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PROJECTIVE PLANE AND PLANAR QUANTUM CODES

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ABSTRACT

Cellulations of the projective plane $\mathbb{R}P^2$ define single qubit topological quantum error correcting codes since there is a unique essential cycle in $H_1(\mathbb{R}P^2;\mathbb{Z}_2)$. We construct three of the smallest such codes, show they are inequivalent, and identify one of them as Shor's original 9 qubit repetition code. We observe that Shor's code can be constructed in a planar domain and generalize to planar constructions of higher genus codes for multiple qubits.

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Kitaev has constructed a class of quantum error correcting codes using qubits arranged on the edges of square lattices embedded in the two dimensional torus [1]. While these *toric* codes are not particularly efficient—they do not come close to saturating the quantum Hamming bound [2]—they are nevertheless interesting for several reasons: Toric codes have local stabilizers, which means that the code subspace can be identified as the (degenerate) ground state subspace of a local Hamiltonian; thus there would be some level of automatic error correction in such a quantum system. Furthermore, fault-tolerant quantum computation can be performed using elementary excitations of the Hamiltonian [3]; universal quantum computation is possible if the qubits (lying in $\mathbb{C}^2 = \mathbb{C}^{\mathbb{Z}_2}$) are replaced in the model with states in $\mathbb{C}^{60} = \mathbb{C}^{A_5}$ [4].

Kitaev also remarks that cellulations with $|E|$ edges of genus g compact orientable surfaces generally encode $2g$ qubits using $|E|$ qubits [3]. The toric codes, for example, encode 2 qubits. In this note we observe that, as is the case for percolation, there is something to be learned from studying the problem on the projective plane $\mathbb{R}P^2$ [5]. Since there is a unique essential cycle in $H_1(\mathbb{R}P^2;\mathbb{Z}_2)$, cellulations of $\mathbb{R}P^2$ encode a single qubit. Here we consider the smallest such quantum error correcting codes and compare them with single qubit codes obtained otherwise.

We begin by reviewing the construction of (two dimensional) *topological* quantum error correcting codes. A cellulation $\mathcal{C}$ of a surface defines sets F, E and V of *faces*, *edges* and *vertices*, respectively. For each face $f \in F$, let $E_f \subset E$ be the set of edges in the boundary of f; define (our construction is dual to Kitaev's [1], but equivalent)

$$A_f := \bigotimes_{e\in E} \sigma_x^{\delta(e\in E_f)}. \tag{1}$$

Similarly, for each vertex $v \in V$, let $E_v \subset E$ be the set of edges in whose boundary v lies; define

$$B_v := \bigotimes_{e\in E} \sigma_z^{\delta(e\in E_v)}. \tag{2}$$

Here σ_x and σ_z are the usual Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

the exponents in equations (1) and (2) are 1(0) according to the truth(falsity) of the argument of $\delta(\cdot)$, and the *stabilizer* operators A_f and B_v act on the Hilbert space $\mathcal{H} = (\mathbb{C}^2)^{\otimes|E|}$ with qubit tensor factors labelled by the edges of the cellulation. These stabilizer operators form an overcomplete set of generators for the stabilizer group; there are two relations:

$$\prod_f A_f = \mathrm{id} = \prod_v B_v. \tag{3}$$

As usual, let 0 and 1 denote the eigenvectors of σ_z with eigenvalues 1 and -1, respectively. The $2^{|E|}$ configurations of 0s and 1s on the edges of $\mathcal{C}$ form a basis for $\mathcal{H}$. There is a natural bijection between this basis and the set of $\mathbb{Z}_2$-linear combinations of elements in E, the 1-chains of $\mathcal{C}$ with coefficients in $\mathbb{Z}_2$, $C_1(\mathcal{C};\mathbb{Z}_2)$; thus we identify $\mathcal{H} = \mathbb{C}^{C_1(\mathcal{C};\mathbb{Z}_2)}$. The subspace of $\mathcal{H}$ which is the intersection of the eigenvalue 1 eigenspaces of the B_v is spanned (over $\mathbb{C}$) by configurations with an even number of edges labelled by 1s incident at each vertex; these are chains in $C_1(\mathcal{C};\mathbb{Z}_2)$ without boundary, *i.e.*, the cycles $Z_1(\mathcal{C};\mathbb{Z}_2)$. So the subspace of $\mathcal{H}$ fixed by all the B_v is $\mathbb{C}^{Z_1(\mathcal{C};\mathbb{Z}_2)}$, the functions from $\mathbb{Z}_2$ 1-cycles to $\mathbb{C}$. Since σ_x acting at an edge exchanges 0 and 1, each A_f corresponds to the order 2 automorphism of $Z_1(\mathcal{C};\mathbb{Z}_2)$ which changes each cycle by the $\mathbb{Z}_2$-addition of the cycle bounding f. Thus the subspace of $\mathcal{H}$ fixed by *both* all the B_v and all the A_f is the set of functions on $Z_1(\mathcal{C};\mathbb{Z}_2)$ which are invariant under the $\mathbb{Z}_2$-addition of cycles bounding 2-chains in $\mathcal{C}$, the boundaries $B_1(\mathcal{C};\mathbb{Z}_2)$. This *code subspace* is therefore $\mathbb{C}^{H_1(\mathcal{C};\mathbb{Z}_2)}$, where $H_1(\mathcal{C};\mathbb{Z}_2) = Z_1(\mathcal{C};\mathbb{Z}_2)/B_1(\mathcal{C};\mathbb{Z}_2)$ is the first homology group of $\mathcal{C}$ over $\mathbb{Z}_2$. For a genus g compact orientable surface Σ, $H_1(\Sigma;\mathbb{Z}_2) = \bigoplus_{i=1}^{2g}\mathbb{Z}_2$, so a corresponding code subspace would have dimension 2^{2g} and thus encode $2g$ qubits. But the real projective plane $\mathbb{R}P^2$ has a unique essential cycle, so $H_1(\mathbb{R}P^2;\mathbb{Z}_2) = \mathbb{Z}_2$ and any corresponding code subspace has dimension 2 and encodes a single qubit.

Consider a bit flip error in the qubit on some edge e, *i.e.*, multiplication by σ_x on the corresponding tensor factor of $\mathcal{H}$. This error will be detected by the eigenvalues of the stabilizer operators B_v for the two vertices in the boundary of e—unless there are bit flip errors on an even number of the edges incident at one or the other vertex. More generally, bit flip errors in the qubits on any collection of edges corresponding to a chain $c_1 \in C_1(\mathcal{C};\mathbb{Z}_2)$ will be detected by the B_v for the vertices in the boundary of c_1. Error correction by choosing any chain $c_2 \in C_1(\mathcal{C};\mathbb{Z}_2)$ with the observed boundary vertices and acting by σ_x on the corresponding qubits succeeds unless $c_1 + c_2$ contains an essential cycle. The length of the shortest essential cycle in $\mathcal{C}$ is thus the (bit flip error) distance of the code [6].

Similarly, phase errors are detected by the eigenvalues of the stabilizer operators A_f. The observed faces correspond to vertices in the dual cellulation $\mathcal{C}^*$ bounding a chain $c_1^* \in C_1(\mathcal{C}^*;\mathbb{Z}_2)$ of edges at which phase errors have occurred (edges in $\mathcal{C}^*$ are dual to edges in $\mathcal{C}$). Error correction by choosing any $c_2^* \in C_1(\mathcal{C}^*;\mathbb{Z}_2)$ with the observed boundary and acting by σ_z on the corresponding qubits succeeds unless $c_1^* + c_2^*$ contains an essential cycle in $C_1(\mathcal{C}^*;\mathbb{Z}_2)$. The length of the shortest essential cycle in the dual cellulation $\mathcal{C}^*$ is thus the (phase error) distance of the code.

The smallest square lattice toric code correcting an arbitrary single error, *i.e.*, with distance 3, uses 18 qubits to encode two qubits. Figure 1 shows a similarly regular triangulation of $\mathbb{R}P^2$ with 15 edges, defining a 15 qubit code for one qubit. In this diagram antipodal points on the circle are identified and it is easy to see that while the minimal length of any essential cycle is 3, it is 5 in the dual cellulation. By considering more general cellulations we can construct smaller codes correcting 1 error. Figure 2 shows a cellulation of $\mathbb{R}P^2$ with 9 edges, defining a distance 3 code for one qubit using 9 qubits; the minimal length of any essential cycle is 3 in both this cellulation and its dual. The dual cellulation also defines a code, with the stabilizers A_f and B_v replaced by corresponding B_{f^*} and

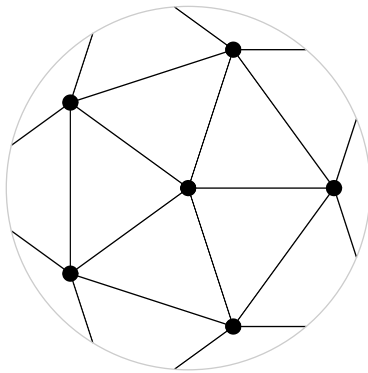

Figure 1. A triangulation of the projective plane with 15 edges, all minimal essential (dual) cycles of length 3 (5). Antipodal points on the circle are identified.

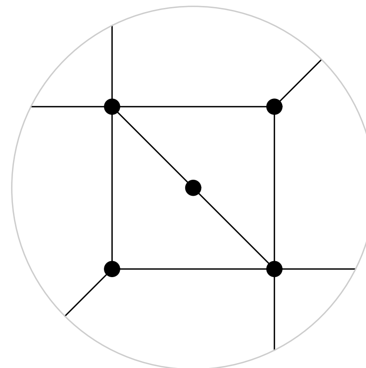

Figure 2. A less regular, but more efficient, cellulation of the projective plane using 9 edges. Both the minimal essential cycles and the minimal essential dual cycles have length 3.

A_{v^*}. The resulting code, however, is *equivalent* [7] to the original under multiplication of each tensor factor by the Hadamard transform

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

since $\sigma_z = H\sigma_x H^{-1}$.

But there are other cellulations of $\mathbb{R}P^2$ which define distinct codes. Figure 3 shows a cellulation obtained from the one in Figure 2 by identifying two vertices. The resulting 9 qubit code still has distance 3 and is not equivalent to the code derived from the cellulation in Figure 2. To demonstrate that these codes are inequivalent, we consider the projections P_1 and P_2 onto the respective code subspaces—and their polynomial invariants under the adjoint actions of $U(2)^{\otimes 9}$ and the permutation group S_9. There are, of course, many such invariants [8] but for our purpose it suffices to consider the coefficients of the characteristic polynomials of the reduced density matrices obtained by tracing over pairs of tensor factors. For P_1 exactly two of these reduced density matrices have rank 2 (the two obtained by tracing over either the qubits on the two edges incident at the valence 2 vertex, or the qubits on the edges bounding the 2-gon in the cellulation). For P_2, however, there are three rank 2 reduced density matrices—corresponding to the presence of the three 2-gons in Figure 3. Thus the two codes are inequivalent.

Finally, let us consider the cellulation shown in Figure 4, obtained from the one in Figure 3 by identifying two vertices—which identifies the endpoints of an edge—and then sliding the endpoints of this edge to the two other vertices. This cellulation again defines a 9 qubit code with distance 3. The resulting code is inequivalent to either of our first two by the same argument: the six reduced density matrices corresponding to tracing the projection P_3 over the qubits on the edges of each 2-gon have rank 2. While the first two codes were new, this code is very familiar—it is exactly Shor's original 9 qubit

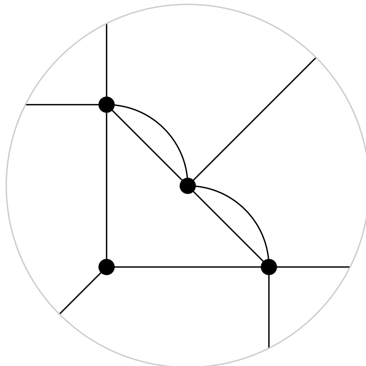

Figure 3. Another cellulation of the projective plane obtained by identifying two vertices of the cellulation shown in Figure 2. Both the minimal essential cycles and the minimal essential dual cycles still have length 3.

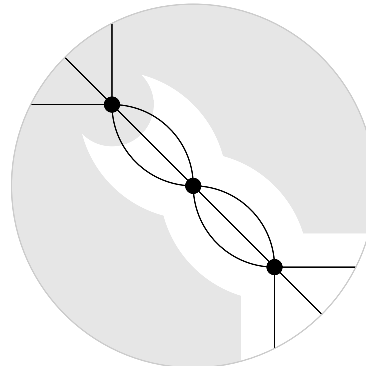

Figure 4. A third cellulation of the projective plane with minimal essential cycle (also dual cycle) of length 3. The interior of one face and a neighborhood of one vertex are shaded; the remaining cellulation is planar.

repetition code [9] as can be seen by comparing their stabilizer operators: The 9 qubits are partitioned into 3 triples according to the endpoints of the edges on which they lie; there are six stabilizers acting by σ_x on pairs of edges in the same triple and three stabilizers acting by σ_z on all the qubits in pairs of triples; one of the latter is redundant, as is the stabilizer corresponding to the hexagonal face of the cellulation. This is the exactly the (dual of the) stabilizer formulation of Shor's code [10, p. 17].

By considering cellulations of the projective plane we have demonstrated the existence of single qubit topological quantum codes. While two of the ones we find are new, the third is Shor's original 9 qubit code [9]; this connects Kitaev's novel perspective [1,3] with the bulk of the work on quantum error correcting codes (see, for example, [10] and the references therein). One might ask whether the 5 qubit [11] and 7 qubit [12] single qubit codes are also equivalent to some projective plane quantum code. They are not—there are no cellulations of $\mathbb{R}P^2$ with 5 or 7 edges and lengths of all essential cycles and dual cycles at least 3. This exemplifies the inefficiency of two dimensional qubit topological quantum error correcting codes, even for cellulations with few edges. We reiterate that their attraction lies in the locality of the stabilizer operators which one might hope to implement with designer (but local) Hamiltonians.

As Kitaev remarks [3], for the purposes of physical implementation one would like to make two dimensional topological quantum error correcting codes *planar* in the sense that the qubits lie in a plane and that each of the necessary stabilizers acts locally, on the qubits at the frontier of one of a set of disjoint regions (*e.g.*, a neighborhood of a vertex or the interior of a face). Notice that the redundancy of the stabilizer operators implied by the relations in equation (3) allows us to disregard one of the faces and one of the vertices and thus make Shor's code planar: removing the interior of the hexagon face and a neighborhood of, say, the upper left vertex in Figure 4 (both shaded), leaves a planar diagram* with a generating set of stabilizers acting locally (imagine the qubits to be located at the midpoints of the edges).

A related observation leads to planar constructions of topological quantum error correcting codes for multiple qubits deriving from higher genus surfaces [3]: the faces of a cellulation need not be disks. For these more general cellulations $\mathcal{C}$, the code subspace corresponds to

$$H_1(\mathcal{C};\mathbb{Z}_2)/\bigoplus_f H_1(f,\partial f;\mathbb{Z}_2)$$

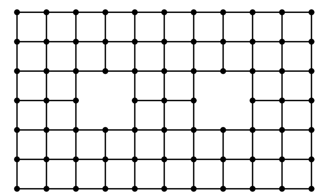

Figure 5. A cellulation of the 2-punctured disk which defines a planar topological quantum code for two qubits correcting 1 phase error and 3 bitflip errors.

To apply this observation to construct a planar code protecting g qubits, cellulate an orientable surface of genus g using one large face with the topology of a g-punctured disk and all other faces disks. Again by the relation (3) we may discard the A_f corresponding to the large face; the remaining faces cellulate a g-punctured disk which is, of course, planar. Particularly simple versions of such planar codes—with all stabilizers involving no more than 4 qubits—can be constructed using subsets of the square lattice. Figure 5 shows such a planar construction for a two qubit topological quantum code correcting 1 phase error and 3 bitflip errors.

Kitaev and Bravyi have discovered a closely related planar construction by a different route [13]. Their planar lattices have "x-boundary" and "z-boundary". Connecting the free edges of the x-boundary to an additional vertex (for which the associated B_v can be discarded) and taking the z-boundary as the boundary of an additional face (for which the associated A_f can also be discarded) defines a cellulation of a closed surface. We greatly appreciate Alexei Kitaev's willingness to describe their preliminary results and his assistance in recognizing the isomorphism between their construction and ours.

We conclude by remarking that higher dimensional manifolds offer the possibility of constructing local codes which are more efficient, in the sense of protecting against more (worst case) errors relative to their size, than any local surface code. Their intrinsic geometry [14] restricts n qubit surface codes for a constant number of qubits to correcting $O(n^{1/2})$ (worst case) errors. But, for example, five dimensional n qubit topological quantum codes for a constant number of qubits can correct $O(n^{32/61})$ (worst case) errors [15].

* Perhaps the simplest way to conceptualize the resulting planarity is to think of the projective plane as formed by the three faces of a cube incident at a single vertex with antipodal identification of the hexagonal boundary. Discarding two of the three faces—these represent the domain of the redundant A_f and B_v—the result is a single square with no boundary identifications, clearly a planar object.

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