Super Greedy Type Algorithms*

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Abstract

We study greedy-type algorithms such that at a greedy step we pick several dictionary elements contrary to a single dictionary element in standard greedy-type algorithms. We call such greedy algorithms *super greedy algorithms*. The idea of picking several elements at a greedy step of the algorithm is not new. Recently, we observed the following new phenomenon. For incoherent dictionaries these new type of algorithms (super greedy algorithms) provide the same (in the sense of order) upper bound for the error as their analogues from the standard greedy algorithms. The super greedy algorithms are computationally simpler than their analogues from the standard greedy algorithms. We continue to study this phenomenon.

Keywords: super greedy algorithms, thresholding, convergence rate, incoherent dictionary

1 Introduction. Weak Super Greedy Algorithm

This paper is a follow up to the paper [3]. We continue to study greedy-type algorithms such that at a greedy step we pick several dictionary elements

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contrary to a single dictionary element in standard greedy-type algorithms. We call such greedy algorithms *super greedy algorithms*. We refer the reader to [5] for a survey of the theory of greedy approximation. The idea of picking several elements at a greedy step of the algorithm is not new. It was used, for instance, in [9]. A new phenomenon that we observed in [3] is the following. For incoherent dictionaries these new type of algorithms (super greedy algorithms) provide the same (in the sense of order) upper bound for the error as their analogues from the standard greedy algorithms. The super greedy algorithms are computationally simpler than their analogues from the standard greedy algorithms. We continue to study this phenomenon here. We note that the idea of applying super greedy algorithm to incoherent dictionaries was used in [2] for building an efficient learning algorithm.

We recall some notations and definitions from the theory of greedy algorithms. Let H be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and the norm $\|x\| := \langle x, x \rangle^{1/2}$ for all $x \in H$. We say a set $\mathcal{D}$ of functions (elements) from H is a dictionary if each $g \in \mathcal{D}$ has a unit norm ($\|g\| = 1$) and $\overline{\mathrm{span}}\mathcal{D} = H$. Let

$$M(\mathcal{D}) := \sup_{\substack{\varphi \neq \psi \\ \varphi, \psi \in \mathcal{D}}} |\langle \varphi, \psi \rangle|$$

be the coherence parameter of dictionary $\mathcal{D}$. We say that a dictionary $\mathcal{D}$ is M-coherent if $M(\mathcal{D}) \leq M$. Main results of this paper concern performance of super greedy algorithms with regard to M-coherent dictionaries. We study two versions of super greedy algorithms: the Weak Super Greedy Algorithm and the Weak Orthogonal Super Greedy Algorithm with Thresholding. We now proceed to the definitions of these algorithms and to the formulations of main results.

Let a natural number s and a weakness sequence $\tau := \{t_k\}_{k=1}^{\infty}$, $t_k \in [0, 1]$, be given. Consider the following Weak Super Greedy Algorithm with parameter s.

WSGA(s, τ). Initialization: $f_0 := f_0^{s,\tau} := f$. Then for each $m \geq 1$ we inductively define:

(1) $\varphi_{(m-1)s+1}, \ldots, \varphi_{ms} \in \mathcal{D}$ are elements of the dictionary $\mathcal{D}$ satisfying the following inequality. Denote $I_m := [(m-1)s+1, ms]$ and assume that

$$\min_{i \in I_m} |\langle f_{m-1}, \varphi_i \rangle| \geq t_m \sup_{g \in \mathcal{D}, g \neq \varphi_i, i \in I_m} |\langle f_{m-1}, g \rangle|.$$

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(2) Let $F_m := F_m(f_{m-1}) := \operatorname{span}(\varphi_i, i \in I_m)$ and let P_{F_m} denote an operator of orthogonal projection onto F_m. Define the residual after mth iteration of the algorithm

$$f_m := f_m^{s,\tau} := f_{m-1} - P_{F_m}(f_{m-1}).$$

(3) Find the approximant

$$G_m^s(f) := G_m^{s,\tau}(f, \mathcal{D}) := \sum_{j=1}^m P_{F_j}(f_{j-1}).$$

In the case $t_k = t$, $k = 1, 2, \ldots$, we write t instead of τ in the notations. If $t = 1$, we call the WSGA$(s, 1)$ the Super Greedy Algorithm with parameter s (SGA(s)). For $s = 1$ the Super Greedy Algorithm coincides with the Pure Greedy Algorithm and the Weak Super Greedy Algorithm coincides with the Weak Greedy Algorithm (see [5]).

For a general dictionary $\mathcal{D}$ we define the class of functions (elements)

$$\mathcal{A}_1^0(\mathcal{D}, B) := \left\{ f \in H : f = \sum_{k \in \Lambda} c_k g_k, \quad g_k \in \mathcal{D}, |\Lambda| < \infty, \quad \sum_{k \in \Lambda} |c_k| \le B \right\}$$

and we define $\mathcal{A}_1(\mathcal{D}, B)$ to be the closure (in H) of $\mathcal{A}_1^0(\mathcal{D}, B)$. For the case $B = 1$, we denote $A_1(\mathcal{D}) := \mathcal{A}_1(\mathcal{D}, 1)$. We define the norm $|f|_{\mathcal{A}_1(\mathcal{D})}$ to be the smallest B such that $f \in \mathcal{A}_1(\mathcal{D}, B)$.

The following open problem (see [10], p.65, Open Problem 3.1) on the rate of convergence of the PGA for the $A_1(\mathcal{D})$ is a central theoretical problem in greedy approximation in Hilbert spaces.

Open problem. Find the order of decay of the sequence

$$\gamma(m) := \sup_{f, \mathcal{D}, \{G_m^1\}} (\|f - G_m^1(f, \mathcal{D})\| |f|_{\mathcal{A}_1(\mathcal{D})}^{-1}),$$

where the supremum is taken over all dictionaries $\mathcal{D}$, all elements $f \in \mathcal{A}_1(\mathcal{D}) \setminus \{0\}$ and all possible choices of $\{G_m^1\}$.

We refer the reader to [5] for a discussion of this open problem. Introduce the following generalization of the quantity $\gamma(m)$ to the case of the Weak Greedy Algorithms

$$\gamma(m, \tau) := \sup_{f, \mathcal{D}, \{G_m^{1,\tau}\}} (\|f - G_m^{1,\tau}(f, \mathcal{D})\| |f|_{\mathcal{A}_1(\mathcal{D})}^{-1}).$$

We prove here the following theorem.

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Theorem 1.1. *Let $\mathcal{D}$ be a dictionary with coherence parameter $M := M(\mathcal{D})$. Then, for $s \leq (2M)^{-1}$, the WSGA(s,t) provides, after m iterations, an approximation of $f \in A_1(\mathcal{D})$ with the following upper bound on the error:*

$$\|f - G_m^{s,t}(f,\mathcal{D})\| \leq Ct^{-1}s^{-\frac{1}{2}}\gamma(m-1,rt),$$

where $r := \left(\frac{1-Ms}{1+Ms}\right)^{\frac{1}{2}}$ and C is an absolute constant.

Theorem 1.1 with $t = 1$ gives the following assertion for the SGA(s).

Corollary 1.2. *Let $\mathcal{D}$ be a dictionary with coherence parameter $M := M(\mathcal{D})$. Then, for $s \leq (2M)^{-1}$, the SGA(s) provides, after m iterations, an approximation of $f \in A_1(\mathcal{D})$ with the following upper bound on the error:*

$$\|f - G_m^s(f,\mathcal{D})\| \leq Cs^{-\frac{1}{2}}\gamma(m,r),$$

where $r := \left(\frac{1-Ms}{1+Ms}\right)^{\frac{1}{2}}$ and C is an absolute constant.

It is interesting to note that even in the case of the SGA(s), when the weakness parameter is 1, we have the upper bound of the error in terms of $\gamma(m,r)$ not in terms of $\gamma(m)$. For estimating $\gamma(m,r)$ we use the following known result from [4].

Theorem 1.3. *Let $\mathcal{D}$ be an arbitrary dictionary in H. Assume $\tau := \{t_k\}_{k=1}^{\infty}$ is a nonincreasing sequence. Then, for $f \in \mathcal{A}_1(D,B)$ we have*

$$\|f - G_m^{1,\tau}(f,\mathcal{D})\| \leq B(1 + \sum_{k=1}^{m} t_k^2)^{-t_m/2(2+t_m)}. \qquad (1.1)$$

For a particular case $t_k = 1$, $k = 1, 2, \ldots$, this theorem gives the following result (see [8]). For each $f \in \mathcal{A}_1(\mathcal{D}, B)$, the PGA provides, after m iterations, an approximant satisfying

$$\|f - G_m(f,\mathcal{D})\| \leq Bm^{-1/6}.$$

For $f \in A_1(\mathcal{D})$, we apply the PGA and the SGA(s). Then after sm iterations of the PGA and m iterations of the SGA(s), both algorithms provide sm-term approximants. For illustration purposes, take $s = N^{1/2}$ and $m = N^{1/2}$. Then the PGA gives

$$\|f_N\| \leq (sm)^{-1/6} = N^{-1/6}$$

and the SGA(s) provides

$$\|f_m\| \leq Cs^{-1/2}m^{-\frac{r}{2(2+r)}} = N^{-\frac{1}{4}-\frac{1}{2}\theta},$$

where $\theta := \frac{r}{2(2+r)}$ and $r = \left(\frac{1-Ms}{1+Ms}\right)^{1/2}$, $\frac{1}{4} + \frac{1}{2}\theta \geq \frac{1}{6}$. Thus, in this particular case, the SGA(s) has a better upper bound for the error than the PGA.

2 Weak Orthogonal Super Greedy Algorithm with Thresholding

In [3] we considered the following algorithm. Let a natural number s and a weakness sequence $\tau := \{t_k\}_{k=1}^{\infty}$, $t_k \in [0,1]$, be given. Consider the following Weak Orthogonal Super Greedy Algorithm with parameter s.

WOSGA(s, τ). Initially, $f_0 := f$. Then, for each $m \geq 1$ we inductively define:

(1) $\varphi_{(m-1)s+1}, \ldots, \varphi_{ms} \in \mathcal{D}$ are elements of the dictionary $\mathcal{D}$ satisfying the following inequality. Denote $I_m := [(m-1)s+1, ms]$ and assume that

$$\min_{i \in I_m} |\langle f_{m-1}, \varphi_i \rangle| \geq t_m \sup_{g \in \mathcal{D}, g \neq \varphi_i, i \in I_m} |\langle f_{m-1}, g \rangle|.$$

(2) Let $H_m := H_m(f) := \operatorname{span}(\varphi_1, \ldots, \varphi_{ms})$ and let P_{H_m} denote an operator of orthogonal projection onto H_m. Define

$$G_m(f) := G_m(f, \mathcal{D}) := G_m^s(f, \mathcal{D}) := P_{H_m}(f).$$

(3) Define the residual after mth iteration of the algorithm

$$f_m := f_m^s := f - G_m(f, \mathcal{D}).$$

In [3] we proved the following error bound for the WOSGA(s, t).

Theorem 2.1. *Let $\mathcal{D}$ be a dictionary with coherence parameter $M := M(\mathcal{D})$. Then, for $s \leq (2M)^{-1}$, the WOSGA(s, t) provides, after m iterations, an approximation of $f \in A_1(\mathcal{D})$ with the following upper bound on the error:*

$$\|f_m\|^2 \leq A(t)(sm)^{-1}, \quad m = 1, 2, \ldots \quad A(t) := (81/8)(1+t)^2 t^{-4}.$$

In this paper we modify the WOSGA in the following way: we replace the greedy step (1) by the thresholding step. Here is the definition of the new algorithm. Let s be a natural number and let a weakness sequence $\tau := \{t_k\}_{k=1}^{\infty}$, $t_k \in [0, 1]$ be given.

WOSGAT(s, τ). Initially, $f_0 := f_0^{s,\tau} := f$. Then for each $m \geq 1$ we inductively define:

(1) $\varphi_i \in \mathcal{D}$ where $i_m \in I_m$ are elements of the dictionary $\mathcal{D}$ satisfying the following inequalities $s_m := |I_m| \leq s$ and

$$\min_{i \in I_m} |\langle f_{m-1}, \varphi_i \rangle| \geq t_m \|f_{m-1}\|^2. \tag{2.1}$$

(2) Let $H_m := H_m(f) := \mathrm{span}(\varphi_i, \ i \in I_1 \cup \ldots \cup I_m)$ and let P_{H_m} denote an operator of orthogonal projection onto H_m. Denote

$$G_m(f) := G_m^{s,\tau}(f, \mathcal{D}) := P_{H_m}(f).$$

(3) Define the residual after mth iteration of the algorithm

$$f_m := f_m^{s,\tau} := f - P_{H_m}(f).$$

For $s = 1$ the WOSGA coincides with the Weak Orthogonal Greedy Algorithm (WOGA) and the WOSGAT coincides with the Modified Weak Orthogonal Greedy Algorithm (MWOGA) (see [10], p. 61). We note that we can run the WOSGA and the WOGA for any $f \in H$. It is proved in [10] that we can run the MWOGA for $f \in A_1(\mathcal{D})$. In the same way one can prove that we can run the WOSGAT for $f \in A_1(\mathcal{D})$. We note that in step (1) of the WOSGAT, if there are more than s φ_i's satisfying

$$|\langle f_{m-1}, \varphi_i \rangle| \geq t_m \|f_{m-1}\|^2,$$

the algorithm may pick any s of them and then make the projection.

If $t_m = t$ for $m = 1, 2, \ldots$, we use t instead of τ in the notation. We will prove an upper bound for the rate of convergence of the WOSGAT for $f \in A_1(\mathcal{D})$ for a more general dictionary than the M-coherent dictionary.

Definition 2.1. We say that a dictionary $\mathcal{D}$ is (N, β)-Bessel if for any N distinct elements $\psi_1, \ldots, \psi_N$ of the dictionary $\mathcal{D}$ we have for any $f \in H$

$$\|P_{\Psi(N)}(f)\|^2 \geq \beta \sum_{i=1}^{N} |\langle f, \psi_i \rangle|^2,$$

where $\Psi(N) := \mathrm{span}\{\psi_1, \ldots, \psi_N\}$.

Theorem 2.2. *Let $\mathcal{D}$ be an (N, β)-Bessel dictionary. Then, for $s \le N$, the WOSGAT(s,t) provides, after m iterations, an approximation of $f \in A_1(\mathcal{D})$ with the following upper bound of the error:*

$$\|f - G_m(f)\| \le (1 + \beta t^2 \sum_{j=1}^{m} s_j)^{-1/2}.$$

We point out that $\sum_{j=1}^{m} s_j$ is the number of elements that the algorithm picked up from $\mathcal{D}$ after m iterations. Therefore, the WOSGAT offers the same error bound (in the sense of order) in terms of the number of elements of the dictionary, used in the approximant, as the WOGA or the MWOGA.

We now give some sufficient conditions for a dictionary $\mathcal{D}$ to be (N, β)-Bessel. We begin with a simple lemma useful in that regard.

Lemma 2.1. *Let a dictionary $\mathcal{D}$ have the following property of (N, A)-stability. For any N distinct elements $\psi_1, \ldots, \psi_N$ of the dictionary $\mathcal{D}$, we have for any coefficients $c_1, \ldots, c_N$*

$$\|\sum_{i=1}^{N} c_i \psi_i\|^2 \le A \sum_{i=1}^{N} |c_i|^2.$$

Then $\mathcal{D}$ is (N, A^{-1})-Bessel.

Proof. Let $f \in H$. We have

$$\|P_{\Psi(N)}(f)\| = \sup_{\psi \in \Psi(N), \|\psi\| \le 1} |\langle P_{\Psi(N)}(f), \psi \rangle|$$

$$= \sup_{(c_1,\ldots,c_N): \|c_1 \psi_1 + \cdots + c_N \psi_N\| \le 1} |\sum_{i=1}^{N} \langle f, \psi_i \rangle c_i|$$

$$\ge \sup_{(c_1,\ldots,c_N): |c_1|^2 + \cdots + |c_N|^2 \le A^{-1}} |\sum_{i=1}^{N} \langle f, \psi_i \rangle c_i| = A^{-1/2} (\sum_{i=1}^{N} |\langle f, \psi_i \rangle|^2)^{1/2}.$$

$\square$

Proposition 2.3. *An M-coherent dictionary is $(N, (1 + M(N-1))^{-1})$-Bessel.*

Proof. By Lemma 2.1 from [1] for an M-coherent dictionary we have

$$\| \sum_{i=1}^{N} c_i \psi_i \|^2 \leq (1 + M(N-1)) \sum_{i=1}^{N} |c_i|^2.$$

Applying the above Lemma 2.1, we obtain the statement of the proposition. $\square$

The following proposition is a direct corollary of Lemma 2.1.

Proposition 2.4. *Let a dictionary $\mathcal{D}$ have the RIP(N, δ): for any distinct $\psi_1, \ldots, \psi_N$*

$$(1 - \delta) \sum_{i=1}^{N} |c_i|^2 \leq \| \sum_{i=1}^{N} c_i \psi_i \|^2 \leq (1 + \delta) \sum_{i=1}^{N} |c_i|^2.$$

Then $\mathcal{D}$ is $(N, (1 + \delta)^{-1})$-Bessel.

3 Proofs

Proof of Theorem 1.1. Let

$$f = \sum_{j=1}^{\infty} c_j g_j, \quad g_j \in \mathcal{D}, \quad \sum_{j=1}^{\infty} |c_j| \leq 1, \quad |c_1| \geq |c_2| \geq \ldots \quad . \tag{3.1}$$

Every element of $A_1(\mathcal{D})$ can be approximated arbitrarily well by elements of the form (3.1). It will be clear from the below argument that it is sufficient to consider elements f of the form (3.1). Suppose ν is such that $|c_\nu| \geq a/s \geq |c_{\nu+1}|$, where $a = \frac{t+3}{2t}$. Then the above assumption on the sequence $\{c_j\}$ implies that $\nu \leq \lfloor s/a \rfloor$ and $|c_{s+1}| < 1/s$. We claim that elements $g_1, \ldots, g_\nu$ will be chosen among $\varphi_1, \ldots, \varphi_s$ at the first iteration. Indeed, for $j \in [1, \nu]$ we have

$$|\langle f, g_j \rangle| \geq |c_j| - M \sum_{k \neq j}^{\infty} |c_k| \geq a/s - M(1 - a/s) > a/s - M.$$

For all g distinct from $g_1, \ldots, g_s$ we have

$$|\langle f, g \rangle| \leq M + 1/s.$$

Our assumption $s \le 1/(2M)$ implies that $M + 1/s \le t(a/s - M)$. Thus, we do not pick any of $g \in \mathcal{D}$ distinct from $g_1, \ldots, g_s$ until we have chosen all $g_1, \ldots, g_\nu$.

Denote

$$f' := f - \sum_{j=1}^{\nu} c_j g_j = \sum_{j=\nu+1}^{\infty} c_j g_j.$$

It is clear from the above argument that

$$f_1 = f - P_{H_1}(f) = f' - P_{H_1}(f') = f' - G_1^s(f'). \tag{3.2}$$

Define a new dictionary

$$\mathcal{D}^s := \left\{ \frac{\sum_{j \in \Lambda} c_j g_j}{\| \sum_{j \in \Lambda} c_j g_j \|} : \quad |\Lambda| = s, \quad g_j \in \mathcal{D}, c_j \in \mathbb{R} \right\}.$$

Let $J_l = [(l-1)s + \nu + 1, ls + \nu]$. We write

$$f' = \sum_{l=1}^{\infty} \sum_{j \in J_l} c_j g_j = \sum_{l=1}^{\infty} \|\psi_l\| \frac{\psi_l}{\|\psi_l\|}, \tag{3.3}$$

where $\psi_l = \sum_{j \in J_l} c_j g_j$. Apparently $\frac{\psi_l}{\|\psi_l\|} \in \mathcal{D}^s$ for $l \ge 1$. Equation (3.3) implies that

$$f' \in \mathcal{A}_1(\mathcal{D}^s, \sum_{l=1}^{\infty} \|\psi_l\|).$$

By Lemma 2.1 from [1] we bound

$$(1 - Ms) \sum_{j \in J_l} c_j^2 \le \|\psi_l\|^2 \le (1 + Ms) \sum_{j \in J_l} c_j^2. \tag{3.4}$$

Then we obtain

$$\sum_{l=1}^{\infty} \|\psi_l\| \le (1 + Ms)^{1/2} \sum_{l=1}^{\infty} (\sum_{j \in J_l} c_j^2)^{1/2}. \tag{3.5}$$

Since the sequence $\{c_j\}$ has the property

$$|c_{\nu+1}| \ge |c_{\nu+2}| \ge \ldots, \quad \sum_{j=\nu+1}^{\infty} |c_j| \le 1, \quad |c_{\nu+1}| \le a/s \tag{3.6}$$

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we may apply the simple inequality,

$$\left(\sum_{j \in J_l} c_j^2\right)^{1/2} \le s^{1/2} |c_{(l-1)s+\nu+1}|,$$

so that we bound the sum in the right side of (3.5)

$$\sum_{l=1}^{\infty} \left(\sum_{j \in J_l} c_j^2\right)^{1/2} \le s^{1/2} \sum_{l=1}^{\infty} |c_{(l-1)s+\nu+1}|$$

$$\le s^{1/2} \left(a/s + \sum_{l=2}^{\infty} s^{-1} \sum_{j \in J_{l-1}} |c_j|\right) \le (a+1)s^{-1/2}. \quad (3.7)$$

Using the above inequality in (3.5), we obtain that

$$f' \in \mathcal{A}_1(\mathcal{D}^s, (3/2)^{1/2}(a+1)s^{-\frac{1}{2}}). \quad (3.8)$$

Assume $u := \sum a_i h_i \in \mathcal{A}_1(\mathcal{D}^s, B)$, where B is an absolute constant. Then for any $\psi \in \mathcal{D}^s$ we have

$$v := u - \langle u, \psi \rangle \psi \in \mathcal{A}_1(\mathcal{D}, 2B),$$

since $|\langle u, \psi \rangle| = |\langle \sum a_i g_i, \psi \rangle| \le \|\psi\| \sum |a_i| \le B$. Along with (3.2) and (3.8) the above argument shows that

$$f_1 \in \mathcal{A}_1(\mathcal{D}^s, 2(3/2)^{1/2}(a+1)s^{-1/2}). \quad (3.9)$$

Consider the following quantity

$$q_s := q_s(f_{m-1}) := \sup_{\substack{h_i \in \mathcal{D} \\ i \in [1,s]}} \|P_{H(s)}(f_{m-1})\|,$$

where $H(s) := \operatorname{span}(h_1, \ldots, h_s)$. It is clear that

$$q_s = \sup_{H(s)} \max_{\psi \in H(s), \|\psi\| \le 1} |\langle f_{m-1}, \psi \rangle| = \sup_{g^s \in \mathcal{D}^s} |\langle f_{m-1}, g^s \rangle|.$$

Let $\psi = \sum_{i=1}^{s} a_i h_i$. Again by Lemma 2.1 from [1] we bound

$$(1 - Ms) \sum_{i=1}^{s} a_i^2 \le \|\psi\|^2 \le (1 + Ms) \sum_{i=1}^{s} a_i^2. \quad (3.10)$$

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Therefore,

$$(1 + Ms)^{-1} \sum_{i=1}^{s} \langle f_{m-1}, h_i \rangle^2 \leq \| P_{H(s)}(f_{m-1}) \|^2 \leq (1 - Ms)^{-1} \sum_{i=1}^{s} \langle f_{m-1}, h_i \rangle^2.$$

(3.11)

Let $p_m := P_{H_m}(f_{m-1})$. In order to relate q_s^2 to $\|p_m\|^2$ we begin with the fact that

$$q_s^2 \leq \sup_{\substack{h_i \in \mathcal{D} \\ i \in [1,s]}} (1 - Ms)^{-1} \sum_{i=1}^{s} \langle f_{m-1}, h_i \rangle^2.$$

Consider an arbitrary set $\{h_i\}_{i=1}^{s}$ of distinct elements of the dictionary $\mathcal{D}$. Let V be a set of all indices $i \in [1, s]$ such that $h_i = \varphi_{k(i)}$, $k(i) \in I_m$. Denote $V' := \{k(i), i \in V\}$. Then

$$\sum_{i=1}^{s} \langle f_{m-1}, h_i \rangle^2 = \sum_{i \in V} \langle f_{m-1}, h_i \rangle^2 + \sum_{i \in [1,s] \setminus V} \langle f_{m-1}, h_i \rangle^2.$$

(3.12)

From the definition of $\{\varphi_k\}_{k \in I_m}$ we get

$$\max_{i \in [1,s] \setminus V} |\langle f_{m-1}, h_i \rangle| \leq t^{-1} \min_{k \in I_m \setminus V'} |\langle f_{m-1}, \varphi_k \rangle|.$$

(3.13)

Using (3.13) we continue (3.12)

$$\leq \sum_{k \in V'} \langle f_{m-1}, \varphi_k \rangle^2 + t^{-2} \sum_{k \in I_m \setminus V'} \langle f_{m-1}, \varphi_k \rangle^2 \leq t^{-2} \sum_{k \in I_m} \langle f_{m-1}, \varphi_k \rangle^2.$$

Therefore,

$$q_s^2 \leq (1 - Ms)^{-1} t^{-2} \sum_{k \in I_m} \langle f_{m-1}, \varphi_k \rangle^2 \leq \frac{1 + Ms}{t^2(1 - Ms)} \|p_m\|^2.$$

This results in the following inequality

$$\|p_m\|^2 \geq \frac{t^2(1 - Ms)}{1 + Ms} q_s^2.$$

(3.14)

Thus we can interpret $\mathrm{WSGA}(s, t)$ as $\mathrm{WGA}(rt)$ with respect to the dictionary $\mathcal{D}^s$, where $r = \left(\frac{1 - Ms}{1 + Ms} \right)^{\frac{1}{2}}$. Using (3.9), we get

$$\|f_m^{s,t}\| = \|(f_1)_{m-1}^{s,t}\| \leq 6^{1/2}(a + 1)s^{-1/2}\gamma(m - 1, rt).$$

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This completes the proof of Theorem 1.1.

Proof of Theorem 2.2. For the $\{\varphi_i\}$ from the definition of the WOSGAT, denote

$$F_m := \text{span}(\varphi_i, i \in I_m).$$

It is easy to see that $H_m = H_{m-1} \oplus F_m$. Therefore,

$$\begin{aligned}
f_m = f - P_{H_m}(f) &= f_{m-1} + G_{m-1}(f) - P_{H_m}(f_{m-1} + G_{m-1}(f)) \\
&= f_{m-1} - P_{H_m}(f_{m-1}).
\end{aligned}$$

Then the inclusion $F_m \subset H_m$ implies

$$\|f_m\| \leq \|f_{m-1} - P_{F_m}(f_{m-1})\|. \tag{3.15}$$

Using the notation $p_m := P_{F_m}(f_{m-1})$, we continue

$$\|f_{m-1}\|^2 = \|f_{m-1} - p_m\|^2 + \|p_m\|^2$$

and by (3.15)

$$\|f_m\|^2 \leq \|f_{m-1}\|^2 - \|p_m\|^2. \tag{3.16}$$

We now prove a lower bound for $\|p_m\|$. By our assumption that the dictionary is (N, β)-Bessel we get

$$\|P_{F_m}(f_{m-1})\|^2 \geq \beta \sum_{i \in I_m} |\langle f_{m-1}, \varphi_i \rangle|^2.$$

Then, by the thresholding condition of the greedy step (1), we obtain

$$\|P_{F_m}(f_{m-1})\|^2 \geq \beta t^2 s_m \|f_{m-1}\|^4. \tag{3.17}$$

Substituting this bound in (3.16), we get

$$\|f_m\|^2 \leq \|f_{m-1}\|^2 \left(1 - \beta t^2 s_m \|f_{m-1}\|^2\right). \tag{3.18}$$

We now apply the following lemma from [4].

Lemma 3.1. *Let $\{a_m\}_{m=0}^{\infty}$ be a sequence of nonnegative numbers satisfying the inequalities*

$$a_0 \leq A, \quad a_m \leq a_{m-1}(1 - \lambda_m^2 a_{m-1}/A), \quad m = 1, 2, \dots$$

Then we have for each m

$$a_m \leq A\left(1 + \sum_{k=1}^{m} \lambda_k^2\right)^{-1}.$$

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It gives us

$$\|f_m\| \le (1 + \beta t^2 \sum_{j=1}^{m} s_j)^{-1/2}.$$

$\square$

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