A Short Note on the Robust Combinatorial Optimization Problems with Cardinality Constrained Uncertainty

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Abstract

Robust combinatorial optimization problems with cardinality constrained uncertainty may be solved by a finite number of nominal problems. In this paper, we show that the number of nominal problems to be solved can be reduced significantly.

Keywords: robust combinatorial optimization; discrete optimization

1 Robust combinatorial optimization problem

Let $X \subset \{0,1\}^n$ be a set of feasible solutions of a combinatorial optimization problem. The nominal combinatorial optimization problem of our interest is defined as follows:

$$\min_{\boldsymbol{x} \in X} \boldsymbol{c}^\top \boldsymbol{x} \tag{1}$$

where $\boldsymbol{c}^\top \boldsymbol{x} = \sum_{j=1}^n c_j x_j$. Bertsimas and Sim (2003) considered uncertainty for objective coefficients such that the cost of item $j \in N = \{1, 2, \cdots, n\}$ takes a value in the interval $[c_j, c_j + d_j]$, where $d_j \geq 0$. A robust combinatorial optimization problem is considered in the following form:

$$Z^* = \min_{\boldsymbol{x} \in X} \left\{ \boldsymbol{c}^\top \boldsymbol{x} + \max_{\{S | S \subset N, |S| \leq \Gamma\}} \sum_{j \in S} d_j x_j \right\} \tag{2}$$

where at most Γ components of the cost coefficients can be $c_j + d_j$; hence, the uncertainty set is called cardinality constrained. The budget of uncertainty Γ is a positive integer and represents the risk attitude of decision makers, and $1 \leq \Gamma \leq n$. Without loss of generality, we assume that the indices are sorted in descending order of the size of d_i and define $d_{n+1} = 0$ so that

$$d_1 \geq d_2 \geq \cdots \geq d_n \geq d_{n+1} = 0. \tag{3}$$

Bertsimas and Sim (2003) showed that (2) is equivalent to

$$Z^* = \min_{\boldsymbol{x} \in X, \theta \geq 0} \left(\Gamma\theta + \boldsymbol{c}^\top \boldsymbol{x} + \sum_{j \in N} \max(d_j - \theta, 0) x_j \right) \tag{4}$$

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and it can be solved by solving $n+1$ nominal problems. In particular,

$$Z^* = \min_{l=1,2,\cdots,n+1} G^l, \tag{5}$$

where for $l = 1, 2, \ldots, n+1$:

$$G^l = \Gamma d_l + \min_{\boldsymbol{x}\in X} \left(\boldsymbol{c}^\top \boldsymbol{x} + \sum_{j=1}^{l} (d_j - d_l) x_j \right) \tag{6}$$

We let $x_{n+1} = 0$, so that (6) is well-defined.

This result is very useful, because we can solve the robust optimization problem by solving a finite number of nominal problems. If the nominal problem can be solved in polynomial time, we can also solve the corresponding robust problem in polynomial time. Park and Lee (2007) showed that the number of nominal problems to be solved can be reduced to $n - \Gamma + 1$, and Álvarez-Miranda et al. (2013) to $n - \Gamma + 2$ independently. In this paper, we show that the number of nominal problems to be solved can be further reduced to $\left\lceil \frac{n-\Gamma}{2} \right\rceil + 1$.

2 New Results

For a feasible solution $\boldsymbol{x} \in X$, let

$$G^l(\boldsymbol{x}) = \Gamma d_l + \boldsymbol{c}^\top \boldsymbol{x} + \sum_{j=1}^{l} (d_j - d_l) x_j \qquad \forall l = 1, 2, ..., n+1 \tag{7}$$

then G^l in (6) can be written as

$$G^l = \min_{\boldsymbol{x}\in X} G^l(\boldsymbol{x}) \tag{8}$$

We let $G^0(\boldsymbol{x}) = G^1(\boldsymbol{x})$ for notational simplicity. We also define $\boldsymbol{x}^l \in X$ such that

$$G^l = G^l(\boldsymbol{x}^l) \leq G^l(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in X \tag{9}$$

We first consider $G^{l+1}(\boldsymbol{x}) - G^l(\boldsymbol{x})$ for $l = 1, ..., n$ and $\boldsymbol{x} \in X$:

$$\begin{aligned} G^{l+1}(\boldsymbol{x}) - G^l(\boldsymbol{x}) &= \Gamma d_{l+1} + \sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{l+1} (d_j - d_{l+1}) x_j \\ &\quad - \Gamma d_l - \sum_{j=1}^{n} c_j x_j - \sum_{j=1}^{l} (d_j - d_l) x_j \\ &= \Gamma (d_{l+1} - d_l) + \sum_{j=1}^{l} (d_l - d_{l+1}) x_j \\ &= (d_{l+1} - d_l) \left(\Gamma - \sum_{j=1}^{l} x_j \right) \end{aligned} \tag{10}$$

Similarly, we consider for $l = 2, ..., n+1$ and $\boldsymbol{x} \in X$:

$$G^l(\boldsymbol{x}) - G^{l-1}(\boldsymbol{x}) = (d_l - d_{l-1}) \left(\Gamma - \sum_{j=1}^{l-1} x_j \right) \tag{11}$$

Using (10) and (11), we provide the following lemmas.

Lemma 1. For $l = 1, 2, ..., n$ and for any $\boldsymbol{x} \in X$, the following holds:

1. If $\sum_{j=1}^{l} x_j \leq \Gamma$, then $G^{l-1}(\boldsymbol{x}) \geq G^l(\boldsymbol{x}) \geq G^{l+1}(\boldsymbol{x})$.

2. If $\sum_{j=1}^{l} x_j > \Gamma$, then $G^{l-1}(\boldsymbol{x}) \leq G^{l}(\boldsymbol{x}) \leq G^{l+1}(\boldsymbol{x})$.

Proof. Recall that $\boldsymbol{x}$ is binary.

1. We prove the first part.
   (a) If $\sum_{j=1}^{l} x_j \leq \Gamma$, we obtain $G^{l+1}(\boldsymbol{x}) \leq G^{l}(\boldsymbol{x})$ from (10), since $d_{l+1} \leq d_l$.
   (b) If $\sum_{j=1}^{l} x_j \leq \Gamma$, then $\sum_{j=1}^{l-1} x_j \leq \Gamma$. Therefore we obtain $G^{l}(\boldsymbol{x}) \leq G^{l-1}(\boldsymbol{x})$ from (11), since $d_l \leq d_{l-1}$.

   Hence, the first part is proved.

2. We can similarly prove the second part.
   (a) If $\sum_{j=1}^{l} x_j > \Gamma$, we obtain $G^{l+1}(\boldsymbol{x}) \geq G^{l}(\boldsymbol{x})$ from (10), since $d_{l+1} \leq d_l$.
   (b) If $\sum_{j=1}^{l} x_j > \Gamma$, then $\sum_{j=1}^{l} x_j \geq \Gamma + 1$, and consequently, $\sum_{j=1}^{l-1} x_j \geq \Gamma$. Therefore we obtain $G^{l}(\boldsymbol{x}) \geq G^{l-1}(\boldsymbol{x})$ from (11), since $d_l \leq d_{l-1}$.

This completes the proof. □

Lemma 2. For any $\boldsymbol{x} \in X$, we have $G^1(\boldsymbol{x}) \geq G^2(\boldsymbol{x}) \geq \cdots \geq G^{\Gamma}(\boldsymbol{x}) \geq G^{\Gamma+1}(\boldsymbol{x})$. Furthermore, $G^1 \geq G^2 \geq ... \geq G^{\Gamma} \geq G^{\Gamma+1}$.

Proof. 1. For any $l \leq \Gamma$, we have $\sum_{j=1}^{l} x_j \leq \Gamma$, and therefore $G^{l}(\boldsymbol{x}) \geq G^{l+1}(\boldsymbol{x})$ by the first part of Lemma 1. This completes the proof for the first part.

2. By definition (9), we have

$$G^{\Gamma+1} = G^{\Gamma+1}(\boldsymbol{x}^{\Gamma+1}) \leq G^{\Gamma+1}(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in X$$

   Choosing $\boldsymbol{x} = \boldsymbol{x}^{\Gamma}$ and applying the first part of this lemma, we obtain

$$G^{\Gamma+1}(\boldsymbol{x}^{\Gamma+1}) \leq G^{\Gamma+1}(\boldsymbol{x}^{\Gamma}) \leq G^{\Gamma}(\boldsymbol{x}^{\Gamma})$$

   consequently, $G^{\Gamma+1} \leq G^{\Gamma}$. By repeating the same procedure for $\Gamma - 1, \Gamma - 2, ..., 1$, we obtain the lemma. □

Lemma 2 indicates that the $l = 1, 2, ..., \Gamma$ cases are no better than the $l = \Gamma + 1$ case. Therefore the $l = 1, 2, ..., \Gamma$ cases need not be examined in (5), if the $l = \Gamma + 1$ case is ensured to be examined.

Lemma 3. For any $l = 1, 2, ..., n$, we have either $G^{l} \geq G^{l+1}$ or $G^{l} \geq G^{l-1}$.

Proof. 1. Suppose $\sum_{j=1}^{l} x_j^l \leq \Gamma$. By definition (9)

$$G^{l+1} = G^{l+1}(\boldsymbol{x}^{l+1}) \leq G^{l+1}(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in X$$

   Choose $\boldsymbol{x} = \boldsymbol{x}^{l}$ and apply the first part of Lemma 1. Then,

$$G^{l+1}(\boldsymbol{x}^{l+1}) \leq G^{l+1}(\boldsymbol{x}^{l}) \leq G^{l}(\boldsymbol{x}^{l})$$

   Therefore, $G^{l+1} \leq G^{l}$.

2. If $\sum_{j=1}^{l} x_j^l > \Gamma$, we can similarly show that $G^{l-1} \leq G^{l}$, by considering G^{l-1} and applying the second part of Lemma 1.

Since the two cases are mutually exclusive, we obtain the lemma. □

Lemma 3 provides a way to significantly reduce the number of nominal problems to be solved; it indicates that any l is no better than either $l-1$ or $l+1$. This also indicates that the minimum of G^l occurs at two or more consecutive indices l, unless it does at $l = \Gamma + 1$ or $l = n + 1$. Our main result follows.

Table 1: Comparison of $\mathcal{L}$, with an example of $n = 20$ and $\Gamma = 5$

Authors	$\mathcal{L}$	$\lvert\mathcal{L}\rvert$
Bertsimas and Sim (2003)	$\{1, 2, ..., n+1\}$ $= \{1, 2, 3, ..., 19, 20, 21\}$	$n + 1 = 21$
Álvarez-Miranda et al. (2013)	$\{\Gamma, \Gamma+1, ..., n-1, n, n+1\}$ $= \{5, 6, 7, ..., 19, 20, 21\}$	$n + 2 - \Gamma = 17$
Park and Lee (2007)	$\{\Gamma, \Gamma+1, ..., n-1, n+1\}$ $= \{5, 6, 7, ..., 19, 21\}$	$n + 1 - \Gamma = 16$
Theorem 1	$\{\Gamma+1, \Gamma+3, \Gamma+5, ..., \Gamma+\gamma, n+1\}$ $= \{6, 8, 10, 12, 14, 16, 18, 20, 21\}$	$\left\lceil \frac{n-\Gamma}{2} \right\rceil + 1 = 9$

Theorem 1. The robust combinatorial optimization problem (2) can be solved by $\left\lceil \frac{n-\Gamma}{2} \right\rceil + 1$ number of nominal problems. In particular,

$$Z^* = \min_{l \in \mathcal{L}} G^l \tag{12}$$

where $\mathcal{L} = \{\Gamma+1, \Gamma+3, \Gamma+5, ..., \Gamma+\gamma, n+1\}$ and γ is the largest odd integer such that $\Gamma + \gamma < n + 1$.

Proof. The set of indices $\mathcal{L}$ is obtained by Lemmas 2 and 3. We prove the number of nominal problems to be solved. If we let $\gamma = 2k - 1$, then k is the largest integer such that $\Gamma + (2k - 1) < n + 1$, or $k < \frac{n-\Gamma}{2} + 1$; therefore $k = \left\lceil \frac{n-\Gamma}{2} \right\rceil$. Consequently, the cardinality of the set $\mathcal{L}$ is $k + 1 = \left\lceil \frac{n-\Gamma}{2} \right\rceil + 1$. This completes the proof. $\square$

Note that in Theorem 1, the set $\mathcal{L}$ includes the two boundary indices $l = \Gamma + 1$ and $l = n + 1$. We compared our result with the previous results in Table 1 with an example of $n = 20$ and $\Gamma = 5$.

3 Concluding Remarks

In this short note, we showed that the number of nominal problems to be solved can be significantly reduced to obtain a solution of robust combinatorial problems. We would like to close this note by providing a small tip for further reduction that is suggested in Kwon et al. (2013) for the case when the cost vector $\boldsymbol{c}$ is nonnegative. Suppose $G^\sharp$ is the smallest G^l found so far. Then, there is no need to consider any indices l such that $\Gamma d_l \geq G^\sharp$, since the objective function value of the corresponding nominal problem is nonnegative, hence there is no chance of improving. Therefore, by examining the set $\mathcal{L}$ in descending order, i.e., first considering $n + 1$ and then $\Gamma + \gamma$ to $\Gamma + 1$, we can stop when we encounter the case of $\Gamma d_l \geq G^\sharp$ for the first time.

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