MULTIDIMENSIONAL PERSISTENCE AND NOISE

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ABSTRACT. In this paper we study multidimensional persistence modules [5, 13] via what we call tame functors and noise systems. A noise system leads to a pseudo-metric topology on the category of tame functors. We show how this pseudo-metric can be used to identify persistent features of compact multidimensional persistence modules. To count such features we introduce the feature counting invariant and prove that assigning this invariant to compact tame functors is a 1-Lipschitz operation. For 1-dimensional persistence, we explain how, by choosing an appropriate noise system, the feature counting invariant identifies the same persistent features as the classical barcode construction.

1. INTRODUCTION

The aim of this paper is to present a new perspective on *multidimensional per*sistence [5] and introduce a tool for creating numerous new invariants for multidimensional persistence modules. This new tool helps in extracting information by purposely defining what is not wanted. We do that by introducing the concept of a noise system and show how it leads to a continuous invariant. For one-dimensional persistence [19] and an appropriate choice of a noise system this invariant turns out to be closely related to the well-studied *barcode*. The barcode in one-dimensional persistence has proven itself to be a valuable tool for analysing data from a variety of different research areas (see e.g. [21], [22], [23] and [24]). Multidimensional persistence however has not yet had as much use in data analysis, even though its potential is even exceeding that of one-dimensional persistence. As an example whenever one has multiple measurements and wants to understand the relations between them this naturally translates into a multidimensional persistent module. Furthermore when one studies a space using a sampling, i.e. point cloud data, multidimensional persistence can provide additional insight into the geometrical properties of the space. These types of applications can be found for example in [1], where hepatic lesions were classified using multidimensional persistence, or in [20], where multidimensional persistence was used to help with content-based image retrieval.

The pipeline for using multidimensional persistence for data analysis starts with a choice of multiple measurements on a data set. These measurements are used (via for example the Čech or Vietoris-Rips constructions, see [4]) to get a topological

²⁰¹⁰ Mathematics Subject Classification. Primary: 55, 18, 68.

 $Key\ words\ and\ phrases.$ Multidimensional persistence, persistence modules, noise systems, stable invariants.

W. Chachólski is a corresponding author.

W. Chachólski was partially supported by Göran Gustafsson Stiftelse and VR grants. Communicated by Peter Olver.

space or a sequence of such, resulting in a functor $X: \mathbf{Q}^r \to \text{Spaces}$ where \mathbf{Q}^r is the poset of r-tuples of non-negative rational numbers (see 2.6) (we use rational instead of real numbers in order to avoid certain technical difficulties). The aim is to gain new insight into the data by extracting homological information out of these spaces. Applying the *i*-th homology with coefficients in a field K gives us a functor $H_i(X,K): \mathbf{Q}^r \to \operatorname{Vect}_K$ called an r-dimensional persistence module. The functors obtained in this way are often *tame* (see Definition 4.4). The category of tame functors $\operatorname{Tame}(\mathbf{Q}^r, \operatorname{Vect}_K)$ has very similar properties to the category of graded modules over the r-graded polynomial ring $K[x_1,\ldots,x_r]$. In the case r=1 this translates into the barcode being a complete discrete invariant for one-dimensional compact and tame persistence modules. For r > 1, it is known that no such discrete invariant can exist, as in this case the moduli space of r-dimensional compact and tame persistence modules is a positive dimensional algebraic variety, (see [5]). Furthermore, this variety is complicated enough that there is simply no realistic hope to find easily visualisable and continuous invariants completely describing their compact objects. In our opinion however looking for complete invariants of algebraic objects such as the multidimensional persistence modules is not the main goal of topological data analysis. For data analysis it is much more useful to be able to extract out of such modules their *continuous* features. That is why we propose that instead of focusing on the objects in $\text{Tame}(\mathbf{Q}^r, \text{Vect}_K)$ we study relations between them using topology and metrics as main tools.

By defining collections of tame functors that are ϵ -small, for every non-negative rational number ϵ , we create what we call a noise system in Definition 6.1. These noise systems help us to tell the size of tame functors and thus also which of these functors we can disregard and consider as noise. This leads us to define a pseudo-metric on the category $\text{Tame}(\mathbf{Q}^r, \text{Vect}_K)$ in Definition 8.6 and hence also induce a topology on that category. Equipped with this topology we define invariants called feature counting functions in Section 9. These invariants are functions $\operatorname{bar}(F): \mathbf{Q} \to \mathbf{N}$ (with values in the set of natural numbers) which for a given functor F and a positive rational number ϵ return the smallest rank of a functor in an ϵ -neighbourhood of F. We then show in Proposition 9.3 that the assignment $F \mapsto \operatorname{bar}(F)$ is not just continuous, but also 1-Lipschitz with respect to the topology we just introduced. A standard way of producing invariants of multidimensional persistence modules is a reduction to one-dimensional case by restricting the modules to one parameter submodules and then using one persistence. This is the key idea behind invariants such as the rank invariant [5] and more generally multidimensional PBNs in [7]. By an appropriate choice of a noise system, these can be recovered as feature counting functions. For arbitrary noises however, feature counting functions provide a much wider set of stable invariants for multidimensional persistence modules, invariants that go beyond the reduction to the one-dimensional case.

Organisation of the Paper. In Section 2 we go through the notation and background needed for the paper. This continues in Section 3 which contains some background on functors indexed by r-tuples of the natural numbers. These results will be crucial in Section 4 where we instead look at functors indexed by r-tuples of the rational numbers and introduce the concept of tameness for functors. Although related, our notion of tameness for functors is not exactly analogous to the concept of tame functions described in [12]. In Section 5 we prove some fundamental properties and show how to compute certain homological invariants of tame and compact functors. Tame and compact functors are our main object of study. Such functors are less general than q-tame persistence modules as defined in [8] since compact functors have finite dimensional values. Nevertheless many applications, as for example the ones defined in [5], can be modelled through such objects.

Section 6 contains the definition of a noise system and several examples of different explicit noise systems. We explore this further in Section 7 where we look at under which circumstances a noise system is closed under direct sums. Section 8 then uses the notion of a noise system to define a pseudo-metric on tame functors inducing a topology on such functors. This allows us to define noise dependent invariants, called feature counting invariants in Section 9, which we prove are 1-Lipschitz with respect to the pseudo-metric. In Section 10 we show that the feature counting invariants generalise the barcode from one-dimensional persistence. We provide a simplified description of the feature counting invariant for the standard noise in the multidimensional case in Section 11. This noise system is the most natural one with respect to what is typically considered to be noise in multipersistence via the interleaving distance of [13]. In Section 12 we describe the notion of denoising and how it (hopefully) can help with computing the feature counting invariants. In section 13 we outline some possible directions for future results. Lastly in the Appendix (Section 14) we prove, for completeness, properties of the category of vector space valued functors indexed by \mathbf{N}^r and construct minimal covers in this category.

We would like to thank Claudia Landi for inspiring discussions about stability.

2. NOTATION AND BACKGROUND

2.1. The symbols Sets and Vect_K denote the categories of respectively sets and K-vector spaces where K is always assumed to be a field. Given a K-vector space, we denote its dimension by $\dim_K V$. The linear span functor, denoted by $K : \text{Sets} \to \text{Vect}_K$, assigns to a set S the vector space $K(S) := \bigoplus_S K$ with base S and to a function $f: S \to S'$ the homomorphism $K(f): \bigoplus_S K \to \bigoplus_{S'} K$ given by f on the bases.

2.2. Let I be a small category and C be a category. The symbol C^{op} denotes the opposite category to C and $\operatorname{Fun}(I, C)$ denotes the category of functors indexed by I with values in C and natural transformations as morphisms (see [15]). We use the symbol $\operatorname{Nat}(F, G)$ to denote the set of natural transformations between two functors $F, G: I \to C$. If C is abelian, then so is $\operatorname{Fun}(I, C)$. A sequence of composable morphisms in $\operatorname{Fun}(I, C)$ is exact if and only if its values at any object iin I form an exact sequence in C. If C has enough projective objects, then so does $\operatorname{Fun}(I, C)$.

2.3. Let X be a set. A **multiset** on X is a function $\beta: X \to \mathbf{N}$ of sets where **N** denotes the set of natural numbers. A multiset β is **finite** if $\beta(x) \neq 0$ for only finitely many x in X. If β is a finite multiset on X, then its **size** is defined as $\sum_{x \in X} \beta(x)$. We say that $\beta: X \to \mathbf{N}$ is a **subset** of $\gamma: X \to \mathbf{N}$ if $\beta(x) \leq \gamma(x)$ for any x in X.

2.4. Let *i* be an object in a small category *I*. The symbol $K_I(i, -): I \to \operatorname{Vect}_K$ denotes the composition of the representable functor $\operatorname{mor}_I(i, -): I \to \operatorname{Sets}$ with

the linear span functor $K: \text{Sets} \to \text{Vect}_K$. This functor is called **free on one generator**. We often omit the subscript I and write K(i, -).

Let $\{V_i\}_{i\in I}$ be a sequence of K-vector spaces indexed by objects in I. Functors of the form $\bigoplus_{i\in I} K(i, -) \otimes V_i$ are called **free**. Two free functors $\bigoplus_{i\in I} K(i, -) \otimes V_i$ and $\bigoplus_{i\in I} K(i, -) \otimes W_i$ are isomorphic if and only if, for any i in I, the vector spaces V_i and W_i are isomorphic. Let $F = \bigoplus_{i\in I} K(i, -) \otimes V_i$ be a free functor. The vector spaces V_i are called the **components** of F. If all component are finite dimensional, then F is called of **finite type**. The **support** of a free functor $\bigoplus_{i\in I} K(i, -) \otimes V_i$ is the subset of the set of objects of I consisting of those i in I for which $V_i \neq 0$. A free functor is said to be of **finite rank** if has finite support and is of finite type. If $F = \bigoplus_{i\in I} K(i, -) \otimes V_i$ is of finite rank, the number $\operatorname{rank}(F) := \sum_{i\in I} \dim_K V_i$ is called the **rank** of F.

Consider a free functor $F = \bigoplus_{i \in I} K(i, -) \otimes V_i$ of finite type. The 0-**Betti diagram** of F is defined to be the multiset on the set of objects of I given by $\beta_0 F(i) := \dim_K V_i$. The 0-Betti diagram of a free finite type functor determines its isomorphism type. Note that if F is free and of finite rank, then the multiset $\beta_0 F$ is finite of size rank(F).

2.5. A morphism $\phi: X \to Y$ in a category \mathcal{C} is called **minimal** if any morphism $f: X \to X$ satisfying $\phi = \phi f$ is an isomorphism. A natural transformation $\phi: F \to G$ in Fun $(I, \operatorname{Vect}_K)$ is called a **minimal cover** of G, if F is free and ϕ is both minimal and an epimorphism. Minimal covers are unique up to isomorphism: if $\phi: F \to G$ and $\phi': F' \to G$ are minimal covers of G, then there is an isomorphism (non necessarily unique) $f: F \to F'$ such that $\phi = \phi' f$. Furthermore any $g: F \to F'$ for which $\phi = \phi' g$ is an isomorphism (minimality).

Consider a functor $G: I \to \operatorname{Vect}_K$ that admits a minimal cover $\phi: F \to G$. If F is of finite type, then we say that G is of **finite type**. If F is of finite rank, then we say that G is of **finite rank** and define the **rank** of G to be the rank of the free functor F and denote it by $\operatorname{rank}(G)$. We define the **support** of G to be the support of F and denote it by $\operatorname{supp}(G)$. Note that G is of finite rank if and only if it has finite support and is of finite type. If G is of finite type, we define the **0-Betti diagram** of G to be the multiset on the set of objects of I given by the 0-Betti diagram of the free finite type functor F (see 2.4) and denote it by $\beta_0 G$. Being of finite type, of finite rank, and the invariants $\operatorname{rank}(G)$, $\operatorname{supp}(G)$, and $\beta_0 G$ do not depend on the choice of the minimal cover of G.

Consider a functor $G: I \to \operatorname{Vect}_K$, recall that an element g in G(i) induces a unique natural transformation, denoted by the same symbol $g: K(i, -) \to G$, that maps the element id_i in $\operatorname{Kmor}_I(i, i) = K(i, i)$ to g in G(i). A **minimal set of generators** for G is a sequence of elements $\{g_1 \in G(i_1), \ldots, g_n \in G(i_n)\}$ such that the induced natural transformation $\bigoplus_{k=1}^n g_k: \bigoplus_{k=1}^n K(i_k, -) \to G$ is a minimal cover. A functor has a minimal set of generators if and only if it is of finite rank, in which case the number of minimal generators is given by $\operatorname{rank}(G)$. If $\{g_1 \in G(i_1), \ldots, g_n \in G(i_n)\}$ is a minimal set of generators of G, then the multiset $\beta_0 G: I \to \mathbf{N}$ assigns to an object i in I the number of generators that belong to G(i).

2.6. Let r be a positive natural number and $v = (v_1, \ldots, v_r)$ and $w = (w_1, \ldots, w_r)$ be r-tuples of non negative rational numbers. Define:

- $||v w|| := \max\{|v_i w_i| \mid 1 \le i \le r\},\$
- $v \leq w$ if $v_i \leq w'_i$ for any i,

• v < w if $v \le w$ and $v \ne w$.

We call the number ||w|| the **norm** of w. The relation \leq is a partial order. We use the symbol \mathbf{Q}^r to denote the category associated to this poset, i.e., the category whose objects are r-tuples of non negative rational numbers with the sets of morphisms $\operatorname{mor}_{\mathbf{Q}^r}(v, w)$ being either empty if $v \not\leq w$, or consisting of only one element in the case $v \leq w$. Note that if $v \leq w \leq u$ in \mathbf{Q}^r , then $||v - w|| \leq ||v - u||$. The full subcategory of \mathbf{Q}^r whose objects are r-tuples of natural numbers is denoted by \mathbf{N}^r . Both posets \mathbf{Q}^r and \mathbf{N}^r are lattices. This means that for any finite set of elements Sin \mathbf{Q}^r (respectively \mathbf{N}^r), there are elements meet(S) and join(S) in \mathbf{Q}^r (respectively \mathbf{N}^r) with the following properties. First, for any v in S, meet(S) $\leq v \leq$ join(S). Second, if u and w are elements in \mathbf{Q}^r (respectively \mathbf{N}^r) for which $u \leq v \leq w$, for any v in S, then $u \leq$ meet(S) and join(S) $\leq w$. Observe that the elements meet(S) and join(S) may not belong to S.

Let S be a subset in \mathbf{Q}^r (respectively \mathbf{N}^r). An element v in S is called **minimal** if, for any w < v, w does not belong to S. The set of minimal elements of any non-empty subset of \mathbf{N}^r is never empty and is finite. Neither of these statements are true for \mathbf{Q}^r .

The element in \mathbf{Q}^r whose coordinates are all 0 is called either the **origin** or the **zero** element and is denoted simply by 0. The element in \mathbf{Q}^r whose coordinates are all 0 except the *i*-th one which is 1 is called the *i*-th **standard vector** and denoted by e_i .

2.7. The set of all linear combinations of elements g_1, \ldots, g_n in \mathbf{Q}^r with nonnegative rational coefficients is called the **cone generated** by g_1, \ldots, g_n and denoted by $\operatorname{Cone}(g_1, g_2, \ldots, g_n)$. A **cone** in \mathbf{Q}^r is by definition a subset of \mathbf{Q}^r of the form $\operatorname{Cone}(g_1, g_2, \ldots, g_n)$ for some non-empty sequence of elements g_1, \ldots, g_n in \mathbf{Q}^r . A **ray** is a cone in \mathbf{Q}^r generated by one non-zero element.

2.8. Let w be in \mathbf{Q}^r and \mathcal{C} be a category. Consider the functor $- + w : \mathbf{Q}^r \to \mathbf{Q}^r$ that maps $v \leq u$ to $v + w \leq u + w$. The composition of - + w with a functor $F: \mathbf{Q}^r \to \mathcal{C}$ is called the *w*-translation of F and is denoted by $F(-+w): \mathbf{Q}^r \to \mathcal{C}$. Let τ be in \mathbf{Q}^r . Two functors F, G in $\operatorname{Fun}(\mathbf{Q}^r, \mathcal{C})$ are τ -interleaved if there exist natural transformations $\phi: F \to G(-+\tau)$ and $\psi: G \to F(-+\tau)$ such that $\psi_{v+\tau} \circ \phi_v = F(v \leq v + 2\tau)$ and $\phi_{v+\tau} \circ \psi_v = G(v \leq v + 2\tau)$ for any v in \mathbf{Q}^r . This definition follows the definition of interleaving given in [3] for functors indexed by a preordered set.

2.9. The symbol \mathbf{R}_{∞} denotes the poset whose underlying set is the disjoint union of the set of non-negative real numbers and the singleton $\{\infty\}$. The order on \mathbf{R}_{∞} is given by the usual being smaller or equal relation for non-negative real numbers and assuming that $x \leq \infty$ for any x in \mathbf{R}_{∞} .

An extended pseudometric on a set X is a function $d: X \times X \to \mathbf{R}_{\infty}$ such that:

(1) for any x and y in X, d(y, x) = d(x, y);

(2) for any x in X, d(x, x) = 0;

(3) for any x, y, and z in X, $d(x, z) \leq d(x, y) + d(y, z)$.

A set equipped with an extended pseudometric is called an extended pseudometric space.

Let d be an extended pseudometric on a set X. For any positive real number t and any x in X, B(x,t) denotes the subset of X consisting of these elements y for which d(x, y) < t. This subset is called the open ball around x with radius t. These sets form a base of a topology on X.

Let k be a positive real number. Given two extended pseudometric spaces (X, d_X) and (Y, d_Y) , a function $f: X \to Y$ is called k-Lipschitz if $d_Y(f(x_1), f(x_2)) \le k d_X(x_1, x_2)$ for any x_1, x_2 in X.

3. Functors indexed by \mathbf{N}^r

In this section we recall how to determine that a functor $F: \mathbf{N}^r \to \operatorname{Vect}_K$ is of finite rank and how to, for a such a functor, compute its support, rank and 0-Betti diagram. The idea of using the 0-Betti diagram as an informative invariant in the context of multidimensional persistence was first introduced in [5].

We will also recall the classification of finite rank functors in the case r = 1. These are standard results as the category $\operatorname{Fun}(\mathbf{N}^r, \operatorname{Vect}_K)$ is equivalent to the category of *r*-graded modules over the polynomial ring in *r* variables with the standard *r*-grading (see [5]). In the appendix we present a classical way of analysing basic properties of the category $\operatorname{Fun}(\mathbf{N}^r, \operatorname{Vect}_K)$, where we in particular identify its compact and projective objects, and discuss minimal covers. We do that for self containment of the paper and to illustrate that this material including all the proofs and the classification of compact functors for r = 1 can be presented on less than 4 pages.

The radical of a functor $G: \mathbf{N}^r \to \operatorname{Vect}_K$ is the key tool to determine its rank, support, and 0-Betti diagram. Recall that the radical of G is a subfunctor $\operatorname{rad}(G) \subset G$ whose value $\operatorname{rad}(G)(v)$ is the subspace of G(v) given by the sum of all the images of $G(u < v): G(u) \to G(v)$ for all u < v (see 14.2). The quotient $G/\operatorname{rad}(G)$ is isomorphic to a functor of the form $\bigoplus_{v \in \mathbf{N}^r} (U_v \otimes V_v)$, where $\{V_v\}_{v \in \mathbf{N}^r}$ is a sequence of vector spaces and $U_v: \mathbf{N}^r \to \operatorname{Vect}_K$ is the unique functor such that $U_v(v) = K$ and $U_v(w) = 0$ if $w \neq v$ (see 14.1). We can now state (for the proofs see 14.4 and 14.8):

- Any functor in $\operatorname{Fun}(\mathbf{N}^r, \operatorname{Vect}_K)$ admits a minimal cover.
- G is of finite type if V_v is finite dimensional for any v.
- $\operatorname{supp}(G) = \{ v \in \mathbf{N}^r \mid V_v \neq 0 \}.$
- G is of finite rank if and only if {v ∈ N^r |V_v ≠ 0} is a finite set and V_v is finite dimensional for any v, in which case rank(G) := ∑_{v∈N^r} dim_KV_v.
 β C(v) = dim vV

• $\beta_0 G(v) = \dim_K V_v.$

Let $w \leq u$ be in **N**. Recall that the bar starting in w and ending in u is a functor $[w, u): \mathbf{N} \to \operatorname{Vect}_K$ given by the cokernel of the unique inclusion $K(u, -) \subset K(w, -)$ (see 14.14). The classification of finite rank functors in $\operatorname{Fun}(\mathbf{N}, \operatorname{Vect}_K)$ states (14.15):

• Any functor of finite rank $F: \mathbf{N} \to \operatorname{Vect}_K$ is isomorphic to a finite direct sum of functors of the form [w, u) and K(v, -). Moreover the isomorphism types of these summands are uniquely determined by the isomorphism type of F.

The above theorem, also known as the structure theorem for finitely generated graded modules over PID's, allows us to decompose any functor of finite rank $F: \mathbf{N} \to \operatorname{Vect}_K$ as a direct sum of bars and free functors. Such decomposition in persistent homology is commonly visualised through a barcode where each bar represents an indecomposable summand (see [19]).

4. TAMENESS

In this section we introduce the category of tame functors indexed by \mathbf{Q}^r . Intuitively, a tame functor is an extension of a functor indexed by \mathbf{N}^r to a functor indexed by \mathbf{Q}^r , which is constant on regions we call fundamental domains. In the following we will be particularly interested in tame and compact functors with values in Vect_K (for compactness see 14.11). On one side, computing homological invariants of such functors, can be recasted to computing analogous invariants of functors indexed by \mathbf{N}^r . On the other, the indexing category \mathbf{Q}^r offers new ways of comparing and measuring distances between tame functors. One could possibly use real number and define tame functors to be indexed by \mathbf{R}^r . For technical reasons however we decided to use the rational numbers \mathbf{Q} . This in our view is not a restrictive choice, as many functors are coming from data sets which are obtained by incremental and discrete measurements, for instance see the examples presented in [5].

4.1. Fundamental domain. If a group acts on a topological space X, a connected subspace of X which contains exactly one representative of each orbit is called a fundamental domain. For example the product of half open intervals $[0,1)^r$ is an example of a fundamental domain of the action of \mathbf{Z}^r on \mathbf{R}^r by translations. For studying multidimensional persistence we would like to replace the group with the monoid \mathbf{N}^r acting by various translations on \mathbf{Q}^r . In this article we are interested in actions given by the following different embeddings of \mathbf{N}^r in \mathbf{Q}^r . Let α be a positive rational number and let the same symbol $\alpha \colon \mathbf{N}^r \to \mathbf{Q}^r$ denote the unique functor that maps an object w in \mathbf{N}^r to αw (the multiplication of all the coordinates of w by α) in \mathbf{Q}^r . Then for v in \mathbf{Q}^r , consider the following finite subset $B_\alpha v := \{w \in \mathbf{N}^r \mid \alpha w \leq v\}$ of \mathbf{N}^r and define $b_\alpha v := \mathrm{join } B_\alpha(v)$ (see 2.6). For any w in \mathbf{N}^r we call the subset $D_\alpha w = \{v \in \mathbf{Q}^r \mid b_\alpha v = w\} \subset \mathbf{Q}^r$ the fundamental domain of α at w.

4.2. **Example.** Let $\alpha = 1$, so that $\alpha : \mathbf{N}^2 \to \mathbf{Q}^2$ is the standard inclusion $x \mapsto x$. Then for any $v \in [2,3) \times [2,3) \subset \mathbf{Q}^2$, we have that $B_{\alpha}v = \{w \in \mathbf{N}^2 \mid w \leq v\} = \{(x,y) \in \mathbf{N}^2 \mid x, y \leq 2\}$ and $b_{\alpha}v = (2,2)$. Conversely the fundamental domain (of 1) at (2,2) is $D_1(2,2) = [2,3) \times [2,3) \subset \mathbf{Q}^2$. This is indicated in Figure 1.

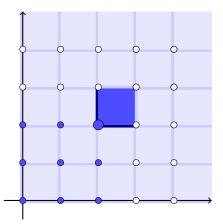


FIGURE 1. $D_1(2,2)$, B_1v and b_1v for $v \in D_1(2,2)$

Note that we have the following properties:

- (1) the fundamental domain $D_{\alpha}w$ consists of elements v in \mathbf{Q}^{r} such that $\alpha w \leq v$ and $||v - \alpha w|| < \alpha$ i.e. all the coordinates of $v - \alpha w$ are non-negative and are strictly smaller than α ;
- (2) for any v in \mathbf{Q}^r , $\alpha b_{\alpha} v \leq v$ and $||v \alpha b_{\alpha} v|| < \alpha$;
- (3) for any w in \mathbf{N}^r , $w = b_\alpha \alpha w$;
- (4) if $v \leq u$ in \mathbf{Q}^r , then $B_{\alpha}v \subset B_{\alpha}u$ and hence $b_{\alpha}v \leq b_{\alpha}u$ in \mathbf{N}^r .

We use these properties to extend functors indexed by \mathbf{N}^r to functors indexed by \mathbf{Q}^r along the embedding $\alpha \colon \mathbf{N}^r \to \mathbf{Q}^r$. For $F \colon \mathbf{N}^r \to C$, define a new functor $\alpha^! F \colon \mathbf{Q}^r \to C$, by letting $\alpha^! F(v) := F(b_\alpha v)$ for v in \mathbf{Q}^r and setting $\alpha^! F(v \le u) \colon \alpha^! F(v) \to \alpha^! F(u)$ to be $F(b_\alpha v \le b_\alpha u)$. By construction this functor is constant on all the fundamental domains, i.e., $F(b_\alpha v \le v)$ is the identity for any v in \mathbf{Q}^r .

4.3. **Example.** Consider the semi-simple functor $U_{(2,2)}$: $\mathbf{N}^2 \to \operatorname{Vect}_K$ (see 14.1). By definition the value of $1^!U_{(2,2)}$: $\mathbf{Q}^2 \to \operatorname{Vect}_K$ is K on the fundamental domain $D_1(2,2) = [2,3) \times [2,3)$ and zero otherwise. Furthermore $1^!U_{(2,2)}(v \leq u)$ is the identity morphism if $b_1v = b_1u = (2,2)$ and the zero morphism otherwise.

We claim that the construction $(F: \mathbf{N}^r \to \mathcal{C}) \mapsto (\alpha^! F: \mathbf{Q}^r \to \mathcal{C})$ is natural. To see this let the restriction functor along α be denoted by $\alpha^*: \operatorname{Fun}(\mathbf{Q}^r, \mathcal{C}) \to \operatorname{Fun}(\mathbf{N}^r, \mathcal{C})$. It assigns to a functor $F: \mathbf{Q}^r \to \mathcal{C}$ the composition $F\alpha: \mathbf{N}^r \to \mathcal{C}$. Note that, for any natural transformation $\psi: F \to G$ in $\operatorname{Fun}(\mathbf{N}^r, \mathcal{C})$, there is a unique natural transformation $\alpha^! \psi: \alpha^! F \to \alpha^! G$ for which $\alpha^* \alpha^! \psi = \psi$ (as $\alpha^! F$ is constant on the fundamental domains). Because of this uniqueness it is clear that $\alpha^! (\psi\phi) = (\alpha^! \psi) (\alpha^! \phi)$ and so $\alpha^!$ is a functor.

Let $F: \mathbf{N}^r \to \mathcal{C}$ be a functor. Since $\alpha^! F$ is constant on fundamental domains, a natural transformation $\psi: \alpha^! F \to G$ into any functor $G: \mathbf{Q}^r \to \mathcal{C}$, is uniquely determined by its restriction $\alpha^* \psi: F = \alpha^* \alpha^! F \to \alpha^* G$. This gives a bijection between $\operatorname{Nat}_{\mathbf{Q}^r}(\alpha^! F, G)$ and $\operatorname{Nat}_{\mathbf{N}^r}(F, \alpha^* G)$. In categorical terms this means that $\alpha^!: \operatorname{Fun}(\mathbf{N}^r, \mathcal{C}) \to \operatorname{Fun}(\mathbf{Q}^r, \mathcal{C})$ is left adjoint to the restriction functor $\alpha^*: \operatorname{Fun}(\mathbf{Q}^r, \mathcal{C}) \to \operatorname{Fun}(\mathbf{N}^r, \mathcal{C})$. Recall that such left adjoints are also called left Kan extensions (see [15]). In particular, since $\alpha^* \alpha^! G = G$ for any $G: \mathbf{N}^r \to \mathcal{C}$, the function $(\psi: F \to G) \mapsto (\alpha^! \psi: \alpha^! F \to \alpha^! G)$ is a bijection between $\operatorname{Nat}(F, G)$ and $\operatorname{Nat}(\alpha^! F, \alpha^! G)$.

Let $G: \mathbf{Q}^r \to \operatorname{Vect}_K$ be a functor. The natural transformation adjoint to the identity id: $\alpha^* G \to \alpha^* G$ is denoted by $\omega: \alpha^! \alpha^* G \to G$. Explicitly, for v in \mathbf{Q}^r , the morphism $\omega_v: \alpha^! \alpha^* G(v) = G(\alpha b_\alpha v) \to G(v)$ is given by $G(\alpha b_\alpha v \leq v)$.

4.4. **Definition.** Let α be in **Q**. A functor $G: \mathbf{Q}^r \to \mathcal{C}$ is called α -tame if it is isomorphic to a functor of the form $\alpha^! F$ for some $F: \mathbf{N}^r \to \mathcal{C}$. A functor is tame if it is α -tame for some $\alpha \in \mathbf{Q}$.

We will use the symbol $\text{Tame}(\mathbf{Q}^r, \mathcal{C})$ to denote the full subcategory of $\text{Fun}(\mathbf{Q}^r, \mathcal{C})$ whose objects are tame functors.

Note $G: \mathbf{Q}^r \to \mathcal{C}$ is α -tame if and only if $\omega: \alpha^! \alpha^* G \to G$ is an isomorphism, i.e., if $G(\alpha b_{\alpha} v \leq v)$ is an isomorphism for any v in \mathbf{Q}^r . Furthermore $G: \mathbf{Q}^r \to \mathcal{C}$ is α -tame if and only if $G(\alpha w \leq v)$ is an isomorphism for any w in \mathbf{N}^r and v in \mathbf{Q}^r such that $\alpha w \leq v$ and $||v - \alpha w|| < \alpha$. In other words, tame functors are exactly the functors that are constant on fundamental domains. Note that the following diagram commutes for any positive rational α :

$$\mathbf{N}^r \xrightarrow{n} \mathbf{N}^r \xrightarrow{\alpha/n} \mathbf{Q}^r$$

Thus the functors $\alpha' F$ and $(\alpha/n)! n' F$ are naturally isomorphic proving:

4.5. **Proposition.** If $G: \mathbf{Q}^r \to \mathcal{C}$ is α -tame, then it is also α/n -tame for any positive natural number n.

Another operation on functors that preserve tameness is translation. Let w be in \mathbf{Q}^r . Recall that the *w*-translation of $F: \mathbf{Q}^r \to \mathcal{C}$, denoted by F(-+w), is the composition of F with the functor $-+w: \mathbf{Q}^r \to \mathbf{Q}^r$ that maps $v \leq u$ to $v + w \leq u + w$ (see 2.8).

4.6. **Proposition.** If $F: \mathbf{Q}^r \to \mathcal{C}$ is tame, then so is F(-+w) for any w in \mathbf{Q}^r .

Proof. Assume F is α -tame. Consider $w = (w_1, \ldots, w_r)$ in \mathbf{Q}^r . Since α and the coordinates of w are rational, there are natural numbers m and n_1, \ldots, n_r such that $\alpha/m = w_1/n_1 = \cdots w_r/n_r = \mu$. We claim that F(-+w) is μ -tame. For that we need to show that, for any v in \mathbf{Q}^r , $F(\mu b_\mu v + w \leq v + w)$ is an isomorphism. Set $u := (n_1, \ldots, n_r)$ in \mathbf{N}^r . Note that $w = \mu u$, and hence $b_\mu(v + w) = b_\mu v + u$, which implies $\mu b_\mu(v + w) = \mu b_\mu v + w$. The morphism $F(\mu b_\mu v + w \leq v + w)$ is then an isomorphism since F is also μ -tame (see 4.5).

5. The CATEGORY $\operatorname{Tame}(\mathbf{Q}^r, \operatorname{Vect}_K)$

In this section we describe basic properties of the category $\text{Tame}(\mathbf{Q}^r, \text{Vect}_K)$ including invariants called the 0-Betti diagrams. For that we need to discuss the restriction α^* and the Kan extension $\alpha^!$ for functors with values in Vect_K (similar properties hold for functors with values in any abelian category). Note that Proposition 5.2 and Corollary 5.3 are false if $\text{Tame}(\mathbf{Q}^r, \text{Vect}_K)$ is replaced by $\text{Tame}(\mathbf{R}^r, \text{Vect}_K)$.

5.1. **Proposition.** Let α be a positive rational number.

- (1) The left Kan extension $\alpha^! K(v, -) \colon \mathbf{Q}^r \to \operatorname{Vect}_K \text{ of } K(v, -) \colon \mathbf{N}^r \to \operatorname{Vect}_K$ is isomorphic to $K(\alpha v, -)$ and hence is free.
- (2) The restriction of K(v, -): $\mathbf{Q}^r \to \operatorname{Vect}_K$ along $\alpha \colon \mathbf{N}^r \to \mathbf{Q}^r$ is also free and isomorphic to $K(\operatorname{meet}\{w \in \mathbf{N}^r \mid v \leq \alpha w\}, -)$.
- (3) Both functors α^* : Fun(\mathbf{Q}^r , Vect_K) \rightleftharpoons Fun(\mathbf{N}^r , Vect_K) : $\alpha^!$ commute with arbitrary colimits and in particular with direct sums.
- (4) A sequence of morphisms in $\operatorname{Fun}(\mathbf{N}^r, \operatorname{Vect}_K)$ is exact if and only if $\alpha^!$ transforms it into an exact sequence in $\operatorname{Fun}(\mathbf{Q}^r, \operatorname{Vect}_K)$.
- (5) If a sequence of morphisms in $\operatorname{Fun}(\mathbf{Q}^r, \operatorname{Vect}_K)$ is exact, then so is its restriction via α in $\operatorname{Fun}(\mathbf{N}^r, \operatorname{Vect}_K)$.
- (6) A functor $F: \mathbf{N}^r \to \operatorname{Vect}_K$ is free if and only if $\alpha^! F: \mathbf{Q}^r \to \operatorname{Vect}_K$ is free.
- (7) If $\phi: F \to G$ is a minimal cover in Fun $(\mathbf{N}^r, \operatorname{Vect}_K)$, then $\alpha' \phi: \alpha' F \to \alpha' G$ is a minimal cover in Fun $(\mathbf{Q}^r, \operatorname{Vect}_K)$.
- (8) If F in $\operatorname{Fun}(\mathbf{N}^r, \operatorname{Vect}_K)$ is compact (see 14.11), then so is the functor $\alpha^! F$ in $\operatorname{Fun}(\mathbf{Q}^r, \operatorname{Vect}_K)$.
- (9) Let $F: \mathbf{Q}^r \to \operatorname{Vect}_K$ be α -tame. Then F is compact in $\operatorname{Fun}(\mathbf{Q}^r, \operatorname{Vect}_K)$ if and only if $\alpha^* F$ is compact in $\operatorname{Fun}(\mathbf{N}^r, \operatorname{Vect}_K)$.

(10) Let $0 \to F \to G \to H \to 0$ be an exact sequence of tame functors in $\operatorname{Fun}(\mathbf{Q}^r, \operatorname{Vect}_K)$. Then G is compact if and only if F and H are compact.

Proof. Statement (1) and (2) are clear. Statement (3) follows from the construction and the fact that colimits in functor categories are formed object-wise. Same argument gives (4) and (5). Statement (6) is implied by (1), (2), and (3). To prove (7) note that by (6) and (4) we know $\alpha^! F$ is free and $\alpha^! \phi : \alpha^! F \to \alpha^! G$ is an epimorphism. Minimality of $\alpha^! \phi$ follows from the fact that $\alpha^!$ induces a bijection between Nat(F, F) and Nat $(\alpha^! F, \alpha^! F)$. Since the same argument can be used to prove both (8) and (9), we present the details of how to show (8) only. Consider a compact functor F in Fun $(\mathbf{N}^r, \operatorname{Vect}_K)$ and a sequence of subfunctors $A_0 \subset A_1 \subset \cdots \subset \alpha^! F$ in Fun $(\mathbf{Q}^r, \operatorname{Vect}_K)$ such that colim $A_i = \alpha^! F$. By taking the restriction along $\alpha \colon \mathbf{N}^r \to \mathbf{Q}^r$ and using (3) we obtain a filtration $\alpha^* A_0 \subset \alpha^* A_1 \subset \cdots \subset \alpha^* \alpha^! F = F$ such that colim $(\alpha^* A_i) = F$. As F is compact, there is n such that $\alpha^* A_n = F$. Apply the Kan extension to get $\alpha^! \alpha^* A_n \to A_n \subset \alpha^! F$. The composition of these two natural transformations is an isomorphism. It follows $A_n = \alpha^! F$ and consequently $\alpha^! F$ is compact. Statement (10) follows from (9) and Proposition 14.12.

5.2. **Proposition.** Let $\phi: F \to G$ be a natural transformation in $\operatorname{Fun}(\mathbf{Q}^r, \operatorname{Vect}_K)$. If F and G are tame, then so are $\operatorname{ker}(\phi)$ and $\operatorname{coker}(\phi)$.

Proof. Let F be α -tame and G be β -tame. Since α and β are rational numbers, there are natural numbers m and n such that $\alpha/n = \beta/m$. The functors F and G are therefore $\mu = \alpha/n$ -tame (see 4.5). Since Kan extensions preserve exactness (see 5.1), $\ker(\phi)$ is isomorphic to $\mu!(\ker(\mu^*\phi))$ and $\operatorname{coker}(\phi)$ is isomorphic to $\mu!(\operatorname{coker}(\mu^*\phi))$.

As a Corollary of 5.2, we get:

5.3. Corollary.

(1) Consider an exact sequence in $\operatorname{Fun}(\mathbf{Q}^r, \operatorname{Vect}_K)$:

$$0 \to F \to G \to H \to 0$$

- If two out of F, G, and H are tame, then so is the third.
- (2) If F and G in Fun(\mathbf{Q}^r , Vect_K) are tame, then so is $F \oplus G$.
- (3) $\operatorname{Tame}(\mathbf{Q}^r, \operatorname{Vect}_K)$ is an abelian subcategory of $\operatorname{Fun}(\mathbf{Q}^r, \operatorname{Vect}_K)$.

Corollary 5.3 states in principle that $\text{Tame}(\mathbf{Q}^r, \text{Vect}_K)$ is an abelian category. Note that even though $\text{Tame}(\mathbf{Q}^r, \text{Vect}_K)$ is closed under finite direct sums, infinite direct sums however do not in general preserve tameness.

We finish the section by explaining how to compute the support, rank and 0-Betti diagram of a tame functor $G: \mathbb{Q}^r \to \operatorname{Vect}_K$. Here is the procedure:

- Choose α in \mathbf{Q} , such that $\omega : \alpha^! \alpha^* G \to G$ is an isomorphism. In this step we choose a scale α for which G is α -tame.
- Find a sequence of vector spaces $\{V_w\}_{w\in \mathbf{N}^r}$ such that $\alpha^*G/\operatorname{rad}(\alpha^*G)$ is isomorphic to $\bigoplus_{w\in \mathbf{N}^r} U_w \otimes V_w$.

We have now all the needed information to compute $\operatorname{supp}(G)$, $\operatorname{rank}(G)$ and $\beta_0 G$:

5.4. **Proposition.** Let α and $\{V_w\}_{w \in \mathbf{N}^r}$ be defined as above.

- (1) $\operatorname{supp}(G) = \{ \alpha w \mid w \in \operatorname{supp}(\alpha^* G) \} = \{ \alpha w \mid w \in N^r \text{ and } V_w \neq 0 \};$
- (2) $\operatorname{rank}(G) = \operatorname{rank}(\alpha^* G) = \sum_{w \in \mathbf{N}^r} \dim_K V_w;$

(3) $\beta_0 G \colon \mathbf{Q}^r \to \mathbf{N}$ is given by:

$$\beta_0 G(v) = \begin{cases} \beta_0(\alpha^* G)(w) = \dim_K V_w & \text{if } v = \alpha w \text{ for } w \in \mathbf{N}^r \\ 0 & \text{otherwise} \end{cases}$$

Proof. This is a consequence of two facts: first $\alpha^! K(w, -) = K(\alpha w, -)$ (see 5.1) and second if $F \to \alpha^* G$ is a minimal cover of $\alpha^* G$ in $\operatorname{Fun}(N^r, \operatorname{Vect}_K)$, then $\alpha^! F \to \alpha^! \alpha^* G = G$ is a minimal cover of G (see 5.1.(7)).

The right sides of the equalities in the above proposition a priori depend on the choice of a scale α for which G is α -tame. However since the left sides are independent of α , then so are the right sides.

5.5. **Example.** Let $w \leq u$ be two elements in \mathbf{Q}^r . There is a unique inclusion $K(u, -) \subset K(w, -)$. The cokernel of this inclusion is denoted by [w, u). Numerical invariants for functors of this type are studied in [18]. Since the free functors are tame, according to 5.3.(1) [w, u) is tame. It is clear that [w, u) is also compact. Note that $\supp([w, u)) = \{w\}$, $\operatorname{rank}([w, u)) = 1$, and:

$$\beta_0[w,u)(v) = \begin{cases} 1 & \text{if } v = w\\ 0 & \text{if } v \neq w. \end{cases}$$

Similarly to functors indexed by \mathbf{N} (see 14.14), there is a classification for compact and tame functors indexed by \mathbf{Q} .

5.6. **Proposition.** Any compact object in $\text{Tame}(\mathbf{Q}, \text{Vect}_K)$ is isomorphic to a finite direct sum of functors of the form [w, u) and K(v, -). Moreover the isomorphism types of these summands are uniquely determined by the isomorphism type of the functor.

Proof. Let $G: \mathbf{Q} \to \operatorname{Vect}_K$ be a compact and tame functor. Choose α in \mathbf{Q} such that $G = \alpha^! \alpha^* G$. Since $\alpha^* G: \mathbf{N} \to \operatorname{Vect}_K$ is compact, it is isomorphic to a finite direct sum of bars and free functors (see 14.14). As $\alpha^!$ commutes with direct sums, we get the desired decomposition of G. Uniqueness is shown in the same way. \Box

Note that Proposition 5.6 is a direct extension of the classification theorem of graded modules over a PID (see 14.15).

6. Noise

An important step in extracting topological features from a data set is to ignore noise. Depending on the situation, noise can mean different things. In this section we discuss what we mean by noise for vector space valued tame functors. Our objective is to be able to mark some functors as small. Thus for any non-negative rational number ϵ , we should have a collection S_{ϵ} of tame functors which we consider to be ϵ -small. This collection is called the ϵ -component of a noise system and its members are called noise of size at most ϵ . Noise systems should satisfy certain natural constrains. Here is a formal definition:

6.1. **Definition.** A noise system in $\text{Tame}(\mathbf{Q}^r, \text{Vect}_K)$ is a collection $\{S_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ of sets of tame functors, indexed by rational non-negative numbers ϵ , such that:

- the zero functor belongs to S_{ϵ} for any ϵ ;
- if $0 \leq \tau < \epsilon$, then $\mathcal{S}_{\tau} \subseteq \mathcal{S}_{\epsilon}$;

- if $0 \to F \to G \to H \to 0$ is an exact sequence in Tame(\mathbf{Q}^r , Vect_K), then - if G is in \mathcal{S}_{ϵ} , then so are F and H;
 - if F is in \mathcal{S}_{ϵ} and H is in \mathcal{S}_{τ} , then G is in $\mathcal{S}_{\epsilon+\tau}$.

The last requirement for a noise system is called additivity.

Let $\{S_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ and $\{\mathcal{T}_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ be noise systems. If, for any ϵ in $\mathbf{Q}, S_{\epsilon} \subset \mathcal{T}_{\epsilon}$, then we write $\{S_{\epsilon}\}_{\epsilon \in \mathbf{Q}} \leq \{\mathcal{T}_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$. With this relation, noise systems in $\operatorname{Tame}(\mathbf{Q}^{r}, \operatorname{Vect}_{K})$ form a poset. This poset has the unique minimal element given by the sequence whose components contain only the zero functor. It has also the unique maximal element given by the sequence whose components contain all tame functors. Note that the intersection of any family of noise systems is again a noise system. This implies for example that the poset of noise systems is a lattice. Moreover, for any sequence of sets $\{S_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ of tame functors, the intersection of all the noise systems $\{\mathcal{T}_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ for which $S_{\epsilon} \subset \mathcal{T}_{\epsilon}$, for any ϵ in \mathbf{Q} , is the smallest noise system containing S_{ϵ} in its ϵ -component. We call it the noise system **generated by the sequence** $\{S_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ and denote it by $\langle\{S_{\epsilon}\}_{\epsilon \in \mathbf{Q}}\rangle$.

Let $\{S_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ be a noise system. Define $S_{\epsilon}^{c} := \{F \in S_{\epsilon} \mid F \text{ is compact}\}$. One can use Proposition 5.1(10) to see that $\{S_{\epsilon}^{c}\}_{\epsilon \in \mathbf{Q}}$ is also a noise system. We call it the compact part of $\{S_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$. It follows that if $\{S_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ is a sequence of sets of compact tame functors, then $\langle \{S_{\epsilon}\}_{\epsilon \in \mathbf{Q}} \rangle$ consists of compact functors.

By definition the 0-component S_0 of any noise system $\{S_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ is a Serre subcategory of Tame(\mathbf{Q}^r , Vect_K) (see [17]). In particular the direct sum of two functors in S_0 is again a functor in S_0 . Since this is not true in general for other components, we need to introduce a definition: a component S_{ϵ} of a noise system is said to be **closed under direct sums** if, for any F and G in S_{ϵ} , the direct sum $F \oplus G$ also belongs to S_{ϵ} . Being closed under direct sums is important for some of our constructions as in this case, for any $\epsilon > 0$, any compact and tame functor has a unique maximal subfunctor that belongs to S_{ϵ} (see Proposition 7.1). In Section 7 we try to understand under what circumstances a noise system is closed under direct sums.

We now present several examples of noise systems. The first two generalise what we interpret as noise in the context of persistent homology induced by interleaving distance (see [19]).

6.2. Standard Noise in the direction of a cone. Let $V \subset \mathbf{Q}^r$ be a subset. Set:

$$V_{\epsilon} := \left\{ F \in \operatorname{Tame}(\mathbf{Q}^{r}, \operatorname{Vect}_{K}) \middle| \begin{array}{c} \text{for any } u \text{ in } \mathbf{Q}^{r} \text{ and for any } x \text{ in } F(u), \\ \text{there is } w \text{ in } V \text{ such that} \\ ||w|| = \epsilon \text{ and } x \text{ is in } \ker \left(F(u \le u + w)\right) \end{array} \right\}$$

We claim that if V is a cone (see 2.7), then the sequence $\{V_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ is a noise system which we call the **standard noise in the direction of the cone** V. It is clear that the zero functor belongs to V_{ϵ} for any ϵ . Let $0 < \tau < \epsilon$. If x is in ker $(F(u \le u+w))$, then x is also in ker $(F(u \le u + \frac{\epsilon}{\tau}w))$, since $w \le \frac{\epsilon}{\tau}w$. As $||\frac{\epsilon}{\tau}w|| = \frac{\epsilon}{\tau}||w||$, the inclusion $V_{\tau} \subset V_{\epsilon}$ follows. Consider now an exact sequence $0 \to F \to G \to H \to 0$ of tame functors. If G is in V_{ϵ} , then, by naturality of $F \hookrightarrow G$ and $G \twoheadrightarrow H$, both functors F and H are also in V_{ϵ} . Assume F is in V_{ϵ} and H is in V_{τ} . Take an element $x \in G(u)$. Its image x_1 in H(u) is therefore in ker $(H(u \le u+w))$ for some w in V with $||w|| = \tau$. This means that $G(u \le u+w)$ takes x to an element x_2 in $F(u+w) \subset G(u+w)$. We can thus find w' in V with $||w'|| = \epsilon$ such that x_2 is in ker $(F(u+w \le u+w+w'))$. It follows that x is in ker $(G(u \le u+w+w'))$. Since $||w+w'|| \le ||w||+||w'|| = \tau+\epsilon$, x is therefore also in ker $(G(u \le u+w+w'))$. The assumption that V is a cone guarantees that $\frac{\tau+\epsilon}{||w+w'||}(w+w')$ belongs to V. We can conclude G belongs to $V_{\epsilon+\tau}$.

Note that the hypothesis that $V \subseteq \mathbf{Q}^r$ is a cone is fundamental for $\{V_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ to be a noise system. To illustrate this consider for example r = 2, V to be the set union of two axes $\{(a,0) \mid a \in \mathbf{Q}\} \cup \{(0,b) \mid b \in \mathbf{Q}\}$, and u = (0,0). The tame functors $F, H: \mathbf{Q}^r \to \operatorname{Vect}_K$ given respectively by:

$$F(v_1, v_2) = \begin{cases} K & \text{if } v_2 < 1\\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad H(v_1, v_2) = \begin{cases} K & \text{if } v_1 < 1\\ 0 & \text{otherwise} \end{cases}$$

are in V_1 . The functor $G: \mathbf{Q}^r \to \operatorname{Vect}_K$ defined as:

$$G(v_1, v_2) = \begin{cases} K & \text{if } v_1 < 1 \text{ or } v_2 < 1 \\ 0 & \text{otherwise} \end{cases}$$

fits into an exact sequence $0 \to F \to G \to H \to 0$ but is not in V_{ϵ} for any positive rational number ϵ .

In general, neither V_{ϵ} nor its compact part V_{ϵ}^c are closed under direct sums. For example consider w = (1, 0, 1) and w' = (1/2, 1, 0) in \mathbf{Q}^3 . Define $F: \mathbf{Q}^3 \to \operatorname{Vect}_K$ to be the tame functor such that F(v) = 0 if $v \ge w$ and F(v) = K otherwise, with $F(u \le v)$ being the identity if F(u) and F(v) are non zero. Similarly, let $G: \mathbf{Q}^3 \to \operatorname{Vect}_K$ be a tame functor such that G(v) = 0 if $v \ge w'$ and G(v) = Kotherwise, with $G(u \le v)$ being the identity if G(u) and G(v) are non zero. Note that F is in $\operatorname{Cone}(w)_1^c$ and G is in $\operatorname{Cone}(w')_1^c$. Although F and G are both in $\operatorname{Cone}(w, w')_1^c$, the functor $F \oplus G$ is not since there is no vector z in $\operatorname{Cone}(w, w')$ such that $||z|| = 1, z \ge w$ and $z \ge w'$.

Note that in the case r = 1, the ϵ -component of the standard noise in the direction of the ray **Q** coincides with the set of ϵ -trivial persistence modules as defined in [2].

6.3. The compact part of the standard noise in the direction of a ray. Let V be a ray (a cone generated by one element, see 2.7). Then there is a unique w in V such that ||w|| = 1. In this case F belongs to V_{ϵ} if and only if $F(v) = V_{\epsilon}$ $\ker(F(v \leq v + \epsilon w))$, for any v in \mathbf{Q}^r (the map $F(v \leq v + \epsilon w)$ is the zero map). For example the cokernel of the unique inclusion $K(v + \epsilon w, -) \subset K(v, -)$ belongs to V_{ϵ} . Recall that this cokernel is denoted by $[v, v + \epsilon w)$ (see 5.5). Note that this cokernel is compact and hence it belongs to V_{ϵ}^{c} . Furthermore any finite direct sum $\bigoplus_{i=1}^{n} [v_i, v_i + \epsilon w]$ is also a member of V_{ϵ}^c . We claim that $\{V_{\epsilon}^c\}_{\epsilon \in \mathbf{Q}}$ is the smallest noise system containing all such finite direct sums in its ϵ -component for any ϵ . In other words $\{V_{\epsilon}^{c}\}_{\epsilon \in \mathbf{Q}}$ is the noise system generated by a sequence of sets $\{S_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$, where S_{ϵ} is the set of all functors of the form $\bigoplus_{i=1}^{n} [v_i, v_i + \epsilon w]$. We have just explained the relation $\langle \{S_{\epsilon}\}_{\epsilon \in \mathbf{Q}} \rangle \leq \{V_{\epsilon}^{c}\}_{\epsilon \in \mathbf{Q}}$. Let F be in V_{ϵ}^{c} . Recall that any element x in F(v) induces a unique natural transformation $x: K(v, -) \to F$ (see 2.5). Since its precomposition with $K(v + \epsilon w, -) \subset K(v, -)$ is trivial, $x \colon K(v, -) \to F$ factors as $K(v, -) \rightarrow [v, v + \epsilon w) \rightarrow F$. This, together with compactness, implies that F is a quotient of a finite direct sum of functors of the form $[v, v + \epsilon w]$ which implies F is in the ϵ -component of $\langle \{S_{\epsilon}\} \rangle_{\epsilon \in \mathbf{Q}}$.

Since a direct sum of zero maps is a zero map, the collections V_{ϵ} and V_{ϵ}^{c} are preserved by direct sums for any ϵ .

6.4. Standard Noise in the direction of a sequence of vectors. Let us choose a finite sequence $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ of elements in \mathbf{Q}^r . For any w in $\operatorname{Cone}(\mathcal{V}) = \operatorname{Cone}(v_1, v_2, \ldots, v_n)$, consider the set T(w) of sequences (a_1, \ldots, a_n) of non-negative rational numbers such that $w = a_1v_1 + \cdots + a_nv_n$. Define the \mathcal{V} -norm as:

$$||w||_{\mathcal{V}} = \min_{\{a_1, \dots, a_n\} \in T(w)} ||(a_1, \dots, a_n)|| = \min_{\{a_1, \dots, a_n\} \in T(w)} \max_{1 \le i \le n} a_i$$

Set:

$$\mathcal{V}_{\epsilon} := \left\{ F \in \operatorname{Tame}(\mathbf{Q}^{r}, \operatorname{Vect}_{K}) \middle| \begin{array}{c} \text{for any } v \text{ in } \mathbf{Q}^{r} \text{ and for any } x \text{ in } F(v), \\ \text{there is } w \text{ in } \operatorname{Cone}(\mathcal{V}) \text{ s.t. } ||w||_{\mathcal{V}} = \epsilon \\ \text{and } x \text{ is in } \ker \left(F(v \le v + w) \right) \end{array} \right\}$$

One can check that $||aw||_{\mathcal{V}} = a||w|||_{\mathcal{V}}$ and $||u + w|||_{\mathcal{V}} \leq ||u|||_{\mathcal{V}} + ||w|||_{\mathcal{V}}$ for any v and w in Cone(\mathcal{V}) and any a in \mathbf{Q} . Exactly the same arguments as in 6.2 can be then used to prove that $\{\mathcal{V}_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ is also a noise system. We call it the **standard** noise in the direction of the sequence \mathcal{V} .

For example let v in \mathbf{Q}^r be non-zero and $\mathcal{V} = \{v\}$. In this case for any w in $\operatorname{Cone}(v)$, $||w||_{\mathcal{V}} = ||w||/||v||$ and $\mathcal{V}_{\epsilon} = \operatorname{Cone}(v)_{\epsilon/||v||}$ for any ϵ in \mathbf{Q} .

6.5. **Domain noise.** Let $\mathcal{X} = \{X_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ be a sequence of subsets of \mathbf{Q}^{r} with the property that if $0 \leq \tau < \epsilon$, then $X_{\tau} \subseteq X_{\epsilon}$. For a tame functor $F : \mathbf{Q}^{r} \to \operatorname{Vect}_{K}$ we define domain $(F) := \{v \in \mathbf{Q}^{r} \mid F(v) \neq 0\}$ and call it the **domain** of F. For example the domain of the zero functor is empty. Set:

$$\mathcal{X}_{\epsilon} := \{F \in \operatorname{Tame}(\mathbf{Q}^r, \operatorname{Vect}_K) \mid \operatorname{domain}(F) \subseteq X_{\epsilon}\}.$$

The fact that $\{\mathcal{X}_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ is a noise system is a direct consequence of the fact that $X_{\tau} \subseteq X_{\epsilon}$ for any $0 \leq \tau < \epsilon$. This noise system satisfies an extra condition. For any exact sequence $0 \to F \to G \to H \to 0$ of tame functors, if F is in \mathcal{X}_{ϵ} and H in \mathcal{X}_{τ} , then G is in $\mathcal{X}_{\max\{\epsilon,\tau\}}$. This implies in particular that both \mathcal{X}_{ϵ} and $\mathcal{X}_{\epsilon}^{c}$ are closed under direct sums.

6.6. Dimension noise. Let $\mathcal{N} = \{n_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ be a sequence of natural numbers such that $n_0 = 0$ and $n_{\tau} + n_{\epsilon} \leq n_{\tau+\epsilon}$ for any τ and ϵ in \mathbf{Q} . Set:

 $\mathcal{N}_{\epsilon} := \{ F \in \operatorname{Tame}(\mathbf{Q}^r, \operatorname{Vect}_K) \mid \text{ for any } v \text{ in } \mathbf{Q}^r, \dim_K F(v) \le n_{\epsilon} \}.$

The proof that $\{\mathcal{N}_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ is a noise system is straightforward and only depends on the facts that $n_0 = 0$, $\{n_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ is a non decreasing sequence of non negative numbers, and that an sequence of tame functors $0 \to F \to G \to H \to 0$ is exact if it is object wise exact.

More examples of noise systems can be produced using the property that the intersection of an arbitrary family of noise systems is a noise system. For example:

6.7. Intersection noise. Let L^1, \ldots, L^n be rays in \mathbf{Q}^r . Choose the unique w_i in L^i such that $||w_i|| = 1$. The intersection of $\{L^i_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ for $1 \leq i \leq n$ is the noise system whose ϵ -component consists of these tame functors $F : \mathbf{Q}^r \to \operatorname{Vect}_K$ for which $F(v \leq v + \epsilon w_i)$ is the zero map for any v and any $1 \leq i \leq n$.

6.8. Noise generated by a functor. Let M be a tame functor and α a positive rational number. Let $\langle M, \alpha \rangle$ be the smallest noise system such that M is in $\langle M, \alpha \rangle_{\alpha}$. This might be interesting in cases in which one wants to declare some functor as

noise of a certain size. The collection $\langle M, \alpha \rangle_{\epsilon}$ can be described inductively as follows:

$$\langle M, \alpha \rangle_{\epsilon} = \begin{cases} \{0\} & \text{if } 0 \leq \epsilon < \alpha \\ [M] & \text{if } \epsilon = \alpha \\ \left[\bigcup_{i=1}^{n-1} \langle M, \alpha \rangle_{i\alpha} \ \cup \ \text{Ext}(\langle M, \alpha \rangle_{(n-i)\alpha}) \right] & \text{if } \epsilon = n\alpha \text{ for } n > 1 \\ \langle M, \alpha \rangle_{n\alpha} & \text{if } n\alpha \leq \epsilon < (n+1)\alpha \end{cases}$$

where [K] denotes the set of all tame subfunctors and quotients of K and Ext(K) denotes the collection of all tame functors G that fit into an exact sequence of the form $0 \to F \to G \to H \to 0$ where F and H are in [K].

7. Noise systems closed under direct sums

Let $\{S_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ be a noise system in Tame $(\mathbf{Q}^r, \operatorname{Vect}_K)$ and $F: \mathbf{Q}^r \to \operatorname{Vect}_K$ be a tame and compact functor. Consider the collection of all subfunctors of F that belong to S_{ϵ} . Because of the compactness of F, Kuratowski-Zorn lemma implies that this collection has maximal elements with respect to the inclusion. In general however there could be many such maximal elements. In this section we discuss under what circumstances there is only one maximal element in this collection. The subfunctor corresponding to this element is the unique maximal noise of size ϵ inside F and we will use it to denoise F. If it exists, we denote this maximal subfunctor by $F[\mathcal{S}_{\epsilon}] \subset F$. By definition the inclusion $F[\mathcal{S}_{\epsilon}] \subset F$ satisfies the following universal property: $F[\mathcal{S}_{\epsilon}]$ belongs to \mathcal{S}_{ϵ} and, for any G in \mathcal{S}_{ϵ} , any natural transformation $G \to F$ maps G into $F[\mathcal{S}_{\epsilon}] \subset F$. Thus for any G in \mathcal{S}_{ϵ} , the inclusion $F[\mathcal{S}_{\epsilon}] \subset F$ induces a bijection between $\operatorname{Nat}(G, F)$ and $\operatorname{Nat}(G, F[\mathcal{S}_{\epsilon}])$.

7.1. **Proposition.** Let $\{S_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ be a noise system in $\operatorname{Tame}(\mathbf{Q}^{r}, \operatorname{Vect}_{K})$. The component S_{ϵ}^{c} is closed under direct sums if and only if $F[S_{\epsilon}] \subset F$ exists for any tame and compact functor $F: \mathbf{Q}^{r} \to \operatorname{Vect}_{K}$.

Proof. Assume S_{ϵ}^{c} is closed under direct sums. Let $F: \mathbf{Q}^{r} \to \operatorname{Vect}_{K}$ be a tame and compact functor and $G \subset F$ and $H \subset F$ be maximal, with respect to the inclusion, subfunctors such that G and H are in S_{ϵ}^{c} . Since $G \oplus H$ is in S_{ϵ}^{c} , then so is its quotient $G + H \subset F$. Using maximality of $G \subset F$ and $H \subset F$, we obtain equalities G = G + H = H. We can conclude that there is a unique maximal subfunctor of F that belongs to S_{ϵ} .

Assume now that, for any tame and compact functor $F : \mathbf{Q}^r \to \operatorname{Vect}_K$, there is a unique maximal $F[\mathcal{S}_{\epsilon}] \subset F$ such that $F[\mathcal{S}_{\epsilon}]$ belongs to \mathcal{S}_{ϵ} . Let G and H be in \mathcal{S}_{ϵ}^c . Consider $(G \oplus H)[\mathcal{S}_{\epsilon}]$. By the maximality, we have inclusions $G \subset (G \oplus H)[\mathcal{S}_{\epsilon}] \supset H$, and thus $G \oplus H = (G \oplus H)[\mathcal{S}_{\epsilon}]$. The functor $G \oplus H$ is therefore in \mathcal{S}_{ϵ}^c .

7.2. Corollary. Let $\{S_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ be a noise system in $\operatorname{Tame}(\mathbf{Q}^{r}, \operatorname{Vect}_{K})$. Assume that the component S_{ϵ}^{c} is closed under direct sums. Then, for any tame and compact functors $F, G: \mathbf{Q}^{r} \to \operatorname{Vect}_{K}, F[S_{\epsilon}] \oplus G[S_{\epsilon}] = (F \oplus G)[S_{\epsilon}]$.

Proof. For any H in S_{ϵ} , we have a sequence of bijections induced by the appropriate inclusions:

$$\begin{split} \operatorname{Nat}(H,(F\oplus G)[\mathcal{S}_{\epsilon}]) &= \operatorname{Nat}(H,F\oplus G) = \operatorname{Nat}(H,F) \oplus \operatorname{Nat}(H,G) = \\ &= \operatorname{Nat}(H,F[\mathcal{S}_{\epsilon}]) \oplus \operatorname{Nat}(H,G[\mathcal{S}_{\epsilon}]) = \operatorname{Nat}(H,F[\mathcal{S}_{\epsilon}]\oplus G[\mathcal{S}_{\epsilon}]) \\ \end{split}$$

This shows $F[\mathcal{S}_{\epsilon}] \oplus G[\mathcal{S}_{\epsilon}] \subset (F\oplus G)[\mathcal{S}_{\epsilon}]$ is an isomorphism. \Box

Based on the above proposition, we are going to look for noise systems whose compact parts are closed under direct sums. The key example of such a noise system is the standard noise in a direction of a ray (see 6.3) or a vector (see 6.4). More generally:

7.3. **Proposition.** Let V be a cone in \mathbf{Q}^r and $\{V_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ be the standard noise in the direction of the cone V. The following are equivalent:

- (1) The collection V_{ϵ} (see 6.2) is closed under direct sums.
- (2) The collection V_{ϵ}^{c} (see 6.2) is closed under direct sums.
- (3) For any w_1 and w_2 in the cone V whose norm is ϵ ($||w_1|| = ||w_2|| = \epsilon$), there is an element w in V of norm ϵ ($||w|| = \epsilon$) such that $w_1 \leq w, w_2 \leq w$.

Proof. The implication $(1) \Rightarrow (2)$ is clear. Assume (2). We show (3). Let w_1 and w_2 be elements of the cone V such that $||w_1|| = ||w_2|| = \epsilon$. Consider the functors $[0, w_1)$ and $[0, w_2)$ (see 5.5). Since they belong to V_{ϵ}^c , then, by the assumption, so does their direct sum $F := [0, w_1) \oplus [0, w_2)$. Consider x in $F(0) = ([0, w_1) \oplus [0, w_2))(0) =$ $[0, w_1)(0) \oplus [0, w_2](0) = K \oplus K$ given by the diagonal element (1, 1). As F is in V_{ϵ} , there is w in V such that $||w|| = \epsilon$ and x is in the kernel of $F(0 \le w)$. This can happen only if $w_1 \leq w$ and $w_2 \leq w$. Thus w is the desired element.

Assume (3). We prove (1). Let F and G be in V_{ϵ} . We need to show that $F \oplus G$ also belongs to V_{ϵ} . Choose (x, y) in $F(v) \oplus G(v)$. Let w_x and w_y be two elements in the cone V of norm ϵ such that x is in ker $(F(v \leq v + w_x))$ and y is in $\ker(G(v \leq v + w_y))$. By the assumption there is w in V of norm ϵ with $w_x \leq w$ and $w_y \leq w$. Thus x is in ker $(F(v \leq v + w))$ and y in ker $(G(v \leq v + w))$. It follows that (x, y) is in ker $((F \oplus G)(v \le v + w))$. As this happens for any (x, y), the direct sum $F \oplus G$ belongs to V_{ϵ} .

Exactly the same argument as in the proof of 7.3, can be also applied to show an analogous statement for the standard noise in the direction of a sequence of vectors in \mathbf{Q}^r :

7.4. **Proposition.** Let $\mathcal{V} = \{v_1, \ldots, v_n\}$ be a sequence of vectors in \mathbf{Q}^r and $\{\mathcal{V}_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ be the standard noise in the direction of the sequence \mathcal{V} . The following are equivalent:

- (1) The collection \mathcal{V}_{ϵ} (see 6.4) is closed under direct sums.
- (2) The collection \mathcal{V}^c_{ϵ} (see 6.4) is closed under direct sums.
- (3) For any w_1 and w_2 in Cone(\mathcal{V}) whose \mathcal{V} -norm is ϵ ($||w_1||_{\mathcal{V}} = ||w_2||_{\mathcal{V}} = \epsilon$), there is an element w in V of V-norm ϵ ($||w||_{\mathcal{V}} = \epsilon$) such that $w_1 \leq w$, $w_2 \leq w$.

We finish this section with:

7.5. **Proposition.** Let $\mathcal{V} = \{v_1, \ldots, v_n\}$ be a sequence of elements in \mathbf{Q}^r .

- (1) Let $V = \text{Cone}(v_1, \ldots, v_n)$ and $L = \text{Cone}(v_1 + \cdots + v_n)$. If $||v_1|| = \cdots =$ $||v_n|| = ||v_1 + \dots + v_n||, \text{ then } \{V_\epsilon\}_{\epsilon \in \mathbf{Q}} = \{L_\epsilon\}_{\epsilon \in \mathbf{Q}}.$ (2) Let $\mathcal{W} = \{v_1 + \dots + v_n\}$. If v_1, \dots, v_n are linearly independent as vectors
- over the field of rational numbers, then $\{\mathcal{V}_{\epsilon}\}_{\epsilon \in \mathbf{Q}} = \{\mathcal{W}_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$.

Proof. (1): Set $w := (\epsilon/||v_1||)(v_1 + \cdots + v_n)$. By the assumption $||w|| = \epsilon$. Let u in V be of norm ϵ . Thus u can be written as $u = a_1v_1 + \ldots + a_nv_n$ where $0 \le a_i \le \epsilon/||v_1||$ for every $1 \le i \le n$. It follows that $u \le (\epsilon/||v_1||)(v_1 + \dots + v_n) = w$. Let F be in V_{ϵ} . This means that, for any x in F(v), there exists w_x in V of norm ϵ such that x is in the kernel of $F(v \leq v + w_x)$. We have already shown that $w_x \leq w$. It thus follows that x is also in the kernel of $F(v \leq v + w)$. As this is true for any x, $F(v \leq v + w)$ is the zero map. This means $V_{\epsilon} = \text{Cone}(w)_{\epsilon}$

(2): Set $w = v_1 + \dots + v_n$. Since $\{v_1, \dots, v_n\}$ are linearly independent, $||\epsilon w||_{\mathcal{V}} = \epsilon$. Let u be in $\operatorname{Cone}(v_1, \dots, v_n)$ of \mathcal{V} -norm ϵ . Thus u can be written as $u = a_1v_1 + \dots + a_nv_n$ where $0 \le a_i \le \epsilon$. It follows that $u \le \epsilon(v_1 + \dots + v_n) = \epsilon w$.

Let F be in \mathcal{V}_{ϵ} . This means that, for any x in F(v), there exists w_x in V of \mathcal{V} -norm ϵ such that x is in the kernel of $F(v \leq v + w_x)$. We have already shown that $w_x \leq \epsilon w$. It thus follows that x is also in the kernel of $F(v \leq v + \epsilon w)$. As this is true for any x, $F(v \leq v + \epsilon w)$ is the zero map. This means $V_{\epsilon} = \mathcal{W}_{\epsilon}$. \Box

The special case of 7.5 we are most interested in is when the vectors v_1, \ldots, v_n are among the standard vectors of \mathbf{Q}^r (see 2.6).

8. TOPOLOGY ON TAME FUNCTORS

In this section we describe how a noise system leads to a pseudo metric and hence a topology on the set of tame functors with values in Vect_K . This metric can be used to measure how close or how far apart tame functors can be relative to the chosen noise system. Let us choose and fix a noise system $\{S_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ in $\operatorname{Tame}(\mathbf{Q}^r, \operatorname{Vect}_K)$.

We are going to compare tame functors using natural transformations. Let $\phi: F \to G$ be a natural transformation between functors in $\text{Tame}(\mathbf{Q}^r, \text{Vect}_K)$. We say that ϕ is an ϵ -equivalence if, there are τ and μ in \mathbf{Q} such that $\tau + \mu \leq \epsilon$, ker (ϕ) belongs to S_{τ} and coker (ϕ) belongs to S_{μ} . Before we define a pseudometric and a topology on the set of tame functors, we need to prove two fundamental properties of being an ϵ -equivalence. The first one is the preservation of ϵ -equivalences by both push-outs and pull-backs:

8.1. **Proposition.** Consider the following commutative square in Tame($\mathbf{Q}^r, \operatorname{Vect}_K$):



- (1) Assume that the square is a push-out. If ϕ is an ϵ -equivalence, the same holds for ϕ' .
- (2) Assume that the square is a pull-back. If φ' is an ε-equivalence, the same holds for φ.

Proof. As the proofs of (1) and (2) are analogous, we present only a sketch of the argument for (1). We claim that the assumption that the square is a push-out implies that $\operatorname{coker}(\phi)$ is isomorphic to $\operatorname{coker}(\phi')$ and $\operatorname{ker}(\phi')$ is a quotient of $\operatorname{ker}(\phi)$. The proposition then clearly follows as a component of a noise system is closed under quotients. As push-outs in the functor category are formed object-wise, it is then enough to show the claim in the category of vector spaces Vect_K . This is left as an easy exercise.

The second property of ϵ -equivalences is additivity with respect to the scale ϵ :

8.2. **Proposition.** Let $\phi: F \to G$ and $\psi: G \to H$ be natural transformations between functors in Tame($\mathbf{Q}^r, \operatorname{Vect}_K$). If ϕ is an ϵ_1 -equivalence, and ψ is an ϵ_2 equivalence, then $\psi \phi$ is an $(\epsilon_1 + \epsilon_2)$ -equivalence.

Proof. The key is to observe that the natural transformations ϕ and ψ induce the following exact sequences:

$$0 \longrightarrow \ker(\phi) \longrightarrow \ker(\psi\phi) \longrightarrow \ker(\psi)$$
$$\operatorname{coker}(\phi) \longrightarrow \operatorname{coker}(\psi\phi) \longrightarrow \operatorname{coker}(\psi) \longrightarrow 0$$

Let τ_1 and μ_1 in \mathbf{Q} be such that $\tau_1 + \mu_1 \leq \epsilon_1$ and $\ker(\phi)$ belongs to \mathcal{S}_{τ_1} and $\operatorname{coker}(\phi)$ belongs to \mathcal{S}_{μ_1} . Similarly let τ_2 and μ_2 in \mathbf{Q} be such that $\tau_2 + \mu_2 \leq \epsilon_2$ and $\ker(\psi)$ belongs to \mathcal{S}_{τ_2} and $\operatorname{coker}(\psi)$ belongs to \mathcal{S}_{μ_2} . Therefore, the image of $\ker(\psi\phi) \to \ker(\psi)$, as a subfunctor in $\ker(\psi)$, belongs to \mathcal{S}_{τ_2} and the kernel of $\operatorname{coker}(\psi\phi) \to \operatorname{coker}(\psi)$, as a quotient of $\operatorname{coker}(\phi)$, belongs to \mathcal{S}_{μ_1} . We can then $\operatorname{conclude}$ that $\ker(\psi\phi)$ belongs to $\mathcal{S}_{\tau_1+\tau_2}$ and $\operatorname{coker}(\psi\phi)$ belongs to $\mathcal{S}_{\mu_1+\mu_2}$. Since $\tau_1 + \tau_2 + \mu_1 + \mu_2 \leq \epsilon_1 + \epsilon_2$, the transformation $\psi\phi$ is an $(\epsilon_1 + \epsilon_2)$ -equivalence. \Box

We can use the above fundamental properties of ϵ -equivalences to prove:

8.3. Corollary. Let F and G be tame functors and τ and μ be non-negative rational numbers. Then the following statements are equivalent:

- (1) There are natural transformations $F \leftarrow H : \phi$ and $\psi : H \rightarrow G$ such that ϕ is a τ -equivalence and ψ is a μ -equivalence.
- (2) There are natural transformations $\psi' \colon F \to P \leftarrow G : \phi'$ such that ϕ' is a τ -equivalence and ψ' is a μ -equivalence.

Proof. Assume (1) and apply 8.1.(1) to the following push-out square to get (2):



The opposite implication $(2) \Rightarrow (1)$ follows from a similar argument by applying 8.1.(2) to an appropriate pull-back square.

We are now ready for our key definition:

8.4. **Definition.** Let ϵ be in **Q**. Two tame functors F and G are ϵ -close if there are natural transformations $F \leftarrow H : \phi$ and $\psi : H \rightarrow G$ such that ϕ is a τ -equivalence, ψ is a μ -equivalence, and $\tau + \mu \leq \epsilon$.

Note that if there is an ϵ -equivalence $\phi \colon F \to G$ or an ϵ -equivalence $\phi \colon G \to F$ then F and G are ϵ -close.

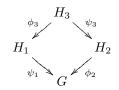
According to 8.3, two tame functors F and G are ϵ -close if and only if there are natural transformations $\psi' \colon F \to P \leftarrow G : \phi'$ such that ϕ' is a τ -equivalence and ψ' is a μ -equivalence and $\tau + \mu \leq \epsilon$.

It is not true that any two tame functors are ϵ -close for some ϵ in \mathbf{Q} . For example let r = 1 and consider the standard noise in the direction of \mathbf{Q} . The free functor $K(0, -): \mathbf{Q} \to \operatorname{Vect}_K$ is not ϵ -close to the zero functor for any ϵ in \mathbf{Q} . We say that two functors are **close** if they are ϵ -close for some ϵ in \mathbf{Q} . If S_0 contains a non-zero functor, then this functor is 0-close to the zero functor. The intersection $\cap_{\epsilon>0}S_{\epsilon}$ consists of the functors which are ϵ -close to the zero functor for any $\epsilon > 0$. If this intersection contains only the zero functor, then a natural transformation is an isomorphism if and only if it is an ϵ -equivalence for any $\epsilon > 0$. In particular two functor are isomorphic if and only if they are ϵ -close for any $\epsilon > 0$.

Being ϵ -close is a reflexive and symmetric relation. It is however not transitive in general. Instead it is additive with respect to the scale ϵ .

8.5. **Proposition.** Let $F, G, H: \mathbb{Q}^r \to \operatorname{Vect}_K$ be tame functors. If F is ϵ_1 -close to G and G is ϵ_2 -close to H, then F is $(\epsilon_1 + \epsilon_2)$ -close to H.

Proof. If F is ϵ_1 -close to G then we have morphisms $F \leftarrow H_1 : \phi_1$ and $\psi_1 : H_1 \rightarrow G$ where ϕ_1 is an α -equivalence, ψ_1 is a β -equivalence, and $\alpha + \beta \leq \epsilon_1$. Similarly if G is ϵ_2 -close to H we have morphisms $G \leftarrow H_2 : \phi_2$ and $\psi_2 : H_2 \rightarrow H$ where ϕ_2 is a γ -equivalence, ψ_2 is a δ -equivalence, and $\gamma + \delta \leq \epsilon_2$. Consider the pull-back square:



From Proposition 8.1 it follows that ϕ_3 is a γ -equivalence and ψ_3 is a β -equivalence. The natural transformation $F \leftarrow H_3 : \phi_1 \phi_3$ is a $(\gamma + \alpha)$ -equivalence and the natural transformation $\psi_2 \psi_3 : H_3 \to H$ is a $(\beta + \delta)$ -equivalence by Proposition 8.2. The claim follows from the fact that $\alpha + \beta + \gamma + \delta \leq \epsilon_1 + \epsilon_2$.

We use the relation of being ϵ -close to define a pseudometric (see 2.9) on tame functors:

8.6. **Definition.** Let $F, G: \mathbf{Q}^r \to \operatorname{Vect}_K$ be tame functors. If F and G are close we define $d(F, G) := \inf\{\epsilon \in \mathbf{Q} \mid F \text{ and } G \text{ are } \epsilon\text{-close}\}$ and otherwise $d(F, G) := \infty$.

8.7. **Proposition.** The function d, defined in 8.6, is an extended pseudometric on the set of tame functors with values in Vect_K .

Proof. The symmetry d(F,G) = d(G,F) follows from the fact that being ϵ -close is a symmetric relation. Since the zero functor is in S_{ϵ} for any ϵ , then d(F,F) = 0. The triangle inequality $d(F,H) \leq d(F,G) + d(G,H)$ is a consequence of Proposition 8.5.

Let $\tau > 0$ be a positive real number and F and G be tame functors. By definition, $d(F,G) < \tau$ if and only if F and G are ϵ -close for some $\epsilon < \tau$. Thus the open ball $B(F,\tau)$, around F with radius τ (see 2.9), consists of all tame functors which are ϵ -close to F for some $\epsilon < \tau$. These sets form a base of the topology induced by the pseudometric defined in 8.6.

For the standard noise in the direction of full cone \mathbf{Q}^r , the pseudometric defined above is related to the interleaving pseudometric introduced by M. Lesnick in [13], and for the case of persistent homology (r = 1) in [9]. This work in fact inspired us towards the formulation of our noise systems. For example one can show that if two functors are ϵ -interleaved, then they are 6ϵ -close and vice versa if two functors are ϵ -close, then they are ϵ -interleaved.

9. FEATURE COUNTING INVARIANT

In this section we describe a pseudometric space of feature counting functions. This space is the range for our invariant which we call a feature counting invariant. This invariant is a continuous function associated to a noise system in Tame(\mathbf{Q}^r , Vect_K). Its domain is the space of tame functors, and its range is the space of feature counting functions. Our aim in this section is to construct this feature counting invariant and show that it is 1-Lipschitz. Throughout this section let us choose and fix a noise system { S_{ϵ} }_{$\epsilon \in \mathbf{Q}$} in Tame(\mathbf{Q}^r , Vect_K). All the distances and neighbourhoods in Tame(\mathbf{Q}^r , Vect_K) are relative to this choice, as defined in Section 8.

By definition a **feature counting function** is a functor $f: \mathbf{Q} \to \mathbf{N}^{\text{op}}$. We write f_t to denote the value of f at t in \mathbf{Q} . Thus a feature counting function is simply a non-increasing sequence of natural numbers indexed by non-negative rational numbers \mathbf{Q} . In particular a feature counting function has only finitely many values. Note that the category $\operatorname{Fun}(\mathbf{Q}, \mathbf{N}^{\operatorname{op}})$ is a poset, and there is a natural transformation between $f: \mathbf{Q} \to \mathbf{N}^{\operatorname{op}}$ and $g: \mathbf{Q} \to \mathbf{N}^{\operatorname{op}}$ if and only if, for any t in $\mathbf{Q}, f_t \geq g_t$.

Let ϵ be in **Q**. We say that two feature counting functions $f, g: \mathbf{Q} \to \mathbf{N}^{\text{op}}$ are ϵ -interleaved if, for any t in **Q**, $f_t \geq g_{t+\epsilon}$ and $g_t \geq f_{t+\epsilon}$ (this definition follows the definition given in [3]).

It is not true that any two feature counting functions are ϵ -interleaved for some ϵ . For example the constant feature counting functions $0, 1: \mathbf{Q} \to \mathbf{N}^{\text{op}}$ with values respectively 0 and 1, are not ϵ -interleaved for any ϵ . Two feature counting functions f and g are called **interleaved** if they are ϵ -interleaved for some ϵ . Note that f and g are 0-interleaved if and only if f = g. It is however not true that if f and g are ϵ -interleaved for any $\epsilon > 0$, then f = g. For example, let:

$$f_t = \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 0 & \text{if } t \ge 1 \end{cases} \qquad g_t = \begin{cases} 1 & \text{if } 0 \le t \le 1 \\ 0 & \text{if } t > 1 \end{cases}$$

Then, although $f \neq g$, the feature counting functions f and g are ϵ -interleaved for any $\epsilon > 0$.

Being ϵ -interleaved is a reflexive and symmetric relation. However it is not transitive. Instead it is additive with respect to the scale: if f and g are τ -interleaved while g and h are μ -interleaved, then f and h are $(\tau + \mu)$ -interleaved.

We use the notion of being interleaved to define a pseudometric on the set of feature counting functions:

9.1. **Definition.** Let $f, g: \mathbf{Q} \to \mathbf{N}^{\text{op}}$ be feature counting functions. If f and g are interleaved we define $d(f,g) := \inf\{\epsilon \mid f \text{ and } g \text{ are } \epsilon\text{-interleaved}\}$ and otherwise $d(f,g) := \infty$.

The discussion before Definition 9.1 proves:

9.2. **Proposition.** The distance d, defined in 9.1, is an extended pseudometric on the set of feature counting functions.

For example, let (here π denotes the length of the circle of diameter 1):

$$f_t = \begin{cases} 1 & \text{if } 0 \le t < \pi \\ 0 & \text{if } t > \pi \end{cases}$$

Then $d(f, 0) = \pi$, where 0 denotes the constant function with value 0.

Let $\tau > 0$ be a positive real number and f and g be feature counting functions. By definition, $d(f,g) < \tau$ if and only if f and g are ϵ -interleaved for some $\epsilon < \tau$. Thus the open ball $B(f,\tau)$ around f with radius τ , consists of all feature counting functions which are ϵ -interleaved with f for some $\epsilon < \tau$. These sets form a base of the topology induced by the pseudometric defined in 9.1.

Recall that we have chosen a noise system $\{S_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ in $\operatorname{Tame}(\mathbf{Q}^r, \operatorname{Vect}_K)$. Together with the rank (see 2.5) we are going to use this noise system to associate a feature counting function to a tame and compact functor. To make the association continuous, we minimize the rank over the neighbourhoods B(F, t) of a given compact and tame functor $F : \mathbf{Q}^r \to \operatorname{Vect}_K$. For t in \mathbf{Q} , define:

$$\operatorname{bar}(F)_t := \begin{cases} \operatorname{rank}(F) & \text{if } t = 0\\ \min\{\operatorname{rank}(G) \mid G \in B(F, t)\} & \text{if } t > 0 \end{cases}$$

Since F is tame and compact, $\operatorname{bar}(F)_t$ is a natural number. Note that $\operatorname{rank}(F) \geq \operatorname{bar}(F)_t$ for any t. Furthermore, if $0 < t \leq s$, then $B(F,t) \subset B(F,s)$ and hence $\operatorname{bar}(F)_t \geq \operatorname{bar}(F)_s$. Thus the association $t \mapsto \operatorname{bar}(F)_t$ defines a functor $\operatorname{bar}(F) \colon \mathbf{Q} \to \mathbf{N}^{\operatorname{op}}$ which we call the **feature counting invariant** of F (with respect to the noise system $\{\mathcal{S}_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$).

It is important to be aware that the feature counting invariant of F depends on the choice of a noise system defining its neighbourhoods B(F, t). The fundamental fact about the feature counting invariant is that it is a 1-Lipschitz function (see 2.9):

9.3. **Proposition.** Let $F, G : \mathbf{Q}^r \to \operatorname{Vect}_K$ be tame and compact. Then:

 $d(\operatorname{bar}(F), \operatorname{bar}(G)) \le d(F, G)$

Proof. It is enough to show that, for any ϵ in \mathbf{Q} , if F and G are ϵ -close then the corresponding feature counting invariants $\operatorname{bar}(F)$ and $\operatorname{bar}(G)$ are ϵ -interleaved. If F and G are ϵ -close, by Proposition 8.5, any functor in B(F,t) is $(t_0 + \epsilon)$ -close to G for some $t_0 < t$. This implies that $B(F,t) \subset B(G,t+\epsilon)$ and therefore $\operatorname{bar}(F)_t \geq \operatorname{bar}(G)_{t+\epsilon}$. In the same way, if F and G are ϵ -close, any functor in B(G,t) is $(t_0 + \epsilon)$ -close to F, for some $t_0 < t$, and therefore $\operatorname{bar}(G)_t \geq \operatorname{bar}(F)_{t+\epsilon}$. As this happens for any t, we get that $\operatorname{bar}(F)$ and $\operatorname{bar}(G)$ are ϵ -interleaved. \Box

Note that the minimal rank in the neighborhood B(F,t) of a given functor F can be obtained by non isomorphic functors, as can be seen in the following example.

9.4. **Example.** Consider the compact and 1-tame functor $F : \mathbf{Q}^2 \to \operatorname{Vect}_K$ whose restriction to the sub-poset $\mathbf{N}^2 \subset \mathbf{Q}^2$ is described as follows. On the square $\{v \leq (2,2)\} \subset \mathbf{N}^2$, F is given by the commutative diagram:

$$\begin{array}{ccc} K \longrightarrow K \longrightarrow K \\ \uparrow & \uparrow & \uparrow \\ K \longrightarrow K \longrightarrow K \\ \uparrow & \uparrow & \uparrow \\ 0 \longrightarrow K \longrightarrow K \end{array}$$

where the homomorphisms between non-zero entries are the identities. For w in $\mathbf{N}^2 \setminus \{v \leq (2,2)\}, F(\text{meet}(w,(2,2)) \leq w)$ is an isomorphism. The feature counting invariant associated to F, using the standard noise in the direction of the vector

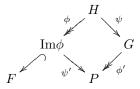
 $(1,1) \in \mathbf{Q}^2$, has values $\operatorname{bar}(F)_t = 2$, for $0 \le t \le 1$, and $\operatorname{bar}(F)_t = 1$ for every t > 1. The set B(F,2) contains the following non isomorphic subfunctors of F of rank one K((1,0),-), K((0,1),-) and K((1,1),-).

Computing the feature counting invariant of a functor $F: \mathbf{Q}^r \to \operatorname{Vect}_K$ is in general not an easy task, since we do not have a formula or an algorithm which explicitly describes the sets B(F,t) for t in \mathbf{Q} . One strategy to calculate the value $\operatorname{bar}(F)_t$ is to find proper subsets of the neighbourhood B(F,t) where the minimal rank is achieved. Here is one such a subset. Let B'(F,t) be the collection of those tame functors G for which there are natural transformations $F \leftarrow H : \phi$ and $\psi: H \to G$ such that ϕ is a τ -equivalence and a monomorphism, ψ is a μ equivalence, and $\tau + \mu < t$. Then:

9.5. **Proposition.** Let $F: \mathbf{Q}^r \to \operatorname{Vect}_K$ be a tame and compact functor. Then:

$$\operatorname{bar}(F)_t = \min\{\operatorname{rank}(G) \mid G \in B(F, t)\} = \min\{\operatorname{rank}(G) \mid G \in B'(F, t)\}$$

Proof. Let t be in **Q** and $F \leftarrow H : \phi$ and $\psi : H \to G$ be natural transformations of tame functors such that ϕ is a τ -equivalence, ψ is a μ -equivalence, $\tau + \mu < t$, and $\operatorname{bar}(F)_t = \operatorname{rank}(G)$. Form the following push-out diagram:



By Proposition 8.1 the morphism ψ' is a μ -equivalence. Since ϕ is a τ -equivalence, the same is true about the inclusion $\operatorname{Im} \phi \hookrightarrow F$ and hence P belongs to B(F,t). As ϕ' is an epimorphism, $\operatorname{rank}(G) \ge \operatorname{rank}(P)$. Therefore, by the minimality of the rank of G, $\operatorname{rank}(P) = \operatorname{rank}(G)$.

9.6. Corollary. Let $F: \mathbf{Q}^r \to \operatorname{Vect}_K$ be a compact and tame functor and t a positive rational number. Then $\operatorname{bar}(F)_t = 0$ if and only if F is contained in \mathcal{S}_{ϵ} for some $\epsilon < t$.

Proof. Let $0: \mathbf{Q}^r \to \operatorname{Vect}_K$ be the functor whose values are all zero. If $F: \mathbf{Q}^r \to \operatorname{Vect}_K$ is in \mathcal{S}_ϵ for $\epsilon < t$ then the morphism $0 \to F$ is an ϵ -equivalence and therefore 0 is contained in B(F,t) and consequently $\operatorname{bar}(F)_t = 0$. On the other hand, if $\operatorname{bar}(F)_t = 0$, then by Proposition 9.5, there is a monomorphism $F \leftrightarrow H: \phi$ such that ϕ is a τ -equivalence, $H \to 0$ is a μ -equivalence, and $\tau + \mu < t$. By the additivity of noise systems, we can then conclude F is in $\mathcal{S}_{\tau+\mu}$.

Instead of trying to calculate the precise values of bar(F), one might try first to estimate them. This can be done using the following propositions:

9.7. **Proposition.** Let $F, F' : \mathbf{Q}^r \to \operatorname{Vect}_K$ be compact and tame functors. If there is an epimorphism $\zeta : F \twoheadrightarrow F'$, then $\operatorname{bar}(F)_t \ge \operatorname{bar}(F')_t$ for any t.

Proof. Let t be in **Q**. Let $F \leftarrow H : \phi$ and $\psi : H \to G$ be natural transformations of tame functors such that ϕ is a τ -equivalence, ψ is a μ -equivalence, $\tau + \mu < t$, and $\operatorname{bar}(F)_t = \operatorname{rank}(G)$. Form the following commutative diagram, where the square

containing ξ and ψ is a push-out:

$$\begin{array}{c|c} F & \stackrel{\phi}{\longleftarrow} H & \stackrel{\psi}{\longrightarrow} G \\ \varsigma \downarrow & \downarrow & \downarrow \\ F' & \stackrel{(\zeta\phi) & \xi}{\longleftarrow} G' \end{array}$$

By Proposition 8.1 the morphism ξ is a μ -equivalence. Since ζ is an epimorphism, it induces an epimorphism between $\operatorname{coker}(\phi)$ and the quotient $F'/\operatorname{Im}(\zeta\phi)$. It then follows that the inclusion $\operatorname{Im}(\zeta\phi) \subset F'$ is a τ -equivalence. This means that G'belongs to B(F', t). As G' is a quotient of G, then $\operatorname{rank}(G) \geq \operatorname{rank}(G')$ proving the inequality $\operatorname{bar}(F)_t \geq \operatorname{bar}(F')_t$.

9.8. Corollary. Let $F, F': \mathbf{Q}^r \to \operatorname{Vect}_K$ be compact and tame functors. Then $\operatorname{bar}(F \oplus F')_t \geq \max\{\operatorname{bar}(F)_t, \operatorname{bar}(F')_t\}$ for any t in \mathbf{Q} . If the components of the noise system $\{S_\epsilon\}_{\epsilon \in \mathbf{Q}}$ are closed under direct sums (see Section 6), then $\operatorname{bar}(F)_t + \operatorname{bar}(F')_t \geq \operatorname{bar}(F \oplus F')_t$ for any t in \mathbf{Q} .

Proof. The first statement is a consequence of Proposition 9.7. The second inequality, in the case the noise system is closed under direct sums, follows from the fact that if G and G' belong respectively to B(F,t) and B(F',t), then $G \oplus G'$ belongs to $B(F \oplus F',t)$.

10. The Feature counting Invariant for r = 1

Let us choose a noise system $\{S_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ in Tame(\mathbf{Q} , Vect_K). Let $F: \mathbf{Q} \to \text{Vect}_K$ be a tame and compact functor. The main result of this section is Proposition 10.2, which holds for any noise system closed under direct sums. It states that $\text{bar}(F)_t$ counts the number of bars in the barcode (see [19]) of the quotient of F by the maximal subfunctor that belongs to S_{ϵ} . As in the classical case the main tool in obtaining this result is the classification theorem of modules over a PID (see 14.15).

Our first step is to show that to compute the value of $\operatorname{bar}(F)_t$ it is enough to only minimise the rank over subfunctors of F. That is the reason we define B''(F,t) to be the collection of tame subfunctors $G \subset F$ for which this inclusion is an ϵ -equivalence, for some $\epsilon < t$ in \mathbb{Q} .

10.1. **Proposition.** Let $F: \mathbf{Q} \to \operatorname{Vect}_K$ be a tame and compact functor. Then for any noise system in $\operatorname{Tame}(\mathbf{Q}, \operatorname{Vect}_K)$:

$$\operatorname{bar}(F)_t = \min\{\operatorname{rank}(G) \mid G \in B(F, t)\} = \min\{\operatorname{rank}(G) \mid G \in B''(F, t)\}$$

Proof. The key property of Tame(**Q**, Vect_K) we use is that, for any subfunctor $G \subset F$, rank(G) ≤ rank(F). This is a consequence of the fact that in Tame(**Q**, Vect_K) any subfunctor of a free functor is also free. We can use this to get natural transformations $F \leftarrow H : \phi$ and $\psi : H \twoheadrightarrow G$ such that ϕ is an inclusion and a τ -equivalence, ψ is an epimorphism and a μ -equivalence, $\tau + \mu < t$, and rank(G) = bar(F)_t =: n. Note that ψ can be assumed to be an epimorphism by replacing G with Im(ψ) if necessary. Let { $g_1 \in G(v_1), \ldots, g_n \in G(v_n)$ } be a minimal set of generators for G (see 2.5) and h_i be any element in $H(v_i)$ which is mapped via ψ to g_i . Since $H/\langle h_1, \ldots, h_n \rangle$ is a quotient of the kernel of ψ , the inclusion $\langle h_1, \ldots, h_n \rangle \subset H$ is a μ -equivalence. It follows that $\langle h_1, \ldots, h_n \rangle \subset F$ is a ($\tau + \mu$)-equivalence. As the rank of $\langle h_1, \ldots, h_n \rangle$ is n, the proposition follows. □

Recall that any compact and tame functor $F: \mathbf{Q} \to \operatorname{Vect}_K$ is isomorphic to a finite direct sum of the form $\bigoplus_{i \in I} [w_i, u_i) \oplus \bigoplus_{j \in J} K(v_j, -)$ (see 5.6). Furthermore the isomorphism types of these summands are uniquely determined by the isomorphism type of F. For a positive t in \mathbf{Q} , define:

$$I_t := \{ i \in I \mid [w_i, u_i) \notin \mathcal{S}_{\epsilon} \text{ for any } \epsilon < t \}$$
$$J_t := \{ j \in J \mid K(v_j, -) \notin \mathcal{S}_{\epsilon} \text{ for any } \epsilon < t \}$$

10.2. **Proposition.** Assume $\{S_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ is a noise system in Tame(\mathbf{Q} , Vect_K) whose components are closed under direct sums. Let $F: \mathbf{Q} \to \text{Vect}_K$ be a compact and tame functor. Then $\text{bar}(F)_t = |I_t| + |J_t|$.

Proof. Let:

$$F_1 := \bigoplus_{i \in I_t} [w_i, u_i) \oplus \bigoplus_{j \in J_t} K(v_j, -) \qquad F_2 := \bigoplus_{i \in I \setminus I_t} [w_i, u_i) \oplus \bigoplus_{j \in J \setminus J_t} K(v_j, -)$$

The functor F is isomorphic to $F_1 \oplus F_2$ and F_2 belongs to S_{ϵ} for some $\epsilon < t$. We can then use Corollary 9.8 to conclude $\operatorname{bar}(F)_t = \operatorname{bar}(F_1)_t$. Without loss of generality, we can therefore assume $F = F_1$, i.e., $I = I_t$ and $J = J_t$.

In that case one shows that for any surjection $\phi: F \to G$ where G is in \mathcal{S}_{ϵ} for some $\epsilon < t$, the kernel of ϕ has the same rank as F. The proposition then follows from 10.1.

Assume the components of the noise system $\{S_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ in Tame(\mathbf{Q} , Vect_K) are closed under direct sums. This implies that, for any ϵ in \mathbf{Q} , there exists the unique maximal subfunctor $F[S_{\epsilon}] \subset F$ with respect to the property that $F[S_{\epsilon}]$ is in S_{ϵ} (see 7.1). Let t > 0 be in \mathbf{Q} . For any $\tau \leq \mu < t$, since $S_{\tau} \subset S_{\mu}$, we have inclusions $F[S_{\tau}] \subset F[S_{\mu}] \subset F$. Define $F[S_{\epsilon t}] := \bigcup_{\tau < t} F[S_{\tau}] \subset F$.

10.3. Corollary. Let $\{S_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ be a noise system in $\operatorname{Tame}(\mathbf{Q}, \operatorname{Vect}_K)$ whose components are closed under direct sums and $F \colon \mathbf{Q} \to \operatorname{Vect}_K$ a compact and tame functor. Then $\operatorname{bar}(F)_t = \operatorname{rank}(\operatorname{coker}(F[S_{< t}] \subset F)).$

11. The Feature Counting Invatiant for the standard noise

The strategy for computing the feature counting function can be further improved in the case of the standard noise $\{V_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ in the direction of a cone $V \subset \mathbf{Q}^r$ (see 6.2). Such a noise system is fixed throughout this section. As in Section 10, define B''(F,t) to be the collection of tame subfunctors $G \subset F$ for which this inclusion is an ϵ -equivalence for some $\epsilon < t$ in \mathbf{Q} .

11.1. **Proposition.** Let $F: \mathbf{Q}^r \to \operatorname{Vect}_K$ be a tame and compact functor. Then for the standard noise in the direction of a cone:

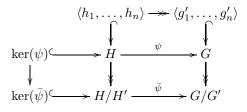
$$\operatorname{bar}(F)_t = \min\{\operatorname{rank}(G) \mid G \in B(F, t)\} = \min\{\operatorname{rank}(G) \mid G \in B''(F, t)\}$$

Proof. By Proposition 9.5 there are natural transformations $F \leftrightarrow H : \phi$ and $\psi: H \to G$ such that $\operatorname{coker}(\phi) \in V_{\tau}$, $\operatorname{coker}(\psi) \in V_a$, $\operatorname{ker}(\psi) \in V_b$, $\tau + a + b < t$, and $\operatorname{rank}(G) = \operatorname{bar}(F)_t =: n$. Let $\{g_1 \in G(v_1), \ldots, g_n \in G(v_n)\}$ be a minimal set of generators for G (see 2.5). Since $\operatorname{coker}(\psi) \in V_a$, there are vectors w_1, \ldots, w_n in the cone V such that $||w_i|| = a$ and the element $g'_i := G(v_i < v_i + w_i)(g_i)$ is in the image of $\psi_{v_i+w_i}: H(v_i+w_i) \to G(v_i+w_i)$. Let $h_i \in H(v_i+w_i)$ be any element that is mapped via $\psi_{v_i+w_i}$ to g'_i . Consider the subfunctor $F' := \langle \phi(h_1), \ldots, \phi(h_n) \rangle$ of F. We claim that the inclusion $F' \subset F$ is $(\tau + a + b)$ -equivalence, and hence F'

belongs to B(F,t). If the claim holds, since $\operatorname{rank}(F') \leq n$, by the minimality of the rank of G, we can conclude that $\operatorname{rank}(F') = n$ and the proposition follows. The inclusion $F' \subset F$ is the image of the composition:

$$\langle h_1, \dots, h_n \rangle \xrightarrow{\longleftarrow} H \xrightarrow{\phi} F$$

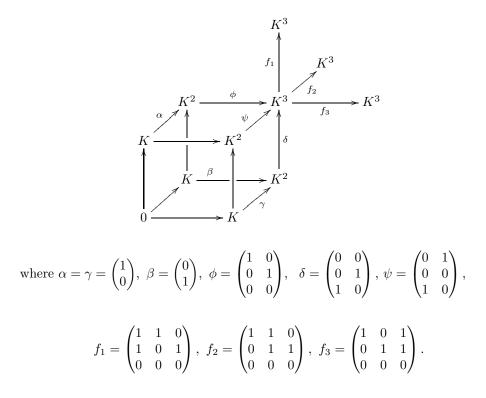
As ϕ is a τ -equivalence, it is enough to show that $\langle h_1, \ldots, h_n \rangle \subset H$ is an (a + b)-equivalence (see 8.2). Set $H' := \langle h_1, \ldots, h_n \rangle \subset H$ and $G' := \langle g'_1, \ldots, g'_n \rangle \subset G$ and consider the following commutative diagram:



The square containing ψ and $\bar{\psi}$ is a push-out square and therefore the natural transformation $\ker(\psi) \to \ker(\bar{\psi})$ is an epimorphism. It follows that $\ker(\bar{\psi})$ belongs to V_b . Furthermore by definition, G/G' is in V_a and hence so is the image of $\bar{\psi}$. We can then use the additivity property of noise systems to conclude that H/H' belongs to V_{a+b} .

According to 11.1, to find the value $\operatorname{bar}(F)_t$, we need to find the minimum rank of subfunctors of $G \subset F$ for which this inclusion is an ϵ -equivalence for some $\epsilon < t$. Consider the functor given in Example 9.4 where $\operatorname{bar}(F)_2 = 1$. Note that a minimal set of generators for F is given by any $g_1 \neq 0$ in F(1,0) and $g_2 \neq 0$ in F(0,1). In this case both $\langle g_1 \rangle$ and $\langle g_2 \rangle$ are 2-close to F. In this example, to get the minimum rank among functors in B(F,2), we do not need to consider all subfunctors $G \subset F$, but only the subfunctors which are generated by subsets of a given set of minimal generators for F. This is also the case when restricting to functors with one-dimensional domain. However the corresponding statement fails in the general setting as illustrated by the following examples:

11.2. **Example.** Assume the characteristic of K is not 2. Let us consider the standard noise in Tame(\mathbf{Q}^3 , Vect_K) in the direction of the cone Cone(1, 1, 1). Consider any compact and 1-tame functor $F: \mathbf{Q}^3 \to \text{Vect}_K$ of rank 3 whose restriction to the sub-poset { $v \leq (1,1,1)$ } \cup {(1,1,2), (1,2,1), (2,1,1)} $\subset \mathbf{N}^3 \subset \mathbf{Q}^3$ is given by the following commutative diagram:



Let $g_1 \in F(1,0,0)$, $g_2 \in F(0,1,0)$ and $g_3 \in F(0,0,1)$ be non-zero vectors in these 1-dimensional vector spaces. These vectors form a minimal set of generators for F. Note that the vectors:

$$h_1 := F((1,0,0) \le (1,1,1))(g_1)$$

$$h_2 := F((0,1,0) \le (1,1,1))(g_2)$$

$$h_3 := F((0,0,1) \le (1,1,1))(g_3)$$

form the standard basis for $F(1,1,1) = K^3$. The subfunctor $\langle h_1 + h_2 + h_3 \rangle \subset F$ is 1-close to F. This follows from the following equalities:

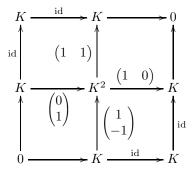
$$f_1(h_1) = 1/2f_1(h_1 + h_2 + h_3)$$

$$f_2(h_2) = 1/2f_2(h_1 + h_2 + h_3)$$

$$f_3(h_3) = 1/2f_3(h_1 + h_2 + h_3)$$

Note further that $f_1(h_3)$, $f_1(h_1)$ and $f_1(h_2)$ are pair-wise linearly independent. This implies that $\langle g_2 \rangle$ and $\langle g_3 \rangle$ are not 1-close to F. In the same way, $f_2(h_2)$ and $f_2(h_1)$ are not parallel and therefore $\langle g_1 \rangle$ is not 1-close to F either.

11.3. **Example.** Let us consider the standard noise in $\text{Tame}(\mathbf{Q}^2, \text{Vect}_K)$ in the direction of the cone Cone(1, 1). Consider any compact and 1-tame functor $F : \mathbf{Q}^2 \to \text{Vect}_K$ of rank 2 whose restriction to the sub-poset $\{v \leq (2, 2)\} \subset \mathbf{N}^2 \subset \mathbf{Q}^2$ is given by the following commutative diagram:



Let $g_1 \in F(1,0)$ and $g_2 \in F(0,1)$ be non-zero vectors in these 1-dimensional vector spaces. These vectors form a minimal set of generators for F. Note that neither $\langle g_1 \rangle \subset F$ nor $\langle g_2 \rangle \subset F$ are 1-close to F. However the subfunctor of F generated by the element (1,0) in $K^2 = F(1,1)$ is 1-close to F.

12. Denoising

The aim of this section is to introduce a notion of **denoising** for tame and compact functors. Intuitively, a denoising is an approximation and hopefully a simplification of such a functor that can be performed at different scales.

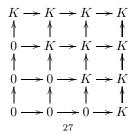
12.1. **Definition.** Let $\{S_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ be a noise system in $\operatorname{Tame}(\mathbf{Q}^r, \operatorname{Vect}_K)$ and $F \colon \mathbf{Q}^r \to \operatorname{Vect}_K$ be a tame and compact functor. A denoising of F is a sequence of functors $\{\operatorname{denoise}(F)_t\}_{0 < t \in \mathbf{Q}}$, indexed by positive rational numbers, such that for any t:

- denoise $(F)_t$ is in B(F, t),
- $\operatorname{rank}(\operatorname{denoise}(F)_t) = \operatorname{bar}(F)_t.$

Thus a denoising of F at scale t is a choice of a functor in the neighborhood B(F,t) that realizes the minimum value of the rank, which is given by $bar(F)_t$. There are of course many such choices and there seems not to be a canonical one in general for r > 1. Different denoising algorithms highlight different properties of the functor. Here we present some examples of denoisings.

12.2. Minimal subfunctor denoising. Let $\{V_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ be the standard noise in the direction of a cone $V \subset \mathbf{Q}^r$, (see 6.2). In this case $\operatorname{bar}(F)_t$ can be obtained as the rank of some G in B(F,t) which is a subfunctor of F (see Proposition 11.1). Among these subfunctors with minimal rank we can then choose one which is also minimal with respect to the inclusion. We call such a choice, a minimal subfunctor denoising of F at t. For example:

12.3. **Example.** Consider the compact and 1-tame functor $F: \mathbb{Q}^2 \to \operatorname{Vect}_K$ whose restriction to the sub-poset $\mathbb{N}^2 \subset \mathbb{Q}^2$ is described as follows. On the square $\{v \leq (3,3)\} \subset \mathbb{N}^2$, F is given by the commutative diagram:



where the maps between non-zero entries are the identities. For w in $\mathbb{N}^2 \setminus \{v \leq (3,3)\}$, the map $F(\text{meet}\{w, (3,3)\} \leq w)$ is an isomorphism. Consider the standard noise $\{V_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ in the direction of Cone(1,1). For that noise system $\text{bar}(F)_2 = 2$. Let $\{g_1 \in F(0,3), g_2 \in F(2,1), g_3 \in F(1,2), g_4 \in F(3,0)\}$ be a minimal set of generators for F. Set $g'_1 := F((0,3) \leq (1,3))(g_1)$ and $g'_4 := F((3,0) \leq (3,1))(g_4)$. The subfunctor $\langle g'_1, g'_4 \rangle \subset F$ is in B(F,2), has the required rank 2 and is the minimal element, with respect to inclusion, among subfunctors of F with rank 2 in B(F,2).

In general the minimal subfunctor denoising is not unique:

12.4. **Example.** Consider the compact and 1-tame functor $F : \mathbf{Q}^2 \to \operatorname{Vect}_K$ whose restriction to the sub-poset $\mathbf{N}^2 \subset \mathbf{Q}^2$ is described as follows. For $\{v \leq (3,5)\} \subset \mathbf{N}^2$, F is given by the commutative diagram:

$K \longrightarrow K$	$\rightarrow K -$	$\rightarrow K -$	$\rightarrow K -$	$\rightarrow K$
1 1	1	1	1	ł
$0 \longrightarrow K$	$\rightarrow K -$	$\rightarrow K -$	$\rightarrow K -$	$\succ K$
1 1	1	ſ	1	1
$0 \longrightarrow 0$ -	$\rightarrow 0 -$	$\rightarrow K -$	$\succ K -$	≻ K
1 1	1	1	1	ł
$0 \longrightarrow 0$ -	$\rightarrow 0 -$	→ 0 —	<u>→ 0 —</u>	≻ K

where the maps between non-zero entries are the identities. For w in $\mathbb{N}^2 \setminus \{v \leq (3,5)\}$, the map $F(\text{meet}\{w, (3,5)\} \leq w)$ is an isomorphism. Consider the standard noise $\{V_{\epsilon}\}_{\epsilon \in \mathbb{Q}}$ in the direction of Cone(1,1). For that noise $\text{bar}(F)_2 = 2$. Let $\{g_1 \in F(0,3), g_2 \in F(1,2), g_3 \in F(3,1), g_4 \in F(5,0)\}$ be a minimal set of generators for F. Let $g'_1 = F((0,3) < (1,3))(g_1)$ and $g'_4 = F((5,0) < (5,1))(g_4)$. The subfunctors of F given by $\langle g'_1, g_3 \rangle$ and $\langle g_2, g'_4 \rangle$ are in B(F,2) and they have required rank 2. Thus they are examples of 2-denoising of F. Furthermore they are both minimal with respect to inclusion.

12.5. Quotient denoising. Let $\{S_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ be a noise system in $\operatorname{Tame}(\mathbf{Q}^r, \operatorname{Vect}_K)$ whose compact part is closed under direct sums (see 7.3 and 7.4 for when a standard noise in the direction of a cone $V \subset \mathbf{Q}^r$ or a sequence of vectors $\mathcal{V} = \{v_1, \ldots, v_n\}$ in \mathbf{Q}^r is closed under direct sums). For any $F: \mathbf{Q}^r \to \operatorname{Vect}_K$ there exists a unique maximal noise of size ϵ contained in $F, F[\mathcal{S}_{\epsilon}] \subset F$ (see 7.1). Given a tame and compact functor $F: \mathbf{Q}^r \to \operatorname{Vect}_K$, define the subfunctor $F[\mathcal{S}_{<t}] := \bigcup_{\tau < t} F[\mathcal{S}_{\tau}] \subset F$. One can ask if the sequence $\{\operatorname{coker}(F[\mathcal{S}_{<t}] \subset F)\}_{0 < t \in \mathbf{Q}}$ is a denoising of F. If it is a denoising, we call it the quotient denoising of F. Corollary 10.3 states that in the case r = 1, this procedure always gives a denoising of F.

12.6. Example. Consider the compact and 1-tame functor $F: \mathbf{Q} \to \operatorname{Vect}_K$ whose restriction to the sub-poset $\mathbf{N} \subset \mathbf{Q}$ is described as follows. On the segment $\{v \leq 4\} \subset \mathbf{N}, F$ is given by the commutative diagram:

$$K^{3} \xrightarrow{\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}} K^{2} \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} K^{2} \xrightarrow{\begin{pmatrix} 1 & 1 \\ 0 \end{pmatrix}} K \xrightarrow{\begin{pmatrix} 1 & 1 \\ 0 \end{pmatrix}} K$$

For any $w \ge 4$, $F(4 \le w)$ is an isomorphism. Let $\{V_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ be standard noise in the direction of the full cone \mathbf{Q} . The basic barcode of F has values:

$$\operatorname{bar}(F)_t = \begin{cases} 4 & \text{if } 0 \le t \le 1\\ 2 & \text{if } 1 < t \le 2\\ 1 & \text{if } t > 2. \end{cases}$$

The cokernel of the inclusion $F[S_{\leq 2}] \subset F$ is isomorphic to:

$$K^2 \xrightarrow{(1 \quad 0)} K \xrightarrow{1} K \xrightarrow{1} K \xrightarrow{1} K$$

For t > 2, the cokernel of the inclusion $F[S_{\leq t}] \subset F$ is isomorphic to:

$$K \xrightarrow{1} K \xrightarrow{1} K \xrightarrow{1} K \xrightarrow{1} K$$

Note that for r > 1, the family of functors $\{\operatorname{coker}(F[\mathcal{S}_{< t}] \subset F)\}_{0 < t \in \mathbf{Q}}$ is not always a denoising.

12.7. **Example.** Let $F: \mathbf{Q}^2 \to \operatorname{Vect}_K$ be the functor defined in Example 9.4 and $\{V_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ is standard noise in the direction of $\operatorname{Cone}(1,1)$. Since $F(v \leq w)$ is a monomorphism for any $v \leq w$ in \mathbf{Q}^2 , the quotient denoising of F at scale t is isomorphic to F, for any t in \mathbf{Q} . It follows that the rank of $\operatorname{coker}(F[\mathcal{S}_{\leq 2}] \subset F)$ is 2, while $\operatorname{bar}(F)_2 = 1$.

Given a denoising $\{\text{denoise}(F)_t\}_{0 < t \in \mathbf{Q}}$, we can consider the family of multisets $\{\beta_0 \text{denoise}(F)_t\}_{0 < t \in \mathbf{Q}}$. Note that in the case of quotient denoising such family of multisets has the property that if s < t in \mathbf{Q} then $\beta_0 \text{denoise}(F)_t$ is a subset of $\beta_0 \text{denoise}(F)_s$, (see 2.3). It will be the focus of future work to study the stability of families of multisets associated to a denoising and how such invariants identify persistent features (see [5]).

13. FUTURE DIRECTIONS

13.1. The feature counting invariant is convenient in estimating the number of significant features of a mutidimensional persistence module relative to a given noise system. However for identifying these features an appropriate denoising scheme (see Section 12) is crucial. Existence of such a denoising follows for a given noise system whenever the following is true:

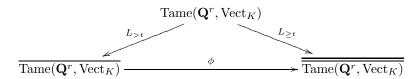
for G_1 and G_2 in $B(F, \epsilon)$ that have the minimal rank, given necessarily by $bar(F)_{\epsilon}$, the Betti diagrams $\beta_0 G_1, \beta_0 G_2: \mathbf{Q}^r \to \mathbf{N}$ are 2ϵ -interleaved.

As of the writing of this paper we do not know for what noise system the above statement holds true. We believe however that characterising such systems is a worthwhile pursuit and we hope to return to it in the near future. A related and possible easier question is if there exist some natural class of multidimensional persistence modules for which the above statement holds true for the standard noise.

13.2. A second possible avenue for future work is the construction and subsequent implementation of algorithms for computing feature counting invariants. An implementation of the feature counting invariant will help us in understanding which type of noise best detects persistent features given a specific application or construction. As the computation of feature counting invariants involves rank minimisation one would expect that this is an NP-hard problem. However for applications to data analysis determining exactly the feature counting invariant is not necessarily the

ultimate aim. Indeed it is rather the order of magnitude of the values of this invariant that is important. Intuitively one may think of feature counting invariants as a measure of the complexity of a space and showing that one space is ever so slightly more complex than another is probably of neglectable interest for applications. On the other hand a big difference between the values of the feature counting invariants for a given ϵ (for example 1, 10 or 10000) has more drastic implications on the underlying geometrical structures of the spaces at hand. Therefore the computability of the more feasible question of bounding or approximating the feature counting function is more interesting.

13.3. Serre localization. Let $\{S_{\epsilon}\}_{\epsilon \in \mathbf{Q}}$ be a noise system. Recall that the 0-th component S_0 is always a Serre subcategory in Tame $(\mathbf{Q}^r, \operatorname{Vect}_K)$. Similarly, for any t in \mathbf{Q} , so are the unions $\bigcup_{\epsilon \geq t} S_{\epsilon}$ and $\bigcup_{\epsilon > t} S_{\epsilon}$. Being Serre means that we can quotient out these subcategories and localize Tame $(\mathbf{Q}^r, \operatorname{Vect}_K)$ away from them. This process is well explained in [10]. Since $\bigcup_{\epsilon > t} S_{\epsilon} \subset \bigcup_{\epsilon \geq t} S_{\epsilon}$, we get the following commutative diagram of functors where the vertical maps denote the appropriate localizations and ϕ is given by the universal property:



We believe that understanding the relation between denoising at scale t and the functor ϕ is a problem worth pursuing. As of the writing of this paper we were unable to use these Serre localizations to give what we could call a "better" conceptual explanation for the feature counting invariants or to construct new continuous invariants.

14. Appendix: functors indexed by \mathbf{N}^r

The aim of this appendix is to recall basic properties of the category of functors $\operatorname{Fun}(\mathbf{N}^r, \operatorname{Vect}_K)$: we identify its compact objects, projective objects, and discuss minimal covers. Although it is standard, we decided to include this material for self containment.

14.1. Semisimplicity. A functor $F: \mathbf{N}^r \to \operatorname{Vect}_K$ is semi-simple if F(v < w) is the zero homomorphism for any v < w. For example the unique functor $U_v: \mathbf{N}^r \to \operatorname{Vect}_K$ such that $U_v(v) = K$ and $U_v(w) = 0$ if $w \neq v$ is semi-simple. Any semisimple functor is a direct sum of functors of the form U_v and hence can be described uniquely as $\bigoplus_{v \in \mathbf{N}^r} (U_v \otimes V_v)$ for some sequence of vector spaces V_v .

14.2. Radical. Let $F: \mathbf{N}^r \to \operatorname{Vect}_K$ be a functor. Define $\operatorname{rad}(F)(v)$ to be the subspace of F(v) given by the sum of all the images of $F(u < v): F(u) \to F(v)$ for all u < v. For any $v \leq w$, the homomorphism $F(v \leq w): F(v) \to F(w)$ maps the subspace $\operatorname{rad}(F)(v) \subset F(v)$ into $\operatorname{rad}(F)(w) \subset F(w)$. Thus these subspaces form a subfunctor denoted by $\operatorname{rad}(F) \subset F$. A natural transformation $\phi: F \to G$ maps the subfunctor $\operatorname{rad}(F) \subset F$ into the subfunctor $\operatorname{rad}(G) \subset G$. The resulting natural transformation is denoted by $\operatorname{rad}(\phi): \operatorname{rad}(F) \to \operatorname{rad}(G)$. Note that for any functor $F: \mathbf{N}^r \to \operatorname{Vect}_K, F/\operatorname{rad}(F)$ is semisimple.

14.3. **Proposition.** A natural transformation $\phi: F \to G$ in $\operatorname{Fun}(\mathbf{N}^r, \operatorname{Vect}_K)$ is an epimorphism if and only if its composition with the quotient $\pi: G \to G/\operatorname{rad}(G)$ is an epimorphism.

Proof. Since π is an epimorphism, if ϕ is an epimorphism, then so is $\pi\phi$. Assume that $\pi\phi$ is an epimorphism. If ϕ is not an epimorphism, then the set of v in \mathbf{N}^r for which $\phi(v): F(v) \to G(v)$ is not an epimorphism is not empty. Let us choose a minimal element u in this set (see 2.6) and consider a commutative diagram with exact rows:

$$\begin{array}{ccc} 0 & \longrightarrow \operatorname{rad}(F)(u) & \longrightarrow F(u) \longrightarrow F(u)/\operatorname{rad}(F)(u) \longrightarrow 0 \\ & & & & \downarrow \\ & & & \downarrow \\ 0 & \longrightarrow \operatorname{rad}(G)(u) & \longrightarrow G(u) \xrightarrow{\pi} G(u)/\operatorname{rad}(G)(u) \longrightarrow 0 \end{array}$$

Minimality of u implies that $rad(\phi)(u)$ is an epimorphism. Since the right side vertical homomorphism is also an epimorphism, it follows that so is the middle one, contradicting the assumption about $\phi(u)$.

Applying 14.3 to $0 \hookrightarrow F$, we get that rad(F) = F if and only if F = 0. The very same argument as in the proof of 14.3 can also be used to show:

14.4. **Proposition.** A functor $G: \mathbb{N}^r \to \operatorname{Vect}_K$ has finite dimensional values if and only if $G/\operatorname{rad}(G)$ has finite dimensional values.

14.5. Minimal covers. Recall that $\phi: F \to G$ in Fun $(\mathbf{N}^r, \operatorname{Vect}_K)$ is called minimal if any $f: F \to F$, such that $\phi = \phi f$, is an isomorphism (see 2.5). For example:

14.6. **Proposition.** Let $\phi: F \to G$ in $\operatorname{Fun}(\mathbf{N}^r, \operatorname{Vect}_K)$ be a natural transformation such that the induced morphism on the quotients $[\phi]: F/\operatorname{rad}(F) \to G/\operatorname{rad}(G)$ is an isomorphism and either the values of F are finite dimensional or F is free. Then ϕ is minimal.

Proof. Let $f: F \to F$ be such that $\phi = \phi f$. By quotienting out radicals we obtained a commutative triangle:

$$F/\mathrm{rad}(F) \xrightarrow[\phi]{[\phi]} F/\mathrm{rad}(F)$$

Since $[\phi]$ is an isomorphism, then so is [f]. We can then use 14.3 to conclude that $f: F \to F$ is an epimorphism. As epimorphisms of finite dimensional vector spaces are isomorphisms, under the assumption that F has finite dimensional values, we can conclude that f is an isomorphism. In the case F is free, the map f splits and F is isomorphic to $F \oplus \ker(f)$. It follows that $\ker(f)/\operatorname{rad}(\ker(f)) = 0$ and hence $\ker(f) = 0$ showing that in this case f is also an isomorphism. \Box

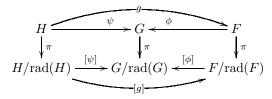
14.7. Let V be a vector space. Consider the functor $K(v, -) \otimes V \colon \mathbf{N}^r \to \operatorname{Vect}_K$. The Yoneda isomorphism (see [15] or [25]) states that, the function assigning to a natural transformation $\phi \colon K(v, -) \otimes V \to G$ the homomorphism $V \to G(i)$ given by $x \mapsto \phi(v)(\operatorname{id}_v \otimes x)$ is an isomorphism between $\operatorname{Nat}(K(v, -) \otimes V, G)$ and $\operatorname{Hom}(V, G(v))$. A direct consequence of this isomorphism is the fact that $G \mapsto$ $\operatorname{Nat}(K(v, -) \otimes V, G)$ is an exact operation (i.e., $K(v, -) \otimes V$ is a projective object in $\operatorname{Fun}(\mathbf{N}^r, \operatorname{Vect}_K)$, see [25, Section 2.2]). 14.8. Consider a functor $G: \mathbf{N}^r \to \operatorname{Vect}_K$. Since $G/\operatorname{rad}(G)$ is semisimple, it is isomorphic to $\oplus_{v \in \mathbf{N}^r}(U_v \otimes V_v)$ for some sequence of vector spaces V_v . Let $F = \bigoplus_{v \in \mathbf{N}^r} K(v, -) \otimes V_v$. Note that $F/\operatorname{rad}(F)$ is also isomorphic to $\bigoplus_{v \in \mathbf{N}^r} (U_v \otimes V_v)$. Since F is projective (see 2.4), then there is a natural transformation $\phi: F \to G$ making the following diagram commutative:

$$\begin{array}{c} F & \xrightarrow{\phi} & G \\ \pi \bigvee & & & \downarrow \pi \\ F/\mathrm{rad}(F) & \xrightarrow{\simeq} & G/\mathrm{rad}(G) \end{array}$$

where π 's are the quotient transformations. According to 14.6 the natural transformation $\phi: F \to G$ is minimal. Since it is also an epimorphism (see 14.3) and F is free, this map is a minimal cover (see 2.5). Thus all functors in Fun(\mathbf{N}^r , Vect_K) admit a minimal cover. Moreover G is of finite type (see 2.5) if and only if V_v is finite dimensional for any v in \mathbf{N}^r . Its support (see 2.5) is given by the subset of \mathbf{N}^r of all elements v for which $V_v \neq 0$. If G is of finite rank (see 2.5), then its rank is given by $\sum_{v \in \mathbf{N}^r} \dim_K V_v$. If G is of finite type, then the multiset $\mathbf{N}^r \ni v \mapsto \dim_K V_v \in \mathbf{N}$ is the 0-Betti diagram of G (see 2.5). Moreover F has a finite set of generators if and only if it is of finite rank. Note that being of finite type, of finite rank, and the invariants supp(G), rank(G) and $\beta_0 G$ depend only on $G/\operatorname{rad}(G)$. However, the choice of a set of minimal generators for G is equivalent to a choice of a minimal cover $F \to G$ and hence it contains much more information, than the semisimple functor $G/\operatorname{rad}(G)$.

14.9. Corollary. A natural transformation $\psi: H \to G$ in $\operatorname{Fun}(\mathbf{N}^r, \operatorname{Vect}_K)$ is a minimal cover if and only if H is free and the induced morphism on the quotients $[\psi]: H/\operatorname{rad}(H) \to G/\operatorname{rad}(G)$ is an isomorphism.

Proof. Assume $\psi: H \to G$ is a minimal cover. Let V_v be a sequence of vector spaces such that $G/\operatorname{rad}(G)$ is isomorphic to $\bigoplus_{v \in \mathbf{N}^r} (U_v \otimes V_v)$. Consider $F = \bigoplus_{v \in \mathbf{N}^r} K(v, -) \otimes V_v$ and a minimal cover $\phi: F \to G$ described above. All these natural transformations fit into the following commutative diagram:



Note that $[\phi]$ is an isomorphism by construction and g is an isomorphism by the minimality assumption on ψ and ϕ . It then follows that [g] is an isomorphism and consequently so is ψ . That proves one implication of the corollary. The other implication follows from 14.6 and 14.3.

14.10. Corollary. Any projective object in $\operatorname{Fun}(\mathbf{N}^r, \operatorname{Vect}_K)$ is free.

Proof. Let $P: \mathbf{N}^r \to \operatorname{Vect}_K$ be projective and $\phi: F \to P$ be a minimal cover. As P is projective, there is a natural transformation $s: P \to F$ such that $\phi s = \operatorname{id}_P$. This implies that s is a monomorphism. According to 14.3 and 14.9 s is also an epimorphism. We can conclude that s is an isomorphism and hence P is free. \Box

14.11. Compact objects. Recall that an object A in an abelian category is compact if, for any sequence of monomorphisms $A_1 \subset A_2 \subset \cdots \subset A$ such that $A = \operatorname{colim} A_i$, there is k for which $A_k = A$ (see [26]).

14.12. **Proposition.** Let $F: \mathbb{N}^r \to \operatorname{Vect}_K$ be a functor. The following are equivalent:

- (a) F is compact in Fun(\mathbf{N}^r , Vect_K);
- (b) F is of finite rank;
- (c) F fulfills the following two conditions: (1) F(v) is finite dimensional for any v; (2) there is a natural number n such that, for any $v = (v_1, \ldots, v_r)$ in \mathbf{N}^r , the homomorphism $F((\min(n, v_1), \ldots, \min(n, v_r)) \leq v)$ is an isomorphism.

Proof. Assume F is compact. If F is not of finite rank, then there is a sequence of proper subfunctors $G_1 \subset G_2 \subset \cdots \subset F/\operatorname{rad}(F)$ such that $\operatorname{colim} G_i = F/\operatorname{rad}(F)$. Let $F_i := \pi^{-1}(G_i)$ where $\pi \colon F \to F/\operatorname{rad}(F)$ is the quotient transformation. We obtain a sequence of proper subfunctors $F_1 \subset F_2 \subset \cdots \subset F$ such that $\operatorname{colim} F_i = F$ contradicting the fact that F is compact. This proves the implication (a) \Rightarrow (b).

Assume F is of finite rank. The first condition in (c) follows from 14.4. Let i be a natural number. For $v = (v_1, \ldots, v_r)$ in \mathbf{N}^r , set $v^i := (\min(i, v_1), \ldots, \min(i, v_r))$. Note if $v \leq w$, then $v^i \leq w^i$. Define $F^i(v) := F(v^i)$ and $F^i(v \leq w) := F(v^i \leq w^i)$. In this way we obtain a functor $F^i : \mathbf{N}^r \to \operatorname{Vect}_K$. If $i \leq j$, then $v^i \leq v^j \leq v$. Let $F^{i \leq j} : F^i \to F^j$ be the natural transformation given by the homomorphisms $F(v^i \leq v^j)$ and $F^{i<\infty} : F^i \to F$ be the natural transformation given by the homomorphisms $F(v^i \leq v^j)$ and $F^{i<\infty} : F^i \to F$ be the natural transformation given by the homomorphisms $F(v^i \leq v)$. Since $\operatorname{colim}_i(F^{i<\infty}) : \operatorname{colim}(F^i) \to F$ is an isomorphism and F is of finite rank, there is m such that $F^{m<\infty} : F^m \to F$ is surjective. From the definition it follows that $F^{m\leq i} : F^m \to F_i$ is also surjective. Let $T := \{v \in \mathbf{N}^r \mid v = v^m\}, v \in T$, and $i \geq m$. Note that T is a finite set of all elements v in \mathbf{N}^r such that $v \leq (m, \ldots, m)$. Define:

$$K_{i,v} = \bigcup_{v \le w} \ker \left(F^m(v) = F(v) \xrightarrow{F(v \le w^i)} F(w^i) = F^i(w) \right)$$
$$K_i := \bigoplus_{v \in T} K_{i,v} \subset \bigoplus_{v \in T} F(v)$$

Note that $K_i \subset K_j$ if $i \leq j$. Since T is finite, the space $\bigoplus_{v \in T} F(v)$ is finite dimensional, and hence there is n such that $K_n = K_i$ for any $i \geq n$. For this n, the natural transformation $F^{n \leq i} \colon F^n \to F^i$ is an isomorphism. It follows that so is $F^{n < \infty} \colon F^n \to F$ proving the second condition in (c).

Assume (c). Let *n* be the number given by second condition in (c). To prove *F* is compact consider a sequence $F_1 \subset F_2 \subset \cdots \subset F$ of sub-functors such that $F = \operatorname{colim} F_i$. Since *F* has finite dimensional values, there is *m* such that, for any $w = (w_1, \ldots, w_r)$ with $w = w^n$, $F_m(w) = F(w)$. This together with condition (2) in (c) implies that, for any $v, F_m(v) = F(v)$. The functor *F* is therefore compact proving the implication (c) \Rightarrow (a).

As already mentioned in Section 3, the category $\operatorname{Fun}(\mathbf{N}^r, \operatorname{Vect}_K)$ is equivalent to the category of *r*-graded modules over the polynomial ring in *r*-variables with the standard *r*-grading. In this context (see [14]) Proposition 14.12 can be rephrased as:

14.13. **Proposition.** Let F be an \mathbf{N}^r -graded module over the polynomial ring $K[x_1, \ldots, x_r]$. The following are equivalent:

- (a) F is Notherian;
- (b) F is finitely generated;
- (c) F is positively (n, \ldots, n) -determined for some $n \in \mathbf{N}$.

While the equivalence between (a) and (b) depends on the fact that $K[x_1, \ldots, x_r]$ is a Notherian ring, the equivalence of (c) with (b) is proved in Proposition 2.5 of [14].

A direct consequence of 14.12 is that all the quotients and all the subfunctors of a compact functor in $\operatorname{Fun}(\mathbf{N}^r, \operatorname{Vect}_K)$ are compact.

We finish this section with recalling (see [19]) the classification of compact objects in $Fun(\mathbf{N}, Vect_K)$:

14.14. Compact objects in Fun(N, Vect_K). Let $w \leq u$ be in N. There is a unique inclusion $K(u, -) \subset K(w, -)$. The cokernel of this inclusion is denoted by [w, u) and called the bar starting in w and ending in u. Note that such functors are compact of rank 1 whose 0-Betti diagram is given by:

$$\beta_0[u,w)(v) = \begin{cases} 1 & \text{if } v = u \\ 0 & \text{if } v \neq u \end{cases}$$

14.15. **Proposition.** Any compact object in $\text{Fun}(\mathbf{N}, \text{Vect}_K)$ is isomorphic to a finite direct sum of functors of the form [w, u) and K(v, -). Moreover the isomorphism types of these summands are uniquely determined by the isomorphism type of the functor.

Proof. Assume not all the compact functors can be expressed as a direct sum of bars and free functors and let G be a such a functor with minimal rank. Since Gis compact, there is l in N, such that $G(l \leq v)$ is an isomorphism for any $v \geq l$. Let $w := \min\{v \in \mathbf{N} \mid G(v) \neq 0\}$. Such w exists since G can not be the zero functor. Note that $w \leq l$. Choose an element $x \neq 0$ in G(w). Consider the set $\{v \in \mathbf{N} \mid v > w \text{ and } G(w < v)(x) \neq 0\}$. If this set does not have a maximum, define $G(l) \to K$ to be any map that maps the element $G(w \leq l)(x)$ to 1. This linear map can be extended uniquely to a surjective map $\phi: G \to K(w, -)$. Since K(w, -) is projective, G is a direct sum of K(w, -) and ker (ϕ) . As ker (ϕ) has a smaller rank than G, it can be expressed as a direct sum of bars and free functors. It would then follow G itself is such a direct sum, contradicting the assumption. We can then define $u = \max\{v \in \mathbf{N} \mid v \ge w \text{ and } G(w \le v)(x) \ne 0\}$ and set $G(u) \to K$ to be any map that maps the element $G(w \leq u)(x)$ to 1. This linear map can be extended uniquely to a surjective map $\phi: G \to [w, u)$. This map has a section given by the inclusion $[w, u) \subset G$ which maps 1 in [w, u)(w) = K to x. The functor G can be then expressed as a direct sum $[w, u) \oplus \ker(\phi)$. That leads to a contradiction by the same argument as before. That means that such a G does not exists and all compact functors can be expressed as direct sums of bars and free functors.

For the uniqueness, note that if G is isomorphic to $\bigoplus [w_i, u_i) \oplus \bigoplus K(v_j, -)$, then $\beta_0 G$ determines the starting points w_i 's and v_j 's and hence these numbers are uniquely determined by G. Let us choose a minimal free cover $F \to G$. The ends u_i 's are determined by $\beta_0 \operatorname{ker}(F \to G)$ and hence again they depend only on the isomorphism type of G.

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