the cost of computing one function value, and let the cost of performing one arithmetic operation be taken as unity. Then the total cost of the algorithm  $A_{n,d}$  is of order

$$\mathbf{c}(d) \left(\frac{1}{\varepsilon}\right)^{p+2} + d \left(\frac{1}{\varepsilon}\right)^{2p+2} \quad \forall d = 1, 2, \dots, \quad \forall \varepsilon \in (0, 1).$$

Hence, the only dependence on  $d$  is through  $\mathbf{c}(d)$  and  $d$ . Clearly, if  $d$  is fixed and  $\varepsilon$  goes to zero then the second term dominates and the total cost of  $A_{n,d}$  is of order

$$d \left(\frac{1}{\varepsilon}\right)^{2p+2}.$$

The essence of these results is that in the randomized setting there is no difference between tractability conditions when we use functionals from  $\Lambda^{\text{all}}$  or from  $\Lambda^{\text{std}}$ . This is especially important when  $s_\gamma > 1$ , since approximation is then intractable in the worst case setting for the class  $\Lambda^{\text{std}}$  independently of  $\alpha$ , and is strongly tractable in the randomized setting for the class  $\Lambda^{\text{std}}$ . Hence for  $s_\gamma > 1$ , randomization breaks intractability of approximation in the worst case setting for the class  $\Lambda^{\text{std}}$ .

In Section 5 we study the quantum setting. We consider quantum algorithms that run on a (hypothetical) quantum computer. Our analysis in this section is based on the framework

for quantum algorithms introduced in [8] that is relevant for the approximate solution of problems of analysis.

We only consider upper bounds for the class  $\Lambda^{\text{std}}$  and weighted Korobov spaces with  $\alpha > 1$  and  $s_\gamma < \infty$ . We present a quantum algorithm with error at most  $\varepsilon$  whose total cost is of order

$$(\mathbf{c}(d) + d) \left( \frac{1}{\varepsilon} \right)^{1+3p/2} \quad \forall d = 1, 2, \dots, \quad \forall \varepsilon \in (0, 1)$$

with  $p \approx p^*(\Lambda^{\text{all}})$  being roughly the exponent of strong tractability in the worst case setting.

The quantum algorithm uses about  $d + \log \varepsilon^{-1}$  qubits. Hence, for moderate  $d$  and even for large  $\varepsilon^{-1}$ , the number of qubits is quite modest. This is especially important, since the number of qubits will be a limiting resource for the foreseeable future.

It is interesting to compare the results in the quantum setting with the results in the randomized setting for the class  $\Lambda^{\text{std}}$ . The number of quantum queries is of order  $\varepsilon^{-1-3p^*(\Lambda^{\text{all}})/2}$  which is smaller than the corresponding number  $\varepsilon^{-2-p}$  of function values in the randomized setting only if  $p^*(\Lambda^{\text{all}}) = 2 \max(s_\gamma, \alpha^{-1}) < 2$ . This holds when  $s_\gamma < 1$ , since  $\alpha > 1$  has been already assumed. However, the number of quantum combinatory operations is always *significantly* smaller than the corresponding number of combinatory operations in the randomized settings. If  $d$  is fixed and  $\varepsilon$  goes to zero then the total cost bound in the randomized setting is of order  $d\varepsilon^{-2p-2}$  which is significantly larger than the total cost bound of order  $(\mathbf{c}(d) + d)\varepsilon^{-1-3p/2}$  in the quantum setting. This means that the exponent of  $\varepsilon^{-1}$  in the cost bound in the quantum setting is  $1 + p/2$  less than the exponent in the randomized setting. We do not know whether our upper bounds for the quantum computer can be improved.

The speedup of the quantum setting over the randomized setting, defined as the ratio of the corresponding randomized and quantum costs, is of order

$$\frac{d}{\mathbf{c}(d) + d} \left( \frac{1}{\varepsilon} \right)^{1+p/2}.$$

Hence, we have a polynomial speedup of order  $\varepsilon^{-(1+p/2)}$ . If  $p^*(\Lambda^{\text{all}})$  is close to zero, we may also take  $p$  close to zero and then the speedup is roughly  $\varepsilon^{-1}$ . But  $p^*(\Lambda^{\text{all}})$  can be arbitrarily large. This holds for large  $s_\gamma$ . In this case  $p$  is also large and the speedup is huge.

We finish our paper with two appendices. The first is about a general framework for quantum algorithms and the second contains a proof of the fact that weighted Korobov spaces are algebras. This fact is crucial for our upper bounds for quantum algorithms and hence for Theorem 4.

## 2 Approximation for Weighted Korobov Spaces

In this section we define approximation for periodic functions from the weighted Korobov space  $H_d$ . The space  $H_d$  is a Hilbert space of complex-valued  $L_2$  functions defined on  $[0, 1]^d$  that are periodic in each variable with period 1. The inner product and norm of  $H_d$  are defined as follows. We take a sequence  $\gamma = \{\gamma_j\}$  of weights such that

$$1 \geq \gamma_1 \geq \gamma_2 \geq \cdots > 0.$$

Let  $\alpha \geq 0$ . For  $h = [h_1, h_2, \dots, h_d] \in \mathbb{Z}^d$  define

$$r_\alpha(\gamma, h) = \prod_{j=1}^d r_\alpha(\gamma_j, h_j) \quad \text{with} \quad r_\alpha(\gamma_j, h_j) = \begin{cases} 1 & \text{if } h_j = 0, \\ \gamma_j^{-s} |h_j|^\alpha & \text{if } h_j \neq 0, \end{cases}$$

where  $s = 1$  for  $\alpha > 0$ , and  $s = 0$  for  $\alpha = 0$ . Note that  $r_\alpha(\gamma, h) \geq 1$  for all  $h \in \mathbb{Z}^d$ , and the smallest  $r_\alpha(\gamma, h)$  is achieved for  $h = 0$  and has the value 1.

The inner product in  $H_d$  is given by

$$\langle f, g \rangle_d = \sum_{h \in \mathbb{Z}^d} r_\alpha(\gamma, h) \hat{f}(h) \overline{\hat{g}(h)},$$

where  $h = (h_1, \dots, h_d)$ , and  $\hat{f}(h)$  is the Fourier coefficient

$$\hat{f}(h) = \int_{[0,1]^d} \exp(-2\pi i h \cdot x) f(x) dx,$$

with  $h \cdot x = h_1 x_1 + \cdots + h_d x_d$ . The inner product in  $H_d$  can be also written as

$$\langle f, g \rangle_d = \hat{f}(0) \overline{\hat{g}(0)} + \sum_{h \in \mathbb{Z}^d, h \neq 0} r_\alpha(\gamma, h) \hat{f}(h) \overline{\hat{g}(h)},$$

thus the zeroth Fourier coefficient is unweighted. The norm in  $H_d$  is

$$\|f\|_d = \left( \sum_{h \in \mathbb{Z}^d} r_\alpha(\gamma, h) |\hat{f}(h)|^2 \right)^{1/2}.$$

Note that for  $\alpha = 0$  we have  $r_0(\gamma, h) \equiv 1$ , and

$$\langle f, g \rangle_d = \sum_{h \in \mathbb{Z}^d} \hat{f}(h) \overline{\hat{g}(h)} = \int_{[0,1]^d} f(x) \overline{g(x)} dx.$$



Hence, in this case  $H_d = L_2([0, 1]^d)$  is the space of square integrable functions. Observe that for any  $\alpha \geq 0$  we have  $H_d \subset L_2([0, 1]^d)$  and

$$\|f\|_{L_2([0,1]^d)} \leq \|f\|_d \quad \forall f \in H_d.$$

For  $\alpha > 1$ , the space  $H_d$  is a reproducing kernel Hilbert space, see [1, 26]. That is, there exists a function  $K_d : [0, 1]^d \times [0, 1]^d \rightarrow \mathbb{C}$ , called the reproducing kernel, such that  $K_d(\cdot, y) \in H_d$  for all  $y \in [0, 1]^d$ , and

$$f(y) = \langle f, K_d(\cdot, y) \rangle_d \quad \forall f \in H_d, \forall y \in [0, 1]^d.$$

The essence of the last formula is that the linear functional  $L_y(f) = f(y)$  for  $f \in H_d$  is continuous and its norm is

$$\|L_y\| = K_d^{1/2}(y, y) \quad \forall y \in [0, 1]^d.$$

It is known, see e.g. [23], that the reproducing kernel  $K_d$  is

$$K_d(x, y) = \sum_{h \in \mathbb{Z}^d} \frac{\exp(2\pi i h \cdot (x - y))}{r_\alpha(\gamma, h)}. \quad (2)$$

This can be rewritten as

$$K_d(x, y) = \prod_{j=1}^d \sum_{h=-\infty}^{\infty} \frac{\exp(2\pi i h(x_j - y_j))}{r_\alpha(\gamma_j, h)} = \prod_{j=1}^d \left( 1 + 2\gamma_j \sum_{h=1}^{\infty} \frac{\cos(2\pi h(x_j - y_j))}{h^\alpha} \right).$$

Hence,  $K_d(x, y)$  depends on  $x - y$  and takes only real values. From this we have

$$K_d(y, y) = \prod_{j=1}^d (1 + 2\gamma_j \zeta(\alpha)),$$

where  $\zeta$  is the Riemann zeta function,  $\zeta(\alpha) = \sum_{h=1}^{\infty} h^{-\alpha}$ . Hence,  $\alpha > 1$  guarantees that  $K_d(y, y)$  is well defined and that  $\|L_y\|$  is finite.

We return to the general case for  $\alpha \geq 0$ . For  $\gamma_j \equiv 1$ , the space  $H_d$  is the  $L_2$  version of the (unweighted) Korobov space of periodic functions. For general weights  $\gamma_j$ , the space  $H_d$  is called a *weighted* Korobov space.

We now explain the role of weights  $\gamma_j$ . Take  $f \in H_d$  with  $\|f\|_d \leq 1$ . For small values of  $\gamma_j$  we must have small Fourier coefficients  $\hat{f}(h)$  with  $h_j \neq 0$ . Indeed,  $\|f\|_d \leq 1$  implies

that  $r_\alpha(\gamma, h)|\hat{f}(h)|^2 \leq 1$ , and for  $h_j \neq 0$  this implies that  $|\hat{f}(h)|^2 \leq \gamma_j/|h_j|^\alpha \leq \gamma_j$ , as claimed. Thus, small  $\gamma_j$ 's correspond to smoother functions in the unit ball of  $H_d$  in the sense that the Fourier coefficients  $\hat{f}(h)$  with  $h_j \neq 0$  must scale like  $\gamma_j^{1/2}$  in order to keep  $\|f\|_d \leq 1$ .

The spaces  $H_d$  are related to each other when we vary  $d$ . Indeed, it is easy to check that for  $d_1 \leq d_2$  we have

$$H_{d_1} \subseteq H_{d_2} \quad \text{and} \quad \|f\|_{d_1} = \|f\|_{d_2} \quad \forall f \in H_{d_1}.$$

That is, a function of  $d_1$  variables from  $H_{d_1}$ , when treated as a function of  $d_2$  variables with no dependence on the last  $d_2 - d_1$  variables, also belongs to  $H_{d_2}$  with the same norm as in  $H_{d_1}$ . This means that we have an increasing sequence of spaces  $H_1 \subset H_2 \subset \cdots \subset H_d$ , and an increasing sequence of the unit balls of  $H_d$ ,  $B_1 \subset B_2 \subset \cdots \subset B_d$ , and  $H_{d_1} \cap B_{d_2} = B_{d_1}$  for  $d_1 \leq d_2$ .

So far we assumed that all weights  $\gamma_j$  are positive. We can also take zero weights as the limiting case of positive weights when we adopt the convention that  $0/0 = 0$ . Indeed, if one of the weights tends to zero, say  $\gamma_d \rightarrow 0$ , then  $r_\alpha(\gamma, h)$  goes to infinity for all  $h$  with  $h_d \neq 0$ . Thus to guarantee that  $\|f\|_d$  remains finite we must have  $\hat{f}(h) = 0$  for all  $h$  with  $h_d \neq 0$ . This means that  $f$  does not depend on the  $x_d$  coordinate. Similarly, if all the weights  $\gamma_j$  are zero for  $j \geq k$  then a function  $f$  from  $H_d$  does not depend on the coordinates  $x_k, x_{k+1}, \dots, x_d$ .

We are ready to define *multivariate approximation* (simply called approximation) as the operator  $\text{APP}_d : H_d \rightarrow L_2([0, 1]^d)$  given by

$$\text{APP}_d f = f.$$

Hence,  $\text{APP}_d$  is the embedding from the Korobov space  $H_d$  to the space  $L_2([0, 1]^d)$ . It is easy to see that  $\|\text{APP}_d\| = 1$ ; moreover  $\text{APP}_d$  is a compact embedding iff  $\alpha > 0$ . Indeed, consider the operator  $W_d := \text{APP}_d^* \text{APP}_d : H_d \rightarrow H_d$ , where  $\text{APP}_d^* : L_2([0, 1]^d) \rightarrow H_d$  is the adjoint operator to  $\text{APP}_d$ . Then for all  $f, g \in H_d$  we have

$$\langle W_d f, g \rangle_d = \langle \text{APP}_d f, \text{APP}_d g \rangle_{L_2([0, 1]^d)} = \langle f, g \rangle_{L_2([0, 1]^d)}.$$

From this we conclude that

$$W_d f_h = r_\alpha^{-1}(\gamma, h) f_h \quad \forall h \in \mathbb{Z}^d,$$

where  $f_h(x) = \exp(2\pi i h \cdot x) / r_\alpha^{1/2}(\gamma, h)$ . We have  $\|f_h\|_d = 1$  and  $\text{span}(f_h : h \in \mathbb{Z}^d)$  is dense in  $L_2([0, 1]^d)$ . This yields that  $W_d$  has the form

$$(W_d f)(x) = \sum_{h \in \mathbb{Z}^d} r_\alpha^{-1}(\gamma, h) \hat{f}(h) \exp(2\pi i h \cdot x) \quad \forall f \in H_d, \quad (3)$$

where for  $\alpha \in [0, 1]$  the convergence of the last series is understood in the  $L_2$  sense.

Thus,  $H_d$  has an orthonormal basis consisting of eigenvectors of  $W_d$ , and  $r_\alpha^{-1}(\gamma, h)$  is the eigenvalue of  $W_d$  corresponding to  $f_h$  for  $h \in \mathbb{Z}^d$ . Clearly,

$$\|\text{APP}_d f\|_{L_2([0,1]^d)} = \langle W_d f, f \rangle_d^{1/2} \quad \forall f \in H_d,$$

and therefore, since  $W_d$  is self adjoint,

$$\|\text{APP}_d\| = \|W_d\|^{1/2} = \left( \max_{h \in \mathbb{Z}^d} r_\alpha^{-1}(\gamma, h) \right)^{1/2} = 1.$$

For  $\alpha = 0$  we have  $\text{APP}_d = W_d$  and both are the identity operator on  $L_2([0,1]^d)$ , and therefore they are *not* compact. In contrast, for  $\alpha > 0$ , the eigenvalues of  $W_d$  go to zero as  $|h| = |h_1| + |h_2| + \dots + |h_d|$  goes to infinity, and therefore the operator  $W_d$  is compact and  $\text{APP}_d$  is a compact embedding.

### 3 Worst Case Setting

In this section we deal with tractability of approximation in the worst case setting. To recall the notion of tractability we proceed as follows. We approximate  $\text{APP}_d$  by algorithms<sup>1</sup> of the form

$$A_{n,d}(f) = \sum_{k=1}^n a_k L_k(f).$$

Here, the  $a_k$ 's are some elements of  $L_2([0,1]^d)$ , and the  $L_k$ 's are some continuous linear functionals defined on  $H_d$ . Observe that the functions  $a_k$  do not depend on  $f$ , they form the fixed output basis of the algorithm, see [18]. For all the algorithms in this paper we use the optimal basis consisting of the eigenvectors of  $W_d$ . We assume that  $L_k \in \Lambda$ , and consider two classes of information  $\Lambda$ . The first class is  $\Lambda = \Lambda^{\text{all}} = H_d^*$  which consists of *all* continuous linear functionals. That is,  $L \in \Lambda^{\text{all}}$  iff there exists  $g \in H_d$  such that  $L(f) = \langle f, g \rangle_d$  for all  $f \in H_d$ . The class  $\Lambda^{\text{all}}$  is well defined for all  $\alpha \geq 0$ . The second class  $\Lambda = \Lambda^{\text{std}}$  is called standard information and is defined only for  $\alpha > 1$ ,

$$\Lambda = \Lambda^{\text{std}} = \{L_x : x \in [0,1]^d \text{ with } L_x(f) = f(x) \forall f \in H_d\}.$$

Hence, the class  $\Lambda^{\text{std}}$  consists of function evaluations. They are continuous linear functionals since  $H_d$  is a reproducing kernel Hilbert space whenever  $\alpha > 1$ .

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<sup>1</sup>It is known that nonlinear algorithms as well as adaptive choice of  $L_k$  do not help in decreasing the worst case error, see e.g., [24].

The worst case error of the algorithm  $A_{n,d}$  is defined as

$$e^{\text{wor}}(A_{n,d}) = \sup \left\{ \|f - A_{n,d}(f)\|_{L_2([0,1]^d)} : f \in H_d, \|f\|_d \leq 1 \right\} = \left\| \text{APP}_d - \sum_{k=1}^n a_k L_k(\cdot) \right\|.$$

Let  $\text{comp}^{\text{wor}}(\varepsilon, H_d, \Lambda)$  be the minimal  $n$  for which we can find an algorithm  $A_{n,d}$ , i.e., find elements  $a_k \in L_2([0,1]^d)$  and functionals  $L_k \in \Lambda$ , with worst case error at most  $\varepsilon \|\text{APP}_d\|$ , that is,

$$\text{comp}^{\text{wor}}(\varepsilon, H_d, \Lambda) = \min \left\{ n : \exists A_{n,d} \text{ such that } e^{\text{wor}}(A_{n,d}) \leq \varepsilon \|\text{APP}_d\| \right\}.$$

Observe that in our case  $\|\text{APP}_d\| = 1$  and this represents the initial error that we can achieve by the zero algorithm  $A_{n,d} = 0$  without sampling the function. Therefore  $\varepsilon \|\text{APP}_d\| = \varepsilon$  can be interpreted as reducing the initial error by a factor  $\varepsilon$ . Obviously, it is only of interest to consider  $\varepsilon < 1$ .

This minimal number  $\text{comp}^{\text{wor}}(\varepsilon, H_d, \Lambda)$  of functional evaluations is closely related to the worst case complexity of the approximation problem, see e.g., [24]. This explains our choice of notation.

We are ready to define tractability, see [29]. We say that approximation is *tractable* in the class  $\Lambda$  iff there exist nonnegative numbers  $C$ ,  $p$  and  $q$  such that

$$\text{comp}^{\text{wor}}(\varepsilon, H_d, \Lambda) \leq C \varepsilon^{-p} d^q \quad \forall \varepsilon \in (0, 1), \forall d \in \mathbb{N}. \quad (4)$$

The essence of tractability is that the minimal number of functional evaluations is bounded by a polynomial in  $\varepsilon^{-1}$  and  $d$ .

We say that approximation is *strongly tractable* in the class  $\Lambda$  iff  $q = 0$  in (4). Hence, strong tractability means that the minimal number of functional evaluations has a bound independent of  $d$  and polynomially dependent on  $\varepsilon^{-1}$ . The infimum of  $p$  in (4) is called the *exponent* of strong tractability and denoted by  $p^* = p^*(\Lambda)$ . That is, for any positive  $\delta$  there exists a positive  $C_\delta$  such that

$$\text{comp}^{\text{wor}}(\varepsilon, H_d, \Lambda) \leq C_\delta \varepsilon^{-(p^*+\delta)} \quad \forall \varepsilon \in (0, 1), \forall d \in \mathbb{N}$$

and  $p^*$  is the smallest number with this property.

Necessary and sufficient conditions on tractability of approximation in the worst case setting easily follow from [12, 27, 28]. In order to present them we need to recall the notion of the sum-exponent  $s_\gamma$  of the sequence  $\gamma$ , see [27], which is defined as

$$s_\gamma = \inf \left\{ s > 0 : \sum_{j=1}^{\infty} \gamma_j^s < \infty \right\}, \quad (5)$$

with the convention that the infimum of the empty set is taken as infinity. Hence, for the unweighted case,  $\gamma_j \equiv 1$ , we have  $s_\gamma = \infty$ . For  $\gamma_j = \Theta(j^{-\kappa})$  with  $\kappa > 0$ , we have  $s_\gamma = 1/\kappa$ . On the other hand, if  $s_\gamma$  is finite then for any positive  $\delta$  there exists a positive  $M_\delta$  such that  $k \gamma_k^{s_\gamma + \delta} \leq \sum_{j=1}^{\infty} \gamma_j^{s_\gamma + \delta} \leq M_\delta$ . Hence,  $\gamma_k = O(k^{-1/(s_\gamma + \delta)})$ . This shows that  $s_\gamma$  is finite iff  $\gamma_j$  goes to zero polynomially fast in  $j^{-1}$ , and the reciprocal of  $s_\gamma$  roughly measures the rate of this convergence.

We begin with the class  $\Lambda^{\text{all}}$ . Complexity and optimal algorithms are well known in this case, see e.g., [24]. Let us define

$$R(\varepsilon, d) = \{ h \in Z^d : r_\alpha^{-1}(\gamma, h) > \varepsilon^2 \}. \quad (6)$$

as the set of indices  $h$  for which the eigenvalues of  $W_d$ , see (3), are greater than  $\varepsilon^2$ . Then the complexity  $\text{comp}^{\text{wor}}(\varepsilon, H_d, \Lambda^{\text{all}})$  is equal to the cardinality of the set  $R(\varepsilon, d)$ ,

$$\text{comp}^{\text{wor}}(\varepsilon, H_d, \Lambda^{\text{all}}) = |R(\varepsilon, d)|, \quad (7)$$

and the algorithm

$$A_{n,d}(f)(x) = \sum_{h \in R(\varepsilon, d)} \hat{f}(h) \exp(2\pi i h \cdot x) \quad (8)$$

with  $n = |R(\varepsilon, d)|$  is optimal and has worst case error at most  $\varepsilon$ . This simply means that the truncation of the Fourier series to terms corresponding to the largest eigenvalues of  $W_d$  is the best approximation of the function  $f$ .

For  $\alpha = 0$  all eigenvalues of  $W_d$  have the value 1. Thus for  $\varepsilon < 1$  we have infinitely many eigenvalues greater than  $\varepsilon^2$  even for  $d = 1$ . Therefore the cardinality of the set  $R(\varepsilon, 1)$  and the complexity are infinite, which means that approximation is not even solvable, much less tractable. For  $\alpha > 0$  and  $d = 1$ , we obtain

$$\text{comp}^{\text{wor}}(\varepsilon, H_1, \Lambda^{\text{all}}) \approx 2 \gamma_1^{1/\alpha} \varepsilon^{-2/\alpha}.$$

It is proven in [27] that strong tractability and tractability are equivalent, and this holds iff  $s_\gamma$  is finite. Furthermore, the exponent of strong tractability is  $p^*(\Lambda^{\text{all}}) = 2 \max(s_\gamma, \alpha^{-1})$ . We stress that the exponent of strong tractability is determined by the weight sequence  $\gamma$  if  $s_\gamma > \alpha^{-1}$ . On the other hand, if  $s_\gamma \leq \alpha^{-1}$  then  $p^*(\Lambda^{\text{all}}) = 2\alpha^{-1}$ , and this exponent appears in the complexity even when  $d = 1$ . For such weights, i.e.,  $s_\gamma \leq \alpha^{-1}$ , multivariate approximation in any number of variables  $d$  requires roughly the same number of functional evaluations as for  $d = 1$ .

We now turn to the class  $\Lambda^{\text{std}}$  and assume that  $\alpha > 1$ . Formally, tractability of approximation in the class  $\Lambda^{\text{std}}$  has not been studied; however, it is easy to analyze this problem

based on the existing results. First, observe that approximation is not easier than *multivariate integration* (or simply integration) defined as

$$\text{INT}_d(f) = \int_{[0,1]^d} f(x) dx = \hat{f}(0) \quad \forall f \in H_d.$$

Indeed,  $\|\text{INT}_d\| = 1$ , and for any algorithm  $A_{n,d}(f) = \sum_{k=1}^n a_k f(x_k)$  for some  $a_k \in L_2([0,1]^d)$  and some  $x_k \in [0,1]^d$ , we have

$$\|\text{APP}_d f - A_{n,d}(f)\|_{L_2([0,1]^d)}^2 = \sum_{h \in \mathbb{Z}^d} |\hat{f}(h) - \widehat{A_{n,d}(f)}(h)|^2 \geq \left| \hat{f}(0) - \sum_{k=1}^n b_k f(x_k) \right|^2,$$

with  $b_k = \int_{[0,1]^d} a_k(x) dx$ . Hence, it is not easier to approximate  $\text{APP}_d$  than  $\text{INT}_d$ , and necessary conditions on tractability of integration are also necessary conditions on tractability for approximation. It is known, see [12], that integration is strongly tractable iff  $\sum_{j=1}^{\infty} \gamma_j < \infty$ , and is tractable iff  $a := \limsup_{d \rightarrow \infty} \sum_{j=1}^d \gamma_j / \ln d < \infty$ . Hence, the same conditions are also necessary for tractability of approximation. Due to [28], it turns out that these conditions are also sufficient for tractability of approximation. More precisely, if  $\sum_{j=1}^{\infty} \gamma_j < \infty$ , then approximation is strongly tractable and its exponent  $p^*(\Lambda^{\text{std}}) \in [p^*(\Lambda^{\text{all}}), p^*(\Lambda^{\text{all}}) + 2]$ , see Corollary 2 (i) of [28]. Clearly, in this case  $p^*(\Lambda^{\text{all}}) \leq 2$ .

Assume that  $a \in (0, \infty)$ . Then there exists a positive  $M$  such that

$$d \gamma_d / \ln d \leq \sum_{j=1}^d \gamma_j / \ln d < M$$

for all  $d$ . Hence,  $\gamma_j = O(j^{-1} \ln j)$ , and clearly  $s_\gamma = 1$ . Once more, by Corollary 2 (i) of [28], we know that for any positive  $\delta$  there exists a positive number  $C_\delta$  such that the worst case complexity of approximation is bounded by  $C_\delta \varepsilon^{-(2+\delta)} d^{4\zeta(\alpha) a + \delta}$ . This proves tractability of approximation. We summarize this analysis in the following theorem.

**Theorem 1** *Consider approximation  $\text{APP}_d : H_d \rightarrow L_2([0,1]^d)$  in the worst case setting.*

1. *Let  $\alpha \geq 0$ . Strong tractability and tractability of approximation in the class  $\Lambda^{\text{all}}$  are equivalent, and this holds iff  $s_\gamma < \infty$  and  $\alpha > 0$ . In this case, the exponent of strong tractability is*

$$p^*(\Lambda^{\text{all}}) = 2 \max(s_\gamma, \alpha^{-1}).$$

2. Let  $\alpha > 1$ . Strong tractability of approximation in the class  $\Lambda^{\text{std}}$  holds iff

$$\sum_{j=1}^{\infty} \gamma_j < \infty.$$

When this holds, then  $p^*(\Lambda^{\text{all}}) \leq 2$  and the exponent of strong tractability

$$p^*(\Lambda^{\text{std}}) \in [p^*(\Lambda^{\text{all}}), p^*(\Lambda^{\text{all}}) + 2].$$

3. Let  $\alpha > 1$ . Tractability of approximation in the class  $\Lambda^{\text{std}}$  holds iff

$$a := \limsup_{d \rightarrow \infty} \frac{\sum_{j=1}^d \gamma_j}{\ln d} < \infty.$$

When this holds, for any positive  $\delta$  there exists a positive  $C_\delta$  such that

$$\text{comp}^{\text{wor}}(\varepsilon, H_d, \Lambda^{\text{std}}) \leq C_\delta \varepsilon^{-(2+\delta)} d^{4\zeta(\alpha)a+\delta} \quad \forall d = 1, 2, \dots, \quad \forall \varepsilon \in (0, 1),$$

where  $\zeta$  is the Riemann zeta function.

## 4 Randomized Setting

In this section we deal with tractability of approximation in the randomized setting for the two classes  $\Lambda^{\text{all}}$  and  $\Lambda^{\text{std}}$ . The randomized setting is precisely defined in [24]. Here we only mention that we consider randomized algorithms

$$A_{n,d}(f, \omega) = \varphi_\omega \left( L_{1,\omega}(f), L_{2,\omega}(f), \dots, L_{n,\omega}(f) \right),$$

where  $\omega$  is a random element that is distributed according to a probability measure  $\varrho$ , and  $L_{k,\omega} \in \Lambda$  with  $\varphi_\omega$  being a mapping from  $\mathbb{C}^n$  into  $L_2([0, 1]^d)$ . The essence of randomized algorithms is that the evaluations, as well the way they are combined, may depend on a random element. The primary example of a randomized algorithm is the standard Monte Carlo for approximating multivariate integration which is of the form

$$A_{n,d}(f, \omega) = \frac{1}{n} \sum_{k=1}^n f(\omega_k),$$

where  $\omega = [\omega_1, \omega_2, \dots, \omega_n]$  with independent and uniformly distributed  $\omega_k$  over  $[0, 1]^d$  which requires  $nd$  random numbers from  $[0, 1]$ . In this case,  $L_{k,\omega}(f) = f(\omega_k)$  are function values at

random sample points, and  $\varphi_\omega(y_1, y_2, \dots, y_n) = n^{-1} \sum_{k=1}^n y_k$  does not depend on  $\omega$  and is a deterministic mapping.

The randomized error of the algorithm  $A_{n,d}$  is defined as

$$e^{\text{ran}}(A_{n,d}) = \sup \left\{ \mathbb{E}^{1/2} \left( \|f - A_{n,d}(f, \omega)\|_{L_2([0,1]^d)}^2 \right) : f \in H_d, \|f\|_d \leq 1 \right\}.$$

Hence, we first take the square root of the average value of the error  $\|f - A_{n,d}(f, \omega)\|_{L_2([0,1]^d)}^2$  with respect to  $\omega$  according to the probability measure  $\varrho$ , and then take the worst case with respect to  $f$  from the unit ball of  $H_d$ .

Let  $\text{comp}^{\text{ran}}(\varepsilon, H_d, \Lambda)$  be the minimal  $n$  for which we can find an algorithm  $A_{n,d}$ , i.e., a measure  $\varrho$ , functionals  $L_{k,\omega}$  and a mapping  $\varphi_\omega$ , with randomized error at most  $\varepsilon$ . That is,

$$\text{comp}^{\text{ran}}(\varepsilon, H_d, \Lambda) = \min \left\{ n : \exists A_{n,d} \text{ such that } e^{\text{ran}}(A_{n,d}) \leq \varepsilon \right\}.$$

Then tractability in the randomized setting is defined as in the paragraph containing (4), with the replacement of  $\text{comp}^{\text{wor}}(\varepsilon, H_d, \Lambda)$  by  $\text{comp}^{\text{ran}}(\varepsilon, H_d, \Lambda)$ .

We are ready to discuss tractability in the randomized setting for the class  $\Lambda^{\text{all}}$ . It is proven in [15] that randomization does not really help for approximating linear operators over Hilbert space for the class  $\Lambda^{\text{all}}$  since

$$\text{comp}^{\text{wor}}(2^{1/2}\varepsilon, H_d, \Lambda^{\text{all}}) \leq \text{comp}^{\text{ran}}(\varepsilon, H_d, \Lambda^{\text{all}}) \leq \text{comp}^{\text{wor}}(\varepsilon, H_d, \Lambda^{\text{all}}),$$

and these estimates hold for all  $\varepsilon \in (0, 1)$  and for all  $d \in \mathbb{N}$ .

This means that tractability in the randomized setting is equivalent to tractability in the worst case setting, and we can use the first part of Theorem 1 to characterize tractability also in the randomized setting.

We now turn to the class  $\Lambda^{\text{std}}$ . It is well known that randomization may significantly help for some problems. The most known example is the standard Monte Carlo for multivariate integration of  $d$  variables, which requires at most  $\varepsilon^{-2}$  random function values if the  $L_2$  norm of a function is at most one, independently of how large  $d$  is.

We now show that randomization also helps for approximation over Korobov spaces, and may even break intractability of approximation in the worst case setting. As we shall see, this will be achieved by a randomized algorithm using the standard Monte Carlo for approximating the Fourier coefficients corresponding to the largest eigenvalues of the operator  $W_d$  defined by (3). To define such an algorithm we proceed as follows.

We assume that  $\alpha > 1$  so that the class  $\Lambda^{\text{std}}$  is well defined. Without loss of generality we also assume that approximation is tractable in the class  $\Lambda^{\text{all}}$ , which is equivalent to assuming that  $s_\gamma < \infty$ .



We know from Section 2 that  $R(\varepsilon/2^{1/2}, d)$  is the set of indices  $h$  for which the eigenvalues of  $W_d$  are greater than  $\varepsilon^2/2$ , see (6). We also know that the cardinality of the set  $R(\varepsilon/2^{1/2}, d)$  is exactly equal to  $\text{comp}^{\text{wor}}(\varepsilon/2^{1/2}, H_d, \Lambda^{\text{all}})$  and that for any positive  $\delta$  there exists a positive  $C_\delta$  such that

$$|R(\varepsilon/2^{1/2}, d)| = \text{comp}^{\text{wor}}(\varepsilon/2^{1/2}, H_d, \Lambda^{\text{all}}) \leq C_\delta \varepsilon^{-(p^*(\Lambda^{\text{all}})+\delta)} \quad \forall d = 1, 2, \dots, \forall \varepsilon \in (0, 1),$$

with  $p^*(\Lambda^{\text{all}}) = 2 \max(s_\gamma, \alpha^{-1})$ .

We want to approximate

$$f(x) = \sum_{h \in \mathbb{Z}^d} \hat{f}(h) \exp(2\pi i h \cdot x)$$

for  $f \in H_d$ . The main idea of our algorithm is to approximate the Fourier coefficients  $\hat{f}(h)$  for  $h \in R(\varepsilon/2^{1/2}, d)$  by the standard Monte Carlo, whereas the Fourier coefficients  $\hat{f}(h)$  for  $h \notin R(\varepsilon/2^{1/2}, d)$  are approximated simply by zero. That is, the algorithm  $A_{n,d}$  takes the form

$$A_{n,d}(f, \omega)(x) = \sum_{h \in R(\varepsilon/2^{1/2}, d)} \left( \frac{1}{n} \sum_{k=1}^n f(\omega_k) \exp(-2\pi i h \cdot \omega_k) \right) \exp(2\pi i h \cdot x), \quad (9)$$

where, as for the standard Monte Carlo,  $\omega = (\omega_1, \omega_2, \dots, \omega_n)$  with independent and uniformly distributed  $\omega_k$  over  $[0, 1]^d$ .

The last formula can be rewritten as

$$A_{n,d}(f, \omega)(x) = \frac{1}{n} \sum_{k=1}^n f(\omega_k) \left( \sum_{h \in R(\varepsilon/2^{1/2}, d)} \exp(-2\pi i h \cdot (x - \omega_k)) \right). \quad (10)$$

From (10) it is clear that the randomized algorithm  $A_{n,d}$  uses  $n$  random function values.

We are ready to analyze the randomized error of the algorithm  $A_{n,d}$ . First of all observe that

$$\int_{[0,1]^d} |f(x) - A_{n,d}(f, \omega)|^2 dx = \sum_{h \in R(\varepsilon/2^{1/2}, d)} \left| \hat{f}(h) - \frac{1}{n} \sum_{k=1}^n f(\omega_k) e^{-2\pi i h \cdot \omega_k} \right|^2 + \sum_{h \notin R(\varepsilon/2^{1/2}, d)} |\hat{f}(h)|^2.$$

We now compute the average value of the last formula with respect to  $\omega$ . Using the well known formula for the Monte Carlo randomized error we obtain

$$\sum_{h \in R(\varepsilon/2^{1/2}, d)} \frac{\text{INT}_d(|f|^2) - |\hat{f}(h)|^2}{n} + \sum_{h \notin R(\varepsilon/2^{1/2}, d)} |\hat{f}(h)|^2.$$

Since  $\text{INT}_d(|f|^2) = \sum_{h \in \mathbb{Z}^d} |\hat{f}(h)|^2 \leq \|f\|_d^2$ , and

$$\begin{aligned} \sum_{h \notin R(\varepsilon/2^{1/2}, d)} |\hat{f}(h)|^2 &= \sum_{h \notin R(\varepsilon/2^{1/2}, d)} r_\alpha(\gamma, h) |\hat{f}(h)|^2 / r_\alpha(\gamma, h) \\ &\leq \frac{1}{2} \varepsilon^2 \sum_{h \notin R(\varepsilon/2^{1/2}, d)} r_\alpha(\gamma, h) |\hat{f}(h)|^2 \leq \frac{1}{2} \varepsilon^2 \|f\|_d^2, \end{aligned}$$

the error of  $A_{n,d}$  satisfies

$$e^{\text{ran}}(A_{n,d})^2 \leq \frac{|R(\varepsilon/2^{1/2}, d)|}{n} + \frac{\varepsilon^2}{2}.$$

Taking

$$n = \frac{2|R(\varepsilon/2^{1/2}, d)|}{\varepsilon^2} = O\left(\varepsilon^{-(2+p^*(\Lambda^{\text{all}})+\delta)}\right) \quad (11)$$

we conclude that the error of  $A_{n,d}$  is at most  $\varepsilon$ . This is achieved for  $n$  given by (11), which does *not* depend on  $d$ , and which depends polynomially on  $\varepsilon^{-1}$  with an exponent that exceeds the exponent of strong tractability in the class  $\Lambda^{\text{all}}$ , roughly speaking, by at most two. This means that approximation is strongly tractable in the class  $\Lambda^{\text{std}}$  under exactly the same conditions as in the class  $\Lambda^{\text{all}}$ .

We now discuss the total cost of the algorithm  $A_{n,d}$ . This algorithm requires  $n$  function evaluations  $f(\omega_k)$ . Since  $\omega_k$  is a vector with  $d$  components, it seems reasonable to assume that the cost of one such function evaluation depends on  $d$  and is, say,  $\mathbf{c}(d)$ . Obviously,  $\mathbf{c}(d)$  should not be exponential in  $d$  since for large  $d$  we could not even compute one function value. On the other hand,  $\mathbf{c}(d)$  should be at least linear in  $d$  since our functions may depend on all  $d$  variables. Let us also assume that we can perform combinatory operations such as arithmetic operations over complex numbers, comparisons of real numbers, and evaluations of exponential functions. For simplicity assume that the cost of one combinatory operation is taken as unity. Hence, for given  $h$  and  $\omega_k$ , we can compute the inner product  $h \cdot \omega_k$  and then  $\exp(-2\pi i h \cdot \omega_k)$  in cost of order  $d$ .

The implementation of the algorithm  $A_{n,d}$  can be done as follows. We compute and output

$$y_h = \frac{1}{n} \sum_{k=1}^n f(\omega_k) \exp(-2\pi i h \cdot \omega_k)$$

for all  $h \in R(\varepsilon/2^{1/2}, d)$ . This is done in cost of order

$$n \mathbf{c}(d) + n d |R(\varepsilon/2^{1/2}, d)|.$$

Knowing the coefficients  $y_h$  we can compute the algorithm  $A_{n,d}$  at any vector  $x \in [0, 1]^d$  as

$$A_{n,d}(f, \omega)(x) = \sum_{h \in R(\varepsilon/2^{1/2}, d)} y_h \exp(2\pi i h \cdot x)$$

with cost of order  $d |R(\varepsilon/2^{1/2}, d)|$ . Using the estimates on  $|R(\varepsilon/2^{1/2}, d)|$  and  $n$  given by (11), we conclude that the total cost of the algorithm  $A_{n,d}$  is of order

$$\left(\frac{1}{\varepsilon}\right)^{p+2} \mathbf{c}(d) + \left(\frac{1}{\varepsilon}\right)^{2p+2} d$$

with  $p = p^*(\Lambda^{\text{all}}) + \delta$ . Hence, the only dependence on  $d$  is through  $\mathbf{c}(d)$  and  $d$ . We stress the difference in the exponents of the number of function values and the number of combinatory operations used by the algorithm  $A_{n,d}$ . For a fixed  $\varepsilon$  and varying  $d$ , the first term of the cost will dominate the second term when  $\mathbf{c}(d)$  grows more than linearly in  $d$ . In this case the first exponent  $p + 2$  determines the total cost of the algorithm  $A_{n,d}$ . On the other hand, for a fixed  $d$  and  $\varepsilon$  tending to zero, the opposite is true, and the second term dominates the first term of the cost, and the second exponent  $2p + 2$  determines the cost of  $A_{n,d}$ . We summarize this analysis in the following theorem.

**Theorem 2** *Consider approximation  $\text{APP}_d : H_d \rightarrow L_2([0, 1]^d)$  in the randomized setting.*

1. *Let  $\alpha \geq 0$ . Strong tractability and tractability of approximation in the class  $\Lambda^{\text{all}}$  are equivalent, and this holds iff  $s_\gamma < \infty$  and  $\alpha > 0$ . When this holds, the exponent of strong tractability is*

$$p^*(\Lambda^{\text{all}}) = 2 \max(s_\gamma, \alpha^{-1}).$$

2. *Let  $\alpha > 1$ . Strong tractability and tractability of approximation in the class  $\Lambda^{\text{std}}$  are equivalent, and this holds under the same conditions as in the class  $\Lambda^{\text{all}}$ , that is, iff  $s_\gamma < \infty$ . When this holds, the exponent of strong tractability  $p^*(\Lambda^{\text{std}}) \in [p^*(\Lambda^{\text{all}}), p^*(\Lambda^{\text{all}}) + 2]$ .*
3. *The algorithm  $A_{n,d}$  defined by (9) with  $n$  given by (11) of order roughly  $\varepsilon^{-(p^*(\Lambda^{\text{all}})+2)}$  approximates  $\text{APP}_d$  with randomized error at most  $\varepsilon$ . For any positive  $\delta$  there exists a positive number  $K_\delta$  such that the total cost of the algorithm  $A_{n,d}$  is bounded by*

$$K_\delta \left( \left(\frac{1}{\varepsilon}\right)^{p+2} \mathbf{c}(d) + \left(\frac{1}{\varepsilon}\right)^{2p+2} d \right) \quad \forall d = 1, 2, \dots, \quad \forall \varepsilon \in (0, 1),$$

*with  $p = p^*(\Lambda^{\text{all}}) + \delta$ .*

We now comment on the assumption  $\alpha > 1$  that is present for the class  $\Lambda^{\text{std}}$ . As we know from Section 3, this assumption is necessary to guarantee that function values are continuous linear functionals and it was essential when we dealt with the worst case setting. In the randomized setting, the situation is different since we are using random function values, and the randomized error depends only on function values in the average sense. This means that  $f(x)$  does not have to be well defined everywhere, and continuity of the linear functional  $L_x(f) = f(x)$  is irrelevant. Since for any  $\alpha \geq 0$ , the Korobov space  $H_\alpha$  is a subset of  $L_2([0, 1]^d)$ , we can treat  $f$  as a  $L_2$  function. This means that in the randomized setting we can consider the class  $\Lambda^{\text{std}}$  for all  $\alpha \geq 0$ .

**Remark 1** This is true only if we allow the use of random numbers from  $[0, 1]$ . If we only allow the use of random bits (coin tossing as a source of randomness) then again we need function values to be continuous linear functionals, which is guaranteed by the condition  $\alpha > 1$ , see [16] for a formal definition of such “restricted” Monte Carlo algorithms. We add that it is easy to obtain random bits from a quantum computer while it is not possible to obtain random numbers from  $[0, 1]$ .

Observe that the algorithm  $A_{n,d}$  is well defined for any  $\alpha \geq 0$  since the standard Monte Carlo algorithm is well defined for functions from  $L_2([0, 1]^d)$ . Furthermore, the randomized error analysis did not use the fact that  $\alpha > 1$ , and is valid for all  $\alpha > 0$ . For  $\alpha = 0$  the analysis breaks down since  $n$  given by (11) would then be infinite. Even if we treat functions in the  $L_2$  sense tractability requires that  $s_\gamma$  be finite. Indeed, for  $s_\gamma = \infty$  we must approximate exponentially<sup>2</sup> many Fourier coefficients which, obviously, contradicts tractability. We summarize this comment in the following corollary.

**Corollary 1** *Consider approximation  $\text{APP}_d : H_d \rightarrow L_2([0, 1]^d)$  in the randomized setting with  $\alpha \in [0, 1]$  in the class  $\Lambda^{\text{std}}$ .*

1. *Strong tractability and tractability of approximation are equivalent, and this holds iff  $\alpha > 0$  and  $s_\gamma < \infty$ . When this holds, the exponent of strong tractability is in the interval  $[p, p + 2]$ , where  $p = p^*(\Lambda^{\text{all}}) = 2 \max(s_\gamma, \alpha^{-1})$ .*
2. *The algorithm  $A_{n,d}$  defined by (9) with  $n$  given by (11) of order roughly  $\varepsilon^{-(p^*(\Lambda^{\text{all}})+2)}$  approximates  $\text{APP}_d$  with randomized error at most  $\varepsilon$ .*

The essence of these results is that in the randomized setting there is no difference between tractability conditions when we use functionals from  $\Lambda^{\text{all}}$  and when we use random function

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<sup>2</sup>We follow a convention of complexity theory that if the function grows faster than polynomial then we say it is exponential.

values. This is especially important when  $s_\gamma > 1$ , since approximation is then intractable in the worst case setting for the class  $\Lambda^{\text{std}}$  independently of  $\alpha$ . Thus we have the following corollary.

**Corollary 2** *Let  $s_\gamma > 1$ . For the class  $\Lambda^{\text{std}}$ , randomization breaks intractability of approximation in the worst case setting.*

## 5 Quantum Setting

Our analysis in this section is based on the framework introduced in [8] of quantum algorithms for the approximate solution of problems of analysis. We refer the reader to the surveys [4], [21], and to the monographs [7], [14], and [20] for general reading on quantum computation.

This approach is an extension of the framework of information-based complexity theory (see [24] and, more formally, [16]) to quantum computation. It also extends the binary black box model of quantum computation (see [2]) to situations where mappings on spaces of functions have to be computed. Some of the main notions of quantum algorithms can be found in Appendix 1. For more details and background discussion we refer to [8].

### 5.1 Quantum Summation of a Single Sequence

We need results about the summation of finite sequences on a quantum computer. The summation problem is defined as follows. For  $N \in \mathbb{N}$  and  $1 \leq p \leq \infty$ , let  $L_p^N$  denote the space of all functions  $g : \{0, 1, \dots, N-1\} \rightarrow \mathbb{R}$ , equipped with the norm

$$\|g\|_{L_p^N} = \left( \frac{1}{N} \sum_{j=0}^{N-1} |g(j)|^p \right)^{1/p} \quad \text{if } p < \infty, \quad \text{and} \quad \|g\|_{L_\infty^N} = \max_{0 \leq j \leq N-1} |g(j)|.$$

Define  $S_N : L_p^N \rightarrow \mathbb{R}$  by

$$S_N(g) = \frac{1}{N} \sum_{j=0}^{N-1} g(j)$$

and let

$$F = \mathcal{B}_p^N := \{g \in L_p^N \mid \|g\|_{L_p^N} \leq 1\}.$$

Observe that  $S_N(\mathcal{B}_p^N) = [-1, 1]$  for all  $p$  and  $N$ . We wish to compute  $A(g, \varepsilon)$  which approximates  $S_N(g)$  with error  $\varepsilon$  and with probability at least  $\frac{3}{4}$ . That is,  $A(g, \varepsilon)$  is a random variable which is computed by a quantum algorithm such that the inequality  $|S_N(g) - A(g, \varepsilon)| \leq \varepsilon$

holds with probability at least  $\frac{3}{4}$ . The performance of a quantum algorithm can be summarized by the number of quantum queries, quantum operations and qubits. These notions are defined in Appendix 1. Here we only mention that the quantum algorithm obtains information on the function values  $g(j)$  by using only quantum queries. The number of quantum operations is defined as the total number of bit operations performed by the quantum algorithm. The number of qubits is defined as  $m$  if all quantum operations are performed in the Hilbert space of dimension  $2^m$ . It is important to seek algorithms that require as small a number of qubits as possible.

We denote by  $e_n^q(S_N, F)$  the minimal error (in the above sense, of probability  $\geq \frac{3}{4}$ ) that can be achieved by a quantum algorithm using only  $n$  queries. The query complexity is defined for  $\varepsilon > 0$  by

$$\text{comp}^{\text{qq}}(\varepsilon, S_N, F) = \min\{n \mid e_n^q(S_N, F) \leq \varepsilon\}.$$

The total (quantum) complexity  $\text{comp}^{\text{qua}}(\varepsilon, S_N, F)$  is defined as the minimal total cost of a quantum algorithm that solves the summation problem to within  $\varepsilon$ . The total cost of a quantum algorithm is defined by counting the total number of quantum queries plus quantum operations used by the quantum algorithm. Let  $\mathbf{c}$  be the cost of one evaluation of  $g(j)$ . It is reasonable to assume that the cost of one quantum query is taken as  $\mathbf{c} + m$  since  $g(j)$ 's are computed and  $m$  qubits are processed by a quantum query, see Appendix 1 for more details.

The quantum summation is solved by the Grover search and amplitude estimation algorithm which can be found in [6] and [3]. This algorithm enjoys almost minimal error and will be repetitively used for approximation as we shall see in Sections 5.2 and 5.3.

Let us summarize the known results about the order of  $e_n^q(S_N, \mathcal{B}_p^N)$  for  $p = \infty$  and  $p = 2$ . The case  $p = \infty$  is due to [6], [3] (upper bounds) and [13] (lower bounds). The results in the case  $p = 2$  are due to [8]. Further results for arbitrary  $1 \leq p \leq \infty$  can be also found in [8] and [11]. In what follows, by “log” we mean the logarithm to the base 2.

**Theorem 3** *There are constants  $c_j > 0$  for  $j \in \{1, \dots, 9\}$  such that for all  $n, N \in \mathbb{N}$  with  $2 < n \leq c_1 N$  we have*

$$e_n^q(S_N, \mathcal{B}_\infty^N) \asymp n^{-1}$$

and

$$c_2 n^{-1} \leq e_n^q(S_N, \mathcal{B}_2^N) \leq c_3 n^{-1} \log^{3/2} n \cdot \log \log n.$$

For  $\varepsilon \leq \varepsilon_0 < \frac{1}{2}$ , we have

$$\text{comp}^{\text{qq}}(\varepsilon, S_N, \mathcal{B}_\infty^N) \asymp \min(N, \varepsilon^{-1})$$

and

$$c_4 \min(N, \varepsilon^{-1}) \leq \text{comp}^{\text{qq}}(\varepsilon, S_N, \mathcal{B}_2^N) \leq c_5 \min(N, \varepsilon^{-1} \log^{3/2} \varepsilon^{-1} \cdot \log \log \varepsilon^{-1}).$$

For  $N \geq \varepsilon^{-1}$ , the algorithm for the upper bound uses about  $\log N$  qubits and the total complexity is bounded by

$$c_6 \mathbf{c} \varepsilon^{-1} \leq \text{comp}^{\text{qua}}(\varepsilon, S_N, \mathcal{B}_\infty^N) \leq c_7 \mathbf{c} \varepsilon^{-1} \cdot \log N$$

and

$$c_8 \mathbf{c} \varepsilon^{-1} \leq \text{comp}^{\text{qua}}(\varepsilon, S_N, \mathcal{B}_2^N) \leq c_9 \mathbf{c} \varepsilon^{-1} \log^{3/2} \varepsilon^{-1} \cdot \log \log \varepsilon^{-1} \cdot \log N.$$

So far we required that the error is no larger than  $\varepsilon$  with probability at least  $\frac{3}{4}$ . To decrease the probability of failure from  $\frac{1}{4}$  to, say,  $e^{-\ell/8}$  one can repeat the algorithm  $\ell$  times and take the median as the final result. See Lemma 3 of [8] for details.

We also assumed so far that  $\|g\|_{L_p^N} \leq 1$ . If this bound is changed to, say,  $\|g\|_{L_p^N} \leq M$  then it is enough to rescale the problem and replace  $g(j)$  by  $g(j)/M$ . Then we multiply the computed result by  $M$  and obtain the results as in the last theorem with  $\varepsilon$  replaced by  $M\varepsilon$ .

## 5.2 The Idea of the Algorithm for Approximation

The starting point of our quantum algorithm for approximation is a deterministic algorithm on a classical computer that is similar to the randomized algorithm given by (9), namely

$$A_{N,d}(f)(x) = \sum_{h \in R(\varepsilon/3, d)} \left( \frac{1}{N} \sum_{j=1}^N f(x_j) \exp(-2\pi i h \cdot x_j) \right) \exp(2\pi i h \cdot x), \quad (12)$$

where the  $x_1, \dots, x_N$  come from a suitable deterministic rule, and  $R(\cdot, d)$  is defined by (6). The error analysis of  $A_{N,d}$  will be based on three types of errors. The first error arises from replacing the infinite Fourier series by a finite series over the set  $R(\varepsilon/3, d)$ ; this error is  $\varepsilon/3$ . The second error is made since we replace the Fourier coefficients which are integrals by a quadrature formulas. We will choose  $N$  and the deterministic rule for computing  $x_j$  in such a way that the combination of these two errors yields

$$\|A_{N,d}(f) - f\|_{L_2([0,1]^d)} \leq \frac{2}{3} \varepsilon \quad \forall f \in H_d, \quad \|f\|_d \leq 1. \quad (13)$$

This will be possible (see (22) below) if  $N$  is, in general, exponentially large in  $d$ . This may look like a serious drawback, but the point is that we do *not* need to exactly compute the sums in (12). Instead, the sums

$$\left( \frac{1}{N} \sum_{j=1}^N f(x_j) \exp(-2\pi i h \cdot x_j) \right)_{h \in R(\varepsilon/3, d)} \quad (14)$$

will be approximately computed by a quantum algorithm whose cost depends only logarithmically on  $N$ . We have to guarantee that this third (quantum) error is bounded by  $\varepsilon/3$ , with probability at least  $\frac{3}{4}$ . As we shall see,  $\log N$  will be at most linear in  $d$  and polynomial in  $\log \varepsilon^{-1}$ , which will allow us to have good bounds on the total cost of the quantum algorithm.

**Remark 2** Observe that the  $|R(\varepsilon/3, d)|$  sums given by (14) depend only on  $N$  function values of  $f$ , whereas  $h$  takes as many values as the cardinality of the set  $R(\varepsilon/3, d)$ . Since each function value costs  $\mathbf{c}(d)$ , and since  $\mathbf{c}(d)$  is usually much larger than the cost of one combinatory operation, it seems like a good idea to compute all sums in (14) simultaneously. We do not know how to do this efficiently on a quantum computer and therefore compute these sums sequentially.

### 5.3 Quantum Summation Applied to our Sequences

As outlined in the previous subsection, for the approximation problem we need to compute  $S_N(g_h)$  for several sequences  $g_1, g_2, \dots, g_R$  each of length  $N$  with  $R = |R(\varepsilon/3, d)|$ . We assume that  $g_h \in L_p^N$  for  $p = 2$  or  $p = \infty$ , and  $\|g_h\|_p \leq M$ . We now want to compute  $A(g_h, \varepsilon)$  on a quantum computer such that (with  $\varepsilon/3$  now replaced by  $\varepsilon$ )

$$\sum_{h=1}^R |S_N(g_h) - A(g_h, \varepsilon)|^2 \leq \varepsilon^2 \quad (15)$$

with probability at least  $\frac{3}{4}$ . In our case the sequences  $g_h$  are the terms of (14) and we assume that we can compute  $g_h(j) = f(x_j) \exp(-2\pi i h \cdot x_j)$ . The cost  $\mathbf{c}$  of computing one function value  $g_h(j)$  is now equal to  $\mathbf{c}(d) + 2d + 2$ , since we can compute  $g_h(j)$  using one evaluation of  $f$  and  $2d + 2$  combinatory operations needed to compute the inner product  $y = h \cdot x_j$  and  $f(x_j) \exp(-2\pi i y)$ . The cost of one call of the oracle is roughly

$$\log N + \mathbf{c}(d) + 2d + 2, \quad (16)$$

since we need about  $\log N$  qubits and the cost of computing  $g_h$  is  $\mathbf{c}(d) + 2d + 2$ .

This summation problem can be solved by the Grover search or amplitude amplification algorithm mentioned in Section 5.1. To guarantee that the bound (15) holds it is enough to compute an approximation for each component with error  $\delta = \varepsilon R^{-1/2}$ . We will assume that

$$M \delta^{-1} = \frac{M R^{1/2}}{\varepsilon} \leq N. \quad (17)$$

We can satisfy (17) by computing each  $S_N(g_h)$  independently for each  $h$ .



We begin with the case  $p = \infty$ . To compute one sum with error  $\delta$  with probability at least  $1 - \eta$  we need roughly  $\log \eta^{-1}$  repetitions of the algorithm and this requires about  $(M/\delta) \log \eta^{-1}$  queries. We put  $\eta R = \frac{1}{4}$  to obtain an algorithm that computes each sum in such a way that (15) holds. Hence we need roughly  $\frac{M\sqrt{R}}{\varepsilon} \log R$  queries for each  $g_h$ . Together we need roughly

$$R \cdot \frac{M\sqrt{R}}{\varepsilon} \cdot \log R \text{ queries.} \quad (18)$$

The case  $p = 2$  is similar and we need roughly

$$R \cdot \frac{M\sqrt{R}}{\varepsilon} \cdot \log^{3/2} \frac{M\sqrt{R}}{\varepsilon} \cdot \log \log \frac{M\sqrt{R}}{\varepsilon} \cdot \log R \text{ queries.} \quad (19)$$

The total cost is of order

$$(\log N + \mathbf{c}(d) + 2d + 2) R \frac{M\sqrt{R}}{\varepsilon} \log R \quad \text{for } p = \infty, \quad (20)$$

$$(\log N + \mathbf{c}(d) + 2d + 2) R \frac{M\sqrt{R}}{\varepsilon} \log^{3/2} \frac{M\sqrt{R}}{\varepsilon} \cdot \log \log \frac{M\sqrt{R}}{\varepsilon} \cdot \log R \quad \text{for } p = 2. \quad (21)$$

## 5.4 Results on Tractability

We only consider upper bounds for the class  $\Lambda^{\text{std}}$  and weighted Korobov spaces for  $\alpha > 1$  and  $s_\gamma < \infty$ . We combine the idea from Subsection 5.2 together with the upper bounds from Subsection 5.3. We need estimates for the numbers  $N$ ,  $M$ , and  $R$ .

We know from Section 2 that  $R(\varepsilon/3, d)$  is the set of indices  $h$  for which the eigenvalues of  $W_d$  are greater than  $\varepsilon^2/9$ , see (6). We also know from (7) that the cardinality of the set  $R(\varepsilon/3, d)$  is exactly equal to  $\text{comp}^{\text{wor}}(\varepsilon/3, H_d, \Lambda^{\text{all}})$  and that for any positive  $\eta$  there exists a positive  $C_\eta$  such that

$$R = |R(\varepsilon/3, d)| = \text{comp}^{\text{wor}}(\varepsilon/3, H_d, \Lambda^{\text{all}}) \leq C_\eta \varepsilon^{-(p^*(\Lambda^{\text{all}}) + \eta)} \quad \forall d = 1, 2, \dots, \quad \forall \varepsilon \in (0, 1).$$

For  $f \in H_d$  with  $\|f\|_d \leq 1$  we know that

$$|f(y)| = |\langle f, K_d(\cdot, y) \rangle| \leq K_d(y, y)^{1/2} = \prod_{j=1}^d \left(1 + 2\gamma_j \zeta(\alpha)\right)^{1/2},$$

where  $\zeta$  is the Riemann zeta function, and hence

$$|f(y)| \leq \exp \left( \zeta(\alpha) \sum_{j=1}^d \gamma_j \right).$$

This means that when  $\sum_{j=1}^{\infty} \gamma_j < \infty$  we can apply the results from Section 5.3 with  $p = \infty$  and  $M$  independent of  $d$  and of order one.

If  $\sum_{j=1}^{\infty} \gamma_j = \infty$ , which happens when  $s_\gamma > 1$  and could happen if  $s_\gamma = 1$ , we use the quantum results for  $p = 2$  and need estimates not only for  $N$  in (14) but also for  $M$  that bounds the  $L_2^N$ -norms of the terms in (14).

We know from Lemma 2 (ii) in [23] that there are lattice rules  $Q_{N,d}(f) = N^{-1} \sum_{j=1}^N f(x_j)$  with prime  $N$  and  $x_j = \{j z/N\}$  for some non-zero integer  $z \in [-N/2, N/2]^d$  and with  $\{\cdot\}$  denoting the fractional part, for which

$$|\text{INT}_d(f) - Q_{N,d}(f)| \leq \frac{\prod_{j=1}^d (1 + 2\gamma_j)^{1/2}}{\sqrt{N}} \cdot \|f\|_d. \quad (22)$$

As in Section 5.2, we have to guarantee an error  $\delta = \varepsilon R^{-1/2} = O(\varepsilon^{1+(p^*(\Lambda^{\text{all}})+2)/2})$  for all integrands  $x \mapsto f_h(x) = f(x) \exp(-2\pi i h \cdot x)$  with  $h \in R(\varepsilon/3, d)$ . For these integrands  $f_h$  we have

$$\begin{aligned} \|f_h\|_d^2 &= \sum_{j \in \mathbb{Z}^d} |\hat{f}(h+j)|^2 r_\alpha(\gamma, j) = \sum_{j \in \mathbb{Z}^d} |\hat{f}(h+j)|^2 r_\alpha(\gamma, h+j) \frac{r_\alpha(\gamma, j)}{r_\alpha(\gamma, h+j)} \\ &\leq \left( \sum_{j \in \mathbb{Z}^d} |\hat{f}(h+j)|^2 r_\alpha(\gamma, h+j) \right) \max_{j \in \mathbb{Z}^d} \frac{r_\alpha(\gamma, j)}{r_\alpha(\gamma, h+j)} \\ &= \|f\|_d^2 \max_{j \in \mathbb{Z}^d} \frac{r_\alpha(\gamma, j)}{r_\alpha(\gamma, h+j)}. \end{aligned}$$

We now show that

$$\frac{r_\alpha(\gamma, j)}{r_\alpha(\gamma, h+j)} \leq r_\alpha(\gamma, h) \prod_{m=1}^d \max(1, \gamma_m 2^\alpha) \quad \forall j, h \in \mathbb{Z}^d. \quad (23)$$

Indeed, since  $r_\alpha$  is a product, it is enough to check (23) for all components of  $r_\alpha$ . For the  $m$ th component it is easy to check that

$$\frac{r_\alpha(\gamma_m, j_m)}{r_\alpha(\gamma_m, h_m + j_m)} \leq \max(1, \gamma_m 2^\alpha) r_\alpha(\gamma_m, h_m),$$

From which (23) follows.

In our case  $s_\gamma < \infty$  which implies that  $\gamma_m$  tends to zero and therefore  $\prod_{m=1}^{\infty} \max(1, \gamma_m 2^\alpha)$  is finite. Furthermore, for  $h \in R(\varepsilon/3, d)$  we have  $r_\alpha(\gamma, h) \leq 9/\varepsilon^2$ . Hence,  $\|f_h\|_d = O(1/\varepsilon)$  for all  $h \in R(\varepsilon/3, d)$ . We replace  $\gamma_j$  by 1 in (22) and have

$$|\text{INT}_d(f_h) - Q_{N,d}(f_h)| = O\left(\frac{3^{d/2}}{\varepsilon \sqrt{N}}\right) = O(\varepsilon^{1+(p^*(\Lambda^{\text{all}})+\eta)/2})$$

if we take  $N$  at least of order

$$N \asymp 3^d \left( \frac{1}{\varepsilon} \right)^{4+p^*(\Lambda^{\text{all}})+\eta}$$

or

$$\log N \asymp d + \log \varepsilon^{-1}.$$

To bound  $M$  we need to consider the  $L_2^N$ -norms of the terms  $f_h(x_j) = g_h(j)$  in (14). Since the Korobov space  $H_d$  is an algebra, see Appendix 2, we know that  $|f_h|^2 \in H_d$  and

$$\| |f_h|^2 \|_d \leq C(d) \cdot \|f_h\|_d^2 = O(C(d) \varepsilon^{-2}),$$

where  $C(d)$  is given in Appendix 2. Applying the bound (22) to the function  $|f_h|^2$ , we obtain a bound, in the  $L_2^N$ -norm, of the sequence  $z_h = (g_h(j))_{j=1,\dots,N} = (f_h(x_j))_{j=1,\dots,N}$ . This is the number  $M$  that we need in our estimates. We obtain

$$\|z_h\|_{L_2^N}^2 \leq M^2 = \text{INT}_d(|f_h|^2) + O(3^{d/2} C(d) \varepsilon^{-2} N^{-1/2}).$$

Obviously,

$$\text{INT}_d(|f_h|^2) = \widehat{|f_h|^2}(0) = \sum_{j \in \mathbb{Z}^d} |\hat{f}(h+j)|^2 \leq \|f\|_d^2 \leq 1 \quad \forall h \in \mathbb{Z}^d.$$

To guarantee that  $M$  does not depend on  $d$  and is of order 1, we take  $N$  such that

$$\log N \asymp d + \log C(d) + \log \varepsilon^{-1} \asymp d + \log \varepsilon^{-1},$$

since  $\log C(d)$  is of order  $d$  due to Appendix 2.

Putting these estimates together, we obtain estimates for the quantum algorithm. We use about  $d + \log \varepsilon^{-1}$  qubits. The total cost of the algorithm is of order

$$(\mathbf{c}(d) + d) \left( \frac{1}{\varepsilon} \right)^{1+3(p^*(\Lambda^{\text{all}})+\eta)/2}.$$

Hence, the only dependence on  $d$  is through  $\mathbf{c}(d)$  and  $d$ . We summarize this analysis in the following theorem.

**Theorem 4** *Consider approximation  $\text{APP}_d : H_d \rightarrow L_2([0, 1]^d)$  in the quantum setting with  $\alpha > 1$  in the class  $\Lambda^{\text{std}}$ . Assume that  $s_\gamma < \infty$ . Then we have strong tractability. The quantum algorithm solves the problem to within  $\varepsilon$  with probability at least  $\frac{3}{4}$  and uses about*

$d + \log \varepsilon^{-1}$  qubits. For any positive  $\delta$  there exists a positive number  $K_\delta$  such that the total cost of the algorithm is bounded by

$$K_\delta \left( (\mathbf{c}(d) + d) \left( \frac{1}{\varepsilon} \right)^{1+3(p^*(\Lambda^{\text{all}})+\delta)/2} \right) \quad \forall d = 1, 2, \dots, \forall \varepsilon \in (0, 1).$$

It is interesting to compare the results in the quantum setting with the results in the worst case and randomized settings for the class  $\Lambda^{\text{std}}$ . We ignore the small parameter  $\delta$  in Theorems 1, 2, 4 and 6. Then if  $s_\gamma > 1$ , the quantum setting (as well as the randomized setting) breaks intractability of approximation in the worst case setting (again for the class  $\Lambda^{\text{std}}$ ). The number of quantum queries and quantum combinatory operations is of order  $\varepsilon^{-1-3p^*(\Lambda^{\text{all}})/2}$ , which is smaller than the corresponding number of function values in the randomized setting only if  $p^*(\Lambda^{\text{all}}) < 2$ . However, the number of quantum combinatory operations is always significantly smaller than the corresponding number of combinatory operations in the randomized settings.

## 6 Appendix 1: Quantum Algorithms

We present a framework for quantum algorithms, see [8] for more details. Let  $D, K$  be nonempty sets, and let  $\mathcal{F}(D, K)$  denote the set of all functions from  $D$  to  $K$ . Let  $\mathbb{K}$ , the scalar field, be either the field of real numbers  $\mathbb{R}$  or the field of complex numbers  $\mathbb{C}$ , and let  $G$  be a normed space with scalar field  $\mathbb{K}$ . Let  $S : F \rightarrow G$  be a mapping, where  $F \subset \mathcal{F}(D, K)$ . We approximate  $S(f)$  for  $f \in F$  by means of quantum computations. Let  $H_1$  be the two-dimensional complex Hilbert space  $\mathbb{C}^2$ , with its unit vector basis  $\{e_0, e_1\}$ , and let

$$H_m = H_1 \otimes \cdots \otimes H_1$$

be the  $m$ -fold tensor product of  $H_1$ , endowed with the tensor Hilbert space structure. It is convenient to let

$$\mathbb{Z}[0, N) := \{0, \dots, N-1\}$$

for  $N \in \mathbb{N}$  (as usual,  $\mathbb{N} = \{1, 2, \dots\}$  and  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ ). Let  $\mathcal{C}_m = \{|i\rangle : i \in \mathbb{Z}[0, 2^m)\}$  be the canonical basis of  $H_m$ , where  $|i\rangle$  stands for  $e_{j_0} \otimes \cdots \otimes e_{j_{m-1}}$ , and  $i = \sum_{k=0}^{m-1} j_k 2^{m-1-k}$  is the binary expansion of  $i$ . Denote the set of unitary operators on  $H_m$  by  $\mathcal{U}(H_m)$ .

A quantum query on  $F$  is given by a tuple

$$Q = (m, m', m'', Z, \tau, \beta), \tag{24}$$

where  $m, m', m'' \in \mathbb{N}$ ,  $m' + m'' \leq m$ ,  $Z \subseteq \mathbb{Z}[0, 2^{m'})$  is a nonempty subset, and

$$\tau : Z \rightarrow D$$

$$\beta : K \rightarrow \mathbb{Z}[0, 2^{m'')}$$

are arbitrary mappings. Denote  $m(Q) := m$ , the number of qubits of  $Q$ .

Given such a query  $Q$ , we define for each  $f \in F$  the unitary operator  $Q_f$  by setting for  $|i\rangle |x\rangle |y\rangle \in \mathcal{C}_m = \mathcal{C}_{m'} \otimes \mathcal{C}_{m''} \otimes \mathcal{C}_{m-m'-m''}$ :

$$Q_f |i\rangle |x\rangle |y\rangle = \begin{cases} |i\rangle |x \oplus \beta(f(\tau(i)))\rangle |y\rangle & \text{if } i \in Z, \\ |i\rangle |x\rangle |y\rangle & \text{otherwise,} \end{cases} \quad (25)$$

where  $\oplus$  means addition modulo  $2^{m''}$ . Hence the query uses  $m'$  bits to represent the index  $i$  which is used to define the argument  $\tau(i)$  at which the function is evaluated. We assume that the cost of one evaluation of  $f$  is  $\mathbf{c}$ . The value of  $f(\tau(i))$  is then coded by the mapping  $\beta$  using  $m''$  bits. Usually, the mapping  $\beta$  is chosen in a such a way that the  $m''$  most significant bits of  $\beta(f(\tau(i)))$  are stored. The number of bits that are processed is  $m' + m'' \leq m$ , and usually  $m' + m''$  is insignificantly less than  $m$ . That is why we define the cost of one query as  $m + \mathbf{c}$ .

A quantum algorithm on  $F$  with no measurement is a tuple  $A = (Q, (U_j)_{j=0}^n)$ , where  $Q$  is a quantum query on  $F$ ,  $n \in \mathbb{N}_0$  and  $U_j \in \mathcal{U}(H_m)$  ( $j = 0, \dots, n$ ), with  $m = m(Q)$ . Given  $f \in F$ , we let  $A_f \in \mathcal{U}(H_m)$  be defined as

$$A_f = U_n Q_f U_{n-1} \dots U_1 Q_f U_0. \quad (26)$$

We denote by  $n_q(A) := n$  the number of queries and by  $m(A) = m = m(Q)$  the number of qubits of  $A$ . Let  $(A_f(x, y))_{x, y \in \mathcal{C}_m}$  be the matrix of the transformation  $A_f$  in the canonical basis  $\mathcal{C}_m$ ,  $A_f(x, y) = \langle x | A_f | y \rangle$ .

A quantum algorithm on  $F$  with output in  $G$  (or shortly, from  $F$  to  $G$ ) with  $k$  measurements is a tuple

$$A = ((A_\ell)_{\ell=0}^{k-1}, (b_\ell)_{\ell=0}^{k-1}, \varphi),$$

where  $k \in \mathbb{N}$ , and  $A_\ell$  ( $\ell = 0, \dots, k-1$ ) are quantum algorithms on  $F$  with no measurements,

$$b_0 \in \mathbb{Z}[0, 2^{m_0}),$$

for  $1 \leq \ell \leq k-1$ ,  $b_\ell$  is a function

$$b_\ell : \prod_{i=0}^{\ell-1} \mathbb{Z}[0, 2^{m_i}) \rightarrow \mathbb{Z}[0, 2^{m_\ell}),$$

where we denoted  $m_\ell := m(A_\ell)$ , and  $\varphi$  is a function

$$\varphi : \prod_{\ell=0}^{k-1} \mathbb{Z}[0, 2^{m_\ell}] \rightarrow G$$

with values in  $G$ . The output of  $A$  at input  $f \in F$  will be a probability measure  $A(f)$  on  $G$ , defined as follows: First put

$$\begin{aligned} p_{A,f}(x_0, \dots, x_{k-1}) &= |A_{0,f}(x_0, b_0)|^2 |A_{1,f}(x_1, b_1(x_0))|^2 \dots \\ &\dots |A_{k-1,f}(x_{k-1}, b_{k-1}(x_0, \dots, x_{k-2}))|^2. \end{aligned} \quad (27)$$

Then define  $A(f)$  by setting

$$A(f)(C) = \sum_{\varphi(x_0, \dots, x_{k-1}) \in C} p_{A,f}(x_0, \dots, x_{k-1}) \quad \forall C \subseteq G. \quad (28)$$

We let  $n_q(A) := \sum_{\ell=0}^{k-1} n_q(A_\ell)$  denote the number of queries used by  $A$ . For brevity we say  $A$  is a quantum algorithm if  $A$  is a quantum algorithm with  $k$  measurements for  $k \geq 0$ .

Informally, such an algorithm  $A$  starts with a fixed basis state  $b_0$  and function  $f$ , and applies in an alternating way unitary transformations  $U_j$  (not depending on  $f$ ) and the operator  $Q_f$  of a certain query. After a fixed number of steps the resulting state is measured, which gives a (random) basis state, say  $\xi_0$ . This state is memorized and then transformed (e.g., by a classical computation, which is symbolized by  $b_1$ ) into a new basis state  $b_1(\xi_0)$ . This is the starting state to which the next sequence of quantum operations is applied (with possibly another query and number of qubits). The resulting state is again measured, which gives the (random) basis state  $\xi_1$ . This state is memorized,  $b_2(\xi_0, \xi_1)$  is computed (classically), and so on. After  $k$  such cycles, we obtain  $\xi_0, \dots, \xi_{k-1}$ . Then finally an element  $\varphi(\xi_0, \dots, \xi_{k-1})$  of  $G$  is computed (e.g., again on a classical computer) from the results of all measurements. The probability measure  $A(f)$  is its distribution.

The error of  $A$  is defined as follows: Let  $0 \leq \theta < 1$ ,  $f \in F$ , and let  $\zeta$  be any random variable with distribution  $A(f)$ . Then put  $e(S, A, f, \theta) = \inf \{ \varepsilon \mid \mathbf{P}\{\|S(f) - \zeta\| > \varepsilon\} \leq \theta \}$ . Associated with this we introduce

$$e(S, A, F, \theta) = \sup_{f \in F} e(S, A, f, \theta),$$

$$e(S, A, f) = e(S, A, f, \tfrac{1}{4}),$$

and

$$e(S, A, F) = e(S, A, F, \tfrac{1}{4}) = \sup_{f \in F} e(S, A, f).$$

Of course one could easily replace here  $\frac{1}{4}$  by another positive number  $a < \frac{1}{2}$ . The  $n$ th minimal query error is defined for  $n \in \mathbb{N}_0$  as

$$e_n^q(S, F) = \inf\{e(S, A, F) \mid A \text{ is any quantum algorithm with } n_q(A) \leq n\}.$$

This is the minimal error which can be reached using at most  $n$  queries. The quantum query complexity is defined for  $\varepsilon > 0$  by

$$\text{comp}^{\text{qq}}(\varepsilon, S, F) = \min\{n_q(A) \mid A \text{ is any quantum algorithm with } e(S, A, F) \leq \varepsilon\}.$$

The quantities  $e_n^q(S, F)$  and  $\text{comp}^{\text{qq}}(\varepsilon, S, F)$  are inverse to each other in the following sense: For all  $n \in \mathbb{N}_0$  and  $\varepsilon > 0$ ,  $e_n^q(S, F) \leq \varepsilon$  if and only if  $\text{comp}^{\text{qq}}(\varepsilon_1, S, F) \leq n$  for all  $\varepsilon_1 > \varepsilon$ . Thus, determining the query complexity is equivalent to determining the  $n$ th minimal query error. The total (quantum) complexity  $\text{comp}^{\text{qua}}(\varepsilon, S, F)$  is defined similarly. Here we count the number of quantum gates that are used by the algorithm; if function values are needed then we put  $\mathbf{c}$  as the cost of one function evaluation. From a practical point of view, the number of available qubits in the near future will be severely limited. Hence it is a good idea to present algorithms that only use a small number of qubits.

## 7 Appendix 2: Korobov Spaces are Algebras

We show that the Korobov space  $H_d$  is an algebra for  $\alpha > 1$ . More precisely, we prove that if  $f, g \in H_d$  then  $fg \in H_d$  and

$$\|fg\|_d \leq C(d) \|f\|_d \|g\|_d, \quad (29)$$

with

$$C(d) = 2^{d \max(1, \alpha/2)} \prod_{j=1}^d \left(1 + 2\gamma_j \zeta(\alpha)\right)^{1/2}.$$

For  $f(x) = \sum_j \hat{f}(j) \exp(2\pi i j \cdot x)$  and  $g(x) = \sum_k \hat{g}(k) \exp(2\pi i k \cdot x)$ , with  $j$  and  $k$  varying through  $\mathbb{Z}^d$ , we have

$$f(x)g(x) = \sum_j \sum_k \hat{f}(j) \hat{g}(k) \exp(2\pi i (j+k) \cdot x) = \sum_h \left( \sum_j \hat{f}(j) \hat{g}(h-j) \right) \exp(2\pi i h \cdot x).$$

Hence, we need to estimate

$$\|fg\|_d^2 = \sum_h \left| \sum_j \hat{f}(j) \hat{g}(h-j) r_\alpha^{1/2}(\gamma, h) \right|^2.$$

Observe that

$$r_\alpha^{1/2}(\gamma_m, h_m) \leq c \left( r_\alpha^{1/2}(\gamma_m, k_m) + r_\alpha^{1/2}(\gamma_m, h_m - k_m) \right) \quad \forall k_m \in \mathbb{Z},$$

with  $c = 2^{\max(0, (\alpha-2)/2)}$ . This holds for  $h_m = 0$  since  $c \geq 1$  and  $r_\alpha(\gamma_m, k_m) \geq 1$ , and is also true for  $h_m \neq 0$  and  $k_m = 0$ . For other values of  $h_m$  and  $k_m$ , the inequality is equivalent to  $|h_m|^{\alpha/2} \leq c(|k_m|^{\alpha/2} + |h_m - k_m|^{\alpha/2})$  which holds with  $c = 1$  for  $\alpha/2 \leq 1$ , and with  $c = 2^{(\alpha-2)/2}$  for  $\alpha/2 > 1$  by the use of the standard argument. Applying this inequality  $d$  times we get

$$r_\alpha^{1/2}(\gamma, h) \leq c^d \prod_{m=1}^d \left( r_\alpha^{1/2}(\gamma_m, k_m) + r_\alpha^{1/2}(\gamma_m, h_m - k_m) \right) \quad \forall k \in \mathbb{Z}^d.$$

Let  $D = \{1, 2, \dots, d\}$  and let  $u \subset D$ . By  $\bar{u} = D - u$  we denote the complement of  $u$ . Define

$$r_\alpha(\gamma, h_u) = \prod_{m \in u} r_\alpha(\gamma_m, h_m), \quad r_\alpha(\gamma, h_{\bar{u}}) = \prod_{m \in \bar{u}} r_\alpha(\gamma_m, h_m).$$

Then we can rewrite the last inequality as

$$r_\alpha^{1/2}(\gamma, h) \leq c^d \sum_{u \subset D} r_\alpha^{1/2}(\gamma, k_u) r_\alpha^{1/2}(\gamma, h_{\bar{u}} - k_{\bar{u}}) \quad \forall k \in \mathbb{Z}^d.$$

For  $u \subset D$ , we define

$$\begin{aligned} F_u(x) &= \sum_j |\hat{f}(j)| r_\alpha^{1/2}(\gamma, j_u) \exp(2\pi i j \cdot x), \\ G_{\bar{u}}(x) &= \sum_j |\hat{g}(j)| r_\alpha^{1/2}(\gamma, j_{\bar{u}}) \exp(2\pi i j \cdot x). \end{aligned}$$

Observe that  $F_u$  and  $G_{\bar{u}}$  are well defined functions in  $L_2([0, 1]^d)$  since  $r_\alpha(\gamma, j_u) \leq r_\alpha(\gamma, j)$  for all  $u$  and since  $f$  and  $g$  are from  $H_d$ . In terms of these functions we see that

$$\begin{aligned} \left| \sum_j \hat{f}(j) \hat{g}(h - j) r_\alpha^{1/2}(\gamma, h) \right| &\leq \sum_j |\hat{f}(j)| |\hat{g}(h - j)| r_\alpha^{1/2}(\gamma, h) \\ &\leq c^d \sum_{u \subset D} \sum_j |\hat{f}(j)| r_\alpha^{1/2}(\gamma, j_u) |\hat{g}(h - j)| r_\alpha^{1/2}(\gamma, h_{\bar{u}} - j_{\bar{u}}) \\ &= c^d \sum_{u \subset D} \sum_j \hat{F}_u(j) \hat{G}_{\bar{u}}(h - j). \end{aligned}$$



Therefore

$$\|f g\|_d^2 \leq c^{2d} \sum_h \left( \sum_{u \subset D} \sum_j \hat{F}_u(j) \hat{G}_{\bar{u}}(h-j) \right)^2.$$

Since the sum with respect to  $u$  has  $2^d$  terms, we estimate the square of the sum of these  $2^d$  terms by the sum of the squared terms multiplied by  $2^d$ , and obtain

$$\|f g\|_d^2 \leq 2^d c^{2d} \sum_{u \subset D} a_u,$$

where

$$a_u = \sum_h \left( \sum_j \hat{F}_u(j) \hat{G}_{\bar{u}}(h-j) \right)^2.$$

We now estimate  $a_u$ . Each  $h$  and  $j$  may be written as  $h = (h_u, h_{\bar{u}})$  and  $j = (j_u, j_{\bar{u}})$ , and therefore

$$\begin{aligned} a_u &= \sum_{h_u} \sum_{h_{\bar{u}}} \left( \sum_{j_u} \sum_{j_{\bar{u}}} \hat{F}_u(j_u, j_{\bar{u}}) \hat{G}_{\bar{u}}(h_u - j_u, h_{\bar{u}} - j_{\bar{u}}) \right)^2 \\ &= \sum_{h_u} \sum_{h_{\bar{u}}} \left( \sum_{j_u} \sum_{j_{\bar{u}}} \hat{F}_u(h_u - j_u, j_{\bar{u}}) \hat{G}_{\bar{u}}(j_u, h_{\bar{u}} - j_{\bar{u}}) \right)^2 \\ &= \sum_{h_u} \sum_{h_{\bar{u}}} \sum_{j_u} \sum_{j_{\bar{u}}} \sum_{k_u} \sum_{k_{\bar{u}}} \hat{F}_u(h_u - j_u, j_{\bar{u}}) \hat{F}_u(h_u - k_u, k_{\bar{u}}) \hat{G}_{\bar{u}}(j_u, h_{\bar{u}} - j_{\bar{u}}) \hat{G}_{\bar{u}}(k_u, h_{\bar{u}} - k_{\bar{u}}). \end{aligned}$$

Note that

$$\sum_{h_{\bar{u}}} \hat{G}_{\bar{u}}(j_u, h_{\bar{u}} - j_{\bar{u}}) \hat{G}_{\bar{u}}(k_u, h_{\bar{u}} - k_{\bar{u}}) \leq G(j_u) G(k_u),$$

where

$$G(j_u) = \left( \sum_{h_{\bar{u}}} \hat{G}_{\bar{u}}(j_u, h_{\bar{u}})^2 \right)^{1/2}.$$

Similarly,

$$\sum_{h_u} \hat{F}_u(h_u - j_u, j_{\bar{u}}) \hat{F}_u(h_u - k_u, k_{\bar{u}}) \leq F(j_{\bar{u}}) F(k_{\bar{u}}),$$

where

$$F(j_{\bar{u}}) = \left( \sum_{h_u} \hat{F}_u(h_u, j_{\bar{u}})^2 \right)^{1/2}.$$

We obtain

$$a_u \leq \sum_{j_u} \sum_{j_{\bar{u}}} \sum_{k_u} \sum_{k_{\bar{u}}} F(j_{\bar{u}}) F(k_{\bar{u}}) G(j_u) G(k_u) = \left( \sum_{j_{\bar{u}}} F(j_{\bar{u}}) \right)^2 \left( \sum_{k_u} G(k_u) \right)^2.$$

Observe that

$$\begin{aligned} \left( \sum_{j_{\bar{u}}} F(j_{\bar{u}}) \right)^2 &= \left( \sum_{j_{\bar{u}}} \left( \sum_{j_u} \hat{F}_u(j_u, j_{\bar{u}})^2 \right)^{1/2} r_{\alpha}^{1/2}(\gamma, j_{\bar{u}}) r_{\alpha}^{-1/2}(\gamma, j_{\bar{u}}) \right)^2 \\ &\leq \sum_{j_{\bar{u}}} \left( \sum_{j_u} \hat{F}_u(j_u, j_{\bar{u}})^2 r_{\alpha}(\gamma, j_u) \right) \left( \sum_{j_{\bar{u}}} r_{\alpha}^{-1}(\gamma, j_{\bar{u}}) \right) \\ &= \left( \sum_{j_{\bar{u}}} \sum_{j_u} |\hat{f}(j_u, j_{\bar{u}})|^2 r_{\alpha}(\gamma, j_u) \right) \left( \sum_{j_{\bar{u}}} r_{\alpha}^{-1}(\gamma, j_{\bar{u}}) \right) \\ &= \left( \sum_j |\hat{f}(j)|^2 r_{\alpha}(\gamma, j) \right) \left( \sum_{j_{\bar{u}}} r_{\alpha}^{-1}(\gamma, j_{\bar{u}}) \right) \\ &= \|f\|_d^2 \sum_{j_{\bar{u}}} r_{\alpha}^{-1}(\gamma, j_{\bar{u}}). \end{aligned}$$

For the last sum we have

$$\sum_{j_{\bar{u}}} r_{\alpha}^{-1}(\gamma, j_{\bar{u}}) = \prod_{m \in \bar{u}} \left( 1 + \gamma_m \sum_{j \neq 0} |j|^{-\alpha} \right) = \prod_{m \in \bar{u}} (1 + 2\gamma_m \zeta(\alpha)).$$

Similarly,

$$\left( \sum_{k_u} G(k_u) \right)^2 \leq \|g\|_d^2 \sum_{k_u} r_{\alpha}^{-1}(\gamma, k_u) = \|g\|_d^2 \prod_{m \in u} (1 + 2\gamma_m \zeta(\alpha)).$$

Putting all these estimates together we conclude that

$$\begin{aligned} \|f g\|_d^2 &\leq 2^d c^{2d} \sum_{u \subset D} \|f\|_d^2 \|g\|_d^2 \prod_{m \in \bar{u}} (1 + 2\gamma_m \zeta(\alpha)) \prod_{m \in u} (1 + 2\gamma_m \zeta(\alpha)) \\ &= 2^d c^{2d} \sum_{u \subset D} \|f\|_d^2 \|g\|_d^2 \prod_{m=1}^d \left( 1 + 2\gamma_m \zeta(\alpha) \right) \\ &= 4^d c^{2d} \prod_{m=1}^d \left( 1 + 2\gamma_m \zeta(\alpha) \right) \|f\|_d^2 \|g\|_d^2, \end{aligned}$$

From which (29) easily follows.

For the quantum setting, we need to consider the function  $w(x) = f(x)\overline{f(x)} = |f(x)|^2$  for  $f \in H_d$ . Note that  $\overline{f}$  also belongs to  $H_d$  and  $\|\overline{f}\|_d = \|f\|_d$ , since  $\widehat{\overline{f}}(h) = \widehat{f}(-h)$  and  $r_\alpha(\gamma, h) = r_\alpha(\gamma, -h)$  for all  $h \in \mathbb{Z}^d$ . Then (29) guarantees that  $w \in H_d$  and

$$\| |f|^2 \|_d \leq C(d) \|f\|_d^2 \quad \forall f \in H_d. \quad (30)$$

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