# Condition length and complexity for the solution of polynomial systems 

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#### Abstract

Smale's 17th problem asks for an algorithm which finds an approximate zero of polynomial systems in average polynomial time (see Smale [17]). The main progress on Smale's problem is Beltrán-Pardo [6] and BürgisserCucker [9]. In this paper we will improve on both approaches and we prove an important intermediate result. Our main results are Theorem 1 on the complexity of a randomized algorithm which improves the result of [6], Theorem 2 on the average of the condition number of polynomial systems


[^0]which improves the estimate found in [9], and Theorem 3 on the complexity of finding a single zero of polynomial systems. This last Theorem is the main result of [9]. We give a proof of it relying only on homotopy methods, thus removing the need for the elimination theory methods used in [9]. We build on methods developed in Armentano et al. [2].

## 1 Introduction

Homotopy or continuation methods to solve a problem which might depend on parameters start with a problem instance and known solution and try to continue the solution along a path in parameter space ending at the problem we wish to solve. We recall how this works for the solutions of polynomial systems using a variant of Newton's method to accomplish the continuation.

Let $\mathcal{H}_{d}=\mathcal{H}_{d}^{n+1}$ be the complex vector space of degree $d$ complex homogeneous polynomials in $n+1$ variables. For $\alpha=\left(\alpha_{0}, \ldots, \alpha_{n}\right) \in \mathbb{N}^{n+1}, \sum_{j=0}^{n} \alpha_{j}=d$, and the monomial $z^{\alpha}=z_{0}^{\alpha_{0}} \cdots z_{n}^{\alpha_{n}}$, the Weyl Hermitian structure on $\mathcal{H}_{d}$ makes $\left\langle z^{\alpha}, z^{\beta}\right\rangle:=0$, for $\alpha \neq \beta$ and

$$
\left\langle z^{\alpha}, z^{\alpha}\right\rangle:=\binom{d}{\alpha}^{-1}=\left(\frac{d!}{\alpha_{0}!\cdots \alpha_{n}!}\right)^{-1}
$$

Now for $(d)=\left(d_{1}, \ldots, d_{n}\right)$ we let $\mathcal{H}_{(d)}=\prod_{k=1}^{n} \mathcal{H}_{d_{k}}$. This is a complex vector space of dimension

$$
N:=\sum_{i=1}^{n}\binom{n+d_{i}}{n} .
$$

That is, $N$ is the size of a system $f \in \mathcal{H}_{(d)}$, understood as the number of complex numbers needed to describe $f$.

We endow $\mathcal{H}_{(d)}$ with the product Hermitian structure

$$
\langle f, g\rangle:=\sum_{k=1}^{n}\left\langle f_{i}, g_{i}\right\rangle,
$$

where $f=\left(f_{1}, \ldots, f_{n}\right)$, and $g=\left(g_{1}, \ldots, g_{n}\right)$. This Hermitian structure is sometimes called the Weyl, Bombieri-Weyl, or Kostlan Hermitian structure. It is invariant under unitary substitution $f \mapsto f \circ U^{-1}$, where $U$ is a unitary transformation of $\mathbb{C}^{n+1}$ (see Blum et al. [8, p. 118] for example).

On $\mathbb{C}^{n+1}$ we consider the usual Hermitian structure

$$
\langle x, y\rangle:=\sum_{k=0}^{n} x_{k} \overline{y_{k}} .
$$

Given $0 \neq \zeta \in \mathbb{C}^{n+1}$, let $\zeta^{\perp}$ denotes Hermitian complement of $\zeta$,

$$
\zeta^{\perp}:=\left\{v \in \mathbb{C}^{n+1}:\langle v, \zeta\rangle=0\right\}
$$

The subspace $\zeta^{\perp}$ is a model for the tangent space, $T_{\zeta} \mathbb{P}\left(\mathbb{C}^{n+1}\right)$, of the projective space $\mathbb{P}\left(\mathbb{C}^{n+1}\right)$ at the equivalence class of $\zeta$ (which we also denote by $\zeta$ ). The space $T_{\zeta} \mathbb{P}\left(\mathbb{C}^{n+1}\right)$ inherits an Hermitian structure from $\langle\cdot, \cdot\rangle$ given by

$$
\langle v, w\rangle_{\zeta}:=\frac{\langle v, w\rangle}{\langle\zeta, \zeta\rangle}
$$

The group of unitary transformations $\mathbb{U}$ acts naturally on $\mathbb{C}^{n+1}$ by $\zeta \mapsto U^{-1} \zeta$ for $U \in \mathbb{U}$, and the Hermitian structure of $\mathbb{C}^{n+1}$ is invariant under this action.

A zero of the system of equations $f$ is a point $x \in \mathbb{C}^{n+1}$ such that $f_{i}(x)=0$, $i=1, \ldots, n$. If we think of $f$ as a mapping $f: \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n}$, it is a point $x$ such that $f(x)=0$.

For a generic system (that is, for a Zariski open set of $\left.f \in \mathcal{H}_{(d)}\right)$, Bézout's theorem states that the set of zeros consist of $\mathcal{D}:=\prod_{k=1}^{n} d_{k}$ complex lines through 0 . These $\mathcal{D}$ lines are $\mathcal{D}$ points in projective space $\mathbb{P}\left(\mathbb{C}^{n+1}\right)$. So our goal will be to approximate one of these points, and we will use the so-called homotopy or continuation methods.

These methods for the solution of a system $f \in \mathcal{H}_{(d)}$ proceed as follows. Choose $g \in \mathcal{H}_{(d)}$ and a zero $\zeta \in \mathbb{P}\left(\mathbb{C}^{n+1}\right)$ of $g$. Connect $g$ to $f$ by a path $f_{t}$, $0 \leq t \leq 1$, in $\mathcal{H}_{(d)}$ such that $f_{0}=g, f_{1}=f$, and try to continue $\zeta_{0}=\zeta$ to $\zeta_{t}$ such that $f_{t}\left(\zeta_{t}\right)=0$, so that $f_{1}\left(\zeta_{1}\right)=0$ (see Beltrán-Shub [7] for details or [10] for a complete discussion).

So homotopy methods numerically approximate the path $\left(f_{t}, \zeta_{t}\right)$. One way to accomplish the approximation is via (projective) Newton's methods. Given an approximation $x_{t}$ to $\zeta_{t}$, define

$$
x_{t+\Delta t}:=N_{f_{t+\Delta t}}\left(x_{t}\right),
$$

where for $h \in \mathcal{H}_{(d)}$ and $y \in \mathbb{P}\left(\mathbb{C}^{n+1}\right)$ we define the projective Newton's method $N_{h}(y)$ following [14]:

$$
N_{h}(y):=y-\left(\left.D h(y)\right|_{y^{\perp}}\right)^{-1} h(y) .
$$

Note that $N_{h}$ is defined on $\mathbb{P}\left(\mathbb{C}^{n+1}\right)$ at those points where $\left.D h(y)\right|_{y^{\perp}}$ is invertible.
That $x_{t}$ is an approximate zero of $f_{t}$ with associated (exact) zero $\zeta_{t}$ means that the sequence of Newton iterations $N_{f_{t}}^{k}\left(x_{t}\right)$ converges immediately and quadratically to $\zeta_{t}$.

Let us assume that $\left\{f_{t}\right\}_{t \in[0,1]}$ is a path in the sphere $\mathbb{S}\left(\mathcal{H}_{(d)}\right):=\left\{h \in \mathcal{H}_{(d)}\right.$ : $\|h\|=1\}$. The main result of Shub [13] ${ }^{1}$ is that the $\Delta t_{k}$ may be chosen so that $t_{0}=0, t_{k}=t_{k-1}+\Delta t_{k}$ for $k=1, \ldots, K$ with $t_{K}=1$, such that for all $k, x_{t_{k}}$ is an approximate zero of $f_{t_{k}}$ with associated zero $\zeta_{t_{k}}$, and the number $K$ of steps can be bounded as follows:

$$
\begin{equation*}
K=K(f, g, \zeta) \leq C D^{3 / 2} \int_{0}^{1} \mu\left(f_{t}, \zeta_{t}\right)\left\|\left(\dot{f}_{t}, \dot{\zeta}_{t}\right)\right\| d t \tag{1.1}
\end{equation*}
$$

[^1]Here $C$ is a universal constant, $D=\max _{i} d_{i}$,

$$
\mu(f, \zeta):= \begin{cases}\|f\|\left\|\left(\left.D f(\zeta)\right|_{\zeta^{\perp}}\right)^{-1} \operatorname{diag}\left(\|\zeta\|^{d_{i}-1} \sqrt{d_{i}}\right)\right\| & \text { if }\left.D f(\zeta)\right|_{\zeta^{\perp}} \text { is invertible } \\ \infty & \text { otherwise }\end{cases}
$$

is the condition number of $f \in \mathcal{H}_{(d)}$ at $\zeta \in \mathbb{P}\left(\mathbb{C}^{n+1}\right)$, and

$$
\left\|\left(\dot{f}_{t}, \dot{\zeta}_{t}\right)\right\|=\left(\left\|\dot{f}_{t}\right\|^{2}+\left\|\dot{\zeta}_{t}\right\|_{\zeta_{t}}^{2}\right)^{1 / 2}
$$

is the norm of the tangent vector to the curve in $\left(f_{t}, \zeta_{t}\right)$. The result in [13] is not fully constructive, but specific constructions have been given, see [3] and [12], and even programmed [4]. These constructions are similar to those given in ShubSmale [16] and Armentano et al. [2] (this last, for the eigenvalue-eigenvector problem case).

The right-hand side in expression (1.1) is known as the condition length of the path $\left(f_{t}, \zeta_{t}\right)$. We will call (1.1) the condition length estimate of the number of steps.

Taking derivatives w.r.t. $t$ in the equality $f_{t}\left(\zeta_{t}\right)=0$ it is easily seen that

$$
\begin{equation*}
\dot{\zeta}_{t}=\left(\left.D f_{t}\left(\zeta_{t}\right)\right|_{\zeta_{t}^{\perp}}\right)^{-1} \dot{f}_{t}\left(\zeta_{t}\right), \tag{1.2}
\end{equation*}
$$

and with some work (see [8, Lemma 12, p. 231] one can prove that

$$
\|\dot{\zeta}\|_{\zeta_{t}} \leq \mu\left(f_{t}, \zeta_{t}\right)\left\|\dot{f_{t}}\right\| .
$$

It is known that $\mu(f, \zeta) \geq 1$, e,g., see [10, Prop. 16.19]. So the estimate (1.1) may be bounded from above by

$$
\begin{equation*}
K(f, g, \zeta) \leq C^{\prime} D^{3 / 2} \int_{0}^{1} \mu^{2}\left(f_{t}, \zeta_{t}\right)\left\|\dot{f}_{t}\right\| d t \tag{1.3}
\end{equation*}
$$

where $C^{\prime}=\sqrt{2} C$. Let us call this estimate the $\mu^{2}$-estimate.
The condition length estimate is better than the $\mu^{2}$-estimate, but algorithms achieving the smaller number of steps are more subtle and the proofs of correctness more difficult.

Indeed in Beltrán-Pardo [5] and Bürgisser-Cucker [9] the authors rely on the $\mu^{2}$-estimate. At the times of these papers the algorithms achieving the condition length bound where in development, and [9] includes a construction which achieves the $\mu^{2}$-estimate.

Yet, in a random situation, one might expect the improvement to be similar to the improvement given by the average of $\|A(x)\|$, in all possible directions, compared with $\|A\|$ (here, $A: \mathbb{C}^{n} \rightarrow \mathbb{C}^{n}$ denotes a linear operator), which according to Armentano [1] should give an improvement by a factor of the square root of the domain dimension. We have accomplished this for the eigenvalue-eigenvector
problem in Armentano et al. [2]. Here we use an argument similar to that of [2] to improve the estimate for the randomized algorithm in Beltrán-Pardo [6].

The Beltrán-Pardo randomized algorithm works as follows (see BeltránPardo [6], and also Bürgisser-Cucker [9]): on input $f \in \mathcal{H}_{(d)}$,

1. Choose $f_{0}$ at random and then a zero $\zeta_{0}$ of $f_{0}$ at random. Beltrán and Pardo [6] describe a general scheme to do so (roughly speaking, one first draws the "linear" part of $f_{0}$, computes $\zeta_{0}$ from it, and then draws the "nonlinear" part of $f_{0}$ ). An efficient implementation of this scheme, having running time $\mathcal{O}(n D N)$, is fully described and analyzed in [10, Section 17.6].
2. Then connect $f_{0} /\left\|f_{0}\right\|$ to $f /\|f\|$ by an arc of a great circle in the sphere, and invoke the continuation strategy above.

The main result of [6] is that the average number of steps of this procedure is bounded by $O\left(D^{3 / 2} n N\right)$, and its total average complexity is then $\mathcal{O}\left(D^{3 / 2} n N^{2}\right)$ (since the cost of an iteration of Newton's method, assuming all $d_{i} \geq 2$, is $\mathcal{O}(N)$, see [10, Proposition 16.32]).

Our first main result is the following improvement of this last bound.
Theorem 1 (Randomized algorithm) The average number of steps of the randomized algorithm with the condition length estimate is bounded by

$$
C D^{3 / 2} n N^{1 / 2},
$$

where $C$ is a universal constant.
The constant $C$ can be taken as $\frac{\pi}{\sqrt{2}} C^{\prime}$ with $C^{\prime}$ not more than 400 even accounting for input and round-off error, cf. Dedieu-Malajovich-Shub [12].

Remark 1 Theorem 1 is an improvement by a factor of $1 / N^{1 / 2}$ of the bound in [6], which results from using the condition length estimate in place of the $\mu^{2}$-estimate.

Before proceeding with the proof of Theorem 1, we introduce some useful notation. We define the solution variety

$$
\mathcal{V}:=\left\{(f, \zeta) \in \mathcal{H}_{(d)} \times \mathbb{P}\left(\mathbb{C}^{n+1}\right) \mid f(\zeta)=0\right\}
$$

and consider the projections


The set of ill-posed pairs is the subset

$$
\Sigma^{\prime}:=\left\{(f, \zeta) \in \mathcal{V}|D f(\zeta)|_{\zeta^{\perp}} \text { is not invertible }\right\}=\{(f, \zeta) \in \mathcal{V} \mid \mu(f, \zeta)=\infty\}
$$

and its projection $\Sigma:=\pi_{1}\left(\Sigma^{\prime}\right)$ is the set of ill-posed systems. The number of iterations of the homotopy algorithm, $K(f, g, \zeta)$, is finite if and only if the lifting $\left\{\left(f_{t}, \zeta_{t}\right)\right\}_{t \in[0,1]}$ of the segment $\left\{f_{t}\right\}_{t \in[0,1]}$ does not cut $\Sigma^{\prime}$.

## 2 Proof of Theorem 1

### 2.1 Preliminaries

Let us start this section with a few general facts we will use from Gaussian measures.

Given a finite dimensional real vector space $V$ of dimension $m$, with an inner product, we define two natural objects.

- The unit sphere $\mathbb{S}(V)$ with the induced Riemannian structure and volume form: the volume of $\mathbb{S}(V)$ is $\frac{2 \pi^{m / 2}}{\Gamma\left(\frac{m}{2}\right)}$.
- The Gaussian measure centered at $c \in V$, with variance $\frac{\sigma^{2}}{2}>0$, whose density is

$$
\begin{equation*}
\frac{1}{\sigma^{m} \pi^{m / 2}} e^{-\|x-c\|^{2} / \sigma^{2}} \tag{2.5}
\end{equation*}
$$

We will denote by $N_{V}\left(c, \sigma^{2} \mathrm{Id}\right)$ the density given in (2.5). We will skip the notation of the underlying space when it is understood. Furthermore, we will denote by $\mathbb{E}_{x \in V}$ the average in the case $\sigma=1$ (that is, variance $1 / 2$ ).

Lemma 2 If $\varphi: V \rightarrow[0,+\infty]$ is measurable and homogeneous of degree $p>-m$, then

$$
\underset{x \in V}{\mathbb{E}}(\varphi(x))=\frac{\Gamma\left(\frac{m+p}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} \underset{u \in \mathbb{S}(V)}{\mathbb{E}}(\varphi(u))
$$

where

$$
\underset{u \in \mathbb{S}(V)}{\mathbb{E}}(\varphi(u))=\frac{1}{\operatorname{vol}(\mathbb{S}(V))} \int_{\mathbb{S}(V)} \varphi(u) d u
$$

Proof. Integrating in polar coordinates we have

$$
\begin{aligned}
\underset{x \in V}{\mathbb{E}}(\varphi(x)) & =\frac{1}{\pi^{m / 2}} \int_{x \in V} \varphi(x) e^{-\|x\|^{2}} d x \\
& =\frac{1}{\pi^{m / 2}} \int_{0}^{+\infty} \rho^{m+p-1} e^{-\rho^{2}} d \rho \cdot \int_{u \in \mathbb{S}(V)} \varphi(u) d u \\
& =\frac{\Gamma\left(\frac{m+p}{2}\right)}{2 \pi^{m / 2}} \int_{u \in \mathbb{S}(V)} \varphi(u) d u=\frac{\Gamma\left(\frac{m+p}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} \underset{u \in \mathbb{S}(V)}{\mathbb{E}}(\varphi(u))
\end{aligned}
$$

where we have used that $\int_{0}^{+\infty} \rho^{k} e^{-\rho^{2}} d \rho=\frac{1}{2} \Gamma\left(\frac{k+1}{2}\right)$.
The next results follows immediately from Fubini's theorem.
Lemma 3 Let $E$ be a linear subspace of $V$, and let $\Pi: V \rightarrow E$ be the orthogonal projection. Then, for any integrable function $\psi: E \rightarrow \mathbb{R}$ and for any $c \in V$, $\sigma>0$, we have

$$
\underset{x \sim N_{V}\left(c, \sigma^{2} \mathrm{Id}\right)}{\mathbb{E}}(\psi(\Pi(x)))=\underset{y \sim N_{E}\left(\Pi(c), \sigma^{2} \mathrm{Id}\right)}{\mathbb{E}}(\psi(y))
$$

When $V$ is a finite dimensional Hermitian vector space of complex dimension $m$, then the complex Gaussian measure on $V$ with variance $\sigma^{2}$ is defined by the real Gaussian measure with variance $\sigma^{2} / 2$ of the $2 m$-dimensional real vector space associated to $V$ with inner product the real part of the Hermitian product.

In this fashion, for any fixed $g \in \mathcal{H}_{(d)}$ and $\sigma>0$, the Hermitian space $\left(\mathcal{H}_{(d)},\langle\cdot, \cdot\rangle\right)$ is equipped with the complex Gaussian measure $N\left(g, \sigma^{2} \mathrm{Id}\right)$. The expected value of a function $\phi: \mathcal{H}_{(d)} \rightarrow \mathbb{R}$ with respect to this measure is given by

$$
\begin{equation*}
\underset{f \sim N\left(g, \sigma^{2} \mathrm{Id}\right)}{\mathbb{E}}(\phi)=\frac{1}{\sigma^{2 N} \pi^{N}} \int_{f \in \mathcal{H}_{(d)}} \phi(f) e^{-\|f-g\|^{2} / \sigma^{2}} d f . \tag{2.6}
\end{equation*}
$$

Fix any $\zeta \in \mathbb{P}\left(\mathbb{C}^{n+1}\right)$. Following [10, Sect. 16.3], the space $\mathcal{H}_{(d)}$ is orthogonally decomposed into the sum $C_{\zeta} \oplus \mathcal{V}_{\zeta}$, where

$$
C_{\zeta}=\left\{\operatorname{diag}\left(\frac{\langle\cdot, \zeta\rangle^{d_{i}}}{\langle\zeta, \zeta\rangle^{d_{i}}}\right) a: a \in \mathbb{C}^{n}\right\}
$$

and

$$
\mathcal{V}_{\zeta}=\pi_{2}^{-1}(\zeta)=\left\{f \in \mathcal{H}_{(d)}: f(\zeta)=0\right\}
$$

Note that $C_{\zeta}$ and $\mathcal{V}_{\zeta}$ are linear subspaces of $\mathcal{H}_{(d)}$ of respective (complex) dimensions $n$ and $N-n$. Note also that

$$
f_{0}=f-\operatorname{diag}\left(\frac{\langle\cdot, \zeta\rangle^{d_{i}}}{\langle\zeta, \zeta\rangle^{d_{i}}}\right) f(\zeta)
$$

is the orthogonal projection $\Pi_{\zeta}(f)$ of $f$ onto the fiber $\mathcal{V}_{\zeta}$. This follows from the reproducing kernel property of the Weyl Hermitian product on $\mathcal{H}_{d_{i}}$, namely,

$$
\begin{equation*}
\left\langle g,\langle\cdot, \zeta\rangle^{d_{i}}\right\rangle=g(\zeta) \tag{2.7}
\end{equation*}
$$

for all $g \in \mathcal{H}_{d_{i}}$ and $i=1, \ldots, n$. In particular, the norm of $\langle\cdot, \zeta\rangle^{d_{i}} \in \mathcal{H}_{d_{i}}$ is equal to $\|\zeta\|^{d_{i}}$.

### 2.2 Average condition numbers

In this section we revisit the average value of the operator and Frobenius condition numbers on $\mathcal{H}_{(d)}$. The Frobenius condition number of $f$ at $\zeta$ is given by

$$
\begin{equation*}
\mu_{F}(f, \zeta):=\|f\|\left\|\left(\left.D f(\zeta)\right|_{\zeta^{+}}\right)^{-1} \operatorname{diag}\left(\|\zeta\|^{d_{i}-1} d_{i}^{1 / 2}\right)\right\|_{F} \tag{2.8}
\end{equation*}
$$

that is, $\mu_{F}$ is defined as $\mu$ but using Frobenius instead of operator norm. Note that $\mu \leq \mu_{F} \leq \sqrt{n} \mu$.

Given $f \in \mathcal{H}_{(d)} \backslash \Sigma$, the average of the condition numbers over the fiber is

$$
\mu_{\mathrm{av}}^{2}(f):=\frac{1}{\mathcal{D}} \sum_{\zeta: f(\zeta)=0} \mu^{2}(f, \zeta), \quad \mu_{F, \mathrm{av}}^{2}(f):=\frac{1}{\mathcal{D}} \sum_{\zeta: f(\zeta)=0} \mu_{F}^{2}(f, \zeta)
$$

(or $\infty$ if $f \in \Sigma$ ). For simplicity, in what follows we write $\mathbb{S}:=\mathbb{S}\left(\mathcal{H}_{(d)}\right)$.
Estimates on the probability distribution of the condition number $\mu$ are known since [15]. The exact expected value of $\mu_{\mathrm{av}}^{2}(f)$ when $f$ is in the sphere $\mathbb{S}$ was found in [6] and the following estimate for the expected value of $\mu_{\mathrm{av}}^{2}(f)$ when $f$ is noncentered Gaussian was proved in [9]: for all $\widehat{f} \in \mathcal{H}_{(d)}$ and all $\sigma>0$,

$$
\begin{equation*}
\underset{f \sim N\left(\widehat{f}, \sigma^{2} \mathrm{Id}\right)}{\mathbb{E}} \frac{\mu_{\mathrm{av}}^{2}(f)}{\|f\|^{2}} \leq \frac{e(n+1)}{2 \sigma^{2}} \tag{2.9}
\end{equation*}
$$

The following result slightly improves (2.9), even though it is computed for $\mu_{F}$.
Theorem 2 (Average condition number) For every $\widehat{f} \in \mathcal{H}_{(d)}$ and $\sigma>0$,

$$
\underset{f \sim N\left(\widehat{f}, \sigma^{2} \mathrm{Id}\right)}{\mathbb{E}} \frac{\mu_{F, \mathrm{av}}^{2}(f)}{\|f\|^{2}} \leq \frac{n}{\sigma^{2}}
$$

and equality holds in the centered case.
Remark 4 The equality of Theorem 2 implies from Lemma 2 with $p=-2$ that

$$
\underset{f \in \mathbb{S}}{\mathbb{E}} \mu_{F, \mathrm{av}}^{2}(f)=(N-1) n
$$

Remark 5 In the proof of Theorem 2 we use the double-fibration technology, a strategy based on the use of the classical Coarea Formula, see for example [8, p. 241]. In order to integrate some real-valued function over $\mathcal{H}_{(d)}$ whose value at some point $f$ is an average over the fiber $\pi_{1}^{-1}(f)$, we lift it to $\mathcal{V}$ and then pushforward to $\mathbb{P}\left(\mathbb{C}^{n+1}\right)$ using the projections given in (1.4). The original expected value in $\mathcal{H}_{(d)}$ is then writen as an integral over $\mathbb{P}\left(\mathbb{C}^{n+1}\right)$ which involves the quotient of normal Jacobians of the projections $\pi_{1}$ and $\pi_{2}$. More precisely,

$$
\begin{equation*}
\int_{f \in \mathcal{H}_{(d)}} \sum_{\zeta: f(\zeta)=0} \phi(f, \zeta) d f=\int_{\zeta \in \mathbb{P}\left(\mathbb{C}^{n+1}\right)} \int_{(f, \zeta) \in \pi_{2}^{-1}(\zeta)} \phi(f, \zeta) \frac{\mathrm{NJ}_{\pi_{1}}}{\mathrm{NJ}_{\pi_{2}}}(f, \zeta) d \pi_{2}^{-1}(\zeta) d \zeta \tag{2.10}
\end{equation*}
$$

where

$$
\frac{\mathrm{NJ}_{\pi_{1}}}{\mathrm{NJ}_{\pi_{2}}}(f, \zeta)=\left|\operatorname{det}\left(\left.D f(\zeta)\right|_{\zeta^{\perp}}\right)\right|^{2}
$$

(see [8, Section 13.2], [10, Section 17.3], or [2, Theorem 6.2] for further details and other examples of use).

We point out that the proof of Theorem 2 can also be achieved using the (slightly) different method of [6] and [10, Chapter 18] based on the mapping taking $(f, \zeta)$ to $(D f(\zeta), \zeta)$ whose Jacobian is known to be constant (see [6, Main Lemma]).

Proof of Theorem 2. By the definition of non-centered Gaussian, and the double-fibration formula (2.10), we have

$$
\begin{align*}
& \underset{f \sim N\left(\hat{f}, \sigma^{2} \mathrm{Id}\right)}{\mathbb{E}} \frac{\mu_{F, \mathrm{av}}^{2}(f)}{\|f\|^{2}}=\frac{1}{\mathcal{D}} \int_{f \in \mathcal{H}_{(d)}}\left(\sum_{\zeta: f(\zeta)=0} \frac{\mu_{F}^{2}(f, \zeta)}{\|f\|^{2}}\right) \frac{e^{\frac{-\|f-\hat{f}\|^{2}}{\sigma^{2}}}}{\sigma^{2 N} \pi^{N}} d f  \tag{2.11}\\
& \left.=\frac{1}{\mathcal{D}} \frac{1}{\sigma^{2 n} \pi^{n}} \int_{\zeta \in \mathbb{P}\left(\mathbb{C}^{n+1}\right)} e^{\frac{-\left\|\hat{f}-\Pi_{\zeta}(\hat{f})\right\|^{2}}{\sigma^{2}}} \int_{f \in \mathcal{V}_{\zeta}} \frac{\mu_{F}^{2}(f, \zeta)}{\|f\|^{2}} \right\rvert\, \operatorname{det}\left(\left.\left.D f(\zeta)\right|_{\zeta^{\perp}}\right|^{2} \frac{e^{\frac{-\left\|f-\Pi_{\zeta}(\hat{f})\right\|^{2}}{\sigma^{2}}}}{\left(\sigma^{2} \pi\right)^{N-n}} d f d \zeta,\right.
\end{align*}
$$

where we have used that $\|f-\widehat{f}\|^{2}=\left\|f-\Pi_{\zeta}(\widehat{f})\right\|^{2}+\left\|\widehat{f}-\Pi_{\zeta}(\widehat{f})\right\|^{2}$ for every $f \in \mathcal{V}_{\zeta}$.

We simplify now the integral $I_{\zeta}$ over the fiber $\mathcal{V}_{\zeta}$. Let $U_{\zeta}$ be a unitary transformation of $\mathbb{C}^{n+1}$ such that $U_{\zeta}(\zeta /\|\zeta\|)=e_{0}$. Then, by the invariance under unitary substitution of each term under the integral sign, we have by the change of variable formula that

$$
\begin{aligned}
I_{\zeta} & :=\int_{f \in \mathcal{V}_{\zeta}} \frac{\mu_{F}^{2}(f, \zeta)}{\|f\|^{2}} \left\lvert\, \operatorname{det}\left(\left.\left.D f(\zeta)\right|_{\zeta^{\perp}}\right|^{2} \frac{e^{\frac{-\left\|f-\Pi_{\zeta}(\hat{f})\right\|^{2}}{\sigma^{2}}}}{\sigma^{2(N-n)} \pi^{N-n}} d f\right.\right. \\
& =\int_{h \in \mathcal{V}_{e_{0}}} \frac{\mu_{F}^{2}\left(h, e_{0}\right)}{\|h\|^{2}} \left\lvert\, \operatorname{det}\left(\left.\left.D h\left(e_{0}\right)\right|_{e_{0}^{\perp}}\right|^{2} \frac{e^{\frac{-\left\|h-\Pi_{e_{0}}(\widehat{h})\right\|^{2}}{\sigma^{2}}}}{\sigma^{2(N-n)} \pi^{N-n}} d h,\right.\right.
\end{aligned}
$$

where $\widehat{h}_{\zeta}:=\widehat{f} \circ U_{\zeta}^{-1}$. We project now $h \in \mathcal{V}_{e_{0}}$ orthogonally onto the vector space

$$
L_{e_{0}}:=\left\{g \in \mathcal{H}_{(d)}: g\left(e_{0}\right)=0, D^{k} g\left(e_{0}\right)=0 \text { for } k \geq 2\right\}
$$

obtaining $g \in L_{e_{0}}$. Since $\left.\operatorname{Dh}\left(e_{0}\right)\right|_{e_{0}^{\perp}}$ coincides with $\left.D g\left(e_{0}\right)\right|_{e_{0}^{\perp}}$ (see for example [10, Prop. 16.16]), we conclude by Fubini that

$$
\begin{aligned}
I_{\zeta} & =\int_{h \in \mathcal{V}_{e_{0}}} \frac{\mu_{F}^{2}\left(h, e_{0}\right)}{\|h\|^{2}} \left\lvert\, \operatorname{det}\left(\left.\left.D h\left(e_{0}\right)\right|_{e_{0}^{\perp}}\right|^{2} \frac{e^{\frac{-\left\|h-\Pi_{e_{0}}\left(\widehat{h}_{C}\right)\right\|^{2}}{\sigma^{2}}}}{\sigma^{2(N-n)} \pi^{N-n}} d g\right.\right. \\
& =\int_{g \in L_{e_{0}}} \frac{\mu_{F}^{2}\left(g, e_{0}\right)}{\|g\|^{2}} \left\lvert\, \operatorname{det}\left(\left.\left.D g\left(e_{0}\right)\right|_{e_{0}^{\perp}}\right|^{2} \frac{e^{\frac{-\| g-\Pi_{L_{e_{0}}\left(\widehat{g}_{\zeta}\right) \|^{2}}^{\sigma^{2}}}{\sigma^{2} \pi^{n^{2}}}} d h,}{} d h,\right.\right.
\end{aligned}
$$

where $\widehat{g}_{\zeta}:=\Pi_{L_{0}}\left(\widehat{h}_{\zeta}\right)$. By the change of variable given by

$$
L_{e_{0}} \rightarrow \mathbb{C}^{n \times n}, g \mapsto A:=\left.\operatorname{diag}\left(d_{i}^{-1 / 2}\right) D g\left(e_{0}\right)\right|_{e_{\bar{⿺}}^{\perp}}
$$

we have $\frac{\mu_{F \text { av }}^{2}(g)}{\|g\|^{2}}=\left\|A^{-1}\right\|_{F}^{2}$ and denoting by $\widehat{A}_{\zeta}$ the image of $\widehat{g}_{\zeta}$, we obtain that

$$
I_{\zeta}=\underset{A \in N\left(\widehat{A}_{\zeta}, \sigma^{2} \mathrm{Id}_{n}\right)}{\mathbb{E}}\left\|A^{-1}\right\|_{F}^{2}|\operatorname{det}(A)|^{2}
$$

We thus conclude from (2.11) that

$$
\begin{equation*}
\underset{f \sim N\left(\widehat{f}, \sigma^{2} \mathrm{Id}\right)}{\mathbb{E}} \frac{\mu_{F, \text { av }}^{2}(f)}{\|f\|^{2}}=\frac{1}{\mathcal{D}} \frac{1}{\sigma^{2 n} \pi^{n}} \int_{\zeta \in \mathbb{P}\left(\mathbb{C}^{n+1}\right)} e^{\frac{-\left\|\hat{f}-\Pi_{\zeta}(\hat{f})\right\|^{2}}{\sigma^{2}}} I_{\zeta} d \zeta \tag{2.12}
\end{equation*}
$$

If we replace $\mu_{F, \text { av }}(f)^{2} /\|f\|^{2}$ by the constant function 1 on $\mathcal{H}_{(d)}$, the same argument leading to (2.12) now leads to

$$
\begin{equation*}
1=\frac{1}{\mathcal{D}} \frac{1}{\sigma^{2 n} \pi^{n}} \int_{\zeta \in \mathbb{P}\left(\mathbb{C}^{n+1}\right)} e^{\frac{-\left\|\hat{f}-\Pi_{\zeta}(\hat{f})\right\|^{2}}{\sigma^{2}}} \underset{A \in N\left(\underset{\left.\hat{A}_{\zeta}, \sigma^{2} \mathrm{Id}_{n}\right)}{\mathbb{E}}|\operatorname{det}(A)|^{2} d \zeta . ~ . . ~\right.}{\text {. }} \text {. } \tag{2.13}
\end{equation*}
$$

From Proposition 7.1 of Armentano et al. [2], we can bound

$$
\begin{equation*}
I_{\zeta}=\underset{A \in N\left(\left.\underset{\left.\widehat{A}_{\zeta}, \sigma^{2} \mathrm{Id}_{n}\right)}{\mathbb{E}}\left\|A^{-1}\right\|_{F}^{2}|\operatorname{det}(A)|^{2} \leq \frac{n}{\sigma^{2}} \underset{A \in N\left(\widehat{A_{\zeta}}, \sigma^{2} \mathrm{Id}_{n}\right)}{\mathbb{E}}|\operatorname{det}(A)|^{2} \right\rvert\, \text {. }{ }^{2} \mid\right.}{ } \tag{2.14}
\end{equation*}
$$

(with equality if $\widehat{A}_{\zeta}=0$ ). By combining (2.12), (2.14), and (2.13) we obtain

$$
\underset{f \sim N\left(\widehat{f}, \sigma^{2} \mathrm{Id}\right)}{\mathbb{E}} \frac{\mu_{F, \mathrm{av}}^{2}(f)}{\|f\|^{2}} \leq \frac{n}{\sigma^{2}} .
$$

Moreover, equality holds if $\widehat{f}=0$ and hence $\widehat{A}_{\zeta}=0$ for all $\zeta$.

### 2.3 Complexity of the randomized algorithm

The goal of this section is to prove Theorem 1. To do so, we begin with some preliminaries.

For $f \in \mathbb{S}$ we denote by $T_{f} \mathbb{S}$ the tangent space at $f$ of $\mathbb{S}$. This space is equipped with the real part of the Hermitian structure of $\mathcal{H}_{(d)}$, and coincides with the (real) orthogonal complement of $f \in \mathcal{H}_{(d)}$.

We consider the map $\phi: \mathbb{S} \times \mathcal{H}_{(d)} \rightarrow[0, \infty]$ defined for $f \notin \Sigma$ by

$$
\phi(f, \dot{f}):=\frac{1}{\mathcal{D}} \sum_{\zeta: f(\zeta)=0} \mu(f, \zeta)\|(\dot{f}, \dot{\zeta})\|
$$

where $\dot{\zeta}=\left(\left.D f(\zeta)\right|_{\zeta^{\perp}}\right)^{-1} \dot{f}(\zeta)$, and by $\phi(f, \dot{f}):=\infty$ if $f \in \Sigma$. Note that $\phi$ satisfies $\phi(f, \lambda \dot{f})=\lambda \phi(f, \dot{f})$ for $\lambda \geq 0$.

Suppose that $f_{0}, f \in \mathbb{S}$ are such that $f_{0} \neq \pm f$ and denote by $\mathcal{L}_{f_{0}, f}$ the shorter great circle segment with endpoints $f_{0}$ and $f$. Moreover, let $\alpha=d_{\mathbb{S}}\left(f_{0}, f\right)$ denote the angle between $f_{0}$ and $f$. If $[0,1] \rightarrow \mathbb{S}, t \mapsto f_{t}$ is the constant speed parametrization of $\mathcal{L}_{f_{0}, f}$ with endpoints $f_{0}$ and $f_{1}=f$, then $\left\|\dot{f}_{t}\right\|=\alpha$. We may also parametrize $\mathcal{L}_{f_{0}, f}$ by the arc-length $s=\alpha t$, setting $F_{s}:=f_{\alpha^{-1} s}$, in which case $\dot{F}_{s}=\alpha^{-1} \dot{f}_{t}$ is the unit tangent vector (in the direction of the parametrization) to $\mathcal{L}_{f_{0}, f}$ at $F_{s}$. Moreover,

$$
\int_{0}^{1} \phi\left(f_{t}, \dot{f}_{t}\right) d t=\int_{0}^{\alpha} \phi\left(F_{s}, \dot{F}_{s}\right) d s
$$

Consider the compact submanifold $\mathcal{S}$ of $\mathbb{S} \times \mathbb{S}$ given by

$$
\mathcal{S}=\left\{(f, \dot{f}) \in \mathbb{S} \times \mathbb{S}: \dot{f} \in T_{f} \mathbb{S}\right\}
$$

where $T_{f} \mathbb{S}$ denotes the real tangent space of $\mathbb{S}$ at $f$. We endow $\mathbb{S}$ with the Riemannian metric induced from the real part of the Hermitian product of $\mathcal{H}_{(d)}$, and therefore $\mathcal{S}$ inherits the product Riemannian structure.

The following lemma has been proven in Armentano et al. [2].
Lemma 6 We have

$$
I_{\phi}:=\underset{f_{0}, f \in \mathbb{S}}{\mathbb{E}}\left(\int_{0}^{1} \phi\left(f_{t}, \dot{f}_{t}\right) d t\right)=\frac{\pi}{2} \underset{(f, \dot{f}) \in \mathcal{S}}{\mathbb{E}}(\phi(f, \dot{f}))
$$

where the expectation on the right hand-side refers to the uniform distribution on $\mathcal{S}$.

We proceed with a further auxiliary result. For $f \in \mathbb{S}$ we consider the unit sphere $\mathcal{S}_{f}:=\left\{\dot{f} \in T_{f} \mathbb{S}:(f, \dot{f}) \in \mathcal{S}\right\}$ in $T_{f} \mathbb{S}$.

Lemma 7 Fix $f \in \mathbb{S}$ and $\zeta \in \mathbb{P}\left(\mathbb{C}^{n+1}\right)$ with $f(\zeta)=0$. For $\dot{f} \in \mathcal{S}_{f}$ let $\dot{\zeta}$ be the function of $(f, \dot{f})$ and $\zeta$ given by $\dot{\zeta}=\left(\left.D f(\zeta)\right|_{\zeta^{\perp}}\right)^{-1} \dot{f}(\zeta)$. Then we have

$$
\underset{\dot{f} \in \mathcal{S}_{f}}{\mathbb{E}}\left(\|\dot{\zeta}\|^{2}\right)=\frac{1}{N-\frac{1}{2}}\left\|\left(\left.D f(\zeta)\right|_{\zeta^{\perp}}\right)^{-1}\right\|_{F}^{2}
$$

where the expectation is with respect to the uniform probability distribution on $\mathcal{S}_{f}$.
Proof. $\quad$ Since the map $T_{f} \mathbb{S} \rightarrow \mathbb{R}, \dot{f} \mapsto\|\dot{\zeta}(\dot{f})\|^{2}$ is quadratic, we get from Lemma 2 (recall that $\operatorname{dim} T_{f} \mathbb{S}=2 N-1$ )

$$
\underset{\dot{f} \in T_{f} \mathbb{S}}{\mathbb{E}}\left(\|\dot{\zeta}(\dot{f})\|^{2}\right)=\left(N-\frac{1}{2}\right) \underset{\dot{f} \in \mathcal{S}_{f}}{\mathbb{E}}\left(\|\dot{\zeta}(\dot{f})\|^{2}\right)
$$

Note that the mapping $\mathcal{H}_{(d)} \rightarrow C_{\zeta}$ given by $\dot{f} \mapsto \Pi_{C_{\zeta}} \dot{f}$ is an orthogonal projection, and furthermore $C_{\zeta} \rightarrow \mathbb{C}^{n}$ given by $\dot{f} \mapsto \dot{f}(\zeta)$ is a linear isometry. Then from Lemma 3, and the change of variables formula we obtain

$$
\underset{\dot{f \in T_{f} \mathbb{S}}}{\mathbb{E}}\left(\|\dot{\zeta}(\dot{f})\|^{2}\right)=\underset{\dot{w} \in \mathbb{C}^{n}}{\mathbb{E}}\left\|\left(\left.D f(\zeta)\right|_{\zeta^{\perp}}\right)^{-1} \dot{w}\right\|^{2}=\left\|\left(\left.D f(\zeta)\right|_{\zeta^{\perp}}\right)^{-1}\right\|_{F}^{2},
$$

where the last equality is straightforward.
Proof of Theorem 1. From (1.1), using the notation from there, we know that the number of Newton steps of the homotopy with starting pair $\left(f_{0}, \zeta_{0}\right)$ and target system $f$ is bounded as

$$
K\left(f, f_{0}, \zeta_{0}\right) \leq C D^{3 / 2} \int_{0}^{1} \mu\left(f_{t}, \zeta_{t}\right)\left\|\left(\dot{f}_{t}, \dot{\zeta}_{t}\right)\right\| d t
$$

Hence we get for $f, f_{0} \in \mathbb{S}$,

$$
\begin{aligned}
\frac{1}{\mathcal{D}} \sum_{\zeta_{0}: f_{0}\left(\zeta_{0}\right)=0} K\left(f, f_{0}, \zeta_{0}\right) & \leq C D^{3 / 2} \int_{0}^{1} \frac{1}{\mathcal{D}} \sum_{\zeta_{0}: f_{0}\left(\zeta_{0}\right)=0} \mu\left(f_{t}, \zeta_{t}\right)\left\|\left(\dot{f}_{t}, \dot{\zeta}_{t}\right)\right\| d t \\
& =C D^{3 / 2} \int_{0}^{1} \phi\left(f_{t}, \dot{f}_{t}\right) d t
\end{aligned}
$$

Therefore, by Lemma 6,

$$
\begin{equation*}
\underset{f, f_{0} \in \mathbb{S}}{\mathbb{E}}\left(\frac{1}{\mathcal{D}} \sum_{\zeta_{0}: f_{0}\left(\zeta_{0}\right)=0} K\left(f, f_{0}, \zeta_{0}\right)\right) \leq C D^{3 / 2} \frac{\pi}{2} \underset{(f, \dot{f}) \in \mathcal{S}}{\mathbb{E}}(\phi(f, \dot{f})) \tag{2.15}
\end{equation*}
$$

It is easy to check that the projection $\mathcal{S} \rightarrow \mathbb{S},(f, \dot{f}) \mapsto f$, has the Normal Jacobian $1 / \sqrt{2}$. From the coarea formula, we therefore obtain

$$
\begin{aligned}
\underset{(f, \dot{f}) \in \mathcal{S}}{\mathbb{E}}(\phi(f, \dot{f})) & =\sqrt{2} \underset{f \in \mathbb{S}}{\mathbb{E}} \underset{f \in \mathcal{S}_{f}}{\mathbb{E}}(\phi(f, \dot{f})) \\
& =\sqrt{2} \underset{f \in \mathbb{S}}{\mathbb{E}}\left(\frac{1}{\mathcal{D}} \sum_{\zeta: f(\zeta)=0} \mu(f, \zeta) \underset{\dot{f} \in \mathcal{S}_{f}}{\mathbb{E}}(\|(\dot{f}, \dot{\zeta})\|)\right)
\end{aligned}
$$

In order to estimate this last quantity, note first that from the Cauchy-Schwartz inequality

$$
\begin{aligned}
\underset{\dot{f} \in \mathcal{S}_{f}}{\mathbb{E}}\left(\left(1+\|\dot{\zeta}\|^{2}\right)^{\frac{1}{2}}\right) & \leq\left(1+\underset{\dot{f} \in \mathcal{S}_{f}}{\mathbb{E}}\left(\|\dot{\zeta}\|^{2}\right)\right)^{1 / 2} \\
& \leq\left(1+\frac{1}{N-\frac{1}{2}}\left\|\left(\left.D f(\zeta)\right|_{\zeta^{+}}\right)^{-1}\right\|_{F}^{2}\right)^{1 / 2}
\end{aligned}
$$

the last by Lemma 7. Now we use $\left\|\left(\left.D f(\zeta)\right|_{\zeta^{\perp}}\right)^{-1}\right\|_{F} \leq \mu_{F}(f, \zeta)$ and $\mu(f, \zeta) \leq$ $\mu_{F}(f, \zeta)$ to deduce

$$
\begin{aligned}
\frac{1}{\sqrt{2}} \underset{(f, \dot{f}) \in \mathcal{S}}{\mathbb{E}}(\phi(f, \dot{f})) & \leq \underset{f \in \mathbb{S}}{\mathbb{E}}\left(\frac{1}{\mathcal{D}} \sum_{\zeta: f(\zeta)=0} \mu_{F}(f, \zeta)\left(1+\frac{\mu_{F}^{2}(f, \zeta)}{N-\frac{1}{2}}\right)^{\frac{1}{2}}\right) \\
& \leq \underset{f \in \mathbb{S}}{\mathbb{E}}\left(\frac{1}{\mathcal{D}} \sum_{\zeta: f(\zeta)=0}\left(\frac{\left(N-\frac{1}{2}\right)^{\frac{1}{2}}}{2}+\frac{\mu_{F}^{2}(f, \zeta)}{\left(N-\frac{1}{2}\right)^{\frac{1}{2}}}\right)\right) \\
& =\frac{\left(N-\frac{1}{2}\right)^{\frac{1}{2}}}{2}+\underset{f \in \mathbb{S}}{\mathbb{E}}\left(\frac{\mu_{F, \mathrm{av}}^{2}(f)}{\left(N-\frac{1}{2}\right)^{\frac{1}{2}}}\right)
\end{aligned}
$$

the second inequality since for all $x \geq 0$ and $a>0$ we have

$$
x^{1 / 2}\left(1+a^{2} x\right)^{1 / 2} \leq \frac{1}{2 a}+a x
$$

A call to Remark 4 finally yields

$$
\frac{1}{\sqrt{2}} \underset{(f, \dot{f}) \in \mathcal{S}}{\mathbb{E}}(\phi(f, \dot{f})) \leq \frac{\left(N-\frac{1}{2}\right)^{\frac{1}{2}}}{2}+\frac{(N-1) n}{\left(N-\frac{1}{2}\right)^{\frac{1}{2}}} \leq \sqrt{N}\left(\frac{1}{2}+n\right)
$$

Replacing this bound in (2.15) finishes the proof.

## 3 A Deterministic Algorithm

A deterministic solution for Smale's 17th problem is yet to be found. The state of the art for this theme is given in [9] where the following result is proven.

Theorem 3 There is a deterministic real-number algorithm that on input $f \in$ $\mathcal{H}_{(d)}$ computes an approximate zero of $f$ in average time $N^{\mathcal{O}(\log \log N)}$. Moreover, if we restrict data to polynomials satisfying

$$
D \leq n^{\frac{1}{1+\varepsilon}} \quad \text { or } \quad D \geq n^{1+\varepsilon}
$$

for some fixed $\varepsilon>0$, then the average time of the algorithm is polynomial in the input size $N$.

The algorithm exhibited in [9] uses two algorithmic strategies according to whether $D \leq n$ or $D>n$. In the first case, it applies a homotopy method and in the second an adaptation of a method coming from symbolic computation.

The goal of this section is to show that a more unified approach, where homotopy methods are used in both cases, yields a proof of Theorem 3 as well. Besides a gain in expositional simplicity, this approach can claim for it the wellestablished numerical stability of homotopy methods.

In all what follows we assume the simpler homotopy algorithm in [9] (as opposed to those in $[3,12]$ ). Its choice of step length at the $k$ th iteration is proportional to $\mu^{-2}\left(f_{t_{k}}, x_{t_{k}}\right)$ (which, in turn, is proportional to $\mu^{-2}\left(f_{t_{k}}, \zeta_{t_{k}}\right)$ ). For this algorithm, we have the $\mu^{2}$-estimate (1.3) but not the finer estimate (1.1).

To understand the technical requirements of the analysis of a deterministic algorithm, let us summarize an analysis (simpler than the one in the preceding section because of the assumption above) for the randomized algorithm. Recall, the latter draws an initial pair $(g, \zeta)$ from a distribution which amounts to first draw $g$ from the distribution on $\mathbb{S}$ and then draw $\zeta$ uniformly among the $\mathcal{D}$ zeros $\left\{\zeta^{(1)}, \ldots, \zeta^{(\mathcal{D})}\right\}$ of $g$. The $\mu^{2}$-estimate (1.3) provides an upper bound for the number of steps needed to continue $\zeta$ to a zero of $f$ following the great circle from $g$ to $f$ (assuming $\|f\|=\|g\|=1$ and $f \neq \pm g$ ). Now (1.3) does not change if we reparametrize $\left\{f_{t}\right\}_{t \in[0,1]}$ by arc-length, so we can also write it as

$$
K(f, g, \zeta) \leq C^{\prime} D^{3 / 2} \int_{0}^{d_{s}(g, f)} \mu^{2}\left(f_{s}, \zeta_{s}\right) d s
$$

where $d_{\mathbb{S}}(g, f)$ is the spherical distance from $g$ to $f$. Thus, the average number of homotopy iterations satisfies

$$
\begin{align*}
\underset{f \in \mathbb{S}}{\mathbb{E}} \underset{g \in \mathbb{S}}{\mathbb{E}} \frac{1}{\mathcal{D}} \sum_{i=1}^{\mathcal{D}} K\left(f, g, \zeta^{(i)}\right) & \leq C^{\prime} D^{3 / 2} \underset{f \in \mathbb{S}}{\mathbb{E}} \underset{g \in \mathbb{S}}{\mathbb{E}} \frac{1}{\mathcal{D}} \sum_{i=1}^{\mathcal{D}} \int_{0}^{d_{\mathbb{S}}(g, f)} \mu^{2}\left(f_{s}, \zeta_{s}\right) d s \\
& \leq C^{\prime} D^{3 / 2} \underset{f \in \mathbb{S}}{\mathbb{E}} \underset{g \in \mathbb{S}}{\mathbb{E}} \int_{0}^{d_{S}(g, f)} \mu_{F, \mathrm{av}}^{2}\left(f_{s}\right) d s \tag{3.16}
\end{align*}
$$

Let $P_{s}$ denote the set of pairs $(f, g) \in \mathbb{S}^{2}$ such that $d_{\mathbb{S}}(g, f) \geq s$. Rewriting the above integral using Fubini, we get

$$
\underset{f \in \mathbb{S}}{\mathbb{E}} \underset{g \in \mathbb{S}}{\mathbb{E}} \int_{0}^{d_{\mathbb{S}}(g, f)} \mu_{F, \mathrm{av}}^{2}\left(f_{s}\right) d s=\int_{0}^{\pi} \int_{P_{s}} \mu_{F, \mathrm{av}}^{2}\left(f_{s}\right) d f d g d s=\frac{\pi}{2} \underset{h \in \mathbb{S}}{\mathbb{E}} \mu_{F, \mathrm{av}}^{2}(h),
$$

the second equality holding since for a fixed $s \in[0, \pi]$ and uniformly distributed $(f, g) \in P_{s}$, one can show that the system $f_{s}$ is uniformly distributed on $\mathbb{S}$. Summarizing, we get

$$
\underset{f \in \mathbb{S}}{\mathbb{E}} \underset{g \in \mathbb{S}}{\mathbb{E}} \frac{1}{\mathcal{D}} \sum_{i=1}^{\mathcal{D}} K\left(f, g, \zeta^{(i)}\right) \leq C^{\prime} D^{3 / 2} \frac{\pi}{2} \underset{h \in \mathbb{S}}{\mathbb{E}} \mu_{F, \mathrm{av}}^{2}(h) \underset{\text { Rmk. } 4}{=} C^{\prime} D^{3 / 2} \frac{\pi}{2}(N-1) n
$$

This constitutes an elegant derivation of the previous $\mathcal{O}\left(n D^{3 / 2} N\right)$ bound (but not of the sharper bound of our Theorem 1).
Proof of Theorem 3. If the initial pair $(g, \zeta)$ is not going to be random we face two difficulties. Firstly - as $g$ is not random - the intermediate systems $f_{t}$ are not going to be uniformly distributed on $\mathbb{S}$. Secondly -as $\zeta$ is not random-
we will need a bound on a given $\mu^{2}\left(f_{t}, \zeta_{t}\right)$ rather than one on the mean of these quantities (over the $\mathcal{D}$ possible zeros of $f_{t}$ ), as provided by Theorem 2 .

Consider a fixed initial pair $(g, \zeta)$ with $g \in \mathbb{S}$ and let $s_{1}$ be the step length of the first step of the algorithm (see for example the definition of Algorithm ALH in [9]), which satisfies

$$
\begin{equation*}
s_{1} \geq \frac{c}{D^{3 / 2} \mu^{2}(g, \zeta)} \quad(c \text { a constant }) \tag{3.17}
\end{equation*}
$$

Note that this bound on the length $s_{1}$ of the first homotopy step depends on the condition $\mu(g, \zeta)$ only and is thus independent of the condition at the other zeros of $g$. Any of the mentioned versions of the continuation algorithm, not only that of [9], satisfies (3.17).

Consider also the (small) portion of great circle contained in $\mathbb{S}$ with endpoints $g$ and $f /\|f\|$, which we parametrize by arc length and call $h_{s}$ (that is, $h_{0}=g$ and $h_{\alpha}=f /\|f\|$ where $\left.\alpha=d_{\mathbb{S}}(g, f /\|f\|)\right)$ defined for $s \in[0, \alpha]$. Thus, after the first step of the homotopy, the current pair is $\left(h_{s_{1}}, x_{1}\right)$ and we denote by $\zeta^{\prime}$ the zero of $h_{s_{1}}$ associated to $x_{1}$. For a time to come, we will focus on bounding the quantity

$$
H:=H(g, \zeta):=\underset{f \in \mathcal{H}_{(d)}}{\mathbb{E}} \frac{1}{\mathcal{D}} \sum_{i=1}^{\mathcal{D}} K\left(f /\|f\|, h_{s_{1}}, \zeta^{(i)}\right)
$$

where the sum is over all the zeros $\zeta^{(i)}$ of $h_{s_{1}}$. This is the average of the number of homotopy steps over both the system $f$ and the $\mathcal{D}$ zeros of $h_{s_{1}}$. We will be interested in this average even though we will not consider algorithms following a path randomly chosen: the homotopy starts at the pair $(g, \zeta)$, moves to $\left(h_{s_{1}}, x_{1}\right)$ and proceeds following this path.

From (1.3) applied to $\left(h_{s_{1}}, \zeta^{\prime}\right)$,

$$
\begin{equation*}
K\left(f /\|f\|, h_{s_{1}}, \zeta^{(i)}\right) \leq C^{\prime} D^{3 / 2} \int_{s_{1}}^{\alpha} \mu^{2}\left(h_{s}, \zeta_{s}^{(i)}\right)\left\|\dot{h}_{s}\right\| d s \tag{3.18}
\end{equation*}
$$

Reparametrizing $\left\{h_{s}: s_{1} \leq s \leq \alpha\right\}$ by $\left\{f_{t} /\left\|f_{t}\right\|: t_{1} \leq t \leq 1\right\}$ where $f_{t}=$ $(1-t) g+t f$ and $t_{1}$ is such that $f_{t_{1}} /\left\|f_{t_{1}}\right\|=h_{s_{1}}$ (see Lemma 8 below) does not change the value of the path integral in (3.18). Moreover, a simple computation shows that

$$
\left\|\frac{d}{d t}\left(\frac{f_{t}}{\left\|f_{t}\right\|}\right)\right\| \leq \frac{\|f\|\|g\|}{\left\|f_{t}\right\|^{2}}=\frac{\|f\|}{\left\|f_{t}\right\|^{2}}
$$

so we have

$$
\begin{equation*}
K\left(f /\|f\|, h_{s_{1}}, \zeta^{(i)}\right) \leq C^{\prime} D^{3 / 2}\|f\| \int_{t_{1}}^{1} \frac{\mu^{2}\left(f_{t}, \zeta_{t}^{(i)}\right)}{\left\|f_{t}\right\|^{2}} d t \tag{3.19}
\end{equation*}
$$

Because of scale invariance, the quantity $H$ satisfies

$$
H=\underset{f \in \mathcal{H}_{(d)}^{\sqrt{2 N}}}{\mathbb{E}} \frac{1}{\mathcal{D}} \sum_{i=1}^{\mathcal{D}} K\left(f, h_{s_{1}}, \zeta^{(i)}\right)
$$

where the second expectation is taken over a truncated Gaussian (that only draws systems $f$ with $\|f\| \leq \sqrt{2 N}$ ) with density function given by

$$
\rho(f):= \begin{cases}\frac{1}{p} \varphi(f) & \text { if }\|f\| \leq \sqrt{2 N} \\ 0 & \text { otherwise }\end{cases}
$$

Here $\varphi$ is the density function of the standard Gaussian on $\mathcal{H}_{(d)}$ and $p:=$ $\operatorname{Prob}\{\|f\| \leq \sqrt{2 N}\}$. Then, using (3.19),

$$
H \leq \sqrt{2 N} C^{\prime} D^{3 / 2} \underset{f \in \mathcal{H} \mathcal{H}_{(d)}^{2 N}}{\mathbb{E}} \frac{1}{\mathcal{D}} \sum_{i=1}^{\mathcal{D}} \int_{t_{1}}^{1} \frac{\mu^{2}\left(f_{t}, \zeta_{t}^{(i)}\right)}{\left\|\dot{f}_{t}\right\|^{2}} d t
$$

From Lemma 8 we have

$$
t_{1} \geq \frac{c^{\prime}}{D^{3 / 2} \sqrt{N} \mu^{2}(g, \zeta)}
$$

for a constant $c^{\prime}$ (different from, but close to, $c$ ). We thus have proved that there are constants $C^{\prime \prime}, c^{\prime}$ such that

$$
\begin{aligned}
H & \leq C^{\prime \prime} \sqrt{N} D^{3 / 2} \underset{f \in \mathcal{H}_{(d)}^{\sqrt{2 N}}}{\mathbb{E}} \frac{1}{\mathcal{D}} \sum_{i=1}^{\mathcal{D}} \int_{\frac{c^{\prime}}{D^{3 / 2} \sqrt{N} \mu^{2}(g, \zeta)}}^{1} \frac{\mu^{2}\left(f_{t}, \zeta_{t}\right)}{\left\|\dot{f}_{t}\right\|^{2}} d t \\
& =C^{\prime \prime} \sqrt{N} D^{3 / 2} \underset{f \in \mathcal{H}_{(d)}^{\sqrt{2 N}}}{\mathbb{E}} \int_{\frac{c^{\prime}}{D^{3 / 2} \sqrt{N} \mu^{2}(g, \zeta)}}^{1} \frac{\mu_{\mathrm{av}}^{2}\left(f_{t}\right)}{\left\|\dot{f}_{t}\right\|^{2}} d t .
\end{aligned}
$$

The $2 N$ in the cut-off for the truncated Gaussian is the expectation of $\|f\|^{2}$ for a standard Gaussian $f \in \mathcal{H}_{(d)}$. Since this expectation is not smaller than the median of $\|f\|^{2}$ (see [11, Cor. 6]) we have $\frac{1}{p} \leq 2$. Using this inequality and the fact that the random variable we are taking the expectation of is nonnegative, we deduce that

$$
\begin{aligned}
H & \leq 2 C^{\prime \prime} \sqrt{N} D^{3 / 2} \underset{f \in \mathcal{H}_{(d)}}{\mathbb{E}} \int_{\frac{c^{\prime}}{D^{3 / 2} \sqrt{N} \mu^{2}(g, \zeta)}}^{1} \frac{\mu_{\mathrm{av}}^{2}\left(f_{t}\right)}{\left\|\dot{f}_{t}\right\|^{2}} d t \\
& \leq 2 C^{\prime \prime} \sqrt{N} D^{3 / 2} \int_{\frac{d^{\prime}}{D^{3 / 2} \sqrt{N} \mu(g, \zeta)^{2}}}^{1} \underset{f \in \mathcal{H}_{(d)}}{\mathbb{E}} \frac{\mu_{F, \mathrm{av}}^{2}\left(f_{t}\right)}{\left\|\dot{f}_{t}\right\|^{2}} d t
\end{aligned}
$$

We next bound the expectation in the right-hand side using Theorem 2 and the fact that $f_{t} \sim N\left((1-t) g, t^{2} \mathrm{Id}\right)$ and obtain

$$
\begin{align*}
H & \leq 2 n C^{\prime \prime} \sqrt{N} D^{3 / 2} \int_{\frac{c^{\prime}}{D^{3 / 2} \sqrt{N} \mu^{2}(g, \zeta)}}^{1} \frac{1}{t^{2}} d t \\
& \leq C^{\prime \prime \prime} D^{3} n N \mu^{2}(g, \zeta), \tag{3.20}
\end{align*}
$$

with $C^{\prime \prime \prime}$ yet another constant.
Having reached thus far, the major obstacle we face is that the quantity $H$, for which we derived the bound (3.20), is an average over all initial zeros of $h_{s_{1}}$ (as well as over $f$ ). For this obstacle, none of the two solutions below is fully satisfactory, but each of them is so for a broad choice of pairs $(n, D)$.
Case 1: $D>n$. Consider any $g \in \mathbb{S}, \zeta$ a well-posed zero of $g$, and let $\zeta^{(1)}, \ldots, \zeta^{(\mathcal{D})}$ be the zeros of $h_{s_{1}}$. Note that when $f$ is Gaussian, these are $\mathcal{D}$ different zeros almost surely. Clearly,

$$
\begin{aligned}
\underset{f \in \mathcal{H}_{(d)}}{\mathbb{E}} K(f, g, \zeta) & \leq 1+\underset{f \in \mathcal{H}_{(d)}}{\mathbb{E}} \sum_{i=1}^{\mathcal{D}} K\left(f, h_{s_{1}}, \zeta^{(i)}\right) \\
& =1+\mathcal{D} H=\mathcal{O}\left(\mathcal{D} D^{3} N n \mu^{2}(g, \zeta)\right)
\end{aligned}
$$

the last by (3.20). We now take as initial pair $(g, \zeta)$ the pair $\left(\bar{g}, e_{0}\right)$ where $\bar{g}=$ $\left(\bar{g}_{1}, \ldots, \bar{g}_{n}\right)$ is given by

$$
\bar{g}_{i}=\left(d_{1}^{-1}+\cdots+d_{n}^{-1}\right)^{-1 / 2} X_{0}^{d_{i}-1} X_{i}, \quad \text { for } i=1, \ldots, n
$$

(the scaling factor guaranteeing that $\|\bar{g}\|=1$ ) and $e_{0}=(1,0, \ldots, 0) \in \mathbb{C}^{n+1}$. It is known that $\mu\left(\bar{g}, e_{0}\right)=1$ (see [10, Rem. 16.18]) (and that all other zeros of $\bar{g}$ are ill-posed, but this is not relevant for our argument). Replacing this equality in the bound above we obtain

$$
\begin{equation*}
\underset{f \in \mathcal{H}_{(d)}}{\mathbb{E}} K\left(f, \bar{g}, e_{0}\right)=\mathcal{O}\left(\mathcal{D} D^{3} N n\right) \tag{3.21}
\end{equation*}
$$

which implies an average cost of $\mathcal{O}\left(\mathcal{D} D^{3} N^{2} n\right)$ since the number of operations at each iteration of the homotopy algorithm is $\mathcal{O}(N)$ (see [10, Proposition 16.32]).

For any $\varepsilon>0$ this quantity is polynomially bounded in $N$ provided $D \geq n^{1+\varepsilon}$ and is bounded as $N^{\mathcal{O}(\log \log N)}$ when $D$ is in the range $\left[n, n^{1+\varepsilon}\right]([9$, Lemma 11.1]).

Case 2: $D \leq n$. The occurrence of $\mathcal{D}$ makes the bound in (3.21) too large when $D$ is small. In this case, we consider the initial pair $\left(\bar{U}, \mathbf{z}_{1}\right)$ where $\bar{U} \in \mathcal{H}_{(d)}$ is given by

$$
\bar{U}_{1}=\frac{1}{\sqrt{2 n}}\left(X_{0}^{d_{1}}-X_{1}^{d_{1}}\right), \ldots, \bar{U}_{n}=\frac{1}{\sqrt{2 n}}\left(X_{0}^{d_{n}}-X_{n}^{d_{n}}\right)
$$

(the scaling factor guaranteeing that $\|\bar{U}\|=1$ ) and $\mathbf{z}_{1}=(1,1, \ldots, 1)$. We denote by $\mathbf{z}_{1}, \ldots, \mathbf{z}_{\mathcal{D}}$ the zeros of $\bar{U}$.

The reason for this choice is a strong presence of symmetries. These symmetries guarantee that, for all $1 \leq i, j \leq \mathcal{D}$,

$$
\begin{equation*}
\mu\left(\bar{U}, \mathbf{z}_{i}\right)=\mu\left(\bar{U}, \mathbf{z}_{j}\right) \tag{3.22}
\end{equation*}
$$

and, consequently, that the value of $s_{1}$ is the same for all the zeros of $\bar{U}$. Hence,

$$
\begin{align*}
\underset{f \in \mathcal{H}_{(d)}}{\mathbb{E}} K\left(f, \bar{U}, \mathbf{z}_{1}\right) & =\frac{1}{\mathcal{D}} \sum_{j=1}^{\mathcal{D}} \underset{f \in \mathcal{H}_{(d)}}{\mathbb{E}} K\left(f, \bar{U}, \mathbf{z}_{j}\right)=\underset{f \in \mathcal{H}_{(d)}}{\mathbb{E}} \frac{1}{\mathcal{D}} \sum_{j=1}^{\mathcal{D}} K\left(f, \bar{U}, \mathbf{z}_{j}\right) \\
& =\underset{f \in \mathcal{H}_{(d)}}{\mathbb{E}} \frac{1}{\mathcal{D}} \sum_{j=1}^{\mathcal{D}}\left(1+K\left(f, h_{s_{1}}, \zeta^{(j)}\right)\right)  \tag{3.23}\\
& =1+H\left(\bar{U}, \mathbf{z}_{1}\right) .
\end{align*}
$$

That is, the average (w.r.t. f) number of homotopy steps with initial system $\bar{U}$ is the same no matter whether the zero of $\bar{U}$ is taken at random or set to be $\mathbf{z}_{1}$. Also,

$$
\begin{equation*}
\mu^{2}\left(\bar{U}, \mathbf{z}_{1}\right) \leq 2(n+1)^{D} \tag{3.24}
\end{equation*}
$$

(actually such bound holds for all zeros of $\bar{U}$ but, again, this is not relevant for our argument). Both (3.22) and (3.24) are proved in [9, Section 10.2]. It follows from (3.23), (3.20), and (3.24) that

$$
\begin{equation*}
\underset{f \in \mathcal{H}_{(d)}}{\mathbb{E}} K\left(f, \bar{U}, \mathbf{z}_{1}\right)=\mathcal{O}\left(D^{3} N n^{D+1}\right) \tag{3.25}
\end{equation*}
$$

As above, for any fixed $\varepsilon>0$ this bound is polynomial in $N$ provided $D \leq n^{\frac{1}{1+\varepsilon}}$ and is bounded by $N^{\mathcal{O}(\log \log N)}$ when $D \in\left[n^{\frac{1}{1+\varepsilon}}, n\right]$.

Lemma 8 With the notations of the proof of Theorem 3 we have

$$
t_{1}=\frac{1}{\|f\| \sin \alpha \cot \left(s_{1} \alpha\right)-\|f\| \cos \alpha+1} \geq \frac{c^{\prime}}{D^{3 / 2} \sqrt{N} \mu^{2}(g, \zeta)}
$$

$c^{\prime}$ a constant.
Proof. The formula for $t_{1}$ is shown in [9, Prop. 5.2]. For the bound, we have

$$
\begin{aligned}
\|f\| \sin \alpha \cot \left(s_{1} \alpha\right)-\|f\| \cos \alpha+1 & \leq\|f\| \sin \alpha\left(s_{1} \alpha\right)^{-1}+\|f\|+1 \\
& \leq \sqrt{2 N} \frac{1}{s_{1}}+\sqrt{2 N}+1 \\
& \leq \sqrt{2 N}\left(\frac{D^{3 / 2} \mu^{2}(g, \zeta)}{c}+1+\frac{1}{\sqrt{2 N}}\right) \\
& \leq \frac{\sqrt{N} D^{3 / 2} \mu^{2}(g, \zeta)}{c^{\prime}}
\end{aligned}
$$

for an appropriately chosen $c^{\prime}$.

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[^1]:    ${ }^{1}$ In Shub [13] the theorem is actually proven in the projective space instead of the sphere, which is sharper, but we only use the sphere version in this paper.

