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On a full discretisation for nonlinear second-order evolution equations with monotone damping: construction, convergence, and error estimates

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Abstract Convergence of a full discretisation method is studied for a class of nonlinear second-order in time evolution equations, where the nonlinear operator acting on the first-order time derivative of the solution is supposed to be hemicontinuous, monotone, coercive and to satisfy a certain growth condition, and the operator acting on the solution is assumed to be linear, bounded, symmetric, and strongly positive. The numerical approximation combines a Galerkin spatial discretisation with a novel time discretisation obtained from a reformulation of the second-order evolution equation as a first-order system and an application of the two-step backward differentiation formula with constant time stepsizes. Convergence towards the weak solution is shown for suitably chosen piecewise polynomial in time prolongations of the resulting fully discrete solutions, and an a priori error estimate ensures convergence of second-order in time provided that the exact solution to the problem fulfills certain regularity requirements. A numerical example for a model problem describing the displacement of a vibrating membrane in a viscous medium illustrates the favourable error behaviour of the proposed full discretisation method in situations where regular solutions exist.

Keywords Nonlinear evolution equation of second-order in time · Monotone operator · Weak solution · Galerkin method · Time discretisation · Backward differentiation formula · Stability · Convergence · Error estimate

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Contents

1 Introduction

Nonlinear second-order evolution equations. In this work, we study the initial value problem for a nonlinear evolution equation of second-order in time

$$\begin{cases} u'' + Au' + Bu = f \quad \text{in } (0,T)\,, \\ u(0) = u_0\,, \quad u'(0) = v_0\,. \end{cases} \tag{1.1}$$

Here, we assume the nonlinear operator $A : V_A \to V_A^*$ acting on the first-order time derivative of the solution to be hemicontinuous, monotone, coercive with exponent $p \geq 2$ and to satisfy a suitable growth condition with exponent $p-1$, and we suppose $(V_A, \|\cdot\|_{V_A})$ to form a real, reflexive, and separable Banach space, which is dense as well as continuously embedded in a Hilbert space $(H, (\cdot|\cdot)_H, \|\cdot\|_H)$. Furthermore, we require the operator $B : V_B \to V_B^*$ acting on the solution to be linear, bounded, symmetric, and strongly positive, which implies that $(V_B, \|\cdot\|_{V_B})$ forms a Hilbert space. We assume V_B to be separable and dense as well as continuously embedded in H. In addition, we suppose the intersection $V = V_A \cap V_B$ to be separable and dense in each of the spaces V_A and V_B, which yields the following scales of Banach spaces with dense and continuous embeddings

$$\begin{aligned} V_A \cap V_B = V \subset V_A \subset H = H^* \subset V_A^* \subset V^* = V_A^* + V_B^*\,, \\ V_A \cap V_B = V \subset V_B \subset H = H^* \subset V_B^* \subset V^* = V_A^* + V_B^*\,. \end{aligned}$$

We assume the function defining the right-hand side of the evolution equation and the initial values to satisfy the requirements $f \in (L^p(0,T;V_A))^*$ and $(u_0, v_0) \in V_B \times H$.

Scope of applications. Our assumptions on (1.1) comply with the functional analytic framework employed for various nonlinear partial differential equations of second-order in time, arising in different fields of application such as mechanics, quantum mechanics, molecular dynamics, and elastodynamics. A typical example describes the displacement $u : \overline{\Omega} \times [0,T] \to \mathbb{R}$ of a vibrating membrane in a viscous medium

$$\begin{cases} \partial_{tt} u + |\partial_t u|^{p-2}\, \partial_t u - \Delta u = f \quad \text{in } \Omega \times (0,T)\,, \\ u = 0 \quad \text{on } \partial\Omega \times (0,T)\,, \\ u(\cdot,0) = u_0\,, \quad \partial_t u(\cdot,0) = v_0 \quad \text{in } \Omega\,, \end{cases} \tag{1.2}$$

see for instance ANDREASSI, TORELLI [1] and LIONS [24].

Analytical results. Results on the existence, uniqueness, and regularity of weak solutions to second-order evolution equations of the form (1.1) have been established in the literature under different assumptions on the defining operators A and B. The rather restrictive case that the spaces V_A and V_B coincide and form a Hilbert space has been studied in GAJEWSKI, GRÖGER, ZACHARIAS [22, Kap. 7], ZEIDLER [31, Ch. 33], and ROUBÍČEK [28, pp. 296ff., 342ff.]; in applications this leads to evolution equations (1.1) involving a linearly bounded operator A. Results including the case $V_A \neq V_B$ have been given in the seminal work of LIONS, STRAUSS [26], see also BARBU [3, Ch. V]. A special class of problems as well as particular examples are treated in FRIEDMAN, NEČAS [21] as well as ANDREASSI, TORELLI [1] and LIONS [24].

Convergence of discretisations and a priori error estimates. Concerning a theoretical analysis of space and time discretisation methods for (1.1), relatively few results are available in the literature. A convergence result provided in COLLI, FAVINI [9] for a semi-discretisation in time applies to the considerably less involved case, where the domains V_A and V_B coincide, the nonlinear operator A is maximal monotone, and the linear operator B is bounded, symmetric, as well as strongly positive, up to an additive shift. More recently, convergence of a time discretisation and hence also existence of a weak solution to (1.1) has been proven in EMMRICH, THALHAMMER [19] under the requirement that V_A is dense and continuously embedded in V_B, and in EMMRICH, THALHAMMER [20] the convergence analysis has been extended to a full discretisation method, which allows to include problems where $V_A \neq V_B$ as well as second-order evolution equations involving non-monotone perturbations and thereby generalises the existence result given in LIONS, STRAUSS [26]. These results have been complemented and illustrated in EMMRICH, ŠIŠKA [16]. In the above mentioned contributions, convergence of approximate solutions towards a weak solution is proven under hypotheses close to the existence theory of nonlinear second-order evolution equations. A main motivation of the present work has been to identify the essential features of time discretisations for (1.1) needed to establish stability and convergence and thus to describe a class of appropriate numerical schemes. In situations, where the exact solution admits additional higher-order regularity, it is of interest to construct and analyse numerical methods of higher-order that are often superior to lower-order discretisations in regard to accuracy and efficiency. In the generic case it is, however, not known which assumptions on the data of the problem guarantee additional regularity of the exact solution. It is, therefore, important to prove convergence of the higher-order numerical scheme towards the weak solution as well as to provide error estimates.

Time discretisation methods. A major difficulty in the convergence analysis of time discretisation methods for the considered class of problems lies in the treatment of the nonlinear damping term. In the context of nonlinear first-order in time evolution equations it is well known that certain stability properties of the numerical approximation are essential in order to be able to utilise the monotone and coercive structure of the underlying operator and to establish a convergence result. As a small excerpt of contributions for nonlinear first-order evolution equations governed by a monotone operator, we mention EMMRICH [14, 15] providing a convergence analysis of the two-step backward differentiation formula and EMMRICH, THALHAMMER [18] introducing a stability criterium which enables to treat a subclass of stiffly accurate Runge–Kutta methods, see also the references given therein. Contrary, the construction of numerical methods for second-order evolution equations, especially for problems where no first-order damping term is present, primarily aims at the preservation of geometric structures such as symplecticity. Favourable time integration methods include the Störmer–Verlet scheme or, more generally, partitioned Runge–Kutta methods as well as the related Nyström methods; further examples are the Newmark and Cowell–Numerov schemes. However, it does not seem possible to establish a convergence result for these classes of time discretisations when applied to the second-order evolution equation (1.1) involving a monotone damping term. Within the analytical framework introduced before we thus restrict ourselves to one-leg methods, since for this type of time discretisations the monotonicity and coercivity of the underlying operator can be utilised to deduce suitable a priori estimates for the discrete solution and to prove convergence in a weak sense. More precisely, emerging from the two-step backward differentiation formula whose favourable stability and accuracy behaviour for nonlinear first-order evolution equations governed by a monotone operator has been confirmed for instance in EMMRICH [14, 15], we construct a time discretisation method that is appropriate for nonlinear second-order evolution equations (1.1). In particular, in situations where sufficiently regular solutions exist, this novel time discretisation method retains the full convergence order.

Full discretisation method. We propose the following full discretisation method for nonlinear second-order evolution equations of the form (1.1) based on the Galerkin method for the space discretisation and the two-step backward differentiation formula for the time discretisation. To the best knowledge of the authors, this numerical approximation has not yet been studied in the literature. Results for the Galerkin method can easily be applied to conforming finite element methods. For notational simplicity, as long as the discretisation parameters $(M,N) \in \mathbb{N} \times \mathbb{N}$ are fixed, we do not indicate the dependence of the numerical approximations on (M,N).

In the sequel, we denote by $(V_M)_{M\in\mathbb{N}}$ a Galerkin scheme for the underlying separable Banach space V, and for any $N \in \mathbb{N}$ we denote by $\tau = \frac{T}{N} > 0$ the associated stepsize defining an equidistant time grid $0 = t_0 < \cdots < t_n = n\tau < \cdots < t_N = T$. For a given pair of integers $(M,N) \in \mathbb{N} \times \mathbb{N}$, we choose a suitable approximation $(f^n)_{n=1}^N \subset V_M$ of the right-hand side of the evolution equation as well as certain initial approximations $(u^0, v^0) \in V_M \times V_M$. The fully discrete solution $(u^n)_{n=1}^N \subset V_M$ is determined such that the relation ($n = 4, 5, \dots, N$)

$$\big(D_2 D_2 u^n \,\big|\, \varphi\big)_H + \big\langle A D_2 u^n \,\big|\, \varphi\big\rangle_{V_A^* \times V_A} + \big\langle B u^n \,\big|\, \varphi\big\rangle_{V_B^* \times V_B} = \big\langle f^n \,\big|\, \varphi\big\rangle_{V_A^* \times V_A} \tag{1.3a}$$

holds for all $\varphi \in V_M$. Here, the divided difference operator D_2 corresponds to the two-step backward differentiation formula ($n = 2, 3, \dots, N$)

$$D_2 u^n = \tau^{-1} \left(\tfrac{3}{2} u^n - 2 u^{n-1} + \tfrac{1}{2} u^{n-2} \right), \tag{1.3b}$$

which further implies ($n = 4, 5, \dots, N$)

$$D_2 D_2 u^n = \tau^{-2} \left(\tfrac{9}{4} u^n - 6 u^{n-1} + \tfrac{11}{2} u^{n-2} - 2 u^{n-3} + \tfrac{1}{4} u^{n-4} \right). \tag{1.3c}$$

The first iterates are determined through the relation $u^1 = u^0 + \tau v^1$ and an application of the one-step backward differentiation formula, which coincides with the implicit Euler method.

Objective. Our main objective is to prove weak convergence of suitably chosen piecewise constant and piecewise linear in time prolongations of the fully discrete solution (1.3) towards the weak solution to the nonlinear second-order evolution equation (1.1) whenever the discretisation parameters tend to infinity. We also provide a result on strong convergence. Moreover, under additional regularity requirements on the exact solution, we deduce an a priori error estimate that implies convergence of second-order in time.

Extensions and future work. In order to avoid additional technicalities that would overburden the present manuscript, we restrict ourselves to the consideration of a time-independent operator $A : V_A \to V_A^*$ that is coercive with exponent $p \in [2, \infty)$, but we conjecture that it is possible to extend our convergence analysis to the case of exponents in the range $p \in (1, 2)$ and operator families $(A(t))_{t \in [0,T]}$ as well as $(B(t))_{t \in [0,T]}$. Although the a priori estimates obtained for the fully discrete solution would allow to study functions $f \in (L^p(0,T; V_A))^* + L^1(0,T; H^*)$, we do not exploit this generalisation, see LIONS, STRAUSS [26]. Contrary, the incorporation of non-monotone perturbations occuring in the description of lower order spatial derivatives seems to be a demanding objective and shall be carried out in a future work, see also EMMRICH, THALHAMMER [20, Section 3].

2 Analytical framework

In this section, we describe in detail the employed functional analytical framework ensuring existence and uniqueness of a weak solution to the nonlinear second-order evolution equation under consideration and recapitulate standard definitions and notations. Furthermore, we introduce formally equivalent reformulations of the problem which are utilised in the construction and convergence analysis of the proposed full discretisation method. Henceforth, we denote by $c > 0$ a generic positive constant, possibly with different values at different occurrences.

2.1 Underlying spaces and operators

Underlying Gelfand triple. We recall that the underlying spaces $(V_A, \|\cdot\|_{V_A})$ and $(V_B, \|\cdot\|_{V_B})$ are assumed to be real, reflexive, and separable Banach spaces, which are dense and continuously embedded in a Hilbert space $(H, (\cdot|\cdot)_H, \|\cdot\|_H)$. Throughout this work, we identify H with its dual space. The intersection $V = V_A \cap V_B$ is assumed to be separable and dense in both spaces V_A and V_B. We endow the space V and its dual V^*, which can be identified with $V_A^* + V_B^*$, with the norms

$$\|v\|_V = \|v\|_{V_A} + \|v\|_{V_B}\,,$$
$$\|f\|_{V^*} = \inf\Big\{\max\Big(\|f_A\|_{V_A^*}, \|f_B\|_{V_B^*}\Big) : f = f_A + f_B,\, f_A \in V_A^*,\, f_B \in V_B^*\Big\}\,,$$

which implies that V is continuously embedded in V_A as well as V_B and thus in H. In particular, the spaces $V \subset H \subset V^*$ form a Gelfand triple.

Underlying operators. In the sequel, we employ the following assumptions on the operators A and B defining the second-order evolution equation in (1.1).

Assumption (A) The operator $A : V_A \to V_A^*$ satisfies the following hypotheses.

(i) *Hemicontinuity.* For arbitrary $v, \widetilde{v}, w \in V_A$, the mapping

$$[0,1] \longrightarrow \mathbb{R} : \sigma \longmapsto \big\langle A(v + \sigma\widetilde{v})\big|w\big\rangle_{V_A^*\times V_A}$$

is continuous.

(ii) *Monotonicity.* For any $v, \widetilde{v} \in V_A$, the relation

$$\big\langle Av - A\widetilde{v}\big|v - \widetilde{v}\big\rangle_{V_A^*\times V_A} \geq 0$$

holds.

(iii) *Coercivity.* There exist $p \in [2,\infty)$, $\mu_A > 0$, and $\lambda_A \geq 0$ such that for all $v \in V_A$

$$\langle Av|v\rangle_{V_A^*\times V_A} \geq \mu_A\,\|v\|_{V_A}^p - \lambda_A\,.$$

(iv) *Growth condition.* There exists $c > 0$ such that for all $v \in V_A$

$$\|Av\|_{V_A^*} \leq c\,\Big(1 + \|v\|_{V_A}^{p-1}\Big)\,.$$

Assumption (B) The operator $B : V_B \to V_B^*$ is linear, bounded, symmetric, and strongly positive. In particular, there exist $\mu_B > 0$ and $c_B > 0$ such that for all $u \in V_B$

$$\langle Bu|u\rangle_{V_B^*\times V_B} \geq \mu_B\,\|u\|_{V_B}^2\,, \qquad \|Bu\|_{V_B^*} \leq c_B\,\|u\|_{V_B}\,.$$

Equivalent norms. Occasionally, it is convenient to make use of the fact that the above hypotheses on the linear operator $B : V_B \to V_B^*$ imply that V_B forms a Hilbert space and that the mapping

$$V_B \longrightarrow [0,\infty) : u \longmapsto \|u\|_B = \sqrt{\langle Bu|u\rangle_{V_B^*\times V_B}} \tag{2.1}$$

defines a norm on V_B, which is equivalent to the norm $\|\cdot\|_{V_B}$.

2.2 Extensions of the underlying operators

Spaces of abstract functions. Let $(W, \|\cdot\|_W)$ be a real, reflexive, and separable Banach space. As usual, for exponents $r \in [1,\infty]$, we denote by $L^r(0,T;W)$ the Bochner–Lebesgue space, equipped with the standard norm $\|\cdot\|_{L^r(0,T;W)}$, see DIESTEL,UHL [11] and EDWARDS [12] for further details. We recall that the space $W^{q,r}(0,T;W)$ ($q \in \mathbb{N}$) comprises all abstract functions $w \in L^r(0,T;W)$ whose distributional time derivatives satisfy $w', \dots, w^{(q)} \in L^r(0,T;W)$, see also GAJEWSKI, GRÖGER, ZACHARIAS [22] and ROUBÍČEK [28]. The space of abstract functions $w : [0,T] \to W$ that are continuous with respect to the weak topology in W is denoted by $\mathscr{C}_w([0,T];W)$. We note that any element $w \in \mathscr{C}_w([0,T];W)$ is a demicontinuous function mapping $[0,T]$ into W. Moreover, we denote by $\mathscr{C}([0,T];W)$ and $\mathscr{AC}([0,T];W)$ the spaces of uniformly continuous and absolutely continuous abstract functions, respectively. In particular, we utilise the embedding $W^{1,1}(0,T;W) \subset \mathscr{AC}([0,T];W)$ and the continuous embedding $W^{1,1}(0,T;W) \hookrightarrow \mathscr{C}([0,T];W)$.

Duality pairings. For $p \in [2,\infty)$, we denote by $p^* = \frac{p}{p-1} \in (1,2]$ the conjugated exponent. The duality pairing between the Bochner–Lebesgue space $L^p(0,T;V)$ and its dual $(L^p(0,T;V))^* = L^{p^*}(0,T;V^*) = L^{p^*}(0,T;V_A^*) + L^{p^*}(0,T;V_B^*)$ is given by

$$\begin{aligned}\langle f | v \rangle_{L^{p^*}(0,T;V^*)\times L^p(0,T;V)} &= \int_0^T \big\langle f(t) \big| v(t) \big\rangle_{V^* \times V} \,\mathrm{d}t \\ &= \int_0^T \big\langle f_A(t) \big| v(t) \big\rangle_{V_A^* \times V_A} \,\mathrm{d}t + \int_0^T \big\langle f_B(t) \big| v(t) \big\rangle_{V_B^* \times V_B} \,\mathrm{d}t\,,\end{aligned}$$

independent of the particular decomposition $f = f_A + f_B \in L^{p^*}(0,T;V^*)$. Moreover, due to the relation $\big(L^1(0,T;H)\big)^* = L^\infty(0,T;H)$, we obtain the duality pairing

$$\langle f | v \rangle_{L^\infty(0,T;H)\times L^1(0,T;H)} = \int_0^T \big(f(t) \big| v(t)\big)_H \,\mathrm{d}t\,.$$

Extension of the underlying operators. The operators $A : V_A \to V_A^*$ and $B : V_B \to V_B^*$ naturally extend to operators governing the nonlinear second-order evolution equation (1.1). These extensions are defined on the associated Bochner–Lebesgue spaces

$$\begin{aligned} A : L^p(0,T;V_A) &\longrightarrow L^{p^*}(0,T;V_A^*) : v \longmapsto [t \mapsto (Av)(t) = Av(t)]\,, \\ B : L^r(0,T;V_B) &\longrightarrow L^r(0,T;V_B^*) : u \longmapsto [t \mapsto (Bu)(t) = Bu(t)]\,, \quad r \in [1,\infty]\,, \end{aligned} \tag{2.2}$$

and inherit the basic hypotheses given in Assumptions (A) and (B). For simplicity, we do not distinguish in notation between the defining operators and their extensions.

(i) *Well-definedness.* In order to justify that the extended operator A is well-defined, we first show that the function $Av : [0,T] \to V_A^*$ associated with a Bochner-measurable function $v : [0,T] \to V_A$ is Bochner-measurable. By assumption the defining operator $A : V_A \to V_A^*$ is monotone as well as hemicontinuous and thus demicontinuous, see for instance GAJEWSKI, GRÖGER, ZACHARIAS [22]. Due to the fact that there exists a sequence $(v_k)_{k\in\mathbb{N}}$ of simple functions $v_k : [0,T] \to V_A$ such that $v_k(t) \to v(t)$ in V_A for almost every $t \in [0,T]$, the associated sequence $(Av_k)_{k\in\mathbb{N}}$ of simple functions satisfies $(Av_k)(t) \rightharpoonup (Av)(t)$ in V_A^* for almost every $t \in [0,T]$, which implies that the function $Av : [0,T] \to V_A^*$ is weakly measurable. According to (2.2) we here set $Av_k : [0,T] \to V_A^* : t \mapsto (Av_k)(t) = Av_k(t)$ as well as $Av : [0,T] \to V_A^* : t \mapsto (Av)(t) = Av(t)$ and utilise that for a reflexive Banach space V_A the notions of weak and weak* convergence coincide. As the dual space V_A^* is separable, by Pettis' theorem, $Av : [0,T] \to V_A^*$ is also Bochner-measurable, see for instance DIESTEL,UHL [11]. It remains to prove that the function $Av : [0,T] \to V_A^*$ associated with a function $v \in L^p(0,T;V_A)$ satisfies $Av \in L^{p^*}(0,T;V_A^*)$. For any $v \in L^p(0,T;V_A)$, the required growth condition on the defining operator yields the bound

$$\left\|(Av)(t)\right\|_{V_A^*}^{p^*} \leq c\,\left(1+\left\|v(t)\right\|_{V_A}^{p-1}\right)^{p^*} \leq c\,\left(1+\left\|v(t)\right\|_{V_A}^{p}\right),$$

which implies that the mapping $[0,T] \to \mathbb{R} : t \to \|(Av)(t)\|_{V_A^*}$ is Lebesgue-integrable, more precisely, we obtain $\|Av\|_{V_A^*} \in L^{p^*}(0,T;\mathbb{R})$. Together with the above considerations this yields $Av \in L^{p^*}(0,T;V_A^*)$.

(ii) *Hemicontinuity.* For arbitrary $v, \widetilde{v}, w \in L^p(0,T;V_A)$, the mapping

$$[0,1] \longrightarrow \mathbb{R} : \sigma \longmapsto \left\langle A(v+\sigma\widetilde{v})\,\middle|\,w\right\rangle_{L^{p^*}(0,T;V_A^*)\times L^p(0,T;V_A)}$$

is continuous. Namely, by the definition of the duality pairing, the required hemicontinuity of the underlying operator $A : V_A \to V_A^*$, and Lebesgue's theorem on dominated convergence, the continuity of the mapping

$$\sigma \mapsto \left\langle A(v+\sigma\widetilde{v})\,\middle|\,w\right\rangle_{L^{p^*}(0,T;V_A^*)\times L^p(0,T;V_A)} = \int_0^T \left\langle A(v(t)+\sigma\widetilde{v}(t))\,\middle|\,w(t)\right\rangle_{V_A^*\times V_A}\,\mathrm{d}t$$

follows, where the applicability of Lebesgue's theorem is ensured by the growth condition on $A : V_A \to V_A^*$.

(iii) *Monotonicity.* For any $v, \widetilde{v} \in L^p(0,T;V_A)$, the relation

$$\left\langle Av - A\widetilde{v}\,\middle|\,v-\widetilde{v}\right\rangle_{L^{p^*}(0,T;V_A^*)\times L^p(0,T;V_A)} \geq 0$$

holds. Evidently, the monotonicity of $A : V_A \to V_A^*$ implies the relation

$$\begin{aligned}&\left\langle Av - A\widetilde{v}\,\middle|\,v-\widetilde{v}\right\rangle_{L^{p^*}(0,T;V_A^*)\times L^p(0,T;V_A)}\\ &\quad= \int_0^T \left\langle Av(t) - A\widetilde{v}(t)\,\middle|\,v(t)-\widetilde{v}(t)\right\rangle_{V_A^*\times V_A}\,\mathrm{d}t \geq 0\,.\end{aligned}$$

(iv) *Coercivity.* For any $v \in L^p(0,T;V_A)$, the relation

$$\langle Av|v\rangle_{L^{p^*}(0,T;V_A^*)\times L^p(0,T;V_A)} \geq \mu_A\,\|v\|_{L^p(0,T;V_A)}^p - \lambda_A T$$

holds. Indeed, the coercivity of $A : V_A \to V_A^*$ yields the relation

$$\begin{aligned}\langle Av|v\rangle_{L^{p^*}(0,T;V_A^*)\times L^p(0,T;V_A)} &= \int_0^T \left\langle Av(t)\,\middle|\,v(t)\right\rangle_{V_A^*\times V_A}\,\mathrm{d}t\\ &\geq \int_0^T \left(\mu_A\,\|v(t)\|_{V_A}^p - \lambda_A\right)\,\mathrm{d}t = \mu_A\,\|v\|_{L^p(0,T;V_A)}^p - \lambda_A T\,.\end{aligned}$$

(v) *Growth condition.* There exists $c > 0$ such that for all $v \in L^p(0,T;V_A)$

$$\|Av\|_{L^{p^*}(0,T;V_A^*)} \leq c\left(1 + \|v\|_{L^p(0,T;V_A)}^{p-1}\right).$$

Namely, by the growth condition on $A : V_A \to V_A^*$, the estimate

$$\begin{aligned}\|Av\|_{L^{p^*}(0,T;V_A^*)}^{p^*} &= \int_0^T \left\|Av(t)\right\|_{V_A^*}^{p^*} \mathrm{d}t \leq c \int_0^T \left(1 + \left\|v(t)\right\|_{V_A}^{p-1}\right)^{p^*} \mathrm{d}t \\ &\leq c \int_0^T \left(1 + \left\|v(t)\right\|_{V_A}^{p}\right) \mathrm{d}t = c\left(T + \|v\|_{L^p(0,T;V_A)}^p\right)\end{aligned}$$

and thus the stated bound with a constant depending in particular on the final time is obtained. We further note that the growth condition ensures boundedness of the operator A in the sense that bounded subsets are mapped into bounded subsets.

Similar considerations show that the hypotheses on the linear and bounded operator $B : V_B \to V_B^*$ are inherited by its time-continuous extension (2.2). Well-definedness is first justified for $r = 1$ and then generalised to arbitrary exponents $r \in [1,\infty]$. Evidently, for every $r \in [1,\infty]$, the operator $B : L^r(0,T;V_B) \to L^r(0,T;V_B^*)$ is linear and satisfies the bound

$$\|Bu\|_{L^r(0,T;V_B^*)} \leq c_B \|u\|_{L^r(0,T;V_B)},$$

provided that $u \in L^r(0,T;V_B)$. In particular, for the exponent $r = 2$ the extended operator is symmetric and strongly positive such that the relation

$$\langle Bu|u\rangle_{L^2(0,T;V_B^*)\times L^2(0,T;V_B)} \geq \mu_B \|u\|_{L^2(0,T;V_B)}^2$$

holds for any $u \in L^2(0,T;V_B)$.

2.3 Existence and uniqueness result

Existence and uniqueness result. The following result deduced in LIONS, STRAUSS [26] ensures existence and uniqueness of a solution to the initial value problem (1.1), see also EMMRICH, THALHAMMER [20] for a generalisation obtained via a full discretisation.

Theorem 2.1 *([26, Thm. 2.1]) Assume $(u_0,v_0) \in V_B \times H$ and $f \in L^{p^*}(0,T;V_A^*)$. Under the hypotheses of Assumptions (A) and (B) there exists a unique solution $u \in L^\infty(0,T;V_B)$ to (1.1) such that $u' \in L^\infty(0,T;H) \cap L^p(0,T;V_A)$ and $u'' \in L^{p^*}(0,T;V_A^*) + L^\infty(0,T;V_B^*) \subset L^{p^*}(0,T;V^*)$. In addition, the said solution fulfills $u \in \mathscr{C}_w([0,T];V_B)$ and $u' \in \mathscr{C}_w([0,T];H)$ and further $u - u_0 \in \mathscr{AC}([0,T];V_A)$ as well as $u \in \mathscr{AC}([0,T];H)$ and $u' \in \mathscr{AC}([0,T];V^*)$. The initial conditions are satisfied in the sense that $u(t) \rightharpoonup u_0$ in V_B and $u'(t) \rightharpoonup v_0$ in H as $t \to 0$.*

2.4 Reformulations as first-order system and integro-differential equation

Reformulations of second-order evolution equation. In view of the construction and convergence analysis of the full discretisation method (1.3) for second-order evolution equations of the form (1.1), it is of importance to introduce the following formally equivalent reformulations as first-order system and integro-differential equation. Setting $v = u'$ yields

$$\begin{cases} v' + Av + Bu = f & \text{in } (0,T)\,, \quad v(0) = v_0\,, \\ u' = v & \text{in } (0,T)\,, \quad u(0) = u_0\,. \end{cases} \tag{2.3}$$

By integrating the second evolution equation in (2.3) and inserting the resulting relation into the first one, the first-order system may be rewritten as the integro-differential equation

$$v' + Av + B\,(u_0 + Kv) = f \quad \text{in } (0,T)\,, \quad v(0) = v_0\,, \tag{2.4a}$$

involving the nonlocal Volterra operator

$$(Kv)(t) = \int_0^t v(s)\,\mathrm{d}s\,. \tag{2.4b}$$

3 Full discretisation method

In this section, we detail the construction of the proposed full discretisation method for nonlinear second-order evolution equations and specify the approximation of the function defining the right-hand side of the equation as well as the starting process. For this purpose, we first introduce a fully discrete approximation combining a Galerkin method with the two-step backward differentiation formula for the formally equivalent first-order system and in addition state the resulting full discretisation method for the related integro-differential equation. Besides, we deduce an auxiliary result on the approximation rate of the arising finite difference operators which is needed in the proof of the a priori error estimate.

3.1 Divided differences

Divided difference operators. In the following, we denote by $(W, \|\cdot\|_W)$ a normed linear space. For simplicity, we do not distinguish between a time-continuous function $w : [0,T] \to W$, the corresponding time discrete values $w = (w(t_n))_{n=0}^N \in W^{N+1}$, or a time grid function $w = (w^n)_{n=0}^N \in W^{N+1}$, and with a minor abuse of notation we write $D_1 w^n = (D_1 w)^n$ etc. for short. For a time grid function $w = (w^n)_{n=0}^N \in W^{N+1}$, the divided difference operators related to the one- and two-step backward differentiation formula, respectively, are defined through ($n = 1,2,\dots,N$ and $n = 2,3,\dots,N$, respectively)

$$\begin{aligned} D_1 w^n &= \tau^{-1}\left(w^n - w^{n-1}\right), \\ D_2 w^n &= \tau^{-1}\left(\tfrac{3}{2} w^n - 2 w^{n-1} + \tfrac{1}{2} w^{n-2}\right). \end{aligned} \tag{3.1a}$$

Applying the operators twice leads to ($n = 2,3,\dots,N$ and $n = 4,5,\dots,N$, respectively)

$$\begin{aligned} D^2 w^{n-1} &= D_1 D_1 w^n = \tau^{-2}\left(w^n - 2w^{n-1} + w^{n-2}\right), \\ D_2 D_2 w^n &= \tau^{-2}\left(\tfrac{9}{4} w^n - 6w^{n-1} + \tfrac{11}{2} w^{n-2} - 2w^{n-3} + \tfrac{1}{4} w^{n-4}\right). \end{aligned} \tag{3.1b}$$

For later use, we note that the first divided differences and the central divided differences are related through the identity ($n = 2,3,\dots,N$)

$$D^2 w^{n-1} = 2\,\tau^{-1}\left(D_2 w^n - D_1 w^n\right), \tag{3.2}$$

see also EMMRICH [13]. The following result, which we state without proof, implies that the divided differences yield approximations to the first- and second-order time derivatives, provided that the time discrete values are related to a sufficiently regular function.

Lemma 3.1 *Provided that the underlying function $w : [0,T] \to W$ defining the time discrete values $w = (w^n)_{n=0}^N = (w(t_n))_{n=0}^N \in W^{N+1}$ satisfies $w \in W^{2,1}(0,T;W) \subset \mathscr{C}^1([0,T];W)$ and $w \in W^{3,1}(0,T;W) \subset \mathscr{C}^2([0,T];W)$, respectively, the estimates*

$$\begin{aligned} \left\|D_1 w^n - w'(t_n)\right\|_W &\le \int_{t_{n-1}}^{t_n} \left\|w''(s)\right\|_W \,\mathrm{d}s, \quad n = 1,2,\dots,N, \\ \left\|D_2 w^n - w'(t_n)\right\|_W &\le \tau \int_{t_{n-2}}^{t_n} \left\|w'''(s)\right\|_W \,\mathrm{d}s, \quad n = 2,3,\dots,N, \end{aligned}$$

hold true, and hence

$$\begin{aligned} \tau \sum_{n=1}^N \left\|D_1 w^n - w'(t_n)\right\|_W &\le \tau \left\|w''\right\|_{L^1(0,T;W)}, \\ \tau \sum_{n=2}^N \left\|D_2 w^n - w'(t_n)\right\|_W &\le 2\tau^2 \left\|w'''\right\|_{L^1(0,T;W)}. \end{aligned}$$

If in addition $w \in W^{4,1}(0,T;W) \subset \mathscr{C}^3([0,T];W)$, then the bounds

$$\begin{aligned} \left\|D^2 w^n - w''(t_n)\right\|_W &\le \tau \int_{t_{n-1}}^{t_{n+1}} \left\|w^{(4)}(s)\right\|_W \,\mathrm{d}s, \quad n = 1,2,\dots,N-1, \\ \left\|D_2 D_2 w^n - w''(t_n)\right\|_W &\le c\,\tau \int_{t_{n-4}}^{t_n} \left\|w^{(4)}(s)\right\|_W \,\mathrm{d}s, \quad n = 4,5,\dots,N, \end{aligned}$$

are valid with some constant $c > 0$, and hence

$$\begin{aligned} \tau \sum_{n=1}^{N-1} \left\|D_2 w^n - w''(t_n)\right\|_W &\le 2\tau^2 \left\|w^{(4)}\right\|_{L^1(0,T;W)}, \\ \tau \sum_{n=4}^{N} \left\|D_2 D_2 w^n - w''(t_n)\right\|_W &\le 4c\tau^2 \left\|w^{(4)}\right\|_{L^1(0,T;W)}. \end{aligned}$$

3.2 Full discretisation method and reformulations

Approximation of right-hand side. In the proof of Theorem 5.1 we restrict our considerations to approximations of the right-hand side of the evolution equation in (1.1) that are obtained by restriction onto the time grid

$$\begin{aligned} f^1 = R_1^1 f = \tau^{-1} \int_0^\tau f(s)\,\mathrm{d}s\,, \\ f^n = R_2^n f = \tau^{-1}\left(\tfrac{3}{2}\int_{t_{n-1}}^{t_n} f(s)\,\mathrm{d}s - \tfrac{1}{2}\int_{t_{n-2}}^{t_{n-1}} f(s)\,\mathrm{d}s\right), \quad n = 2,3,\dots,N\,; \end{aligned} \tag{3.3}$$

this naturally corresponds to the chosen divided differences (3.1), since

$$R_1^1 w' = D_1 w(t_1)\,, \qquad R_2^n w' = D_2 w(t_n)\,, \quad n = 2,3,\dots,N\,,$$

if $w \in W^{1,1}(0,T;W)$. However, in situations where f is even continuous such that pointwise evaluation is possible, we may instead set $f^n = f(t_n)$ for $n = 1,2,\dots,N$.

Full discretisation method for first-order system. The proposed full discretisation method (1.3) for nonlinear second-order evolution equations of the form (1.3) results from a fully discrete approximation of the formally equivalent first-order system (2.3) combining the Galerkin method for the space discretisation with the two-step backward differentiation formula for the time discretisation. That is, the fully discrete solution $(u^n, v^n)_{n=1}^N \subset V_M \times V_M$ is determined such that the relations $(n = 2,3,\dots,N)$

$$\begin{cases} \big(D_2 v^n \,\big|\, \varphi\big)_H + \big\langle A v^n \,\big|\, \varphi\big\rangle_{V_A^*\times V_A} + \big\langle B u^n \,\big|\, \varphi\big\rangle_{V_B^*\times V_B} = \big\langle f^n \,\big|\, \varphi\big\rangle_{V_A^*\times V_A}\,, \\ \big(D_2 u^n \,\big|\, \varphi\big)_H = \big(v^n \,\big|\, \varphi\big)_H \end{cases} \tag{3.4a}$$

hold for all $\varphi \in V_M$. The first iterate (u^1, v^1) is instead determined by means of the one-step backward differentiation formula (implicit Euler method), which yields

$$\begin{cases} \big(D_1 v^1 \,\big|\, \varphi\big)_H + \big\langle A v^1 \,\big|\, \varphi\big\rangle_{V_A^*\times V_A} + \big\langle B u^1 \,\big|\, \varphi\big\rangle_{V_B^*\times V_B} = \big\langle f^1 \,\big|\, \varphi\big\rangle_{V_A^*\times V_A}\,, \\ \big(D_1 u^1 \,\big|\, \varphi\big)_H = \big(v^1 \,\big|\, \varphi\big)_H \end{cases} \tag{3.4b}$$

for all $\varphi \in V_M$. We recall the definition (3.1) of the divided difference operators.

Practical realisation of the full discretisation method. The practical realisation of the fully discrete approximation (3.4) relies on the solution of nonlinear equations for the numerical solution values v^n for $n = 1,2,\dots,N$. That is, starting from an initial approximation $(u^0, v^0) \approx (u_0, v_0)$, the relation $u^1 = u^0 + \tau v^1$ (recall that $v^1 = D_1 u^1 = \tau^{-1}(u^1 - u^0)$) is inserted into the first equation in (3.4b) and the resulting nonlinear equation (for all $\varphi \in V_M$)

$$\begin{aligned} \tau^{-1}\big(v^1 - v^0 \,\big|\, \varphi\big)_H + \big\langle A v^1 \,\big|\, \varphi\big\rangle_{V_A^*\times V_A} \\ + \big\langle B\big(u^0 + \tau v^1\big) \,\big|\, \varphi\big\rangle_{V_B^*\times V_B} = \big\langle f^1 \,\big|\, \varphi\big\rangle_{V_A^*\times V_A} \end{aligned} \tag{3.5a}$$

is resolved for v^1. The new iterate u^1 is then determined through the relation

$$u^1 = u^0 + \tau v^1 \,. \tag{3.5b}$$

For the remaining steps, the relation $v^n = D_2 u^n = \tau^{-1}\big(\frac{3}{2}u^n - 2u^{n-1} + \frac{1}{2}u^{n-2}\big)$ is used in order to express $u^n = \frac{4}{3}u^{n-1} - \frac{1}{3}u^{n-2} + \frac{2}{3}\tau v^n$. Resolving the nonlinear equation (for all $\varphi \in V_M$)

$$\begin{aligned} &\tau^{-1}\big(\tfrac{3}{2}v^n - 2v^{n-1} + \tfrac{1}{2}v^{n-2}\,\big|\,\varphi\big)_H + \big\langle Av^n\,\big|\,\varphi\big\rangle_{V_A^*\times V_A} \\ &\quad + \big\langle B\big(\tfrac{4}{3}u^{n-1} - \tfrac{1}{3}u^{n-2} + \tfrac{2}{3}\tau v^n\big)\,\big|\,\varphi\big\rangle_{V_B^*\times V_B} = \big\langle f^n\,\big|\,\varphi\big\rangle_{V_A^*\times V_A} \end{aligned} \tag{3.5c}$$

for v^n determines the new iterate

$$u^n = \tfrac{4}{3}u^{n-1} - \tfrac{1}{3}u^{n-2} + \tfrac{2}{3}\tau v^n \tag{3.5d}$$

for every $n = 2, 3, \dots, N$.

Full discretisation method for second-order evolution equation. The full discretisation scheme (3.4) is equivalent to (1.3) with the starting process (3.5a)-(3.5b) and (3.5c)-(3.5d) for $n = 2, 3$ and all $\varphi \in V_M$

$$\left\{\begin{aligned} &\tau^{-1}\big(v^1 - v^0\,\big|\,\varphi\big)_H + \big\langle Av^1\,\big|\,\varphi\big\rangle_{V_A^*\times V_A} \\ &\qquad + \big\langle B\big(u^0 + \tau v^1\big)\,\big|\,\varphi\big\rangle_{V_B^*\times V_B} = \big\langle f^1\,\big|\,\varphi\big\rangle_{V_A^*\times V_A}\,, \\ &u^1 = u^0 + \tau v^1\,, \end{aligned}\right.$$

$$\left\{\begin{aligned} &\tau^{-1}\big(\tfrac{3}{2}v^2 - 2v^1 + \tfrac{1}{2}v^0\,\big|\,\varphi\big)_H + \big\langle Av^2\,\big|\,\varphi\big\rangle_{V_A^*\times V_A} \\ &\qquad + \big\langle B\big(\tfrac{4}{3}u^1 - \tfrac{1}{3}u^0 + \tfrac{2}{3}\tau v^2\big)\,\big|\,\varphi\big\rangle_{V_B^*\times V_B} = \big\langle f^2\,\big|\,\varphi\big\rangle_{V_A^*\times V_A}\,, \\ &u^2 = \tfrac{4}{3}u^1 - \tfrac{1}{3}u^0 + \tfrac{2}{3}\tau v^2\,, \end{aligned}\right.$$

$$\left\{\begin{aligned} &\tau^{-1}\big(\tfrac{3}{2}v^3 - 2v^2 + \tfrac{1}{2}v^1\,\big|\,\varphi\big)_H + \big\langle Av^3\,\big|\,\varphi\big\rangle_{V_A^*\times V_A} \\ &\qquad + \big\langle B\big(\tfrac{4}{3}u^2 - \tfrac{1}{3}u^1 + \tfrac{2}{3}\tau v^3\big)\,\big|\,\varphi\big\rangle_{V_B^*\times V_B} = \big\langle f^3\,\big|\,\varphi\big\rangle_{V_A^*\times V_A}\,, \\ &u^3 = \tfrac{4}{3}u^2 - \tfrac{1}{3}u^1 + \tfrac{2}{3}\tau v^3\,. \end{aligned}\right.$$

For the following considerations, we assume that $u^n = u(t_n)$ holds for $n = 0, 1, \dots, N$ with u being a sufficiently regular function. We point out that by construction

$$\begin{aligned} D_2 v^n = D_2 D_2 u^n &= \tau^{-2}\big(\tfrac{9}{4}u^n - 6u^{n-1} + \tfrac{11}{2}u^{n-2} - 2u^{n-3} + \tfrac{1}{4}u^{n-4}\big) \\ &= u''(t_n) + \mathscr{O}(\tau^2)\,, \qquad n = 4, 5, \dots, N\,, \end{aligned}$$

see also Lemma 3.1, but that $D_1 v^1$, $D_2 v^2$, and $D_2 v^3$ may not approximate $u''(t_1)$, $u''(t_2)$, and $u''(t_3)$, respectively. Indeed, a straightforward calculation shows that

$$\begin{aligned} D_1 v^1 &= \tau^{-1}\big(v^1 - v^0\big) = \tau^{-1}\big(D_1 u^1 - v^0\big) \\ &= \tau^{-2}\big(u^1 - u_0 - \tau v^0\big)\,, \\ D_2 v^2 &= \tau^{-1}\big(\tfrac{3}{2}v^2 - 2v^1 + \tfrac{1}{2}v^0\big) = \tau^{-1}\big(\tfrac{3}{2}D_2 u^2 - 2D_1 u^1 + \tfrac{1}{2}v^0\big) \\ &= \tau^{-2}\big(\tfrac{9}{4}u^2 - 5u^1 + \tfrac{11}{4}u^0 + \tfrac{1}{2}\tau v^0\big)\,, \\ D_2 v^3 &= \tau^{-1}\big(\tfrac{3}{2}v^3 - 2v^2 + \tfrac{1}{2}v^1\big) = \tau^{-1}\big(\tfrac{3}{2}D_2 u^3 - 2D_2 u^2 + \tfrac{1}{2}D_1 u^1\big) \\ &= \tau^{-2}\big(\tfrac{9}{4}u^3 - 6u^2 + \tfrac{21}{4}u^1 - \tfrac{3}{2}u^0\big)\,. \end{aligned}$$

As a consequence, $D_1 v^1$ yields an approximation to $u''(t_1)$ only if the initial value satisfies $v^0 = u'(0) - \frac{1}{2}\tau u''(0) + o(\tau)$, which is in contradiction with the relation $v^0 = u'(0) - 2\tau u''(0) + o(\tau)$ required for $D_2 v^2$ approximating $u''(t_2)$. Furthermore, $D_2 v^3 = \frac{3}{4}u''(0) + \mathscr{O}(\tau)$ does not yield an approximation to $u''(t_3)$. Despite the fact that the starting procedure does not lead to consistent approximations of u'', we show convergence. The phenomenon that an inconsistent scheme (or a scheme of low-order consistency) can still be convergent (or convergent of high order) is well-known from the analysis of finite difference methods on irregular meshes, see for instance LEVERMORE, MANTEUFFEL, WHITE [23] and the references given therein. Moreover, in MANTEUFFEL, WHITE [27] a discretisation scheme that is convergent of order two for second-order ordinary differential equations is constructed via a symmetric finite difference method for the equivalent first-order system.

Reformulation of full discretisation method. In our convergence analysis, we also utilise an equivalent reformulation of the full discretisation method (3.4) for the first-order system (2.3) that is related to the integro-differential equation (2.4). We rewrite the two-step relation (3.5d) for the new iterate as a one-step relation for two subsequent iterates ($n = 2, 3, \dots, N$)

$$\begin{bmatrix} u^n \\ u^{n-1} \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & -\frac{1}{3} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u^{n-1} \\ u^{n-2} \end{bmatrix} + \frac{2}{3}\,\tau \begin{bmatrix} v^n \\ 0 \end{bmatrix}$$

and resolve the resulting recurrence ($n = 2, 3, \dots, N$)

$$\begin{bmatrix} u^n \\ u^{n-1} \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & -\frac{1}{3} \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} u^1 \\ u^0 \end{bmatrix} + \frac{2}{3}\,\tau \sum_{j=2}^{n} \begin{bmatrix} \frac{4}{3} & -\frac{1}{3} \\ 1 & 0 \end{bmatrix}^{n-j} \begin{bmatrix} v^j \\ 0 \end{bmatrix}.$$

Recalling that the roots of the characteristic polynomial associated with the two-step backward differentiation formula are given by $1, \frac{1}{3}$, an eigenvalue decomposition of the arising matrix implies ($k \in \mathbb{N}$)

$$\begin{bmatrix} \frac{4}{3} & -\frac{1}{3} \\ 1 & 0 \end{bmatrix}^{k} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3^{-k} \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix},$$

and, consequently, we obtain ($n = 2, 3, \dots, N$)

$$\begin{aligned} \begin{bmatrix} u^n \\ u^{n-1} \end{bmatrix} &= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3^{-n+1} \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u^1 \\ u^0 \end{bmatrix} \\ &\quad + \frac{1}{3}\,\tau \sum_{j=2}^{n} \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3^{-n+j} \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v^j \\ 0 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} \big(3 - 3^{-n+1}\big)\, u^1 + \big(3^{-n+1} - 1\big)\, u^0 \\ \big(3 - 3^{-n+2}\big)\, u^1 + \big(3^{-n+2} - 1\big)\, u^0 \end{bmatrix} + \tau \sum_{j=2}^{n} \begin{bmatrix} 1 - 3^{-n-1+j} \\ 1 - 3^{-n+j} \end{bmatrix} v^j . \end{aligned}$$

Inserting $u^1 = u^0 + \tau v^1$ leads to ($n = 1, 2, \dots, N$)

$$u^n = u^0 + \tau \sum_{j=1}^{n} v^j + \frac{1}{2}\big(1 - 3^{-n+1}\big)\,\tau v^1 - \tau \sum_{j=2}^{n} 3^{-n-1+j}\, v^j . \tag{3.6}$$

Altogether, replacing in (3.4) the quantity u^n by the above representation, we obtain the following reformulation. The fully discrete solution $(v^n)_{n=1}^N \subset V_M$ is determined such that the relation ($n = 2,3,\dots,N$)

$$\big(D_2 v^n \,\big|\, \varphi\big)_H + \big\langle A v^n \,\big|\, \varphi\big\rangle_{V_A^* \times V_A} + \big\langle B\left(u^0 + K_\tau v^n\right) \big|\, \varphi\big\rangle_{V_B^* \times V_B} = \big\langle f^n \,\big|\, \varphi\big\rangle_{V_A^* \times V_A} \tag{3.7a}$$

holds for all $\varphi \in V_M$, where the discrete sum operator K_τ is defined through ($n = 1,2,\dots,N$)

$$K_\tau v^n = \tau \sum_{j=1}^{n} v^j + \widetilde{K}_\tau v^n\,, \quad \widetilde{K}_\tau v^n = \tfrac{1}{2}\left(1 - 3^{-n+1}\right)\tau v^1 - \tau \sum_{j=2}^{n} 3^{-n-1+j}\, v^j\,. \tag{3.7b}$$

As before, we write $K_\tau v^n = (K_\tau v)^n$ etc. for short. In particular, the first iterate is determined such that the identity

$$\big(D_1 v^1 \,\big|\, \varphi\big)_H + \big\langle A v^1 \,\big|\, \varphi\big\rangle_{V_A^* \times V_A} + \big\langle B\left(u^0 + K_\tau v^1\right) \big|\, \varphi\big\rangle_{V_B^* \times V_B} = \big\langle f^1 \,\big|\, \varphi\big\rangle_{V_A^* \times V_A} \tag{3.7c}$$

is fulfilled for all $\varphi \in V_M$, where $\widetilde{K}_\tau v^1 = 0$ and thus $K_\tau v^1 = \tau v^1$.

Remark. Arguments close to the proof of Lemma 3.1 show that the discrete sum operator K_τ yields an approximation to the integral operator K defined in (2.4). Indeed, provided that the underlying function $w : [0,T] \to W$ defining the time discrete values $w = (w^n)_{n=1}^N = (w(t_n))_{n=1}^N \in W^N$ satisfies $w \in W^{1,1}(0,T;W)$, the expansion ($n = 1,2,\dots,N$)

$$\tau \sum_{j=1}^{n} w^j - \int_0^{t_n} w(t)\,\mathrm{d}t = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \big(w(t_j) - w(t)\big)\,\mathrm{d}t = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \int_t^{t_j} w'(s)\,\mathrm{d}s\,\mathrm{d}t$$

holds. Besides, the estimate ($n = 1,2,\dots,N$)

$$\big\|\widetilde{K}_\tau w(t_n)\big\|_W \leq \tau \max\left\{\big\|w(t_j)\big\|_W : j = 1,2,\dots,n\right\} \leq \tau\, \|w\|_{\mathscr{C}([0,T];W)}$$

is obtained by means of the partial sum of the geometric sequence in the estimate ($n = 1,2,\dots,N$)

$$\begin{aligned}
\big\|\widetilde{K}_\tau w^n\big\|_W &\leq \tau \left(\tfrac{1}{2}\left(1 - 3^{-n+1}\right) + \sum_{j=2}^{n} 3^{-n-1+j}\right) \max\left\{\big\|w^j\big\|_W : j = 1,2,\dots,n\right\} \\
&= \tau\left(1 - 3^{-n+1}\right) \max\left\{\big\|w^j\big\|_W : j = 1,2,\dots,n\right\}.
\end{aligned}$$

Altogether, this implies ($n = 1,2,\dots,N$)

$$\big\|K_\tau w(t_n) - K w(t_n)\big\|_W \leq \tau \left(\int_0^{t_n} \|w'(s)\|_W\,\mathrm{d}s + \|w\|_{\mathscr{C}([0,T];W)}\right),$$

and hence, due to the continuous embedding $W^{1,1}(0,T;W) \hookrightarrow \mathscr{C}([0,T];W)$, the bound ($n = 1,2,\dots,N$)

$$\big\|K_\tau w(t_n) - K w(t_n)\big\|_W \leq c\,\tau\, \|w\|_{W^{1,1}(0,T;W)}$$

follows; this estimate also shows that K_τ defines a convergent quadrature formula for K.

3.3 Auxiliaries

Auxiliary relations. As verified by brief calculations, the auxiliary relations

$$(a-b)\,a = \tfrac{1}{2}\left(a^2 - b^2 + (a-b)^2\right), \tag{3.8a}$$
$$\left(\tfrac{3}{2}a - 2b + \tfrac{1}{2}c\right) a = \tfrac{1}{4}\left(a^2 - b^2 + (2a-b)^2 - (2b-c)^2 + (a-2b+c)^2\right), \tag{3.8b}$$
$$\left(\tfrac{3}{2}\,a - 2\,b + \tfrac{1}{2}\,c\right)(a - 2\,b + c) = \tfrac{1}{2}\left((a-b)^2 - (b-c)^2\right) + (a - 2\,b + c)^2 \tag{3.8c}$$

hold true for arbitrary $a, b, c \in \mathbb{R}$; we note that these algebraic identities reflect the stability of the time discretisation, see also EMMRICH [13–15].

4 Solvability of the discrete problem and a priori estimates

In this section, we prove existence and uniqueness of the fully discrete solution to the considered nonlinear second-order evolution equation and deduce a basic result providing a priori estimates for the fully discrete solution.

Galerkin method. We recall that $(V_M)_{M\in\mathbb{N}}$ denotes a Galerkin scheme for the separable Banach space V. For the subsequent considerations, we further suppose that the family $(\varphi_M)_{M\in\mathbb{N}}$ forms a Galerkin basis of V such that

$$V = \operatorname{clos}_{\|\cdot\|_V} \bigcup_{M\in\mathbb{N}} V_M\,, \quad V_M = \operatorname{span}\{\varphi_1, \dots, \varphi_M\}\,,$$

which ensures that $V_M \subset V_{M+1}$. However, considering a generalised internal approximation and using a suitable restriction operator would permit to avoid this inclusion, see TEMAM [30, p. 25ff.]. All the results then apply, in particular, to conforming finite element methods. Due to the fact that the underlying space V is dense and continuously embedded in the domains of the defining operators, the family $(V_M)_{M\in\mathbb{N}}$ also forms a Galerkin scheme for V_A with limited completeness with respect to the norm $\|\cdot\|_{V_A}$ and as well for V_B with limited completeness with respect to $\|\cdot\|_{V_B}$, respectively.

4.1 Existence and uniqueness

The following auxiliary result is utilised in order to establish existence and uniqueness of fully discrete solutions.

Lemma 4.1 *Let $\Phi : \mathbb{R}^M \to \mathbb{R}^M$ be a continuous function and assume that there is $R > 0$ such that $\Phi(\mathbf{v}) \cdot \mathbf{v} \geq 0$ for all $\mathbf{v} \in \mathbb{R}^M$ with $\|\mathbf{v}\|_{\mathbb{R}^M} = R$. Then there exists an element $\overline{\mathbf{v}} \in \mathbb{R}^M$ with $\|\overline{\mathbf{v}}\|_{\mathbb{R}^M} \leq R$ and $\Phi(\overline{\mathbf{v}}) = 0$.*

Proof The assertion follows by contradiction from Brouwer's fixed point theorem, see for instance GAJEWSKI, GRÖGER, ZACHARIAS [22, Lemma 2.1, p. 74]. ♢

We are now ready to state a result ensuring the existence and uniqueness of a solution to the fully discrete scheme (3.7) related to the integro-differential equation (2.4). This also proves the unique solvability of the equivalent formulations (1.3) and (3.4), which correspond to the considered nonlinear second-order evolution equation (1.1) and the associated first-order system (2.3), respectively.

Theorem 4.2 (Existence and uniqueness of a fully discrete solution) *Suppose that Assumptions (A) and (B) hold, and let $(u^0, v^0) \in V_M \times V_M$ as well as $(f^n)_{n=1}^N \subset V_A^*$ be given. Then there exists a unique solution $(u^n, v^n)_{n=1}^N \subset V_M \times V_M$ to the full discretisation scheme* (3.7)*.*

Proof We prove existence and uniqueness of a solution $(v^n)_{n=1}^N \subset V_M$ to the full discretisation scheme (3.7) for the integro-differential equation (2.4). To this purpose, proceeding step-by-step, we construct the new iterate v^n from given discrete solution values $v^0, \dots, v^{n-1}$ for $n = 1, 2, \dots, N$. The associated solution $(u^n)_{n=1}^N \subset V_M$ is then uniquely determined by relation (3.6).

(i) *Representation in Galerkin basis.* For any element $v \in V_M$, the representation with respect to the Galerkin basis

$$v = \sum_{m=1}^{M} v_m \, \varphi_m \in V_M \,, \quad \mathbf{v} = (v_m)_{m=1}^M \in \mathbb{R}^M \,,$$

defines a bijection between V_M and $\mathbb{R}^M$, and the mapping

$$\|\cdot\|_{\mathbb{R}^M} : \mathbb{R}^M \longrightarrow \mathbb{R} : \mathbf{v} \longmapsto \|\mathbf{v}\|_{\mathbb{R}^M} = \|v\|_{V_A}$$

defines a norm on $\mathbb{R}^M$.

(ii) *Reformulation of numerical scheme.* We recall the full discretisation scheme (3.7) and in particular the defining relation for the discrete sum operator, which in the first time step simplifies to $K_\tau v^1 = \tau \, v^1$ and otherwise may be rewritten as ($n = 2, 3, \dots, N$)

$$\begin{aligned} K_\tau v^n &= \widehat{K}_\tau v^{n-1} + \tfrac{2}{3} \, \tau \, v^n \,, \\ \widehat{K}_\tau v^{n-1} &= \tfrac{3}{2} \left(1 - 3^{-n}\right) \tau \, v^1 + \tau \sum_{j=2}^{n-1} \left(1 - 3^{-n-1+j}\right) v^j \,. \end{aligned} \tag{4.1}$$

Setting $\varphi = \varphi_m$ for $m = 1, 2, \dots, M$, the full discretisation method reduces to the finite dimensional problem of determining the coefficients $(\mathbf{v}^n)_{n=1}^N \subset \mathbb{R}^M$ associated with representations in the Galerkin basis. In particular, in the first step this leads to the nonlinear equation

$$\begin{aligned} \Phi_1\left(\mathbf{v}^1\right) = \Big(&\left(D_1 v^1 \,\middle|\, \varphi_m\right)_H + \left\langle A v^1 \,\middle|\, \varphi_m \right\rangle_{V_A^* \times V_A} + \left\langle B\left(u^0 + \tau v^1\right) \,\middle|\, \varphi_m \right\rangle_{V_B^* \times V_B} \\ &- \left\langle f^1 \,\middle|\, \varphi_m \right\rangle_{V_A^* \times V_A} \Big)_{m=1}^M = 0 \,, \end{aligned}$$

and in the subsequent steps this yields ($n = 2, 3, \dots, N$)

$$\begin{aligned} \Phi_n\left(\mathbf{v}^n\right) = \Big(&\left(D_2 v^n \,\middle|\, \varphi_m\right)_H + \left\langle A v^n \,\middle|\, \varphi_m \right\rangle_{V_A^* \times V_A} + \left\langle B\left(u^0 + \widehat{K}_\tau v^{n-1} + \tfrac{2}{3} \, \tau \, v^n\right) \,\middle|\, \varphi_m \right\rangle_{V_B^* \times V_B} \\ &- \left\langle f^n \,\middle|\, \varphi_m \right\rangle_{V_A^* \times V_A} \Big)_{m=1}^M = 0 \,. \end{aligned}$$

(iii) *Existence of fully discrete solution.* In order to ensure the existence of a fully discrete solution to (3.7) it remains to prove the existence of a zero of the function $\Phi_n : \mathbb{R}^M \to \mathbb{R}^M$ ($n = 1,2,\dots,N$) by means of Lemma 4.1. As similar arguments apply to Φ_1, we focus on Φ_n for $n = 2,3,\dots,N$.

(a) On the one hand, the nonlinear operator $A : V_A \to V_A^*$ is hemicontinuous, monotone, and thus demicontinuous, and on the other hand, the linear operator $B : V_B \to V_B^*$ is bounded and thus continuous. This ensures that $\Phi_n : \mathbb{R}^M \to \mathbb{R}^M$ is a continuous function.

(b) Our aim is to show that there exists a constant $R > 0$ such that the relation

$$\begin{aligned}\Phi_n(\mathbf{v}^n)\cdot\mathbf{v}^n = \big(D_2 v^n \,\big|\, v^n\big)_H + \big\langle A v^n \,\big|\, v^n\big\rangle_{V_A^*\times V_A} + \big\langle B\big(u^0 + \widehat{K}_\tau v^{n-1} + \tfrac{2}{3}\tau v^n\big) \,\big|\, v^n\big\rangle_{V_B^*\times V_B} \\ - \big\langle f^n \,\big|\, v^n\big\rangle_{V_A^*\times V_A} \geq 0\end{aligned}$$

holds true for all elements $\mathbf{v}^n \in \mathbb{R}^M$ with $\|\mathbf{v}^n\|_{\mathbb{R}^M} = R$. We observe that

$$\begin{aligned}\big(D_2 v^n \,\big|\, v^n\big)_H &= \tau^{-1}\big(\tfrac{3}{2} v^n - 2v^{n-1} + \tfrac{1}{2} v^{n-2} \,\big|\, v^n\big)_H \\ &\geq \tfrac{3}{2}\tau^{-1}\big\|v^n\big\|_H^2 - \tau^{-1}\big\|2v^{n-1} - \tfrac{1}{2}v^{n-2}\big\|_{V_A^*}\big\|v^n\big\|_{V_A}.\end{aligned}$$

The coercivity condition on A stated in Assumption (A) yields

$$\big\langle A v^n \,\big|\, v^n\big\rangle_{V_A^*\times V_A} \geq \mu_A \big\|v^n\big\|_{V_A}^p - \lambda_A .$$

Furthermore, the boundedness and strong positivity of B as well as Young's inequality imply

$$\begin{aligned}&\big\langle B\big(u^0 + \widehat{K}_\tau v^{n-1} + \tfrac{2}{3}\tau v^n\big) \,\big|\, v^n\big\rangle_{V_B^*\times V_B} \\ &\quad = \tfrac{2}{3}\tau\big\langle B v^n \,\big|\, v^n\big\rangle_{V_B^*\times V_B} + \big\langle B\big(u^0 + \widehat{K}_\tau v^{n-1}\big) \,\big|\, v^n\big\rangle_{V_B^*\times V_B} \\ &\quad \geq \tfrac{2}{3}\mu_B\tau\big\|v^n\big\|_{V_B}^2 - c_B\big\|u^0 + \widehat{K}_\tau v^{n-1}\big\|_{V_B}\big\|v^n\big\|_{V_B} \\ &\quad \geq \tfrac{1}{3}\mu_B\tau\big\|v^n\big\|_{V_B}^2 - c\tau^{-1}\big\|u^0 + \hat{K}_\tau v^{n-1}\big\|_{V_B}^2 ,\end{aligned}$$

see also Assumption (B). Altogether, we find

$$\begin{aligned}\Phi_n(\mathbf{v}^n)\cdot\mathbf{v}^n \geq \Big(\mu_A\big\|v^n\big\|_{V_A}^{p-1} - \tau^{-1}\big\|2v^{n-1} - \tfrac{1}{2}v^{n-2}\big\|_{V_A^*} - \big\|f^n\big\|_{V_A^*}\Big)\big\|v^n\big\|_{V_A} \\ + \tfrac{3}{2}\tau^{-1}\big\|v^n\big\|_H^2 + \tfrac{1}{3}\mu_B\tau\big\|v^n\big\|_{V_B}^2 - c\tau^{-1}\big\|u^0 + \hat{K}_\tau v^{n-1}\big\|_{V_B}^2 - \lambda_A .\end{aligned}$$

Choosing $\|\mathbf{v}^n\|_{\mathbb{R}^m} = \|v^n\|_{V_A} = R$ sufficiently large, ensures $\Phi_n(\mathbf{v}^n)\cdot\mathbf{v}^n \geq 0$. Thus, by Lemma 4.1, this proves the existence of a solution to the nonlinear equation $\Phi_n(\mathbf{v}^n) = 0$ for each $n = 2,3,\dots,N$.

(iv) *Uniqueness of fully discrete solution.* The uniqueness of the solution to the full discretisation scheme (3.7) or, equivalently, to the nonlinear equations $\Phi_n(\mathbf{v}^n) = 0$ for $n = 1,2,\dots,N$ follows by contradiction; we again focus on the cases

$n = 2,3,\dots,N$. For integers $n \in \{2,3,\dots,N\}$ fixed, let $(v^j)_{j=0}^{n-1}$ and f^n be given, and assume that v^n and $\widetilde{v}^n$ are two different solutions satisfying

$$\begin{aligned}
\Phi_n(\mathbf{v}^n) = \Big(&\tau^{-1}\big(\tfrac{3}{2}v^n - 2v^{n-1} + \tfrac{1}{2}v^{n-2}\,\big|\,\varphi_m\big)_H + \big\langle Av^n\,\big|\,\varphi_m\big\rangle_{V_A^*\times V_A} \\
&+ \big\langle B\big(u^0 + \widehat{K}_\tau v^{n-1} + \tfrac{2}{3}\tau v^n\big)\,\big|\,\varphi_m\big\rangle_{V_B^*\times V_B} - \big\langle f^n\,\big|\,\varphi_m\big\rangle_{V_A^*\times V_A}\Big)_{m=1}^M = 0\,, \\
\Phi_n(\widetilde{\mathbf{v}}^n) = \Big(&\tau^{-1}\big(\tfrac{3}{2}\widetilde{v}^n - 2v^{n-1} + \tfrac{1}{2}v^{n-2}\,\big|\,\varphi_m\big)_H + \big\langle A\widetilde{v}^n\,\big|\,\varphi_m\big\rangle_{V_A^*\times V_A} \\
&+ \big\langle B\big(u^0 + \widehat{K}_\tau v^{n-1} + \tfrac{2}{3}\tau \widetilde{v}^n\big)\,\big|\,\varphi_m\big\rangle_{V_B^*\times V_B} - \big\langle f^n\,\big|\,\varphi_m\big\rangle_{V_A^*\times V_A}\Big)_{m=1}^M = 0\,.
\end{aligned}$$

Taking the difference, testing with $v^n - \widetilde{v}^n$, and utilising the monotonicity of A as well as the strong positivity of B leads to a contradiction whenever $v^n \neq \widetilde{v}^n$, since then

$$\begin{aligned}
0 &= \big(\Phi_n(\mathbf{v}^n) - \Phi_n(\widetilde{\mathbf{v}}^n)\big)\cdot(\mathbf{v}^n - \widetilde{\mathbf{v}}^n) \\
&= \tfrac{3}{2}\tau^{-1}\big\|v^n - \widetilde{v}^n\big\|_H^2 + \big\langle Av^n - A\widetilde{v}^n\,\big|\,v^n - \widetilde{v}^n\big\rangle_{V_A^*\times V_A} + \tfrac{2}{3}\tau\big\langle B\,(v^n - \widetilde{v}^n)\,\big|\,v^n - \widetilde{v}^n\big\rangle_{V_B^*\times V_B} \\
&\geq \tfrac{3}{2}\tau^{-1}\big\|v^n - \widetilde{v}^n\big\|_H^2 + \tfrac{2}{3}\mu_B\,\tau\,\big\|v^n - \widetilde{v}^n\big\|_{V_B}^2 > 0\,,
\end{aligned}$$

see also Assumptions (A) and (B). Again, the first time step only requires minor modifications of the above considerations. ♢

We shall remark that continuous dependence of the fully discrete solution on the initial approximations and the approximation to the right-hand side can be shown by standard techniques.

4.2 A priori estimates

The following a priori estimates for the fully discrete solution are a basic ingredient in our convergence analysis. In what follows, we restrict our considerations to approximations $(f^n)_{n=1}^N \subset V_A^*$ obtained by restriction of f onto the time grid, see (3.3). Provided that $f \in L^{p^*}(0,T;V_A^*)$, a straightforward calculation then shows that

$$\tau \sum_{n=1}^N \big\|f^n\big\|_{V_A^*}^{p^*} \leq c\,\big\|f\big\|_{L^{p^*}(0,T;V_A^*)}^{p^*}\,.$$

We note that the arising quantities $C(u^0,v^0,f)$ and $\widetilde{C}(u^0,v^0,f)$ remain bounded with respect to $(u^0,v^0) \in V_B \times H$ on bounded subsets of $V_B \times H$.

Theorem 4.3 (A priori estimates for the fully discrete solution) *Suppose that Assumptions (A) and (B) are satisfied. Furthermore, let $(u^0,v^0) \in V_M \times V_M$ be given, and set*

$$C(u^0,v^0,f) = \big\|u^0\big\|_{V_B}^2 + \big\|v^0\big\|_H^2 + \lambda_A T + \big\|f\big\|_{L^{p^*}(0,T;V_A^*)}^{p^*}\,.$$

(i) The estimate ($N_0 = 1, 2, \ldots, N$)

$$\begin{aligned}
&\big\|u^{N_0}\big\|_{V_B}^2 + \big\|u^{N_0} - u^{N_0-1}\big\|_{V_B}^2 + \sum_{n=2}^{N_0} \big\|\tau^2 D^2 u^{n-1}\big\|_{V_B}^2 \\
&\qquad + \big\|v^{N_0}\big\|_H^2 + \big\|v^{N_0} - v^{N_0-1}\big\|_H^2 + \sum_{n=2}^{N_0} \big\|\tau^2 D^2 v^{n-1}\big\|_H^2 + \mu_A \tau \sum_{n=1}^{N_0} \big\|v^n\big\|_{V_A}^p \\
&\quad \leq c\, C(u^0, v^0, f)
\end{aligned}$$

is valid.

(ii) Moreover, the bound

$$\tau \sum_{n=2}^{N_0} \big\|D_2 v^n\big\|_{V^*}^{p^*} + \tau \big\|D_1 v^1\big\|_{V^*}^{p^*} \leq c\, \|P_M\|_{V\leftarrow V}^{p^*}\, \widetilde{C}(u^0, v^0, f)$$

holds with $\widetilde{C}(u^0, v^0, f)$ depending on $C(u^0, v^0, f)$. Here, $P_M : H \to V_M \subset H$ denotes the H-orthogonal projection onto V_M and $\|P_M\|_{V\leftarrow V}$ its norm as a linear and bounded operator in V.

Proof In order to deduce the asserted a priori estimates, we employ the formulation (3.4) related to the first-order system (2.3).

(i) (a) *Initial step.* In the first relation in (3.4b) for the initial time step, we set $\varphi = v^1 = \tau^{-1}(u^1 - u^0) \in V_M$ to obtain

$$\tau^{-1}\big(v^1 - v^0 \,\big|\, v^1\big)_H + \big\langle A v^1 \,\big|\, v^1\big\rangle_{V_A^* \times V_A} + \tau^{-1}\big\langle B u^1 \,\big|\, u^1 - u^0\big\rangle_{V_B^* \times V_B} = \big\langle f^1 \,\big|\, v^1\big\rangle_{V_A^* \times V_A},$$

and utilise (3.8a) together with the coercivity of A to obtain

$$\begin{aligned}
&\tfrac{1}{2}\, \tau^{-1} \Big(\big\|v^1\big\|_H^2 - \big\|v^0\big\|_H^2 + \big\|v^1 - v^0\big\|_H^2 + \big\|u^1\big\|_B^2 - \big\|u^0\big\|_B^2 + \big\|u^1 - u^0\big\|_B^2\Big) \\
&\qquad + \mu_A \big\|v^1\big\|_{V_A}^p - \lambda_A \\
&\quad \leq \big\langle f^1 \,\big|\, v^1\big\rangle_{V_A^* \times V_A},
\end{aligned}$$

see also Assumption (A) and recall (2.1). Applying Young's inequality leads to

$$\big\langle f^1 \,\big|\, v^1\big\rangle_{V_A^* \times V_A} \leq \big\|f^1\big\|_{V_A^*} \big\|v^1\big\|_{V_A} \leq c\, \big\|f^1\big\|_{V_A^*}^{p^*} + \tfrac{1}{2}\, \mu_A \big\|v^1\big\|_{V_A}^p .$$

By absorption, we thus find

$$\begin{aligned}
&\big\|v^1\big\|_H^2 + \big\|v^1 - v^0\big\|_H^2 + \mu_A \tau \big\|v^1\big\|_{V_A}^p + \big\|u^1\big\|_B^2 + \big\|u^1 - u^0\big\|_B^2 \\
&\quad \leq \big\|v^0\big\|_H^2 + \big\|u^0\big\|_B^2 + 2\,\lambda_A \tau + c\,\tau \big\|f^1\big\|_{V_A^*}^{p^*} .
\end{aligned} \tag{4.2}$$

(b) *Subsequent steps and first a priori bound.* Similar arguments as before apply to the subsequent time steps. Inserting $\varphi = v^n = \tau^{-1}(\frac{3}{2}u^n - 2u^{n-1} + \frac{1}{2}u^{n-2}) \in V_M$ into the first relation in (3.4a) and employing (3.8b) as well as the coercivity condition on A leads to the estimate ($n = 2,3,\dots,N$)

$$\begin{aligned}
&\tfrac{1}{4}\,\tau^{-1}\Big(\big\|v^n\big\|_H^2 + \big\|2v^n - v^{n-1}\big\|_H^2 + \big\|u^n\big\|_B^2 + \big\|2u^n - u^{n-1}\big\|_B^2 \\
&\qquad - \big\|v^{n-1}\big\|_H^2 - \big\|2v^{n-1} - v^{n-2}\big\|_H^2 - \big\|u^{n-1}\big\|_B^2 - \big\|2u^{n-1} - u^{n-2}\big\|_B^2 \\
&\qquad + \big\|v^n - 2v^{n-1} + v^{n-2}\big\|_H^2 + \big\|u^n - 2u^{n-1} + u^{n-2}\big\|_B^2\Big) + \mu_A \big\|v^n\big\|_{V_A}^p - \lambda_A \\
&\quad \leq \big\langle f^n \big| v^n \big\rangle_{V_A^*\times V_A}\,.
\end{aligned}$$

By means of Young's inequality

$$\big\langle f^n \big| v^n \big\rangle_{V_A^*\times V_A} \leq c\,\big\|f^n\big\|_{V_A^*}^{p^*} + \tfrac{3}{4}\,\mu_A \big\|v^n\big\|_{V_A}^p$$

and absorption, we further obtain

$$\begin{aligned}
&\big\|v^n\big\|_H^2 + \big\|2v^n - v^{n-1}\big\|_H^2 + \big\|u^n\big\|_B^2 + \big\|2u^n - u^{n-1}\big\|_B^2 \\
&\qquad - \big\|v^{n-1}\big\|_H^2 - \big\|2v^{n-1} - v^{n-2}\big\|_H^2 - \big\|u^{n-1}\big\|_B^2 - \big\|2u^{n-1} - u^{n-2}\big\|_B^2 \\
&\qquad + \big\|v^n - 2v^{n-1} + v^{n-2}\big\|_H^2 + \big\|u^n - 2u^{n-1} + u^{n-2}\big\|_B^2 + \mu_A\,\tau\,\big\|v^n\big\|_{V_A}^p \\
&\quad \leq 4\,\lambda_A\,\tau + c\,\tau\,\big\|f^n\big\|_{V_A^*}^{p^*}\,.
\end{aligned}$$

Summation over $n = 2,3,\dots,N_0$ together with a telescopic identity and the stated bound (4.2) for the initial time step yields ($N_0 = 1,2,\dots,N$)

$$\begin{aligned}
&\big\|v^{N_0}\big\|_H^2 + \big\|2v^{N_0} - v^{N_0-1}\big\|_H^2 + \sum_{n=2}^{N_0} \big\|v^n - 2v^{n-1} + v^{n-2}\big\|_H^2 + \mu_A\,\tau \sum_{n=1}^{N_0} \big\|v^n\big\|_{V_A}^p \\
&\qquad + \big\|u^{N_0}\big\|_B^2 + \big\|2u^{N_0} - u^{N_0-1}\big\|_B^2 + \sum_{n=2}^{N_0} \big\|u^n - 2u^{n-1} + u^{n-2}\big\|_B^2 \\
&\quad \leq \big\|v^1\big\|_H^2 + \big\|2v^1 - v^0\big\|_H^2 + \mu_A\,\tau\,\big\|v^1\big\|_{V_A}^p + \big\|u^1\big\|_B^2 + \big\|2u^1 - u^0\big\|_B^2 + 4\,\lambda_A T \\
&\qquad + c\,\tau \sum_{n=2}^{N_0} \big\|f^n\big\|_{V_A^*}^{p^*} \\
&\quad \leq c\,\Big(\big\|v^0\big\|_H^2 + \big\|u^0\big\|_B^2 + \lambda_A T + \tau \sum_{n=1}^{N_0} \big\|f^n\big\|_{V_A^*}^{p^*}\Big)
\end{aligned}$$

and thus proves the first assertion.

(ii) (a) *Initial step.* In order to deduce a suitable estimate for the discrete first time derivative $D_1 v^1 \in V_M$ with respect to the dual norm, we employ the orthogonal projection $P_M : H \to V_M \subset H$. Noting that $D_1 v^1 \in V_M \subset H$, by the definition of the dual norm and the orthogonal projection, we find

$$\begin{aligned}
\left\|D_1 v^1\right\|_{V^*} &= \sup_{w\in V\setminus\{0\}} \tfrac{1}{\|w\|_V} \left\langle D_1 v^1 \,\middle|\, w\right\rangle_{V^*\times V} \\
&= \sup_{w\in V\setminus\{0\}} \tfrac{1}{\|w\|_V} \left(D_1 v^1 \,\middle|\, P_M w\right)_H \\
&= \sup_{w\in V\setminus\{0\}} \tfrac{1}{\|w\|_V} \left\langle f^1 - Av^1 - Bu^1 \,\middle|\, P_M w\right\rangle_{V^*\times V} \\
&\leq \sup_{w\in V\setminus\{0\}} \tfrac{\|P_M w\|_V}{\|w\|_V} \left\|f^1 - Av^1 - Bu^1\right\|_{V^*} \\
&= \|P_M\|_{V\leftarrow V} \left\|f^1 - Av^1 - Bu^1\right\|_{V^*} \\
&\leq \|P_M\|_{V\leftarrow V} \Big(\left\|f^1\right\|_{V^*} + \left\|Av^1\right\|_{V^*} + \left\|Bu^1\right\|_{V^*}\Big) \\
&\leq \|P_M\|_{V\leftarrow V} \Big(\left\|f^1\right\|_{V_A^*} + \left\|Av^1\right\|_{V_A^*} + \left\|Bu^1\right\|_{V_B^*}\Big)\,,
\end{aligned}$$

see also (3.4b) and recall the definition of $\|\cdot\|_{V^*}$. Together with the growth conditions on A and B this implies

$$\left\|D_1 v^1\right\|_{V^*} \leq c\,\|P_M\|_{V\leftarrow V} \Big(1 + \left\|f^1\right\|_{V_A^*} + \left\|v^1\right\|_{V_A}^{p-1} + \left\|u^1\right\|_{V_B}\Big)\,,$$

see Assumptions (A) and (B), and as a consequence, we obtain

$$\left\|D_1 v^1\right\|_{V^*}^{p^*} \leq c\,\|P_M\|_{V\leftarrow V}^{p^*} \Big(1 + \left\|f^1\right\|_{V_A^*}^{p^*} + \left\|v^1\right\|_{V_A}^{p} + \left\|u^1\right\|_{V_B}^{p^*}\Big)\,.$$

(b) *Subsequent steps and second a priori bound.* In an analogous manner, the bound ($n = 2,3,\dots,N$)

$$\left\|D_2 v^n\right\|_{V^*}^{p^*} \leq c\,\|P_M\|_{V\leftarrow V}^{p^*} \Big(1 + \left\|f^n\right\|_{V_A^*}^{p^*} + \left\|v^n\right\|_{V_A}^{p} + \left\|u^n\right\|_{V_B}^{p^*}\Big)$$

for the discrete time derivative $D_2 v^n \in V_M$ results. Summing up and applying the bound for the initial time step yields

$$\begin{aligned}
&\tau \sum_{n=2}^{N_0} \left\|D_2 v^n\right\|_{V^*}^{p^*} + \tau \left\|D_1 v^1\right\|_{V^*}^{p^*} \\
&\qquad \leq c\,\|P_M\|_{V\leftarrow V}^{p^*} \Big(T + \tau \sum_{n=1}^{N_0} \left\|f^n\right\|_{V_A^*}^{p^*} + \tau \sum_{n=1}^{N_0} \left\|v^n\right\|_{V_A}^{p} + \tau \sum_{n=1}^{N_0} \left\|u^n\right\|_{V_B}^{p^*}\Big)\,.
\end{aligned}$$

Altogether, invoking the first a priori bound proves the assertion. $\diamondsuit$

5 Convergence towards the weak solution

In this section, we establish a convergence result for piecewise constant and piecewise linear in time prolongations of the fully discrete solution towards the weak solution to the nonlinear second-order evolution equation.

5.1 Piecewise polynomial prolongations

Piecewise polynomial prolongations. Our construction of piecewise constant and piecewise linear in time prolongations is according to EMMRICH [13–15]. We recall that $(V_M)_{M\in\mathbb{N}}$ denotes a Galerkin scheme for the underlying Banach space V and that $N \in \mathbb{N}$ defines a constant time stepsize $\tau = \frac{T}{N} > 0$ with corresponding time grid points $(t_n)_{n=0}^N$ given by $t_n = n\tau$ for $n = 0, 1, \ldots, N$. For $(M,N) \in \mathbb{N}\times\mathbb{N}$ and certain initial approximations $(u^0, v^0) \in V_M \times V_M$, we consider the solution $(u^n, v^n)_{n=1}^N \subset V_M \times V_M$ to the full discretisation method (3.4) related to the first-order system (2.3). The piecewise constant in time interpolant $\overline{u} : [0,T] \to V_M$ as well as the piecewise linear in time prolongation $\hat{u} : [0,T] \to V_M$ associated with $(u^n)_{n=0}^N$ are defined through

$$
\begin{aligned}
\overline{u}(t) &= \begin{cases} u^1 & \text{if } t \in [0,t_1]\,, \\ u^n & \text{if } t \in (t_{n-1},t_n]\,,\ n = 2,3,\ldots,N\,, \end{cases} \\
\hat{u}(t) &= \begin{cases} \frac{3}{2}u^1 - \frac{1}{2}u^0 - (t_1 - t)\,D_1 u^1 & \text{if } t \in [0,t_1]\,, \\ \frac{3}{2}u^n - \frac{1}{2}u^{n-1} - (t_n - t)\,D_2 u^n & \text{if } t \in (t_{n-1},t_n]\,,\ n = 2,3,\ldots,N\,. \end{cases}
\end{aligned}
\tag{5.1a}
$$

We note that $\hat{u}$ does not interpolate the discrete values $(t_n, u^n)_{n=0}^N$. However, by construction, the piecewise linear prolongation is continuous on $[0,T]$ with

$$
\hat{u}(t_n) = \begin{cases} \frac{1}{2}\left(u^1 + u^0\right) & \text{if } n = 0\,, \\ \frac{3}{2}u^n - \frac{1}{2}u^{n-1} & \text{if } n = 1,2,\ldots,N\,, \end{cases}
\tag{5.1b}
$$

and weakly differentiable with

$$
\hat{u}'(t) = \begin{cases} D_1 u^1 & \text{if } t \in [0,t_1]\,, \\ D_2 u^n & \text{if } t \in (t_{n-1},t_n]\,,\ n = 2,3,\ldots,N\,. \end{cases}
\tag{5.1c}
$$

In an analogous manner, the piecewise constant and piecewise linear prolongations $\overline{v}, \hat{v} : [0,T] \to V_M$ associated with $(v^n)_{n=0}^N$ are defined through

$$
\begin{aligned}
\overline{v}(t) &= \begin{cases} v^1 & \text{if } t \in [0,t_1]\,, \\ v^n & \text{if } t \in (t_{n-1},t_n]\,,\ n = 2,3,\ldots,N\,, \end{cases} \\
\hat{v}(t) &= \begin{cases} \frac{3}{2}v^1 - \frac{1}{2}v^0 - (t_1 - t)\,D_1 v^1 & \text{if } t \in [0,t_1]\,, \\ \frac{3}{2}v^n - \frac{1}{2}v^{n-1} - (t_n - t)\,D_2 v^n & \text{if } t \in (t_{n-1},t_n]\,,\ n = 2,3,\ldots,N\,. \end{cases}
\end{aligned}
\tag{5.1d}
$$

We point out that the identity

$$
\hat{u}' = \overline{v}
\tag{5.1e}
$$

holds, due to the relations $v^1 = D_1 u^1$ and $v^n = D_2 u^n$ for $n = 2,3,\ldots,N$.

Auxiliary relation. In the proof of Theorem 5.1 we utilise the estimates

$$
\begin{aligned}
\int_0^T \big(\hat{v}'(t)\,\big|\,\overline{v}(t)\big)_H \,\mathrm{d}t &\geq \tfrac{1}{2}\Big(\big\|\hat{v}(T)\big\|_H^2 - \big\|\hat{v}(0)\big\|_H^2\Big) - \tfrac{1}{8}\,\big\|v^1 - v^0\big\|_H^2\,,\\
\int_0^T \big\langle \big(B\overline{u}\big)(t)\,\big|\,\overline{v}(t)\big\rangle_{V_B^*\times V_B} \,\mathrm{d}t &\geq \tfrac{1}{2}\Big(\big\|\hat{u}(T)\big\|_B^2 - \big\|\hat{u}(0)\big\|_B^2\Big) - \tfrac{1}{8}\,\big\|u^1 - u^0\big\|_B^2\,,
\end{aligned}
\tag{5.2}
$$

see also also EMMRICH [14].

(i) In order to deduce the first relation, we employ the reformulation

$$
\int_0^T \big(\hat{v}'(t)\,\big|\,\overline{v}(t)\big)_H \,\mathrm{d}t = \int_0^T \big(\hat{v}'(t)\,\big|\,\hat{v}(t)\big)_H \,\mathrm{d}t + \int_0^T \big(\hat{v}'(t)\,\big|\,\big(\overline{v}-\hat{v}\big)(t)\big)_H \,\mathrm{d}t
$$

of the left-hand side. Observing that, for a fixed time increment, the piecewise linear prolongation is sufficiently regular to carry out integration by parts, we obtain

$$
\int_0^T \big(\hat{v}'(t)\,\big|\,\hat{v}(t)\big)_H \,\mathrm{d}t = \tfrac{1}{2}\Big(\big\|\hat{v}(T)\big\|_H^2 - \big\|\hat{v}(0)\big\|_H^2\Big)\,.
$$

Thus, it remains to inspect the remaining contribution which equals

$$
\begin{aligned}
\int_0^T \big(\hat{v}'(t)\,\big|\,\big(\overline{v}-\hat{v}\big)(t)\big)_H \,\mathrm{d}t &= \sum_{n=2}^{N}\int_{t_{n-1}}^{t_n} \big(D_2 v^n\,\big|\,\tfrac{1}{2}\big(v^{n-1}-v^n\big) + (t_n - t)\,D_2 v^n\big)_H \,\mathrm{d}t\\
&\qquad + \int_0^\tau \big(D_1 v^1\,\big|\,\tfrac{1}{2}\big(v^0 - v^1\big) + (t_1 - t)\,D_1 v^1\big)_H \,\mathrm{d}t\\
&= \tfrac{1}{2}\tau^2 \sum_{n=2}^{N} \big(D_2 v^n\,\big|\,D_2 v^n - D_1 v^n\big)_H\\
&= \tfrac{1}{4}\sum_{n=2}^{N} \big(\tfrac{3}{2}v^n - 2v^{n-1} + \tfrac{1}{2}v^{n-2}\,\big|\,v^n - 2v^{n-1} + v^{n-2}\big)_H\\
&= \tfrac{1}{8}\sum_{n=2}^{N} \Big(\big\|v^n - v^{n-1}\big\|_H^2 - \big\|v^{n-1} - v^{n-2}\big\|_H^2\Big)\\
&\qquad + \tfrac{1}{4}\sum_{n=2}^{N} \big\|\tau^2 D^2 v^{n-1}\big\|_H^2\,,
\end{aligned}
$$

due to the identities $D_2 v^n - D_1 v^n = \frac{1}{2}\tau D^2 v^{n-1}$ and (3.8c), see also (3.2). We thus obtain

$$
\begin{aligned}
\int_0^T \big(\hat{v}'(t)\,\big|\,\big(\overline{v}-\hat{v}\big)(t)\big)_H \,\mathrm{d}t &\geq \tfrac{1}{8}\sum_{n=2}^{N} \Big(\big\|v^n - v^{n-1}\big\|_H^2 - \big\|v^{n-1} - v^{n-2}\big\|_H^2\Big)\\
&= \tfrac{1}{8}\,\Big(\big\|v^N - v^{N-1}\big\|_H^2 - \big\|v^1 - v^0\big\|_H^2\Big)\\
&\geq -\tfrac{1}{8}\,\big\|v^1 - v^0\big\|_H^2\,.
\end{aligned}
$$

(ii) Recalling the identity $\overline{v} = \hat{u}'$ and using that the operator $B : V_B \to V_B^*$ is assumed to be symmetric, similar arguments yield the second estimate.

5.2 Statement of the main result

Limiting process. In the following, we consider situations where the positive integers characterising the space and time discretisation tend to infinity. More precisely, we consider a sequence $(M_\ell, N_\ell)_{\ell \in \mathbb{N}} \subset \mathbb{N} \times \mathbb{N}$ such that $M_\ell \to \infty$ and $N_\ell \to \infty$ as $\ell \to \infty$. For this reason, we henceforth indicate the dependences of the piecewise polynomial prolongations defined in (5.1) on the discretisation parameters $(M_\ell, N_\ell) \in \mathbb{N} \times \mathbb{N}$ by writing $\overline{u}_\ell = \overline{u}_{(M_\ell, N_\ell)}$ etc. for short. Furthermore, we denote by $\tau_\ell = \frac{T}{N_\ell}$ the corresponding time stepsize and by $\big(u_\ell^0, v_\ell^0\big) \in V_{M_\ell} \times V_{M_\ell}$ the prescribed initial approximations. However, for ease of notation, we occasionally do not indicate the dependence of the associated time grid $(t_n)_{n=0}^{N_\ell}$ and the fully discrete solution values $(u^n, v^n)_{n=1}^{N_\ell} \subset V_{M_\ell} \times V_{M_\ell}$ on (M_ℓ, N_ℓ).

Assumptions. We employ the following assumptions on the discretisation parameters, the initial approximations, and the H-orthogonal projection.

Assumption (IC, P) The sequence $(M_\ell, N_\ell)_{\ell \in \mathbb{N}} \subset \mathbb{N} \times \mathbb{N}$ is chosen such that $M_\ell, N_\ell \to \infty$ as $\ell \to \infty$. The corresponding sequence $(u_\ell^0, v_\ell^0)_{\ell \in \mathbb{N}}$ of approximations to the intial values $(u_0, v_0) \in V_B \times H$ satisfies $\big(u_\ell^0, v_\ell^0\big) \in V_{M_\ell} \times V_{M_\ell}$ for any $\ell \in \mathbb{N}$ and

$$\big(u_\ell^0, v_\ell^0\big) \to (u_0, v_0) \ \text{ in } V_B \times H\,.$$

There exists a constant $c > 0$ such that for every $\ell \in \mathbb{N}$ the H-orthogonal projection onto $V_{M_\ell} \subset V \subset H$ satisfies

$$\big\| P_{M_\ell} \big\|_{V \leftarrow V} \le c\,.$$

H-orthogonal projection. It seems to be an open question which assumptions on V and H ensure the existence of a Galerkin scheme for V such that the above assumption on the projection is fulfilled. Nevertheless, this property on the projection has been studied by many authors in the context of the finite element method, see for instance BOMAN [4] and CROUZEIX, THOMÉE [10] for $V = L^p(\Omega)$ or $V = W^{1,p}(\Omega)$, BANK, YSERENTANT [2] and CARSTENSEN [6] for $V = H^1(\Omega)$, STEINBACH [29] for $V = H^s(\Omega)$ $(s \in (0,1])$, COCKBURN [8] for $V = \mathrm{BV}(\Omega)$, in all cases with $H = L^2(\Omega)$, as well as EMMRICH, ŠIŠKA [17] for $H = H^{-1}(\Omega)$, $V = L^p(\Omega)$ in the one-dimensional case. We note that if the projection is stable as a linear and bounded operator in V_A as well as in V_B then it is also stable in V.

Main result. Provided that the stated assumptions on the operators defining the nonlinear second-order evolution equation (1.1) and on the full discretisation method (1.3) are fulfilled, the following result ensures convergence of the piecewise polynomial prolongations (5.1) towards the weak solution of the problem, characterised in Theorem 2.1. Again, the proof of the statement relies on the equivalent formulation (3.4) related to the first-order system (2.3).

Theorem 5.1 (Convergence towards the weak solution) *Assume $(u_0, v_0) \in V_B \times H$ as well as $f \in L^{p^*}(0,T;V_A^*)$, and let $(\overline{u}_\ell, \overline{v}_\ell)_{\ell \in \mathbb{N}}$ and $(\hat{u}_\ell, \hat{v}_\ell)_{\ell \in \mathbb{N}}$ denote the sequences* *of piecewise constant and piecewise linear prolongations associated with the full discretisation method* (3.4)*. Provided that Assumptions (A), (B), and (IC, P) are satisfied, the sequences* $(\overline{u}_\ell)_{\ell\in\mathbb{N}}$ *and* $(\hat{u}_\ell)_{\ell\in\mathbb{N}}$ *converge weakly* in* $L^\infty(0,T;V_B)$ *towards the exact solution u to the nonlinear second-order evolution equation* (1.3)*, characterised by the regularity properties* $u\in\mathscr{C}_w([0,T];V_B)$*,* $u'\in\mathscr{C}_w([0,T];H)\cap L^p(0,T;V_A)$*, and* $u''\in L^{p^*}(0,T;V^*)$*. Moreover, the sequences* $(\overline{v}_\ell)_{\ell\in\mathbb{N}}$ *and* $(\hat{v}_\ell)_{\ell\in\mathbb{N}}$ *converge weakly* towards* u' *in* $L^\infty(0,T;H)$*; the sequence* $(\overline{v}_\ell)_{\ell\in\mathbb{N}}$ *converges also weakly towards* u' *in* $L^p(0,T;V_A)$*. Finally, the sequence* $(\hat{v}'_\ell)_{\ell\in\mathbb{N}}$ *converges weakly towards* u'' *in* $L^{p^*}(0,T;V^*)$*.*

5.3 Auxiliary results

The proof of Theorem 5.1 relies on two auxiliary results, which we deduce next. For the piecewise polynomial prolongations introduced in (5.1), we first prove the existence of weakly convergent subsequences and verify fundamental properties of the corresponding limits, see Lemma 5.2. To this purpose, we employ the results that a bounded sequence in a reflexive Banach space possesses a weakly convergent subsequence and that a bounded sequence in the dual of a separable normed space possesses a weakly* convergent subsequence, see for instance BRÉZIS [5]. As ensured by Lemma 5.3, it is also possible to establish a strong convergence result for the first iterates. In addition, in order to verify that the obtained limiting function is indeed a solution to the nonlinear second-order evolution equation, we utilise an appropriate analogue to the integration-by-parts formula, which is found in EMMRICH, THALHAMMER [20] and similarly in LIONS, STRAUSS [26]. For the convenience of the reader, we restate this result in Lemma 5.4.

Lemma 5.2 (Weak convergence of subsequences) *Under the requirements of Theorem 5.1 there exists a function* $u:[0,T]\to V_B$ *with the properties*

$$\begin{gathered}
u\in\mathscr{AC}([0,T];H)\cap\mathscr{C}_w([0,T];V_B)\,,\quad u-u_0\in\mathscr{AC}([0,T];V_A)\,,\\
u'\in\mathscr{AC}([0,T];V^*)\cap\mathscr{C}_w([0,T];H)\cap L^p(0,T;V_A)\,,\quad u''\in L^{p^*}(0,T;V^*)\,,\\
u(0)=u_0\,,\quad u'(0)=v_0\,,
\end{gathered}\tag{5.3}$$

and a subsequence $(\ell')\subset(\ell)$ *such that the corresponding sequences of piecewise constant and piecewise linear prolongations, denoted by* $(\overline{u}_{\ell'},\overline{v}_{\ell'})$ *and* $(\hat{u}_{\ell'},\hat{v}_{\ell'})$ *satisfy*

$$\begin{gathered}
\overline{u}_{\ell'},\hat{u}_{\ell'}\overset{*}{\rightharpoonup}u\ \text{ in } L^\infty(0,T;V_B)\,,\quad \hat{u}'_{\ell'}=\overline{v}_{\ell'}\overset{*}{\rightharpoonup}u'\ \text{ in } L^\infty(0,T;H)\cap L^p(0,T;V_A)\,,\\
\overline{v}_{\ell'},\hat{v}_{\ell'}\overset{*}{\rightharpoonup}u'\ \text{ in } L^\infty(0,T;H)\,,\quad \hat{v}'_{\ell'}\rightharpoonup u''\ \text{ in } L^{p^*}(0,T;V^*)\,,\\
\big(\overline{u}_{\ell'}(0),\overline{v}_{\ell'}(0)\big),\big(\hat{u}_{\ell'}(0),\hat{v}_{\ell'}(0)\big)\rightharpoonup\big(u(0),u'(0)\big)=(u_0,v_0)\ \text{ in } V_B\times H\,,\\
\big(\hat{u}_{\ell'}(T),\hat{v}_{\ell'}(T)\big)\rightharpoonup\big(u(T),u'(T)\big)\ \text{ in } V_B\times H\,,
\end{gathered}$$

as $\ell'\to\infty$*. Furthermore, there exists an element* $a\in L^{p^*}(0,T;V_A^*)$ *such that*

$$A\overline{v}_{\ell'}\rightharpoonup a\ \text{ in } L^{p^*}(0,T;V_A^*)\,,\quad B\overline{u}_{\ell'}\overset{*}{\rightharpoonup}Bu\ \text{ in } L^\infty(0,T;V_B^*)\,,$$

where $Bu\in\mathscr{C}_w([0,T];V_B^*)$*.*

Proof We first prove the existence of convergent subsequences of the sequences of prolongations (5.1) and then verify the stated regularity properties of the resulting limiting function.

(i) *Existence of convergent subsequences.* Evidently, the piecewise constant and piecewise linear prolongations to the fully discrete solutions satisfy ($\ell \in \mathbb{N}$)

$$\begin{gathered}
\left\|\overline{u}_\ell\right\|_{L^\infty(0,T;V_B)} = \max_{n=1,2,\dots,N_\ell} \left\|u^n\right\|_{V_B}, \quad \left\|\hat{u}_\ell\right\|_{L^\infty(0,T;V_B)} \leq 2 \max_{n=0,1,\dots,N_\ell} \left\|u^n\right\|_{V_B}, \\
\left\|\overline{v}_\ell\right\|_{L^\infty(0,T;H)} = \max_{n=1,2,\dots,N_\ell} \left\|v^n\right\|_{H}, \quad \left\|\hat{v}_\ell\right\|_{L^\infty(0,T;H)} \leq 2 \max_{n=0,1,\dots,N_\ell} \left\|v^n\right\|_{H}, \\
\left\|\overline{v}_\ell\right\|^p_{L^p(0,T;V_A)} = \sum_{n=1}^{N_\ell} \int_{t_{n-1}}^{t_n} \left\|\overline{v}_\ell(t)\right\|^p_{V_A} \,\mathrm{d}t = \tau_\ell \sum_{n=1}^{N_\ell} \left\|v^n\right\|^p_{V_A}, \\
\left\|\hat{v}'_\ell\right\|^{p^*}_{L^{p^*}(0,T;V^*)} = \tau_\ell \sum_{n=2}^{N_\ell} \left\|D_2 v^n\right\|^{p^*}_{V^*} + \tau_\ell \left\|D_1 v^1\right\|^{p^*}_{V^*}.
\end{gathered}$$

Regarding Theorem 4.3, we note that the sequences $(C(u^0_\ell, v^0_\ell, f))_{\ell\in\mathbb{N}}$ and $(\widetilde{C}(u^0, v^0, f))_{\ell\in\mathbb{N}}$ are bounded, since the sequence $(u^0_\ell, v^0_\ell)_{\ell\in\mathbb{N}}$ is bounded in $V_B \times H$. The a priori estimates provided by Theorem 4.3 together with Assumption (IC, P) on the initial approximations and the H-orthogonal projection thus ensure boundedness of the sequences $(\overline{u}_\ell)_{\ell\in\mathbb{N}}$ and $(\hat{u}_\ell)_{\ell\in\mathbb{N}}$ in $L^\infty(0,T;V_B)$, of $(\overline{v}_\ell)_{\ell\in\mathbb{N}}$ and $(\hat{v}_\ell)_{\ell\in\mathbb{N}}$ in $L^\infty(0,T;H)$, of $(\overline{v}_\ell)_{\ell\in\mathbb{N}}$ in $L^p(0,T;V_A)$, and of $(\hat{v}'_\ell)_{\ell\in\mathbb{N}}$ in $L^{p^*}(0,T;V^*)$. We recall that $\overline{v}_\ell = \hat{u}'_\ell$ holds. As a consequence, a common subsequence $(\ell') \subset (\ell)$ of positive integers and elements $\overline{u}, \hat{u} \in L^\infty(0,T;V_B)$, $\overline{v} \in L^\infty(0,T;H) \cap L^p(0,T;V_A)$, $\hat{v} \in L^\infty(0,T;H)$, and $w \in L^{p^*}(0,T;V^*)$ exist such that the corresponding sequences of piecewise constant and piecewise linear prolongations fulfill

$$\begin{gathered}
\overline{u}_{\ell'} \overset{*}{\rightharpoonup} \overline{u} \ \text{ in } L^\infty(0,T;V_B), \quad \hat{u}_{\ell'} \overset{*}{\rightharpoonup} \hat{u} \ \text{ in } L^\infty(0,T;V_B), \\
\overline{v}_{\ell'} \overset{*}{\rightharpoonup} \overline{v} \ \text{ in } L^\infty(0,T;H) \cap L^p(0,T;V_A), \quad \hat{v}_{\ell'} \overset{*}{\rightharpoonup} \hat{v} \ \text{ in } L^\infty(0,T;H), \\
\hat{v}'_{\ell'} \rightharpoonup w \ \text{ in } L^{p^*}(0,T;V^*).
\end{gathered}$$

(ii) *Coincidences of limiting functions.* In the sequel, we show that certain limiting functions coincide. We observe that $\overline{v} = \hat{u}'$ as well as $w = \hat{v}'$ holds, due to the identity $\hat{u}'_\ell = \overline{v}_\ell$ for $\ell \in \mathbb{N}$ and the definition of the weak time derivative. Employing the relations $D_1 u^n = D_2 u^n - \frac{1}{2}\tau_\ell D^2 u^{n-1}$ and $v^1 = D_1 u^1$ as well as $v^n = D_2 u^n$, a straightforward calculation yields ($\ell \in \mathbb{N}$)

$$\begin{aligned}
\big(\overline{u}_\ell - \hat{u}_\ell\big)(t) &= \begin{cases} \big(\frac{1}{2}t_1 - t\big)\, D_1 u^1 & \text{if } t \in [0,t_1], \\ (t_n - t)\, D_2 u^n - \frac{1}{2}\tau_\ell D_1 u^n & \text{if } t \in (t_{n-1},t_n],\ n = 2,3,\dots,N, \end{cases} \\
&= \begin{cases} \big(\frac{1}{2}t_1 - t\big)\, v^1 & \text{if } t \in [0,t_1], \\ (t_n - \frac{1}{2}\tau_\ell - t)\, v^n + \frac{1}{4}\tau_\ell^2 D^2 u^{n-1} & \text{if } t \in (t_{n-1},t_n],\ n = 2,3,\dots,N, \end{cases}
\end{aligned}$$

see also (3.2) and (3.4). The a priori estimates given in Theorem 4.3 together with the continuous embedding $V_B \hookrightarrow H$ ensure that

$$\begin{aligned}
\big\|\overline{u}_\ell - \hat{u}_\ell\big\|_{L^2(0,T;H)}^2 &= \int_0^T \big\|\big(\overline{u}_\ell - \hat{u}_\ell\big)(t)\big\|_H^2 \,\mathrm{d}t = \sum_{n=1}^{N_\ell} \int_{t_{n-1}}^{t_n} \big\|\big(\overline{u}_\ell - \hat{u}_\ell\big)(t)\big\|_H^2 \,\mathrm{d}t \\
&= \sum_{n=2}^{N_\ell} \int_{t_{n-1}}^{t_n} \big\|\big(t_n - \tfrac{1}{2}\tau_\ell - t\big)\, v^n + \tfrac{1}{4}\tau_\ell^2 D^2 u^{n-1}\big\|_H^2 \,\mathrm{d}t \\
&\quad + \int_0^{t_1} \big(\tfrac{1}{2}t_1 - t\big)^2 \big\|v^1\big\|_H^2 \,\mathrm{d}t \\
&\le c\,\tau_\ell \left(\tau_\ell^2 \sum_{n=1}^{N_\ell} \big\|v^n\big\|_H^2 + \sum_{n=2}^{N_\ell} \big\|\tau_\ell^2 D^2 u^{n-1}\big\|_B^2\right)
\end{aligned}$$

converges towards zero as $\tau_\ell \to 0$, which proves that the weak limits of $(\overline{u}_{\ell'})$ and $(\hat{u}_{\ell'})$ coincide, i.e. $\overline{u} = \hat{u}$. An analogous calculation implies ($\ell \in \mathbb{N}$)

$$\big(\overline{v}_\ell - \hat{v}_\ell\big)(t) = \begin{cases} \big(\tfrac{1}{2}t_1 - t\big)\, D_1 v^1 & \text{if } t \in [0,t_1]\,, \\ \big(t_n - \tfrac{1}{2}\tau_\ell - t\big)\, D_2 v^n + \tfrac{1}{4}\tau_\ell^2 D^2 v^{n-1} & \text{if } t \in (t_{n-1},t_n]\,,\ n = 2,3,\dots,N\,. \end{cases}$$

As $p \in [2,\infty)$ and $p^* \in (1,2]$ holds by assumption, Hölder's inequality and the continuous embedding $H \hookrightarrow V^*$ yield ($\ell \in \mathbb{N}$)

$$\begin{aligned}
\big\|\overline{v}_\ell - \hat{v}_\ell\big\|_{L^{p^*}(0,T;V^*)}^{p^*} &= \int_0^T \big\|\big(\overline{v}_\ell - \hat{v}_\ell\big)(t)\big\|_{V^*}^{p^*} \,\mathrm{d}t = \sum_{n=1}^{N_\ell} \int_{t_{n-1}}^{t_n} \big\|\big(\overline{v}_\ell - \hat{v}_\ell\big)(t)\big\|_{V^*}^{p^*} \,\mathrm{d}t \\
&\le c\,\tau_\ell^{p^*} \left(\tau_\ell \sum_{n=2}^{N_\ell} \big\|D_2 v^n\big\|_{V^*}^{p^*} + \tau_\ell \big\|D_1 v^1\big\|_{V^*}^{p^*}\right) + c\,\tau_\ell \sum_{n=2}^{N_\ell} \big\|\tau_\ell^2 D^2 v^{n-1}\big\|_{V^*}^{p^*} \\
&\le c\,\tau_\ell^{p^*} \left(\tau_\ell \sum_{n=2}^{N_\ell} \big\|D_2 v^n\big\|_{V^*}^{p^*} + \tau_\ell \big\|D_1 v^1\big\|_{V^*}^{p^*}\right) + c\,\tau_\ell^{\frac{1}{2}p^*} \left(\sum_{n=2}^{N_\ell} \big\|\tau_\ell^2 D^2 v^{n-1}\big\|_H^2\right)^{\frac{1}{2}p^*}\,.
\end{aligned}$$

A further application of Theorem 4.3 and the assumption on the orthogonal projection imply $\overline{v}_\ell - \hat{v}_\ell \to 0$ in $L^{p^*}(0,T;V^*)$ as $\tau_\ell \to 0$, which shows that the weak limits of $(\overline{v}_{\ell'})$ and $(\hat{v}_{\ell'})$ coincide such that $\overline{v} = \hat{v}$. Altogether, this implies

$$\overline{u} = \hat{u}\,, \quad \overline{u}' = \hat{u}' = \overline{v} = \hat{v}\,, \quad \overline{u}'' = \hat{u}'' = \overline{v}' = \hat{v}' = w\,.$$

(iii) *Regularity of limiting function.* So far, we know that the limiting function $u := \overline{u} = \hat{u}$ fulfills

$$u \ \text{ in } L^\infty(0,T;V_B)\,, \quad u' \ \text{ in } L^\infty(0,T;H) \cap L^p(0,T;V_A)\,, \quad u'' \ \text{ in } L^{p^*}(0,T;V^*)\,.$$

Recalling once more the continuous embeddings $V_B \hookrightarrow H$ as well as $V^* \hookrightarrow H$ and applying the embeddings $W^{1,1}(0,T;H) \subset \mathscr{AC}([0,T];H)$ and $W^{1,1}(0,T;V^*) \subset \mathscr{AC}([0,T];V^*)$ as well as LIONS, MAGENES [25, Ch. 3, Lemme 8.1], we conclude that

$$u \in \mathscr{AC}([0,T];H) \cap \mathscr{C}_w([0,T];V_B)\,, \quad u' \in \mathscr{AC}([0,T];V^*) \cap \mathscr{C}_w([0,T];H)\,,$$

which further implies

$$u - u_0 = \int_0^{(\cdot)} u'(s)\,\mathrm{d}s \in L^\infty(0,T;V_A)$$

with $(u-u_0)' \in L^p(0,T;V_A)$ and thus $u - u_0 \in \mathscr{AC}([0,T];V_A)$.

(iv) *Initial values.* We first show that $u(0)$ coincides with the prescribed initial value u_0. For this purpose, we rewrite the difference $u_0 - u(0)$ as ($\ell \in \mathbb{N}$)

$$u_0 - u(0) = u_0 - u_\ell^0 + u_\ell^0 - \hat{u}_\ell(0) + \hat{u}_\ell(0) - u(0)\,.$$

By Assumption (IC, P), the contribution $u_0 - u_\ell^0$ converges strongly towards zero in $V_B \hookrightarrow H$ as $\ell \to \infty$, and, with the help of the first a priori estimate in Theorem 4.3, we obtain

$$\big\|u_\ell^0 - \hat{u}_\ell(0)\big\|_H = \tfrac{1}{2}\tau_\ell\,\big\|v^1\big\|_H \to 0\,.$$

To justify that $\hat{u}_\ell(0) - u(0)$ converges weakly towards zero in H, we consider the continuous embedding $W^{1,1}(0,T;H) \hookrightarrow \mathscr{C}([0,T];H)$, which implies that the trace operator

$$\Gamma_0 : W^{1,1}(0,T;H) \longrightarrow H : g \longmapsto \Gamma_0 g = g(0)$$

is linear as well as bounded and thus weakly-weakly-continuous. Due to the fact that the sequence $(\hat{u}_{\ell'})$ converges weakly* towards u in $L^\infty(0,T;V_B)$ and that $(\hat{u}_{\ell'}')$ converges weakly* towards u' in $L^\infty(0,T;H)$, we find that $(\hat{u}_{\ell'})$ converges weakly towards u in $W^{1,1}(0,T;H)$, and as a consequence we find $\hat{u}_{\ell'}(0) \rightharpoonup u(0)$ in H. Altogether, these considerations yield $u(0) = u_0$. A further application of Theorem 4.3 ensures that there exists an element $\xi \in V_B$ and a subsequence of (ℓ'), again denoted by (ℓ'), such that

$$\hat{u}_{\ell'}(0) = \tfrac{1}{2}\big(u_{\ell'}^1 + u_{\ell'}^0\big) \rightharpoonup \xi \ \text{ in } V_B\,.$$

This limit ξ, however, has to coincide with $u(0) = u_0$.

Similar arguments are used to verify $u'(0) = v_0$. Namely, rewriting the difference $v_0 - u'(0)$ as ($\ell \in \mathbb{N}$)

$$v_0 - u'(0) = v_0 - v_\ell^0 + v_\ell^0 - \hat{v}_\ell(0) + \hat{v}_\ell(0) - u'(0)\,,$$

Assumption (IC, P) ensures that $v_0 - v_\ell^0$ converges strongly towards zero in H as $\ell \to \infty$, and an application of Theorem 4.3 yields

$$\big\|v_\ell^0 - \hat{v}_\ell(0)\big\|_{V^*} = \tfrac{1}{2}\tau_\ell\,\big\|D_1 v^1\big\|_{V^*} = \tfrac{1}{2}\,\tau_\ell^{\frac{1}{p}}\,\Big(\tau_\ell\,\big\|D_1 v^1\big\|_{V^*}^{p^*}\Big)^{\frac{1}{p^*}} \to 0\,.$$

In the present situation, we utilise the continuous embedding $W^{1,1}(0,T;V^*) \hookrightarrow \mathscr{C}([0,T];V^*)$, which shows that the associated trace operator $\Gamma_0 : W^{1,1}(0,T;V^*) \to V^* : g \mapsto \Gamma_0 g = g(0)$ is linear as well as bounded and thus weakly-weakly-continuous. As $(\hat{v}_{\ell'})$ converges weakly* towards u' in $L^\infty(0,T;H)$ and $(\hat{v}_{\ell'}')$ converges weakly towards u'' in $L^{p^*}(0,T;V^*)$, we conclude that $(\hat{v}_{\ell'})$ converges weakly towards u' in $W^{1,1}(0,T;V^*)$ such that $\hat{v}_{\ell'}(0) \rightharpoonup u'(0)$ in V^*, which implies $u'(0) = v_0$. Moreover, for a subsequence of (ℓ'), again denoted by (ℓ'), this yields

$$\hat{v}_{\ell'}(0) = \tfrac{1}{2}\big(v_{\ell'}^1 + v_{\ell'}^0\big) \rightharpoonup u'(0) = v_0 \ \text{ in } H\,.$$

As by assumption $u_\ell^0 \to u_0$ in V_B and $v_\ell^0 \to v_0$ in H, it immediately follows $\overline{u}_{\ell'}(0) = u_{\ell'}^1 = 2\hat{u}_{\ell'}(0) - u_{\ell'}^0 \rightharpoonup u_0$ in V_B and $\overline{v}_{\ell'}(0) = v_{\ell'}^1 = 2\hat{v}_{\ell'}(0) - v_{\ell'}^0 \rightharpoonup v_0$ in H.

(v) *Final values.* Passing again to a subsequence $\big(\hat{u}_{\ell'}, \hat{v}_{\ell'}\big)$ if necessary, the a priori estimates given in Theorem 4.3 ensure the existence of an element $(\xi, \zeta) \in V_B \times H$ such that

$$\big(\hat{u}_\ell(T), \hat{v}_\ell(T)\big) \rightharpoonup (\xi, \zeta) \ \text{ in } V_B \times H\,.$$

Employing the trace operators $W^{1,1}(0,T;H) \to H : g \mapsto g(T)$ and $W^{1,1}(0,T;V^*) \to V^* : g \mapsto g(T)$, which evaluate a function at the final time, we find that

$$\big(\hat{u}_{\ell'}(T), \hat{v}_{\ell'}(T)\big) \rightharpoonup (\xi, \zeta) = \big(u(T), u'(T)\big) \ \text{ in } H \times V^*\,.$$

(vi) *Application of A and B.* With the help of the growth condition on the nonlinear operator $A : V_A \to V_A^*$ and Theorem 4.3, we observe that the sequence $(A\overline{v}_\ell)_{\ell \in \mathbb{N}}$ remains bounded in $L^{p^*}(0,T;V_A^*)$, since ($\ell \in \mathbb{N}$)

$$\begin{aligned} \big\|A\overline{v}_\ell\big\|_{L^{p^*}(0,T;V_A^*)}^{p^*} &= \int_0^T \big\|A\overline{v}_\ell(t)\big\|_{V_A^*}^{p^*} \,\mathrm{d}t = \tau_\ell \sum_{n=1}^{N_\ell} \big\|Av^n\big\|_{V_A^*}^{p^*} \leq c\,\tau_\ell \sum_{n=1}^{N_\ell} \Big(1 + \|v\|_{V_A}^{p-1}\Big)^{p^*} \\ &\leq c\,T + c\,\tau_\ell \sum_{n=1}^{N_\ell} \|v\|_{V_A}^{p}\,, \end{aligned}$$

see also Assumption (A). This implies that there exists an element $a \in L^{p^*}(0,T;V_A^*)$ and a subsequence of (ℓ'), again denoted by (ℓ'), such that $A\overline{v}_{\ell'} \rightharpoonup a$ in $L^{p^*}(0,T;V_A^*)$.

It remains to verify weak* convergence of $(B\overline{u}_{\ell'})$ towards Bu in $L^\infty(0,T;V_B^*)$. For this purpose, we utilise the symmetry of B, which leads to the identity ($\ell \in \mathbb{N}$)

$$\int_0^T \big\langle (B\overline{u}_\ell)(t) \big| w(t)\big\rangle_{V_B^* \times V_B} \,\mathrm{d}t = \int_0^T \big\langle (Bw)(t) \big| \overline{u}_\ell(t)\big\rangle_{V_B^* \times V_B} \,\mathrm{d}t\,,$$

valid for any $w \in L^1(0,T;V_B)$ such that $Bw \in L^1(0,T;V_B^*)$. Utilising that for a certain subsequence, again denoted by (ℓ'), $(\overline{u}_{\ell'})$ converges weakly* to u in $L^\infty(0,T;V_B)$ and identifying the dual of $L^1(0,T;V_B^*)$ with $L^\infty(0,T;V_B^{**}) = L^\infty(0,T;V_B)$, this implies

$$\begin{aligned} \big\langle B\overline{u}_{\ell'} \big| w\big\rangle_{L^\infty(0,T;V_B^*) \times L^1(0,T;V_B)} &= \int_0^T \big\langle (B\overline{u}_{\ell'})(t) \big| w(t)\big\rangle_{V_B^* \times V_B} \,\mathrm{d}t \\ &= \int_0^T \big\langle (Bw)(t) \big| \overline{u}_{\ell'}(t)\big\rangle_{V_B^* \times V_B} \,\mathrm{d}t = \big\langle \overline{u}_{\ell'} \big| Bw\big\rangle_{L^\infty(0,T;V_B) \times L^1(0,T;V_B^*)} \end{aligned}$$

and further shows that

$$\big\langle B\overline{u}_{\ell'} - Bu \big| w\big\rangle_{L^\infty(0,T;V_B^*) \times L^1(0,T;V_B)} = \big\langle \overline{u}_{\ell'} - u \big| Bw\big\rangle_{L^\infty(0,T;V_B) \times L^1(0,T;V_B^*)}$$

converges to zero as $\ell' \to \infty$ such that $B\overline{u}_{\ell'} \overset{*}{\rightharpoonup} Bu$ in $L^\infty(0,T;V_B^*)$. Altogether, this concludes the proof. $\diamondsuit$

Remark. Under the additional assumption

$$\sup_{\ell \in \mathbb{N}} \tau_\ell \left\|v_\ell^0\right\|_{V_A}^p < \infty\,,$$

it can easily be shown that also the subsequence $(\hat{v}_{\ell'})$ converges weakly towards u' in $L^p(0,T;V_A)$. As V_A is dense in $H \ni v_0$, this assumption can always be fulfilled.

Strong convergence of first iterates. In the present situation, contrary to the convergence analysis given in EMMRICH, THALHAMMER [20] for a full discretisation method based on the one-step backward differentiation formula, we also need to prove strong convergence of the first iterates towards the initial approximations. This is related to EMMRICH [14], where the two-step backward differentiation formula has been studied for the time discretisation of non-Newtonian fluid flows. For clarity, we meanwhile indicate the dependence of the fully discrete solution values on $(M_\ell, N_\ell) \subset \mathbb{N} \times \mathbb{N}$ for some $\ell \in \mathbb{N}$, that is, we write $(u_\ell^n, v_\ell^n)_{n=0}^{N_\ell} \subset V_{M_\ell} \times V_{M_\ell}$ for short.

Lemma 5.3 (Strong convergence of first iterates) *In the situation of Lemma 5.2, there exists a subsequence of (ℓ'), again denoted by (ℓ'), such that*

$$\left(u_{\ell'}^1 - u_{\ell'}^0, v_{\ell'}^1 - v_{\ell'}^0\right) \to 0 \ \text{ in } V_B \times H$$

as $\ell' \to \infty$. In particular, this implies

$$\left(\hat{u}_{\ell'}(0), \hat{v}_{\ell'}(0)\right) \to \left(u(0), u'(0)\right) = (u_0, v_0) \ \text{ in } V_B \times H$$

as $\ell' \to \infty$.

Proof The assertion is proven in two steps, showing first weak convergence and then strong convergence.

(i) *Weak convergence.* By Assumption (IC, P) the sequence of initial approximations $(u_\ell^0, v_\ell^0)_{\ell \in \mathbb{N}}$ converges in $V_B \times H$ and thus is bounded in $V_B \times H$. Together with the first a priori estimate in Theorem 4.3 this ensures that the sequence $(u_\ell^1 - u_\ell^0, v_\ell^1 - v_\ell^0)_{\ell \in \mathbb{N}}$ remains bounded in $V_B \times H$, which further implies that there exists an element $(\xi, \zeta) \in V_B \times H$ and a subsequence, denoted by $(u_{\ell'}^1 - u_{\ell'}^0, v_{\ell'}^1 - v_{\ell'}^0)$, such that

$$(u_{\ell'}^1 - u_{\ell'}^0, v_{\ell'}^1 - v_{\ell'}^0) \rightharpoonup (\xi, \zeta) \ \text{ in } V_B \times H\,.$$

On the other hand, Theorem 4.3 implies that the quantities

$$\left\|u_\ell^1 - u_\ell^0\right\|_H = \tau_\ell \left\|v_\ell^1\right\|_H, \quad \left\|v_\ell^1 - v_\ell^0\right\|_{V^*} = \tau_\ell \left\|D_1 v_\ell^1\right\|_{V^*} = \tau_\ell^{\frac{1}{p}} \left(\tau_\ell \left\|D_1 v_\ell^1\right\|_{V^*}^{p^*}\right)^{\frac{1}{p^*}}$$

converge to zero as $\tau_\ell \to 0$, which yields $(\xi, \zeta) = (0, 0)$.

(ii) *Strong convergence.* In order to show that even strong convergence takes place in $V_B \times H$, we employ Assumptions (A) and (B).

(a) Let $w \in V_A$. By the definition of the initial approximation f_ℓ^1 and an application of Hölder's inequality, we obtain ($\ell \in \mathbb{N}$)

$$\begin{aligned}
\tau_\ell \left\langle f_\ell^1 \,\middle|\, v_\ell^1 \right\rangle_{V_A^* \times V_A} &\le \tau_\ell \left\|f_\ell^1\right\|_{V_A^*} \left\|v_\ell^1\right\|_{V_A} = \tau_\ell^{\frac{1}{p^*}} \left\|f_\ell^1\right\|_{V_A^*} \left(\tau_\ell \left\|v_\ell^1\right\|_{V_A}^p\right)^{\frac{1}{p}} \\
&\le \tau_\ell^{\frac{1}{p^*}-1} \int_0^{\tau_\ell} \left\|f(s)\right\|_{V_A^*} \mathrm{d}s \left(\tau_\ell \left\|v_\ell^1\right\|_{V_A}^p\right)^{\frac{1}{p}} \le \left\|f\right\|_{L^{p^*}(0,\tau_\ell;V_A^*)} \left(\tau_\ell \left\|v_\ell^1\right\|_{V_A}^p\right)^{\frac{1}{p}},
\end{aligned}$$

see also (3.3). By the required monotonicity of A and the growth condition on A, the estimate ($\ell \in \mathbb{N}$)

$$\begin{aligned}
-\,\tau_\ell \left\langle A v_\ell^1 \,\middle|\, v_\ell^1 \right\rangle_{V_A^*\times V_A} &\leq \tau_\ell \left\langle A v_\ell^1 - Aw \,\middle|\, v_\ell^1 - w \right\rangle_{V_A^*\times V_A} - \tau_\ell \left\langle A v_\ell^1 \,\middle|\, v_\ell^1 \right\rangle_{V_A^*\times V_A} \\
&= -\,\tau_\ell \left\langle Aw \,\middle|\, v_\ell^1 - w \right\rangle_{V_A^*\times V_A} - \tau_\ell \left\langle A v_\ell^1 \,\middle|\, w \right\rangle_{V_A^*\times V_A} \\
&\leq \tau_\ell \left\| Aw \right\|_{V_A^*} \left\| v_\ell^1 - w \right\|_{V_A} + \tau_\ell \left\| A v_\ell^1 \right\|_{V_A^*} \|w\|_{V_A} \\
&\leq \tau_\ell^{\frac{1}{p^*}} \left\| Aw \right\|_{V_A^*} \left(\tau_\ell \left\| v_\ell^1 - w \right\|_{V_A}^p \right)^{\frac{1}{p}} + c\,\tau_\ell \left(1 + \|v_\ell^1\|_{V_A}^{p-1} \right) \|w\|_{V_A} \\
&= \tau_\ell^{\frac{1}{p^*}} \left\| Aw \right\|_{V_A^*} \left(\tau_\ell \left\| v_\ell^1 - w \right\|_{V_A}^p \right)^{\frac{1}{p}} + c\,\tau_\ell \|w\|_{V_A} + c\,\tau_\ell^{\frac{1}{p}} \left(\tau_\ell \left\| v_\ell^1 \right\|_{V_A}^p \right)^{\frac{1}{p^*}}
\end{aligned}$$

follows. We further note that the positivity of B implies ($\ell \in \mathbb{N}$)

$$\begin{aligned}
-\left\langle B u_\ell^1 \,\middle|\, u_\ell^1 - u_\ell^0 \right\rangle_{V_B^*\times V_B} &= -\left\langle B\big(u_\ell^1 - u_\ell^0\big) \,\middle|\, u_\ell^1 - u_\ell^0 \right\rangle_{V_B^*\times V_B} - \left\langle B u_\ell^0 \,\middle|\, u_\ell^1 - u_\ell^0 \right\rangle_{V_B^*\times V_B} \\
&\leq -\left\langle B u_\ell^0 \,\middle|\, u_\ell^1 - u_\ell^0 \right\rangle_{V_B^*\times V_B}.
\end{aligned}$$

(b) On the one hand, employing the defining relations for the initial iterates as well as the bounds deduced before, we find that ($\ell \in \mathbb{N}$)

$$\begin{aligned}
\left\| v_\ell^1 - v_\ell^0 \right\|_H^2 &= \big(v_\ell^1 - v_\ell^0 \,\big|\, v_\ell^1 \big)_H - \big(v_\ell^1 - v_\ell^0 \,\big|\, v_\ell^0 \big)_H \\
&= \tau_\ell \left\langle f_\ell^1 \,\middle|\, v_\ell^1 \right\rangle_{V_A^*\times V_A} - \tau_\ell \left\langle A v_\ell^1 \,\middle|\, v_\ell^1 \right\rangle_{V_A^*\times V_A} - \left\langle B u_\ell^1 \,\middle|\, u_\ell^1 - u_\ell^0 \right\rangle_{V_B^*\times V_B} - \big(v_\ell^1 - v_\ell^0 \,\big|\, v_\ell^0 \big)_H \\
&\leq \|f\|_{L^{p^*}(0,\tau_\ell;V_A^*)} \left(\tau_\ell \left\| v_\ell^1 \right\|_{V_A}^p \right)^{\frac{1}{p}} + \tau_\ell^{\frac{1}{p^*}} \left\| Aw \right\|_{V_A^*} \left(\tau_\ell \left\| v_\ell^1 - w \right\|_{V_A}^p \right)^{\frac{1}{p}} + c\,\tau_\ell \|w\|_{V_A} \\
&\qquad + c\,\tau_\ell^{\frac{1}{p}} \left(\tau_\ell \left\| v_\ell^1 \right\|_{V_A}^p \right)^{\frac{1}{p^*}} - \left\langle B u_\ell^0 \,\middle|\, u_\ell^1 - u_\ell^0 \right\rangle_{V_B^*\times V_B} - \big(v_\ell^1 - v_\ell^0 \,\big|\, v_\ell^0 \big)_H,
\end{aligned}$$

see also (3.4b) and the fundamental bounds for the first iterates $(v_\ell^1)_{\ell\in\mathbb{N}}$ provided by Theorem 4.3. We note that by assumption $f \in L^{p^*}(0,T;V_A^*)$ such that $\|f\|_{L^{p^*}(0,\tau_\ell;V_A^*)} \to 0$ as $\tau_\ell \to 0$. The above considerations ensure weak convergence of a subsequence $(u_{\ell'}^1 - u_{\ell'}^0)$ towards zero in V_B, and by assumption the sequence $(u_\ell^0)_{\ell\in\mathbb{N}}$ converges strongly towards u_0 in V_B such that in particular $\langle B u_{\ell'}^0 \,|\, u_{\ell'}^1 - u_{\ell'}^0 \rangle_{V_B^*\times V_B} \to 0$ as $\ell' \to \infty$. Furthermore, $(v_{\ell'}^1 - v_{\ell'}^0)$ converges weakly towards zero in H, and by assumption the sequence $(v_\ell^0)_{\ell\in\mathbb{N}}$ converges strongly towards v_0 in H. Altogether, this proves $\|v_{\ell'}^1 - v_{\ell'}^0\|_H \to 0$ as $\ell' \to \infty$.

On the other hand, utilising in addition the equivalence of the norms $\|\cdot\|_{V_B}$ and $\|\cdot\|_B$ defined in (2.1), we obtain ($\ell \in \mathbb{N}$)

$$\begin{aligned}
c^{-1}\,\big\|u_\ell^1-u_\ell^0\big\|_{V_B}^2 &\leq \big\|u_\ell^1-u_\ell^0\big\|_B^2 = \big\langle B\left(u_\ell^1-u_\ell^0\right)\big|\, u_\ell^1-u_\ell^0\big\rangle_{V_B^*\times V_B}\\
&= \big\langle Bu_\ell^1\big|\, u_\ell^1-u_\ell^0\big\rangle_{V_B^*\times V_B} - \big\langle Bu_\ell^0\big|\, u_\ell^1-u_\ell^0\big\rangle_{V_B^*\times V_B}\\
&= \tau_\ell\,\big\langle Bu_\ell^1\big|\, v_\ell^1\big\rangle_{V_B^*\times V_B} - \big\langle Bu_\ell^0\big|\, u_\ell^1-u_\ell^0\big\rangle_{V_B^*\times V_B}\\
&= \tau_\ell\,\big\langle f_\ell^1\big|\, v_\ell^1\big\rangle_{V_A^*\times V_A} - \big(v_\ell^1-v_\ell^0\big|\, v_\ell^1\big)_H - \tau_\ell\,\big\langle Av_\ell^1\big|\, v_\ell^1\big\rangle_{V_A^*\times V_A} - \big\langle Bu_\ell^0\big|\, u_\ell^1-u_\ell^0\big\rangle_{V_B^*\times V_B}\\
&\leq \|f\|_{L^{p^*}(0,\tau_\ell;V_A^*)}\,\Big(\tau_\ell\,\big\|v_\ell^1\big\|_{V_A}^p\Big)^{\frac{1}{p}} + \tau_\ell^{\frac{1}{p^*}}\,\|Aw\|_{V_A^*}\,\Big(\tau_\ell\,\big\|v_\ell^1-w\big\|_{V_A}^p\Big)^{\frac{1}{p}} + c\,\tau_\ell\,\|w\|_{V_A}\\
&\qquad + c\,\tau_\ell^{\frac{1}{p}}\,\Big(\tau_\ell\,\big\|v_\ell^1\big\|_{V_A}^p\Big)^{\frac{1}{p^*}} - \big\langle Bu_\ell^0\big|\, u_\ell^1-u_\ell^0\big\rangle_{V_B^*\times V_B} - \big(v_\ell^1-v_\ell^0\big|\, v_\ell^1\big)_H
\end{aligned}$$

for some $c>0$. Combining the above arguments with the established strong convergence of a subsequence $(v_{\ell'}^1-v_{\ell'}^0)$ towards zero in H thus implies $\|u_{\ell'}^1-u_{\ell'}^0\|_{V_B}\to 0$.

(c) Moreover, making use of the fact that the identities ($\ell \in \mathbb{N}$)

$$\hat{u}_\ell(0) = \tfrac{1}{2}\left(u_\ell^1+u_\ell^0\right) = u_\ell^0 + \tfrac{1}{2}\left(u_\ell^1-u_\ell^0\right), \quad \hat{v}_\ell(0) = v_\ell^0 + \tfrac{1}{2}\left(v_\ell^1-v_\ell^0\right)$$

hold, strong convergence of the subsequence $(\hat{u}_{\ell'}(0),\hat{v}_{\ell'}(0))$ towards $(u(0),u'(0)) = (u_0,v_0)$ in $V_B\times H$ is an immediate consequence of the former considerations. $\diamondsuit$

Integration-by-parts formula. A particular integration-by-parts formula is deduced in EMMRICH, THALHAMMER [20], see also LIONS, STRAUSS [26]. For the convenience of the reader, we restate the result, adapted to the present situation.

Lemma 5.4 *([20, Lemma 6]) Provided that $u_0\in V_B$ and $w\in L^p(0,T;V_A)$ satisfies*

$$Kw = \int_0^{(\cdot)} w(s)\,\mathrm{d}s \in L^2(0,T;V_B)\,, \quad w' + B(u_0+Kw) \in L^{p^*}(0,T;V_A^*)\,,$$

the identity

$$\begin{aligned}
&\int_\alpha^\beta \big\langle \big(w'+B(u_0+Kw)\big)(t)\,\big|\,w(t)\big\rangle_{V_A^*\times V_A}\,\mathrm{d}t\\
&\quad = \tfrac{1}{2}\Big(\big\|w(\beta)\big\|_H^2 - \big\|w(\alpha)\big\|_H^2 + \big\|u_0+(Kw)(\beta)\big\|_B^2 - \big\|u_0+(Kw)(\alpha)\big\|_B^2\Big)
\end{aligned}$$

holds for almost all $\alpha,\beta\in[0,T]$ with $\alpha<\beta$. If in addition $w\in\mathscr{C}_w([0,T];H)$ and $Kw\in\mathscr{C}_w([0,T];V_B)$ is fulfilled, the relation

$$\begin{aligned}
&\int_0^\beta \big\langle \big(w'+B(u_0+Kw)\big)(t)\,\big|\,w(t)\big\rangle_{V_A^*\times V_A}\,\mathrm{d}t\\
&\quad \leq \tfrac{1}{2}\Big(\big\|w(\beta)\big\|_H^2 - \big\|w(0)\big\|_H^2 + \big\|u_0+(Kw)(\beta)\big\|_B^2 - \big\|u_0\big\|_B^2\Big)
\end{aligned}$$

is valid for almost all $\beta\in[0,T]$.

5.4 Proof of the main result

We are now ready to prove the main result.

Proof (of Theorem 5.1) For a suitable subsequence $(\ell') \subset (\ell)$, Lemma 5.2 and 5.3 ensure the existence of a limiting function with the properties specified in (5.3). It remains to verify that this limiting function is a solution to the nonlinear second-order evolution equation (1.1) or equivalently to the first-order system (2.3).

(i) *Reformulation of numerical scheme.* Our starting point is the full discretisation scheme (3.4) related to the first-order system. By means of the piecewise constant and piecewise linear prolongations $(\overline{u}_\ell, \overline{v}_\ell)_{\ell\in\mathbb{N}}$ and $(\hat{u}_\ell, \hat{v}_\ell)_{\ell\in\mathbb{N}}$, making use of the identity $\hat{u}_\ell' = \overline{v}_\ell$, the numerical scheme may be rewritten as $(\ell \in \mathbb{N})$

$$\big(\hat{v}_\ell'(t)\,\big|\,\varphi\big)_H + \big\langle \big(A\overline{v}_\ell\big)(t)\,\big|\,\varphi\big\rangle_{V_A^*\times V_A} + \big\langle \big(B\overline{u}_\ell\big)(t)\,\big|\,\varphi\big\rangle_{V_B^*\times V_B} = \big\langle \overline{f}_\ell(t)\,\big|\,\varphi\big\rangle_{V_A^*\times V_A} \tag{5.4}$$

for almost all $t \in [0,T]$ and all $\varphi \in V_{M_\ell}$. Here, the piecewise constant prolongation $(\overline{f}_\ell)_{\ell\in\mathbb{N}}$ is defined analogously to (5.1). Testing with $\psi \in \mathscr{C}_c^\infty((0,T);\mathbb{R})$ and applying integration by parts leads to the relation $(\ell \in \mathbb{N})$

$$\begin{aligned} -\int_0^T \big(\hat{v}_\ell(t)\,\big|\,\varphi\big)_H\, \psi'(t)\,\mathrm{d}t + \int_0^T \big\langle \big(A\overline{v}_\ell\big)(t)\,\big|\,\varphi\big\rangle_{V_A^*\times V_A} \psi(t)\,\mathrm{d}t \\ + \int_0^T \big\langle \big(B\overline{u}_\ell\big)(t)\,\big|\,\varphi\big\rangle_{V_B^*\times V_B} \psi(t)\,\mathrm{d}t = \int_0^T \big\langle \overline{f}_\ell(t)\,\big|\,\varphi\big\rangle_{V_A^*\times V_A} \psi(t)\,\mathrm{d}t\,. \end{aligned}$$

This relation holds for all $\varphi \in V_{M_\ell}$ and thus also for all $\varphi \in V_k$ with $k \le M_\ell$.

(ii) *Relation for limiting function.* We note that, by standard arguments, the sequence of piecewise constant prolongations $(\overline{f}_\ell)_{\ell\in\mathbb{N}}$ converges strongly towards f in $L^{p^*}(0,T;V_A^*)$, see also (3.3). Fixing k and passing to the limit along the subsequence as specified in Lemma 5.2 and 5.3 thus yields

$$\begin{aligned} -\int_0^T \big(u'(t)\,\big|\,\varphi\big)_H\, \psi'(t)\,\mathrm{d}t + \int_0^T \Big(\big\langle a(t)\,\big|\,\varphi\big\rangle_{V_A^*\times V_A} + \big\langle (Bu)(t)\,\big|\,\varphi\big\rangle_{V_B^*\times V_B}\Big)\, \psi(t)\,\mathrm{d}t \\ = \int_0^T \big\langle f(t)\,\big|\,\varphi\big\rangle_{V_A^*\times V_A} \psi(t)\,\mathrm{d}t\,, \end{aligned}$$

which now holds for all $\varphi \in \bigcup_{k\in\mathbb{N}} V_k$. Because of the limited completeness of the Galerkin scheme, this relation indeed holds for all $\varphi \in V$. Due to the fact that by assumption $f \in L^{p^*}(0,T;V_A^*)$ and by Lemma 5.2 the limiting functions satisfy $a \in L^{p^*}(0,T;V_A^*)$, $Bu \in L^\infty(0,T;V_B^*)$, $u' \in L^p(0,T;V_A)$, and $u'' \in L^{p^*}(0,T;V^*)$, this shows, by definition of the weak time derivative, that $f - a - Bu \in L^{p^*}(0,T;V_A^*) + L^\infty(0,T;V_B^*) \subset L^{p^*}(0,T;V^*)$ coincides with $u'' \in L^{p^*}(0,T;V^*)$, that is $u'' = f - a - Bu$ in $L^{p^*}(0,T;V^*)$. Moreover, using that $u'' + Bu = f - a \in L^{p^*}(0,T;V_A^*)$, we obtain

$$u'' + a + Bu = f \;\text{ in } L^{p^*}(0,T;V_A^*)\,. \tag{5.5}$$

(iii) *Identification of a.* In order to identify the element $a \in L^{p^*}(0,T;V_A^*)$, we set $\varphi = \overline{v}_\ell = \hat{u}_\ell'$ in (5.4), integrate, and utilise the monotonicity of A to obtain ($\ell \in \mathbb{N}$)

$$\begin{aligned}
&\int_0^T \Big(\big\langle \overline{f}_\ell(t) \big| \overline{v}_\ell(t) \big\rangle_{V_A^* \times V_A} - \big(\hat{v}_\ell'(t) \big| \overline{v}_\ell(t) \big)_H - \big\langle \big(B\overline{u}_\ell\big)(t) \big| \overline{v}_\ell(t) \big\rangle_{V_B^* \times V_B} \Big) \,\mathrm{d}t \\
&\quad = \int_0^T \big\langle \big(A\overline{v}_\ell\big)(t) \big| \overline{v}_\ell(t) \big\rangle_{V_A^* \times V_A} \,\mathrm{d}t \\
&\quad \geq \int_0^T \Big(\big\langle \big(A\overline{v}_\ell\big)(t) \big| \overline{v}_\ell(t) \big\rangle_{V_A^* \times V_A} - \big\langle \big(A\overline{v}_\ell - Aw\big)(t) \big| \big(\overline{v}_\ell - w\big)(t) \big\rangle_{V_A^* \times V_A} \Big) \,\mathrm{d}t \\
&\quad = \int_0^T \Big(\big\langle \big(A\overline{v}_\ell\big)(t) \big| w(t) \big\rangle_{V_A^* \times V_A} + \big\langle \big(Aw\big)(t) \big| \big(\overline{v}_\ell - w\big)(t) \big\rangle_{V_A^* \times V_A} \Big) \,\mathrm{d}t
\end{aligned}$$

for any $w \in L^p(0,T;V_A)$. Together with the relations (5.2), this implies ($\ell \in \mathbb{N}$)

$$\begin{aligned}
&\int_0^T \big\langle \overline{f}_\ell(t) \big| \overline{v}_\ell(t) \big\rangle_{V_A^* \times V_A} \,\mathrm{d}t + \tfrac{1}{8} \Big(\big\| u_\ell^1 - u_\ell^0 \big\|_B^2 + \big\| v_\ell^1 - v_\ell^0 \big\|_H^2 \Big) \\
&\qquad - \tfrac{1}{2} \Big(\big\| \hat{u}_\ell(T) \big\|_B^2 - \big\| \hat{u}_\ell(0) \big\|_B^2 + \big\| \hat{v}_\ell(T) \big\|_H^2 - \big\| \hat{v}_\ell(0) \big\|_H^2 \Big) \\
&\quad \geq \int_0^T \big\langle \big(A\overline{v}_\ell\big)(t) \big| \overline{v}_\ell(t) \big\rangle_{V_A^* \times V_A} \,\mathrm{d}t \\
&\quad \geq \int_0^T \Big(\big\langle \big(A\overline{v}_\ell\big)(t) \big| w(t) \big\rangle_{V_A^* \times V_A} + \big\langle \big(Aw\big)(t) \big| \big(\overline{v}_\ell - w\big)(t) \big\rangle_{V_A^* \times V_A} \Big) \,\mathrm{d}t \,.
\end{aligned}$$

Next, we employ Lemma 5.2 as well as 5.3 combined with the above considerations. In particular, we utilise that a common subsequence exists such that $(\overline{v}_{\ell'})$ converges weakly towards u' in $L^p(0,T;V_A)$ and that $(\overline{f}_{\ell'})$ converges strongly towards $f = u'' + a + Bu$ in $L^{p^*}(0,T;V_A^*)$. In addition, we make use of the fact that strong convergence of $(u_{\ell'}^1 - u_{\ell'}^0, v_{\ell'}^1 - v_{\ell'}^0)$ towards zero in $V_B \times H$, weak convergence of $(\hat{u}_{\ell'}(T), \hat{v}_{\ell'}(T))$ towards $(u(T), u'(T))$ in $V_B \times H$, and strong convergence of $(\hat{u}_{\ell'}(0), \hat{v}_{\ell'}(0))$ towards $(u(0), u'(0)) = (u_0, v_0)$ in $V_B \times H$ are ensured. Recall also that $(A\overline{v}_{\ell'})$ converges weakly to a in $L^{p^*}(0,T;V_A^*)$. Passing to the subsequence (ℓ') and using the weak sequential lower semicontinuity of the norm, we thus obtain

$$\begin{aligned}
&\int_0^T \Big(\big\langle a(t) \big| w(t) \big\rangle_{V_A^* \times V_A} + \big\langle \big(Aw\big)(t) \big| \big(u' - w\big)(t) \big\rangle_{V_A^* \times V_A} \Big) \,\mathrm{d}t \\
&\quad \leq \limsup_{\ell' \in \mathbb{N}} \int_0^T \big\langle \big(A\overline{v}_\ell\big)(t) \big| \overline{v}_\ell(t) \big\rangle_{V_A^* \times V_A} \,\mathrm{d}t \\
&\quad \leq \int_0^T \big\langle \big(u'' + a + Bu\big)(t) \big| u'(t) \big\rangle_{V_A^* \times V_A} \,\mathrm{d}t \\
&\qquad - \tfrac{1}{2} \Big(\big\| u(T) \big\|_B^2 - \big\| u(0) \big\|_B^2 + \big\| u'(T) \big\|_H^2 - \big\| u'(0) \big\|_H^2 \Big) \,.
\end{aligned}$$

As a consequence, we obtain

$$\begin{aligned}&\int_0^T \big\langle \big(a - Aw\big)(t) \,\big|\, \big(w-u'\big)(t)\big\rangle_{V_A^*\times V_A}\,\mathrm{d}t\\&\quad\le \int_0^T \big\langle \big(u''+Bu\big)(t)\,\big|\,u'(t)\big\rangle_{V_A^*\times V_A}\,\mathrm{d}t\\&\qquad - \tfrac{1}{2}\Big(\big\|u(T)\big\|_B^2 - \big\|u(0)\big\|_B^2 + \big\|u'(T)\big\|_H^2 - \big\|u'(0)\big\|_H^2\Big)\end{aligned}$$

for all $w \in L^p(0,T;V_A)$. If the second assertion of Lemma 5.4 holds for $\beta = T$, then the right-hand side of the foregoing estimate is less or equal zero. Otherwise, there is a sequence $(\beta_k)_{k\in\mathbb{N}}$ converging towards T such that the second assertion of Lemma 5.4 holds for any β_k. One may then revisit the previous steps with final time β_k instead of T; as described in LIONS, STRAUSS [26, p. 80 f.], a diagonal sequence argument then leads to the desired result. We note that the requirements of Lemma 5.4 are indeed fulfilled, since

$$\begin{gathered}u' \in \mathscr{C}_w([0,T];H)\,, \quad u - u_0 = Ku' = \int_0^{(\cdot)} u'(s)\,\mathrm{d}s \in \mathscr{C}_w([0,T];V_B)\,,\\ u'' + Bu = u'' + B\left(u_0 + Ku'\right) \in L^{p^*}(0,T;V_A^*)\,.\end{gathered}$$

Setting $w = u' \pm \sigma\widetilde{w}$ for $\widetilde{w} \in L^p(0,T;V_A)$ and $\sigma \in (0,1]$ leads to

$$\pm \int_0^T \big\langle \big(a - A(u' \pm \sigma\widetilde{w})\big)(t)\,\big|\,\widetilde{w}(t)\big\rangle_{V_A^*\times V_A}\,\mathrm{d}t \le 0\,.$$

Employing the hemicontinuity of $A : L^p(0,T;V_A) \to L^{p^*}(0,T;V_A^*)$ implies

$$Au' = a \ \text{ in } L^{p^*}(0,T;V_A^*)\,.$$

(iv) *Final conclusion.* Revisiting (5.5), we finally conclude that the limiting function u satisfies the nonlinear second-order evolution equation

$$u'' + Au' + Bu = f \ \text{ in } L^{p^*}(0,T;V_A^*)$$

and thus is a solution in the sense of Theorem 2.1. As the weak solution to the initial value problem (1.1) is unique, the whole sequence of the numerical approximations converges, which can be shown by contradiction. ♢

5.5 Strong convergence

In this section, we establish a result on strong convergence under the additional assumption that the nonlinear operator $A : V_A \to V_A^*$ is uniformly monotone in the sense that there exists $\widetilde{\mu}_A > 0$ such that for all $v, \widetilde{v} \in V_A$

$$\big\langle Av - A\widetilde{v}\,\big|\,v - \widetilde{v}\big\rangle_{V_A^*\times V_A} \ge \widetilde{\mu}_A \left\|v - \widetilde{v}\right\|_{V_A}^p\,. \tag{5.6}$$

We note that strong convergence may also be obtained under the weaker assumption of d-monotonicity if V_A is uniformly convex, see EMMRICH, ŠIŠKA [16].

Theorem 5.5 *Under the assumptions of Theorem 5.1 and the additional monotonicity property, the sequence $(\overline{v}_\ell)_{\ell\in\mathbb{N}}$ converges strongly towards u' in $L^p(0,T;V_A)$ and the sequence $(\hat{u}_\ell - \hat{u}_\ell(0))_{\ell\in\mathbb{N}}$ converges strongly towards $u - u_0$ in $\mathscr{C}([0,T];V_A)$.*

Proof The monotonicity assumption (5.6) implies

$$\begin{aligned}0 \le \widetilde{\mu}_A \big\|\overline{v}_\ell - u'\big\|^p_{L^p(0,T;V_A)} &\le \big\langle A\overline{v}_\ell - Au' \big| \overline{v}_\ell - u'\big\rangle_{L^{p^*}(0,T;V_A^*)\times L^p(0,T;V_A)} \\ &= -\big\langle A\overline{v}_\ell \big| u'\big\rangle_{L^{p^*}(0,T;V_A^*)\times L^p(0,T;V_A)} - \big\langle Au' \big| \overline{v}_\ell - u'\big\rangle_{L^{p^*}(0,T;V_A^*)\times L^p(0,T;V_A)} \\ &\quad + \big\langle A\overline{v}_\ell \big| \overline{v}_\ell\big\rangle_{L^{p^*}(0,T;V_A^*)\times L^p(0,T;V_A)}\,.\end{aligned}$$

From Theorem 5.1, we already know that the sequence $(\overline{v}_\ell)_{\ell\in\mathbb{N}}$ converges weakly towards u' in $L^p(0,T;V_A)$. Moreover, in the course of the proof, we also have shown that the sequence $(A\overline{v}_\ell)_{\ell\in\mathbb{N}}$ converges weakly towards Au' in $L^{p^*}(0,T;V_A^*)$. This shows that the first two terms on the right-hand side of the foregoing estimate converge towards

$$-\big\langle Au' \big| u'\big\rangle_{L^{p^*}(0,T;V_A^*)\times L^p(0,T;V_A)}\,.$$

For the last term, we utilise the numerical scheme (5.4) and the estimates (5.2) to find

$$\begin{aligned}&\big\langle A\overline{v}_\ell \big| \overline{v}_\ell\big\rangle_{L^{p^*}(0,T;V_A^*)\times L^p(0,T;V_A)} \\ &\quad= \big\langle \overline{f}_\ell \big| \overline{v}_\ell\big\rangle_{L^{p^*}(0,T;V_A^*)\times L^p(0,T;V_A)} - \big(\hat{v}_\ell' \big| \overline{v}_\ell\big)_{L^2(0,T;H)} - \big\langle B\overline{u}_\ell \big| \overline{v}_\ell\big\rangle_{L^2(0,T;V_B^*)\times L^2(0,T;V_B)} \\ &\quad\le \big\langle \overline{f}_\ell \big| \overline{v}_\ell\big\rangle_{L^{p^*}(0,T;V_A^*)\times L^p(0,T;V_A)} - \tfrac{1}{2}\left(\big\|\hat{v}_\ell(T)\big\|_H^2 - \big\|\hat{v}_\ell(0)\big\|_H^2\right) + \tfrac{1}{8}\big\|v_\ell^1 - v_\ell^0\big\|_H^2 \\ &\qquad - \tfrac{1}{2}\left(\big\|\hat{u}_\ell(T)\big\|_B^2 - \big\|\hat{u}_\ell(0)\big\|_B^2\right) + \tfrac{1}{8}\big\|u_\ell^1 - u_\ell^0\big\|_B^2\,.\end{aligned}$$

The first term on the right-hand side converges towards

$$\begin{aligned}\big\langle f \big| u'\big\rangle_{L^{p^*}(0,T;V_A^*)\times L^p(0,T;V_A)} &= \big\langle Au' \big| u'\big\rangle_{L^{p^*}(0,T;V_A^*)\times L^p(0,T;V_A)} \\ &\quad + \big\langle u'' + Bu \big| u'\big\rangle_{L^{p^*}(0,T;V_A^*)\times L^p(0,T;V_A)}\,,\end{aligned}$$

since $(\overline{f}_\ell)_{\ell\in\mathbb{N}}$ converges strongly towards $f = u'' + Au' + Bu$ in $L^{p^*}(0,T;V_A^*)$. For the remaining terms, we apply Lemma 5.2 and 5.3, and the weak lower semicontinuity of the norm to obtain the estimate

$$\begin{aligned}&\limsup_{\ell\in\mathbb{N}} \big\langle A\overline{v}_\ell - Av \big| \overline{v}_\ell - v\big\rangle_{L^{p^*}(0,T;V_A^*)\times L^p(0,T;V_A)} \\ &\quad\le \big\langle u'' + Bu \big| u'\big\rangle_{L^{p^*}(0,T;V_A^*)\times L^p(0,T;V_A)} - \tfrac{1}{2}\left(\big\|u'(T)\big\|_H^2 - \big\|u'(0)\big\|_H^2\right) \\ &\qquad - \tfrac{1}{2}\left(\big\|u(T)\big\|_B^2 - \big\|u(0)\big\|_B^2\right)\,.\end{aligned}$$

An application of Lemma 5.4 shows that the right-hand side of the foregoing estimate is less or equal zero; if necessary, combined with a diagonal sequence argument as described in the proof of Theorem 5.1, see also LIONS, STRAUSS [26, p. 80 f.]. This proves the first assertion.

The second assertion on the strong convergence of $(\hat{u}_\ell - \hat{u}_\ell(0))_{\ell\in\mathbb{N}}$ immediately follows from the relation

$$\big(\hat{u}_\ell(t) - \hat{u}_\ell(0)\big) - \big(u(t) - u(0)\big) = \int_0^t \big(\hat{u}_\ell'(s) - u'(s)\big)\,\mathrm{d}s$$

together with $\hat{u}_\ell' = \overline{v}_\ell$. $\diamondsuit$

6 Estimate of the time discretisation error

In this section, presuming a sufficiently regular solution to the considered nonlinear evolution equation, we establish an a priori error estimate of second-order in time. In what follows, we focus on the time discretisation error and neglect the spatial discretisation. Moreover, we restrict ourselves to the case $f \in \mathscr{C}([0,T];V_A^*)$. Instead of the fully discrete scheme (3.4), we study the semi-discrete problem of finding $(u^n, v^n)_{n=1}^N \subset V_B \times V$ such that, for given $(u^0, v^0) \in V_B \times H$, the relations $(n = 2,3,\dots,N)$

$$\begin{cases} \big(D_1 v^1 \,\big|\, \varphi\big)_H + \big\langle Av^1 \,\big|\, \varphi\big\rangle_{V_A^*\times V_A} + \big\langle Bu^1 \,\big|\, \varphi\big\rangle_{V_B^*\times V_B} = \big\langle f(t_1) \,\big|\, \varphi\big\rangle_{V_A^*\times V_A}\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad \big(D_1 u^1 \,\big|\, \varphi\big)_H = \big(v^1 \,\big|\, \varphi\big)_H\,, \end{cases}$$
$$\begin{cases} \big(D_2 v^n \,\big|\, \varphi\big)_H + \big\langle Av^n \,\big|\, \varphi\big\rangle_{V_A^*\times V_A} + \big\langle Bu^n \,\big|\, \varphi\big\rangle_{V_B^*\times V_B} = \big\langle f(t_n) \,\big|\, \varphi\big\rangle_{V_A^*\times V_A}\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad \big(D_2 u^n \,\big|\, \varphi\big)_H = \big(v^n \,\big|\, \varphi\big)_H \end{cases} \tag{6.1}$$

hold for all $\varphi \in V$. These relations are equivalent to finding $(v^n)_{n=1}^N \subset V$ such that

$$\begin{aligned} &\tau^{-1}v^1 + Av^1 + \tau Bv^1 = \tau^{-1}v^0 - Bu^0 + f(t_1)\,, \\ &\tfrac{3}{2}\tau^{-1}v^n + Av^n + \tfrac{2}{3}\tau Bv^n = \tau^{-1}\big(2v^{n-1} - \tfrac{1}{2}v^{n-2}\big) - B\big(u^0 + \widehat{K}_\tau v^{n-1}\big) + f(t_n)\,, \end{aligned} \tag{6.2}$$

where $\widehat{K}_\tau v^{n-1}$ is given by (4.1) and $(u^n)_{n=1}^N \subset V_B$ is determined by

$$u^1 = u^0 + \tau v^1\,, \quad u^n = \tfrac{4}{3}u^{n-1} - \tfrac{1}{3}u^{n-2} + \tfrac{2}{3}\tau v^n\,.$$

Existence and uniqueness of $(v^n)_{n=1}^N \subset V$ follows step-by-step from the famous Browder–Minty theorem, see for instance GAJEWSKI, GRÖGER, ZACHARIAS [22], since the operators

$$\tau^{-1}I + A + \tau B : V \to V^*\,, \quad \tfrac{3}{2}\tau^{-1}I + A + \tfrac{2}{3}\tau B : V \to V^* \tag{6.3}$$

are hemicontinuous, strictly monotone, and coercive. In particular, the coercivity of the operators as mappings of V into V^* follows from the p-coercivity of $A : V_A \to V_A^*$ and the strong positivity of $B : V_B \to V_B^*$, see Assumptions (A) and (B). We note that $(u^n)_{n=1}^N$ is only in V_B, since u^0 is only in V_B.

Theorem 6.1 (Error estimate) *Suppose that Assumptions (A) and (B) hold. Furthermore, let $(u^0, v^0) \in V_B \times H$ be given approximations to the initial values. Provided that the exact solution satisfies*

$$u \in W^{4,1}(0,T;H) \cap W^{3,1}(0,T;V_B) \hookrightarrow \mathscr{C}^3([0,T];H) \cap \mathscr{C}^2([0,T];V_B)\,,$$
$$u' \in \mathscr{C}([0,T];V)\,,$$

the error estimate ($N_0 = 1,2,\dots,N$)

$$\begin{aligned}&\big\|u^{N_0} - u(t_{N_0})\big\|_{V_B} + \big\|v^{N_0} - u'(t_{N_0})\big\|_H \\ &\quad \leq c\left(\big\|u^0 - u_0\big\|_{V_B} + \big\|v^0 - v_0\big\|_H + \tau^2\left(\big\|u''\big\|_{W^{1,1}(0,T;V_B)} + \big\|u'''\big\|_{W^{1,1}(0,T;H)}\right)\right)\end{aligned}$$

is valid for the time discrete solution to (3.4).

Proof We aim at a suitable estimate for the time discretisation errors ($n = 0,1,\dots,N$)

$$d^n = u^n - u(t_n)\,, \quad e^n = v^n - v(t_n)\,,$$

associated with the time discretisation scheme (6.1) for the first-order system (2.3).

(i) *Consistency errors.* Inserting the exact solution values into the numerical scheme defines the associated consistency errors ($n = 2,3,\dots,N$)

$$\begin{cases}\varepsilon^1 = v'(\tau) - \tau^{-1}\left(v(\tau) - v(0)\right), \\ \delta^1 = u'(\tau) - \tau^{-1}\left(u(\tau) - u(0)\right),\end{cases}$$
$$\begin{cases}\varepsilon^n = v'(t_n) - \tau^{-1}\left(\frac{3}{2}v(t_n) - 2v(t_{n-1}) + \frac{1}{2}v(t_{n-2})\right), \\ \delta^n = u'(t_n) - \tau^{-1}\left(\frac{3}{2}u(t_n) - 2u(t_{n-1}) + \frac{1}{2}u(t_{n-2})\right).\end{cases}$$

With these abbreviations, the numerical scheme (6.1) implies

$$\begin{aligned}&\left(D_1 e^1 \,\middle|\, \varphi\right)_H + \left\langle Av^1 - Av(t_1) \,\middle|\, \varphi\right\rangle_{V_A^* \times V_A} + \left\langle Bd^1 \,\middle|\, \varphi\right\rangle_{V_B^* \times V_B} = \left(\varepsilon^1 \,\middle|\, \varphi\right)_H, \\ &\left(D_2 e^n \,\middle|\, \varphi\right)_H + \left\langle Av^n - Av(t_n) \,\middle|\, \varphi\right\rangle_{V_A^* \times V_A} + \left\langle Bd^n \,\middle|\, \varphi\right\rangle_{V_B^* \times V_B} = \left(\varepsilon^n \,\middle|\, \varphi\right)_H,\end{aligned} \tag{6.4}$$

With the help of Taylor series expansions, we obtain ($n = 2,3,\dots,N$)

$$\begin{aligned}\varepsilon^1 &= \tau^{-1}\int_0^\tau s\, u'''(s)\,\mathrm{d}s\,, \\ \varepsilon^n &= -\tfrac{1}{4}\,\tau^{-1}\left(3\int_{t_{n-1}}^{t_n}\left(s - t_{n-1} + \tfrac{\tau}{3}\right)(s - t_n)\, u^{(4)}(s)\,\mathrm{d}s - \int_{t_{n-2}}^{t_{n-1}}(s - t_{n-2})^2\, u^{(4)}(s)\,\mathrm{d}s\right), \\ \delta^1 &= \tau^{-1}\int_0^\tau s\, u''(s)\,\mathrm{d}s\,, \\ \delta^n &= -\tfrac{1}{4}\,\tau^{-1}\left(3\int_{t_{n-1}}^{t_n}\left(s - t_{n-1} + \tfrac{\tau}{3}\right)(s - t_n)\, u'''(s)\,\mathrm{d}s - \int_{t_{n-2}}^{t_{n-1}}(s - t_{n-2})^2\, u'''(s)\,\mathrm{d}s\right).\end{aligned}$$

We may also use the representation ($n = 2,3,\dots,N$)

$$\varepsilon^n = -\,\tau^{-1}\left(\int_{t_{n-1}}^{t_n}(s - t_{n-1})^2\, u^{(4)}(s)\,\mathrm{d}s - \tfrac{1}{4}\int_{t_{n-2}}^{t_n}(s - t_{n-2})^2\, u^{(4)}(s)\,\mathrm{d}s\right),$$

and analogously for δ^n. A straightforward estimation shows that

$$\begin{aligned}
&\tau \left\| \varepsilon^1 \right\|_H \leq c\,\tau^2 \left\| u''' \right\|_{\mathscr{C}([0,T];H)}\,, \quad \tau \sum_{n=2}^{N_0} \left\| \varepsilon^n \right\|_H \leq c\,\tau^2 \left\| u^{(4)} \right\|_{L^1(0,T;H)}\,,\\
&\tau \left\| \delta^1 \right\|_{V_B} \leq c\,\tau^2 \left\| u'' \right\|_{\mathscr{C}([0,T];V_B)}\,, \quad \tau \sum_{n=2}^{N_0} \left\| \delta^n \right\|_{V_B} \leq c\,\tau^2 \left\| u''' \right\|_{L^1(0,T;V_B)}\,.
\end{aligned} \tag{6.5}$$

(iii) *A priori error estimate.* Testing the first equation in (6.4) with $\varphi = e^1 = D_1 d^1 - \delta^1$ and the subsequent equations with $\varphi = e^n = D_2 d^n - \delta^n$, leads to the identities ($n = 2,3,\dots,N$)

$$\begin{aligned}
&\big(D_1 e^1 \,\big|\, e^1\big)_H + \big\langle Av^1 - Av(t_1) \,\big|\, v^1 - v(t_1)\big\rangle_{V_A^* \times V_A} + \big\langle Bd^1 \,\big|\, D_1 d^1\big\rangle_{V_B^* \times V_B}\\
&\quad = \big(\varepsilon^1 \,\big|\, e^1\big)_H + \big\langle Bd^1 \,\big|\, \delta^1\big\rangle_{V_B^* \times V_B}\,,\\
&\big(D_2 e^n \,\big|\, e^n\big)_H + \big\langle Av^n - Av(t_n) \,\big|\, v^n - v(t_n)\big\rangle_{V_A^* \times V_A} + \big\langle Bd^n \,\big|\, D_2 d^n\big\rangle_{V_B^* \times V_B}\\
&\quad = \big(\varepsilon^n \,\big|\, e^n\big)_H + \big\langle Bd^n \,\big|\, \delta^n\big\rangle_{V_B^* \times V_B}\,;
\end{aligned}$$

here, we employ the assumption $u' \in \mathscr{C}([0,T];V)$ to ensure $e^n \in V$ for $n = 1,2,\dots N$. We next employ (3.8) and the monotonicity of A to obtain

$$\begin{aligned}
\left\|e^1\right\|_H^2 + \left\|d^1\right\|_B^2 &\leq \left\|e^0\right\|_H^2 + \left\|d^0\right\|_B^2 + 2\tau \left\|\varepsilon^1\right\|_H \left\|e^1\right\|_H + 2\tau \left\|\delta^1\right\|_B \left\|d^1\right\|_B\\
&\leq \left\|e^0\right\|_H^2 + \left\|d^0\right\|_B^2 + 4\tau \big(\left\|\varepsilon^1\right\|_H + \left\|\delta^1\right\|_B\big) \Big(\left\|e^1\right\|_H^2 + \left\|d^1\right\|_B^2\Big)^{\frac{1}{2}}\,,
\end{aligned}$$

and ($n = 2,3,\dots,N$)

$$\begin{aligned}
&\left\|e^n\right\|_H^2 + \left\|2e^n - e^{n-1}\right\|_H^2 + \left\|d^n\right\|_B^2 + \left\|2d^n - d^{n-1}\right\|_B^2\\
&\quad \leq \left\|e^{n-1}\right\|_H^2 + \left\|2e^{n-1} - e^{n-2}\right\|_H^2 + \left\|d^{n-1}\right\|_B^2 + \left\|2d^{n-1} - d^{n-2}\right\|_B^2\\
&\qquad + 4\tau \left\|\varepsilon^n\right\|_H \left\|e^n\right\|_H + 4\tau \left\|\delta^n\right\|_B \left\|d^n\right\|_B\\
&\quad \leq \left\|e^{n-1}\right\|_H^2 + \left\|2e^{n-1} - e^{n-2}\right\|_H^2 + \left\|d^{n-1}\right\|_B^2 + \left\|2d^{n-1} - d^{n-2}\right\|_B^2\\
&\qquad + 8\tau \big(\left\|\varepsilon^n\right\|_H + \left\|\delta^n\right\|_B\big)\\
&\qquad\quad \times \Big(\left\|e^n\right\|_H^2 + \left\|2e^n - e^{n-1}\right\|_H^2 + \left\|d^n\right\|_B^2 + \left\|2d^n - d^{n-1}\right\|_B^2\Big)^{\frac{1}{2}}\,.
\end{aligned}$$

Resolving the above quadratic inequalities yields

$$\Big(\left\|e^1\right\|_H^2 + \left\|d^1\right\|_B^2\Big)^{\frac{1}{2}} \leq \Big(\left\|e^0\right\|_H^2 + \left\|d^0\right\|_B^2\Big)^{\frac{1}{2}} + 4\tau \big(\left\|\varepsilon^1\right\|_H + \left\|\delta^1\right\|_B\big)$$

and

$$\begin{aligned}
&\Big(\left\|e^n\right\|_H^2 + \left\|2e^n - e^{n-1}\right\|_H^2 + \left\|d^n\right\|_B^2 + \left\|2d^n - d^{n-1}\right\|_B^2\Big)^{\frac{1}{2}}\\
&\quad \leq \Big(\left\|e^{n-1}\right\|_H^2 + \left\|2e^{n-1} - e^{n-2}\right\|_H^2 + \left\|d^{n-1}\right\|_B^2 + \left\|2d^{n-1} - d^{n-2}\right\|_B^2\Big)^{\frac{1}{2}}\\
&\qquad + 8\tau \big(\left\|\varepsilon^n\right\|_H + \left\|\delta^n\right\|_B\big)\,.
\end{aligned}$$

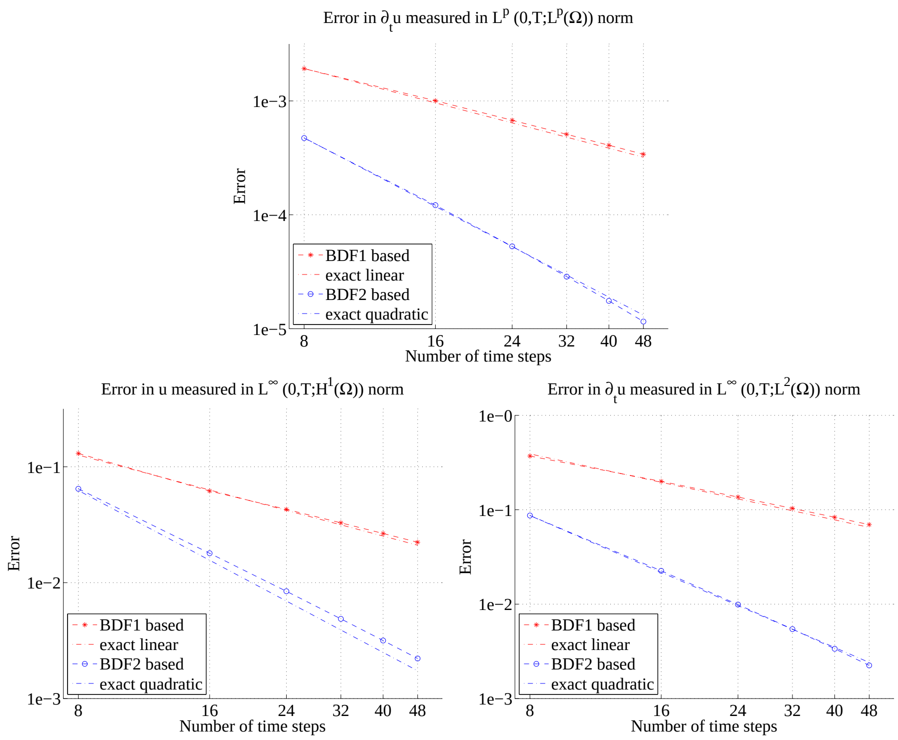

Fig. 1 Time discretisation errors for model problem with prescribed exact solution

Summation together with a telescopic identity finally implies

$$\|e^{N_0}\|_H + \|d^{N_0}\|_B \leq c \left(\|e^0\|_H + \|d^0\|_B + \tau \sum_{n=1}^{N_0} \left(\|\varepsilon^n\|_H + \|\delta^n\|_B \right) \right).$$

Combining this estimate with (6.5) and the continuous embeddings $W^{1,1}(0,T;H) \hookrightarrow \mathscr{C}([0,T];H)$ as well as $W^{1,1}(0,T;V_B) \hookrightarrow \mathscr{C}([0,T];V_B)$ thus proves the assertion. $\diamondsuit$

Remark. If the operator $A: V_A \to V_A^*$ is uniformly monotone in the sense of (5.6), one may also deduce estimates for $(e_n)_{n=1}^N$ in the discrete $L^p(0,T;V_A)$-norm under somewhat different regularity assumptions, see EMMRICH [15].

7 Model problem and numerical illustration

In this section, we study a model problem describing a vibrating membrane in a viscous medium and illustrate convergence as well as the favourable error behaviour of the proposed full discretisation method in a situation where a regular solution exists.

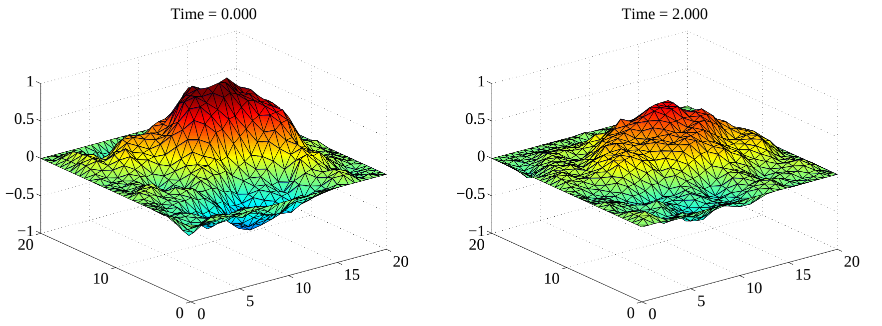

Fig. 2 Random u_0 (left) and the numerical solution at time $T = 2$, projected onto a coarse mesh

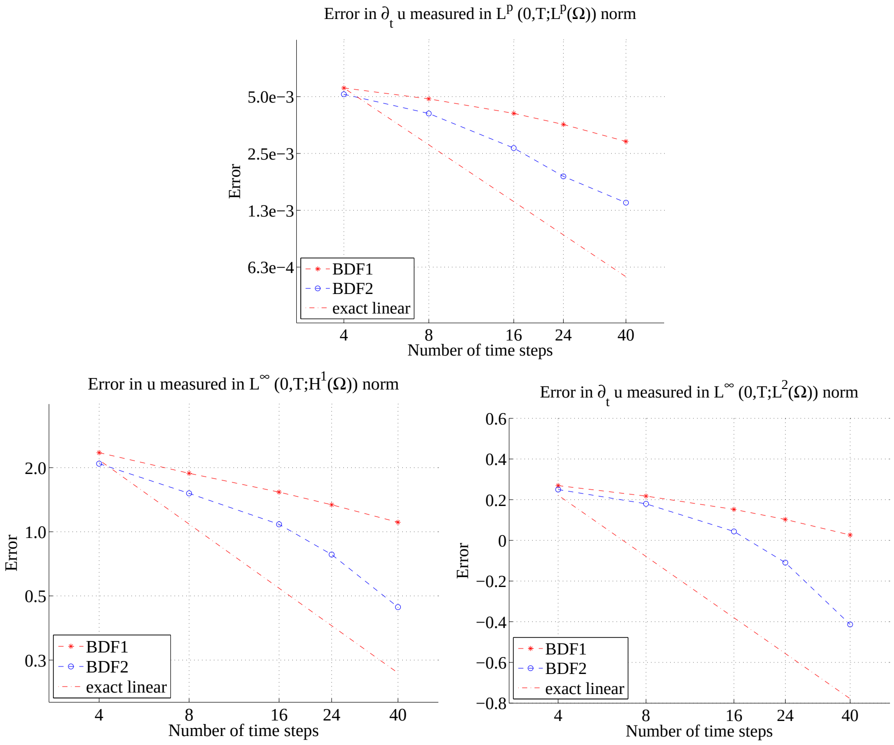

Fig. 3 Time discretisation errors for model problem with random initial datum

Model problem. We consider the initial-boundary value problem (1.2) for a function $u : \overline{\Omega} \times [0,T] \to \mathbb{R}$ describing the displacement of a vibrating membrane in a viscous medium, where we assume $p \geq 2$ and denote by $\Omega \subset \mathbb{R}^d$ a bounded domain with Lipschitz boundary $\partial\Omega$. This model problem may be cast into the form of an abstract second-order evolution equation (1.1) that complies with the functional ana-

N	Error in v in $L^p(0,T;L^p(\Omega))$	Error in v in $L^\infty(0,T;L^2(\Omega))$	Error in u in $L^\infty(0,T;H_0^1(\Omega))$
8	$1.92e-03$	$3.71e-01$	$1.30e-01$
16	$1.00e-03$	$2.00e-01$	$6.19e-02$
24	$6.76e-04$	$1.36e-01$	$4.29e-02$
32	$5.09e-04$	$1.03e-01$	$3.28e-02$
40	$4.08e-04$	$8.32e-02$	$2.66e-02$
48	$3.40e-04$	$6.95e-02$	$2.23e-02$

Table 1 Time discretisation errors for BDF1 based method for model problem with prescribed exact solution

N	Error in v in $L^p(0,T;L^p(\Omega))$	Error in v in $L^\infty(0,T;L^2(\Omega))$	Error in u in $L^\infty(0,T;H_0^1(\Omega))$
8	$4.72e-04$	$8.69e-02$	$6.46e-02$
16	$1.22e-04$	$2.25e-02$	$1.79e-02$
24	$5.28e-05$	$9.91e-03$	$8.43e-03$
32	$2.87e-05$	$5.44e-03$	$4.88e-03$
40	$1.75e-05$	$3.37e-03$	$3.17e-03$
48	$1.16e-05$	$2.25e-03$	$2.22e-03$

Table 2 Time discretisation errors for BDF2 based method for model problem with prescribed exact solution

N	Error in v in $L^p(0,T;L^p(\Omega))$	Error in v in $L^\infty(0,T;L^2(\Omega))$	Error in u in $L^\infty(0,T;H_0^1(\Omega))$
4	$5.55e-03$	$1.86e+00$	$2.35e+00$
8	$4.86e-03$	$1.65e+00$	$1.88e+00$
16	$4.08e-03$	$1.42e+00$	$1.54e+00$
24	$3.56e-03$	$1.27e+00$	$1.34e+00$
40	$2.89e-03$	$1.06e+00$	$1.11e+00$

Table 3 Time discretisation errors for BDF1 based method for model problem with random initial datum

N	Error in v in $L^p(0,T;L^p(\Omega))$	Error in v in $L^\infty(0,T;L^2(\Omega))$	Error in u in $L^\infty(0,T;H_0^1(\Omega))$
4	$5.14e-03$	$1.78e+00$	$2.09e+00$
8	$4.07e-03$	$1.51e+00$	$1.52e+00$
16	$2.67e-03$	$1.11e+00$	$1.08e+00$
24	$1.89e-03$	$7.77e-01$	$7.82e-01$
40	$1.37e-03$	$3.86e-01$	$4.43e-01$

Table 4 Time discretisation errors for BDF2 based method for model problem with random initial datum

lytic framework introduced in Section 2. Namely, defining the operators $A : V_A \to V_A^*$ and $B : V_B \to V_B^*$ related to the weak formulation of (1.2) by

$$\begin{aligned} \langle Av \,|\, w\rangle_{V_A^*\times V_A} &= \int_\Omega \big|v(x)\big|^{p-2}\, v(x)\, w(x)\,\mathrm{d}x\,, \quad v,w \in V_A = L^p(\Omega)\,, \\ \langle Bu \,|\, w\rangle_{V_B^*\times V_B} &= \int_\Omega \nabla u(x)\cdot \nabla w(x)\,\mathrm{d}x\,, \quad u,w \in V_B = H_0^1(\Omega)\,, \end{aligned}$$

the basic hypotheses given in Assumptions (A) and (B) and the uniform monotonicity condition (5.6) are satisfied. The pivot space is $H = L^2(\Omega)$. For $(u_0, v_0) \in H_0^1(\Omega) \times L^2(\Omega)$ and $f \in L^{p^*}(0,T;L^{p^*}(\Omega)) = L^{p^*}(\Omega\times(0,T))$, existence and uniqueness of a weak solution is thus ensured by Theorem 2.1.

Numerical solution. For the numerical solution of the initial-boundary value problem (1.2), we apply the novel full discretisation method (1.3). For the purpose of comparison, we also consider a full discretisation method based on the implicit Euler scheme applied to both equations in the first-order system (3.4). For the sake of simplicity, we henceforth restrict ourselves to a squared domain $\Omega \subset \mathbb{R}^2$ and use Courant linear finite elements on a uniform triangular mesh. This mesh is constructed by an equidistant partition of each side into 2^M intervals and cutting the resulting squares along a diagonal. This leads to a partition into 2^{2M+1} triangles and $\overline{M} = (2^M - 1)^2$ interior nodes. The corresponding finite element space then forms $V_M = \operatorname{span}\{\varphi_1, \dots, \varphi_{\overline{M}}\}$ with a global basis $(\varphi_m)_{m=1}^{\overline{M}}$. We note that here $\dim V_M = \overline{M}$ and $V_M \subset V_{M+1}$. We also remark that the assumption on the H-orthogonal projection onto V_M (Assumption (IC,P)) is satisfied, see CROUZEIX, THOMÉE [10].

Let $(i, j = 1, 2, \dots, \overline{M}, n = 1, \dots, N)$

$$G_{ij} = \int_\Omega \varphi_i \varphi_j \,\mathrm{d}x, \quad S_{ij} = \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \,\mathrm{d}x, \quad f_i^n = \int_\Omega f^n \varphi_i \,\mathrm{d}x.$$

Moreover, for $\mathbf{w} = [w_1, \dots, w_{\overline{M}}]^T \in \mathbb{R}^{\overline{M}}$ corresponding to $w = \sum_{m=1}^{\overline{M}} w_m \varphi_m \in V_M$, we define

$$\overline{A}(\mathbf{w})_{ij} = \int_\Omega |w|^{p-2} \varphi_i \varphi_j \,\mathrm{d}x, \qquad i, j = 1, 2, \dots, \overline{M}.$$

The numerical scheme then reads as $(n = 2, 3, \dots, N)$

$$\begin{cases} \mathbf{G}\big(\mathbf{v}^1 - \mathbf{v}^0\big) + \tau\overline{\mathbf{A}}(\mathbf{v}^1)\mathbf{v}^1 + \tau\mathbf{S}\big(\mathbf{u}^0 + \tau\mathbf{v}^1\big) = \tau\mathbf{f}^1, \\ \mathbf{u}^1 = \mathbf{u}^0 + \tau\mathbf{v}^1, \end{cases}$$

$$\begin{cases} \mathbf{G}\left(\tfrac{3}{2}\mathbf{v}^n - 2\mathbf{v}^{n-1} + \tfrac{1}{2}\mathbf{v}^{n-2}\right) + \tau\overline{\mathbf{A}}(\mathbf{v}^n)\mathbf{v}^n + \tau\mathbf{S}\big(\tfrac{4}{3}\mathbf{u}^{n-1} - \tfrac{1}{3}\mathbf{u}^{n-2} + \tfrac{2}{3}\tau\mathbf{v}^n\big) = \tau\mathbf{f}^n, \\ \mathbf{u}^n = \tfrac{4}{3}\mathbf{u}^{n-1} - \tfrac{1}{3}\mathbf{u}^{n-2} + \tfrac{2}{3}\tau\mathbf{v}^n. \end{cases}$$

In each time step, for the numerical computation of $\mathbf{v}^n$, we apply Newton's method with initial guess $\mathbf{v}^{n-1}$, where the exact Jacobians $\mathbf{G} + \tau\mathbf{J}(\cdot) + \tau^2\mathbf{S}$ and $\tfrac{3}{2}\mathbf{G} + \tau\mathbf{J}(\cdot) + \tfrac{2}{3}\tau^2\mathbf{S}$, respectively, with $(i, j = 1, 2, \dots, \overline{M})$

$$J(\mathbf{w})_{ij} = (p-1)\int_\Omega |w|^{p-2}\, \varphi_i \varphi_j \,\mathrm{d}x$$

are used.

Numerical results. In what follows, we choose $p = 10$, $\Omega = (0, L) \times (0, L) \subset \mathbb{R}^2$ with $L = 20$, and $T = 8$. In a first numerical experiment, we prescribe as exact solution the solution to the corresponding homogeneous linear wave equation and choose the right-hand side f of (1.2) appropriately to compensate for the nonlinear damping,

$$u(x,t) = \cos(\sqrt{2}\pi L^{-1} t)\, \sin(\pi L^{-1} x_1)\, \sin(\pi L^{-1} x_2), \quad x = (x_1, x_2) \in \Omega, \quad t \in [0, T],$$
$$f = |\partial_t u|^{p-2} \partial_t u.$$

We focus on the time discretisation error and therefore choose a very fine mesh for the spatial discretisation. The results obtained for the time discretisation based on the implicit Euler method (BDF1) and the two-step backward differentiation formula (BDF2) are shown in Figure 1 (together with reference lines for first and second order) as well as Table 1 and 2. These results in particular confirm the second-order error estimate provided by Theorem 6.1. We also present results for the velocity error measured in $L^p(0,T;L^p(\Omega))$.

In a second numerical experiment, we choose $T = 2$, $f = 0$, $v_0 = 0$, and let $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ be the unique solution to the homogeneous Dirichlet problem for the Poisson equation with random right-hand side $g \in L^2(\Omega)$, see Figure 2. A numerical reference solution is computed on a fine time grid with $N = 1024$. The results can be seen in Figure 3 as well as Table 3 and 4. In this situation, strong convergence is ensured by Theorem 5.5; second-order error estimates as provided by Theorem 6.1 cannot be expected because of the possible lack of regularity.

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