Stationary points at infinity for analytic combinatorics

Abstract: On complex algebraic varieties, height functions arising in combinatorial applications fail to be proper. This complicates the description and computation via Morse theory of key topological invariants. Here we establish checkable conditions under which the behavior at infinity may be ignored, and the usual theorems of classical and stratified Morse theory may be applied. This allows for simplified arguments in the field of analytic combinatorics in several variables, and forms the basis for new methods applying to problems beyond the reach of previous techniques.

Yuliy Baryshnikov, University of Illinois, Department of Mathematics, 273 Altgeld Hall 1409 W. Green Street (MC-382), Urbana, IL 61801, ymb@illinois.edu, partially supported by NSF grant DMS-1622370.

Stephen Melczer, Department of Combinatorics & Optimization, University of Waterloo, 200 University Avenue West, Waterloo, ON N2L 3G1, Canada, smelczer@uwaterloo.ca, partially supported by an NSERC postdoctoral fellowship.

Robin Pemantle, University of Pennsylvania, Department of Mathematics, 209 South 33rd Street, Philadelphia, PA 19104, pemantle@math.upenn.edu, partially supported by NSF grant DMS-1612674.

Subject classification: 05A16, 32Q55; secondary 14F45, 57Q99.

Keywords: Analytic combinatorics, stratified Morse theory, computer algebra, critical point, intersection cycle, ACSV.

1 Introduction

1.1 Motivation from Analytic Combinatorics

Analytic combinatorics in several variables (ACSV) studies coefficients of multivariate generating functions via analytic methods; see, for example, [PW13]. The most developed part of the theory is the asymptotic determination of coefficients of multivariate series $F(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ where the coefficients $a_{\mathbf{r}}$ are defined by a multivariate Cauchy integral

$$a_{\mathbf{r}} = \frac{1}{(2\pi i)^d} \int_T \mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}}\,, \tag{1.1}$$

for an appropriate torus of integration $T \subset \mathbb{C}^d$. In many applications, $F(\mathbf{z}) = P(\mathbf{z})/Q(\mathbf{z})$ is rational function with a power series expansion whose coefficients are indexed by $\mathbf{r}$, an integer vector. More generally, one often looks for formulas valid as $\mathbf{r}$ varies over some cone.

Let $|\mathbf{r}| := \sum_{j=1}^d |r_j|$ denote the ℓ_1 norm and let $\hat{\mathbf{r}}$ denote the scaled vector $\mathbf{r}/|\mathbf{r}|$. As a slight abuse of notation, we will sometimes consider $\hat{\mathbf{r}}$ to be an element of $\mathbb{RP}^{d-1}$ rather than the ℓ^1 unit ball, when the implicit identification of $\pm\hat{\mathbf{r}}$ leads to no ambiguity.

Given any $\mathbf{r} \in \mathbb{R}^d$ we define the *phase function*, or *height function*, depending only on $\hat{\mathbf{r}}$, by

$$h(\mathbf{z}) = h_{\mathbf{r}}(\mathbf{z}) = h_{\hat{\mathbf{r}}}(\mathbf{z}) = -\Re(\hat{\mathbf{r}} \cdot \log \mathbf{z}) = -\sum_{j=1}^d \hat{r}_j \log |z_j|\,, \tag{1.2}$$

where the logarithm is taken coordinate-wise and $\Re(z)$ denotes the real part of complex z. The height function is useful because it captures the behaviour of the term $|\mathbf{z}^{-\mathbf{r}}| = \exp(|\mathbf{r}| h_{\mathbf{r}}(\mathbf{z}))$ in the Cauchy integral that grows with $\mathbf{r}$. Note that even though the ratio $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$ is always rational, a sequence of such direction vectors may converge to any real $\hat{\mathbf{r}}$ and our results often hold uniformly for $\hat{\mathbf{r}}$ in cones of $\mathbb{RP}^{d-1}$. We will be clear when results do not require $\hat{\mathbf{r}}$ to be rational.

Typically, the Cauchy integral (1.1) is evaluated by applying the stationary phase (saddle point) method after a series of deformations of the chain of integration. To elaborate, we let $\mathcal{V} := \{\mathbf{z} : Q(\mathbf{z}) = 0\}$ denote the pole variety of rational $F(\mathbf{z}) = P(\mathbf{z})/Q(\mathbf{z})$, with P and Q coprime, let $\mathbb{C}_*$ denote the nonzero complex numbers, and let $\mathcal{M} := \mathbb{C}_*^d \setminus \mathcal{V}$ denote the domain of holomorphy of the Cauchy integrand in (1.1). The Cauchy integral depends only on the homology class[1] $[T]$ of T in $H_d(\mathcal{M})$. Stratified Morse theory for complements of closed, Whitney stratified spaces suggests that any cycle can be deformed downward (in the sense of decreasing $h_{\mathbf{r}}$) until it reaches a topological obstacle at some (stratified) critical point of $\mathcal{V}$. A topological obstacle generally implies that a stationary phase contour has been reached. This leads to the following outline for the asymptotic evaluation of coefficients $a_{\mathbf{r}}$.

(I) Find a basis of the singular homology group with integer coefficients $H_d(\mathcal{M})$, consisting of attachment chains $\{\sigma_k\}$ localized near the near critical points p_k of $h_{\mathbf{r}}$ on the Whitney stratified space $\mathcal{V}$, in the sense that each chain σ_k intersected with the set of points $\{\mathbf{z} : h_{\mathbf{r}}(\mathbf{z}) > h_{\mathbf{r}}(p_k) - \epsilon\}$ is contained in a ball around p_k of radius shrinking with ϵ.

[1] Throughout, we assume integer coefficients for all homology groups.

(II) Compute the coefficients of $[T]$ in this basis; that is, write

$$[T] = \sum_k n_k \sigma_k \tag{1.3}$$

for some integers $\{n_k\}$.

(III) Asymptotically compute the Cauchy integral over each chain of integration σ_k.

This paper concerns pre-conditions for the validity of this program. In reverse order, we discuss parts of the program carried out elsewhere.

Part III of the program varies in difficulty depending on the nature of the critical point. When the point is in a stratum of positive dimension where $\mathcal{V}$ has at worst normal intersections, it is at worst a multivariate residue together with a saddle point integral, the integral being somewhat more tricky if the saddle point is not quadratically nondegenerate. This follows from [Var77] or [Vas77] and is stated explicitly in [PW04] and [PW13, Chapter 10]. The more difficult case is when the critical point p is an isolated singularity of the stratified space $\mathcal{V}$. Homogenizing at p, one reduces to the problem of computing the inverse Fourier transform of a homogeneous hyperbolic function, some instructions for which can be found in [ABG70]. This is carried out in [BP11] for two classes of quadratic singularities, and in [BMP19] for some singularities of lacuna type. For non-isolated singularities, or isolated singularities of higher degree, the technology is still somewhat *ad hoc*.

Step II is a topological computation. In a slightly different context, Malgrange [Mal80] noted the lack of techniques for approaching a similar problem. Effective algorithms exist only in special cases, such as the bivariate case [DvdHP12]. A new computational approach, relying on the results of the present article together with techniques from computer algebra, is discussed in [BMP19] and below.

Step I may fail entirely. It is not always true that such a basis exists (see Examples 2 and 4 below), due to the existence of a topological obstruction at infinity[2]. The focus of the present paper is to find checkable conditions under which the class $[T]$ is indeed representable in the form (1.3). This has been a sticking point up to now in the development of ACSV methods.

1.2 Previous work

Although the methods of ACSV parallel well-established mathematical techniques, the underlying combinatorics often results in constraints which are natural in our context but not covered by existing theory. In this section we discuss related previous work and why the results we need do not follow from it.

To begin, there are several reasons why stratified Morse theory does not immediately imply the existence of the type of basis appearing in Step I. If $h_{\mathbf{r}}$ were a Morse function (in the stratified sense) then Theorems A and B of [GM88] would in fact imply that such a basis exists with some number m_k of generators associated with each critical point p_k. These are given by the rank of a relative homology group of the normal link at p_k of the stratum containing p_k (in the dimension equal to the codimension of the stratum). If $h_{\mathbf{r}}$ fails to be

[2] Here and throughout, "infinity" refers not only to projective points but to points where at least one coordinate vanishes; these are the cases in which the height function may not be well defined. Affine points with coordinates equal to zero may arise as critical points for Laurent series.

Morse by behaving degenerately at p_k such a basis still exists, though it might take a messy perturbation argument to compute the rank at p_k and give cycle representatives for a basis.

A more serious problem for us is that $h_{\mathbf{r}}$ is not a proper function on $\mathcal{V}$. This means that gradient flows of the Cauchy domain of integration may reach infinity or the coordinate planes at some finite height c, and hence that contours may not be deformable to levels below c because they get sucked out to infinity first. As shown in the examples in Section 4, this can indeed happen.

A somewhat generic cure for this is to compactify. In the appendix, we outline how to embed $\mathcal{M}$ in a compactification X to which the phase function $h_{\mathbf{r}}$ and its gradient extend continuously. Applying the results of stratified Morse theory to X then decomposes the topology of X into attachments at critical points of X. Generically, there will be finitely many critical points of X, all lying in $\mathcal{M}$. When this occurs, $\mathcal{M}$ is said to have no stationary points at infinity and the decomposition in Step I follows. Two weaknesses of this approach are the difficulty in computing X (it relies on an unspecified resolution of singularities) and the fact that X depends on $\mathbf{r}$ not continuously but rather through the arithmetic properties of the rational vector $\hat{\mathbf{r}}$, thus failing to deliver asymptotics uniform in a region.

1.2.1 Related notions of singularities at infinity

We now review three streams of prior work where, under some hypotheses of avoidance of critical points at infinity, the topology of a space is shown to decompose similarly to the desired decomposition in Step I.

One setting where such problems have been investigated concerns the Fourier transform of $\exp(f)$ where $f : \mathbb{R}^d \to \mathbb{R}$ is a real polynomial. When f is homogeneous, resolution of singularities puts the phase into a monomial form, after which leading term asymptotics can be read from its Newton diagram [Vas77, Var77]. For general polynomials, critical points of f occur at places other than the origin and these local integrals must be pieced together according to some global topological computation[3]. A key difference from our case is that this is an integral over all of $\mathbb{R}^d$ (there is no polar set). Morse theory enters the picture via the filtration $\{\Im(f) \leq c\}_{-\infty<c<\infty}$, which dictates how the contour of integration may be deformed, where $\Im(f)$ denotes the imaginary part of f.

Many similarities between this case and ours are evident. The height function $\Im\{f\}$ plays a similar role to our height function $h_{\mathbf{r}}$. One may apply methods of saddle point integration at the critical points of f, as done by Fedoryuk [Fed77] and Malgrange [Mal74, Mal80]. Fedoryuk computes in relative homology, which is good enough for the estimation of integrals. Pham [Pha83] uses an absolute homology theory over a family of supports, enabling more precise asymptotic results. Pham's crucial hypothesis H1 is that there are no bifurcation points of the second type or "critical points at infinity". The conclusion is the existence of a basis for the homology of $\mathbb{C}^d$ with downward supports consisting of so-called *Lefschetz thimbles*, along with a dual basis (in the sense of intersections) allowing one to compute the coefficients of an arbitrary cycle in this basis. Unfortunately, we can see no direct connection that would reduce our computation to the type analyzed by Pham and others before him. Even if we could, the issue of spurious critical points at infinity would still need to be addressed. As pointed out in [Pha83, page 330], there is no simple way of telling which of these is relevant.

[3] Pham attributes this idea to Malgrange, "The reduction of a *global* Fourier-like integral to a sum of Laplace-like integrals is the topic of Malgrange's recent paper, motivated by an idea of Balian-Parisi-Voros." [Pha83].

A second stream of work concerns the topology of complex polynomial hypersurfaces. Here there are no integrals, hence no phase functions *per se*, although motivation from [Mal80] and [Pha83] is cited in the introduction of [Bro88]. In this paper, Broughton computes the homotopy type of a generic level set $f^{-1}(c)$, showing it to be a bouquet of n spheres, where n is obtained by summing the Milnor numbers at the critical points of f other than those with critical value c. Examples show that a hypothesis is necessary to rule out "critical points at infinity"; see also [Par95]. Both of these works refer to an assumption of only isolated critical points at infinity or none at all, but do not supply a specific definition of critical points at infinity. Such a definition is supplied in [ST95]. They compactify f by taking the closure of its graph in projective space and taking a Whitney stratification of the resulting relation. Depending on whether there are no critical points at infinity, void (hence ignorable) critical points at infinity, isolated non-void critical points at infinity, or non-isolated points, various conclusions can be drawn about the topologies of the fibers $f^{-1}(c)$. Again, no direct relation allows us to derive from this the decomposition in Steps I and II, and even then, the issue of spurious critical points would remain.

A third stream of work concerning critical values at infinity comes closest to our aims here. The focus of this stream of work is, given a smooth map $F : \mathcal{M} \to \mathbb{N}$ on some sort of space, to find a set $B \subseteq \mathbb{N}$ that is not too big, such that outside of $F^{-1}[B]$, the mapping is a locally trivial fibration. For proper maps one has the Thom isotopy lemma, which states that B can be taken as the set K_0 of critical values, namely the set $F(x)$ where F is not a submersion at x, failing to map the tangent space surjectively. To extend the isotopy lemma to nonproper maps, one needs to add to K_0 an apprrpriately defined set K_∞ of critical values at infinity. When $\mathcal{M}$ is smooth the so-called Palais-Smale Condition, and Rabier's general notion of **asymptotic critical value** [Rab97, Section 6], yields such an isotopy result, valid in a quite general infinite dimensional setting. Further work has shown that under reasonable hypotheses the set of critical values K_∞ is not too big, for example it has measure zero [KOS00].

Our work requires a result along similar lines, but for stratified spaces. More specifically, in the proof of Theorem 2.7 below, we use an isotopy result away from a small computable set of stratified critical values for the height function to describe $H_d(\mathcal{M})$ in terms of stationary phase contours. A similar project is undertaken in the contemporaneous work [DJ21]. There, Dinh and Jelonek prove a version of the (stratified) Thom isotopy lemma away from a computable and nowhere dense set $K_0 \cup K_\infty$ of affine and at-infinity critical values. In comparison to our work, Dinh and Jelonek work in a more general setting but thus use more complicated constructions leading to a less practical algorithm for detecting critical points at infinity. We thus do not use their stratified non-proper isotopy lemma [DJ21, Theorem 3.4, Section 3]; rather, we use a similar but streamlined approach to prove exactly what is needed for the topological decomposition in the first part of our main results. A robust study of these topics from the point of view of efficient algorithms is a promising direction for future work in the computer algebra community.

1.3 Present work

We define a set of projective points which we call **stationary points at infinity (SPAI)**. These are limits at infinity (or the coordinate planes) of sequences of points that are asymptotically converging to criticality for a given height function $h_{\hat{\mathbf{r}}}$. The ultimate goal is to find such sequences on which $h_{\hat{\mathbf{r}}}$ remains bounded, because these indicate trajectories in which gradient like Morse deformations may get pulled out to infinity. Such sequences, together with limit points of the height function, are called heighted SPAI (H-SPAI). Their image under $h_{\hat{\mathbf{r}}}$ coincides with the set of asymptotic critical values in, e.g., [Rab97, DJ21]. Note that we use the term *stationary point* instead of the term critical point (common in the analytic combinatorics literature) as 'critical point ' is overloaded and potentially misleading to readers in some areas of mathematics.

The advantage to working with SPAI is that these are easily computed for any real $\hat{\mathbf{r}}$ or when $\hat{\mathbf{r}}$ is a symbolic parameter. While spurious SPAI do arise (see Examples 3 and 7), all such examples we know of can be ruled out by determining the height function to be unbounded (that is, we do not need to compute the limit set of heights, just to check whether it is empty). As to whether H-SPAI themselves can be spurious, as noted in the introduction of [DJ21], characterization of bifurcation values (where local triviality fails) is open, and in particular these can be a proper subset of critical values. However, in applications to ACSV, in all examples we know of when there are H-SPAI, the attachment cycles do not in fact form a basis of the relevant homology group, therefore the isotopy arguments must fail.

Methodologically, it should be noted that we do not try to show topological trivality at infinity in the absence of SPAI, only that the necessary deformations can avoid infinity. This, we suspect, is why our Section 5 is shorter than [DJ21, Section 3].

Our main results are the following.

(i) Given a direction $\hat{\mathbf{r}}$ and real $a < b$, we define three sets SPAI $\supseteq$ H-SPAI $\supseteq \mathsf{crit}_{[a,b]}$.

(ii) We give an algorithm for computing SPAI.

(iii) Theorem 2.7, which states that cycles may be pushed down until a stationary value is reached in such a way that they remain above the stationary height only in an arbitrarily small neighborhood of the stationary point(s).

As a consequence, when SPAI is empty (which is easily computed) or when H-SPAI is empty (which may be computed more easily than whether $\mathsf{crit}_{[a,b]}$ is empty), any cycle may be decomposed into attachment cycles. We also note that for a generic $\hat{\mathbf{r}}$ the set SPAI is indeed empty, and that computability of SPAI for symbolic $\hat{\mathbf{r}}$ means we can compute a polynomial criterion for the set of directions $\hat{\mathbf{r}}$ in which SPAI is nonempty.

The main value of our work lies in its application to ACSV which, in turn, derives its value from combinatorial applications. The next subsection reviews several combinatorial paradigms in which ACSV yields strong results; a different class of examples is presented in Section 4 below. The purpose of those later examples is (a) to show how Theorem 2.7 can considerably strengthen ACSV analysis, and (b) to illuminate the role of the hypotheses in the main results and the increased efficacy in eliminating spurious stationary points.

Beyond this, Theorem 2.7 pays back a debt in the literature. Previous books and papers on ACSV [PW08, PW13] often use Morse-theoretic heuristics to motivate certain constructions, but cannot use Morse theory outright to prove general results. By ruling out stationary points at infinity, those results can be recast as following from the stratified Morse framework.

The remaining sections of the paper, after reviewing applications of ACSV, are organized as follows. Section 2 sets the notation for the study of stratified spaces and stationary points, formulates definitions, and states the main result. Section 3 shows how to determine all SPAI using a computer algebra system. Some examples are given in Section 4. Section 5 proves Theorem 2.7 by constructing Morse deformations from height b down to height a, remaining in a bounded region provided $\mathsf{crit}_{[a,b]}$ is empty.

1.4 Applications of ACSV

The techniques of analytic combinatorics in several variables find application to a diverse range of topics in mathematics, computer science, and the natural sciences. We briefly summarize some of these applications here; anyone wanting more information can consult, for instance, Pemantle and Wilson [PW08, PW13] or Melczer [Mel21].

Quantum Random Walk: Since their introduction in the early 1990s [ADZ93], quantum variants of random walks have been studied as a computational tool for quantum algorithms (see the introduction of Ambainis et al. [ABN$^+$01] for a listing of quantum algorithms based around quantum random walks, for example). Results obtained by ACSV go well beyond what has been obtained by other methods such as orthogonal polynomials or the univariate Darboux method [CIR03]. In particular, ACSV may be used to analyze one-dimensional quantum walks with arbitrary numbers of quantum states [BGPP10] and families of quantum random walks on the two-dimensional integer lattice [BBBP11]. Both of these results involve *ad hoc* geometric arguments which may be streamlined based on the results of the present paper.

Example 1. *As described in Bressler and Pemantle [BP07], the analysis of quantum random walks on the one-dimensional integer lattice can be reduced to studying asymptotics of coefficients*

$$F(x,y) = \frac{G(x,y)}{1-cy+cxy-xy^2} = \sum_{i,j\geq 0} f_{i,j}x^iy^j,$$

where $c \in [0,1]$ is a parameter depending on the underlying probabilities used to transition between different states in the walk and $G(x,y)$ is a polynomial which depends on the initial state of the system. In particular, for given c one wishes to determine the asymptotic behaviour of the sequence $a_n^\lambda = f_{n,\lceil \lambda n\rceil}$ as $n \to \infty$. A short argument about the roots of $H(x,y) = 1-cy+cxy-xy^2$ implies $a_n^\lambda \sim C_\lambda n^{-1/2}\rho_\lambda^n$ where $0<\lambda<1$; the values of λ such that $\rho_\lambda = 1$ form the feasible region *of study while the values of λ with $\rho_\lambda < 1$, where a_n^λ exponentially decays, form the* nonfeasible region. *For any values of $c, \lambda \in (0,1)$ the height function $h_{\mathbb{C}}(x,y)$ with $\hat{\mathbf{r}} = (1:\lambda)$ has two stationary points. Previously, to determine asymptotic behaviour one needed to check which of these stationary points were in the domain of convergence of $F(x,y)$, a computationally difficult task that requires arguing about inequalities involving the moduli of variables in an algebraic system with parameters. Running our Maple implementation of Algorithm 1 shows that $F(x,y)$ has no stationary points at infinity, meaning Theorem 2.18 applies and asymptotics of a_n^λ can be written as an integer linear combination of two explicitly known asymptotic series. In particular, when $2\lambda \in [1-c, 1+c]$ then both stationary points are on the unit circle and our results immediately imply that λ* must *be in the feasible region. Similarly, if $2\lambda \notin [1-c,1+c]$ it can be shown that λ is not in the feasible region. Although our results ease the derivation of previously known results in this instance, they also allow for the derivation of results outside the scope of previous methods (see, for instance, Example 6 below).*

Queuing Theory and Lattice Walks: Queuing theory–the study of systems in which items enter, exit, and move between various lines–arises naturally in computer networking, telecommunications, and industrial engineering, among other areas. Often, one can derive multivariate generating functions describing the state of a system at point in time, then derive desired information about the underlying model through an asymptotic analysis. Such analyses, using analytic combinatorics methods to analyze queuing models, can be seen in Bertozzi and McKenna [BM93] and Pemantle and Wilson [PW08, Section 4.12], for instance. These systems can often be modeled by (classical) random walks on integer lattices subject to various constraints [FIM99, Ch. 9 & 10], an enumerative problem for which the methods of ACSV are extremely effective [Mel21].

RNA Secondary Structure: The secondary structure of a molecule's RNA, describing base pairings between its elements, encodes important information about the molecule, and predicting such structure is a well-studied topic in bioinformatics. One approach to secondary structure prediction uses stochastic context-free grammars to generate potential pairings; this approach is implemented in the popular Pfold program of Knudsen and Hein [KH99]. To analyze Pfold, Poznanović and Heitsch [PH14] used multivariate generating functions tracking the probability that certain biological features arise. Using classical methods in analytic combinatorics, those authors found distributions for single features (the numbers of base pairs, loops and helices generated by a grammar). Their central limit theorems rely on results of Flajolet and Sedgewick, quoted as [PH14, Thoerem 4.1], whose hypotheses can be replaced by more checkable multivariate hypotheses once one has our Theorem 2.7 below. More recently, Greenwood (see [Gre18] and a forthcoming extension) used ACSV to analyze the probability that certain *combinations* of features appear. Greenwood's hypotheses in his Theorem 1 and Corollary 2 can be weakened and much more easily checked with the Morse-theoretic tools in the present paper.

Sequence alignment: The problem of optimally aligning more than two sequences on a finite alphabet is fundamental to the study of DNA and known in several ways to be mathematically intractable. In [PW08, Section 4.9] several cases are analyzed using techniques of ACSV. At the time of that paper, ACSV could only handle cases where the dominant singularity was at a stationary point all of whose coordinates were known, via Pringsheim's Theorem, to be real. Morse theory allows us in principle to handle further biologically relevant cases.

2 Definitions and results

2.1 Spaces, stratifications and stationary points

Throughout the remainder of the paper, Q is a polynomial and $\mathcal{V}$ is the algebraic hypersurface $\{\mathbf{z} \in \mathbb{C}^d : Q(\mathbf{z}) = 0\}$. The elements of $\mathcal{V}$ with non-zero coordinates is denoted $\mathcal{V}_* := \mathcal{V} \cap \mathbb{C}_*^d$.

2.1.1 Whitney stratifications

The following definitions of stratification and Whitney stratification are taken from [Har75, GM88]. A **stratification** of a space $\mathcal{V} \subseteq \mathbb{C}^d$ is a partition of $\mathcal{V}$ into finitely many disjoint sets $\{\Sigma_\alpha : \alpha \in \mathcal{A}\}$ such that each stratum Σ_α is a real manifold[4] of some dimension at most $2d$. We consider here only algebraic stratifications, meaning that each stratum is an algebraic set, potentially with an algebraic set of lower dimension removed. A **Whitney stratification** furthermore satisfies the following conditions.

1. If a stratum Σ_α intersects the closure $\overline{\Sigma_\beta}$ of another, then it lies entirely inside: $\Sigma_\alpha \subseteq \overline{\Sigma_\beta}$.
2. Whitney's Condition B on tangent planes and secant lines of the strata should hold; because we make use only once of this condition and never need to check it, we refer readers to [GM88, Chapter 1.2] for the definition.

[4] In fact our strata are always complex manifolds and complex algebraic sets, however this is not required in the definition of a stratification.

A stratification of the pair $(\mathbb{C}_*^d, \mathcal{V}_*)$ is a Whitney stratification of $\mathbb{C}_*^d$ in which the only $(2d)$-dimensional stratum is $\mathcal{M} := \mathbb{C}_*^d \setminus \mathcal{V}$. When $\mathcal{V}$ is a complex algebraic variety, such a stratification always exists.

Logarithmic space, natural Riemannian metric, tilde notation

Let $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ denote the one-dimensional torus. We introduce the *logarithmic space (logspace)* $\mathcal{L} \cong \mathbb{T}^d \times \mathbb{R}^d$ together with the exponential diffeomorphism $\exp : \mathcal{L} \to \mathbb{C}_*^d$ defined for $\eta \in \mathbb{T}^d$ and $\xi \in \mathbb{R}^d$ by

$$\exp(\eta, \xi) := \exp(\xi + i\eta).$$

We refer to $\xi \in \mathbb{R}^d$ as the *real* coordinates on the logspace. Mostly we do not need logspace until Section 5, though we refer to coordinates ξ and η occasionally. The reason for using both $\mathbb{C}_*^d$ and the logspace is that deformations and other geometric constructions are more transparent in $\mathcal{L}$ but that polynomial computations via computer algebra involving Q must be carried out in $\mathbb{C}_*^d$.

Directions and affine stationary points

A *direction* is an equivalence class of vectors in $\mathbb{R}^d$ under positive multiples. The direction containing the nonzero vector $\mathbf{r}$ can be canonically identified with the unit vector $\hat{\mathbf{r}}$. Fix a *direction* $\hat{\mathbf{r}}$. The phase function $h_{\hat{\mathbf{r}}}$ from (1.2) is given in logspace by the linear function $\hat{\mathbf{r}} \cdot \xi$. stationary points of $h_{\hat{\mathbf{r}}}$ on any complex-analytic submanifold of $\mathbb{C}_*^d$ are therefore the same as stationary points of a branch of $\mathbf{z}^{\mathbf{r}}$.

Fix a stratification $\{\Sigma_\alpha\}$ of $\mathcal{V}$. For a stratum Σ, we define an *affine* $\mathbf{r}$*-stationary point* as a stationary point of the restriction of $h_{\hat{\mathbf{r}}}$ to Σ. These points are what is referred to in ACSV literature as critical points; we call them *affine stationary points*, in contrast to SPAI, which are stationary points at infinity.

Stationary point relation

Given a stratum Σ, a point $\mathbf{x} \in \Sigma$ and a direction $\hat{\mathbf{r}}$, a necessary and sufficient condition for $h_{\hat{\mathbf{r}}}|_\Sigma$ to have a stationary point at $\mathbf{x}$ is a drop in the rank of a certain matrix of differentials. To make this more precise, choose a neighborhood of $\mathbf{x}$ in which the closure of Σ is locally cut out by c independent polynomials, where c is the codimension of Σ,

$$\overline{\Sigma} = \left\{ \mathbf{z} : f_j^\Sigma(\mathbf{z}) = 0 : 1 \le j \le c \right\} . \tag{2.1}$$

The stratum Σ may be obtained by intersecting $\overline{\Sigma}$ with a set $\{g_i^\Sigma(\mathbf{z}) \neq 0\}$ of polynomial non-equalities to remove points in substrata. By refining the stratification if necessary, we can assume that the polynomials f_j^Σ and g_i^Σ define the whole stratum and that the differentials $\{df_j^\Sigma\}$ are linearly independent everywhere on Σ. For any point $\mathbf{z} \in \Sigma$ and vector $\mathbf{y} \in \mathbb{C}^d$, let $\mathbf{J}(\mathbf{z}, \mathbf{y}) = \mathbf{J}^{\Sigma, \hat{\mathbf{r}}}(\mathbf{z}, \mathbf{y})$ denote the $(c+1) \times d$ matrix with entries

$$\mathbf{J}_{i,j} := z_j \frac{\partial f_i}{\partial z_j} \text{ for } 1 \le i \le c, \text{ and } \mathbf{J}_{c+1,j} = y_j \, . \tag{2.2}$$

Note that after dividing the jth column of $\mathbf{J}(\mathbf{z}, \mathbf{y})$ by z_j its rows become the gradients $(\nabla f_i)(\mathbf{z})$ together with the scaled gradient $(\nabla h_{\mathbf{y}})(\mathbf{z})/h_{\mathbf{y}}(\mathbf{z})$. The rank of $J(\mathbf{z}, \mathbf{w})$ is invariant under the map $\mathbf{w} \mapsto \lambda \mathbf{w}$ for any non-zero λ, thus we define a binary relation $(\mathbf{z}, \mathbf{y}) \in S$ which we interpret as $(\nabla h_{\mathbf{y}})(\mathbf{z})$ lying in the normal space to Σ at $\mathbf{z}$.

Definition 2.1 (binary relation for stationary points)**.** *Define* $S = S(\Sigma) \subseteq \mathbb{C}_*^d \times \mathbb{CP}^{d-1}$ *by* $(\mathbf{z}, \mathbf{y}) \in S$ *if and only if* $\mathrm{rank}(J(\mathbf{z}, \mathbf{y})) \leq c$*.*

Proposition 2.2. *Let* Σ *be a stratum and suppose* $\mathbf{z} \in \Sigma$ *and* $\hat{\mathbf{r}}$ *is real. then* $(\mathbf{z}, \hat{\mathbf{r}}) \in S(\Sigma)$ *if and only if* $\mathbf{z}$ *is a critical point of the height function* $h_{\hat{\mathbf{r}}}$ *on the stratum* Σ*.*

PROOF: The tangent space to a stratum Σ defined by analytic functions is a complex linear subspace of $\mathbb{C}^d$. The function $h_{\hat{\mathbf{r}}}$ is the real part of a (branched) analytic function $h_{\hat{\mathbf{r}}}^{\mathbb{C}} := \sum_{j=1}^d r_j \log z_j$. It follows that the vanishing of $dh_{\hat{\mathbf{r}}}|_\Sigma$ on the real part of the complex tangent space $T\Sigma$ is equivalent to its vanishing on all of $T\Sigma$, which is equivalent to the vanishing of $dh_{\hat{\mathbf{r}}}^{\mathbb{C}}$ on all of $T\Sigma$. Vanishing of the complexified height is equivalent to $\hat{\mathbf{r}}$ being in the complex normal space, which is equivalent to the rank being at most c of any basis for the normal space, together with the vector $\hat{\mathbf{r}}$. $\square$

Stationary points at infinity

Definition 2.3 (SPAI)**.** *Let* $\overline{S}$ *denote the closure of* S *when embedded in* $\mathbb{CP}^d \times \mathbb{CP}^{d-1}$*. Define a SPAI in direction* $\hat{\mathbf{r}}$ *to be an element* $(\mathbf{z}, \hat{\mathbf{r}}) \in \overline{S}$ *where* $\mathbf{z} \notin \mathbb{C}_*^d$*, meaning* $\mathbf{z}$ *either lies in the plane at infinity or has at least one vanishing coordinate. A* witness *to the SPAI* $(\mathbf{z}, \hat{\mathbf{r}})$ *is a sequence* $(\mathbf{z}_n, \hat{\mathbf{r}}_n)$ *in* $\mathbb{C}_*^d \times \mathbb{CP}^{d-1}$ *converging to* $(\mathbf{z}, \hat{\mathbf{r}})$*.*

Definition 2.4 (ternary relation for heighted stationary points)**.** *Fix a direction* $\hat{\mathbf{r}}$*. Let* $R := R(\Sigma, \hat{\mathbf{r}})$ *denote the set of triples* $(\mathbf{z}, \mathbf{y}, \eta) \in \mathbb{C}_*^d \times \mathbb{CP}^{d-1} \times \mathbb{C}$ *such that the following three conditions hold.*

(i) $\mathbf{z} \in \Sigma$ *;*

(ii) $\mathrm{rank}(\mathbf{J}(\mathbf{z}, \mathbf{y})) \leq c$*, where* c *is the co-dimension of* Σ *;*

(iii) $h_{\hat{\mathbf{r}}}(\mathbf{z}) = \eta$*.*

Projecting $R(\Sigma, \hat{\mathbf{r}})$ *to the first two coordinates yields* $S(\Sigma)$*.*

Definition 2.5 (H-SPAI)**.** *Let* $\overline{R}$ *denote the closure of* R *in* $\mathbb{CP}^d \times \mathbb{CP}^{d-1} \times \mathbb{R}$*. A triple* $(\mathbf{z}, \mathbf{y}, \eta) \in \overline{R}$ *with* $\mathbf{z} \notin \mathbb{C}_*^d$ *is called an H-SPAI in* Σ *in direction* $\hat{\mathbf{r}}$ *and is said to have height* η*. A* witness *for the H-SPAI* $(\mathbf{z}, \mathbf{y}, \eta)$ *is a sequences* $(\mathbf{z}_n, \mathbf{y}_n, \eta_n)$ *in* $R(\Sigma, \hat{\mathbf{r}})$ *converging to* $(\mathbf{z}, \mathbf{y}, \eta)$*.*

We say that the real number η is a *generalized stationary value* of $h_{\hat{\mathbf{r}}}$ on the stratum Σ if either it is an affine stationary value (that is, a stationary value of $h_{\hat{\mathbf{r}}}|_\Sigma$) or else it is the third coordinate of some H-SPAI. We denote the set of stationary values by $K(\Sigma, \hat{\mathbf{r}}) = K_0 \cup K_\infty$.

Definition 2.6 (stationary points in an interval)**.** *Fix* $\hat{\mathbf{r}}, \Sigma$*, and fix* $-\infty \leq a < b \leq \infty$*. The* $\mathbf{z}$ *coordinates of all stationary points on* Σ *with heights in* $[a, b]$ *form the set*

$$\mathsf{crit}_{[a,b]}(\Sigma, \hat{\mathbf{r}}) := \{\mathbf{z} \in \mathbb{CP}^d : \exists \mathbf{y}, \eta \text{ with } (\mathbf{z}, \mathbf{y}, \eta) \in \overline{R}, \mathbf{y} = \hat{\mathbf{r}} \text{ and } \eta \in [a, b]\}\,.$$

Omitting the argument Σ *or* $[a, b]$ *denotes a union over all strata and taking* $[a, b] = (-\infty, \infty)$*, respectively. We write* $\mathsf{crit}_{[a,b]}^{\mathrm{aff}}(\Sigma, \hat{\mathbf{r}})$ *for the elements of* $\mathsf{crit}_{[a,b]}(\Sigma, \hat{\mathbf{r}})$ *which are affine stratified points and* $\mathsf{crit}_{[a,b]}^{\infty}(\Sigma, \hat{\mathbf{r}})$ *for the remaining elements.*

Remark 1. Because we sometimes need to refine stratifications, we note that refining the stratification can introduce more stationary points, affine or infinite, but cannot remove any.

2.2 Main topological results

For any space S with height function h and any real b, we define $S_{\le b} = \{\mathbf{x} \in S : h(\mathbf{x}) \le b\}$. We first state our main deformation result, an extension to the nonproper setting of well known stratified Morse theoretic results for proper height functions.

Theorem 2.7 (Morse deformation)**.** *Fix $Q, \mathcal{V}$, $\mathcal{M} := \mathbb{C}_*^d \setminus \mathcal{V}$, and a Whitney stratification $\{\Sigma_\alpha\}$ of $(\mathbb{C}_*^d, \mathcal{V}_*)$. Fix also a direction $\hat{\mathbf{r}}$ and height function $h_{\hat{\mathbf{r}}}(\mathbf{z}) = \mathbf{z}^{\hat{\mathbf{r}}}$.*

(i) Suppose $\mathsf{crit}_{[a,b]}(\Sigma, \hat{\mathbf{r}})$ is empty. Then $\Sigma_{\le b} \cong \Sigma_{\le a}$ for any stratum Σ, and $\mathcal{M}_{\le b} \cong \mathcal{M}_{\le a}$.

(ii) Suppose $\mathsf{crit}_{[a,b]}(\Sigma, \hat{\mathbf{r}}) = \mathsf{crit}^{\mathrm{aff}}_{[a,b]}(\Sigma, \hat{\mathbf{r}}) = \{\mathbf{z}_1, \dots, \mathbf{z}_k\}$ with $h_{\hat{\mathbf{r}}}(\mathbf{z}_j) = c \in (a, b)$ for all $j \le k$. Then for any stratum Σ, any chain $\mathcal{C}$ supported on $\Sigma_{\le b}$ is homotopic in $\Sigma_{\le b}$ to a chain supported on the union of $\Sigma_{<c}$ together with arbitrarily small neighborhoods of the points $\mathbf{z}_k$ in Σ. Taking $\Sigma = \mathcal{M}$, it follows that every homology class in $H_d(\mathcal{M}_{\le b})$ is represented by a cycle supported on this union.

Remark 2. This theorem draws no conclusion about whether the topology of $\mathcal{M}$ or $\mathcal{V}$ can be deduced from the topology near the stationary points, only stating that certain needed deformations exist. In fact, all the topological information necessary to estimate the Cauchy integrals is present in the relative homology group $(\mathcal{M}, \mathcal{M}_{\le -K})$ for some sufficiently large K, hence the topology of $\mathcal{M}$ at sufficiently low heights is irrelevant.

The purpose of these homotopy equivalences is to push the cycle of integration T down to one whose maximum height is as low as possible. For example, because the cycle T in the Cauchy integral of interest can be pushed down at least until hitting the first stationary point corresponding to direction $\hat{\mathbf{r}}$, the magnitude of coefficients in direction $\hat{\mathbf{r}}$ is bounded above by the Cauchy integral over a contour at this height. The following corollary of Theorem 2.7 was given only as a conjecture in [PW13] because it was not known under what conditions T could be pushed down to the stationary height.

Corollary 2.8. *Fix $\hat{\mathbf{r}}$ and a Laurent polynomial Q and Laurent expansion $P(\mathbf{z})/Q(\mathbf{z}) = \sum_{\mathbf{r} \in K} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$. Suppose $K(\Sigma, \hat{\mathbf{r}})$ is finite for all strata and denote the maximum stationary value by c. Then*

$$\limsup_{\substack{\mathbf{r} \to \infty \\ \mathbf{r}/|\mathbf{r}| \to \hat{\mathbf{r}}}} \frac{1}{|\mathbf{r}|} \log |a_{\mathbf{r}}| \le c\,.$$

More generally, we would like to examine how a cycle $\mathcal{C}$ in $\mathcal{M}$ can be represented as the sum of cycles, each of which has been pushed down until reaching an obstacle at some stationary height. In the case where $h_{\hat{\mathbf{r}}}$ is proper, this is a classical result of Morse theory (when $\mathcal{V}$ is smooth) or more generally of stratified Morse theory. We briefly recall the relevant Morse theoretic notions.

Let $h : X \to \mathbb{R}$ be a proper smooth function on a stratified space X. Suppose that h has finitely many (stratified) stationary points $\mathbf{x}_1, \dots, \mathbf{x}_k$. For ease of exposition, assume the stationary values $c_1 := h(\mathbf{x}_1) > \cdots > c_k := h(\mathbf{x}_k)$ are distinct. Let (X_j^+, X_j^-) denote the pair $(X_{\le c_j - \varepsilon} \cup B_{2\varepsilon}(x_j), X_{\le c_j - \varepsilon})$ where ε is sufficiently small. The relative homology group $H_d(X_j^+, X_j^-)$ is called the *attachment group* at x_j. Stratified Morse theory guarantees that the attachment pairs generate all the homology of X. For example, the following proposition is well known (see, e.g., [PW13, Section B.2]).

Proposition 2.9 (attachments generate homology: proper case)**.**

(i) Every integer homology class $\mathcal{C} \in H_d(X)$ can be written as a finite integer combination of classes $\alpha \in H_d(X_j^+)$ projecting to a nonzero class in $H_d(X_j^+, X_j^-)$.

(ii) Let G_j be the image of $H_d(X_j^+)$ under the projection from X_j^+ to (X_j^+, X_j^-), that is, those relative homology classes representable by absolute cycles. If d is the homological dimension of X then $H_d(X) \cong \bigoplus_{j=1}^k G_j$.

(iii) When X is a smooth $2d$-manifold and h is harmonic, each attachment group is a homology d-sphere, each G_j is is the whole attachment group $H_d(X_k^+, X_k^-)$ and $H_d(X) \cong \bigoplus_{j=1}^k H_d(X_k^+, X_k^-)$.

Proof: The first statement follows from the homotopy equivalence between X_k^+ and X_{k-1}^+ (because Theorem 2.7 always holds for proper height functions) and the standard Morse filtration. The second statement follows from the long exact sequence for the pair (X_k^+, X_k^-) and the vanishing of $H_{d+1}(X_k^-)$. The third follows because attachments in smooth Morse theory are d-balls modulo their boundaries, where d is the index of the stationary point. $\square$

Remark 3. When the stationary values are not distinct one may add in the balls $B_{2\varepsilon}(x_j)$ one at a time, arriving at the same result.

Our main topological result is that all of this holds for the nonproper height function $h_{\hat{\mathbf{r}}}$ unless obstructed by stationary points at infinity.

Theorem 2.10 (attachments generate homology: general case)**.** *Fix $Q, \mathcal{V}$, $\mathcal{M} := \mathbb{C}_*^d \setminus \mathcal{V}$, and a Whitney stratification $\{\Sigma_\alpha\}$ of $(\mathbb{C}_*^d, \mathcal{V}_*)$. Fix also a direction $\hat{\mathbf{r}}$ and let $\mathbf{z}_1, \ldots \mathbf{z}_k$ be the affine stationary points with $c_j := h_{\hat{\mathbf{r}}}(\mathbf{z}_j)$ nonincreasing. Let $\mathcal{M}_j^\pm$ be the spaces $X_k^\pm$ from above, with $X = \mathcal{M}$.*

(i) Suppose $K_\infty = \emptyset$, that is, there are no H-SPAI. Then $H_d(\mathcal{M}) \cong \bigoplus_{j=1}^k G_j$ where G_j are the relative cycles in $H_d(\mathcal{M}_j^+, \mathcal{M}_j^-)$ represented by absolute cycles.

(ii) Suppose K_∞ is nonempty, having maximum element c. Let the $\{c_j\}$ be the stratified values as above, with s chosen so that $c_s > c \geq c_{s+1}$. Then any class $\mathcal{C} \in H_d(\mathcal{M})$ may be written as $\mathcal{C} = \beta + \sum_{j=1}^s \alpha_j$ where $\alpha_j \in G_j$ for $j \leq s$ and β is supported on $\mathcal{M}_{\leq c+\varepsilon}$.

Proof: Part (i) follows from Theorem 2.7 and Proposition 2.9 by deforming each $\mathcal{M}_j^-$ to $\mathcal{M}_{j+1}^+$. Part (ii) follows by performing the deformations only down to $j = s$. $\square$

Theorem 2.10 says that in Steps I–III of the asymptotic coefficient evaluation from Section 1.1 the only integrals that need to be evaluated are integrals over classes in each G_j. In general, these attachment classes are the best places to integrate. For example, as mentioned above, if a stationary point $\mathbf{x}$ is a smooth point of $\mathcal{V}_*$ then G will have a single generator which is a saddle point contour and an asymptotic expansion can be deduced automatically.

We also remark that the direct sum in part (i) and the expansion of $\mathcal{C}$ in part (ii) are not natural. Elements of G_j are equivalence classes modulo G_i for all $i > j$, and correspondingly α_j in part (ii) is determined only modulo linear combinations of α_i for $i > j$. However, the pair (j_*, α_{j_*}) is well defined, where j_* is the least index j for which $\alpha_j \neq 0$ (in part (ii), the least among $1, \ldots, s$).

In the remainder of Section 2 we expand on our underlying motivations, describing the application of Theorems 2.7 and 2.10 to the computation of cycles, integrals over these cycles, and coefficient asymptotics of multivariate rational functions. These results, collected from various prior and simultaneous works, can be skipped if one is only interested in examples, proofs and computations of stationary points at infinity.

2.3 Intersection classes on smooth varieties

We assume throughout this section that $\mathcal{V}$ is smooth. It is useful to be able to transfer between $H_d(\mathcal{M})$ and $H_{d-1}(\mathcal{V}_*)$: topologically this is the Thom isomorphism and, when computing integrals, corresponds to taking a single residue. We outline this construction, which goes back at least to Griffiths [Gri69]. Because ∇Q does not vanish on $\mathcal{V}$, the well known Collar Lemma [MS74, Theorem 11.1] states[5] that an open tubular vicinity of $\mathcal{V}$ is diffeomorphic to the space of the normal bundle to $\mathcal{V}$.

It follows that for any k-chain γ in $\mathcal{V}$ we can define a $(k+1)$-chain $\circ\gamma$, obtained by taking the boundary of the union of small disks in the fibers of the normal bundle. The radii of these disk should be small enough to fit into the domain of the collar map, but can (continuously) vary with the point on the base. Different choices of the radii matching over the boundary of the chain lead to homologous tubes. We will be referring to $\circ\gamma$ informally as the *tube around* γ. Similarly, the symbol $\bullet\gamma$ denotes the product with the solid disk. The elementary rules for boundaries of products imply

$$\begin{aligned} \partial(\circ\gamma) &= \circ(\partial\gamma)\,; \\ \partial(\bullet\gamma) &= \circ\gamma \cup \bullet(\partial\gamma)\,. \end{aligned} \tag{2.3}$$

Because $\circ$ commutes with ∂, cycles map to cycles, boundaries map to boundaries, and the map $\circ$ on the singular chain complex of $\mathcal{V}_*$ induces a map on homology $H_*(\mathbb{C}_*^d \setminus \mathcal{V})$; we also denote this map on homology by $\circ$ to simplify notation.

Proposition 2.11 (intersection classes)**.** *Suppose $\mathcal{V} := \{Q = 0\}$ is smooth, and define $\circ : H_{d-1}(\mathcal{V}_*) \to H_d(\mathcal{M})$ as above.*

(i) $\circ$ is injective and its image is the kernel of the map ι_* induced by the inclusion $\mathcal{M} \xrightarrow{\iota} \mathbb{C}_*^d$.

(ii) Given $\alpha \in \ker(\iota_*)$, one may compute the pullback $\mathcal{I}(\alpha) := \circ^{-1}(\alpha)$ by intersecting $\mathcal{V}_*$ with any $(d+1)$-chain in $\mathbb{C}_*^{d+1}$ whose boundary is α, and for which the intersection with $\mathcal{V}_*$ is transverse.

Specializing to $\alpha = \mathbf{T} - \mathbf{T}'$ where $\mathbf{T}$ and $\mathbf{T}'$ are two d-cycles in $\mathcal{M}$ homologous in $\mathbb{C}_d^*$, we call $\mathcal{I}(\mathbf{T} - \mathbf{T}')$ the **intersection class** *of $\mathbf{T}$ and $\mathbf{T}'$.*

Proof: The Thom-Gysin long exact sequence implies exactness in the following diagram,

$$H_{d+1}(\mathbb{C}_*^d) \xrightarrow{I_*} H_{d-1}(\mathcal{V}_*) \xrightarrow{\circ} H_d(\mathcal{M}) \to H_d(\mathbb{C}_*^d). \tag{2.4}$$

This may be found in [Gor75, page 127], taking $W = \mathbb{C}_*^d$, though in the particular situation at hand it goes back to Leray [Ler50]; here, the first mapping, I_*, denotes the map induced by transverse intersection, I. Injectivity of $\circ$ follows from the vanishing of $H_{d+1}(\mathbb{C}_*^d)$. The rest of part (i) follows from exactness at $H_d(\mathcal{M})$.

For part (ii), we begin by showing that I induces a well defined map from $\ker(\iota_*)$ to $H_{d-1}(\mathcal{V}_*)$. Given $\alpha \in \ker(\iota_*)$, because transversality is generic, there exist $(d+1)$-chains intersecting $\mathcal{V}_*$ transversely whose boundary is α. If $\mathcal{D}$ is such a chain and $\mathcal{C} = I(\mathcal{D})$ then $\mathcal{C}$ is a cycle:

$$\partial\mathcal{C} = \partial(\mathcal{D} \cap \mathcal{V}_*) = (\partial\mathcal{D}) \cap \mathcal{V}_* = \alpha \cap \mathcal{V}_* = \emptyset\,.$$

[5] See [Lan02] for a full proof.

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two such chains, and denote $\mathcal{C}_j := \mathcal{D}_j \cap \mathcal{V}_*$. Observe that $\mathcal{D}_1 - \mathcal{D}_2$ is null homologous because there is no $(d+1)$-homology in $\mathbb{C}_*^d$, whence

$$[\mathcal{C}_1 - \mathcal{C}_2] = [I(\mathcal{D}_1 - \mathcal{D}_2)] = 0\,,$$

showing that $[I(\mathcal{D})]$ for $\partial\mathcal{D} = \alpha$ is well defined in $H_{d-1}(\mathcal{V}_*)$.

Finally, if $\alpha = \mathsf{o}(\gamma)$ then taking $\mathcal{D} = \bullet(\gamma)$ gives $I(\mathcal{D}) = \gamma$, showing that I does in fact invert o, hence computes $\mathcal{I}$. $\square$

2.4 Integration

Integrals of holomorphic forms on a space X are well defined on homology classes in $H_*(X)$. Relative homology is useful for us because it defines integrals up to terms of small order. Throughout the remainder of the paper, $F = P/Q$ denotes a quotient of polynomials except when a more general numerator is explicitly noted. Let $\mathsf{amoeba}(Q)$ denote the amoeba $\{\log|\mathbf{z}| : \mathbf{z} \in \mathcal{V}_*\}$ associated to the polynomial Q, where log and $|\cdot|$ are taken coordinatewise[6]. Components B of the complement of $\mathsf{amoeba}(Q)$ are open convex sets and are in correspondence with convergent Laurent expansions $\sum_{\mathbf{r}\in E} a_{\mathbf{r}}\mathbf{z}^{\mathbf{r}}$ of $F(\mathbf{z})$, each expansion being convergent when $\log|\mathbf{z}| \in B$ and determined by the Cauchy integral (1.1) over the torus $\log|\mathbf{z}| = \mathbf{x}$ for any $\mathbf{x} \in B$.

Definition 2.12 (c_* and the pair $(\mathcal{M}, -\infty)$)**.** *Fix $\hat{\mathbf{r}}_*$ and let $c_* = c_*(\hat{\mathbf{r}}_*)$ denote the infimum of heights of stationary points, both affine and at infinity. Denote by $H_d(\mathcal{M}, -\infty)$ the homology of the pair $(\mathcal{M}, \mathcal{M}_{\leq c})$ for any $c < c_*$. By part (i) of Theorem 2.7, these pairs are all naturally homotopy equivalent.*

For functions of $\mathbf{r} \in E \subseteq (\mathbb{Z}^+)^d$, let $\simeq$ denote the relation of differing by a quantity decaying more rapidly than any exponential function of $|\mathbf{r}|$. If E consists of vectors $\mathbf{r}$ whose angle with a fixed $\mathbf{r}_*$ is bounded above by $\pi/2 - \varepsilon$, we note for use below that $h_{\hat{\mathbf{r}}} \leq \varepsilon h_{\hat{\mathbf{r}}_*}$, in other words, $h_{\hat{\mathbf{r}}}$ and $h_{\hat{\mathbf{r}}_*}$ go to $-\infty$ at comparable rates on E. Homology relative to $-\infty$ and equivalence up to superexponentially decaying functions are related by the following result.

Theorem 2.13. *Let $F = G/Q$ with Q rational and G holomorphic. Fix $\hat{\mathbf{r}}_*$ and suppose that $c_*(\hat{\mathbf{r}}_*) > -\infty$. For d-cycles C in $\mathcal{M}$, as $\mathbf{r}$ varies over a set E whose angle with $\hat{\mathbf{r}}_*$ is bounded above by $\pi - \varepsilon$, the $\simeq$ equivalence class of the integral $\displaystyle\int_{\mathcal{C}} \mathbf{z}^{-\mathbf{r}} F(\mathbf{z})\, d\mathbf{z}$ depends only on the relative homology class of $\mathcal{C}$ when projected to $H_d(\mathcal{M}, -\infty)$.*

Proof: Fix any $c < c_*$. Suppose $C_1 = C_2$ in $H_d(\mathcal{M}, -\infty)$. From the exactness of

$$H_d(\mathcal{M}_{\leq -c}) \to H_d(\mathcal{M}) \to H_d(\mathcal{M}, \mathcal{M}_{\leq c})\,,$$

observing that $C_1 - C_2$ projects to zero in $H_d(\mathcal{M}, \mathcal{M}_{\leq c})$, it follows that $C_1 - C_2$ is homologous in $H_d(\mathcal{M})$ to some cycle $C \in \mathcal{M}_{\leq c}$. Homology in $\mathcal{M}$ determines the integral exactly. Therefore, it suffices to show that $\int_C \mathbf{z}^{\mathbf{r}} F(\mathbf{z})\, d\mathbf{z} \simeq 0$.

As a consequence of the homotopy equivalence in part (i) of Theorem 2.7, for any $t < c_*$ there is a cycle C_t supported on $\mathcal{M}_{\leq t}$ and homologous to C in $\mathcal{M}$. Fix such a collection of cycles $\{C_t\}$. Let $M_t := \sup\{|F(\mathbf{z})| :$

[6] One should think of the amoeba as sitting in ξ-space, the real part of logspace.

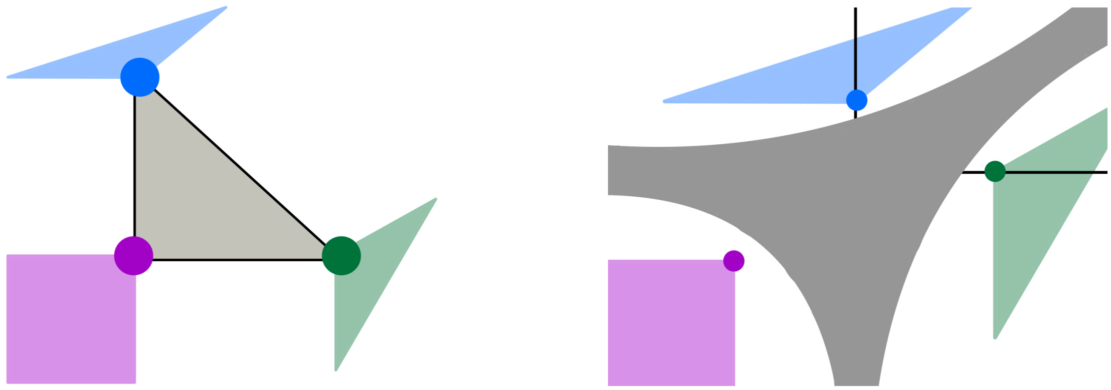

Figure 1: *Left:* The Newton polytope of $Q(x, y) = 1 - x - y$ together with the dual cones at each vertex. *Right:* The amoeba of $Q(x, y)$ together with the recession cones of the complement components.

$\mathbf{z} \in C_t\}$ and let V_t denote the volume of C_t. Observe that $|\mathbf{z}^{-\mathbf{r}}| = \exp(|\mathbf{r}| h_{\hat{\mathbf{r}}}(\mathbf{z})) \le \exp(\varepsilon |\mathbf{r}| h_{\hat{\mathbf{r}}_*}(\mathbf{z})) \le \exp(\varepsilon t |\mathbf{r}|)$ on C_t. It follows that

$$\begin{aligned} \int_C \mathbf{z}^{-\mathbf{r}} F(\mathbf{z})\, d\mathbf{z} &= \int_{C_t} \mathbf{z}^{-\mathbf{r}} F(\mathbf{z})\, d\mathbf{z} \\ &\le V_t M_t \exp(t|\mathbf{r}|). \end{aligned}$$

Because this inequality holds for all $t < c_*$, the integral is thus smaller than any exponential function of $|\mathbf{r}|$. $\square$

When $F = P/Q$ is rational we may strengthen determination up to $\simeq$ to exact equality for all but finitely many coefficients. The **Newton polytope**, denoted $\mathbf{P}$, is defined as the convex hull of degrees $\mathbf{m} \in \mathbb{Z}^d$ of monomials in Q. It is known (see, e.g., [FPT00]) that the components of $\mathsf{amoeba}(Q)^c$ map injectively into the integer points in $\mathbf{P}$, and that to each extreme point $\mathbf{P}$ corresponds a non-empty component. Moreover, this can be done in such a way that the recession cone of a component (collection of directions of rays contained in the component) equals the dual cone of the Newton polytope at the corresponding vertex; see Figure 1. Hence the linear function $\xi \mapsto -(\xi \cdot \mathbf{r})$ is unbounded from below on any component when $\mathbf{r}$ points in the same direction as any element of the dual cone of the Newton polytope $\mathbf{P}(Q)$ at the corresponding integer point. Fix the component B corresponding to the Laurent expansion $F = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ and integer point $\mathbf{v}$ in the Newton polytope. The closed dual cones at extreme points of the Newton polytope cover all of $\mathbb{R}^d$, therefore there exists a component B' of the amoeba complement (probably many components would do) with $h_{\hat{\mathbf{r}}}(\xi) \to -\infty$ linearly in $|\xi|$ as $\xi \to \infty$ in B'.

Proposition 2.14. *Let $\mathbf{T}(\xi)$ denote the centered torus with polyradii $\exp(\xi_1), \ldots, \exp(\xi_d)$. If $F = P/Q$ is rational and $\xi \in B'$, then*

$$\int_{\mathbf{T}(\xi)} \mathbf{z}^{-\mathbf{r}} F(\mathbf{z})\, d\mathbf{z} = 0$$

for all but finitely many $\mathbf{r} \in E$, the support of the Laurent expansion on B.

Proof: By assumption, there is a continuous path moving ξ to infinity within B'. On the corresponding tori, the (constant) value of $h_{\mathbf{r}}$ approaches $-\infty$. Let $\mathbf{T}(\xi_t)$ denote such a torus supported on $\mathcal{M}_{\le t}$. Because the tori are all homotopic in $\mathcal{M}$, the value of the integral

$$\int_{\mathbf{T}(\xi_t)} \mathbf{z}^{-\mathbf{r}} F(\mathbf{z})\, d\mathbf{z} \tag{2.5}$$

cannot change. On the other hand, with M_t and V_t as in the proof of the first part, both M_t and V_t are bounded by polynomials in $|\mathbf{z}|$, the common polyradius of points in $\mathbf{T}(\xi_t)$. Once any coordinate r_j is great enough so that the product of the volume and the maximum grows more slowly than $|z_j|^{r_j}$ the integral for that fixed $\mathbf{r}$ goes to zero as $t \to -\infty$, and thus is identically zero. $\square$

The utility of Proposition 2.14 is to represent the Cauchy integral as a tube integral. Let $\mathbf{T} = \mathbf{T}(\xi)$ for $\xi \in B$, the component of $\mathsf{amoeba}(Q)^c$ defining the Laurent expansion, and choose $\mathbf{T}' = \mathbf{T}(\xi')$ for $\xi' \in B'$ as in Proposition 2.14. By Proposition 2.11, if γ denotes the intersection class $\mathcal{I}(\mathbf{T}, \mathbf{T}')$, we have $\mathbf{T} = \circ\gamma + \mathbf{T}'$ in $H_d(\mathcal{M})$. By Proposition 2.14, the integral over $\mathbf{T}'$ vanishes for all but finitely many $\mathbf{r} \in E$, yielding

Corollary 2.15. *If $F = P/Q$ is rational and $\mathcal{V}$ is smooth, then there exists a $(d-1)$-cycle of integration γ in $\mathcal{V}_*$ such that for all but finitely many $\mathbf{r} \in E$,*

$$a_{\mathbf{r}} = \frac{1}{(2\pi i)^d} \int_{\circ\gamma} \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z})\, d\mathbf{z}\,.$$

$\square$

2.5 Residues on smooth varieties

This section again assumes that $\mathcal{V}_*$ is smooth. Having transferred homology from $\mathcal{M}$ to $\mathcal{V}_*$ via intersection classes, we transfer integration there as well via residues. The point of this is to obtain integrals amenable to a saddle point analysis: pushing down cycles until their maximum height is minimized drives the maximum to occur on $\mathcal{V}_*$, not on $\mathcal{M}$. Thus we need a reduction to saddle point integrals on $\mathcal{V}_*$ rather than on $\mathcal{M}$. In what follows, $H^*(X)$ denotes the holomorphic de Rham complex, whose k-cochains are holomorphic k forms. The following duality between residues and tubes is well known.

Proposition 2.16 (residue theorem)**.** *There is a functor* $\mathrm{Res} : H^d(\mathcal{M}) \to H_{d-1}(\mathcal{V}_*)$ *such that for any class $\gamma \in H_d(\mathcal{V})$ and every $\omega \in H^d(\mathcal{M})$,*

$$\int_{\circ\gamma} \omega = 2\pi i \int_\gamma \mathrm{Res}\,(\omega)\,. \tag{2.6}$$

The residue functor is defined locally and, when Q is squarefree, it commutes with products by any locally holomorphic scalar function. If, furthermore, $F = P/Q$ is rational, there is an implicit formula

$$Q \wedge \mathrm{Res}\,(F\, d\mathbf{z}) = P\, d\mathbf{z}\,.$$

For higher order poles, the residue can be computed by choosing coordinates: if $F = P/Q^k$, and locally $\{Q = 0\}$ defines a graph of a function, $\{z_1 = S(z_2, \ldots, z_d)\}$, then

$$\mathrm{Res}_{\mathcal{V}} \left[\mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}}\right] := \frac{1}{(k-1)!(\partial Q/\partial z_1)^k} \frac{d^{k-1}}{dz_1^{k-1}} \left[\frac{P\mathbf{z}^{-\mathbf{r}}}{\mathbf{z}}\right]\Bigg|_{z_1 = S(z_2,\ldots,z_d)} dz_2 \wedge \cdots \wedge dz_d\,. \tag{2.7}$$

Proof: Restrict to a neighborhood of the support of the cycle γ in the smooth variety $\mathcal{V}_*$ coordinatized so that the last coordinate is Q. The result follows by applying the (one variable) residue theorem, taking the residue in the last variable. $\square$

Applying this to intersection classes and using homology relative to $-\infty$ to simplify integrals yields the following representation.

Theorem 2.17. *Let $F = P/Q$ be the quotient of Laurent polynomials with Laurent series $\sum_{\mathbf{r}\in E} a_\mathbf{r} \mathbf{z}^\mathbf{r}$ converging on $\mathbf{T}(\mathbf{x})$ when $\mathbf{x} \in B$, where B is a component of $\mathsf{amoeba}(Q)^c$. Fix $\hat{\mathbf{r}}_*$ and assume the minimal stationary value $c_*(\hat{\mathbf{r}}_*)$ is finite and $K_\infty(\hat{\mathbf{r}}_*)$ is empty. Let B' denote a component of the complement of $\mathsf{amoeba}(Q)$ on which $h_{\hat{\mathbf{r}}_*}$ goes linearly to $-\infty$, as constructed prior to Proposition 2.14. Then for any $\mathbf{x} \in B$ and $\mathbf{y} \in B'$,*

$$a_\mathbf{r} = \frac{1}{(2\pi i)^{d-1}} \int_{\mathcal{I}(\mathbf{T},\mathbf{T}')} \operatorname{Res}\left(\mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}}\right)$$

for all but finitely many $\mathbf{r}$. If P is replaced by any holomorphic function then the same representation of $a_\mathbf{r}$ holds up to a function decreasing super-exponentially in $|\mathbf{r}|$.

Proof: If P is polynomial then, for all but finitely many $\mathbf{r} \in E$,

$$\begin{aligned}
(2\pi i)^{d-1} a_\mathbf{r} &= \frac{1}{2\pi i} \int_{\mathbf{T}(\mathbf{x})} \mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}} \\
&= \frac{1}{2\pi i} \int_{\circ\mathcal{I}(\mathbf{T},\mathbf{T}')} \mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}} + \frac{1}{2\pi i} \int_{\mathbf{T}(\mathbf{y})} \mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}} \\
&= \frac{1}{2\pi i} \int_{\circ\mathcal{I}(\mathbf{T},\mathbf{T}')} \mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}}\,.
\end{aligned}$$

The first line above is Cauchy's integral formula, the second is Proposition 2.11, and the third is Corollary 2.15 or Proposition 2.14. By the Residue Theorem,

$$\frac{1}{2\pi i} \int_{\circ\mathcal{I}(\mathbf{T},\mathbf{T}')} \mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) = \int_{\mathcal{I}(\mathbf{T},\mathbf{T}')} \mathbf{z}^{-\mathbf{r}} \operatorname{Res}\left(F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}}\right),$$

proving the theorem when P is a polynomial. When P is not polynomial, use Theorem 2.13 in place of Proposition 2.14 in the last line. $\square$

Combining Theorems 2.10 and 2.17 yields the most useful form of the result: a representation of the coefficients $a_\mathbf{r}$ in terms of integrals over relative homology generators produced by the stratified Morse decomposition. In the following theorem we remove the assumption that $\mathcal{V}_*$ is smooth, though we still use residues at the smooth points.

Let $\sigma_1, \ldots, \sigma_m$ enumerate the stationary points of $\mathcal{V}_*$ in weakly decreasing order of height $c_1 \geq c_2 \geq \cdots \geq c_m$. For each j, denote the relevant homology pair by

$$(X_j^+, X_j^+) := (\mathcal{V}_{\leq c_j - \varepsilon} \cup B_j, \mathcal{V}_{\leq c_j - \varepsilon}) \tag{2.8}$$

where B_j is a sufficiently small ball around σ_j in $\mathcal{V}$. Let $k_j := \dim H_{d-1}(X_j^+, X_j^-)$ and let $\beta_{j,1}, \ldots, \beta_{j,k_j}$ denote cycles in $H_{d-1}(X_j^+)$ that project to a basis for $H_{d-1}(X_j^+, X_j^-)$ with integer coefficients.

In the case where σ_j is a smooth point of $\mathcal{V}$, stratified Morse theory [GM88] implies that $k_j = 1$ and $\beta_{j,1} = \circ\gamma_j$ is a cycle agreeing locally with a tube around the unstable manifold γ_j for the downward $h_{\hat{\mathbf{r}}}$ gradient flow on $\mathcal{V}$. This leads to the following decomposition for $a_\mathbf{r}$.

Theorem 2.18 (stratified Morse homology decomposition)**.** *Let $F = P/Q$ be rational. Fix $\hat{\mathbf{r}}_*$, assume $K_\infty(\hat{\mathbf{r}}_*)$ is empty, and enumerate the affine stationary points $\sigma_1 \ldots, \sigma_m$ as above. Then there are integers $\{n_{j,i} : 1 \le j \le m, 1 \le i \le k_j\}$ such that*

$$a_{\mathbf{r}} = \frac{1}{(2\pi i)^{d-1}} \sum_{j=1}^{m} \sum_{i=1}^{k_j} n_{j,i} \int_{\beta_{j,i}} \mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}} \,. \tag{2.9}$$

When σ_j is a smooth point of $\mathcal{V}_*$, then $k_j = 1$, the cycle β_j agrees locally with a tube γ_j around the unstable manifold γ_j at σ_j for the downward gradient flow of $h_{\hat{\mathbf{r}}_*}$ on $\mathcal{V}$ and the corresponding summand in (2.9) is given by

$$\int_{\gamma_j} \operatorname{Res} \left(\mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}} \right) .$$

Theorem 2.18 is the culmination of this Section, and provides a crucial (and highly desired) tool for analytic combinatorics in several variables. Let j_* be the least j for which some $n_{j,i}$ is nonzero. Generically the dominant asymptotic term is then the sum of the terms in (2.9) with $n_{j,i} = 0$ and $h_{\hat{\mathbf{r}}_*}(\sigma_j) = h_{\hat{\mathbf{r}}_*}(\sigma_{j_*})$. The full expansion has several benefits over a statement of the leading term only. First, one might not know j_*. In fact in [BMP19], the asymptotics of the diagonal coefficients are settled only by computing all the residue integrals, then determining the integers $\{n_{j,i}\}$ via rigorous numerics. Secondly, knowing the subdominant terms allows one to compute error estimates in the case where the exponential rates are very close or are converging to one another. Thirdly, sometimes G_j is generated by a local cycle, supported in an arbitrarily small neighborhood of σ_j. This is a natural choice for α_j, whence the next asymptotic terms are meaningful. Fourthly, one may want a trans-series expansion of $a_{\mathbf{r}}$, for which the contributions of all orders are needed.

3 Computation of stationary points at infinity

We begin by recalling some background about stratifications and affine stationary points.

Computing a stratification

To compute stationary points one requires a stratification. Often, there is an obvious stratification; for example, polynomial varieties are generically smooth, in which case the trivial stratification $\{\mathcal{V}\}$ suffices[7] In non-generic cases, however, one must produce a stratification of $\mathcal{V}$ before proceeding with the search for stationary points.

There are two relevant facts to producing a stratification. One is that there is a coarsest possible Whitney stratification, called the *canonical Whitney stratification* of $\mathcal{V}$. It is shown in [Tei82, Proposition VI.3.2] that there are algebraic sets $\mathcal{V} = F_0 \supset F_1 \supset \cdots \supset F_m = \emptyset$ such that the set of all connected components of $F_i \setminus F_{i+1}$ for all i forms a Whitney stratification of $\mathcal{V}$ and such that every Whitney stratification of $\mathcal{V}$ is a refinement of this stratification. This canonical stratification is effectively computable: an algorithm exists, given Q, to determine generators for the radical ideals corresponding to the Zariski closed sets F_i.

[7] Formally one must join with the stratification generated by the coordinate planes, so smooth manifolds intersecting coordinate subspaces nontransversely (which again, is non-generic) might require refinement.

Mostowski and Rannou [MR91] give an algorithm to compute stratifications using quantifier elimination, leading to a bound on the computation time which is doubly exponential in m; an alternative algorithmic approach is presented in [DJ21, Section 2]. In our experience, the large doubly-exponential upper bound is somewhat more pessimistic than what one has to deal with on actual combinatorial examples.

Computing the affine stationary points

Assume now that a Whitney stratification $\{\Sigma_\alpha : \alpha \in A\}$ is given, meaning the index set A is stored, along with, for each $\alpha \in A$, a collection of polynomial generators $f_{\alpha,1}, \ldots, f_{\alpha,m_\alpha}$ for the radical ideal $\mathcal{I}(\Sigma_\alpha)$. The set Σ_α is the algebraic set $V_\alpha := \mathbb{V}(f_{\alpha,1}, \ldots, f_{\alpha,m_\alpha})$ minus the union of varieties V_β of higher codimension. Potentially by replacing $\mathcal{I}(\Sigma_\alpha)$ with its prime components, we may assume that the tangent space of V_α at any smooth point has constant codimension k_α. After computing the canonical Whitney stratification, recall that we refine if necessary to ensure that the defining ideal for each stratum of co-dimension k has k generators with linearly independent differentials at every point.

By Definition 2.4 the set of affine stationary points $\mathsf{crit}^{\text{aff}}(\Sigma_\alpha, \mathbf{y})$ in the direction $\mathbf{y}$ is defined, after removing points in varieties of higher codimension, by the ideal containing the polynomials $f_{\alpha,1}, \ldots, f_{\alpha,m_\alpha}$ together with the $(c+1) \times (c+1)$ minors of $\mathbf{J}(\mathbf{z}, \mathbf{y})$. Taking the union over all strata Σ_α produces all the affine stationary points. This description simply restates the so-called 'critical point equations' given in [PW13, (8.3.1-8.3.2)], or, in the common special case of a stratum of co-dimension one, more explicitly by (8.3.3) therein.

For the fixed integer vector $\mathbf{r}$ and height interval $[a, b]$, the inequalities $h_{\hat{\mathbf{r}}}(\mathbf{z}) \in [a, b]$ impose further semi-algebraic constraints. Unfortunately, these increase the complexity considerably and behave badly under perturbations of the integer vector $\mathbf{r}$. If one can compute open cones of values of $\mathbf{r}$ in which the structure of the computation does not change, one can then pick a single $\mathbf{r}$ in the cone to minimize complexity, making for a feasible computation. Otherwise, one must settle for computations based on a fixed direction $\mathbf{r}$.

Computing stationary points at infinity

To determine whether there exist stationary points at infinity we use ideal quotients, corresponding to the difference of algebraic varieties. Recall that the variety $\mathbb{V}(I : J^\infty)$ defined by the saturation $I : J^\infty$ of two ideals I and J is the Zariski closure of the set difference $\mathbb{V}(I) \setminus \mathbb{V}(J)$ (see [CLO92, Section 4.4]), and can be determined through Gröbner basis computations.

Definition 3.1 (saturated stationary point ideal $\mathfrak{C}_\alpha$).

- *For a stratum Σ_α let C_α be the projective ideal defining $\mathsf{crit}^{\text{aff}}(\Sigma_\alpha, \mathbf{y})$. In other words, taking the homogenizing variable to be z_0, the ideal C_α is generated by the homogenizations in the z variables of both $f_{\alpha,1}(\mathbf{z}, \mathbf{y}), \ldots, f_{\alpha,m_\alpha}(\mathbf{z}, \mathbf{y})$ and the $(c+1) \times (c+1)$ minors of $\mathbf{J}(\mathbf{z}, \mathbf{y})$.*

- *Let D_α denote the ideal generated by $z_0 z_1 \cdots z_d$ and the homogenizations of all polynomials $f_{\beta,j}$ in strata of higher codimension $k_\beta > k_\alpha$.*

- *Define $\mathfrak{C}_\alpha$ to be the result of saturating C_α by the ideal D_α.*

Geometrically, the variety $\mathbb{V}(\mathfrak{C}_\alpha)$ is the Zariski closure of $\mathbb{V}(C_\alpha) \setminus (\mathbb{V}(z_0 z_1 \cdots z_d) \cup \mathbb{V}(\mathfrak{F}_{j+1}))$, that is, the closure of that part of the graph of the relation $\mathbf{z} \in \mathsf{crit}(\mathbf{y})$ in $\mathbb{CP}^d \times \mathbb{CP}^{d-1}$ corresponding to points $(\mathbf{z}, \mathbf{y})$ whose $\mathbf{z}$ component is not on a substratum and not at infinity (including the coordinate planes). Note that in this setting, the Zariski closure equals the classical topological closure [Mum76, Theorem 2.33].

Definition 3.2 (saturated ideals)**.** *Fix a stratum Σ_α.*

- *Let $\mathfrak{C}_\alpha^\infty$ denote the result of substituting $z_0 = 0$ in $\mathfrak{C}_\alpha$.*
- *Let $\mathfrak{C}_\alpha^\infty(\hat{\mathbf{r}})$ denote the result of substituting $\mathbf{y} = \hat{\mathbf{r}}$ in $\mathfrak{C}_\alpha^\infty$.*

The variety $\mathbb{V}(\mathfrak{C}_\alpha^\infty)$ finds all SPAI. The variety $\mathbb{V}(\mathfrak{C}_\alpha^\infty(\hat{\mathbf{r}}))$ finds all SPAI in a given direction $\hat{\mathbf{r}}$, that is, all limits of affine points in $\mathsf{crit}(\Sigma_\alpha, \mathbf{y})$ with $\mathbf{y} \to \hat{\mathbf{r}}$. This is stated in the following proposition, whose proof follows directly from our definitions.

Proposition 3.3 (computability of stationary points at infinity)**.** *(i) The rational function $F(\mathbf{z}) = P(\mathbf{z})/Q(\mathbf{z})$ has SPAI if and only if for some α there is a projective solution to $\mathfrak{C}_\alpha^\infty$, in other words, a solution other than $(0, \ldots, 0)$.*

(ii) The rational function $F(\mathbf{z}) = P(\mathbf{z})/Q(\mathbf{z})$ has SPAI in direction $\hat{\mathbf{r}}$ if and only if for some α there is a projective solution to $\mathfrak{C}_\alpha^\infty(\hat{\mathbf{r}})$.

□

Proposition 3.3 computes a superset of what we need: SPAI regardless of height. We only care about those at finite heights, indeed heights above the least affine stationary value. Unfortunately, we do not know a way to automate the height computation, which is not polynomial; doing so is an interesting problem for further research.

Problem 1. *Find an effective way to compute $\mathsf{crit}_{[a,b]}(\hat{\mathbf{r}})$, for irrational $\hat{\mathbf{r}}$ or as a symbolic computation in $\hat{\mathbf{r}}$.*

When $\hat{\mathbf{r}}$ is rational, we can do a little better. In this case the height(s) may be computed from the start along with the stationary points themselves because the exponentiated heights are polynomial. This gives the following corollary, whose use is illustrated in the upcoming examples.

Corollary 3.4. *Let $\mathbf{r}$ be an integer vector. Introducing one more variable η, let H_α denote the ideal generated by C_α along with $\eta z_0^{|\mathbf{r}|} - \mathbf{z}^\mathbf{r}$. Let $\mathfrak{H}_\alpha$ be the result of saturating by D_α, let $\mathfrak{H}_\alpha^\infty$ be the result of substituting $z_0 = 0$, and let $\mathfrak{H}_\alpha^\infty(\mathbf{r})$ be the result of further substituting $\mathbf{y} = \mathbf{r}$. Then there exists a H-SPAI of height $\log c$ in direction $\mathbf{r}$ if and only if there is a solution to $\mathfrak{H}_\alpha^\infty(\mathbf{r})$ with η-coordinate equal to c.*

4 Examples

When Q is square-free and $\mathcal{V}$ is smooth, $\mathcal{V}$ itself forms a stratification. The pseudocode in Algorithm 1 computes stationary points at infinity in this case (the pseudocode in Algorithm 2 computes stationary points at infinity in the general case but requires the algebraic sets defining the canonical Whitney stratification of

$\mathcal{V}$ as input). We have implemented this algorithm, and more general variants, in Maple. A Maple worksheet with our code and examples is available from ACSVproject.org (search for this paper) and the authors' webpages.

Algorithm 1: Stationary points at infinity (smoothness assumption)

Input: Polynomial $Q \in \mathbb{Z}[\mathbf{z}]$ and direction $\mathbf{r} \in \mathbb{Z}^d$ with $\mathcal{V}(Q)$ smooth

Output: Ideal $\mathfrak{C}'$ in the variables of Q and a homogenizing variable z_0 such that there is a stationary point at infinity if and only if the generators of $\mathfrak{C}'$ have a non-zero solution.

If Q is not square-free replace it with its square-free part (the product of its distinct irreducible factors);

Let $\tilde{Q} = z_0^{\deg Q} Q(z_1/z_0, \ldots, z_d/z_0)$;

Let C be the ideal generated by $\tilde{Q}$ and

$$y_j z_1 (\partial \tilde{Q}/\partial z_1) - y_1 z_j (\partial \tilde{Q}/\partial z_j), \qquad (2 \leq j \leq d);$$

Saturate C by $z_0 z_1 \cdots z_d$ to obtain the ideal $\mathfrak{C}$;

Substitute $y_j = r_j$ for $1 \leq j \leq d$ and return the resulting ideal with the generator $z_0 z_1 \cdots z_d$ added.

Algorithm 2: Stationary points at infinity (no smoothness assumption)

Input: Polynomial $Q \in \mathbb{Z}[\mathbf{z}]$, direction $\mathbf{r} \in \mathbb{Z}^d$ and polynomial generators of algebraic sets $F_0 \supset F_1 \supset \cdots \supset F_m$ defining the canonical Whitney stratification of the zero set of Q

Output: Set of ideals $\mathcal{S}$ in the variables of Q and a homogenizing variable z_0 such that there is a stationary point at infinity if and only if there exists $\mathfrak{C}' \in \mathcal{S}$ whose generators have a non-zero solution.

Set $\mathcal{S} = \emptyset$

For j from 1 to $m - 1$:

  Compute the prime decomposition of ideals $F_j = P_1 \cap \cdots \cap P_r$

  For each $I \in \{P_1, \ldots, P_r\}$:

    Let c be the codimension of I;

    Let C be the ideal generated by I together with the $(c+1) \times (c+1)$ minors of the matrix $\mathbf{J}(\mathbf{z}, \mathbf{y})$ in (2.2) with the f_i polynomials set to the generators of I;

    Homogenize C in the $\mathbf{z}$ variables with the new variable z_0 to obtain the ideal C';

    Saturate C' by $z_0 z_1 \cdots z_d$ to obtain the ideal $\mathfrak{C}$;

    Substitute $y_j = r_j$ into $\mathfrak{C}$ for $1 \leq j \leq d$ and put this ideal with $z_0 z_1 \cdots z_d$ added into $\mathcal{S}$;

Return $\mathcal{S}$

We give three examples in the smooth bivariate case, always computing in the main diagonal direction $\mathbf{r} = (1, 1)$ and examining the coefficients $a_{n,n}$ of $1/Q$. In the first example there is no affine stationary point; it follows that there must be a stationary point at infinity which determines the exponential growth rate. In the second there are both stationary points at infinity and affine stationary points, with the stationary point at infinity being too low to matter. In the third, the stationary point at infinity is higher than all the affine ones and controls the exponential growth rate of diagonal coefficients.

Example 2 (smooth case, stationary point at infinity)**.** *Let $Q(x, y) = 2 - xy^2 - 2xy - x + y$, so that $\mathcal{V}$ is smooth and we can use the above code for the diagonal direction $\mathbf{r} = (1, 1)$. First, an examination of the polynomial system $Q = \partial Q/\partial x - \partial Q/\partial y = 0$ shows there are no affine stationary points. Thus, if there were* *no stationary points at infinity then the diagonal coefficients of $Q(x,y)^{-1}$ would decay super-exponentially. It is easy to see that this does not happen, for example by extracting the diagonal. This may be done via the Hautus-Klarner-Furstenberg method [HK71, BDS17], giving $\Delta Q(x,y)^{-1} = (1-z)^{-1/2}/2$.*

We compute the existence of stationary points at infinity in the diagonal direction using our Maple implementation via the command

```
SPatInfty(2 - x*y^2 - 2*x*y - x + y , [1,1])
```

This returns the ideal

```
 [(H - 1)^2 , Z, y (H - 1), x ]
```

where Z is the homogenizing variable, meaning the projective point $(Z : x : y) = (0 : 0 : 1)$ is a stationary point at infinity, which has height $\log|1| = 0$. The stationary point at infinity is a topological obstruction to the gradient flow across height 0, pulling trajectories to infinity; it suggests that the diagonal power series coefficients of $1/Q$ do not grow exponentially nor decay exponentially (in fact they decay like a constant times $1/\sqrt{n}$). Further geometric analysis of this example is found in [DeV11, pages 120–121].

Example 3 (highest stationary point is affine)**.** *Let $Q(x,y) = 1 - x - y - xy^2$. To look for stationary points at infinity in the diagonal direction, we execute* `SPatInfty(Q,[1,1])` *to obtain the ideal*

```
[(H + 1)(4H^2 + 4H - 1), Z, y(H + 1), x]
```

showing that $(x : y : Z) = (0 : 1 : 0)$ is a stationary point at infinity at height $\log|-1| = 0$. This time, there is an affine stationary point $(1/2, \sqrt{2}-1)$ of greater height. This affine point is easily seen to be a topological obstruction and therefore controls the exponential growth. Theorem 2.17 allows us to write the resulting integral as a saddle point integral in $\mathcal{V}_*$ over a class local to $(1/2, \sqrt{2}-1)$, thereby producing an asymptotic expansion with leading term $a_{n,n} \sim cn^{-1/2}(2+\sqrt{2})^n$.

Example 4 (stationary point at infinity dominates affine points)**.** *Let $Q(x,y) = -x^2y - 10xy^2 - x^2 - 20xy - 9x + 10y + 20$. This time* `SPatInfty(Q,[1,1])` *produces*

```
[(2 H^4 - 11 H^3 + 171 H^2 - 1382 H + 3220) (H-1)^2, Z, y (H-1), x]
```

Again, because x and y cannot both vanish, the only height of a stationary point at infinity is $\log|1| = 0$. There are four of affine stationary points: a Gröbner basis computation produces one conjugate pair with $|xy| \approx 9.486$ and another conjugate pair with $|xy| \approx 4.230$. Both of these lead to exponentially decreasing contributions, meaning the point at infinity could give a topological obstruction for establishing asymptotics. If there is an obstruction, it would increase the exponential growth rate of diagonal coefficients from 4.23^{-n} to no exponential growth or decrease. To settle this, we can compute a linear differential equation satisfied by the sequence of diagonal coefficients. This reveals that the diagonal asymptotics are of order $a_{n,n} \asymp n^{-1/2}$, meaning the exponential growth rate on the diagonal is in fact one.

Example 5 (dominant asymptotics with no SPAI). *If $Q(x,y,z) = 1 - x - y - z - xy$ then running* `SPatInfty(Q,[1,1,1])` *shows there are no SPAI. The two affine stationary points are computed easily,*

$$\sigma_1 = \left(-\frac{3+\sqrt{17}}{4}, -\frac{3+\sqrt{17}}{4}, \frac{7+\sqrt{17}}{8}\right)$$

$$\sigma_2 = \left(-\frac{3-\sqrt{17}}{4}, -\frac{3-\sqrt{17}}{4}, \frac{7-\sqrt{17}}{8}\right),$$

so Theorem 2.18 implies

$$a_{n,n,n} = \frac{1}{(2\pi i)^2} \sum_{j=1}^{2} \int_{\beta_j} \mathbf{z}^{-\mathbf{r}} \mathrm{Res}\left(F(\mathbf{z})\,\frac{d\mathbf{z}}{\mathbf{z}}\right)$$

where β_1 and β_2 are respectively the downward gradient flow arcs on $\mathcal{V}$ at σ_1 and σ_2. Saddle point integration gives an asymptotic series for each, the series for σ_1 dominating the series for σ_2, yielding an asymptotic expansion for $a_{n,n,n}$ beginning

$$a_{n,n,n} = \left(\frac{3+\sqrt{17}}{2}\right)^{2n} \left(\frac{7+\sqrt{17}}{4}\right)^{n} \cdot \frac{2}{n\pi\sqrt{26\sqrt{17}-102}}\left(1 + O\left(\frac{1}{n}\right)\right).$$

The following application concerns an analysis in a case where $\mathcal{V}$ is not smooth. There is an interesting singularity at $(1/3, 1/3, 1/3, 1/3)$ and a geometric analysis involving a lacuna [BMP19], which depends on there being no stationary points at infinity at finite height. This example illustrates shows the full power of the results in Sections 2.3 – 2.5.

Example 6 (application to GRZ function). *In [BMPS18], asymptotics are derived for the diagonal coefficients of several classes of symmetric generating functions, including the family of 4-variable functions $\{1 - x - y - z - w + Cxyzw : C > 0\}$ attributed to Gillis, Reznick and Zeilberger [GRZ83]. The most interesting case is when the parameter C passes through the stationary value 27: diagonal extraction and univariate analysis show the exponential growth rate of the main diagonal to have a discontinuous jump downward at the stationary value [BMPS18, Section 1.4]. This follows from Corollary 2.8 if we show there are in fact no stationary points at infinity above this height. Here there is a single point where the zero set of Q is non-smooth, the point $(1/3, 1/3, 1/3, 1/3)$. Running Algorithm 1 with the modification to remove the single non-smooth point yields the ideal* `[Z,z^4,-z+y,-z+x,-z+w]` *which has only the trivial solution $Z = w = x = y = z = 0$. Thus, there is no stationary point at infinity for the diagonal direction.*

There are three affine stationary points, one at $(1/3, 1/3, 1/3, 1/3)$, one at $(\zeta, \zeta, \zeta, \zeta)$ and one at $(\overline{\zeta}, \overline{\zeta}, \overline{\zeta}, \overline{\zeta})$, where $\zeta = (-1 - i\sqrt{2})/3$. Call these points $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, and $\mathbf{x}^{(3)}$. The stationary point with the greatest value of $h_{\mathbf{r}}$ (in the main diagonal direction) is $(1/3, 1/3, 1/3, 1/3)$, which has height $\log 81$. The other stationary points both have height $\log 9$. By a somewhat involved topological process, it is checked in [BMP19] that $\pi\mathbf{T} = 0$ in $H_4(B \cup \mathcal{M}_{\log 81} - \varepsilon, \mathcal{M}_{\log 81} - \varepsilon)$, where B is a small ball centered at $\mathbf{x}^{(1)}$. Crucially for the analysis after that, it follows from Proposition 2.10 that $\mathbf{T}$ is homologous to a chain supported in $\mathcal{M}_{c_2+\varepsilon}$. In other words, $\mathbf{T}$ can be pushed down until hitting obstructions at $(\zeta, \zeta, \zeta, \zeta)$ and $(\overline{\zeta}, \overline{\zeta}, \overline{\zeta}, \overline{\zeta})$. In fact an alternative analysis using a differential equation satisfied by the diagonal verifies a growth rate of 9^n, not 81^n. By means of Theorem 2.17, the coefficients $a_{n,n,n,n}$ may be represented as a residue integrals and put in standard saddle point form. When this is done, one obtains the more precise asymptotic $Kn^{-3/2}9^n$. The value of K depends on geometric invariants (curvature) and topological invariants (intersection numbers) and can be deduced by rigorous numeric methods. Details are given in [BMP19, Section 8].

We end with a three-dimension example with SPAI that are irrelevant for asymptotics.

Example 7 (irrelevant SPAI in three dimensions)**.** *Consider the diagonal direction and*

$$Q(x, y, z) = 1 - x + y - z - 2xy^2z.$$

There are two affine stationary points, $\mathbf{x}_\pm = \left(\frac{1}{3}, \frac{9\pm\sqrt{105}}{4}, \frac{1}{3}\right)$. *Running our algorithm shows there is also a stationary point at infinity of height* $-\log|-1/2| = \log(2)$. *Since the height of* $\mathbf{x}_-$ *is* $-\log\left|\frac{9-\sqrt{105}}{36}\right| > \log(2)$, *the stationary point at infinity does not affect dominant asymptotics of* $1/Q$. *One can get a mental picture of the situation by examining the Newton polytope of* Q. *Due to the monomials* x, y, *and* z, *the dual cone of the Newton polytope at the origin is the negative orthant. Thus, the component* B *of* $\mathsf{amoeba}(Q)$ *corresponding to the power series expansion of* $1/Q$ *admits the negative orthant as its recession cone. This implies one cannot move along a direction perpendicular to* $(1, 1, 1)$ *and stay in* B, *so the stationary point at infinity comes from other components of the amoeba complement. In fact, the stationary point at infinity lies on the closure of the complements of* $\mathsf{amoeba}(Q)$ *corresponding to the vertices* $(0, 1, 0)$ *and* $(1, 2, 1)$ *of the Newton polytope of* Q*; it can be directly verified that both of these components have a recession cone containing a vector normal to the diagonal direction. Ultimately, the lack of a stationary point at infinity of highest height implies an asymptotic expansion of the diagonal of* $1/Q$ *beginning*

$$a_{n,n,n} = \left(-\frac{27 + 3\sqrt{105}}{2}\right)^n \cdot \frac{\sqrt{3}}{2n\pi}\left(1 + O\left(\frac{1}{n}\right)\right).$$

5 Proof of Theorem 2.7

In the log-space, the phase function h becomes the linear function $\tilde{h}$ mapping $\mathbf{x}$ to $\langle \hat{\mathbf{r}}, \mathbf{x}\rangle$. We denote by $d\tilde{h}_\Sigma(\mathbf{x})$ the tangential differential, meaning the restriction of the differential of the phase function (in log space) to the tangent space to Σ at the point $\mathbf{x} \in \Sigma$.

Lemma 5.1. *Assume that* $\mathsf{crit}^\infty_{[a,b]}(\hat{\mathbf{r}}) = \emptyset$ *for some real interval* $[a, b]$. *Then for every neighborhood* $\mathcal{N}$ *of* $\mathsf{crit}^{\mathrm{aff}}_{[a,b]}(\hat{\mathbf{r}})$, *there is a* $\delta > 0$ *such that*

$$\left|d\tilde{h}_\Sigma(\mathbf{x})\right| \geq \delta \tag{5.1}$$

at every affine point $\mathbf{x} \in \tilde{\mathcal{V}} \setminus \mathcal{N}$ *whose height is in the interval* $[a, b]$. *In particular, if there are no affine stationary points then* $|d\tilde{h}_\Sigma(\mathbf{x})|$ *is globally at least* δ *for some* $\delta > 0$.

Proof: It suffices to prove this separately for each of the finitely many strata. We may therefore fix $\hat{\mathbf{r}}$ and Σ. Also fixing $a < b$, we let $\Sigma_{[a,b]}$ denote the intersection of a stratum Σ in the log space with the set of points having heights in $[a, b]$. Assume towards a contradiction that the norm of the tangential differential is *not* bounded from below on $\Sigma_{[a,b]} \setminus \mathcal{N}$ where $\mathcal{N}$ is a neighborhood of $\mathsf{crit}^{\mathrm{aff}}(\hat{\mathbf{r}})$ in the log space. Let $\mathbf{x}_k$ be a sequence in $\Sigma_{[a,b]} \setminus \mathcal{N}$ for which the left-hand side of (5.1) goes to zero; this sequence has no limit points whose height lies outside of $[a, b]$ and no limit points in $\mathsf{crit}^{\mathrm{aff}}_{[a,b]}(\hat{\mathbf{r}})$. There are also no affine limit points outside of $\mathsf{crit}(\hat{\mathbf{r}})$; this is because if $\mathbf{x} \to \mathbf{y}$ with $\mathbf{y}$ in a stratum $\Sigma_y \subset \overline{\Sigma}$ then $|d\tilde{h}_{\Sigma_y}(\mathbf{y})| \leq \liminf_{\mathbf{x}\to\mathbf{y}} |d\tilde{h}_\Sigma(\mathbf{x})|$, since the projection of the differential onto a substratum is at most the projection onto Σ.

By compactness $\{\mathbf{x}_k\}$ must have a limit point $\mathbf{x} \in \mathbb{CP}^d$. It follows from ruling out noncritical points, and affine stationary points with heights inside or outside of $[a, b]$, that $\mathbf{x} \in H_\infty$. Passing to a subsequence if necessary, $\tilde{h}(\mathbf{x}_k)$ converges to a point $c \in [a, b]$. The differential of the phase on the log space is the constant co-vector $\hat{\mathbf{r}}$. Therefore, convergence of the norm of the tangential projection of the differential to zero is equivalent to the projection $\mathbf{y}_k$ of $\hat{\mathbf{r}}$ onto the normal space to Σ at $\mathbf{x}_k$ converging to $\hat{\mathbf{r}}$ (here we make repeated use of the identification of normal spaces to the phase function at different points of log-space). The points $(\mathbf{x}_k, \mathbf{y}_k, \tilde{h}(\mathbf{x}_k))$ have thus been shown to converge to $(\mathbf{x}, \hat{\mathbf{r}}, c)$. Furthermore, each $(\mathbf{x}_k, \mathbf{y}_k, \tilde{h}(\mathbf{x}_k))$ is in $R(\Sigma, \hat{\mathbf{r}})$ due to the choice of $\mathbf{y}_k$ as a nonzero vector in the normal space to Σ at $\mathbf{x}_k$. The sequence is therefore a witness to a stationary point at infinity of height c, contradicting the hypothesis and proving the lemma. $\square$

Recall from [Mat12, Section 9] the notion of a stratified vector field. Given a Whitney stratification of $\mathcal{L}$, a **stratified vector field** $\mathbf{v}$ is defined to be a collection $\{\mathbf{v}_\Sigma\}$ of smooth sections of $(\Sigma, T\Sigma)$. Greatly summarizing Sections 7 and 8 of [Mat12], a stratified vector field is said to be **controlled** if for any strata $\Sigma \subseteq \partial\Sigma'$,

(i) $\langle d\rho, \mathbf{v}_{\Sigma'} \rangle = 0$ where ρ is the squared distance function to Σ in a given local product structure, and

(ii) projection from Σ' to Σ in this local product structure maps $\mathbf{v}_{\Sigma'}$ to $\mathbf{v}_\Sigma$.

Proposition 9.1 of [Mat12], with $P = \mathbb{R}$, $f = h$ and ζ the constant vector field $-d/dx$ on $\mathbb{R}$, says that there is a controlled lift of ζ to $\mathcal{L}$, that is a controlled stratified vector field $\mathbf{v}$ mapping by h_* to ζ. This will be almost enough to prove Theorem 2.7. What we need in addition is a uniform bound on $|\mathbf{v}|$. Although Mather (and Thom before him) was not interested in bounding $|\mathbf{v}|$ (indeed, their setting did not allow a meaningful metric), his proof in fact gives such a bound, as we now show.

Lemma 5.2. *Let $a < b$ be real and suppose $\{\Sigma_\alpha\}$ is a Whitney stratification of $\tilde{\mathcal{V}}$ for which there are no finite or infinite stationary points with heights in $[a, b]$. Assume a given set of projections and distance functions satisfying the control conditions of [Mat12, Sections 7-8]. Then there is a vector field $\mathbf{v}$ in the log space, in other words a section of $T\mathcal{L}$, over the base set $\mathcal{L} \cap \tilde{h}^{-1}[a, b]$, with the following properties.*

(i) control: $\mathbf{v}$ is a controlled stratified vector field for the stratum Σ.

(ii) unit speed: $\langle d\tilde{h}, \mathbf{v} \rangle \equiv -1$.

(iii) regularity: $\mathbf{v}$ is bounded.

Before beginning the proof we motivate with a shorter argument that does not quite hold water. By Lemma 5.1, the negative unit gradient vector field $\mathbf{v}_\Sigma$ on each stratum has magnitude at least δ. For each point x in a stratum Σ, take a neighborhood $\mathcal{N}_x$ in $\mathbb{C}_*^d$ intersecting no lower dimensional strata and for which $\mathbf{v}_\Sigma$ extends continuously to a vector field $\mathbf{v}_x$ for which $\langle d\tilde{h}, \mathbf{v}_x \rangle \geq \delta/2$. Piece these together with a partition of unity. By convexity the resulting vector field $\mathbf{v}$ has norm at most 1 everywhere. By linearity $\langle d\tilde{h}, \mathbf{v}_x \rangle \geq \delta/2$ everywhere. This is the natural argument. The gap is that the local product structure does not, on the surface, guarantee a bounded continuous extension of $\mathbf{v}_\Sigma$. This must be argued; however as mentioned above, it follows from Whitney's conditions and is implicit in Mather's arguments.

Proof: By Lemma 5.1, the tangential differential $d\tilde{h}_\Sigma$ is globally bounded from below by some positive quantity δ. Hence, the hypotheses of [Mat12, Proposition 9.1] are satisfied with $V = \mathcal{L}$ and the given Whitney stratification, $P = \mathbb{R}$, $f = h$ and ζ the negative unit vector field on $\mathbb{R}$. Following the proof of [Mat12, Proposition 9.1], which already yields conclusions (i) and (ii) of the lemma, the only place further argumentation is needed for boundedness and continuity is at the bottom of page 494. There, $\mathbf{v}_\Sigma$ is constructed inductively given $\mathbf{v}_{\Sigma'}$ on all strata of lower dimension. The word "clearly" in the third line from the bottom of page 494 hides some linear algebra which we now make explicit.

We assume that X and Y are strata, with $X \subseteq \partial Y$ and $\dim X = m < \ell = \dim Y$. First we straighten X near a point $x \in X$. In a neighborhood $\mathcal{N}$ of x in the ambient space $\mathbb{R}^{2d}$ there is a smooth coordinatization such that $\mathbb{R}^m \times \mathbf{0}$ maps to X. Applying another linear change of coordinates if necessary, we can assume that $\mathbf{v}_X(x) = e_1$. This requires a distortion of magnitudes of tangent vectors by $1/|d\tilde{h}_X|$ at the point x; taking $\mathcal{N}$ small enough, we can assume that the distortion on the tangent bundle over $\mathcal{N}$ is bounded by twice this, hence globally by at most $2/\delta$.

Whitney's Condition A stipulates that as $y \in Y$ converges to a point $x \in X$, any limit ℓ-plane of a sequence $T_y(Y)$ must contain $T_x X = \mathbb{R}^m \times \mathbf{0}$. This implies for $1 \leq j \leq m$ that the distance of e_j to $T_y(Y)$ goes to zero (recall we have identified the tangent spaces $T_y(Y)$ for different y). Hence, for $1 \leq k \leq m$ there are vectors $c_k(y) \in (\mathbb{R}^m)^\perp$ going to zero as $y \to x$, such that $e_k + c_k(y) \in T_y(Y)$. These vectors $c_k(y)$ may be chosen as smooth functions of $y \in Y$, continuously as $y \to x \in X$; this follows because the tangent planes $T_y(Y)$ vary continuously with $y \in Y$ and semicontinuously as $y \to x \in X$, meaning that $T_x(X)$ is contained in the liminf of $T_{y_n}(Y)$. The vectors $f_k := e_k + c_k(y)$ are linearly independent for $1 \leq k \leq m$ because their projections to the first m coordinates are linearly independent; hence they may be completed to a basis $\{f_1, \ldots, f_\ell\}$ of $T_y(Y)$.

Let $\{y_n\}$ be a sequence of points in Y converging to $x \in X$. Write y_n as (x_n, z_n) with $x \in \mathbb{R}^m$ in local coordinates and $z \in \mathbb{R}^{2d-m}$. Apply Whitney's Condition B applied to the sequences $\{y_n\}$ in Y and $\{x_n\}$ in X. Passing to a subsequence in which $T_{y_n}(Y) \to \tau$ and $z_n/|z_n| \to u$, Whitney's Condition B asserts that $u \in T_x(Y)$. Because $T_x(Y) \subseteq \tau$, it follows that the distance between $z_n/|z_n|$ and $T_{y_n}(Y)$ goes to zero. Hence there is a sequence $c_{m+1}(y_n) \to \mathbf{0}$ in $(\mathbb{R}^m)^\perp$ as $n \to \infty$ such that $z_n/|z_n| + c_{m+1}(y_n) \in T_{y_n}(Y)$ for all n. Because $c_{m+1} \in (\mathbb{R}^m)^\perp$, it follows that we may choose the basis $\{1_j : 1 \leq j \leq \ell\}$ so that $f_{m+1}(y) = z(y)/|z(y)| + c_{m+1}(y)$.

Now we have what we need to construct a controlled vector field on Y that is controlled and close to e_1. In fact we can construct one that is spanned by $f_1, \ldots, f_{m+1}$ using linear algebra. Write

$$\mathbf{v}(y) = \sum_{j=1}^{m+1} a_j(x(y), z(y)) f_j(y)\,. \tag{5.2}$$

Guessing at the solution, we impose the conditions $\langle \mathbf{v}, e_j \rangle = \delta_{1,j}$ for $1 \leq j \leq m$. This implies the second control condition, namely that $\mathbf{v}$ is preserved by projection from Y to X. The first control condition, preservation of distance along the vector field, means that $\langle \mathbf{v}, z/|z| \rangle = 0$ in local coordinates. Thus we arrive at the following system.

$$\begin{aligned}
\langle \mathbf{v}, e_1 \rangle = \sum_{j=1}^{m+1} a_j \langle f_j, e_1 \rangle = a_1 + a_{m+1} \langle c_{m+1}, e_1 \rangle \quad &= \quad 1 \\
\langle \mathbf{v}, e_2 \rangle = \sum_{j=1}^{m+1} a_j \langle f_j, e_2 \rangle = a_2 + a_{m+1} \langle c_{m+1}, e_2 \rangle \quad &= \quad 0 \\
&\vdots
\end{aligned}$$

$$\begin{aligned}\langle \mathbf{v}, e_m\rangle = \sum_{j=1}^{m+1} a_j \langle f_j, e_m\rangle = a_m + a_{m+1}\langle c_{m+1}, e_m\rangle &= 0 \\ \langle \mathbf{v}, \frac{z}{|z|}\rangle = \sum_{j=1}^{m+1} a_j \langle f_j, \frac{z}{|z|}\rangle = a_{m+1} + \sum_{j=1}^{m} a_j \langle c_j, \frac{z}{|z|}\rangle &= 0\end{aligned}$$

We may write this as $(I+M)\mathbf{a} = (1,0,\ldots,0)^T$ where I is the $(m+1)\times(m+1)$ identity matrix and $M \to 0$ smoothly as $y \to x$. Therefore, in a neighborhood of x in Y, the solution exists uniquely and smoothly and converges to e_1 as $y \to X$. This allows us to extend $\mathbf{v}$ from X to a controlled vector field in a neighborhood of X in Y, varying smoothly on $Y \setminus X$ and extending continuously to X, with the properties that $\langle d\tilde{h}_Y, \mathbf{v}\rangle \equiv -1$ and that the norm on Y is at most a bounded multiple of the norm on X.

The solution of the control conditions together with the condition $\langle d\tilde{h}_Y, \mathbf{v}\rangle \equiv -1$ are a convex set; a partition of unity argument as in [Mat12] preserves continuity, and, by the triangle inequality, global boundedness of $\mathbf{v}$. $\square$

We are now ready to complete the proof of Theorem 2.7.

PROOF: [of conclusion (i) of Theorem 2.7] Choose a vector field $\mathbf{v}$ as in the conclusion of Lemma 5.2. Let $\mathcal{D}$ be the set $\mathcal{L} \cap \tilde{h}^{-1}[a,b]$, inheriting stratification from the pair $(\mathcal{L}, \tilde{\mathcal{V}})$. On each stratum of $\mathcal{D}$ the vector field $\mathbf{v}$ is smooth and bounded. Let $\mathcal{D}'$ be the space-time domain $\{(x,t) \in \mathcal{D} \times \mathbb{R}^+ : t \leq \tilde{h}(x) - a\}$. Let $\Psi : \mathcal{D}' \to \mathcal{D}$ be a solution to the differential equation

$$\frac{d}{dt}\Psi(x,t) = \mathbf{v}\left(\Psi(x,t)\right) \; ; \qquad \Psi(x,0) = x \, .$$

Such a flow exists on each stratum because because $\mathbf{v}$ is smooth and bounded, and the phase exit the interval $[a,b]$ in finite time $\tilde{h}(x) - a \leq b - a$; therefore each trajectory stays within a uniformly bounded vicinity of its starting point. Hence, by compactifying the support of the vector field if necessary, the flow exists for the time necessary for the height to drop below a from any point of the stratum, and is smooth, hence is well-defined everywhere on the stratum. As in [Mat12, Section 10], the local one-parameter group of time-t maps are all injective. Because the flow preserves the squared distance functions to strata, the flow started on a stratum X cannot reach a stratum $Y \subseteq \partial X$; hence the time-t maps preserve strata. Continuity of the vector field implies continuity of the time-t maps.

Extend the flow to $\mathcal{D} \times [0, b-a]$ by setting $\Psi(x,t) = \Psi(x, \tilde{h}(x) - a)$ for $t > \tilde{h}(x) - a$. By construction the flow is tangent to strata, hence the flow is stratum preserving. The flow is continuous because the velocity is bounded and the stopping time $\tilde{h}(x) - a$ is continuous. For any stratum Σ, the flow defines a homotopy within $\Sigma_{\leq b}$ whose final cross section is in $\Sigma_{\leq a}$. $\square$

PROOF: [of conclusion (ii) of Theorem 2.7] Still in the logspace, for any $r \geq 0$, let $\mathcal{N}_r$ denote the union of closed r-balls about the affine stationary points; in particular $\mathcal{N}_0$ is the set of stationary points. Due to the presence of affine stationary points, we may no longer invoke the conclusions of Lemma 5.2. We claim, however, that the conclusions follow if (iii) is replace by (iii'): $\mathbf{v}$ is bounded and continuous outside any neighborhood $\mathcal{N}_r$ of the affine stationary points. This follows from the original proof because the distortions are bounded by constant multiples of the quantities $1/|d\tilde{h}_X|$, while by Lemma 5.1 $|d\tilde{h}_X|$ is uniformly bounded away from zero outside any neighborhood of the affine stationary points. Fix the vector field $\mathbf{v} = \mathbf{v}_r$ satisfying conclusions (i) and (ii) of Lemma 5.2 and (iii') above and let $K = K(r)$ be an upper bound for $|\mathbf{v}(r)|$ outside of $\mathcal{N}_{r/2}$.

We build a deformation in two steps. Fix $r > 0$ and set $\varepsilon = r/K(r)$. Next, apply Lemma 5.2 with $[c+\varepsilon, b]$ in place of $[a, b]$, which we can do because there are no stationary points with heights in $[c+\varepsilon, b]$. This produces a deformation retract of $h^{-1}(-\infty, b]$ to $h^{-1}(-\infty, c+\varepsilon]$.

Now we compose this with a slowed down version of $\mathbf{v}$. The hypotheses of no affine or infinite critical values in $[a, b]$ other than at c imply the same hypotheses hold over a slightly large interval $[a-\theta, b]$. Let $\chi : [0, \infty) \to [0, 1]$ be a smooth nondecreasing function equal to 1 on $[1, \infty)$ and 0 on $[0, 1/2)$. Define the vector field $\mathbf{w} = \mathbf{w}_r$ on $h^{-1}[a-\theta, b]$ by

$$\mathbf{w}(x) := \chi\left(||x\ , \mathcal{N}_0||\ \right) \cdot \chi\left(\frac{h(x) - (a-\theta)}{\theta}\right) \cdot \mathbf{v}(x)\,,$$

where $||\cdot,\cdot||$ denotes distance. Because $\mathbf{v}$ defines a flow, so does $\mathbf{w}$, the trajectories of which are precisely the trajectories of $\mathbf{v}$ slowed down inside $\mathcal{N}_r$ and below height a. Note that the trajectories of $\mathbf{w}$ stop completely inside $\mathcal{N}_{r/2}$ and below height $a-\theta$, however, trajectories not in these regions slow down so as never to reach $\mathcal{N}_{r/2}$ nor height $a-\theta$.

Run the flow defined by $\mathbf{v}$ at time 2ε. Let Σ be any stratum and let x be any point in $\Sigma_{\leq c+\varepsilon}$. Let $\tau_x : [0, \varepsilon] \to \Sigma$ be the trajectory started from x (the trajectory remains in Σ because $\mathbf{v}$ is a controlled stratified vector field). We claim that the time-2ε map takes x to a point in $\Sigma_{<c} \cup \mathcal{N}_{3r}$. Because r is arbitrary, this is enough to prove theorem. The flow decreases height, so we may assume without loss of generality that $x \in h^{-1}[c, c+\varepsilon]$. If the trajectory never enters $\mathcal{N}_r$ then height decreases at speed 1, hence $\tau_x(2\varepsilon) \in \Sigma_{\leq c-\varepsilon}$ and the claim follows. If the $\tau_x(2\varepsilon) \in \mathcal{N}_r$ then trivially the claim is true. Lastly, suppose the trajectory enters $\mathcal{N}_r$ and leaves again. Let $s \in (0, 2\varepsilon)$ be the last time that $\tau_x(s) \in \mathcal{N}_r$. Because $|\mathbf{v}| \leq K$, we have $||\tau(2\varepsilon), \mathcal{N}_0|| \leq r + 2\varepsilon K \leq 3r$, that is, $\tau(2\varepsilon) \in \mathcal{N}_{3r}$, finishing the proof of the claim. $\square$

6 acknowledgements

The authors would like to thank Paul Görlach for his advice on computational methods for determining stationary points at infinity, to Justin Hilburn for ideas on the proof of the compactification result and to Roberta Guadagni for related conversations. The authors also thank anonymous referees for pointing out related results from the literature and helping to clarify and simplify our arguments.

A Appendix

We now give an abstract answer to [Pem10, Conjecture 2.11], in a manner suggested to us by Justin Hilburn and Roberta Guadagni. Let $H(\mathbf{z}) := \mathbf{z^m}$ be the monomial function on C_*^d, and $G \subset C_*^d \times \mathbb{C}^*$, its graph. An easy case of toric resolution of singularities (see, e.g., [Kho78]) implies the following result.

Theorem A.1. *There exists a compact toric manifold K such that C_*^d embeds into it as an open dense stratum and the function H extends from this stratum to a smooth $\mathbb{P}^1$-valued function on K.*

Proof: The graph G is the zero set of the polynomial $P_{\mathbf{m}} := h - \mathbf{z^m}$ on $C_*^d \times \mathbb{C}^*$, where h is the coordinate on the second factor. Theorem 2 in [Kho78] implies that a compactification of $C_*^d \times \mathbb{C}^*$ in which the closure of G is smooth exists if the restrictions of the polynomial $P_{\mathbf{m}}$ to any facet of the Newton polyhedron of $P_{\mathbf{m}}$ is nondegenerate (defines a nonsingular manifold in the corresponding subtorus). In our case, the Newton polytope is a segment, connecting the points $(\mathbf{m}, 0)$ and $(\mathbf{0}, 1)$, and this condition follows immediately. Hence, the closure of G in the compactification of $C_*^d \times \mathbb{C}$ is a compact manifold K. We notice that the projection to C_*^d is an isomorphism on G, and therefore K compactifies C_*^d in such a way that H lifts to a smooth function on K.

Lifting the variety $\mathcal{V}_*$ to $G \subset K$ and taking the closure produces the desired result: a compactification of $\mathcal{V}_*$ in a compact manifold K on which H is smooth. $\square$

A practical realization of the embedding requires construction of a simple fan (partition of $\mathbb{R}^{d+1}$ into simplicial cones with unimodular generators) which subdivides the fan dual to the Newton polytope of $h - \mathbf{z}^{\mathbf{m}}$. While this is algorithmically doable (and implementations exist, for example in `macaulay2`), the resulting fans depend strongly on $\mathbf{m}$, and the resulting compactifications K are hard to work with.

Definition A.2 **(compactified stationary point).** *Define a compactified stationary point of H, with respect to a compactification of C_*^d to which H extends smoothly, as a point $\mathbf{x}$ in the closure of $\mathcal{V}$ such that dH vanishes at $\mathbf{x}$ on the stratum $\overline{S}(\mathbf{x})$, and $H(\mathbf{x})$ is not zero or infinite.*

Applying basic results of stratified Morse theory [GM88] to $\mathcal{K}$ directly yields the following consequence.

Corollary A.3 (no compactified stationary point implies Morse results)**.**

(i) If there are no stationary points or compactified stationary points with heights in $[a, b]$, then $\mathcal{V}_{\leq b}$ is homotopy equivalent to $\mathcal{V}_{\leq a}$ via the downward gradient flow.

(ii) If there is a single stationary point x with critical value in $[a, b]$, and there is no compactified stationary point with height in $[a, b]$, then the homotopy type of the pair $(\mathcal{M}_{\leq b}, \mathcal{M}_{\leq b})$ is determined by a neighborhood of x, with an explicit description following from results in [GM88].

References

[ABG70] M. Atiyah, R. Bott, and L. Gårding. Lacunas for hyperbolic differential operators with constant coefficients, I. *Acta Mathematica*, 124:109–189, 1970.

[ABN+01] Andris Ambainis, Eric Bach, Ashwin Nayak, Ashvin Vishwanath, and John Watrous. One-dimensional quantum walks. In *Proceedings of the Thirty-Third Annual ACM Symposium on Theory of Computing*, pages 37–49. ACM, New York, 2001.

[ADZ93] Y. Aharonov, L. Davidovich, and N. Zagury. Quantum random walks. *Phys. Rev. A*, 48:1687–1690, Aug 1993.

[BBBP11] Yuliy Baryshnikov, Wil Brady, Andrew Bressler, and Robin Pemantle. Two-dimensional quantum random walk. *J. Stat. Phys.*, 142(1):78–107, 2011.

[BDS17] Alin Bostan, Louis Dumont, and Bruno Salvy. Algebraic diagonals and walks: algorithms, bounds, complexity. *J. Symbolic Comput.*, 83:68–92, 2017.

[BGPP10] Andrew Bressler, Torin Greenwood, Robin Pemantle, and Marko Petkovšek. Quantum random walk on the integer lattice: examples and phenomena. In *Algorithmic probability and combinatorics*, volume 520 of *Contemp. Math.*, pages 41–60. Amer. Math. Soc., Providence, RI, 2010.

[BM93] Andrea Bertozzi and James McKenna. Multidimensional residues, generating functions, and their application to queueing networks. *SIAM Rev.*, 35(2):239–268, 1993.

[BMP19] Y. Baryshnikov, S. Melczer, and R. Pemantle. Asymptotics of multivariate sequences in the presence of a lacuna. *Preprint, 24 pages*, 2019.

[BMPS18] Y. Baryshnikov, S. Melczer, R. Pemantle, and A. Straub. Diagonal asymptotics for symmetric rational functions via ACSV. In *29th International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms (AofA 2018)*, volume 110, page 12. Dagstuhl, 2018.

[BP07] A. Bressler and R. Pemantle. Quantum random walks in one dimension via generating functions. In *Proceedings of the 2007 Conference on the Analysis of Algorithms*, volume AofA 07, page 11. LORIA, Nancy, France, 2007.

[BP11] Y. Baryshnikov and R. Pemantle. Asymptotics of multivariate sequences, part III: quadratic points. *Adv. Math.*, 228:3127–3206, 2011.

[Bro88] S. A. Broughton. Milnor numbers and the topology of polynomial hypersurfaces. *Invent. Math.*, 92:217–241, 1988.

[CIR03] Hilary A. Carteret, Mourad E. H. Ismail, and Bruce Richmond. Three routes to the exact asymptotics for the one-dimensional quantum walk. *J. Phys. A*, 36(33):8775–8795, 2003.

[CLO92] D. Cox, J. Little, and D. O'Shea. *Ideals, Varieties and Algorithms.* Undergraduate Texts in Mathematics. Springer-Verlag, Berlin, second edition, 1992.

[DeV11] T. DeVries. *Algorithms for bivariate singularity analysis.* PhD thesis, University of Pennsylvania, 2011.

[DJ21] S. Dinh and Z. Jelonek. Thom isotopy theorem for nonproper maps and computation of sets of stratified generalized critical values. *Discrete Comput. Geom.*, 65(1):279–304, 2021.

[DvdHP12] T. DeVries, J. van der Hoeven, and R. Pemantle. Effective asymptotics for smooth bivariate generating functions. *Online J. Anal. Comb.*, 7:24, 2012.

[Fed77] M. Fedoryuk. *Saddle point method (in Russian).* Nauka, Moscow, 1977.

[FIM99] Guy Fayolle, Roudolf Iasnogorodski, and Vadim Malyshev. *Random walks in the quarter-plane*, volume 40 of *Applications of Mathematics (New York).* Springer-Verlag, Berlin, 1999.

[FPT00] M. Forsberg, M. Passare, and A. Tsikh. Laurent determinants and arrangements of hyperplane amoebas. *Advances in Mathematics*, 151:45–70, 2000.

[GM88] M. Goresky and R. MacPherson. *Stratified Morse Theory.* Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer-Verlag, Berlin, 1988.

[Gor75] G. Gordon. The residue calculus in several complex variables. *Trans. AMS*, 213:127–176, 1975.

[Gre18] Torin Greenwood. Asymptotics of bivariate analytic functions with algebraic singularities. *J. Combin. Theory Ser. A*, 153:1–30, 2018.

[Gri69] P. Griffiths. On the periods of certain rational integrals, I. *Ann. Math.*, 90(3):460–495, 1969.

[GRZ83] J. Gillis, B. Reznick, and D. Zeilberger. On elementary methods in positivity theory. *SIAM J. Math. Anal.*, 14:396–398, 1983.

[Har75] R. Hardt. Stratification of real analytic mappings and images. *Invent. Math.*, 28:193–208, 1975.

[HK71] M. Hautus and D. Klarner. The diagonal of a double power series. *Duke Math. J.*, 23:613–628, 1971.

[KH99] B Knudsen and J Hein. RNA secondary structure prediction using stochastic context-free grammars and evolutionary history. *Bioinformatics*, 15(6):446–454, 06 1999.

[Kho78] A. G. Khovanskii. Newton polyhedra and toroidal varieties. *Functional Analysis and Its Applications*, 11(4):289–296, 1978.

[KOS00] K. Kurdyka, P. Orro, and S. Simon. Semialgebraic Sard theorem for generalized critical values. *J. Differential Geom.*, 56(1):67–92, 2000.

[Lan02] S. Lang. *Introduction to Differentiable Manifolds.* Springer, New York, 1962, 2002.

[Ler50] J. Leray. Le calcul différentiel et intégral sur un variété analytique complexe. *Bull. Soc. Math. France*, 87:81–180, 1950.

[Mal74] B. Malgrange. Intégrales asymptotiques et monodromie. *Ann. Sci. ENS*, 7:405–430, 1974.

[Mal80] B. Malgrange. Méthode de la phase staionnaire et sommation de borel. In *Complex Analysis, Microlocal Calculus and Relativistic Quantum Theory*, volume 126. Springer, Berlin, 1980.

[Mat12] J. Mather. Notes on topological stability. *Bull. AMS*, 49(4):475–506, 2012.

[Mel21] Stephen Melczer. *An Invitation to Analytic Combinatorics: From One to Several Variables.* Texts & Monographs in Symbolic Computation. Springer International Publishing, 2021.

[MR91] T. Mostowski and E. Rannou. Complexity of the computation of the canonical Whitney stratification of an algebraic set in Cn. *Lecture Notes in Computer Science*, 539:281–291, 1991.

[MS74] J. Milnor and J. Stasheff. *Characteristic Classes*, volume 76 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, 1974.

[Mum76] D. Mumford. *Algebraic Geometry I. Complex Algebraic Varieties.* Classics in Mathematics. Springer-Verlag, Berlin, 1995, reprinted from 1976.

[Par95] A. Parusiński. On the bifurcation set of complex polynomial with isolated singularties at infinity. *Comp. Math.*, 97(3):369–384, 1995.

[Pem10] R. Pemantle. Analytic combinatorics in several variables: an overview. In *Algorithmic Probability and Combinatorics*, volume 520, pages 195–220. American Mathematical Society, 2010.

[PH14] Svetlana Poznanović and Christine E. Heitsch. Asymptotic distribution of motifs in a stochastic context-free grammar model of RNA folding. *J. Math. Biol.*, 69(6-7):1743–1772, 2014.

[Pha83] Frédéric Pham. Vanishing homologies and the n variable saddlepoint method. In *Singularities, Part 2 (Arcata, Calif., 1981)*, volume 40 of *Proc. Sympos. Pure Math.*, pages 319–333. Amer. Math. Soc., Providence, RI, 1983.

[PW04] R. Pemantle and M.C. Wilson. Asymptotics of multivariate sequences, II. Multiple points of the singular variety. *Combin. Probab. Comput.*, 13:735–761, 2004.

[PW08] R. Pemantle and M.C. Wilson. Twenty combinatorial examples of asymptotics derived from multivariate generating functions. *SIAM Review*, 50:199–272, 2008.

[PW13] R. Pemantle and M. Wilson. *Analytic Combinatorics in Several Variables*, volume 340 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, New York, 2013.

[Rab97] Patrick J. Rabier. Ehresmann fibrations and Palais-Smale conditions for morphisms of Finsler manifolds. *Ann. of Math. (2)*, 146(3):647–691, 1997.

[ST95] D. Siersma and M. Tibar. Singularities at infinity and their vanishing cycles. *Duke Math. J.*, 80(3):771–783, 1995.

[Tei82] Bernard Teissier. Variétés polaires. II. Multiplicités polaires, sections planes, et conditions de Whitney. In *Algebraic geometry (La Rábida, 1981)*, volume 961 of *Lecture Notes in Math.*, pages 314–491. Springer, Berlin, 1982.

[Var77] A. N. Varchenko. Newton polyhedra and estimation of oscillating integrals. *Functional Anal. Appl.*, 10:175–196, 1977.

[Vas77] V. Vassiliev. Asymptotic exponential integrals, Newton's diagram, and the classification of minimal points. *Functional Anal. Appl.*, 11:163–172, 1977.