Noname manuscript No.
(will be inserted by the editor)

Low rank perturbation of regular matrix pencils with symmetry structures

Fernando De Terán · Christian Mehl · Volker Mehrmann

February 4, 2019

Abstract The generic change of the Weierstraß Canonical Form of regular complex structured matrix pencils under generic structure-preserving additive low-rank perturbations is studied. Several different symmetry structures are considered and it is shown that for most of the structures, the generic change in the eigenvalues is analogous to the case of generic perturbations that ignore the structure. However, for some odd/even and palindromic structures, there is a different behavior for the eigenvalues 0 and ∞, respectively $+1$ and -1. The differences arise in those cases where the parity of the partial multiplicities in the perturbed pencil provided by the generic behavior in the general structure-ignoring case is not in accordance with the restrictions imposed by the structure. The new results extend results for the rank-1 and rank-2 cases that were obtained in [3, 5] for the case of special structure-preserving perturbations. As the main tool, we use decompositions of matrix pencils with symmetry structure into sums of rank-one pencils, as those allow a parametrization of the set of matrix pencils with a given symmetry structure and a given rank.

Key Words: Even matrix pencil, palindromic matrix pencil, Hermitian matrix pencil, symmetric matrix pencil, skew-symmetric matrix pencil, perturbation analysis, generic perturbation, low-rank perturbation, additive decomposition of structured pencils, Weierstraß canonical form.

Mathematics Subject Classification: 15A22, 15A18, 15A21, 15B57.

1 Introduction

The generic change in the Jordan structure of matrices under low-rank perturbations has been established in [21] and was rediscovered later independently in [36, 38, 39]: if a matrix $A \in \mathbb{C}^{n\times n}$ has an eigenvalue λ_0 with partial multiplicities $n_1 \geqslant \cdots \geqslant n_g$ (i.e., these are the sizes of the Jordan blocks associated with λ_0 in the Jordan canonical form of A), then a generic perturbation of rank $r < g$ has the effect that the perturbed matrix still has the eigenvalue λ_0 with partial multiplicities $n_{r+1} \geqslant \cdots \geqslant n_g$, while λ_0 is no longer an eigenvalue of the perturbed matrix if a generic perturbation of rank $r \geqslant g$ is applied.

Starting with [28] a series of papers has studied the generic changes in the Jordan structure of matrices with symmetry structures under structure-preserving low-rank perturbations and it has

Fernando De Terán
Universidad Carlos III de Madrid, Departamento de Matemáticas, Leganés, Madrid, Spain, E-mail: fteran@math.uc3m.es

Christian Mehl · Volker Mehrmann
Technische Universität Berlin, Institut für Mathematik, Sekretariat MA 4-5, 10623 Berlin, E-mail: {mehl,mehrmann}@math.tu-berlin.de

been observed that sometimes the behavior differs from the one under arbitrary low-rank perturbations due to restrictions in the possible Jordan structures of the matrices with symmetry structures, see [6, 15, 22, 28–33].

There are many applications where low-rank perturbations of matrix pencils with or without symmetry structures arise. For example, matrix pencils are the coefficient representations of linear differential-algebraic equations, see e.g. [7,23] and the references therein. Structured low-rank perturbations are then common when power networks or electrical circuits are considered, and the stability is studied when interconnections are interrupted [1, 13, 19, 37]. These are typically perturbations of rank one or two. Another class of problems where the perturbations are of low-rank compared to the system size, but not low-rank in absolute terms, are switched systems which change their states, see e.g. [18, 20, 24, 25, 35]. We will study low-rank perturbations of structured matrix pencils from an abstract matrix-theoretical point of view and do not consider the many concrete applications where this topic has major implications.

A result on the generic change of the Weierstraß structure (namely, the partial multiplicities) under low-rank perturbations of regular pencils without any additional symmetry structures has been established as early as in [12], where *genericity* was understood in the following sense: a subset of a finite-dimensional linear space of perturbations is called generic if it is an open dense subset with respect to the natural topology on the linear space. In contrast to this notion, a stronger concept of genericity had been used in the references starting from [28]: in that sense, a subset $\mathcal{G}$ of $\mathbb{C}^m$ is generic if its complement $\mathbb{C}^m \setminus \mathcal{G}$ is contained in a proper algebraic set, i.e., a set of common zeros of finitely many polynomials in m variables that does not coincide with the full set $\mathbb{C}^m$. The latter concept is not only stronger than the previous one (clearly any generic set in the latter sense is an open dense subset of $\mathbb{C}^m$ while the converse is not true in general), but it also allowed an easy transition from the complex to the real case as it was shown in [30]. This concept requires the parametrization of the set of considered perturbations as a subset of $\mathbb{C}^m$. In [11] such a parametrization of the set of pencils of rank at most r was introduced and the result from [12] could be generalized to the stronger concept of genericity in the sense of its complement being contained in a proper algebraic set. The main result obtained in [11] states that the generic behavior in the case of matrix pencils coincides with the one for matrices. More precisely, if $A + \lambda B$ is a regular pencil and $\lambda_0 \in \mathbb{C} \cup \{\infty\}$ is an eigenvalue of $A + \lambda B$ with partial multiplicities $n_1 \geqslant \cdots \geqslant n_g$, then a generic additive perturbation of $A + \lambda B$ with rank r "destroys" the r largest multiplicities, so that the perturbed pencil has the partial multiplicities $n_{r+1} \geqslant \cdots \geqslant n_g$ at λ_0.

Surprisingly, the case of matrix pencils with some additional symmetry structure has not yet been as well studied as the matrix case. The first attempt to investigate the generic change in the Weierstraß structure of such matrix pencils under structure-preserving low-rank perturbations was undertaken in [3–5], where the cases of rank-1 perturbations and special perturbations of rank two were considered - the restriction to these cases was due to the fact that straightforward parameterizations were available in that case. While it was shown in [6] how the knowledge of the behavior in the rank-one case can be extended to arbitrary rank in the matrix case, a similar transition is not possible in the pencil case, since a structured pencil of small rank can in general not be written as a sum of those rank-1 or rank-2 pencils that were considered in [3–5]. Therefore, the case of structure-preserving perturbations of rank larger than two remained an open problem.

It is our aim to fill this gap by extending the ideas from [11] to develop parameterizations of low-rank pencils with symmetry structures and obtain results on the generic change in the Weierstraß structure of structured matrix pencils under low-rank structure-preserving perturbations. Moreover, we will also consider one aspect that has not been considered in the pencil case so far: the generic multiplicity of newly generated eigenvalues.

Low-rank perturbation of singular matrix pencils has been considered in [9], restricted to the case where the perturbed pencil remains singular. A different generic behavior on the change of the partial multiplicities of eigenvalues is shown in this case. In particular, for generic perturbations, all partial multiplicities of any eigenvalue of the unperturbed pencil stay after perturbation. In this paper, however, we restrict ourselves to regular matrix pencils which remain regular after perturbation (which is a generic condition). Nonetheless, singular pencils naturally appear in the context of the present work, since low-rank pencils are necessarily singular.

The paper is organized as follows. In Section 2 we introduce some notation and recall the Weierstraß canonical form. The symmetry structures considered in the paper are introduced in Section 3, where we also present the rank-1 decomposition of low-rank structured pencils for any of these structures. We consider the Hermitian and $\top$-even cases in full detail, and from the results for these two structures we derive the results for the remaining symmetry structures. Section 4 contains the main results of the paper, namely the description of the generic change of the partial multiplicities of regular pencils with symmetry structures under low-rank structure-preserving perturbations. If we restrict ourselves to pencils with real entries, the approach followed in the manuscript is no longer valid. In the short Section 5 we briefly discuss the case of real matrix pencils with symmetry structures and explain why the results of the previous sections cannot be applied in that case. In Section 6 we summarize the contributions of the paper and we present some lines of further research. Appendix A contains the proof of a couple of technical results used in Section 4.

2 Notation and basic results

By e_i we denote the ith canonical vector of appropriate size, i.e., the ith column of the identity matrix with the appropriate order. By $\mathfrak{i}$ we denote the imaginary unit. The notation $0_{m\times n}$ stands for the $m \times n$ zero matrix. When either $m = 1$ or $n = 1$, then we just write 0_n or 0_m, respectively. Note that we use the same notation for zero rows and zero columns, but which is the right one is clear by the context.

As usual, $\mathbb{C}^{m\times n}$ denotes the set of $m \times n$ matrices with complex entries, and $\mathbb{C}^n$ denotes the set of vectors with n complex coordinates in column form (i.e., $\mathbb{C}^n = \mathbb{C}^{n\times 1}$). Given a matrix $A \in \mathbb{C}^{m\times n}$, we denote by $A(i, j)$ the (i, j) entry of A. By $\mathbb{C}[\lambda]^n$ we denote the set of vector polynomials with n coordinates, i.e., the set of vectors with n coordinates which are polynomials in the variable λ.

We use $L(\lambda)$ for general pencils, as well as for the given (unperturbed) pencil, whereas $E(\lambda)$ will be used for the perturbation pencil. The notation $\star$ is used for either the transpose ($\top$) or the conjugate transpose ($*$) of a matrix. Given a matrix pencil $L(\lambda) = A + \lambda B$ (or just L, for short), by $L(\lambda)^\star$ (or $L^\star$, for short) we denote the pencil $A^\star + \lambda B^\star$. It is important to note that, when $\star = *$, then the operator $*$ does not affect the variable λ, but just the coefficients of the pencil. The pencil is said to be *regular* if it is square and $\det L(\lambda)$ is not identically zero. Otherwise, it is said to be *singular*. The *rank* of $L(\lambda)$, denoted $\operatorname{rank} L$, is the size of the largest non-identically zero minor of $L(\lambda)$ (considering the minors as polynomials in λ), i.e., the rank of $L(\lambda)$ considered as a matrix over the field of rational functions in λ. In other words, it is the quantity $\max_{\lambda\in\mathbb{C}} \operatorname{rank}(A + \lambda B)$. This is sometimes referred to as the *normal rank* in the literature (see, for instance, [14]). Note that, if $A + \lambda B$ is a square $n \times n$ matrix pencil with rank $r < n$, then $A + \lambda B$ is singular.

The *reversal* $\operatorname{rev}(A + \lambda B)$ of a matrix pencil $A + \lambda B$ is the matrix pencil $B + \lambda A$.

By L_α we denote a *right singular block of order* α, i.e., the $\alpha \times (\alpha + 1)$ pencil

$$L_\alpha := \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \end{bmatrix}_{\alpha\times(\alpha+1)} .$$

By $J_k(a - \lambda)$ we denote a pencil corresponding to a $k \times k$ Jordan block associated with the eigenvalue a, namely

$$J_k(a-\lambda) := \begin{bmatrix} a-\lambda & 1 & & \\ & \ddots & \ddots & \\ & & a-\lambda & 1 \\ & & & a-\lambda \end{bmatrix}_{k\times k} ,$$

and R denotes the *reverse identity* matrix, namely

$$R := \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} ,$$

where the size will be clear by the context.

Remark 1 If $w \in \mathbb{C}[\lambda]^n$ is a vector polynomial of degree (at most) 1, and $v \in \mathbb{C}^n$ (i.e., a constant vector) then $\operatorname{rev}(vw^\star) = v \cdot (\operatorname{rev} w)^\star$.

If $A + \lambda B$ is a regular $n \times n$ matrix pencil, then it can be transformed to Weierstraß canonical form (WCF). More precisely, there exist nonsingular matrices $S, T \in \mathbb{C}^{n\times n}$ such that

$$S(A+\lambda B)T = \operatorname{diag}\big(\mathcal{J}_{n_{1,1}}(a_1-\lambda), \dots, \mathcal{J}_{n_{1,g_1}}(a_1-\lambda), \dots, \mathcal{J}_{n_{\kappa,1}}(a_\kappa-\lambda), \dots, \mathcal{J}_{n_{\kappa,g_\kappa}}(a_\kappa-\lambda), \\ \operatorname{rev} J_{n_{\kappa+1,1}}(-\lambda), \dots, \operatorname{rev} J_{n_{\kappa+1,g_{\kappa+1}}}(-\lambda)\big).$$

Here $\kappa \in \mathbb{N}$, and $a_1, \dots, a_\kappa \in \mathbb{C}$ are the finite eigenvalues of $A + \lambda B$ with geometric multiplicities $g_1, \dots, g_\kappa$, respectively. The value $g_{\kappa+1}$ is the geometric multiplicity of the infinite eigenvalue, where we allow $g_{\kappa+1} = 0$ for the case that ∞ is not an eigenvalue of the pencil. The parameters $n_{i,1}, \dots, n_{i,g_i}$ are called the *partial multiplicities* of $A + \lambda B$ at λ_i. Without loss of generality, we may assume that they are ordered non-increasingly, i.e., we have $n_{i,1} \geqslant \cdots \geqslant n_{i,g_i}$.

If $A+\lambda B$ is a singular $m\times n$ matrix pencil, then the corresponding canonical form is the Kronecker canonical form (KCF): there exist nonsingular matrices $S \in \mathbb{C}^{m\times m}$ and $T \in \mathbb{C}^{n\times n}$ such that

$$S(A+\lambda B)T = \operatorname{diag}\big(\widetilde{L}(\lambda), L_{\alpha_1}, \dots, L_{\alpha_\eta}, L_{\beta_1}^\top, \dots, L_{\beta_\xi}^\top\big)$$

with $\widetilde{L}(\lambda)$ in WCF. Here, the parameters $\alpha_1, \dots, \alpha_\eta \in \mathbb{N}$ and $\beta_1, \dots, \beta_\xi \in \mathbb{N}$ are called the *right* or *left minimal indices*, respectively.

3 Representation of structured pencils as a sum of rank-1 pencils

It is well-known, see e. g. [16], that any Hermitian or symmetric matrix $A \in \mathbb{C}^{n\times n}$ with $\operatorname{rank} A = r \leqslant n$ can be written as a sum of rank-1 matrices of the same structure (this is an immediate consequence of the so-called *spectral decomposition*). In particular, if A is symmetric, then it can be written as $A = u_1u_1^\top + \cdots + u_ru_r^\top$ (or $A = s_1u_1u_1^\top + \cdots + s_ru_ru_r^\top$ if we restrict ourselves to real coefficients), whereas if A is Hermitian, then it can be written as $A = s_1u_1u_1^* + \cdots + s_ru_ru_r^*$ where $s_1, \dots, s_r \in \{+1, -1\}$ are *signs*. By *Sylvester's Law of Inertia*, the numbers of positive (resp. negative) signs among $s_1, \dots, s_r$ are uniquely determined.

It is natural to ask whether an analogous decomposition holds for matrix pencils with symmetry structures. The structures we are interested in are compiled in the following list. A matrix pencil $L(\lambda) = A + \lambda B$ with $A, B \in \mathbb{C}^{n\times n}$ is said to be

- *Hermitian* if $A = A^*, B = B^*$;
- *symmetric* if $A = A^\top, B = B^\top$;
- *skew-Hermitian* if $A^* = -A, B^* = -B$;
- *skew-symmetric* if $A^\top = -A, B^\top = -B$;
- *$\star$-even* if $A^\star = A, B^\star = -B$;
- *$\star$-odd* if $A^\star = -A, B^\star = B$;
- *$\star$-palindromic* if $A^\star = B$;
- *$\star$-anti-palindromic* if $A^\star = -B$.

The name *$\star$-alternating* is also used as an umbrella term for both $\star$-even and $\star$-odd.

For the sake of brevity, we will use the following notation for the set of $n \times n$ structured matrix pencils with rank at most r, for each of the previous structures:

structure	notation
Hermitian	$\mathbb{H}_r$
symmetric	Sym_r
skew-Hermitian	$S\mathbb{H}_r$
skew-symmetric	$SSym_r$
$\star$-even	$Even_r^\star$
$\star$-odd	$Odd_r^\star$
$\star$-palindromic	$Pal_r^\star$
$\star$-anti-palindromic	$Apal_r^\star$

Note that, for the ease of notation, and since all matrices considered in this paper are of the same size $n \times n$, there is no explicit mention of the size in the notation introduced above.

We start by showing the existence of a decomposition of structured low-rank pencils as a sum of structured rank-1 pencils. For this, we will use structured canonical forms for these kinds of pencils. These canonical forms comprise the information displayed in the WCF, with the appropriate restrictions imposed by the corresponding symmetry structure. We refer to [4] for these canonical forms, since they are all gathered in this reference, even though all of them were introduced in earlier references. Furthermore, we focus on the case of Hermitian pencils and will give a detailed proof for this case only, while for the cases of other structures we will either reduce them to the Hermitian case or mention in which parts the proofs of the corresponding results differ from the Hermitian case.

3.1 Rank-1 decompositions for the Hermitian case

First, we recall the well-known canonical form for Hermitian pencils under congruence, see, e.g., [4, Theorem 2.20].

Theorem 1 (Canonical form of Hermitian pencils). *Let $E(\lambda)$ be a Hermitian $n \times n$ matrix pencil. Then there exists a nonsingular matrix P such that*

$$P^* E(\lambda) P = \operatorname{diag}\big(E_1(\lambda), \ldots, E_m(\lambda)\big),$$

where each pencil $E_j(\lambda)$, for $j = 1, \ldots, m$, has exactly one of the following four forms:

i) *blocks $\sigma R J_k(a - \lambda)$ associated with a real eigenvalue $a \in \mathbb{R}$ and a sign $\sigma \in \{+1, -1\}$;*

ii) *blocks*

$$\operatorname{rev}\big(\sigma R J_k(-\lambda)\big) = \sigma \begin{bmatrix} & & & -1 \\ & & -1 & \lambda \\ & \iddots & \iddots & \\ -1 & \lambda & & \end{bmatrix}$$

associated with the eigenvalue infinity and a sign $\sigma \in \{+1, -1\}$;

iii) *blocks $R \operatorname{diag}\big(J_k(\overline{\mu} - \lambda), J_k(\mu - \lambda)\big)$ associated with a pair $(\mu, \overline{\mu})$ of conjugate complex eigenvalues, with $\mu \in \mathbb{C}$ having positive imaginary part;*

iv) *blocks*

$$\begin{bmatrix} 0 & L_k^\top \\ L_k & 0 \end{bmatrix}$$

consisting of a pair of one right and one left singular block with the same index k.

The parameters a, k, σ, and μ depend on the particular block $L_j(\lambda)$ and may be distinct in different blocks. Furthermore, the canonical form is unique up to permutation of blocks.

The signs σ in the blocks of type i) and ii) in Theorem 1 are invariant under congruence transformations and their collection is referred to as the *sign characteristic* of the Hermitian pencil following the terminology of [17, 34]. The following result presents a decomposition of a given Hermitian pencil as a sum of rank-1 Hermitian pencils, which extends the one for Hermitian matrices mentioned at the beginning of this section. Hereafter, we deal with polynomial vectors, namely vectors $v(\lambda) \in \mathbb{C}[\lambda]^n$, though, for brevity, in general we will drop the dependence on λ. For a given $v(\lambda) \in \mathbb{C}[\lambda]^n$, by $\deg v$ we denote the largest degree of the entries of v. In order to avoid confusion, it is important to recall that, given a pencil $A + \lambda B$, we write $(A + \lambda B)^*$ to denote the pencil $A^* + \lambda B^*$, i.e., we only apply the conjugate transpose to the coefficients of the pencil, and not to the variable λ.

Theorem 2 (Rank-1 decomposition for Hermitian pencils). *If $E(\lambda)$ is a Hermitian $n \times n$ matrix pencil with* $\operatorname{rank} E = r \leqslant n$, *then it can be written as*

$$E(\lambda) = (a_1 + \lambda b_1) u_1 u_1^* + \cdots + (a_\ell + \lambda b_\ell) u_\ell u_\ell^* + v_1 w_1^* + \cdots + v_s w_s^* + w_1 v_1^* + \cdots + w_s v_s^*, \tag{1}$$

where $a_i, b_i \in \mathbb{R}$, for $i = 1, \ldots, \ell$, and

(i) $\ell + 2s = r$,
(ii) $\deg u_1 = \cdots = \deg u_\ell = 0 = \deg v_1 = \cdots = \deg v_s$ *and* $\deg w_1, \dots, \deg w_s \leqslant 1$.

Proof It suffices to prove the statement for $E(\lambda)$ being in Hermitian canonical form as in Theorem 1. To see this, just notice that if $K_E(\lambda)$ is the Hermitian canonical form of $E(\lambda)$ and if it has a decomposition

$$K_E(\lambda) = (a_1 + \lambda b_1)\widetilde{u}_1\widetilde{u}_1^* + \cdots + (a_\ell + \lambda b_\ell)\widetilde{u}_\ell\widetilde{u}_\ell^* + \widetilde{v}_1\widetilde{w}_1^* + \cdots + \widetilde{v}_s\widetilde{w}_s^* + \widetilde{w}_1\widetilde{v}_1^* + \cdots + \widetilde{w}_s\widetilde{v}_s^*,$$

as in (1), then there exists a nonsingular matrix P such that

$$\begin{aligned} E(\lambda) = PK_E(\lambda)P^* = {} & (a_1 + \lambda b_1)u_1u_1^* + \cdots + (a_\ell + \lambda b_\ell)u_\ell u_\ell^* \\ & + v_1w_1^* + \cdots + v_sw_s^* + w_1v_1^* + \cdots + w_sv_s^* \end{aligned}$$

with $u_i = P\widetilde{u}_i, v_j = P\widetilde{v}_j$, and $w_j = P\widetilde{w}_j$, for $i = 1, \dots, \ell$ and $j = 1, \dots, s$. This gives the desired decomposition (1) for $L(\lambda)$.

So we may assume $E(\lambda)$ to be in Hermitian canonical form, which is a direct sum of blocks of the four different types i)–iv) as in Theorem 1. We will provide a decomposition like (1) for each of these blocks.

1) A $k \times k$ block associated with a real eigenvalue $a \in \mathbb{R}$ and sign $\sigma \in \{+1, -1\}$ can be decomposed as follows, depending on whether k is odd or even. If k is even then

$$\begin{aligned}
&\sigma R J_k(a-\lambda) \\
&= \sigma \begin{bmatrix} & & & & & a-\lambda \\ & & & & 0 & 1/2 \\ & & & a-\lambda & 0 & \\ & & 0 & 1/2 & & \\ & \iddots & \iddots & & & \\ a-\lambda & 0 & & & & \\ 0 \;\; 1/2 & & & & & \end{bmatrix} + \sigma \begin{bmatrix} & & & & & 0 \\ & & & & a-\lambda & 1/2 \\ & & & 0 & 0 & \\ & & a-\lambda & 1/2 & & \\ & \iddots & \iddots & & & \\ 0 & 0 & & & & \\ a-\lambda \;\; 1/2 & & & & & \end{bmatrix} \\
&= \sigma \left(\begin{bmatrix} a-\lambda \\ 1/2 \\ 0_{k-2} \end{bmatrix} e_k^* + \begin{bmatrix} 0_2 \\ a-\lambda \\ 1/2 \\ 0_{k-4} \end{bmatrix} e_{k-2}^* + \cdots + \begin{bmatrix} 0_{k-2} \\ a-\lambda \\ 1/2 \end{bmatrix} e_2^* \right) \\
&\quad + \sigma \left(e_k \begin{bmatrix} a-\lambda \\ 1/2 \\ 0_{k-2} \end{bmatrix}^* + e_{k-2} \begin{bmatrix} 0_2 \\ a-\lambda \\ 1/2 \\ 0_{k-4} \end{bmatrix}^* + \cdots + e_2 \begin{bmatrix} 0_{k-2} \\ a-\lambda \\ 1/2 \end{bmatrix}^* \right),
\end{aligned}$$

which is of the form (1) with $v_i = \sigma e_{2i}$ and $w_i = \begin{bmatrix} 0_{k-2i} \; a-\lambda \; 1/2 \; 0_{2i-2} \end{bmatrix}^*$, for $i = 1, \dots, k/2$. Note that σ can be included either in v_i or w_i, for $i = 1, \dots, k/2$.

If k is odd, then we can split the block in two pieces

$$\begin{aligned}
&\sigma RJ_k(a-\lambda)\\
&= \sigma(a-\lambda)e_{\frac{k+1}{2}}e^*_{\frac{k+1}{2}} + \sigma \left[\begin{array}{ccc|ccc} &&&&& a-\lambda\\ &&&&\iddots&1\\ &&&a-\lambda&\iddots&\\ &&0&1&&\\ \hline &a-\lambda&1&&&\\ \iddots&\iddots&&&&\\ a-\lambda&1&&&&\end{array}\right]\\
&= \sigma(a-\lambda)e_{\frac{k+1}{2}}e^*_{\frac{k+1}{2}} + \sigma\left(\begin{bmatrix} a-\lambda\\ 1\\ 0_{k-2}\end{bmatrix} e_k^* + \begin{bmatrix} 0\\ a-\lambda\\ 1\\ 0_{k-3}\end{bmatrix} e_{k-1}^* + \cdots + \begin{bmatrix} 0_{\frac{k-3}{2}}\\ a-\lambda\\ 1\\ 0_{\frac{k-1}{2}}\end{bmatrix} e^*_{\frac{k-1}{2}}\right)\\
&\qquad + \sigma\left(e_k \begin{bmatrix} a-\lambda\\ 1\\ 0_{k-2}\end{bmatrix}^* + e_{k-1}\begin{bmatrix} 0\\ a-\lambda\\ 1\\ 0_{k-3}\end{bmatrix}^* + \cdots + e_{\frac{k-1}{2}}\begin{bmatrix} 0_{\frac{k-3}{2}}\\ a-\lambda\\ 1\\ 0_{\frac{k-1}{2}}\end{bmatrix}^*\right),
\end{aligned}$$

and proceed as in the previous case with the last two summands.

2) A $k\times k$ block associated with ∞ and sign characteristic σ can be decomposed in a similar way, replacing the roles of $a-\lambda$ and 1 in the previous case by -1 and λ, respectively.

3) A pair of $k\times k$ blocks corresponding to a pair of complex conjugate eigenvalues $\mu, \overline{\mu}$ can be decomposed as

$$\begin{aligned}
R\,\mathrm{diag}(J_k(\overline{\mu}-\lambda), J_k(\mu-\lambda)) &= \left[\begin{array}{ccc|ccc} &&&&& \mu-\lambda\\ &&&&\iddots&1\\ &&&\mu-\lambda&\iddots&\\ &&&\mu-\lambda&1&\\ \hline &&\overline{\mu}-\lambda&&&\\ &\iddots&1&&&\\ \overline{\mu}-\lambda&\iddots&&&&\\ \overline{\mu}-\lambda&1&&&&\end{array}\right]\\
&= \begin{bmatrix} \mu-\lambda\\ 1\\ 0_{2k-2}\end{bmatrix} e^*_{2k} + \cdots + \begin{bmatrix} 0_{k-2}\\ \mu-\lambda\\ 1\\ 0_k\end{bmatrix} e^*_{k+2} + \begin{bmatrix} 0_{k-1}\\ \mu-\lambda\\ 0_k\end{bmatrix} e^*_{k+1}\\
&\quad + e_{k+1}\begin{bmatrix} 0_{k-1}\\ \mu-\lambda\\ 0_k\end{bmatrix}^* + e_{k+1}\begin{bmatrix} 0_{k-2}\\ \mu-\lambda\\ 1\\ 0_k\end{bmatrix}^* + \cdots + e_{2k}\begin{bmatrix} \mu-\lambda\\ 1\\ 0_{2k-2}\end{bmatrix}^*,
\end{aligned}$$

which is of the desired form.

4) Finally, a pair consisting of a left and a right singular block with respective sizes $k\times(k+1)$ and $(k+1)\times k$ can be decomposed as

$$\begin{bmatrix} 0 & L_k^\top\\ L_k & 0\end{bmatrix} = e_{k+1}\begin{bmatrix} \lambda\\ 1\\ 0_{2k-1}\end{bmatrix}^* + \cdots + e_{2k+1}\begin{bmatrix} 0_{k-1}\\ \lambda\\ 1\\ 0_k\end{bmatrix}^* + \begin{bmatrix} \lambda\\ 1\\ 0_{2k-1}\end{bmatrix} e^*_{k+1} + \cdots + \begin{bmatrix} 0_{k-1}\\ \lambda\\ 1\\ 0_k\end{bmatrix} e^*_{2k+1},$$

which is, again, in the desired form.

So each block in the canonical form has a decomposition like (1). Forming this direct sum by padding up with zeroes in the entries of each vector corresponding to the other blocks, we arrive at a decomposition (1) for $E(\lambda)$ given in Hermitian canonical form.

Remark 2 Note that $u_1,\ldots,u_\ell$ and $v_1,\ldots,v_s$ are constant vectors, but $w_1,\ldots,w_s$ are (column) pencils, which means that their entries are polynomials in λ with degree at most 1. Thus writing $w_i(\lambda) = w_{i,A} + \lambda w_{i,B}$ for $i = 0,\ldots,s$ with $w_{1,A},\ldots,w_{s;A},w_{1,B},\ldots,w_{s,B} \in \mathbb{C}^n$ and using the notation

$$\begin{aligned} U &:= \begin{bmatrix} u_1 & \ldots & u_\ell \end{bmatrix}, & V &:= \begin{bmatrix} v_1 & \ldots & v_s \end{bmatrix}, \\ W_A &:= \begin{bmatrix} w_{1,A} & \ldots & w_{s,A} \end{bmatrix}, & W_B &:= \begin{bmatrix} w_{1,B} & \ldots & w_{s,B} \end{bmatrix}, \\ D_A &:= \operatorname{diag}(a_1,\ldots,a_\ell), & D_B &:= \operatorname{diag}(b_1,\ldots,b_\ell), \end{aligned}$$

we can write (1) in the concise form

$$E(\lambda) = U(D_A + \lambda D_B)U^* + V(W_A^* + \lambda W_B^*) + (W_A + \lambda W_B)V^*. \tag{2}$$

Remark 3 By the construction in the proof of Theorem 2, the terms of the form $(a+\lambda b)uu^*$ in the decomposition (1) come either from blocks associated with real eigenvalues or from blocks associated with the infinite eigenvalue, and in both cases the blocks have odd size.

Remark 4 If (1) is a decomposition into rank-1 pencils as in Theorem 2, then the vectors $u_1,\ldots,u_\ell$, $v_1,\ldots,v_s$ are linearly independent. To see this, assume that they are linearly dependent. Let $X := \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix} \in \mathbb{C}^{n\times n}$ be nonsingular such that the columns of $\begin{bmatrix} X_1 & X_2 \end{bmatrix} \in \mathbb{C}^{n\times(p+q)}$ span the orthogonal complement of the span of $v_1,\ldots,v_s$ and the columns of $X_1 \in \mathbb{C}^{n\times p}$ span the orthogonal complement of the span of $u_1,\ldots,u_\ell,v_1,\ldots,v_s$. Then we have $p+q \geqslant n-s$ and, because of the assumed linear dependency, $p > n-(\ell+s)$. Observe that

$$X^*E(\lambda)X = \begin{array}{c} \\ p \\ q \\ n-p-q \end{array} \begin{array}{c} \begin{array}{ccc} p & q & n-p-q \end{array} \\ \begin{bmatrix} 0 & 0 & X_1^*E(\lambda)X_3 \\ 0 & X_2^*E(\lambda)X_2 & X_2^*E(\lambda)X_3 \\ X_3^*E(\lambda)X_1 & X_3^*E(\lambda)X_f & X_3^*E(\lambda)X_3 \end{bmatrix} \end{array}$$

from which we obtain that the rank of $E(\lambda)$ is bounded by

$$2(n-p-q)+q = n-p+n-p-q < \ell+s+s = r,$$

which is in contradiction to the assumption in Theorem 2 that $E(\lambda)$ has rank r.

Unfortunately, the decomposition (1) is far from being unique as the following example illustrates.

Example 1 Consider the Hermitian pencil

$$E(\lambda) := \lambda \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \lambda-1 \\ \lambda-1 & 0 \end{bmatrix}$$

and let $u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$, $u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \end{bmatrix}^\top$, $v_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^\top$, and $w_1 = \begin{bmatrix} 0 & \lambda-1 \end{bmatrix}^\top$. Then we have

$$E(\lambda) = (\lambda-1)u_1u_1^* - (\lambda-1)u_2u_2^* = v_1w_1^* + w_1v_1^*.$$

In particular, Example 1 shows that also the parameters ℓ and s from Theorem 2 are not unique, as in the first decomposition we have $\ell = 2$ and $s = 0$ and in the latter we have $\ell = 0$ and $s = 2$. However, the values of ℓ and s can be fixed by requiring ℓ to be minimal. Interestingly, in that case the minimal parameter ℓ depends on the sign characteristic of the Hermitian pencil. In order to state the following theorem, we recall the definition of the so-called *sign sum* from [27].

Definition 1 Let $E(\lambda)$ be a Hermitian $n \times n$ pencil and let $\mu \in \mathbb{R}$ be an eigenvalue of $E(\lambda)$. Assume that $(n_1, \dots, n_m, n_{m+1}, \dots, n_q)$ are the sizes of the blocks associated with the eigenvalue μ in the Hermitian canonical form of $E(\lambda)$, where $n_1, \dots, n_m$ are odd and $n_{m+1}, \dots, n_q$ are even. Furthermore, let $(\sigma_1, \dots, \sigma_m, \sigma_{m+1}, \dots, \sigma_q)$ be the corresponding signs (of the blocks associated with μ) from the sign characteristic of $E(\lambda)$. Then the *signsum* $\operatorname{sigsum}(E, \mu)$ of μ is defined as

$$\operatorname{sigsum}(E, \mu) := \sum_{j=1}^{m} \sigma_j.$$

If ∞ is an eigenvalue of $E(\lambda)$, then the *signsum* of ∞ is defined as

$$\operatorname{sigsum}(E, \infty) := \operatorname{sigsum}(\operatorname{rev} E, 0).$$

Thus, the signsum of the real eigenvalue μ of a Hermitian pencil is just the sum of the signs that correspond to blocks of odd size associated with μ.

Example 2 Consider the following three Hermitian pencils

$$E_1(\lambda) = \begin{bmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\lambda \\ 0 & 0 & 1-\lambda & 1 \\ 0 & 1-\lambda & 1 & 0 \end{bmatrix},$$

$$E_2(\lambda) = \begin{bmatrix} 1-\lambda & 0 \\ 0 & \lambda-1 \end{bmatrix}, \quad E_3(\lambda) = \begin{bmatrix} 0 & 1-\lambda \\ 1-\lambda & 1 \end{bmatrix},$$

which all have just the single eigenvalue $a = 1$. Then we have $\operatorname{sigsum}(E_1, 1) = 2$, since $E_1(\lambda)$ has two odd-sized blocks associated with $a = 1$ (one of size one and one of size three), both having the sign $+1$. On the other hand $\operatorname{sigsum}(E_2, 1) = 0$ as $E_2(\lambda)$ has two blocks of size one, but with opposite signs $+1$ and -1. For the pencil $E_3(\lambda)$, we also obtain $\operatorname{sigsum}(E_3, 1) = 0$, because it has no odd-sized blocks associated with the eigenvalue $a = 1$, but just one block of size two. In that case, the sum in Definition 1 is empty and thus, by definition, equal to zero.

Theorem 3 *Let $E(\lambda)$ be a Hermitian $n \times n$ pencil and let $\mu_1, \dots, \mu_p \in \mathbb{R} \cup \{\infty\}$ be the pairwise distinct real eigenvalues of $E(\lambda)$. (Infinity is interpreted as a possible real eigenvalue here.) Furthermore, let* (1) *as in Theorem* 2 *be a decomposition of E into rank-*1 *pencils so that the parameter ℓ from Theorem* 2 *is minimal among all possible such decompositions. Then*

$$\ell = \sum_{j=1}^{p} |\operatorname{sigsum}(E, \mu_j)|. \tag{3}$$

Proof In the following, let ℓ_0 denote the right-hand-side of (3), i.e., $\ell_0 = \sum_{j=1}^{p} |\operatorname{sigsum}(E, \mu_j)|$.
"$\leqslant$": We first show that there exists a decomposition as in (1) such that $\ell = \ell_0$. Using the same construction as in the proof of Theorem 2, we see from Remark 3 that in their decomposition into rank-1 pencils only blocks of odd-size that are associated with real eigenvalues (including ∞) have a term of the form $(a + \lambda b)uu^*$ (with $a, b \in \mathbb{R}$ and $u \in \mathbb{C}^n$), and thus only those blocks contribute to the number ℓ in the decomposition (1). Therefore and because it is sufficient to consider each real eigenvalue separately, we may assume, without loss of generality, that $E(\lambda)$ is regular and only has a single eigenvalue μ that is real and finite, such that all blocks in the Hermitian canonical form of $E(\lambda)$ associated with μ have odd size. We then have to show that $E(\lambda)$ has a decomposition as in (1) with $\ell = |\operatorname{sigsum}(E, \mu)|$.

To this end, assume that the Hermitian canonical form of the pencil $E(\lambda)$ consists of m blocks with size $n_1, \dots, n_m$ (which are all odd). Let $\sigma_1, \dots, \sigma_m$ be the signs from the sign characteristic of $E(\lambda)$, where σ_j is associated with n_j for $j = 1, \dots, m$. By the construction in the proof of Theorem 2, we then obtain a decomposition of the form

$$E(\lambda) = \sigma_1(a-\lambda)u_1u_1^* + \cdots + \sigma_m(a-\lambda)u_mu_m^* + v_1w_1^* + \cdots + v_sw_s^* + w_1v_1^* + \cdots + w_sv_s^*. \tag{4}$$

Suppose that $m = m_+ + m_-$, where m_+ is the number of blocks with positive sign σ_j and m_- is the number of blocks with negative sign σ_j. Then $\operatorname{sigsum}(E, a) = |m_+ - m_-|$, i.e., if we try to pair up the blocks into pairs consisting of two blocks with opposite signs (but possibly different sizes) then the signsum of a corresponds to the number of blocks that will remain unpaired. In particular, all of these remaining blocks will have the same sign. Thus, to prove the assertion, it remains to show that in the decomposition (4) each summand

$$(a-\lambda)u_i u_i^* - (a-\lambda)u_j u_j^*$$

(where we have $\sigma_i = 1$ and $\sigma_j = -1$) can be replaced by a summand of the form $v_k w_k^* + w_k v_k^*$ with $v_k \in \mathbb{C}^n$ and w_k being an $n \times 1$ pencil. This goal can be achieved by choosing $v_k = u_i + \mathrm{i}u_j$ and $w_k = \frac{1}{2}(a-\lambda)(u_i - \mathrm{i}u_j)$.

"$\geqslant$": It remains to show that ℓ cannot be chosen smaller than ℓ_0. Thus, let (1) be a decomposition of $E(\lambda)$ into rank-1-pencils with some $\ell < \ell_0$. By Remark 4, the columns of the matrix $\begin{bmatrix} U & V \end{bmatrix}$ with $U = \begin{bmatrix} u_1 & \dots & u_\ell \end{bmatrix}$ and $V = \begin{bmatrix} v_1 & \dots & v_s \end{bmatrix}$ are linearly independent. Thus, let $X \in \mathbb{C}^{n\times(n-s-\ell)}$ be such that $\begin{bmatrix} X & U & V \end{bmatrix}$ is invertible and set $P := \begin{bmatrix} X & U & V \end{bmatrix}^{-1}$. Then we obtain

$$PE(\lambda)P^* = \begin{array}{c} \\ {\scriptstyle n-s-\ell} \\ {\scriptstyle \ell} \\ {\scriptstyle s} \end{array} \!\!\begin{array}{c} \begin{array}{ccc} {\scriptstyle n-s-\ell} & {\scriptstyle \ell} & {\scriptstyle s} \end{array} \\ \begin{bmatrix} 0 & 0 & S_A^* + \lambda S_B^* \\ 0 & D_A + \lambda D_B & * \\ S_A + \lambda S_B & * & * \end{bmatrix} \end{array},$$

where $S_A + \lambda S_B$ are the first $n-s-\ell$ columns of $(W_A^* + \lambda W_B^*)P^*$, and where D_A, D_B, W_A, W_B are as in Remark 2. In particular, all eigenvalues of $D_A + \lambda D_B$ are real and semisimple, because the pencil $D_A + \lambda D_B$ is diagonal. Furthermore, we can assume that if $D_A + \lambda D_B$ has a multiple eigenvalue, say μ, then all signs in the sign characteristic of $D_A + \lambda D_B$ associated with μ are equal. Otherwise, we may use the trick from the part "$\leqslant$" to get a decomposition of the form (1) with an even smaller ℓ.

Note that $S_A + \lambda S_B$ must be of full normal rank s, because otherwise the pencil $E(\lambda)$ would have less than $r = s + \ell + s$ linearly independent columns. Thus, in particular $S_A - \eta S_B$ has rank s for all values $\eta \in \mathbb{C}$ that are not eigenvalues of $E(\lambda)$. This implies that the only eigenvalues of $E(\lambda)$ are the eigenvalues of $D_A + \lambda D_B$. Moreover, if we denote the eigenvalues of $E(\lambda)$ by $\mu_1, \dots, \mu_d$, with respective algebraic multiplicities $m_1, \dots, m_d$, then we have $\ell = \sum_{j=1}^d m_j$. Now, it suffices to prove that $m_j = |\operatorname{sigsum}(E, \mu_j)|$, for $j = 1, \dots, d$. This will prove that $\ell = \ell_0$, a contradiction to the assumption $\ell < \ell_0$. So let μ be one of the eigenvalues of $D_A + \lambda D_B$, i.e., μ is real (or infinite). Suppose first that $\mu \in \mathbb{R}$. Then for sufficiently small $\varepsilon > 0$, we have that no $\widehat{\lambda} \in [\mu - \varepsilon, \mu + \varepsilon] \setminus \{\mu\}$ is an eigenvalue of $E(\lambda)$. Consequently, for all such $\widehat{\lambda}$, there exist a nonsingular matrix $M \in \mathbb{C}^{(n-s-\ell)\times(n-s-\ell)}$ (depending on $\widehat{\lambda}$) such that

$$(S_A + \widehat{\lambda} S_B)M = \begin{array}{c} \\ {\scriptstyle s} \end{array} \!\!\begin{array}{c} \begin{array}{cc} {\scriptstyle n-r} & {\scriptstyle s} \end{array} \\ \begin{bmatrix} 0 & S \end{bmatrix} \end{array},$$

where $S \in \mathbb{C}^{s\times s}$ is invertible (and also depends on $\widehat{\lambda}$). But this implies that

$$\begin{bmatrix} M^* & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} PE(\widehat{\lambda})P^* \begin{bmatrix} M & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{array}{c} \\ {\scriptstyle n-r} \\ {\scriptstyle s} \\ {\scriptstyle \ell} \\ {\scriptstyle s} \end{array} \!\!\begin{array}{c} \begin{array}{cccc} {\scriptstyle n-r} & {\scriptstyle s} & {\scriptstyle \ell} & {\scriptstyle s} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & S^* \\ 0 & 0 & D_A + \widehat{\lambda} D_B & * \\ 0 & S & * & * \end{bmatrix} \end{array},$$

and due to the nonsingularity of S, we can easily read off the inertia index from the Hermitian matrix $E(\widehat{\lambda})$. If $\operatorname{ind}(H) = (\nu_+, \nu_-, \nu_0)$ denotes the inertia index of a given Hermitian matrix H, i.e., ν_+, ν_-, and ν_0 are the numbers of positive, negative, and zero eigenvalues of H (counted with multiplicities), respectively, then we easily obtain (see also [27, Lemma 6]) that

$$\operatorname{ind}\big(E(\widehat{\lambda})\big) = (s, s, n-r) + \operatorname{ind}(D_A + \widehat{\lambda} D_B),$$

where the sum of triples is taken componentwise. Assume that $\operatorname{ind}(D_A + \mu D_B) = (d_+, d_-, m)$, i.e., m is the algebraic multiplicity of the eigenvalue μ of $D_A + \mu D_B$. Then it follows that

$$\operatorname{ind}\big(E(\mu-\varepsilon)\big) = (s+d_+, s+d_-+m, n-r) \quad \text{and} \quad \operatorname{ind}\big(E(\mu+\varepsilon)\big) = (s+d_++m, s+d_-, n-r)$$

if the sign of μ in the sign characteristic of $D_A + \lambda D_B$ is positive (recall that all signs associated with μ in the sign characteristic of $D_A + \lambda D_B$ are equal), or

$$\operatorname{ind}\big(E(\mu-\varepsilon)\big) = (s+d_++m, s+d_-, n-r) \quad \text{and} \quad \operatorname{ind}\big(E(\mu+\varepsilon)\big) = (s+d_+, s+d_-+m, n-r)$$

if the sign of μ in the sign characteristic of $D_A + \lambda D_B$ is negative. Similarly, checking the change of inertia index of $E(\widehat{\lambda})$ based on its Hermitian canonical form, a straightforward computation shows that the number of positive or negative eigenvalues change by the number $\operatorname{sigsum}(\mu)$ when $\widehat{\lambda}$ passes from $\mu-\varepsilon$ to $\mu+\varepsilon$. This shows that we must have $m = |\operatorname{sigsum}(\mu)|$.

Finally, assume that $\mu = \infty$ is an eigenvalue of $D_A + \lambda D_B$ with algebraic multiplicity m. If $\eta > 0$ is sufficiently large such that all finite eigenvalues of $E(\lambda)$ are contained in the interval $]-\eta, \eta[$, then a similar comparison of the inertia indices of $E(\eta)$ and $E(-\eta)$ reveals that the algebraic multiplicity of ∞ as an eigenvalue of $D_A + \lambda D_B$ must be $|\operatorname{sigsum}(\infty)|$.

3.2 Rank-1 decomposition for other structures

Next, we consider a decomposition analogous to (1) for the other structures mentioned at the beginning of this section. For most of these decompositions, observations similar to the ones in Remark 2–4 can be made, but for the sake of brevity we refrain from stating them explicitly.

Theorem 4 (Rank-1 decomposition for symmetric pencils). *If $E(\lambda)$ is a symmetric $n \times n$ matrix pencil with $\operatorname{rank} E = r \leqslant n$, then it can be written as*

$$E(\lambda) = (a_1 + \lambda b_1)u_1u_1^\top + \cdots + (a_\ell + \lambda b_\ell)u_\ell u_\ell^\top + v_1w_1^\top + \cdots + v_sw_s^\top + w_1v_1^\top + \cdots + w_sv_s^\top, \tag{5}$$

where $a_i, b_i \in \mathbb{C}$, for $i = 1, \ldots, \ell$, and

(i) $\ell + 2s = r$,
(ii) $\deg u_1 = \cdots = \deg u_\ell = 0 = \deg v_1 = \cdots = \deg v_s$ *and* $\deg w_1, \ldots, \deg w_s \leqslant 1$.

Proof The proof is similar to the one of Theorem 2 using the canonical form for complex symmetric pencils [4, Theorem 2.17]. The only difference with the Hermitian case is that in the symmetric case complex eigenvalues are not necessarily paired up by conjugation, so terms of the form $(a + \lambda b)vv^\top$ may come also from odd blocks associated with complex eigenvalues.

Remark 5 The minimal value of ℓ is achieved when all eigenvalues of the pencil

$$D_A + \lambda D_B := \operatorname{diag}(a_1, \ldots, a_\ell) + \lambda \operatorname{diag}(b_1, \ldots, b_\ell),$$

as in Remark 2, have algebraic multiplicity equal to 1. If the multiplicity is larger than 1 for some eigenvalue which is given, say, by the ith and jth diagonal entries $a + \lambda b$ and $c(a + \lambda b)$, with some $c \in \mathbb{C} \setminus \{0\}$, then with a similar trick as in the proof of Theorem 3 two summands of the form $(a + \lambda b)u_iu_i^\top + (ca + \lambda cb)u_ju_j^\top$ can be replaced by two summands of the form $v_kw_k^\top + w_kv_k^\top$ by choosing $v_k = \frac{1}{2}(u_i + \mathrm{i}du_j)$ and $w_k = a(u_i - \mathrm{i}du_j) + \lambda b(u_i - \mathrm{i}du_j)$, where $d \in \mathbb{C}$ is a square root of c, i.e., $d^2 = c$. On the other hand, each eigenvalue of $E(\lambda)$ with odd algebraic multiplicity must occur in one of the summands $(a + \lambda b)u_iu_i^\top$. Indeed, similar to Remark 4 we can show that the vectors $u_1, \ldots, u_\ell, v_1, \ldots, v_s$ are linearly independent and with an argument similar to the one in the proof of Theorem 3, we can show that $E(\lambda)$ is congruent to a pencil of the form

$$\begin{array}{c c} & \begin{array}{ccc} n-s-\ell & \ell & s \end{array} \\ \begin{array}{c} n-s-\ell \\ \ell \\ s \end{array} & \begin{bmatrix} 0 & 0 & S_A^\top + \lambda S_B^\top \\ 0 & D_A + \lambda D_B & * \\ S_A + \lambda S_B & * & * \end{bmatrix} \end{array},$$

which shows that any eigenvalue that is not an eigenvalue of $D_A + \lambda D_B$ must have even algebraic multiplicity being an eigenvalue of both $S_A + \lambda S_B$ and $S_A^\top + \lambda S_B^\top$. Thus, we have just shown that the minimal value of ℓ is equal to the number of pairwise distinct eigenvalues of $E(\lambda)$ that have odd algebraic multiplicity.

We highlight in passing that in the case of complex symmetric matrices and other structures that are based on the transpose rather than the Hermitian transpose no sign characteristic is involved.

Theorem 5 (Rank-1 decomposition for skew-symmetric pencils). *If $E(\lambda)$ is a skew-symmetric $n \times n$ matrix pencil with* $\operatorname{rank} E = r \leqslant n$*, then r is even and $E(\lambda)$ can be written as*

$$E(\lambda) = v_1 w_1^\top + \cdots + v_s w_s^\top - w_1 v_1^\top - \cdots - w_s v_s^\top, \tag{6}$$

where $s = \frac{r}{2}$, $\deg v_1 = \cdots = \deg v_s = 0$, *and* $\deg w_1, \ldots, \deg w_s \leqslant 1$.

Proof The proof follows the same steps as the proof of Theorem 2. All blocks in the skew-symmetric canonical form are paired up (see [4, Theorem 2.18]). More precisely, the blocks in this canonical form are of three different kinds, namely: (a) pairs of $k \times k$ blocks associated with the eigenvalue ∞, (b) pairs of $k \times k$ blocks associated with a complex eigenvalue, and (c) pairs of a $k \times (k+1)$ right singular and a $(k+1) \times k$ left singular block. Then, following the proof of Theorem 2, we can decompose any of these blocks as a sum of rank-1 pencils as in (6).

Theorem 6 (Rank-1 decomposition for $\top$-even pencils). *If $E(\lambda)$ is a $\top$-even $n \times n$ matrix pencil with* $\operatorname{rank} E = r \leqslant n$*, then it can be written as*

$$E(\lambda) = \begin{cases} v_1 w_1(\lambda)^\top + \cdots + v_s w_s(\lambda)^\top + w_1(-\lambda) v_1^\top + \cdots + w_s(-\lambda) v_s^\top, & \text{if } r \text{ is even}, \\ u u^\top + v_1 w_1(\lambda)^\top + \cdots + v_s w_s(\lambda)^\top + w_1(-\lambda) v_1^\top + \cdots + w_s(-\lambda) v_s^\top, & \text{if } r \text{ is odd}, \end{cases} \tag{7}$$

where $s = \lfloor r/2 \rfloor$, $\deg u = \deg v_1 = \cdots = \deg v_s = 0$ *and* $\deg w_1, \ldots, \deg w_s \leqslant 1$.

Proof We proceed in a similar way as in the proof of Theorem 2 using the canonical form for $\top$-even pencils [4, Theorem 2.16]. Again, we may assume the $\top$-even pencil $L(\lambda)$ is given in canonical form. Then, it is a direct sum of blocks of six kinds, namely: (a) $(2k+1) \times (2k+1)$ blocks associated with the eigenvalue ∞, (b) pairs of $(2\ell) \times (2\ell)$ blocks associated with the eigenvalue ∞, (c) pairs of $(2m+1) \times (2m+1)$ blocks associated with the eigenvalue 0, (d) $(2p) \times (2p)$ blocks associated with the eigenvalue 0, (e) pairs of $q \times q$ blocks corresponding to a pair of eigenvalues $\mu, -\mu \in \mathbb{C} \setminus \{0\}$, and (f) pairs of a right and a left singular block of size $(r+1) \times r$ and $r \times (r+1)$, respectively. Blocks of type (d) can be written as a sum of two rank-1 pencils of the form $vw^\top + wv^\top$ using the same decomposition as in the proof of Theorem 2. Similarly, paired blocks of types (b)–(c) and (e)–(f) can be written as a sum of paired rank-1 pencils $vw^\top + wv^\top$ using a combined row-column expansion. For instance, a pair of blocks of type (e) has the form

$$\left[\begin{array}{cccc|cccc}
 & & & & & & & \mu+\lambda \\
 & & & & & & \iddots & 1 \\
 & & & & & \mu+\lambda & \iddots & \\
 & & & & \mu+\lambda & 1 & & \\ \hline
 & & & \mu-\lambda & & & & \\
 & & \mu-\lambda & 1 & & & & \\
 & \iddots & \iddots & & & & & \\
\mu-\lambda & 1 & & & & & &
\end{array}\right]_{(2q)\times(2q)}$$

and can be decomposed into a sum $v_1 w_1(\lambda)^\top + \cdots + v_q w_q(\lambda)^\top + w_1(-\lambda) v_1^\top + \cdots + w_q(-\lambda) v_q^\top$ of $2q$ rank-1 pencils with $v_i = e_{2q-i+1}$, for $i = 1, \ldots, q$, and $w_i(\lambda)$ being, up to the sign, the $(2q-i+1)$th column of the whole matrix pencil, namely $w_i(\lambda) = \begin{bmatrix} 0_{i-1} & \mu-\lambda & 1 & 0_{2q-i-1} \end{bmatrix}^\top$ for $i = 1, \ldots, q-1$, and

$w_q(\lambda) = \begin{bmatrix} 0_{q-1} & \mu-\lambda & 0_q \end{bmatrix}^\top$. Blocks of type (a), however, will need one extra term of the form $uu^\top$. To be more precise, the $(2k+1) \times (2k+1)$ block associated with ∞ having the form

$$\begin{bmatrix} & & & & & & 1 \\ & & & & & \iddots & \lambda \\ & & & & 1 & \iddots & \\ & & & 1 & \lambda & & \\ & & \iddots & -\lambda & & & \\ & 1 & \iddots & & & & \\ 1 & -\lambda & & & & & \end{bmatrix}_{(2k+1)\times(2k+1)}$$

can be decomposed as $uu^\top + v_1 w_1(\lambda)^\top + \cdots + v_k w_k(\lambda)^\top + w_1(-\lambda)v_1^\top + \cdots + w_k(-\lambda)v_k^\top$, where $u = e_k, v_i = e_{2k-i+2}$ for $i = 1, \ldots, k$, and where for $i = 1, \ldots, k$, $w_i(\lambda)^\top$ is the $(2k-i+2)$th row of the matrix pencil, namely $w_i(\lambda) = \begin{bmatrix} 0_{1\times(i-1)} & 1 & -\lambda & 0_{1\times(2k-i)} \end{bmatrix}^\top$.

The previous arguments show that $E(\lambda)$ can be written as

$$\begin{aligned} E(\lambda) = u_1 u_1^\top + \cdots + u_\ell u_\ell^\top + v_1 w_1(\lambda)^\top + \cdots + v_s w_s(\lambda)^\top \\ + w_1(-\lambda)v_1^\top + \cdots + w_s(-\lambda)v_s^\top, \end{aligned} \tag{8}$$

with $\ell + 2s = r$, and $\deg u_1 = \ldots = \deg u_\ell = \deg v_1 = \ldots = \deg v_s = 0$. It remains to prove that, given two vectors $u_1, u_2 \in \mathbb{C}^n$, there exist another two vectors v, w, with $\deg v = 0$, such that

$$u_1 u_1^\top + u_2 u_2^\top = vw^\top + wv^\top. \tag{9}$$

Note that, if this is true, then we can group an even number of summands of the form $uu^\top$ in (8) to get a decomposition like in (7).

To get the expression (9), just set $v = u_1 + \mathrm{i}u_2$ and $w = \frac{1}{2}(u_1 - \mathrm{i}u_2)$.

Theorem 7 (Rank-1 decomposition for $\top$-odd pencils). *If $E(\lambda)$ is a $\top$-odd $n \times n$ matrix pencil with* $\operatorname{rank} E = r \leqslant n$*, then it can be written as*

$$E(\lambda) = \begin{cases} v_1 w_1(\lambda)^\top + \cdots + v_s w_s(\lambda)^\top - w_1(-\lambda)v_1^\top - \cdots - w_s(-\lambda)v_s^\top, & \text{if } r \text{ is even,} \\ \lambda uu^\top + v_1 w_1(\lambda)^\top + \cdots + v_s w_s(\lambda)^\top - w_1(-\lambda)v_1^\top - \cdots - w_s(-\lambda)v_s^\top, & \text{if } r \text{ is odd,} \end{cases} \tag{10}$$

where $s = \lfloor r/2 \rfloor$, $\deg u = \deg v_1 = \cdots = \deg v_s = 0$ *and* $\deg w_1, \ldots, \deg w_s \leqslant 1$.

Proof The result follows from Theorem 6 applied to the reversal of $E(\lambda)$ and using Remark 1.

The following decomposition for low-rank $\top$-palindromic pencils has been presented in the recent reference [8, Th. 3.1]. For completeness, we provide a different proof based on Theorem 2.

Theorem 8 (Rank-1 decomposition for $\top$-palindromic pencils). *If $E(\lambda)$ is a $\top$-palindromic $n \times n$ matrix pencil with* $\operatorname{rank} E = r \leqslant n$*, then it can be written as*

$$E(\lambda) = \begin{cases} v_1 w_1^\top + \cdots + v_s w_s^\top + (\operatorname{rev} w_1)v_1^\top + \cdots + (\operatorname{rev} w_s)v_s^\top, & \text{if } r \text{ is even,} \\ (1+\lambda)uu^\top + v_1 w_1^\top + \cdots + v_s w_s^\top + (\operatorname{rev} w_1)v_1^\top + \cdots + (\operatorname{rev} w_s)v_s^\top, & \text{if } r \text{ is odd,} \end{cases} \tag{11}$$

where $s = \lfloor r/2 \rfloor$, $\deg u = \deg v_1 = \cdots = \deg v_s = 0$ *and* $\deg w_1, \ldots, \deg w_s \leqslant 1$.

Proof The result follows from Theorem 2 using Cayley transformations. More precisely, let $\mathcal{C}_{-1}$ and $\mathcal{C}_{+1}$ be the Cayley transformations of a given matrix pencil $P(\lambda)$ defined as

$$\mathcal{C}_{-1}(P)(\lambda) = (1+\lambda)P\left(\frac{\lambda-1}{1+\lambda}\right) \quad \text{and} \quad \mathcal{C}_{+1}(P)(\lambda) = (1-\lambda)P\left(\frac{1+\lambda}{1-\lambda}\right). \tag{12}$$

It is known that, if $E(\lambda)$ is $\top$-palindromic, then $\mathcal{C}_{+1}(E)$ is $\top$-even [26, Theorem 2.7]. It is clear, by definition, that both $\mathcal{C}_{-1}$ and $\mathcal{C}_{+1}$ preserve the rank. Then $\mathcal{C}_{+1}(E)$ is $\top$-even with $\operatorname{rank} \mathcal{C}_{+1}(E) = r$, so it admits a decomposition like (7). We will focus on the case when r is odd, because the case when r is even is analogous. Using that $\mathcal{C}_{-1}(\mathcal{C}_{+1}(P))(\lambda) = 2P(\lambda)$ for any matrix pencil $P(\lambda)$, see [26, Proposition 2.5], it follows that

$$\begin{aligned} 2E(\lambda) &= \mathcal{C}_{-1}\left(uu^\top + \textstyle\sum_{j=1}^s \left(v_j w_j(\lambda)^\top + w_j(-\lambda) v_j^\top\right)\right) \\ &= (1+\lambda)uu^\top + \textstyle\sum_{j=1}^s v_j \left((1+\lambda) w_j\left(\frac{\lambda-1}{1+\lambda}\right)\right)^\top + \textstyle\sum_{j=1}^s \left((1+\lambda) w_j\left(\frac{1-\lambda}{1+\lambda}\right)\right) v_j^\top, \end{aligned}$$

where $s = (r-1)/2$. Now, the result follows from the identity

$$\operatorname{rev}\left((1+\lambda) w\left(\frac{\lambda-1}{1+\lambda}\right)\right) = \lambda\left(1+\frac{1}{\lambda}\right) w\left(\frac{\frac{1}{\lambda}-1}{1+\frac{1}{\lambda}}\right) = (1+\lambda) w\left(\frac{1-\lambda}{1+\lambda}\right). \tag{13}$$

Using again appropriate Cayley transformations and the decomposition for $\top$-even matrix pencils in Theorem 6 we can also get a rank-1 decomposition for $\star$-anti-palindromic pencils.

Theorem 9 (Rank-1 decomposition for $\top$-anti-palindromic pencils). *If $E(\lambda)$ is a $\top$-anti-palindromic $n \times n$ matrix pencil with* $\operatorname{rank} E = r \leqslant n$*, then it can be written as*

$$E(\lambda) = \begin{cases} v_1 w_1^\top + \cdots + v_s w_s^\top - (\operatorname{rev} w_1) v_1^\top - \cdots - (\operatorname{rev} w_s) v_s^\top, & \text{if } r \text{ is even}, \\ (1-\lambda) uu^\top + v_1 w_1^\top + \cdots + v_s w_s^\top - (\operatorname{rev} w_1) v_1^\top - \cdots - (\operatorname{rev} w_s) v_s^\top, & \text{if } r \text{ is odd}, \end{cases} \tag{14}$$

where $s = \lfloor r/2 \rfloor = 0$, $\deg v_1 = \cdots = \deg v_s = 0$, and $\deg w_1, \ldots, \deg w_s \leqslant 1$.

Proof The proof is similar to the one of Theorem 8, but first considering $\mathcal{C}_{-1}(E)$, which is $\top$-even [26, Th. 2.7], and then applying $\mathcal{C}_{+1}$ to get $\mathcal{C}_{+1}(\mathcal{C}_{-1}(E)) = 2E$. The differences between (14) and (11) come from the identities

$$\begin{aligned} &\mathcal{C}_{+1}(uu^\top) = (1-\lambda) uu^\top, \\ &\mathcal{C}_{+1}(v w(\lambda)^\top) = v\left((1-\lambda) w\left(\tfrac{1+\lambda}{1-\lambda}\right)\right)^\top, \ \mathcal{C}_{+1}(w(-\lambda) v^\top) = (1-\lambda) w\left(\tfrac{1+\lambda}{\lambda-1}\right) v^\top, \end{aligned}$$

and

$$\operatorname{rev}\left((1-\lambda) w\left(\frac{1+\lambda}{1-\lambda}\right)\right) = \lambda\left(1-\frac{1}{\lambda}\right) w\left(\frac{1+\frac{1}{\lambda}}{1-\frac{1}{\lambda}}\right) = -(1-\lambda) w\left(\frac{1+\lambda}{\lambda-1}\right).$$

We highlight that the parameter ℓ in the decomposition $r = \ell + 2s$ takes the minimal value zero or one in the decompositions in Theorem 5–9. This is in contrast with Theorem 2 and Theorem 4, where the minimal value for ℓ can be as large as r, for example if the pencil $E(\lambda)$ does only have simple eigenvalues in the symmetric case, or only simple real eigenvalues in the Hermitian case.

The rank-1 decompositions for skew-Hermitian, $*$-even, and $*$-odd pencils can be directly obtained from the decomposition in the Hermitian case, by means of the following observation (see [4, page 80]):

- If $A + \lambda B$ is skew-Hermitian then $\mathrm{i}(A + \lambda B)$ is Hermitian.
- If $A + \lambda B$ is $*$-even then $A + \lambda(\mathrm{i}B)$ is Hermitian.
- $A + \lambda B$ is $*$-odd if and only if $B + \lambda A$ is $*$-even.

For completeness, we explicitly state these decompositions in a similar way as we have done for the previous structures.

Theorem 10 (Rank-1 decomposition for skew-Hermitian pencils). *If $E(\lambda)$ is a skew-Hermitian $n \times n$ matrix pencil with* $\operatorname{rank} E = r \leqslant n$*, then it can be written as*

$$E(\lambda) = \mathrm{i}(a_1 + \lambda b_1) u_1 u_1^* + \cdots + \mathrm{i}(a_\ell + \lambda b_\ell) u_\ell u_\ell^* + v_1 w_1^* + \cdots + v_s w_s^* - w_1 v_1^* - \cdots - w_s v_s^*, \tag{15}$$

where $a_i, b_i \in \mathbb{R}$, for $i = 1, \ldots, \ell$, and

(i) $\ell + 2s = r$,
(ii) $\deg u_1 = \cdots = \deg u_\ell = 0 = \deg v_1 = \cdots = \deg v_s$ *and* $\deg w_1, \ldots, \deg w_s \leqslant 1$.

Theorem 11 (Rank-1 decomposition for $*$-even pencils). *If $E(\lambda)$ is a $*$-even $n \times n$ matrix pencil with* $\operatorname{rank} E = r \leqslant n$, *then it can be written as*

$$\begin{aligned} E(\lambda) = & (a_1 + \lambda(b_1 \mathfrak{i}))u_1 u_1^* + \cdots + (a_\ell + \lambda(b_\ell \mathfrak{i}))u_\ell u_\ell^* \\ & + v_1 w_1(\lambda)^* + \cdots + v_s w_s(\lambda)^* + w_1(-\lambda)v_1^* + \cdots + w_s(-\lambda)v_s^*, \end{aligned} \tag{16}$$

where $a_i, b_i \in \mathbb{R}$, for $i = 1, \ldots, \ell$, and

(i) $\ell + 2s = r$,
(ii) $\deg u_1 = \cdots = \deg u_\ell = 0 = \deg v_1 = \cdots = \deg v_s$ *and* $\deg w_1, \ldots, \deg w_s \leqslant 1$.

Theorem 12 (Rank-1 decomposition for $*$-odd pencils). *If $E(\lambda)$ is a $*$-odd $n \times n$ matrix pencil with* $\operatorname{rank} E = r \leqslant n$, *then it can be written as*

$$\begin{aligned} E(\lambda) = & (a_1 \mathfrak{i} + \lambda b_1)u_1 u_1^* + \cdots + (a_\ell \mathfrak{i} + \lambda b_\ell)u_\ell u_\ell^* \\ & + v_1 w_1(\lambda)^* + \cdots + v_s w_s(\lambda)^* - w_1(-\lambda)v_1^* - \cdots - w_s(-\lambda)v_s^*, \end{aligned} \tag{17}$$

where $a_i, b_i \in \mathbb{R}$, for $i = 1, \ldots, \ell$, and

(i) $\ell + 2s = r$,
(ii) $\deg u_1 = \cdots = \deg u_\ell = 0 = \deg v_1 = \cdots = \deg v_s$ *and* $\deg w_1, \ldots, \deg w_s \leqslant 1$.

The decomposition in (15) follows from (1) after multiplying by $\mathfrak{i}$ and using that, for any pair of vectors $u, v \in \mathbb{C}[\lambda]^n$, we can write $\mathfrak{i}(uw^* + wv^*) = (\mathfrak{i}v)w^* - w(\mathfrak{i}v)^* = \widetilde{v}w^* - w\widetilde{v}^*$, with $\widetilde{v} = \mathfrak{i}v$. Similarly, the expression (16) follows from (1) applied to $E(\mathfrak{i}\lambda)$ and then multiplying the leading coefficient in the decomposition by $-\mathfrak{i}$. Note that, if $A+\lambda(\mathfrak{i}B) = vw(\lambda)^* + w(\lambda)v^* = v(w_0^* + \lambda w_1^*) + (w_0 + \lambda w_1)v^*$ (with $v \in \mathbb{C}^n$ and $w(\lambda) = w_0 + \lambda w_1$, $w_0, w_1 \in \mathbb{C}^n$), then, multiplying the leading coefficient by $-\mathfrak{i}$, we get $A+\lambda B = v(w_0^* - \mathfrak{i}\lambda w_1^*) + (w_0 - \mathfrak{i}\lambda w_1)v^* = v(w_0^* + \lambda(\mathfrak{i}w_1)^*) + (w_0 - \lambda(\mathfrak{i}w_1))v^* = vw(\lambda)^* + w(-\lambda)v^*$. Finally, the decomposition (17) follows from (16) applied to $\operatorname{rev} E(\lambda)$ and then applying the reversal to the decomposition in the right-hand side. Note that, if $\lambda A + B = v\widetilde{w}(\lambda)^* + \widetilde{w}(-\lambda)v^* = v(w_0^* + \lambda w_1^*) + (w_0 - \lambda w_1)v^*$ (with $v \in \mathbb{C}^n$ and $\widetilde{w}(\lambda) = w_0 + \lambda w_1$, $w_0, w_1 \in \mathbb{C}^n$), then $A + \lambda B = v(w_1^* + \lambda w_0^*) - (w_1 - \lambda w_0)v^* = vw(\lambda)^* - w(-\lambda)v^*$, where $w(\lambda) = \operatorname{rev} \widetilde{w}(\lambda)$.

As for the $*$-palindromic structure, the decomposition follows from (16) using appropriate Cayley transforms, like for the $\top$-palindromic structure.

Theorem 13 (Rank-1 decomposition for $*$-palindromic pencils). *If E is a $*$-palindromic $n \times n$ matrix pencil with* $\operatorname{rank} E = r \leqslant n$, *then it can be written as*

$$\begin{aligned} E(\lambda) = & \big((a_1 - b_1\mathfrak{i}) + \lambda(a_1 + b_1\mathfrak{i})\big)u_1 u_1^* + \cdots + \big((a_\ell - b_\ell\mathfrak{i}) + \lambda(a_\ell + b_\ell\mathfrak{i})\big)u_\ell u_\ell^* \\ & + v_1 w_1^* + \cdots + v_s w_s^* + (\operatorname{rev} w_1)v_1^* + \cdots + (\operatorname{rev} w_s)v_s^*, \end{aligned} \tag{18}$$

where $a_i, b_i \in \mathbb{R}$, for $i = 1, \ldots, \ell$, and

(i) $\ell + 2s = r$,
(ii) $\deg u_1 = \cdots = \deg u_\ell = 0 = \deg v_1 = \cdots = \deg v_s$ *and* $\deg w_1, \ldots, \deg w_s \leqslant 1$.

Proof The proof is similar to the one of Theorem 8, but we include it here to illustrate where the difference in the first ℓ summands comes from. In particular, if $E(\lambda)$ is $*$-palindromic as in the statement, then $\mathcal{C}_{+1}(E)$ is $*$-even [26, Theorem 2.7]. Therefore, it admits a decomposition like (16). Now

$$\begin{aligned} 2E(\lambda) = \mathcal{C}_{-1}\big(\mathcal{C}_{+1}(E)\big) &= \mathcal{C}_{-1}\left(\textstyle\sum_{i=1}^{\ell} \big(a_i + \lambda(b_i\mathfrak{i})\big)u_i u_i^*\right) + \mathcal{C}_{-1}\left(\textstyle\sum_{j=1}^{s} \big(v_j w_j(\lambda)^* + w_j(-\lambda)v_j^*\big)\right) \\ &= \textstyle\sum_{i=1}^{\ell} \big((a_i - b_i\mathfrak{i}) + \lambda(a_i + b_i\mathfrak{i})\big)u_i u_i^* + \sum_{j=1}^{s} \big(v_j w_j^* + (\operatorname{rev} w_j)v_j^*\big), \end{aligned}$$

where, for the first sum, we have used that

$$\mathcal{C}_{-1}\big((a + \lambda(b\mathfrak{i}))uu^*\big) = (1+\lambda)\left(a + \frac{\lambda - 1}{1+\lambda}\, b\mathfrak{i}\right) uu^* = \big((a - b\mathfrak{i}) + \lambda(a + b\mathfrak{i})\big)uu^*,$$

and, for the second sum, we have followed exactly the same steps as in the proof of Theorem 8, just replacing $\top$ by $*$.

Note that the first ℓ summands in the right-hand side of (18) come from eigenvalues of $E(\lambda)$ which lie on the unit circle. Moreover, any complex value on the unit circle can be identified as a root of a linear polynomial of the form $(a - b\mathrm{i}) + \lambda(a + b\mathrm{i})$.

Theorem 14 (Rank-1 decomposition for $*$-anti-palindromic pencils). *If $E(\lambda)$ is a $*$-anti–palindromic $n \times n$ matrix pencil with* $\operatorname{rank} E = r \leqslant n$, *then it can be written as*

$$\begin{aligned} E(\lambda) = {} & ((a_1 + b_1\mathrm{i}) + \lambda(-a_1 + b_1\mathrm{i}))u_1u_1^* + \cdots + ((a_\ell + b_\ell\mathrm{i}) + \lambda(-a_\ell + b_\ell\mathrm{i}))u_\ell u_\ell^* \\ & + v_1w_1^* + \cdots + v_sw_s^* - (\operatorname{rev} w_1)v_1^* - \cdots - (\operatorname{rev} w_s)v_s^*, \end{aligned} \tag{19}$$

where $a_i, b_i \in \mathbb{R}$, for $i = 1, \ldots, \ell$, and

(i) $\ell + 2s = r$,

(ii) $\deg u_1 = \ldots = \deg u_\ell = 0 = \deg v_1 = \ldots = \deg v_s$ *and* $\deg w_1, \ldots, \deg w_s \leqslant 1$.

Proof The proof follows the same steps as the proof of Theorem 9.

Concerning minimality of the parameter ℓ, there is a characterization analogous to the one in Theorem 3 involving the signsum of real eigenvalues in the case of skew-Hermitian pencils, of purely imaginary eigenvalues in the case of $*$-even and $*$-odd pencils, or unimodular eigenvalues in the case of $*$-palindromic or $*$-anti-palindromic pencils. We refrain from explicitly stating these characterizations.

4 Structure-preserving low-rank perturbations

In this section, we will develop our main results on the change of the partial multiplicities of eigenvalues of matrix pencils with symmetry structure under generic structure-preserving low-rank perturbations. For this, we follow the approach in [11]. More precisely, let $\mathbb{S}_r$ be the set of matrix pencils with structure $\mathbb{S}$ and with rank at most r, where $\mathbb{S}$ is any of the structures mentioned in Section 3, let $L(\lambda)$ be a regular pencil (with structure $\mathbb{S}$) and let λ_0 be an eigenvalue of $L(\lambda)$ (finite or infinite). The procedure then consists of two main steps:

Step 1. Obtain a (polynomial) parameterization of $\mathbb{S}_r$.

Step 2. Prove that, for a generic set of parameters, all pencils $E(\lambda) \in \mathbb{S}_r$ obtained from the previous parameterization are such that the partial multiplicities of $(L+E)(\lambda)$ at λ_0 are the ones described in the main results (given in Section 4.4).

Step 1 is addressed in Section 4.3, and Step 2 is addressed in Section 4.4. For the realization of Step 2 we will need as a key ingredient a localization result that we develop in Section 4.1, where we will also clarify the notion of genericity.

4.1 A localization result

Let $\mathbb{F}$ denote one of the fields $\mathbb{R}$ or $\mathbb{C}$, we then use the following notion of genericity.

Definition 2 A *generic set* $\mathcal{G}$ of $\mathbb{F}^m$ is a subset of $\mathbb{F}^m$ whose complement is contained in a proper algebraic set, i.e., $\mathcal{G}$ is nonempty and coincides with the complement of a set of common zeros of finitely many polynomials in m variables.

We highlight that even though in this paper we only deal with the case of complex matrix pencils, we have to use the concept of genericity with respect to the real numbers when symmetry structures involving the conjugate transpose are considered, because complex conjugation is not a polynomial map on $\mathbb{C}$. This problem can be circumvented if we identify $\mathbb{C}^m$ with $\mathbb{R}^{2m}$ by considering the real and imaginary parts of each component separately. In this context, complex conjugation is an $\mathbb{R}$-linear map and thus in particular polynomial.

We will need the following result, which is almost identical to [33, Lemma 3.1]. (The parameter μ will be equal to 1 for most cases, which corresponds to simple eigenvalues. However, in the case of skew-symmetric matrix pencils, considered in Theorem 21, we will apply the result with $\mu = 2$.)

Lemma 1 *Let $A \in \mathbb{C}^{n\times n}$ have the pairwise distinct eigenvalues $\lambda_1, \ldots, \lambda_\kappa \in \mathbb{C}$ with algebraic multiplicities $a_1, \ldots, a_\kappa$, and let $\varepsilon > 0$ be such that the discs*

$$D_j := \left\{\mu \in \mathbb{C} : |\lambda_j - \mu| < \varepsilon^{2/n}\right\}, \quad j = 1, \ldots, \kappa$$

are pairwise disjoint. Furthermore, let $U \subseteq \mathbb{F}^m$ be open and let $C : U \to \mathbb{C}^{n\times n}$ be an analytic function with $C(0) = A$, such that the following conditions are satisfied:

1) *For all $x \in U$, the algebraic multiplicity of any eigenvalue of $C(u)$ is always a multiple of $\mu \in \mathbb{N} \setminus \{0\}$.*
2) *There exists a generic set $\mathcal{G} \subseteq \mathbb{F}^m$ such that, for all $x \in \mathcal{G} \cap U$, the matrix $C(x)$ has the eigenvalues $\lambda_1, \ldots, \lambda_\kappa$ with algebraic multiplicities $\widetilde{a}_1, \ldots, \widetilde{a}_\kappa$, where $\widetilde{a}_j \leqslant a_j$ for $j = 1, \ldots, \kappa$. (Here, we allow $a_j = 0$ in the case that λ_j is no longer an eigenvalue of $C(x)$.)*
3) *For each $j = 1, \ldots, \kappa$ there exists $x_j \in U$ with $\|x_j\| < \varepsilon$ such that the matrix $C(x_j)$ has exactly $(a_j - \widetilde{a}_j)/\mu$ pairwise distinct eigenvalues in D_j different from λ_j and each one has algebraic multiplicity exactly μ.*

Then there exists $\varepsilon' > 0$ and a set $\mathcal{G}_0$, open and dense in $\{x \in \mathbb{F}^m \mid \|x\| < \varepsilon'\}$, with $\mathcal{G}_0 \subseteq U$, such that, for all $x \in \mathcal{G}_0$, the pencil $C(x)$ has exactly $\sum_{j=1}^{\kappa} \frac{1}{\mu}(a_j - \widetilde{a}_j)$ eigenvalues that are different from those of A and each of these eigenvalues has algebraic multiplicity exactly μ.

Proof The proof is almost identical to the one of Lemma 3.1 in [33] and therefore omitted. (One just has to replace $\mathbb{R}$ in [33] with $\mathbb{F}$ and remove the final paragraph on the proof which is not needed here, because the statement of Lemma 1 has been adapted correspondingly.) □

The next result generalizes [33, Theorem 3.2] (which itself was an extension of [6, Theorem 2.6]) from the matrix to the pencil case and will be the main tool in Section 4.4.

Theorem 15 *Let $L(\lambda) = A + \lambda B$ be a regular complex $n \times n$ matrix pencil and let $\lambda_1, ..., \lambda_\kappa$ be its pairwise distinct eigenvalues (finite or infinite) with geometric multiplicities g_i, nonzero partial multiplicities $n_{i,1} \geqslant n_{i,2} \geqslant \cdots \geqslant n_{i,g_i} > 0$, and algebraic multiplicities*

$$a_i = \sum_{j=1}^{g_i} n_{i,j},$$

for $i = 1, \ldots, \kappa$, respectively. Let $\Phi : \mathbb{F}^m \to \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n}$ be a polynomial map and, for $x \in \mathbb{F}^m$, let us identify $\Phi(x) = (\Phi_A(x), \Phi_B(x))$ with the pencil $\Phi_A(x) + \lambda\Phi_B(x)$. Furthermore, assume that, for all $x \in \mathbb{F}^m$, we have

(i) *$\Phi(0) = (0, 0)$;*
(ii) *$\operatorname{rank} \Phi(x) \leqslant r$;*
(iii) *if $L + \Phi(x)$ is regular, then the algebraic multiplicity of any eigenvalue of $L + \Phi(x)$ is always a multiple of some $\mu \in \mathbb{N} \setminus \{0\}$.*

Then the following statements hold:

(1) *If $x \in \mathbb{F}^m$ is such that $L + \Phi(x)$ is regular and if $\eta_{i,1} \geqslant \cdots \geqslant \eta_{\widetilde{g}_i}$ are the partial multiplicities associated with λ_i as an eigenvalue of $L + \Phi(x)$, for $i = 1, \ldots, \kappa$ (here we allow $\widetilde{g}_i = 0$ if λ_i is not an eigenvalue of $L + \Phi(x)$), then the list $(\eta_{i,1}, \ldots, \eta_{i,\widetilde{g}_i})$ dominates the list $(n_{i,r+1}, \ldots, n_{i,g_i})$, i.e., we have $\widetilde{g}_i \geqslant g_i - r$ and $\eta_{i,j} \geqslant n_{i,j+r}$, for $j = 1, \ldots, g_i - r$ and $i = 1, \ldots, \kappa$.*
(2) *Assume that, for all $x \in \mathbb{F}^m$ for which $L + \Phi(x)$ is regular, we have that, for each $i = 1, \ldots, \kappa$, the algebraic multiplicity $a_i^{(x)}$ of λ_i as an eigenvalue of $L + \Phi(x)$ satisfies $a_i^{(x)} \geqslant \widetilde{a}_i$, for some $\widetilde{a}_i \in \mathbb{N}$. If, for any $\varepsilon > 0$ and each $i = 1, \ldots, \kappa$, there exists $x_{0,i} \in \mathbb{F}^m$ with $\|x_{0,i}\| < \varepsilon$ such that $L + \Phi(x_{0,i})$ is regular, $a_i^{(x_{0,i})} = \widetilde{a}_i$, and all eigenvalues of $L + \Phi(x_{0,i})$ that are different from those of L have multiplicity precisely μ, then there exists a generic set $\mathcal{G} \subseteq \mathbb{F}^m$ such that, for all $x \in \mathcal{G}$, the following conditions are satisfied:*
   (a) *the pencil $L + \Phi(x)$ is regular;*
   (b) *$a_i^{(x)} = \widetilde{a}_i$ for all $i = 1, \ldots, \kappa$;*

(c) *all eigenvalues of $L+\Phi(x)$ which are different from those of L have multiplicity precisely μ.*
If, in addition, we have $\widetilde{a}_i = n_{i,r+1} + \cdots + n_{i,g_i}$ for some $i \in \{1,\ldots,\kappa\}$, then the partial multiplicities of λ_i as an eigenvalue of $L+\Phi(x)$ are precisely $n_{i,r+1},\ldots,n_{i,g_i}$ for all $x \in \mathcal{G}$.

Proof In order to introduce the dependence on λ in the pencil $\Phi(x)$, we denote $\Phi_x(\lambda) := \Phi(x)$ along the proof. First of all, we may assume that ∞ is not an eigenvalue of $L(\lambda)$. Otherwise, consider instead the pencil $\widehat{L}(\lambda) = A + \lambda\big(\alpha A + B\big)$, for some $\alpha \in]0,1[$ such that ∞ is not an eigenvalue of $\widehat{L}(\lambda)$. Note that this transformation only changes the eigenvalues, but not their corresponding multiplicities and their behavior under perturbation when the perturbation pencil is adapted to $\widehat{\Phi}_x(\lambda) = \Phi_A(x) + \lambda\big(\alpha\Phi_A(x) + \Phi_B(x)\big)$.

Part (1) is a direct consequence of [12, Lemma 2.1] using the fact that the rank of $\Phi_x(\lambda)$ is at most r, for any $x \in \mathbb{F}^m$.

For part (2), we first show that the set

$$\mathcal{G}_{\text{reg}} = \{x \in \mathbb{F}^m \mid (L+\Phi_x)(\lambda) \text{ is regular}\}$$

is a generic set. To see this, let $z \in \mathbb{C}$ be a value which is not an eigenvalue of $L(\lambda)$. Then $p(x) := \det\big((L+\Phi_x)(z)\big)$ is a polynomial in the entries of x that is not the zero polynomial. The set of pencils for which $L+\Phi_x$ is singular is then contained in the set of pencils for which $p(x)=0$, which by definition is an algebraic set. Therefore, $\mathcal{G}_{\text{reg}}$ is generic.

Next, let $Y_i(x)$ be the matrix $Y_i(x) = \big((L+\Phi_x)(\lambda_i)\big)^n$. Then, by assumption, we have $\operatorname{rank} Y_i(x_{0,i}) = n - \widetilde{a}_i$, for some $x_{0,i} \in \mathbb{F}^m$, and it follows from [28, Lemma 2.1] that the set

$$\mathcal{G}_i := \{x \in \mathbb{F}^m \mid \operatorname{rank} Y_i(x) \geqslant n - \widetilde{a}_i\}$$

is a generic set, for $i = 1,\ldots,\kappa$. On the set $\mathcal{G}_i \cap \mathcal{G}_{\text{reg}}$ the condition $\operatorname{rank} Y_i(x) \geqslant n - \widetilde{a}_i$ is equivalent to $a_i^{(x)} \leqslant \widetilde{a}_i$, and since, by assumption, the reverse inequality $a_i^{(x)} \geqslant \widetilde{a}_i$ holds for all $x \in \mathcal{G}_{\text{reg}}$, it follows that we have $a_i^{(x)} = \widetilde{a}_i$ for all $x \in \mathcal{G}_i \cap \mathcal{G}_{\text{reg}}$. Thus, setting $\widetilde{\mathcal{G}} := \mathcal{G}_{\text{reg}} \cap \mathcal{G}_1 \cap \cdots \cap \mathcal{G}_\kappa$, we find that $\widetilde{\mathcal{G}}$ is generic, as being the intersection of finitely many generic sets, and for all $x \in \widetilde{\mathcal{G}}$ the conditions (a) and (b) are satisfied.

Finally, let $\chi_x(\lambda)$ denote the characteristic polynomial of $(L+\Phi_x)(\lambda)$. Then the number of distinct roots of χ_x is given by

$$\operatorname{rank} S\left(\chi_x, \frac{\partial\chi_x}{\partial\lambda}\right) - n + 1,$$

where $S(p_1,p_2)$ denotes the Sylvester resultant matrix (see, for instance [2, p. 290]) of the two polynomials $p_1(\lambda)$, $p_2(\lambda)$. (Recall that $S(p_1,p_2)$ is a square matrix of size $\deg(p_1) + \deg(p_2)$ and that the rank deficiency of $S(p_1,p_2)$ coincides with the degree of the greatest common divisor of the polynomials $p_1(\lambda)$ and $p_2(\lambda)$.) Therefore, the set $\mathcal{G}$ of all $x \in \widetilde{\mathcal{G}}$ on which the number of distinct roots of $\chi(x)$ is maximal, is a generic set. (Again this uses [28, Lemma 2.1], which states that the set where a matrix depending on $x \in \mathbb{C}^m$ has maximal rank is a generic set.) If we can show that this maximal number is equal to $\kappa + \sum_{i=1}^{\kappa} \frac{1}{\mu}(a_i - \widetilde{a}_i)$, then clearly (a)–(c) are satisfied for all $x \in \mathcal{G}$. To this end, observe that P as a polynomial is an analytic function and that, by assumption, $x_{0,i}$ can be chosen to be of arbitrarily small norm. Furthermore, for $\varepsilon_0 > 0$ sufficiently small, the continuity of P guarantees that, for all $x \in \mathbb{C}^m$ with $\|x\| \leqslant \varepsilon_0$, the perturbed pencil $(L+\Phi_x)(\lambda)$ is regular and does not have ∞ as an eigenvalue. But then $B + \Phi_B$ is invertible and we can apply Lemma 1 to the matrix $(B+\Phi_B)^{-1}(A+\Phi_A)$ using the fact that matrix inversion is an analytic function to prove that the maximal number of distinct roots of χ_x is as desired.

The additional part follows from the fact that the only list of partial multiplicities that both dominates $(n_{i,r+1},\ldots,n_{i,g})$ and has $a_i^{(x)} = n_{i,r+1} + \cdots + n_{i,g_i}$ is the list $(n_{i,r+1},\ldots,n_{i,g})$. □

The key consequence of Theorem 15 is the following: If we want to show that a pencil has a particular behavior under perturbations, it is now enough to consider the pencil locally in the following sense: it is sufficient to focus on a single eigenvalue and construct examples of perturbations that provide the desired behavior for that particular eigenvalue. We will use this strategy exhaustively in the following subsections.

4.2 Revisiting the unstructured case

In this subsection, we will briefly revisit the case of general matrix pencils (possibly without additional symmetry structures) and discuss their parameterizations from [11]. This will not only give us an idea on how we can extend this procedure to the case of structured pencils, but also allows us to strengthen the main result in [11], which only considered the generic change in the Weierstraß structure of regular matrix pencils under low-rank perturbations, but did not discuss the multiplicity of newly generated eigenvalues.

As in [11], let us pick an integer $r \leqslant n$ and let us define for each $s = 0, 1, \ldots, r$ the set

$$\mathfrak{C}_s := \left\{ v_1(\lambda)w_1(\lambda)^\top + \cdots + v_r(\lambda)w_r(\lambda)^\top \;\middle|\; \begin{array}{c} v_1, \ldots, v_r, w_1, \ldots, w_r \in \mathbb{C}[\lambda]^n, \\ \deg v_i, \deg w_i \leqslant 1, \text{ for } j = 1, \ldots, r, \\ \deg v_1 = \cdots = \deg v_s = 0, \\ \deg w_{s+1} = \cdots = \deg w_r = 0 \end{array} \right\}.$$

Then using [9, Lemma 2.8] it was shown in [11, Lemma 3.1] that

$$\mathbb{P}_r = \mathfrak{C}_0 \cup \mathfrak{C}_1 \cup \cdots \cup \mathfrak{C}_r, \tag{20}$$

where $\mathbb{P}_r$ denotes the set of $n \times n$ matrix pencils with rank at most r.

Remark 6 It is important to note that the union in (20) is not a partition, as the sets $\mathfrak{C}_0, \mathfrak{C}_1, \ldots, \mathfrak{C}_r$ are not disjoint. In particular, if $A \in \mathbb{C}^{n \times n}$ is a matrix of rank r, then the pencil $A = A + \lambda 0$ is contained in each $\mathfrak{C}_s$ for $s = 0, \ldots, r$.

Definition 3 (Parameterization of the set of pencils with rank at most r). Let $r \in \mathbb{N}$. For each $s = 0, 1, \ldots, r$ we define the map $\Phi_s : \mathbb{C}^{3rn} \longrightarrow \mathfrak{C}_s$ as follows: for $x \in \mathbb{C}^{3rn}$ decomposed as $x = \left[\,\alpha \middle| \beta \middle| \gamma \middle| \delta\,\right]^\top$ with

$$\begin{aligned} \alpha &= \left[\,\alpha_{11} \cdots \alpha_{n1} \middle| \cdots \middle| \alpha_{1r} \cdots \alpha_{nr}\,\right] \in \mathbb{C}^{1 \times rn}, \\ \beta &= \left[\,\beta_{1,s+1} \cdots \beta_{n,s+1} \middle| \cdots \middle| \beta_{1r} \cdots \beta_{nr}\,\right] \in \mathbb{C}^{1 \times (r-s)n}, \\ \gamma &= \left[\,\gamma_{11} \cdots \gamma_{n1} \middle| \cdots \middle| \gamma_{1r} \cdots \gamma_{nr}\,\right] \in \mathbb{C}^{1 \times rn}, \\ \delta &= \left[\,\delta_{11} \cdots \delta_{n1} \middle| \cdots \middle| \delta_{1s} \cdots \delta_{ns}\,\right] \in \mathbb{C}^{1 \times sn}, \end{aligned}$$

we set

$$\Phi_s(x) = v_1(\lambda)w_1(\lambda)^\top + \cdots + v_r(\lambda)w_r(\lambda)^\top,$$

where $v_1, \ldots, v_r, w_1, \ldots, w_r$ are defined via

$$\begin{aligned} v_i &= \left[\,\alpha_{1i} \cdots \alpha_{ni}\,\right]^\top, && \text{for } i = 1, \ldots, s, \\ v_j &= \left[\,\alpha_{1j} + \lambda\beta_{1j} \cdots \alpha_{nj} + \lambda\beta_{nj}\,\right]^\top, && \text{for } j = s+1, \ldots, r, \\ w_i &= \left[\,\gamma_{1i} + \lambda\delta_{1i} \cdots \gamma_{ni} + \lambda\delta_{ni}\,\right]^\top, && \text{for } i = 1, \ldots, s, \\ w_j &= \left[\,\gamma_{1j} \cdots \gamma_{nj}\,\right]^\top, && \text{for } j = s+1, \ldots, r. \end{aligned}$$

With this preparation, we are able to prove the following result, which extends the main result from [11] by adding a statement on the simplicity of newly generated eigenvalues.

Theorem 16 (Generic change under low-rank perturbations of general regular matrix pencils). *Let $L(\lambda)$ be a regular $n \times n$ matrix pencil and let $\lambda_1, \ldots, \lambda_\kappa$ denote the pairwise distinct eigenvalues of $L(\lambda)$ having the partial multiplicities $n_{i,1} \geqslant \ldots \geqslant n_{i,g_i} > 0$, for $i = 1, \ldots, \kappa$, respectively. Furthermore, let r be a positive integer, let $0 \leqslant s \leqslant r$, and let Φ_s be the map in Definition 3. Then, there exists a generic set $\mathcal{G}_s$ in $\mathbb{C}^{3rn}$ such that for all $E(\lambda) \in \Phi_s(\mathcal{G}_s)$, the perturbed pencil $L+E$ is regular and the partial multiplicities of $L+E$ at λ_i are given by $n_{i,r+1} \geqslant \cdots \geqslant n_{i,g_i}$. (In particular, if $r \geqslant g_i$ then λ_i is not an eigenvalue of $L+E$.) Furthermore, all eigenvalues of $L+E$ that are different from those of L are simple.*

Proof By Theorem 15 it is sufficient to focus on a particular eigenvalue λ_i and construct one particular example $E = \Phi_s(x)$ of a pencil such that the partial multiplicities of $L + E$ are as claimed in the theorem and such all eigenvalues that are different from those of L are simple. For the moment, let us suppose that λ_i is finite and, for simplicity, let us write $n_1 \geqslant \cdots \geqslant n_g$ instead of $n_{i,1} \geqslant \ldots \geqslant n_{i,g_i}$ for its partial multiplicities. Since genericity of sets is preserved under multiplication with invertible matrices, we may assume, without loss of generality, that L is in WCF and has the form

$$L(\lambda) = \operatorname{diag}\left(J_{n_1}(\lambda_i - \lambda), \ldots, J_{n_g}(\lambda_i - \lambda), \widetilde{L}(\lambda)\right),$$

where $\widetilde{L}(\lambda)$ consists of all the blocks associated with eigenvalues different from λ_i. As in the proof of [11, Theorem 3.4], let $E_k(\psi)$ be the $k \times k$ matrix that is zero everywhere except for the $(k,1)$-entry which takes the value $\psi \in \mathbb{C}$. Then it is straightforward to check that the pencil $J_m(\lambda_i - \lambda) + E_m(\psi)$ has determinant equal to $\chi(\lambda) = (\lambda_i - \lambda)^m + (-1)^{m-1}\psi$, i.e., its eigenvalues lie on a circle centered around λ_i with radius $|\psi|^{\frac{1}{m}}$. Thus, consider the $n \times n$ pencil

$$E(\lambda) = \operatorname{diag}\left(E_{n_1}(\psi_1), \ldots, E_{n_r}(\psi_r), 0\right).$$

Then $E(\lambda)$ is a constant pencil of rank r and hence, by Remark 6, there exists $x \in \mathbb{C}^{3rn}$ such that $E(\lambda) = \Phi_s(x)$. Moreover, we find that $L + E$ has the partial multiplicities $n_{r+1} \geqslant \cdots \geqslant n_g$ at λ_i. Furthermore, having chosen the values $\psi_1, \ldots, \psi_r \in \mathbb{C}$ appropriately such that all radii $|\psi_j|^{\frac{1}{n_j}}$ are pairwise distinct and smaller than the distance of λ_i to the spectrum of $\widetilde{L}(\lambda)$, we can guarantee that all eigenvalues of $L + E$ that are different from those of L are simple. Finally, by also choosing $\psi_1, \ldots, \psi_r$ to be of sufficiently small modulus, we can guarantee that the norm of x is arbitrarily small. This gives the desired example. For the case $\lambda_i = \infty$ consider the reversal of the pencil $L(\lambda)$ and apply the result for the already proved case $\lambda_i = 0$.

4.3 Parameterization of low-rank structured matrix pencils

In this subsection, we finally consider the generic change in the Weierstraß structure of structured matrix pencils under structure-preserving low-rank perturbations. Following the procedure in [11], we first look for a parameterization of the set of $n \times n$ structured matrix pencils with rank at most r, for any of the structures considered in Section 3. Such a parameterization comes naturally from the decomposition into a sum of rank-1 pencils provided in that section. More precisely, we decompose the set of $n \times n$ structured matrix pencils as the union of subsets given by fixing the value of the parameter s in Theorems 2, 4, 5–13, and 14. Again, we will use the Hermitian case as a model for other structures. Thus, while the Hermitian case will be presented in full detail, we only give a brief remark on how other structures have to be dealt with whenever this is necessary, with one exception: we will add a bit more details in the case of $\top$-even pencils, because the effect of structure-preserving low-rank perturbation needs a more detailed discussion for this structure and related ones. Thus, the set of $\top$-even pencils will be a subordinate case.

For the Hermitian structure, the decomposition outlined in the previous paragraph is as follows. For each $0 \leqslant s \leqslant \lfloor r/2 \rfloor$, let us define

$$\mathfrak{C}_s^{\mathbb{H}} := \left\{ \begin{array}{c|c} & \ell = r - 2s, \\ & u_1, \ldots, u_\ell \in \mathbb{C}^n, \\ (a_1 + \lambda b_1)u_1u_1^* + \cdots + (a_\ell + \lambda b_\ell)u_\ell u_\ell^* & v_1, \ldots, v_s \in \mathbb{C}^n, \\ +v_1w_1^* + \cdots + v_sw_s^* + w_1v_1^* + \cdots + w_sv_s^* & w_1, \ldots, w_s \in \mathbb{C}[\lambda]^n, \\ & \deg w_j \leqslant 1, \text{ for } j = 1, \ldots, s, \\ & a_i, b_i \in \mathbb{R}, \text{ for } i = 1, \ldots, \ell \end{array} \right\}.$$

Then, Theorem 2 states that

$$\mathbb{H}_r = \mathfrak{C}_0^{\mathbb{H}} \cup \mathfrak{C}_1^{\mathbb{H}} \cup \cdots \cup \mathfrak{C}_{\lfloor r/2 \rfloor}^{\mathbb{H}}. \tag{21}$$

We emphasize that, as in the general case without particular structure, the decomposition (21) is not a partition, since the sets $\mathfrak{C}_i^{\mathbb{H}}$ are not disjoint.

The case of the structures $Sym_r, S\mathbb{H}_r, Even_r^*, Odd_r^*, Pal_r^*$, and $Apal_r^*$ is similar and the decomposition is obtained through the same number of subsets as in (21), using (5), (15), (16), (17), (18), and (19), respectively, and replacing $*$ by $\top$ and allowing $a_i, b_i \in \mathbb{C}$ for the case Sym_r.

For the remaining structures $SSym_r$, $Even_r^\top$, $Odd_r^\top$, $Pal_r^\top$, and $Apal_r^\top$, we also have to replace $*$ by $\top$ and allow $a_i, b_i \in \mathbb{C}$. In addition, the decomposition of the set of structured matrices of rank r consists of only one set, since the value of s is fixed by $s = r/2$ if r is even, or by $s = (r-1)/2$ if r is odd.

Next, we introduce a parameterization for the sets of $n \times n$ structured matrix pencils with rank at most r by introducing a parameterization for each of the subsets that give rise to the decompositions above.

Definition 4 (Parameterization of the set of Hermitian matrix pencils with rank at most r).
Let $r \in \mathbb{N}$. For each $s = 0, 1, \ldots, \lfloor r/2 \rfloor$ we define the map $\Phi_s : \mathbb{R}^{2\ell} \times \mathbb{C}^{(r+s)n} \longrightarrow \mathfrak{C}_s^{\mathbb{H}}$ with $\ell = r - 2s$ as follows: For $x \in \mathbb{C}^{(r+s)n}$ decomposed as $x = \left[\,\alpha\middle|\beta\middle|\gamma\middle|\delta\,\right]^\top$ with

$$\begin{aligned}
\alpha &= \left[\,\alpha_{11} \cdots \alpha_{n1}\middle|\cdots\middle|\alpha_{1\ell} \cdots \alpha_{n\ell}\,\right] \in \mathbb{C}^{1\times \ell n},\\
\beta &= \left[\,\beta_{11} \cdots \beta_{n1}\middle|\cdots\middle|\beta_{1s} \cdots \beta_{ns}\,\right] \in \mathbb{C}^{1\times sn},\\
\gamma &= \left[\,\gamma_{11} \cdots \gamma_{n1}\middle|\cdots\middle|\gamma_{1s} \cdots \gamma_{ns}\,\right] \in \mathbb{C}^{1\times sn},\\
\delta &= \left[\,\delta_{11} \cdots \delta_{n1}\middle|\cdots\middle|\delta_{1s} \cdots \delta_{ns}\,\right] \in \mathbb{C}^{1\times sn},
\end{aligned}$$

we set

$$\begin{aligned}
&\Phi_s\left(\left[\,a_1\ b_1 \cdots a_\ell\ b_\ell\,\right]^\top, x\right)\\
&\quad = (a_1 + \lambda b_1)u_1u_1^* + \cdots + (a_\ell + \lambda b_\ell)u_\ell u_\ell^* + v_1w_1^* + \cdots + v_sw_s^* + w_1v_1^* + \cdots + w_sv_s^*,
\end{aligned}$$

where $u_1, \ldots, u_\ell, v_1, \ldots, v_s, w_1, \ldots, w_s$ are defined by

$$\begin{aligned}
u_i &= \left[\,\alpha_{1i} \cdots \alpha_{ni}\,\right]^\top, && \text{for } i = 1, \ldots, \ell,\\
v_j &= \left[\,\beta_{1j} \cdots \beta_{nj}\,\right]^\top, && \text{for } j = 1, \ldots, s,\\
\text{and}\quad w_j &= \left[\,\gamma_{1j} + \lambda\delta_{1j} \cdots \gamma_{nj} + \lambda\delta_{nj}\,\right]^\top, && \text{for } j = 1, \ldots, s.
\end{aligned}$$

Remark 7 For the other structures, the parameterization is defined analogously. More precisely, let $\mathbb{S}_r$ be the set of $n \times n$ matrix pencils with rank at most r having the structure $\mathbb{S}$ and assume that $\mathbb{S}_r = \mathfrak{C}_{i_1}^{\mathbb{S}} \cup \ldots \cup \mathfrak{C}_{i_k}^{\mathbb{S}}$ is a decomposition into smaller subsets, where the number k depends on the structure and on r. Then the parameterization of $\mathbb{S}_r$ is a tuple of continuous, surjective maps $\Phi_s : \mathbb{R}^{p_s} \times \mathbb{C}^{m_s} \longrightarrow \mathfrak{C}_s^{\mathbb{S}}$, for $s \in \{i_1, \ldots, i_k\}$, and where p_s, m_s depend on s. (In fact, these parameterizations are not only continuous, but are polynomials either in the entries of x or in the real and imaginary parts of the entries of x.)

For the Hermitian, skew-Hermitian, $*$-even, $*$-odd, $*$-palindromic, and $*$-anti-palindromic structures, we have $k = \lfloor r/2 \rfloor + 1$, $\{i_1, \ldots, i_k\} = \{0, 1, \ldots, \lfloor r/2 \rfloor\}$, $p_s = 2(r-2s)$, and $m_s = (r+s)n$, while for the symmetric structure, we have $k = \lfloor r/2 \rfloor + 1$, $\{i_1, \ldots, i_k\} = \{0, 1, \ldots, \lfloor r/2 \rfloor\}$, $p_s = 0$, and $m_s = 2(r-2s) + (r+s)n$.

In the remaining structures, we have $k = 1, s = \lfloor r/2 \rfloor$, $p_s = 0$, and $m_s = \lfloor 3r/2 \rfloor n$. For example, for the case of $\top$-even pencils, the map

$$\Phi : \mathbb{C}^{\lfloor \frac{3r}{2} \rfloor n} \longrightarrow Even_r^\top \tag{22}$$

is defined by $\Phi(x) = E(\lambda)$, with $E(\lambda)$ as in (7), and where u, v_j, w_j, for $j = 1, \ldots, \lfloor r/2 \rfloor$, are defined as follows: if $x \in \mathbb{C}^{\lfloor 3r/2 \rfloor n}$ is decomposed as $x = \left[\,\alpha\middle|\beta\middle|\gamma\middle|\delta\,\right]^\top$, where

$$\begin{aligned}
\alpha &= \left[\,\alpha_1 \cdots \alpha_{\ell n}\,\right] \in \mathbb{C}^{1\times \ell n},\\
\beta &= \left[\,\beta_{11} \cdots \beta_{n1}\middle|\cdots\middle|\beta_{1,\lfloor r/2 \rfloor n} \cdots \beta_{n,\lfloor r/2 \rfloor n}\,\right] \in \mathbb{C}^{1\times \lfloor r/2 \rfloor n},\\
\gamma &= \left[\,\gamma_{11} \cdots \gamma_{n1}\middle|\cdots\middle|\gamma_{1,\lfloor r/2 \rfloor n} \cdots \gamma_{n,\lfloor r/2 \rfloor n}\,\right] \in \mathbb{C}^{1\times \lfloor r/2 \rfloor n},\\
\delta &= \left[\,\delta_{11} \cdots \delta_{n1}\middle|\cdots\middle|\delta_{1,\lfloor r/2 \rfloor n} \cdots \delta_{n,\lfloor r/2 \rfloor n}\,\right] \in \mathbb{C}^{1\times \lfloor r/2 \rfloor n},
\end{aligned}$$

with $\ell = r - 2\lfloor r/2 \rfloor$, then

$$\begin{aligned} u &= \begin{bmatrix} \alpha_1 \cdots \alpha_{\ell n} \end{bmatrix}^\top, \\ v_j &= \begin{bmatrix} \beta_{1j} \cdots \beta_{nj} \end{bmatrix}^\top, && \text{for } j = 1, \ldots, \lfloor r/2 \rfloor, \\ w_j &= \begin{bmatrix} \gamma_{1j} + \lambda\delta_{1j} \cdots \gamma_{nj} + \lambda\delta_{nj} \end{bmatrix}^\top, && \text{for } j = 1, \ldots, \lfloor r/2 \rfloor. \end{aligned}$$

Note that α is void if r is even, because we then have $\ell = 0$.

We highlight that, in all cases, the map Φ_s is surjective.

4.4 Generic perturbation theory for pencils with symmetry structures

In this subsection, we will develop the eigenvalue perturbation theory of regular matrix pencils with symmetry structures under structure-preserving perturbations with the help of the parameterizations from Section 4.3. The sets of the form $\mathbb{R}^{p_s} \times \mathbb{C}^{m_s}$ that appear as domains for the parameterizations constructed analogous to Definition 4 will be identified with the set $\mathbb{R}^{p_s+2m_s}$ by splitting the variables in $\mathbb{C}$ into their real and imaginary parts. As noted before, this detour via the reals is necessary when symmetry structures involving complex conjugation are considered. When we deal with symmetry structures only involving the complex transpose, but not complex conjugation, then we have $p_s = 0$ and we can express genericity in terms of complex polynomials only.

Theorem 17 (Generic change under low-rank perturbations of Hermitian pencils). *Let $L(\lambda)$ be a regular $n \times n$ Hermitian matrix pencil and let $\lambda_1, \ldots, \lambda_\kappa$ denote the pairwise distinct eigenvalues of $L(\lambda)$ having the partial multiplicities $n_{i,1} \geqslant \ldots \geqslant n_{i,g_i} > 0$ for $i = 1, \ldots, \kappa$, respectively. Furthermore, let r be a positive integer, let $0 \leqslant s \leqslant \lfloor r/2 \rfloor$, and let Φ_s be the map in Definition 4 and $\ell = r - 2s$. Then, there exists a generic set $\mathcal{G}_s$ in $\mathbb{R}^{2\ell} \times \mathbb{C}^{(r+s)n}$ such that, for all $E(\lambda) \in \Phi_s(\mathcal{G}_s)$, the perturbed pencil $L + E$ is regular and the partial multiplicities of $L + E$ at λ_i are given by $n_{i,r+1} \geqslant \cdots \geqslant n_{i,g_i}$. (In particular, if $r \geqslant g_i$ then λ_i is not an eigenvalue of $L + E$.) Furthermore, all eigenvalues of $L + E$ that are different from those of L are simple.*

Proof By Theorem 15 (applied for the case $\mathbb{F} = \mathbb{R}$ and $m = 2\ell + 2(r + s)n$ in accordance with the identification $\mathbb{R}^{2\ell+2(r+s)n} = \mathbb{R}^{2\ell} \times \mathbb{C}^{(r+s)n}$) it is sufficient to show, for each $i = 1, \ldots, \kappa$, the existence of one particular $x_i \in \mathbb{R}^{2\ell} \times \mathbb{C}^{(r+s)n}$ of arbitrarily small norm such that, with the corresponding perturbation pencil $E(\lambda) = \Phi_s(x_i)$, the perturbed pencil $L+E$ has precisely the partial multiplicities $n_{i,r+1} \geqslant \cdots \geqslant n_{i,g_i}$ at λ_0 and all eigenvalues of $L + E$ that are different from those of L are simple. Since genericity of sets is invariant under multiplication with invertible matrices, it suffices to consider the case when L is given in Hermitian canonical form (Theorem 1). To this end, we distinguish three cases and for the ease of notation we will from now on drop the dependence on i of the geometric multiplicity and partial multiplicities of λ_i, thus writing g and $n_1, \ldots, n_g$ instead of g_i and $n_{i,1}, \ldots, n_{i,g_i}$.

Case (1): $\lambda_i \in \mathbb{R}$. Then we can assume, without loss of generality, that L is of the form

$$L(\lambda) = \operatorname{diag}\left(\sigma_1 R J_{n_1}(\lambda_i - \lambda), \ldots, \sigma_g R J_{n_g}(\lambda_i - \lambda), \widetilde{L}(\lambda)\right),$$

where λ_i is not an eigenvalue of $\widetilde{L}(\lambda)$. Let $F_\nu = \widetilde{u}\widetilde{u}^*$, with $\widetilde{u} = e_1 \in \mathbb{C}^\nu$, and $G_{\nu,\widetilde{\nu}} = \widetilde{v}\widetilde{w}^* + \widetilde{w}\widetilde{v}^*$, with $\widetilde{v} = e_{\widetilde{\nu}+1}, \widetilde{w} = \frac{1}{2}e_1 \in \mathbb{C}^{\nu+\widetilde{\nu}}$, i.e., F_ν is the $\nu \times \nu$ matrix that is everywhere zero except for $F_\nu(1,1) = 1$, and $G_{\nu,\widetilde{\nu}}$ is the $(\nu + \widetilde{\nu}) \times (\nu + \widetilde{\nu})$ matrix which is everywhere zero except for $G_{\nu,\widetilde{\nu}}(1, \widetilde{\nu} + 1) = G_{\nu,\widetilde{\nu}}(\widetilde{\nu} + 1, 1) = 1$. Note that both F_ν and $G_{\nu,\widetilde{\nu}}$ are Hermitian matrices.

First, let us assume that $r \leqslant g$. Then, we set

$$E(\lambda) = \operatorname{diag}(\alpha_1 F_{n_1}, \ldots, \alpha_\ell F_{n_\ell}, \beta_1 G_{n_{\ell+1},n_{\ell+2}}, \ldots, \beta_s G_{n_{r-1},n_r}, 0) + \lambda 0_{n\times n} \tag{23}$$

for some values $\alpha_1, \ldots, \alpha_\ell, \beta_1, \ldots, \beta_s \in \mathbb{R}$ to be specified later. The matrix pencil $E(\lambda)$ has rank r and, from the construction of F_m and $G_{m,\widetilde{m}}$, it is clear that $E(\lambda)$ can be written in the form (1) (e.g., with $a_1 = \ldots = a_\ell = 1, b_1 = \ldots = b_\ell = 0$). Thus, we have $E(\lambda) \in \mathfrak{C}_s^{\mathbb{H}}$. Then, since Φ_s is surjective, there exists some $x \in \mathbb{R}^{2\ell} \times \mathbb{C}^{(r+s)n}$ such that $\Phi_s(x) = E(\lambda)$, and provided that the parameters α_i, β_j are sufficiently small, it is clear that this x can be chosen to be of arbitrarily small norm. (This uses the fact that Φ_s is not injective, i.e., we can "split up" the small values α_i, β_j and put them into the parameters a_i, b_i, u_j, v_k, w_k of Definition 4 in such a way that all entries of x are small.) Moreover, the nonzero partial multiplicities of $L+E$ at λ_i are $(n_{r+1}, \ldots, n_g)$. To see this, note first that only the first r blocks of L are modified so, in particular, $L+E$ contains $g-r$ Jordan blocks associated with λ_i with sizes $(n_{r+1}, \ldots, n_g)$. (If $g = r$, then this means that λ_i is not an eigenvalue of $L+E$.) Furthermore, the part of the pencil $L+E$ corresponding to the first r blocks of L is block diagonal, and with the help of the Laplace expansion it is easy to verify that the characteristic polynomials of its diagonal blocks $RJ_{n_j}(\lambda_i - \lambda) + \alpha_j F_{n_j}$, $j = 1, \ldots, \ell$, and $\mathrm{diag}(RJ_{n_{\ell+2j-1}}(\lambda_i - \lambda), RJ_{n_{\ell+2j}}(\lambda_i - \lambda)) + \beta_j G_{n_{\ell+2j-1}, n_{\ell+2j}}$, for $j = 1, \ldots, s$, are given by

$$(-1)^{\varrho_j}\big((\lambda - \lambda_i)^{n_j} - \alpha_j\big),\ j = 1, \ldots, \ell \quad \text{and} \quad (-1)^{\varrho_{\ell+j}}\big((\lambda - \lambda_i)^{n_{\ell+2j-1}+n_{\ell+2j}} - \beta_j^2\big),\ j = 1, \ldots, s,$$

respectively, where $\varrho_1, \ldots, \varrho_{\ell+s}$ are integers only depending on the sizes $n_1, \ldots, n_r$ and the signs $\sigma_1, \ldots, \sigma_r$. Thus, the eigenvalues of this diagonal blocks lie on circles centered around λ_i with radii $|\alpha_1|^{\frac{1}{n_1}}, \ldots, |\alpha_\ell|^{\frac{1}{n_\ell}}$, $|\beta_1|^{\frac{2}{n_{\ell+1}+n_{\ell+2}}}, \ldots, |\beta_s|^{\frac{2}{n_{r-1}+n_r}}$. Clearly, choosing the parameters $\alpha_1, \ldots, \alpha_\ell$ and $\beta_1, \ldots, \beta_s$ appropriately, we can guarantee that all eigenvalues of $L+E$ that are different from those of L are simple.

Now assume that $g < r$. If $g \leqslant \ell$ or if g has the same parity as ℓ (i.e. $g - \ell$ is even) then we define $E(\lambda)$ as in (23), where we interpret $n_j = 0$ for $j > g$. Then $E(\lambda)$ has rank less than r, but still can be written in the form (1). Indeed, if $g \leqslant \ell$ then we set $u_i = 0$ for $i > g$ and $v_j = w_j = 0$ for $j = 1, \ldots, s$, and if $g > \ell$ then we set $v_j = w_j = 0$ for $j = \frac{g-\ell}{2} + 1, \ldots, s$. If, on the other hand, $g > \ell$ and g has the opposite parity to ℓ, i.e. $g - \ell = 2\kappa + 1$, then we slightly alter the pencil in (23) to

$$E(\lambda) = \mathrm{diag}(\alpha_1 F_{n_1}, \ldots, \alpha_\ell F_{n_\ell}, \beta_1 G_{n_{\ell+1}, n_{\ell+2}}, \ldots, \beta_\kappa G_{n_{\ell+2\kappa-1}, n_{\ell+2\kappa}}, \beta_{\kappa+1} F_{n_g}, 0) + \lambda 0_{n\times n}.$$

Also this pencil can be written in the form (1), noting that a block F_ν can also be represented in the form $\widetilde{v}\widetilde{w}^* + \widetilde{w}\widetilde{v}^*$ by choosing $\widetilde{v} = \widetilde{w} = \frac{1}{2}e_1$. In all cases, the perturbed pencil $L+E$ does not have the eigenvalue λ_i and all eigenvalues different from those of L are simple if the parameters α_i and β_j are chosen appropriately.

Case (2): $\lambda_i = \infty$. This case follows by applying the already proved Case (1) to the reversal of the pencil L.

Case (3): $\lambda_i \in \mathbb{C} \setminus \mathbb{R}$. In the following we denote λ_i by μ, for consistency with the notation used before. In this case, the Hermitian canonical form contains $2k \times 2k$ coupled blocks associated with μ and $\overline{\mu}$, each of size $k \times k$, as indicated in the proof of Theorem 2. Then, we may assume that $L(\lambda)$ is of the form

$$\begin{aligned} L(\lambda) = \mathrm{diag}\,\big(&R\,\mathrm{diag}(J_{n_1}(\overline{\mu} - \lambda), J_{n_1}(\mu - \lambda)), \ldots, \\ &R\,\mathrm{diag}(J_{n_g}(\overline{\mu} - \lambda), J_{n_g}(\mu - \lambda)), \widetilde{L}(\lambda))\Big), \end{aligned}$$

where, again, neither μ nor $\overline{\mu}$ are eigenvalues of $\widetilde{L}(\lambda)$. Furthermore, we assume that $g \geqslant r$. (The subcase $g < r$ can be treated analogously to the corresponding subcase in Case (1).)

Let $\widetilde{F}_{2\nu} = uu^*$, with $u = e_1 + e_{\nu+1} \in \mathbb{C}^{2\nu}$ and $\widetilde{G}_{2\nu, 2\widetilde{\nu}} = vw^* + wv^*$, with $v = e_{2\nu+1} + e_{2\nu+\widetilde{\nu}+1} \in \mathbb{C}^{2(\nu+\widetilde{\nu})}$, $w = \frac{1}{2}(e_1 + e_{\nu+1}) \in \mathbb{C}^{2(\nu+\widetilde{\nu})}$. Thus $\widetilde{F}_{2\nu}$ is the $2\nu \times 2\nu$ matrix whose entries are all zero except for the entries in the positions $(1,1)$, $(1, \nu+1)$, $(\nu+1, 1)$ and $(\nu+1, \nu+1)$, which are all equal to 1, and $\widetilde{G}_{2\nu, 2\widetilde{\nu}}$ is the $2(\nu+\widetilde{\nu}) \times 2(\nu+\widetilde{\nu})$ matrix whose entries are all zero except for the entries in the positions $(1, 2\nu+1)$, $(1, 2\nu+\widetilde{\nu}+1)$, $(\nu+1, 2\nu+1)$, $(\nu+1, 2\nu+\widetilde{\nu}+1)$, $(2\nu+1, 1)$, $(2\nu+1, \nu+1)$, $(2\nu+\widetilde{\nu}+1, 1)$, and $(2\nu+\widetilde{\nu}+1, \nu+1)$ which are all equal to 1. Let $E(\lambda)$ be

$$E(\lambda) = \mathrm{diag}(\alpha_1 \widetilde{F}_{2n_1}, \ldots, \alpha_\ell \widetilde{F}_{2n_\ell}, \beta_1 \widetilde{G}_{2n_{\ell+1}, 2n_{\ell+2}}, \ldots, \beta_s \widetilde{G}_{2n_{r-1}, 2n_r}, 0) + \lambda 0_{n\times n}, \tag{24}$$

where the real parameters $\alpha_1, \ldots, \alpha_\ell, \beta_1, \ldots, \beta_s$ will be specified later.

By construction, $\operatorname{rank} E = r$ and $E(\lambda) \in \mathfrak{C}_s^{\mathbb{H}}$. Again, since Φ_s is surjective, there is some $x \in \mathbb{R}^{2\ell} \times \mathbb{C}^{(r+s)n}$ such that $\Phi_s(x) = E(\lambda)$. (Again, x can be chosen to be of arbitrarily small norm provided that the parameters α_i, β_j are sufficiently small.) It remains to see that the partial multiplicities of $L+E$ at μ are $(n_{r+1}, \ldots, n_g)$ and that all eigenvalues of $L+E$ that are different from those of L are simple. Again, since the smallest $g-r$ Jordan blocks associated with μ in $L(\lambda)$ are not modified by the perturbation $E(\lambda)$, they will stay in the WCF of $L+E$, so $(n_{r+1}, \ldots, n_g)$ is a sublist of the list of partial multiplicities of $L+E$ at λ_0.

With the help of the Laplace expansion, one can easily show that the determinant of each block $R\,\mathrm{diag}(J_{n_i}(\overline{\mu}-\lambda), J_{n_i}(\mu-\lambda)) + \alpha_i \widetilde{F}_{n_i,n_i}$ is given by

$$\chi_i(\lambda) = (-1)^{\varrho_i}\big((\lambda-\mu)^{n_i}(\lambda-\overline{\mu})^{n_i} - \alpha_i(\lambda-\mu)^{n_i} - \alpha_i(\lambda-\overline{\mu})^{n_i}\big),$$

where ϱ_i is an integer only depending on n_i. It was shown in [29, Example 4.2] that such a polynomial has simple roots (and clearly these are different from μ and $\overline{\mu}$) if α_i is chosen such that $|\alpha_i| \leqslant \frac{|\mu-\overline{\mu}|^{n_i}}{2}$.

On the other hand, again with the help of the Laplace expansion and performing tedious but elementary calculations, one finds that the determinant of each block

$$R\,\mathrm{diag}(J_{n_{\ell+2j-1}}(\overline{\mu}-\lambda), J_{n_{\ell+2j-1}}(\mu-\lambda), J_{n_{\ell+2j}}(\overline{\mu}-\lambda), J_{n_{\ell+2j}}(\mu-\lambda)) + \beta_j G_{n_{\ell+2j-1},n_{\ell+2j}}$$

is given by

$$\begin{aligned}
&\chi_{\ell+j}(\lambda)\\
&= (-1)^{\varrho_j}\big((\lambda-\mu)^{n_{\ell+2j-1}+n_{\ell+2j}}(\lambda-\overline{\mu})^{n_{\ell+2j-1}+n_{\ell+2j}} - \beta_j^2(\lambda-\mu)^{n_{\ell+2j-1}}(\lambda-\overline{\mu})^{n_{\ell+2j}}\\
&\qquad - \beta_j^2(\lambda-\mu)^{n_{\ell+2j}}(\lambda-\overline{\mu})^{n_{\ell+2j-1}} - \beta_j^2(\lambda-\mu)^{n_{\ell+2j-1}+n_{\ell+2j}} - \beta_j^2(\lambda-\overline{\mu})^{n_{\ell+2j-1}+n_{\ell+2j}}\big).
\end{aligned}$$

If $|\beta_j|$ is sufficiently small, then $\chi_{\ell+j}$ is guaranteed to have only simple roots (that are clearly all different from μ and $\overline{\mu}$). Indeed, assume that λ is a common root of $\chi_{\ell+j}$ and $\chi'_{\ell+j}$. Then multiplying the equation $\chi_{\ell+j}=0$ with $(\lambda-\mu)(\lambda-\overline{\mu})$ and using twice the equation $\chi_{\ell+j}(\lambda)=0$, we obtain that

$$\beta^2\big((\lambda-\mu)^{n_{\ell+2j-1}+n_{\ell+2j}} + (\lambda-\overline{\mu})^{n_{\ell+2j-1}+n_{\ell+2j}}\big) = 0,$$

which implies $|\lambda-\mu| = |\lambda-\overline{\mu}|$. Using the fact that roots of polynomials depend continuously on the coefficients of the polynomials it follows that β_j can be chosen sufficiently small such that the roots of $\chi_{\ell+j}$ have a distance from either μ or $\overline{\mu}$ less than $\frac{|\mu-\overline{\mu}|}{2}$ which then contradicts $|\lambda-\mu| = |\lambda-\overline{\mu}|$.

Therefore, choosing $\alpha_1, \ldots, \alpha_\ell, \beta_1, \ldots, \beta_s$ appropriately, we can guarantee that there are $n_1 + \cdots + n_r$ simple eigenvalues close to μ or $\overline{\mu}$, respectively, corresponding to the r Jordan blocks that were perturbed by E. Indeed, after having chosen α_1, let δ_1 denote the smallest distance of a root of χ_1 to the set $\{\mu, \overline{\mu}\}$. Then choose α_2 so small that the (simple) roots of χ_2 are located within circles of a radius less then δ_1 around μ or $\overline{\mu}$, respectively. Then let δ_2 be the smallest distance of a root of χ_2 to the set $\{\mu, \overline{\mu}\}$ and continue in this manner choosing $\alpha_3, \ldots, \alpha_\ell, \beta_1, \ldots, \beta_s$ such that all eigenvalues of $L+E$ that are different from the eigenvalues of L are simple.

Theorem 18 (Generic change under low-rank perturbations of symmetric pencils). *Let $L(\lambda)$ be a regular $n\times n$ symmetric matrix pencil and let $\lambda_1, \ldots, \lambda_\kappa$ denote the pairwise distinct eigenvalues of $L(\lambda)$ having the partial multiplicities $n_{i,1} \geqslant \ldots \geqslant n_{i,g_i} > 0$ for $i = 1, \ldots, \kappa$, respectively. Furthermore, let r be a positive integer, let $0 \leqslant s \leqslant \lfloor r/2 \rfloor$ and let Φ_s be the map as in Remark 7 and $\ell = r - 2s$. Then there exists a generic set $\mathcal{G}_s$ in $\mathbb{C}^{2\ell+(r+s)n}$ such that, for all $E(\lambda) \in \Phi_s(\mathcal{G}_s)$, the perturbed pencil $L+E$ is regular and the partial multiplicities of $L+E$ at λ_i are given by $n_{i,r+1} \geqslant \cdots \geqslant n_{i,g_i}$. (In particular, if $r \geqslant g_i$ then λ_i is not an eigenvalue of $L+E$.) Furthermore, all eigenvalues of $L+E$ that are different from those of L are simple.*

Proof The proof is similar to the one of Theorem 17 now applying Theorem 15 for the case $\mathbb{F} = \mathbb{C}$ and $m = 2\ell + (r+s)n$. The only difference comes from the blocks in the symmetric canonical form, which are different to the ones in the Hermitian canonical form. In particular, in the symmetric case there is no need to distinguish between real and complex eigenvalues, so we can follow exactly the same arguments as in the proof of Theorem 17 for an eigenvalue $\lambda_i \in \mathbb{R}$, which now is valid for a general $\lambda_i \in \mathbb{C}$.

Theorem 19 (Generic change under low-rank perturbations of $\top$-alternating pencils). *Let $L(\lambda)$ be a regular $n \times n$ $\top$-alternating matrix pencil and let $\lambda_1, \ldots, \lambda_\kappa$ denote the pairwise distinct eigenvalues of $L(\lambda)$ having the partial multiplicities $n_{i,1} \geqslant \ldots \geqslant n_{i,g_i} > 0$ for $i = 1, \ldots, \kappa$, respectively. Furthermore, let r be a positive integer and let Φ be the map as in Remark 7, i.e., Φ is as in* (22)*. Then, there exists a generic set $\mathcal{G}$ in $\mathbb{C}^{\lfloor \frac{3r}{2} \rfloor n}$ such that for all $E(\lambda) \in \Phi(\mathcal{G})$, the perturbed pencil $L + E$ is regular and the partial multiplicities of $L + E$ at λ_i are the ones given in Table* 1*, where* (P) *is the following property:*

$$n_{i,r} = n_{i,r+1} = \cdots = n_{i,r+d} > n_{i,r+d+1}, \qquad \text{with } d \text{ odd.} \tag{P}$$

Structure	e-val λ_i	case	multiplicities
$\top$-even	$\lambda_i = 0$	$n_{i,r+1}$ odd and (P) holds	$(n_{i,r+1}+1, n_{i,r+2}, \ldots, n_{i,g_i})$
		otherwise	$(n_{i,r+1}, n_{i,r+2}, \ldots, n_{i,g_i})$
	$\lambda_i = \infty$	r even, $n_{i,r+1}$ even, and (P) holds	$(n_{i,r+1}+1, n_{i,r+2}, \ldots, n_{i,g_i})$
		r even, otherwise	$(n_{i,r+1}, n_{i,r+2}, \ldots, n_{i,g_i})$
		r odd, $n_{i,r+1}$ even, and (P) holds	$(n_{i,r+1}+1, n_{i,r+2}, \ldots, n_{i,g_i}, 1)$
		r odd, otherwise	$(n_{i,r+1}, n_{i,r+2}, \ldots, n_{i,g_i}, 1)$
	$\lambda_i \in \mathbb{C} \setminus \{0\}$	all	$(n_{i,r+1}, n_{i,r+2}, \ldots, n_{i,g_i})$
$\top$-odd	$\lambda_i = 0$	r even, $n_{i,r+1}$ even, and (P) holds	$(n_{i,r+1}+1, n_{i,r+2}, \ldots, n_{i,g_i})$
		r even, otherwise	$(n_{i,r+1}, n_{i,r+2}, \ldots, n_{i,g_i})$
		r odd, $n_{i,r+1}$ even, and (P) holds	$(n_{i,r+1}+1, n_{i,r+2}, \ldots, n_{i,g_i}, 1)$
		r odd, otherwise	$(n_{i,r+1}, n_{i,r+2}, \ldots, n_{i,g_i}, 1)$
	$\lambda_i = \infty$	$n_{i,r+1}$ odd and (P) holds	$(n_{i,r+1}+1, n_{i,r+2}, \ldots, n_{i,g_i})$
		otherwise	$(n_{i,r+1}, n_{i,r+2}, \ldots, n_{i,g_i})$
	$\lambda_i \in \mathbb{C} \setminus \{0\}$	all	$(n_{i,r+1}, n_{i,r+2}, \ldots, n_{i,g_i})$

Table 1 Generic partial multiplicities at λ_i for rank-r $\top$-alternating perturbations

In particular, if $r \geqslant g_i$ then λ_i is not an eigenvalue of $L + E$. Furthermore, all eigenvalues of $L + E$ that are different from those of L are simple.

Proof For simplicity, we drop the dependence on i in the geometric and partial multiplicities of λ_i, i.e., we write g instead of g_i and $n_1 \geqslant \ldots \geqslant n_g$ instead of $n_{i,1} \geqslant \ldots \geqslant n_{i,g_i}$. We also replace λ_i by λ_0. We will only prove the case $g \geqslant r$ in full detail. (The case $g < r$ can be treated similarly by constructing an analogous perturbation of rank g instead of rank r, thus showing that λ_i is not an eigenvalue of the perturbed pencil.) We aim to apply Theorem 15 for the case $\mathbb{F} = \mathbb{C}$ to any single eigenvalue of the pencil. Here we make use of the fact that, in contrast to the Hermitian case, the set $Even_r^\top$ need not be decomposed into smaller sets that can be parameterized as in the sense of Definition 4, but the parameterization map Φ as in (22) is already a map onto $Even_r^\top$.

Case 1): property (P) *does not apply.* We first consider all cases except those where property (P) appears in Table 1. In these cases, it is sufficient to prove the existence of one particular perturbation $E(\lambda)$ of arbitrarily small norm which belongs to $Even_r^\top$.

Subcase 1a): $\lambda_0 \in \mathbb{C} \setminus \{0\}$. As in the proof of Theorem 17, we may assume that $L(\lambda)$ is given in $\top$-alternating canonical form. Let us start with the $\top$-even structure. In the $\top$-even canonical form, the blocks associated with λ_0 and $-\lambda_0$ appear in pairs [4, Th. 2.16]. Then, we may assume that $L(\lambda)$ is of the form:

$$L(\lambda) = \operatorname{diag}\big(R\operatorname{diag}(-\lambda I - J_{n_1}(\lambda_0), \lambda I - J_{n_1}(\lambda_0)), \ldots,\\ R\operatorname{diag}(-\lambda I - J_{n_g}(\lambda_0), \lambda I - J_{n_g}(\lambda_0)), \widetilde{L}(\lambda)\big),$$

where λ_0 is not an eigenvalue of $\widetilde{L}(\lambda)$.

Let $\widetilde{F}_{2m}$ and $\widetilde{G}_{2m,2n}$ be the same matrices as in the proof of Theorem 17, and let $E(\lambda)$ be the pencil in (24). Note that the pencil $E(\lambda)$ belongs to $Even_r^\top$. Therefore, there is some $x \in \mathbb{C}^{\lfloor \frac{3r}{2} \rfloor n}$ such that $\Phi(x) = E(\lambda)$, and x can be chosen to be of arbitrarily small norm provided that the parameters α_i, β_j are sufficiently small. Moreover, with similar reasonings to the ones in the proof of Theorem 17, it can be seen that the nonzero partial multiplicities at λ_0 in $L + E$ are $(n_{r+1}, \ldots, n_g)$, and that all eigenvalues of $L+E$ different from those of L are simple, if the parameters α_i, β_j in the pencil (24) have been chosen appropriately.

The case of the $\top$-odd structure can be addressed in a similar way, just multiplying by λ the perturbation blocks $\widetilde{F}_{2m}$ and $\widetilde{G}_{2m,2n}$ in (24).

Subcase 1b): $\lambda_0 = 0$ and $\top$-even structure. Recall that by assumption condition (P) is not satisfied. Then $L(\lambda)$ is of the form

$$L(\lambda) = \operatorname{diag}(L_0(\lambda), \widehat{L}_0(\lambda), \widetilde{L}(\lambda)),$$

where $L_0(\lambda)$ contains the Jordan blocks corresponding to the largest r partial multiplicities at 0 (namely, $n_1 \geqslant \cdots \geqslant n_r$), $\widehat{L}_0$ contains the blocks corresponding to the remaining partial multiplicities at 0, and $\widetilde{L}(\lambda)$ contains the information of the nonzero eigenvalues.

If n_{r+1} is even or n_{r+1} is odd, but $n_r = n_{r+1} = \cdots = n_{r+d} > n_{r+d+1}$ with d even (i.e., (P) does not hold), then the part $L_0(\lambda)$ is a direct sum of blocks of two types:

(i) a $2k \times 2k$ block of the form

$$\begin{bmatrix} & & & & & \lambda \\ & & & & \iddots & 1 \\ & & & \lambda & \iddots & \\ & & -\lambda & 1 & & \\ & \iddots & \iddots & & & \\ -\lambda & 1 & & & & \end{bmatrix}_{2k\times 2k}.$$

(ii) A pair of $(2k+1)\times(2k+1)$ blocks of the form $R\operatorname{diag}(J_{2k+1}(-\lambda), J_{2k+1}(\lambda))$.

This is a consequence of the fact that, in the $\top$-even canonical form, the Jordan blocks with odd size associated with the eigenvalue 0 are paired up, and can be matched up to form pairs as in blocks of the form (ii) (see [4, Th. 2.16]). Therefore, the blocks in $L_0(\lambda)$ with odd size larger than n_r (if any) are paired up, and, since d is even, also those of size n_r (if any) are paired up.

For each block of type (i) we can add a rank-1 perturbation by adding just one entry equal to α in the upper left corner of the block. This perturbation is of the form $uu^\top$ (actually, it is αF_{2k} in the proof of Theorem 17), and it is easily checked that the characteristic polynomial of the resulting perturbed block is given by $\chi = \lambda^{2k} - (-1)^k\alpha$ which means that its eigenvalues are simple and on a circle with center in the origin and radius $|\alpha|^{\frac{1}{2k}}$. For each pair of blocks of type (ii) we can add a rank-2 perturbation by adding entries equal to β in the positions $(1,1)$ and $(2k+2, 2k+2)$. This perturbation is of the form $\beta(vv^\top + ww^\top)$ with $v = e_1$ and $w = e_{2k+2}$, and, again, it is easily checked that the characteristic polynomial of the resulting perturbed block is given by $\chi = \lambda^{4k+2} + \beta^2$ which implies that its eigenvalues are simple and on a circle with center in the origin and radius $|\beta|^{\frac{1}{2k+1}}$. Therefore, choosing the parameters α and β appropriately, we can construct a rank-r perturbation $E(\lambda)$ of arbitrarily small norm which is $\top$-even such that the nonzero partial multiplicities at 0 in $L+E$ are $(n_{r+1}, \ldots, n_g)$ and such that all eigenvalues different from those of L are simple, as desired.

Subcase 1c): $\lambda_0 = 0$ and $\top$-odd structure. The case that r is even can be treated analogously to the previous subcase 1b), by just replacing 1 with λ in the nonzero entries of the perturbation constructed above. However, the case when r is odd deserves some more effort. The reason for this relies on the fact that any generic $\top$-odd perturbation with rank r and $r \leqslant n$ being odd contains 0 as an eigenvalue. This can be seen by looking at the summand $\lambda uu^\top$ in Theorem 7. In this case, the part $L_0(\lambda)$ is a direct sum of blocks of two types:

(i) A pair $2k \times 2k$ blocks of the form $R\operatorname{diag}(J_{2k}(\lambda), -J_{2k}(-\lambda))$.

(ii) A $(2k+1)\times(2k+1)$ block of the form

$$U_k := \begin{bmatrix} &&&&&&\lambda\\ &&&&&\lambda&1\\ &&&&\iddots&\iddots&\\ &&&\lambda&1&&\\ &&\lambda&-1&&&\\ &\iddots&\iddots&&&&\\ \lambda&-1&&&&& \end{bmatrix}_{(2k+1)\times(2k+1)}.$$

Since the $\top$-odd perturbation pencil $E(\lambda) = \lambda E_A + E_B$ has odd rank r, it follows that the skew-symmetric constant coefficient E_B has rank at most $r-1$. Then a straightforward dimension argument implies that the geometric multiplicity of the eigenvalue zero can change at most by $r-1$. Hence, the geometric multiplicity of the eigenvalue zero must be at least $g-r+1$. Since the list of partial multiplicities at zero must dominate the list $(n_{r+1}, \ldots, n_g)$, but also must contain, at least, $g-r+1$ elements, the algebraic multiplicity of $n_{r+1} + \cdots + n_g$ is not possible for the eigenvalue zero. Now, the (unique) list of partial multiplicities with minimal algebraic multiplicity that dominates $(n_{r+1}, \ldots, n_g)$ and is consistent with a geometric multiplicity of, at least, $g-r+1$ is the list $(n_{r+1}, \ldots, n_g, 1)$. Thus, by Theorem 15, it remains to construct one particular perturbation (of arbitrarily small norm) such that the perturbed pencil has this list of partial multiplicities at zero and such that all eigenvalues different from those of the unperturbed pencil are simple to show that this is the generic case.

Now, we are going to show how to construct such a $\top$-odd perturbation, like in the previous case. For each pair of blocks of type (i) we add the pencil $M_k := (\lambda+\alpha)e_1e_{2k+1}^\top + (\lambda-\alpha)e_{2k+1}e_1^\top$, with $e_1, e_{2k+1} \in \mathbb{C}^{4k\times 4k}$. It is straightforward to see that $\det(R\operatorname{diag}(J_{2k}(\lambda), -J_{2k}(-\lambda)) + M_k) = (\lambda^{2k} - \lambda + \alpha)(\lambda^{2k} - \lambda - \alpha)$, and that the roots of this polynomial are simple for $\alpha \neq 0$.

For each pair of blocks of type (ii), U_{k_1} and U_{k_2}, we add a rank-2 perturbation of the form $N_{k_1,k_2} := \beta(e_1e_{2k_1+2}^\top - e_{2k_1+2}e_1^\top)$, with $e_1, e_{2k_1+2} \in \mathbb{C}^{2(k_1+k_2+1)}$. It is straightforward to see that $\det\big(\operatorname{diag}(U_{k_1}, U_{k_2}) + N_{k_1,k_2}\big) = (-1)^{k_1+k_2}\lambda^{2(k_1+k_2+1)} + \beta^2$, so all the eigenvalues of the perturbed pencil are simple for $\beta \neq 0$.

Finally, we must include a rank-1 summand of the form $\lambda uu^\top$ to get a perturbation like in (10). This summand may correspond to either a pair of blocks of type (i) or to a block of type (ii) above. The first case is not possible, since otherwise condition (P) would hold. Therefore, we must have a block of the form $U_{\frac{n_r-1}{2}}$, and we add a perturbation $\gamma e_1 e_1^\top$, with $u_1 \in \mathbb{C}^{n_r}$. It is straightforward to see that $\det(U_{\frac{n_r-1}{2}} + \gamma e_1e_1^\top) = (-1)^{\frac{n_r-1}{2}}\lambda^{n_r} + \lambda\gamma$. Therefore, the perturbed pencil has $\lambda_0 = 0$ as a simple eigenvalue, and the remaining eigenvalues are simple for $\gamma \neq 0$.

As before, choosing the parameters α, β, and γ appropriately, we can construct a rank-r perturbation $E(\lambda)$ of arbitrarily small norm which is $\top$-odd such that the nonzero partial multiplicities at 0 in $L+E$ are $(n_{r+1}, \ldots, n_g, 1)$ and such that all eigenvalues different from those of L are simple.

Subcase 1d) $\lambda_0 = \infty$. For the eigenvalue $\lambda_0 = \infty$ we just apply the result for $\lambda_0 = 0$ in the reversal pencil (recall that $L(\lambda)$ is $\top$-even if and only if $\operatorname{rev} L(\lambda)$ is $\top$-odd).

Case 2) Property (P) *applies.* Note that in this case we must have $\lambda_0 = 0$ or $\lambda_0 = \infty$. We distinguish several subcases.

Subcase 2a) $\lambda_0 = 0$ *and* $\top$*-even structure.* This case corresponds to the first line of Table 1. By part (1) of Theorem 15 we know that, for any $\top$-even rank-r pencil E, there are at least $g-r$ partial multiplicities at 0 in $L+E$, say $m_{r+1} \geqslant \cdots \geqslant m_g$, with $m_i \geqslant n_i$, for $i = r+1, \ldots, g$. However, by the canonical form for $\top$-even pencils (see [4, Th. 2.16]), it is not possible that these partial multiplicities be exactly $n_{r+1} \geqslant \cdots \geqslant n_g$, because $L+E$ is $\top$-even, n_{r+1} is odd, and its value appears an odd number of times in the list $\{n_{r+1}, \ldots, n_g\}$, by property (P). Consequently, the algebraic multiplicity $n_{r+1} + \cdots + n_g$ for the eigenvalue λ_0 of $L+E$ is not possible in this case.

As in the previous case, we will instead construct a $\top$-even perturbation E of rank r and of arbitrarily small norm such that the algebraic multiplicity of $L+E$ at 0 is $\widetilde{a} = n_{r+1} + \cdots + n_g + 1$

and such that all eigenvalues that are different from those of L are simple. Then by part (2) of Theorem 15 there is a generic set $\mathcal{G} \subseteq \mathbb{C}^{\lfloor \frac{3r}{2} \rfloor n}$ such that for all corresponding perturbations E we have the situation outlined above.

As before, let us assume that $L(\lambda)$ is given in $\top$-even canonical form, so we can write it as

$$L(\lambda) = \operatorname{diag}\left(L_1(\lambda), R \operatorname{diag}(J_{n_r}(-\lambda), J_{n_r}(\lambda)), L_2(\lambda), \widetilde{J}(\lambda)\right),$$

where $L_1(\lambda)$ contains the first $r-1$ Jordan blocks associated with 0, $L_2(\lambda)$ contains the Jordan blocks associated with 0 and with sizes $n_{r+2}, \ldots, n_g$, and $\widetilde{J}(\lambda)$ corresponds to the nonzero eigenvalues (including infinity). Here, we used the fact that $n_r = n_{r+1}$ by property (P). Now, let $E(\lambda)$ be of the form

$$E(\lambda) = \operatorname{diag}(E_1, \gamma(e_1 + e_{n_r+2})(e_1 + e_{n_r+2})^\top, 0)$$

where $e_1, e_{n_r+2} \in \mathbb{C}^{2n_r}$ (with e_{n_r+2} interpreted as being the zero vector in the case $n_r = 1$), and where E_1 is of size $(n_1 + \cdots + n_{r-1}) \times (n_1 + \cdots + n_{r-1})$ and is constructed as a direct sum of blocks as explained above for the precedent case associated with the eigenvalue $\lambda_0 = 0$. (Namely, E_1 consists of a direct sum of rank-1 blocks with sizes $n_i \times n_i$ or rank-2 blocks with sizes $(n_i + n_{i+1}) \times (n_i + n_{i+1})$, depending on whether $L_1(\lambda)$ contains a $n_i \times n_i$ block, with n_i even, or a pair of blocks with sizes $n_i \times n_i$ and $n_{i+1} \times n_{i+1}$, with $n_{i+1} = n_i$ odd.) Then

$$\begin{aligned} \det(L + E) = &\det(L_1(\lambda) + E_1) \\ &\cdot \det\left(R \operatorname{diag}(J_{n_r}(-\lambda), J_{n_r}(\lambda)) + \gamma(e_1 + e_{n_r+2})(e_1 + e_{n_r+2})^\top)\right) \\ &\cdot \det L_2(\lambda) \cdot \det \widetilde{J}(\lambda). \end{aligned} \tag{25}$$

With straightforward computations (using again the Laplace expansion) it can be seen that

$$\det(R \operatorname{diag}(J_{n_r}(-\lambda), J_{n_r}(\lambda)) + \gamma(e_1 + e_{n_r+2})(e_1 + e_{n_r+2})^\top)) = \lambda^{n_r+1}(\lambda^{n_r-1} - 2\gamma) \tag{26}$$

if $n_r > 1$, or $\det(R \operatorname{diag}(J_{n_r}(-\lambda), J_{n_r}(\lambda)) + \gamma(e_1 + e_{n_r+2})(e_1 + e_{n_r+2})^\top)) = \lambda^2$ if $n_r = 1$, see Appendix A (see also [3, p. 663]). On the other hand, we have $\det L_2(\lambda) = \lambda^{n_{r+2}+\cdots+n_g}$. Thus, choosing the parameters α_i and β_j in E_1 and the parameter γ appropriately, we can construct a perturbation pencil E of arbitrarily small norm such that the algebraic multiplicity of $L+E$ at zero is $\widetilde{a} = n_{r+1} + \cdots + n_g + 1$ and such that all eigenvalues of $L + E$ that are different from those of L are simple, as desired.

However, the reader should keep in mind that part (2) of Theorem 15 only contains information on the generic algebraic multiplicity of the eigenvalue 0 of $L + E$ for a generic $\top$-even perturbation E. Unlike the previous cases, it is no longer true that combining the parts (1) and (2) of Theorem 15 forces the partial multiplicities of $L + E$ at 0 to be uniquely determined. Therefore, it is necessary to further investigate which lists of partial multiplicities at 0 are possible such that both (1) and (2) of Theorem 15 are satisfied. To this end, there are three possible situations:

(a) If $n_{r+1} - 1 \notin \{n_{r+2}, \ldots, n_g, 0\}$, then the only possible partial multiplicities are $n_{r+1} + 1 > n_{r+2} \geqslant \cdots \geqslant n_g$.
(b) If $n_{r+1} - 1 \in \{n_{r+2}, \ldots, n_g\}$, say $n_{r+1} - 1 = n_{r+d+1}$ (and d being minimal with this property), then there are two possible lists of partial multiplicities:
  (b1) $n_{r+1} + 1 > n_{r+2} \geqslant \cdots \geqslant n_g$, or
  (b2) $n_{r+1} = \cdots = n_{r+d} = n_{r+d+1} + 1 > n_{r+d+2} \geqslant \cdots \geqslant n_g$.
(c) If $n_r = 1$, then there are two possible lists of partial multiplicities:
  (c1) $(2, \underbrace{1, \ldots, 1}_{g-r-1})$, or
  (c2) $(\underbrace{1, \ldots, 1}_{g-r+1})$.

To see this, first note that, for any $x \in \mathcal{G}$, the algebraic multiplicity of $L + \Phi(x)$ at 0 is, exactly, $\widetilde{a}$. Since the partial multiplicities at 0 are $m_{r+1} \geqslant \cdots \geqslant m_g$, with $m_i \geqslant n_i$, for $i = r+1, \ldots, g$, then either one of the partial multiplicities $n_{r+1} \geqslant \cdots \geqslant n_g$ at 0 in L increases one unit, or either a new partial multiplicity equal to 1 appears after adding $E = \Phi(x)$. However, it is not possible to add or remove just one odd partial multiplicity after perturbing by E, since this would imply that the parity in the number of some of the odd-sized Jordan blocks associated with 0 would change, and this is not allowed by the $\top$-even structure. However, when increasing in one unit just one partial multiplicity at 0 in L, say n_i, either one odd partial multiplicity is added or removed, depending on the parity of n_i. In order for the number of each odd-sized Jordan blocks associated with 0 to stay as an even number, the only possibility is that either $n_i = n_{r+1}$ or $n_i = n_{r+1} - 1$. The first case corresponds to cases (a), (b1), and (c1) above, whereas the second one corresponds to cases (b2) and (c2).

With an argument identical to the one used in [4], we are going to prove that the generic partial multiplicities are just the ones in either (a), (b1), or (c1), which essentially reduce to the same behavior, namely, one of the largest remaining partial multiplicities increases in one unit.

Let us focus on case (b) first. By assumption on d being minimal, we have $(n_r =)n_{r+1} = \cdots = n_{r+d} > n_{r+d+1} \geqslant \cdots \geqslant n_g$ and $n_{r+1} - 1 = n_{r+d+1}$. Note that necessarily d is odd as we are in the case of property (P).

Assume that the change in case (b1) is not generic. Then the set $\mathcal{B} \subseteq \mathbb{C}^{\lfloor \frac{3r}{2} \rfloor n}$ of all x for which the partial multiplicities of $L + \Phi(x)$ at 0 are $n_{r+1} = \cdots = n_{r+d} = n_{r+d+1} + 1 > n_{r+d+2} \geqslant \cdots \geqslant n_g$ is not contained in a proper algebraic set. (Note that it must happen that $g - r \geqslant 2$.)

Now, let us define the map

$$\begin{array}{rccl} \widetilde{\Phi}_d : & (\mathbb{C}^n)^d & \longrightarrow & Even_d^\top \\ & u = (u_1, \ldots, u_d) & \mapsto & \widetilde{\Phi}(u) = u_1 u_1^\top + \cdots + u_d u_d^\top. \end{array}$$

and also consider the map

$$\begin{array}{rcl} \widetilde{\Phi} : \mathbb{C}^{\lfloor \frac{3r}{2} \rfloor n} \times (\mathbb{C}^n)^d & \longrightarrow & Even_{r+d}^\top \\ (x, u) & \mapsto & \widetilde{\Phi}(x, u) = \Phi(x) + \widetilde{\Phi}_d(u), \end{array}$$

Observe that the map $\widetilde{\Phi}$ may be different from the corresponding map $\mathbb{C}^{\lfloor \frac{3(r+d)}{2} \rfloor n} \longrightarrow Even_{r+d}^\top$ from (22). (Indeed, the dimensions of the domains do not coincide if r is odd.) Moreover, it is not even clear whether the map $\widetilde{\Phi}$ is surjective. Nevertheless, $\widetilde{\Phi}$ satisfies the hypotheses of Theorem 15 and thus by part (1) of Theorem 15 we have that for any $(x, u) \in \mathcal{B} \times (\mathbb{C}^n)^d$ the list of partial multiplicities of $L + \widetilde{\Phi}(x, u)$ at λ_i dominates the list $n_{r+d+1} + 1 > n_{r+d+2} \geqslant \cdots \geqslant n_g$. The key observation is now that by [5, Lemma 2.2] the set $\mathcal{B} \times (\mathbb{C}^n)^d$ is not contained in a proper algebraic subset of $\mathbb{C}^{\lfloor \frac{3r}{2} \rfloor n} \times (\mathbb{C}^n)^d$. If we can show that there exist (x_0, u_0) of arbitrarily small norm such that the partial multiplicities of $L + \widetilde{\Phi}(x_0, u_0)$ are $n_{r+d+1} \geqslant n_{r+d+2} \geqslant \cdots \geqslant n_g$, then by part (2) of Theorem 15 this hold for all $L + \widetilde{\Phi}(x, u)$ with (x, u) from a generic set $\widetilde{\mathcal{G}} \subseteq \mathbb{C}^{\lfloor \frac{3r}{2} \rfloor n} \times (\mathbb{C}^n)^d$. Since the list $n_{r+d+1} \geqslant n_{r+d+2} \geqslant \cdots \geqslant n_g$ does not dominate the list $n_{r+d+1} + 1 > n_{r+d+2} \geqslant \cdots \geqslant n_g$ this leads to a contradiction, because the sets $\widetilde{\mathcal{G}}$ and $\mathcal{B} \times (\mathbb{C}^n)^d$ must have a nonempty intersection, the first set being generic and the second set not being contained in a proper algebraic set.

Thus it remains to construct one particular example with the properties outlined above. To this end, note that, by assumption on k, the pencil L has the form

$$L(\lambda) = \operatorname{diag}\big(L_1(\lambda), R \operatorname{diag}(J_{n_r}(-\lambda), J_{n_r}(\lambda)), \ldots, R \operatorname{diag}(J_{n_r}(-\lambda), J_{n_r}(\lambda)), L_3(\lambda), \widetilde{J}(\lambda)\big),$$

where the block $R \operatorname{diag}(J_{n_r}(-\lambda), J_{n_r}(\lambda))$ is repeated $\frac{d+1}{2}$ times and $L_3(\lambda)$ contains the blocks associated with the partial multiplicities $n_{r+k+2} \geqslant \cdots \geqslant n_g$. Then the desired example for a perturbation that does the job is given by

$$E(\lambda) = \gamma \operatorname{diag}(E_1, e_1 e_1^\top + e_{n_r+1} e_{n_r+1}^\top, \ldots, e_1 e_1^\top + e_{n_r+1} e_{n_r+1}^\top, 0),$$

where E_1 is as before, the block $e_1 e_1^\top + e_{n_r+1} e_{n_r+1}^\top$ is repeated $\frac{d+1}{2}$ times, and $\gamma > 0$ is chosen sufficiently small. Indeed note that, as before, all blocks in L_1 and all the paired blocks of size n_r are perturbed in such a way that all eigenvalues lie on circles around zero, so that the partial multiplicities of $L + E$ at 0 are given by $n_{r+d+2} \geqslant \cdots \geqslant n_g$. Moreover, $E_1 + e_1 e_1^\top$ is a $\top$-even pencil of rank r and thus, using the surjectivity of Φ, there exists $x \in \mathbb{C}^{\lfloor \frac{3r}{2} \rfloor n}$ with $\Phi(x) = E_1 + e_1 e_1^\top$. Since the remaining part of E is of the form $u_1u_1^\top + \cdots + u_du_d^\top$, this implies the existence of $(x,u) \in \mathbb{C}^{\lfloor \frac{3r}{2} \rfloor n} \times (\mathbb{C}^n)^d$ with $\widetilde{\Phi}(x,u) = E$.

To show that in case (c) the subcase (c1) is generic can be shown by contradiction in a similar way. In this case, there would be two generic sets of $\top$-even perturbations with rank $r+1$ giving different behavior.

Subcase 2b) $\lambda_0 = 0$ and r odd and $\top$-odd structure. In this case, the situation is similar to the one in the previous subcase, but we are also in a situation similar to the one in Subcase 1c), i.e., the geometric multiplicity of the eigenvalue $\lambda_0 = 0$ after perturbation must be at least $g - r + 1$. But then, it is straightforward to show that the algebraic multiplicity $n_{r+1} + \cdots + n_g + 1$ is not possible in this case. Thus, we will construct a perturbation leading to the algebraic multiplicity $\widetilde{a} = n_{r+1} + \cdots + n_g + 2$. As before, let us assume that $L(\lambda)$ is given in $\top$-odd canonical form, so we can write it as

$$L(\lambda) = \operatorname{diag}\left(L_1(\lambda), R \operatorname{diag}(J_{n_r}(-\lambda), J_{n_r}(\lambda)), L_2(\lambda), \widetilde{J}(\lambda)\right),$$

where $L_1(\lambda)$ contains the first $r-1$ Jordan blocks associated with 0, $L_2(\lambda)$ contains the Jordan blocks associated with 0 and with sizes $n_{r+2}, \ldots, n_g$, and $\widetilde{J}(\lambda)$ corresponds to the nonzero eigenvalues (including infinity). Since the pencil $L_1(\lambda)$ does not have the property (P), we can construct a $\top$-odd perturbation $E_1(\lambda)$ as in subcase 1b) such that the eigenvalues of the perturbed pencil $L_1 + E_1$ are all nonzero and simple. It remains to perturb the block $R \operatorname{diag}(J_{n_r}(-\lambda), J_{n_r}(\lambda))$ in an appropriate way. For this we consider the perturbation $\gamma\lambda(e_1 + e_{n_r+2})(e_1 + e_{n_r+2})^\top$, with $e_1, e_{n_r+2} \in \mathbb{C}^{2n_r}$. It is straightforward to see that

$$\det(R \operatorname{diag}(-J_{n_r}(-\lambda), J_{n_r}(\lambda)) + \gamma\lambda(e_1 + e_{n_r+2})(e_1 + e_{n_r+2})^\top = \lambda^{n_r+2}\left(\lambda^{n_r-2} + 2\gamma\right) \tag{27}$$

(a proof of this identity is provided in Appendix A). Moreover, since, for $\lambda_0 = 0$, the perturbed subpencil has the same rank as the original one, the geometric multiplicity of $\lambda_0 = 0$ at the perturbed subpencil is the same one as in the original one, namely 2. Therefore, setting $E(\lambda) = \operatorname{diag}(E_1, \gamma\lambda(e_1 + e_{n_r+2})(e_1 + e_{n_r+2})^\top, 0)$ and choosing γ sufficiently small, the eigenvalues of the perturbed pencil $L+E$ are $\lambda_0 = 0$ with algebraic multiplicity $\widetilde{a}$, geometric multiplicity $g-r+1$, and the remaining eigenvalues are all simple, for $\gamma \neq 0$. The argument that the geometric multiplicities are as claimed in Table 1 is shown in a way that is analogous to the one in Subcase 1c).

Subcase 2c) $\lambda_0 = \infty$. The cases $\lambda_0 = \infty$ where property (P) appears in Table 1 can be proved from the cases $\lambda_0 = 0$ by using the reversal, which exchanges the roles of these two eigenvalues and takes $\top$-even pencils into $\top$-odd ones and viceversa. In particular, the case $\lambda_0 = \infty$ in the $\top$-odd structure can be obtained from the case $\lambda_0 = 0$ in the $\top$-even structure, and the case $\lambda_0 = \infty$ in the $\top$-even case can be obtained from the case $\lambda_0 = 0$ in the $\top$-odd structure.

Theorem 20 (Generic change under low-rank perturbations of $\top$-palindromic pencils). *Let $\lambda_1, \ldots, \lambda_\kappa$ be the pairwise distinct eigenvalues of the regular $n \times n$ $\top$-palindromic or $\top$-anti-palindromic matrix pencil $L(\lambda)$, having the nonzero partial multiplicities $n_{i,1} \geqslant n_{i,2} \geqslant \cdots \geqslant n_{i,g_i} > 0$, for $k = 1, \ldots, \kappa$, respectively. Furthermore, let $r > 0$ be an integer and let Φ be the map as in Remark 7. Then, there is a generic set $\mathcal{G}$ in $\mathbb{C}^{\lfloor \frac{3r}{2} \rfloor n}$ such that, for all $E(\lambda) \in \Phi(\mathcal{G})$, the perturbed pencil $L+E$ is regular and the partial multiplicities of $L+E$ at λ_0 are the ones given in Table 2, where (P) is the same property as in the statement of Theorem 19. (In particular, if $r \geqslant g_i$ then λ_i is not an eigenvalue of $L+E$.) Furthermore, all eigenvalues of $L+E$ different from those of L are simple.*

Proof We just prove the $\top$-palindromic case, since the $\top$-anti-palindromic one follows similar reasonings.

Let $L(\lambda)$ be a given $\top$-palindromic pencil satisfying the conditions in the statement, and let $E(\lambda)$ be another $\top$-palindromic pencil of the form (11). Let $\mathcal{C}_{+1}$ and $\mathcal{C}_{-1}$ be the Cayley transforms in (12). Then

$$\mathcal{C}_{+1}(L+E) = \mathcal{C}_{+1}(L) + \mathcal{C}_{+1}(E),$$

Structure	e-val λ_i	case	multiplicities
$\top$-palindromic	$\lambda_i = 1$	$n_{i,r+1}$ odd and (P) holds	$(n_{i,r+1}+1, n_{i,r+2}, \dots, n_{i,g_i})$
		otherwise	$(n_{i,r+1}, n_{i,r+2}, \dots, n_{i,g_i})$
	$\lambda_i = -1$	r even, $n_{i,r+1}$ even, and (P) holds	$(n_{i,r+1}+1, n_{i,r+2}, \dots, n_{i,g_i})$
		r even, otherwise	$(n_{i,r+1}, n_{i,r+2}, \dots, n_{i,g_i})$
		r odd, $n_{i,r+1}$ even, and (P) holds	$(n_{i,r+1}+1, n_{i,r+2}, \dots, n_{i,g_i}, 1)$
		r odd, otherwise	$(n_{i,r+1}, n_{i,r+2}, \dots, n_{i,g_i}, 1)$
	$\lambda_i \in \mathbb{C} \setminus \{\pm 1\}$	all	$(n_{i,r+1}, n_{i,r+2}, \dots, n_{i,g_i})$
$\top$-anti-palindromic	$\lambda_i = 1$	r even, $n_{i,r+1}$ even, and (P) holds	$(n_{i,r+1}+1, n_{i,r+2}, \dots, n_{i,g_i})$
		r even, otherwise	$(n_{i,r+1}, n_{i,r+2}, \dots, n_{i,g_i})$
		r odd, $n_{i,r+1}$ even, and (P) holds	$(n_{i,r+1}+1, n_{i,r+2}, \dots, n_{i,g_i}, 1)$
		r odd, otherwise	$(n_{i,r+1}, n_{i,r+2}, \dots, n_{i,g_i}, 1)$
	$\lambda_i = -1$	$n_{i,r+1}$ odd and (P) holds	$(n_{i,r+1}+1, n_{i,r+2}, \dots, n_{i,g_i})$
		otherwise	$(n_{i,r+1}, n_{i,r+2}, \dots, n_{i,g_i})$
	$\lambda_i \in \mathbb{C} \setminus \{\pm 1\}$	all	$(n_{i,r+1}, n_{i,r+2}, \dots, n_{i,g_i})$

Table 2 Generic partial multiplicities at λ_0 for rank-r $\top$-alternating perturbations

with both the pencil in the left-hand side and the ones in the right-hand side being $\top$-even [26, Th. 2.7]. Moreover, if $r = \operatorname{rank} E$ is odd, then

$$\begin{aligned} \mathcal{C}_{+1}(E)(\mu) &= \mathcal{C}_{+1}((1+\lambda)uu^\top + v_1 w_1^\top + \cdots + v_{(r-1)/2} w_{(r-1)/2}^\top \\ &\quad +(\operatorname{rev} w_1)v_1^\top + \cdots + (\operatorname{rev} w_{(r-1)/2})v_{(r-1)/2}^\top) \\ &= 2uu^\top + v_1 \widehat{w}_1(\mu)^\top + \cdots + v_{(r-1)/2}\widehat{w}_{(r-1)/2(\mu)}^\top \\ &\quad +(\widehat{w}_1(-\mu))v_1^\top + \cdots + (\widehat{w}_{(r-1)/2}(-\mu))v_{(r-1)/2}^\top), \end{aligned} \tag{28}$$

with $\widehat{w}_i(\mu) = \mathcal{C}_{+1}(w_i)(\mu) = (1-\mu)w(\frac{1+\mu}{1-\mu})$, for $i = 1, \dots, (r-1)/2$. The second sum in the last term of (28) follows by using similar identities to the ones in (13), which allow us to see that

$$\begin{aligned} \mathcal{C}_{+1}(\operatorname{rev} w_i)(\mu) &= \mathcal{C}_{+1}(\lambda w_i(1/\lambda))(\mu) = (1-\mu) \cdot \tfrac{1+\mu}{1-\mu} \cdot w_i\left(\tfrac{1-\mu}{1+\mu}\right) \\ &= (1+\mu)w_i\left(\tfrac{1-\mu}{1+\mu}\right) = \widehat{w}_i(-\mu). \end{aligned}$$

If r is even, then we get a similar expression according to the expression for $E(\lambda)$ in (11). This means that the pencil $\mathcal{C}_{+1}(E)$ is of the form (7). Then, by Theorem 19, there is a generic set $\mathcal{G}$ in $\mathbb{C}^{\lfloor \frac{3r}{2} \rfloor n}$ such that, for all $x \in \mathcal{G}$, the perturbed pencil $(\mathcal{C}_{+1}(L) + \Phi(x))(\mu)$ is regular and the partial multiplicities at μ_0 are the ones given in Table 1, replacing μ_0 by λ_i, with $\mu_0 = (\lambda_i - 1)/(\lambda_i + 1)$ if $\lambda_i \neq 1$, and $\mu_0 = \infty$ if $\lambda_i = 1$, and furthermore, such that all eigenvalues that are different from those of $\mathcal{C}_{+1}(L)$ are simple. Note that Φ is the map that takes a set of parameters $x \in \mathbb{C}^{\lfloor \frac{3r}{2} \rfloor n}$ to a pencil like in (28).

Applying the Cayley transformation $\mathcal{C}_{-1}$ we conclude that, for any $x \in \mathcal{G}$, the pencil $L + \mathcal{C}_{-1}(\Phi(x))$ is regular and has the partial multiplicities at λ_i as given in Table 2, while all eigenvalues that are different from those of L are simple. But, since $\Phi(x) = \mathcal{C}_{+1}(E)(\mu)$, then $\mathcal{C}_{-1}(\Phi(x)) = E(\lambda)$, and this concludes the proof for this case.

For the $\top$-anti-palindromic case just replace $\mathcal{C}_{-1}$ by $\mathcal{C}_{+1}$ and vice versa, and refer to the $\top$-odd case instead of the $\top$-even one.

Next, we turn to the skew-symmetric structure. As it is well known, the algebraic multiplicity of each eigenvalue of a skew-symmetric pencil is necessarily even (see, e.g., [4, Theorem 2.18]). As a consequence, the newly generated eigenvalues by a structure-preserving perturbation will generically be double eigenvalues instead of simple ones.

Theorem 21 (Generic change under low-rank perturbations of skew-symmetric pencils). *Let $L(\lambda)$ be a regular $n \times n$ skew-symmetric matrix pencil and let $\lambda_1, \dots, \lambda_\kappa \in \mathbb{C}$ be its pairwise distinct eigenvalues having the nonzero partial multiplicities $n_{i,1} \geqslant n_{i,2} \geqslant \cdots \geqslant n_{i,g_i} > 0$, for $i = 1, \dots, \kappa$, respectively. (We highlight that both n and all values $n_{i,j}$, $j = 1, \dots, g_i$, $i = 1, \dots, \kappa$ are necessarily even.) Furthermore, let r be a nonzero even integer and let Φ be the map as in Remark 7. Then, there is a generic set $\mathcal{G}$ in*

$\mathbb{C}^{\frac{3rn}{2}}$ such that, for all $E(\lambda) \in \Phi(\mathcal{G})$, the perturbed pencil $L+E$ is regular and the partial multiplicities of $L+E$ at λ_i are $n_{i,r+1} \geqslant \cdots \geqslant n_{i,g_i}$, for $i = 1, \ldots, \kappa$. Furthermore, all eigenvalues of $L+E$ that are not eigenvalues of L have algebraic multiplicity precisely two.

Proof Without loss of generality we may assume that $L(\lambda)$ is of the form

$$L(\lambda) = \begin{bmatrix} 0 & D(\lambda) \\ -D(\lambda) & 0 \end{bmatrix},$$

where $D(\lambda)$ is a regular pencil of size $\frac{n}{2} \times \frac{n}{2}$. This assumption can be made since $L(\lambda)$ is congruent to a pencil in the indicated form - a fact that follows easily by assuming that $L(\lambda)$ is in the canonical form of [4, Theorem 2.18] and then applying simultaneous row and column permutations. Clearly, the eigenvalue λ_i of $D(\lambda)$ has the partial multiplicities $\frac{n_{i,1}}{2} \geqslant \frac{n_{i,2}}{2} \geqslant \cdots \geqslant \frac{n_{i,g_i}}{2}$. By the proof of Theorem 16, there exists $\widetilde{x} \in \mathbb{C}^{\frac{3rn}{4}}$ of arbitrarily small norm such that $\widetilde{E}(\lambda) = \Phi_{\frac{r}{2}}(\widetilde{x})$ (with $\Phi_{\frac{r}{2}}$ being the map from Definition 3) is an $\frac{n}{2} \times \frac{n}{2}$ pencil of rank $\frac{r}{2}$ such that $D + \widetilde{E}$ is regular, has the partial multiplicities $\frac{n_{i,r+1}}{2} \geqslant \cdots \geqslant \frac{n_{i,g_i}}{2}$ at λ_i, for $i = 1, \ldots, \kappa$, and all its eigenvalues that are different from those of D are simple. Then setting

$$E(\lambda) = \begin{bmatrix} 0 & \widetilde{E}(\lambda) \\ -\widetilde{E}(\lambda) & 0 \end{bmatrix},$$

it follows that E is skew-symmetric and has rank r. Furthermore, due to the surjectivity of Φ it follows that there exists $x \in \mathbb{C}^{\frac{3rn}{2}}$ such that $\Phi(x) = E$ and it is straightforward to check that x can be chosen to be of the same norm as $\widetilde{x}$. Obviously, $L+E$ now has the partial multiplicities $n_{i,r+1} \geqslant \cdots \geqslant n_{i,g_i}$ at λ_i for $i = 1, \ldots, \kappa$, and all eigenvalues of $L+E$ that are not eigenvalues of L have algebraic multiplicity precisely two. Then applying Theorem 15 with $\mu = 2$ yields the desired result.

As for the remaining structures (skew-Hermitian, $*$-alternating, $*$-palindromic, and $*$-anti-palindromic) a similar result to Theorem 17 can be obtained either from this result directly using the observations in the paragraph right after Theorem 9 (skew-Hermitian, $*$-alternating) or using appropriate Cayley transformations as in the proof of Theorem 20 ($*$-palindromic, and $*$-anti-palindromic). We gather all these results in just one statement in Theorem 22.

Theorem 22 (Generic change under low-rank perturbations of skew-Hermitian, $*$-alternating, $*$-palindromic, and $*$-anti-palindromic pencils). *Let $\lambda_1, \ldots, \lambda_\kappa$ be the pairwise distinct eigenvalues (finite or infinite) of the regular $n \times n$ skew-Hermitian, $*$-alternating, $*$-palindromic, or $*$-anti-palindromic matrix pencil $L(\lambda)$, with nonzero partial multiplicities $n_{i,1} \geqslant n_{i,2} \geqslant \cdots \geqslant n_{i,g_i} > 0$ for $i = 1, \ldots, \kappa$, respectively. Furthermore, let r be a positive integer and, for each $0 \leqslant s \leqslant \lfloor r/2 \rfloor$, let Φ_s be the map as in Remark 7. Then, there is a generic set $\mathcal{G}_s$ in $\mathbb{R}^\ell \times \mathbb{C}^{(r+s)n}$ such that, for all $E(\lambda) \in \Phi_s(\mathcal{G}_s)$, the perturbed pencil $(L+E)(\lambda)$ is regular and the partial multiplicities of $L+E$ at λ_i are $n_{i,r+1} \geqslant \cdots \geqslant n_{i,g_i}$ for $i = 1, \ldots, \kappa$. In particular, if $g_i \leqslant r$ then λ_i is not an eigenvalue of $L+E$. Furthermore, all eigenvalues of $L+E$ that are not eigenvalues of L are simple.*

The results presented in Theorems 18–20 extend the ones in [3] and [5] from rank-1 and special rank-2 perturbations to low-rank perturbations of matrix pencils with symmetry structures. Even though some of the arguments and techniques in the proof of Theorems 18–20 are analogous to some of the ones used in [3, 5], the main approach, which uses the parameterizations constructed from the rank-1 decompositions given in Section 3, is different to the one followed in [3, 5].

If we compare Theorems 17–22 with Theorem 16, we will realize that, in most cases, the generic behavior for pencils with symmetry structures coincides with the one for general pencils. However, there are several cases in Theorems 19 and 20 where this behavior is different. In these cases, the $\top$-alternating and $\top$-palindromic structures impose additional restrictions that must be fulfilled in the canonical form, which prevent some behaviors, that in the general case are allowed, to occur under structure-preserving perturbations of pencils having these symmetry structures.

5 Outlook on the real case

So far, we have restricted ourselves to the complex case only. The main reason for this is the surprising fact that in general real versions of rank-1 decompositions as in Theorem 2 or Theorem 4 need not exist as the following example shows.

Example 3 Consider the real symmetric pencil

$$E(\lambda) = 2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -2\lambda & 2 \\ 2 & 2\lambda \end{bmatrix}.$$

This pencil has the eigenvalues $\mathrm{i}, -\mathrm{i}$ and a decomposition in complex Hermitian rank-1 pencils is given by

$$E(\lambda) = \begin{bmatrix} 1 \\ -\mathrm{i} \end{bmatrix} \begin{bmatrix} -\lambda & 2+\mathrm{i}\lambda \end{bmatrix} + \begin{bmatrix} -\lambda \\ 2-\mathrm{i}\lambda \end{bmatrix} \begin{bmatrix} 1 & \mathrm{i} \end{bmatrix}$$

while for a decomposition into complex symmetric rank-1 pencil pencils we can take

$$E(\lambda) = (\lambda+\mathrm{i}) \begin{bmatrix} -\mathrm{i} \\ 1 \end{bmatrix} \begin{bmatrix} -\mathrm{i} & 1 \end{bmatrix} + (\lambda-\mathrm{i}) \begin{bmatrix} \mathrm{i} \\ 1 \end{bmatrix} \begin{bmatrix} \mathrm{i} & 1 \end{bmatrix}.$$

However, $E(\lambda)$ does not allow a decomposition of the form

$$E(\lambda) = v(w+\lambda x)^\top + (w+\lambda x)v^\top \quad \text{with } v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},\ w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},\ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2. \tag{29}$$

Indeed, (29) leads to the contradictory equations

$$\begin{bmatrix} 2v_1w_1 & v_1w_2+v_2w_1 \\ v_1w_2+v_2w_1 & 2v_2w_2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2v_1x_1 & v_1x_2+v_2x_1 \\ v_1x_2+v_2x_1 & 2v_2x_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix},$$

since these imply $v_1, v_2 \neq 0$ and thus $w_1 = w_2 = 0$, which contradicts $v_1w_2 + v_2w_1 = 2$. But $E(\lambda)$ does not allow a decomposition of the form

$$E(\lambda) = (a_1+\lambda b_1)uu^\top + (a_2+\lambda b_2)vv^\top \quad \text{with } u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},\ v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

either, because in that case the pencil would have real eigenvalues, which is not the case.

We expect that in the real case one will have to allow summands of rank two in order to obtain a decomposition into low-rank pencils. This will be subject to subsequent research.

6 Conclusions and future work

We have described the generic change of the Weierstraß Canonical Form (given by the partial multiplicities) of regular matrix pencils with symmetry structures under structure-preserving additive low-rank perturbations. In particular, we have considered all the structures indicated at the beginning of Section 3. We have seen that, for most eigenvalues and most of the structures, the generic change coincides with the one in the unstructured case, namely: given an eigenvalue $\lambda_0 \in \mathbb{C} \cup \{\infty\}$ of the pencil $L(\lambda)$, with g associated partial multiplicities, for a generic perturbation, $E(\lambda)$, of rank r, the partial multiplicities of $(L+E)(\lambda)$ at λ_0 are exactly the $g-r$ smallest partial multiplicities of $L(\lambda)$. In particular, if $r \geqslant g$, the value λ_0 is not generically an eigenvalue of $(L+E)(\lambda)$. However, for the $\top$-alternating structures, there is a (generic) different behavior for the eigenvalues $\lambda_0 = 0$ and $\lambda_0 = \infty$, and similarly for the $\top$-palindromic structures with the eigenvalues $\lambda_0 = \pm 1$. These differences arise in those cases where the parity of the partial multiplicities in the perturbed pencil $L+E$ provided by the generic behavior in the unstructured case is not in accordance with the restrictions imposed by the structure (for instance, the even-sized blocks associated with $\lambda_0 = 0$ in $\top$-even pencils must be paired-up).

Our results contain the ones in [3], valid for rank-1 perturbations of pencils with symmetry structure, and extend the ones in [5] that are valid for special rank-2 perturbations of pencils with symmetry structures. However, the main tools and developments used in this work are different to the ones in [3,5]. More precisely, to obtain our main results we have introduced a structure-preserving rank-1 decomposition of low-rank pencils with symmetry structures, for each of the structures considered in the paper.

Several lines of research arise as a natural continuation of this work:

- To analyze the generic change of the partial multiplicities under low-rank perturbations of pencils with symmetry structures that have real coefficients, together with the generic change of the sign characteristic. In this work, we have restricted ourselves to the partial multiplicities, but the sign characteristic is also a key ingredient in the eigenstructure, for instance, of Hermitian pencils. The sign characteristic also appears in matrix pencils with real coefficients, for some of the other structures considered in this work (like the $\top$-even structure, see [40]). So it is natural to address the generic change of the sign characteristic in the context of pencils with symmetry structures having real coefficients.
- To describe the generic change of the partial multiplicities under low-rank perturbations of regular matrix polynomials with symmetry structures of arbitrary degree. The generic change of the partial multiplicities of regular matrix polynomials without additional symmetry structures has been described in [10]. However, the case of structure-preserving perturbations of matrix polynomials with symmetry structures remains open.

A Appendix

This appendix is devoted to prove the identities (26) and (27).

We start with (26). In this case n_r is odd, say $n_r = 2k+1$. The case $k=0$ is straightforward, so we assume $k>0$. Setting $\Delta_k := \det(R \operatorname{diag} J_{2k+1}(-\lambda), (J_{2k+1}(\lambda)) + \gamma(e_1 + e_{2k+3})(e_1 + e_{2k+3})^\top$ we have

$$\Delta_k = \left|\begin{array}{cccc|cccc} \gamma & 0 & \dots & 0 & 0 & \gamma & & \lambda \\ 0 & 0 & \dots & 0 & & & \lambda & 1 \\ \vdots & \ddots & \ddots & \vdots & & \iddots & \iddots & \\ 0 & 0 & \dots & 0 & \lambda & 1 & & \\ \hline 0 & \dots & 0 & -\lambda & 0 & 0 & \dots & 0 \\ \gamma & & -\lambda & 1 & 0 & \gamma & \dots & 0 \\ & \iddots & \iddots & & \vdots & & & \vdots \\ -\lambda & 1 & & & 0 & 0 & \dots & 0 \end{array}\right| .$$

Using the Laplace expansion with respect to the $(2k+1)$st row and column we arrive at

$$\Delta_k = \lambda^2 \cdot \left|\begin{array}{cccc|cccc} \gamma & 0 & \dots & 0 & \gamma & & & \lambda \\ 0 & 0 & \dots & 0 & & & \lambda & 1 \\ \vdots & \vdots & \ddots & \vdots & & \iddots & \iddots & \\ 0 & 0 & \dots & 0 & \lambda & 1 & & \\ \hline \gamma & & & -\lambda & \gamma & 0 & \dots & 0 \\ & & -\lambda & 1 & 0 & 0 & \dots & 0 \\ & \iddots & \iddots & & \vdots & \vdots & \ddots & \vdots \\ -\lambda & 1 & & & 0 & 0 & \dots & 0 \end{array}\right| .$$

Using the Laplace expansion with respect to the last column, we obtain

$$\Delta_k = \lambda^2 \left((-\lambda) \left|\begin{array}{cccc|cccc} & & & & & & & \lambda \\ & & & & & & \lambda & 1 \\ & & & & & \iddots & \iddots & \\ & & & & \lambda & 1 & & \\ \hline \gamma & & & -\lambda & \gamma & & & \\ & & -\lambda & 1 & & & & \\ & \iddots & \iddots & & & & & \\ -\lambda & 1 & & & & & & \end{array}\right| + \left|\begin{array}{cccc|cccc} \gamma & & & & \gamma & & & 0 \\ & & & & & & \lambda & 1 \\ & & & & & \iddots & \iddots & \\ & & & & \lambda & 1 & & \\ \hline \gamma & & & -\lambda & \gamma & & & \\ & & -\lambda & 1 & & & & \\ & \iddots & \iddots & & & & & \\ -\lambda & 1 & & & & & & \end{array}\right| \right) . \tag{30}$$

Computing separately the first and second determinant again via Laplace expansion, the first determinant is equal to

$$
\begin{aligned}
(-1)^{k-1}\lambda^{2k-1}\begin{vmatrix} \gamma & & & -\lambda \\ & & -\lambda & 1 \\ & \iddots & \iddots & \\ -\lambda & 1 & & \end{vmatrix} &= (-1)^{k-1}\lambda^{2k-1}\left((-1)^{k-1}\gamma-(-1)^{k-1}(-\lambda)^{2k}\right) \\
&= \lambda^{2k-1}\left(\gamma-\lambda^{2k}\right),
\end{aligned}
$$

and the second determinant is

$$
\begin{aligned}
(-1)^{k-1}\begin{vmatrix} \gamma & 0 & \dots & 0 & \gamma \\ \gamma & 0 & \dots & -\lambda & \gamma \\ & & \iddots & 1 & \\ & -\lambda & \iddots & & \\ -\lambda & 1 & & & \end{vmatrix} &= (-1)^{k-1}\begin{vmatrix} \gamma & 0 & \dots & 0 & \gamma \\ & & & -\lambda & \\ & & \iddots & 1 & \\ & -\lambda & \iddots & & \\ -\lambda & 1 & & & \end{vmatrix} \\
&= (-1)^{k}\gamma\begin{vmatrix} & & & -\lambda \\ & & -\lambda & 1 \\ & \iddots & \iddots & \\ -\lambda & 1 & & \end{vmatrix} = -\gamma\lambda^{2k}.
\end{aligned}
$$

so that for (30) we get

$$
\varDelta_k = -\lambda^2\left(-\lambda^{2k}(\gamma-\lambda^{2k})-\gamma\lambda^{2k}\right) = \lambda^{2k+2}\left(\lambda^{2k}-2\gamma\right),
$$

as claimed.

The proof of (27) proceeds analogously, with only minor modifications. Now n_r is even, say $n_r = 2k$. Thus, setting $\widetilde{\varDelta}_k := \det(R\mathrm{diag}(-J_{2k}(-\lambda), J_{2k}(\lambda)) + \gamma\lambda(e_1+e_{2k+2})(e_1+e_{2k+2})^\top$ we have

$$
\widetilde{\varDelta}_k = \left|\begin{array}{cccc|cccc}
\gamma\lambda & 0 & \dots & 0 & 0 & \gamma\lambda & & \lambda \\
0 & 0 & \dots & 0 & & & \lambda & 1 \\
\vdots & \ddots & \ddots & \vdots & & \iddots & \iddots & \\
0 & 0 & \dots & 0 & \lambda & 1 & & \\
\hline
0 & \dots & 0 & \lambda & 0 & 0 & \dots & 0 \\
\gamma\lambda & & \lambda & -1 & 0 & \gamma\lambda & \dots & 0 \\
& \iddots & \iddots & & \vdots & & & \vdots \\
\lambda & -1 & & & 0 & 0 & \dots & 0
\end{array}\right|.
$$

Using the Laplace expansion with respect to the $(2k+1)$st row and column we arrive at

$$
\widetilde{\varDelta}_k = -\lambda^2\cdot\left|\begin{array}{cccc|cccc}
\gamma\lambda & 0 & \dots & 0 & \gamma\lambda & & & \lambda \\
0 & 0 & \dots & 0 & & & \lambda & 1 \\
\vdots & \vdots & \ddots & \vdots & & \iddots & \iddots & \\
0 & 0 & \dots & 0 & \lambda & 1 & & \\
\hline
\gamma\lambda & & & \lambda & \gamma\lambda & 0 & \dots & 0 \\
& & \lambda & -1 & 0 & 0 & \dots & 0 \\
& \iddots & \iddots & & \vdots & \vdots & \ddots & \vdots \\
\lambda & -1 & & & 0 & 0 & \dots & 0
\end{array}\right|.
$$

Using the Laplace expansion with respect to the last column, the previous expression is equal to

$$
\widetilde{\varDelta}_k = -\lambda^2\left((-\lambda)\left|\begin{array}{cccc|cccc}
& & & & & & & \lambda \\
& & & & & & \lambda & 1 \\
& & & & & \iddots & \iddots & \\
& & & & \lambda & 1 & & \\
\hline
\gamma\lambda & & & \lambda & \gamma\lambda & & & \\
& & \lambda & -1 & & & & \\
& \iddots & \iddots & & & & & \\
\lambda & -1 & & & & & &
\end{array}\right| + \left|\begin{array}{cccc|cccc}
\gamma\lambda & & & & \gamma\lambda & & & 0 \\
& & & & & & \lambda & 1 \\
& & & & & \iddots & \iddots & \\
& & & & \lambda & 1 & & \\
\hline
\gamma\lambda & & & \lambda & \gamma\lambda & & & \\
& & \lambda & -1 & & & & \\
& \iddots & \iddots & & & & & \\
\lambda & -1 & & & & & &
\end{array}\right|\right). \qquad (31)
$$

Computing separately the first and second determinant again via Laplace expansion, the first determinant is equal to

$$
\begin{aligned}
(-1)^{k-1}\lambda^{2k-2}\begin{vmatrix} \gamma\lambda & & & \lambda \\ & & \lambda & -1 \\ & \iddots & \iddots & \\ \lambda & -1 & & \end{vmatrix} &= (-1)^{k-1}\lambda^{2k-2}\left((-1)^{k-1}\gamma\lambda + (-1)^{k-1}\lambda^{2k-1}\right) \\
&= \lambda^{2k-1}\left(\lambda^{2k-2}+\gamma\right),
\end{aligned}
$$

and the second determinant is

$$
\begin{aligned}
(-1)^{k-1}\begin{vmatrix} \gamma\lambda & 0 & \dots & 0 & \gamma\lambda \\ \gamma\lambda & 0 & \dots & \lambda & \gamma\lambda \\ & & \iddots & -1 & \\ & \lambda & \iddots & & \\ \lambda & -1 & & & \end{vmatrix} &= (-1)^{k-1}\begin{vmatrix} \gamma\lambda & 0 & \dots & 0 & \gamma\lambda \\ & & & \lambda & \\ & & \iddots & -1 & \\ & \lambda & \iddots & & \\ \lambda & -1 & & & \end{vmatrix} \\
&= (-1)^{k}\gamma\lambda\begin{vmatrix} & & & \lambda \\ & & \lambda & -1 \\ & \iddots & \iddots & \\ \lambda & -1 & & \end{vmatrix} = -\gamma\lambda^{2k}.
\end{aligned}
$$

so that for (31) we get

$$
\widetilde{\Delta}_k = -\lambda^2\left(-\lambda^{2k}(\lambda^{2k-2}+\gamma) - \gamma\lambda^{2k}\right) = \lambda^{2k+2}\left(\lambda^{2k-2}+2\gamma\right),
$$

as claimed.

Acknowledgments. The work of Fernando De Terán has been supported by the Ministerio de Economía y Competitividad of Spain through grants MTM2015-68805-REDT, and MTM2015-65798-P, and by the Ministerio de Educación, Cultura y Deportes of Spain through grant PRX16/00128 "Programa de estancias de movilidad de profesores e investigadores en centros de enseñanza superior e investigación "Salvador de Madariaga"".

The work of Volker Mehrmann has been supported by Einstein Foundation Berlin through project OT3 within the Einstein Center ECMath.

References

1. R. Albert, I. Albert, and G.L. Nakarado. Structural vulnerability of the north american power grid. *Phys. Rev. E*, 69:025103, Feb 2004.
2. S. Barnett. *Matrices: Methods and Applications.* Oxford, New York, 1990.
3. L. Batzke. Generic rank-one perturbations of structured regular matrix pencils. *Linear Algebra Appl.*, 458:638–670, 2014.
4. L. Batzke. *Generic Low-Rank Perturbations of Structured Regular Matrix Pencils and Structured Matrices.* PhD thesis, TU Berlin, Berlin, Germany, 2015.
5. L. Batzke. Generic rank-two perturbations of structured regular matrix pencils. *Oper. Matrices*, 10:83–112, 2016.
6. L. Batzke, C. Mehl, A. C. M. Ran, and L. Rodman. Generic rank-k perturbations of structured matrices. *Oper. Theory Adv. Appl.*, 225:27–48, 2016.
7. K. E. Brenan, S. L. Campbell, and L. R. Petzold. *Numerical Solution of Initial-Value Problems in Differential Algebraic Equations.* SIAM Publications, Philadelphia, PA, 2nd edition, 1996.
8. F. De Terán. A geometric description of the set of palindromic and alternating matrix pencils with bounded rank. *SIAM J. Matrix Anal. Appl.*, 39:1116–1134, 2018.
9. F. De Terán and F. M. Dopico. Low rank perturbation of Kronecker structures without full rank. *SIAM J. Matrix Anal. Appl.*, 29:496–529, 2007.
10. F. De Terán and F. M. Dopico. Low rank perturbation of regular matrix polynomials. *Linear Algebra. Appl.*, 430:579–586, 2009.
11. F. De Terán and F. M. Dopico. Generic change of the partial multiplicities of regular matrix pencils under low rank perturbations. *SIAM J. Matrix Anal. Appl.*, 37:823–835, 2016.
12. F. De Terán, F. M. Dopico, and J. Moro. Low rank perturbation of Weierstrass structure. *SIAM J. Matrix Anal. Appl.*, 30:538–547, 2008.
13. N.H. Du, V.H. Linh, and V. Mehrmann. Robust stability of differential-algebraic equations. In *Differential Algebraic Equation Forum, Surveys in Differential-Algebraic Equations I*, pages 63–96, Heidelberg, 2013. Springer Verlag.
14. A. Edelman, E. Elmroth, and B. Kågström. A geometric approach to perturbation theory of matrices and matrix pencils. Part II: A stratification-enhanced staircase algorithm. *SIAM J. Matrix Anal. Appl.*, 20:667–699, 1999.

15. J. H. Fourie, G. J. Groenewald, D. B. Janse van Rensburg, and A. C. M. Ran. Rank one perturbations of H-positive real matrices. *Linear Algebra Appl.*, 439:653–674, 2013.
16. F.R. Gantmacher. *Theory of Matrices*, volume 1. Chelsea, New York, 1959.
17. I. Gohberg, P. Lancaster, and L. Rodman. *Indefinite Linear Algebra and Applications*. Birkhäuser, Basel, 2005.
18. N. Gräbner, V. Mehrmann, S. Quraishi, C. Schröder, and U. von Wagner. Numerical methods for parametric model reduction in the simulation of disc brake squeal. *Z. Angew. Math. Mech.*, 96:1388–1405, 2016.
19. I. Abou Hamad, B. Israels, P.A. Rikvold, and S.V. Poroseva. Spectral matrix methods for partitioning power grids: Applications to the italian and floridian high-voltage networks. In *Computer Simulation Studies in Condensed-Matter Physics XXIII (CSP10)*, volume 4, pages 125–129. Physics Procedia, 2010.
20. P. Hamann and V. Mehrmann. Numerical solution of hybrid differential-algebraic equations. *Comp. Meth. Appl. Mech. Eng.*, 197:693–705, 2008.
21. L. Hörmander and A. Melin. A remark on perturbations of compact operators. *Math. Scand.*, 75:255–262, 1994.
22. D. B. Janse van Rensburg. *Structured matrices in indefinite inner product spaces: simple forms, invariant subspaces and rank-one perturbations*. PhD thesis, North-West University, Potchefstroom, South Africa, 2012.
23. P. Kunkel and V. Mehrmann. *Differential-Algebraic Equations. Analysis and Numerical Solution.* Zürich: European Mathematical Society Publishing House, 2006.
24. D. Liberzon. *Switching in Systems and Control*. Birkhäuser, Boston, 2003.
25. D. Liberzon and S. Trenn. Switched nonlinear differential algebraic equations: Solution theory, Lyapunov functions, and stability. *Automatica*, 48(5):954–963, 2012.
26. D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Structured polynomial eigenvalue problems: good vibrations from good linearizations. *SIAM J. Matrix Anal. Appl.*, 28:1029–1051, 2006.
27. C. Mehl. Anti-triangular and anti-m-Hessenberg forms for Hermitian matrices and pencils. *Linear Algebra Appl.*, 317:143–176, 2000.
28. C. Mehl, V. Mehrmann, A. C. M. Ran, and L. Rodman. Eigenvalue perturbation theory of classes of structured matrices under generic structured rank one perturbations. *Linear Algebra Appl.*, 435:687–716, 2011.
29. C. Mehl, V. Mehrmann, A. C. M. Ran, and L. Rodman. Perturbation theory of selfadjoint matrices and sign characteristics under generic structured rank one perturbations. *Linear Algebra Appl.*, 436:4027–4042, 2012.
30. C. Mehl, V. Mehrmann, A. C. M. Ran, and L. Rodman. Jordan forms of real and complex matrices under rank one perturbations. *Oper. Matrices*, 7:381–398, 2013.
31. C. Mehl, V. Mehrmann, A. C. M. Ran, and L. Rodman. Eigenvalue perturbation theory of symplectic, orthogonal, and unitary matrices under generic structured rank one perturbations. *BIT*, 54:219–255, 2014.
32. C. Mehl, V. Mehrmann, A. C. M. Ran, and L. Rodman. Eigenvalue perturbation theory of structured real matrices under generic structured rank-one perturbations. *Linear and Multilinear Algebra*, 64:527–556, 2016.
33. C. Mehl and A. C. M. Ran. Low rank perturbation of quaternion matrices. *Electron. J. Linear Algebra*, 32:514–530, 2017.
34. V. Mehrmann, V. Noferini, F. Tisseur, and H. Xu. On the sign characteristics of hermitian matrix polynomials. *Linear Algebra Appl.*, 511:328–364, 2016.
35. V. Mehrmann and L. Wunderlich. Hybrid systems of differential-algebraic equations – analysis and numerical solution. *J. Process Control*, 19:1218–1228, 2009.
36. J. Moro and F. Dopico. Low rank perturbation of Jordan structure. *SIAM J. Matrix Anal. Appl.*, 25:495–506, 2003.
37. M.C. Petri. National power grid simulation capability: Needs and issues. Technical report, Argonne National Laboratory, Energy Sciences and Engineering Directorate, U.S. Department of Homeland Security, 2008.
38. S.V. Savchenko. Typical changes in spectral properties under perturbations by a rank-one operator. *Mat. Zametki*, 74:590–602, 2003. (Russian). Translation in Mathematical Notes. 74:557–568, 2003.
39. S.V. Savchenko. On the change in the spectral properties of a matrix under a perturbation of a sufficiently low rank. *Funktsional. Anal. i Prilozhen*, 38:85–88, 2004. (Russian). Translation in Funct. Anal. Appl. 38:69–71, 2004.
40. R. C. Thompson. Pencils of complex and real symmetric and skew matrices. *Linear Algebra Appl.*, 147:323–371, 1991.