# Decompositions of Trigonometric Polynomials with Applications to Multivariate Subdivision Schemes ${ }^{1}$ 

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#### Abstract

We study multivariate trigonometric polynomials, satisfying a set of constraints close to the known Strung-Fix conditions. Based on the polyphase representation of these polynomials relative to a general dilation matrix, we develop a simple constructive method for a special type of decomposition of such polynomials. These decompositions are of interest to the analysis of convergence and smoothness of multivariate subdivision schemes associated with general dilation matrices. We apply these decompositions, by verifying sufficient conditions for the convergence and smoothness of multivariate scalar subdivision schemes, proved here. For the convergence analysis our sufficient conditions apply to arbitrary dilation matrices, while the previously known necessary and sufficient conditions are relevant only in case of dilation matrices with a self similar tiling. For the analysis of smoothness, we state and prove two theorems on multivariate matrix subdivision schemes, which lead to sufficient conditions for $C^{1}$ limits of scalar multivariate subdivision schemes associated with isotropic dilation matrices. Although similar results are stated in the literature, we give here detailed proofs of the results, which we could not find elsewhere.


MSC 41A30, 42C40

## 1. Introduction

In this paper we study multivariate trigonometric polynomials (masks), satisfying a set of constraints, which is close to the known Strung-Fix conditions.

[^0]We are interested in decomposing such a polynomial multiplied by a factor of the form $1-e^{2 \pi i\left(x, \mathbf{e}_{j}\right)}$, into a sum of $d$ trigonometric polynomials, satisfying the same type of constraints, but of one order less, each multiplied by a different factor of the form $1-e^{2 \pi i\left(M^{*} x, \mathbf{e}_{k}\right)}$.

This type of decompositions is of interest in the analysis of convergence and smoothness of multivariate subdivision schemes. The algebraic approach to the analysis of such schemes is investigated in [1] relative to the dilation matrix $2 I$, and in [4], [8], [9], relative to general dilation matrices.

The analogous decompositions for multivariate algebraic polynomials are based on the rich structure of ideals of multivariate polynomials. The trigonometric decompositions are based on the rather simple idea of the representation of a trigonometric polynomial in terms of its unique polyphase trigonometric polynomials relative to a dilation matrix [10].

The outline of the paper is as follows: in Section 2 we introduce our notation and the set of constraints. We also bring basic results related to polyphase representations of trigonometric polynomials. Properties of trigonometric polynomials satisfying the set of constraints of order zero relative to a general dilation matrix are derived in Section 3, in terms of their polyphase trigonometric polynomials.

These results are used in Section 4 to prove the decomposition of trigonometric polynomials, satisfying the set of constraints of order zero. An algorithm for the computation of this decomposition is presented. The decomposition for order zero is used to derive that for order one. The general case of order greater than one, which is much more involved, is also investigated, and an algorithm for this case is given.

Section 5 gives a characterization of a trigonometric polynomial satisfying a set of constraints of a general order, in terms of the values at the origin of the derivatives of its polyphase trigonometric polynomials. This result provides a tool for the construction of such polynomials.

Applications of the decompositions of Section 4 to the convergence and smoothness analysis of multivariate subdivision schemes associated with general dilation matrices are presented in Section 6. The application of the decompositions to the convergence analysis is based on a new result giving sufficient conditions for the convergence of a scalar multivariate subdivision scheme associated with an arbitrary dilation matrix. The result in the literature giving necessary and sufficient conditions for convergence, is limited to dilation matrices having a self similar tiling [4]. For the analysis of smoothness, we state and prove two theorems on multivariate matrix subdivision
schemes. First we give a detailed proof of a sufficient condition for $C^{1}$ limit functions of such a scheme, associated with an isotropic dilation matrix. Although this condition is stated in [9], the sketch of a proof given there, is valid only for dilation matrices which are multiples of the identity matrix. Then we state and prove a condition, in terms of the decompositions, that allows us to check the sufficient condition for $C^{1}$.

## 2. Notation and preliminary information

Let $\mathbb{N}$ be the set of positive integers, $\mathbb{R}^{d}$ denotes the $d$-dimensional Euclidean space, $x=\left(x_{1}, \ldots, x_{d}\right), y=\left(y_{1}, \ldots, y_{d}\right)$ are its elements (vectors), $(x, y)=$ $x_{1} y_{1}+\ldots+x_{d} y_{d},|x|=\max _{j=1, \ldots, d}\left|x_{j}\right|, \mathbf{e}_{j}=(0, \ldots, 1, \ldots, 0)$ is the $j$-th unit vector in $\mathbb{R}^{d}, \mathbf{0}=(0, \ldots, 0) \in \mathbb{R}^{d} ; \mathbb{Z}^{d}$ is the integer lattice in $\mathbb{R}^{d}$. For $x, y \in \mathbb{R}^{d}$, we write $x>y$ if $x_{j}>y_{j}, j=1, \ldots, d ; \mathbb{Z}_{+}^{d}=\left\{x \in Z^{d}: x \geq \mathbf{0}\right\}$. If $\alpha, \beta \in \mathbb{Z}_{+}^{d}$, $a, b \in \mathbb{R}^{d}$, we set $\alpha!=\prod_{j=1}^{d} \alpha_{j}!,\binom{\alpha}{\beta}=\frac{\alpha!}{\beta!(\alpha-\beta)!}, a^{b}=\prod_{j=1}^{d} a_{j}{ }^{b_{j}},[\alpha]=\sum_{j=1}^{d} \alpha_{j}$, $D^{\alpha} f=\frac{\partial^{[\alpha]} f}{\partial^{\alpha_{1} x_{1} \ldots \partial^{\alpha} d x_{d}}} ; \delta_{a b}$ denotes the Kronecker delta; $z:=\left(z_{1}, \ldots, z_{d}\right)$, where $z_{k}:=e^{2 \pi i x_{k}} ; \ell_{\infty}^{N}\left(\mathbb{Z}^{d}\right)=\ell_{\infty}^{N}:=\left\{f=\left\{f_{\alpha}\right\}_{\alpha \in \mathbb{Z}^{d}}: \quad f_{\alpha} \in \mathbb{R}^{N},\left|f_{\alpha}\right| \leq C\right\}$, $\|f\|_{\ell_{\infty}^{N}}=\|f\|_{\infty}:=\sup _{\alpha \in \mathbb{Z}^{d}}\left|f_{\alpha}\right|, \ell_{\infty}:=\ell_{\infty}^{1}$.

Let $M$ be a non-degenerate $d \times d$ integer matrix whose eigenvalues are bigger than 1 in module, $M^{*}$ is the transpose of $M, I_{d}$ and $\mathbb{O}_{d}$ denote respectively the unit and the zero $d \times d$ matrices. We say that the numbers $k, n \in \mathbb{Z}^{d}$ are congruent modulo $M($ write $k \equiv n(\bmod M))$ if $k-n=M \ell$, $\ell \in \mathbb{Z}^{d}$. The integer lattice $\mathbb{Z}^{d}$ is splitted into cosets with respect to the introduced relation of congruence. The number of cosets is equal to $|\operatorname{det} M|$ (see, e.g., [11, p. 107]). Let us take an arbitrary representative from each coset, call them digits and denote the set of digits by $D(M)$. Throughout the paper we consider that such a matrix $M$ (dilation matrix) is fixed, $m=|\operatorname{det} M|$, $D(M)=\left\{s_{0}, \ldots, s_{m-1}\right\}, D\left(M^{*}\right)=\left\{s_{0}^{*}, \ldots, s_{m-1}^{*}\right\}, s_{0}=s_{0}^{*}=\mathbf{0}, r_{k}=M^{-1} s_{k}$, $k=0, \ldots, m-1$.

If $A$ is a $N \times N^{\prime}$ matrix with entries $a_{j k}$, then $A^{[n]}$ denotes its $n$-th Kronecker power, i.e.,

$$
A^{[0]}:=1, \quad A^{[1]}:=A, \quad A^{[n+1]}:=\left(\begin{array}{ccc}
a_{11} A^{[n]} & \ldots & a_{1 N^{\prime}} A^{[n]} \\
\vdots & \ddots & \vdots \\
& & \\
a_{N 1} A^{[n]} & \ldots & a_{N N^{\prime}} A^{[n]}
\end{array}\right)
$$

Proposition A The matrix $\left\{\frac{1}{\sqrt{m}} e^{2 \pi i\left(r_{k}, s_{l}^{*}\right)}\right\}_{k, l=0}^{m-1}$ is unitary. In particular,

$$
\begin{equation*}
\frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} e^{2 \pi i\left(r_{k}, s_{l}^{*}\right)}=\delta_{0 l} \tag{1}
\end{equation*}
$$

A proof of this statement can be found in [6] or in [2].
We will consider 1-periodic trigonometric polynomials in $d$ variables

$$
t(x)=\sum_{n \in \mathbb{Z}^{d}} \widehat{t}(n) e^{2 \pi i(n, x)},
$$

where the set $\left\{n \in \mathbb{Z}^{d}: \widehat{t}(n) \neq 0\right\}$ is finite.
For any trigonometric polynomial $t$, there exists a unique set of trigonometric polynomials $\tau_{\nu}, \nu=0, \ldots, m-1$, (polyphase functions of $t$ ) (see, e.g., [6]) such that

$$
\begin{equation*}
t(x)=\sum_{\nu=0}^{m-1} e^{2 \pi i\left(s_{\nu}, x\right)} \tau_{\nu}\left(M^{*} x\right) \tag{2}
\end{equation*}
$$

It is clear that

$$
\tau_{\nu}(x)=\sum_{n \in \mathbb{Z}^{d}} \widehat{t}\left(M n+s_{\nu}\right) e^{2 \pi i(n, x)}
$$

For any $n \in \mathbb{Z}_{+}$, denote by $\mathcal{Z}^{n}$ the set of trigonometric polynomials $t$ satisfying the following "zero-condition" of order $n:\left.D^{\beta} t\left(M^{*-1} x\right)\right|_{x=s}=0$ for all $s \in D\left(M^{*}\right), s \neq \mathbf{0}$, and for all $\beta \in \mathbb{Z}_{+}^{d},[\beta] \leq n$. It is well known that this "zero-condition" is equivalent to the so-called "sum-rule" and close to the Strang-Fix condition of the same order. It will be convenient for us to define $\mathcal{Z}^{-1}$ as the set of all trigonometric polynomials.

## 3. Auxiliary results

Proposition 1 A trigonometric polynomial $t$ belongs to $\mathcal{Z}^{0}$ if and only if

$$
\begin{equation*}
\tau_{\nu}(\mathbf{0})=\frac{t(\mathbf{0})}{m}, \quad \nu=0, \ldots, m-1 \tag{3}
\end{equation*}
$$

where $\tau_{0}, \ldots, \tau_{m-1}$ are the polyphase functions of $t$.

Proof. Let $s \in D\left(M^{*}\right)$, using (2) we have

$$
t\left(M^{*-1} s\right)=\sum_{\nu=0}^{m-1} e^{2 \pi i\left(r_{\nu}, s\right)} \tau_{\nu}(s)=\sum_{\nu=0}^{m-1} e^{2 \pi i\left(r_{\nu}, s\right)} \tau_{\nu}(\mathbf{0}) .
$$

So, the relation $t \in \mathcal{Z}^{0}$ can be rewritten as

$$
\sum_{\nu=0}^{m-1} e^{2 \pi i\left(r_{\nu}, s_{l}^{*}\right)} \tau_{\nu}(\mathbf{0})=t(\mathbf{0}) \delta_{\mathbf{0} l}, \quad l=0, \ldots, m-1
$$

Consider these equalities as a linear system with unknowns $\tau_{0}(\mathbf{0}), \ldots, \tau_{m-1}(\mathbf{0})$. Due to Proposition A, the system has a unique solution given by (3). $\diamond$

Lemma 2 Let $t$ be a trigonometric polynomial, $t(\mathbf{0})=0$, then

$$
t(x)=\sum_{k=1}^{d} t_{k}(x)\left(1-e^{2 \pi i\left(x, \mathbf{e}_{k}\right)}\right)
$$

where $t_{k}, k=1, \ldots, d$, are trigonometric polynomials.
Proof. There exists $N \in \mathbb{Z}^{d}$ such that $t(x) e^{2 \pi i(N, x)}=p(z)$, where $z=$ $\exp (2 \pi i x)$ and $p$ is an algebraic polynomial. It is clear that $P(1, \ldots, 1)=0$. By Taylor formula it follows that

$$
p(z)=\sum_{k=1}^{d} p_{k}(z)\left(1-z_{k}\right) .
$$

where $p_{k}, k=1, \ldots, d$, are algebraic polynomials. It remains to set $t_{k}(x):=$ $p_{k}(z) e^{-2 \pi i(N, x)} . \diamond$

Lemma 3 Let $t, \tilde{t} \in \mathcal{Z}^{0}, t(\mathbf{0})=\tilde{t}(\mathbf{0})$, then

$$
\begin{equation*}
t(x)-\tilde{t}(x)=\sum_{k=1}^{d} t_{k}(x)\left(1-e^{2 \pi i\left(M^{*} x, \mathbf{e}_{k}\right)}\right) \tag{4}
\end{equation*}
$$

where $t_{k}, k=1, \ldots, d$, are trigonometric polynomials.

Proof. Using (2) for $t, \tilde{t}$, we have

$$
\begin{equation*}
t(x)-\tilde{t}(x)=\sum_{\nu=0}^{m-1} e^{2 \pi i\left(s_{\nu}, x\right)}\left(\tau_{\nu}\left(M^{*} x\right)-\tilde{\tau}_{\nu}\left(M^{*} x\right)\right) \tag{5}
\end{equation*}
$$

Due to Proposition 1, $\tau_{\nu}(\mathbf{0})-\tilde{\tau}_{\nu}(\mathbf{0})=0, \nu=1, \ldots, m-1$. Hence, by Lemma 2,

$$
\begin{equation*}
\tau_{\nu}(y)-\tilde{\tau}_{\nu}(y)=\sum_{l=1}^{d} \tau_{\nu k}(y)\left(1-e^{2 \pi i\left(y, \mathbf{e}_{k}\right)}\right) \tag{6}
\end{equation*}
$$

where $\tau_{\nu k}, k=1, \ldots, d$, are trigonometric polynomials. It remains to set $y=M^{*} x$ and combine (6) with (5). $\diamond$

Corollary 4 Let $t \in \mathcal{Z}^{0}, r \in \mathbb{Z}^{d}, \tilde{t}:=e^{2 \pi i(r, \cdot)} t$, then (4) holds.
Lemma 5 Let $t \in \mathcal{Z}^{-1}, r \in \mathbb{Z}^{d}, \tilde{t}:=e^{2 \pi i(r,)} t$, then the $k$-th polyphase function of $\tilde{t}$ is given by $\tilde{\tau}_{k}(x):=e^{2 \pi i\left(n_{\nu}, x\right)} \tau_{\nu}(x)$, where $\nu$ is the number of the unique digit $s_{\nu} \in D(M)$ such that $s_{\nu}+r=s_{k}+M n_{\nu}, n_{\nu} \in \mathbb{Z}^{d}$.

Proof. By (2),

$$
\tilde{t}(x)=e^{2 \pi i(r, x)} \sum_{\nu=0}^{m-1} e^{2 \pi i\left(s_{\nu}, x\right)} \tau_{\nu}\left(M^{*} x\right)=\sum_{\nu=0}^{m-1} e^{2 \pi i\left(s_{\nu}+r, x\right)} \tau_{\nu}\left(M^{*} x\right) .
$$

It is clear that $s_{\nu}+r=s_{l_{\nu}}+M n_{\nu}, n_{\nu} \in \mathbb{Z}^{d},\left\{l_{0}, \ldots, l_{m-1}\right\}=\{0, \ldots, m-1\}$. So,

$$
\tilde{t}(x)=\sum_{\nu=0}^{m-1} e^{2 \pi i\left(s_{l_{\nu}}, x\right)} e^{2 \pi i\left(n_{\nu}, M^{*} x\right)} \tau_{\nu}\left(M^{*} x\right),
$$

and the trigonometric polynomial $\tilde{\tau}_{l_{\nu}}(x):=e^{2 \pi i\left(n_{\nu}, x\right)} \tau_{\nu}(x)$ is the $l_{\nu}$-th polyphase function of $\tilde{t}$. $\diamond$

Lemma 6 Let $n \in \mathbb{Z}^{d}, 1-e^{2 \pi i\left(M^{-1} n, \mathbf{e}_{j}\right)}=0$ for each $j=1, \ldots, d$. Then $n \equiv \mathbf{0} \quad(\bmod M)$.

Proof. Since $e^{2 \pi i n}=1$ if and only if $n \in \mathbb{Z}$, we have $\left(M^{-1} n\right)_{j}=$ $\left(M^{-1} n, \mathbf{e}_{j}\right) \in \mathbb{Z}$ for each $j=1, \ldots, d$. Hence, $M^{-1} n=l \in \mathbb{Z}^{d}$, i.e. $n=M l . \diamond$

## 4. Decomposition of masks

In this section we prove the existence of a decomposition of a trigonometric polynomial $t \in \mathcal{Z}^{n}$ of the form

$$
\left(1-e^{2 \pi i\left(x, \mathbf{e}_{k}\right)}\right) t(x)=\sum_{j=1}^{d} t_{j k}(x)\left(1-e^{2 \pi i\left(M^{*} x, \mathbf{e}_{j}\right)}\right)
$$

with $t_{j k} \in \mathcal{Z}^{n-1}, j=1, \ldots, d$, for all $k=1, \ldots, d$, and present algorithms for its computation. We start with the simpler case $n=0,1$, and then do the general case $n>1$, which is much more complicated.
4.1 The case $t \in \mathcal{Z}^{n}, n=0,1$

Proposition 7 Let $t \in \mathcal{Z}^{0}$, then

$$
\begin{equation*}
\left(1-e^{2 \pi i\left(x, \mathbf{e}_{k}\right)}\right) t(x)=\sum_{j=1}^{d} t_{j k}(x)\left(1-e^{2 \pi i\left(M^{*} x, \mathbf{e}_{j}\right)}\right) \tag{7}
\end{equation*}
$$

where $t_{j k}, j=1, \ldots, d$, are trigonometric polynomials. Conversely, if (7) holds for a trigonometric polynomialt, then $t \in \mathcal{Z}^{0}$ and $t_{j k}(\mathbf{0})=\left(M^{-1}\right)_{j k} t(\mathbf{0})$.

Proof. If $t \in \mathcal{Z}^{0}$, then (7) follows immediately form Corollary 4. Now let $t$ be an arbitrary trigonometric polynomial such that (7) holds, $s \in D\left(M^{*}\right)$, $s \neq \mathbf{0}$. It follows from (7) that

$$
t\left(M^{*-1} s\right)\left(1-e^{\left.2 \pi i\left(M^{*-1} s, \mathbf{e}_{k}\right)\right)}\right)=\sum_{j=1}^{d} t_{j k}\left(M^{*-1} s\right)\left(1-e^{2 \pi i\left(s, \mathbf{e}_{j}\right)}\right)=0
$$

Taking into account Lemma 6, we obtain $t\left(M^{*-1} s\right)=0$, which means that $t \in \mathcal{Z}^{0}$.

Rewrite identity (7) as follows

$$
\left(1-e^{2 \pi i\left(x, M^{-1} \mathbf{e}_{k}\right)}\right) t\left(M^{*-1} x\right)=\sum_{j=1}^{d} t_{j k}\left(M^{*-1} x\right)\left(1-e^{2 \pi i\left(x, \mathbf{e}_{j}\right)}\right)
$$

Differentiating by $x_{l}$, we have

$$
\begin{array}{r}
\left.-2 \pi i\left(M^{-1} \mathbf{e}_{k}, \mathbf{e}_{l}\right) e^{2 \pi i\left(x, M^{-1} \mathbf{e}_{k}\right)}\right) t\left(M^{*-1} x\right)+\left(1-e^{2 \pi i\left(x, M^{-1} \mathbf{e}_{k}\right)}\right) \frac{\partial}{\partial x_{l}} t\left(M^{*-1} x\right)= \\
\sum_{j=1}^{d} \frac{\partial}{\partial x_{l}} t_{j k}\left(M^{*-1} x\right)\left(1-e^{2 \pi i\left(x, \mathbf{e}_{j}\right)}\right)-2 \pi i t_{l k}\left(M^{*-1} x\right) e^{2 \pi i\left(x, \mathbf{e}_{l}\right)}
\end{array}
$$

Substituting $x=\mathbf{0}$, we obtain

$$
-2 \pi i\left(M^{-1} \mathbf{e}_{k}, \mathbf{e}_{l}\right) t(\mathbf{0})=-2 \pi i t_{l k}(\mathbf{0}) .
$$

It remains to note that $\left(M^{-1} \mathbf{e}_{k}, \mathbf{e}_{l}\right)=\left(M^{-1}\right)_{l k} . \diamond$
Analyzing the proofs of Lemmas 3 and 5 it is not difficult to describe an algorithm for finding the polynomials $t_{j k}, j, k=1, \ldots, d$, in the decomposition (7). We assume that the polyphase functions $\tau_{\nu}$ of $t$ are given, and the algorithm derives the polyphase functions $\tau_{j k \nu}$ of each polynomial $t_{j k}$.

## ALGORITHM 1

Input: $\quad\left\{\widehat{\tau}_{\nu}(l), l \in \mathbb{Z}^{d}, \nu=0, \ldots, m-1\right\}$.
Output: $\quad\left\{\widehat{\tau}_{j k \nu}(l), l \in \mathbb{Z}^{d}, \nu=0, \ldots, m-1, j, k=1, \ldots, d\right\}$.
Step 0. For each $\nu=0, \ldots, m-1$ find the sets

$$
\Omega_{\nu}:=\left\{l \in \mathbb{Z}^{d}: \widehat{\tau}_{\nu}(l) \neq 0\right\}
$$

For each $k=1, \ldots, d$ do
For each $\nu=0, \ldots, m-1$ do
Step 1. Compute $M^{-1}\left(\mathbf{e}_{k}-s_{\nu}+s_{n}\right)=: l_{n}$ for all $n=0, \ldots, m-1$ and denote by $n^{*}$ the unique $n$ such that $l_{n} \in \mathbb{Z}^{d}$.
Step 2. Find the set $\tilde{\Omega}_{k \nu}:=\left\{l=r+l_{n^{*}}, r \in \Omega_{n^{*}}\right\}$, and for each $l \in \tilde{\Omega}_{k \nu}$ put $\widehat{\tilde{\tau}}_{\nu}(l):=\widehat{\tau}_{n^{*}}\left(l-l_{n^{*}}\right)$,

$$
p_{k \nu}(z):=\sum_{l \in \Omega_{\nu} \cup \tilde{\Omega}_{k \nu}}\left(\widehat{\tau}_{\nu}(l)-\widehat{\tilde{\tau}}_{\nu}(l)\right) z^{l}
$$

Step 3. Put

$$
\begin{aligned}
& \tau_{1 k \nu}(x)=\frac{1}{1-z_{1}}\left(p_{k \nu}(z)-p_{k \nu}\left(1, z_{2}, \ldots, z_{d}\right)\right), \\
& \tau_{2 k \nu}(x)=\frac{1}{1-z_{2}}\left(p_{k \nu}\left(1, z_{2}, z_{3}, \ldots, z_{d}\right)-p_{k \nu}\left(1,1, z_{3}, \ldots, z_{d}\right)\right), \\
& \tau_{d k \nu}(x)=\frac{1}{1-z_{d}}\left(p_{k \nu}\left(1,1, \ldots, 1, z_{d}\right)-p_{k \nu}(1,1, \ldots, 1,1)\right) . \diamond
\end{aligned}
$$

Theorem 8 If $t \in \mathcal{Z}^{1}$, then in any decomposition (7) the trigonometric polynomials $t_{j k}, j, k=1, \ldots, d$, belong to $\mathcal{Z}^{0}$.

Proof. Let $l=1, \ldots, d, s \in D\left(M^{*}\right), s \neq \mathbf{0}$. Since $t \in \mathcal{Z}^{(1)}$,

$$
\begin{equation*}
\left.\frac{\partial}{\partial x_{l}}\left(\left(1-e^{\left.2 \pi i\left(M^{*-1} x, \mathbf{e}_{k}\right)\right)}\right) t\left(M^{*-1} x\right)\right)\right|_{x=s}=0, \quad k=1, \ldots, d \tag{8}
\end{equation*}
$$

On the other hand, it follows from (7) that for each $k=1, \ldots, d$ we have

$$
\begin{aligned}
&\left.\frac{\partial}{\partial x_{l}}\left(\left(1-e^{\left.2 \pi i\left(M^{*-1} x, \mathbf{e}_{k}\right)\right)}\right) t\left(M^{*-1} x\right)\right)\right|_{x=s}= \\
&\left.\frac{\partial}{\partial x_{l}}\left(\sum_{j=1}^{d} t_{j k}\left(M^{*-1} x\right)\left(1-e^{2 \pi i x_{j}}\right)\right)\right|_{x=s}= \\
&\left.\sum_{j=1}^{d} t_{j k}\left(M^{*-1} s\right) \frac{\partial}{\partial x_{l}}\left(1-e^{2 \pi i x_{j}}\right)\right|_{x=s}=-2 \pi i t_{l k}\left(M^{*-1} s\right) .
\end{aligned}
$$

So, we proved that $t_{l k}\left(M^{*-1} s\right)=0$ for all $s \in D\left(M^{*}\right), s \neq \mathbf{0}$, which means $t_{l k} \in \mathcal{Z}^{(0)}$. $\diamond$

### 4.2 The general case $t \in \mathcal{Z}^{n}, n>1$

Proposition 7 and Theorem 8 state that for $n=0,1$ the condition $t \in \mathcal{Z}^{n}$ implies $t_{j k} \in \mathcal{Z}^{n-1}$. This fact can not be extended to the case $n>1$. If $t \in \mathcal{Z}^{2}$, there exist decompositions (7) whose elements are not in $\mathcal{Z}^{1}$. Nevertheless, it will be shown that any decomposition of $t \in \mathcal{Z}^{n}$ can be fixed to provide $t_{j k} \in \mathcal{Z}^{n-1}$.

Theorem 9 Let $n \in \mathbb{Z}_{+}$. A trigonometric polynomial $t$ belongs to $\mathcal{Z}^{n}$ if and only if there exists a decomposition (7) with $t_{j k} \in \mathcal{Z}^{n-1}, j, k=1, \ldots, d$.

Proof. Set

$$
\begin{aligned}
a(x)=t\left(M^{*-1} x\right), & a_{j k}(x)=t_{j k}\left(M^{*-1} x\right), \\
b_{k}(x)=1-e^{2 \pi i\left(M^{*-1} x, \mathbf{e}_{k}\right)}, & c_{k}(x)=1-e^{2 \pi i\left(x, \mathbf{e}_{k}\right)}
\end{aligned}
$$

In these notation, (7) can be rewritten as

$$
\begin{equation*}
b_{k}(x) a(x)=\sum_{j=1}^{d} a_{j k}(x) c_{j}(x) \tag{9}
\end{equation*}
$$

Note the following trivial properties of $c_{j}$ :

$$
\begin{align*}
& c_{j}(l)=0 \forall l \in \mathbb{Z}^{d},  \tag{10}\\
& D^{\delta} c_{j} \equiv 0 \forall \delta \neq r \mathbf{e}_{j}, r \in \mathbb{N}, \delta \neq \mathbf{0},  \tag{11}\\
& \frac{\partial c_{j}}{\partial x_{j}}(l) \neq 0 \forall l \in \mathbb{Z}^{d} \tag{12}
\end{align*}
$$

We will prove the theorem by induction on $n$. We have a base for $n=0$ due to Proposition 7. Let us prove the inductive step: $n \rightarrow n+1$.

Assume that each polynomial $t_{j k}$ belongs to $\mathcal{Z}^{n}$, i.e. $D^{\beta} a_{j k}\left(s_{l}^{*}\right)=0$, $l=1, \ldots, m-1$, for all $\beta \in \mathbb{Z}_{+}^{d},[\beta] \leq n$. Let $\alpha \in \mathbb{Z}^{d},[\alpha]=n+1$, $s \in D\left(M^{*}\right), s \neq \mathbf{0}$. It follows from (9) and Leibniz formula that

$$
\sum_{0 \leq \beta \leq \alpha}\binom{\alpha}{\beta} D^{\alpha-\beta} b_{k}(s) D^{\beta} a(s)=D^{\alpha} a_{j k}(s) c_{j}(s)
$$

By the inductive hypotheses, $D^{\beta} a(s)=0$ whenever $\beta<\alpha$. Hence, taking into account (10), we have

$$
b_{k}(s) D^{\alpha} a(s)=c_{j}(s) D^{\alpha} a_{j k}(s)=0
$$

Due to Lemma $6, b_{k}(s) \neq 0$ for at least one $k=1, \ldots, d$. It follows that $D^{\alpha} a(s)=0$.

Now we assume that $t \in \mathcal{Z}^{n+1}$, i.e. $D^{\alpha} a\left(s_{\nu}^{*}\right)=0, \nu=1, \ldots, m-1$, for all $\alpha \in \mathbb{Z}_{+}^{d},[\alpha] \leq n+1$. Due to the inductive hypotheses, there exist polynomials $t_{j k} \in \mathcal{Z}^{n-1}$ satisfying (7), i.e. $D^{\delta} a_{j k}\left(s_{\nu}^{*}\right)=0, \nu=1, \ldots, m-1$, for all $\delta \in \mathbb{Z}_{+}^{d},[\delta] \leq n-1$. We will construct new trigonometric polynomials satisfying (7) and belonging to $\mathcal{Z}^{n}$. Note that for $n=0$ any trigonometric polynomials $t_{j k}$ satisfying (7) belong to $\mathcal{Z}^{0}$, due to Theorem 8.

Let us prove the following statement. If $D^{\delta} a_{j k}\left(s_{\nu}^{*}\right)=0, \nu=1, \ldots, m-1$, for all $\delta \in \mathbb{Z}_{+}^{d},[\delta] \leq n$ and for all $j=1, \ldots, l-1(l=1, \ldots, d)$, then there exist polynomials $\tilde{t}_{j k}, j, k=1, \ldots, d$, such that $\sum_{j=1}^{d} \tilde{a}_{j k} c_{j}=\sum_{j=1}^{d} a_{j k} c_{j}$, where $\tilde{a}_{j k}(x)=\tilde{t}_{j k}\left(M^{*-1} x\right)$, and $D^{\delta} \tilde{a}_{j k}\left(s_{\nu}^{*}\right)=0, \nu=1, \ldots, m-1, k=1, \ldots, d$, for all $\delta \in \mathbb{Z}_{+}^{d},[\delta] \leq n$, and for all $j=1, \ldots, l$.

Let $s \in D\left(M^{*}\right), s \neq \mathbf{0}, \beta \in \mathbb{Z}_{+}^{d},[\beta]=n, \sum_{i=l+1}^{d} \beta_{i}=0$. Set $\alpha=\beta+\mathbf{e}_{l}$ and note that $[\alpha]=n+1$. It follows from (9) that

$$
\begin{equation*}
\left.D^{\alpha}\left(\sum_{j=1}^{d} a_{j k}(x) c_{j}(x)\right)\right|_{x=s}=\left.D^{\alpha}\left(b_{k}(x) a(x)\right)\right|_{x=s}=0 \tag{13}
\end{equation*}
$$

On the other hand, due to (10), (11) and the assumption of the statement, we have

$$
\left.D^{\alpha}\left(\sum_{j=1}^{d} a_{j k}(x) c_{j}(x)\right)\right|_{x=s}=\sum_{j=1}^{l} D^{\alpha-\mathbf{e}_{j}} a_{j k}(s) \frac{\partial c_{j}}{\partial x_{j}}(s)=D^{\beta} a_{l k}(s) \frac{\partial c_{l}}{\partial x_{l}}(s)
$$

Combining this with (12) and (13), we get $D^{\beta} a_{l k}(s)=0$. is proved already for $l=d$ (in this case $\tilde{t}_{j k}=t_{j k}, j, k=1, \ldots, d$ ).

Next let $j=l+1, \ldots, d, \beta \in \mathbb{Z}_{+}^{d},[\beta]=n, \sum_{i=l+1}^{d} \beta_{i}>0$. Define the functions

$$
q_{\beta j}(x)=\frac{1}{-2 \pi i} g_{n-1, \beta-\mathbf{e}_{j}}(x) \sum_{\nu=1}^{m-1} h_{\nu}(x) D^{\beta} a_{l k}\left(s_{\nu}^{*}\right),
$$

where $g_{N \delta}$ is a trigonometric polynomial such that $D^{\gamma} g_{N \delta}(\mathbf{0})=0$ for all $\gamma \in \mathbb{Z}_{+}^{d},[\gamma] \leq N, \gamma \neq \delta$ and $D^{\delta} g_{N \delta}(\mathbf{0})=1 ;$

$$
h_{\nu}(x)=\frac{1}{m} \sum_{\mu=0}^{m-1} e^{2 \pi i\left(x-s_{\nu}^{*}, M^{-1} s_{\mu}\right)} .
$$

Since, by Proposition A, $h_{\nu}\left(s_{\mu}^{*}\right)=\delta_{\mu \nu}$, due to (10), (11) and Leibniz formula, it is not difficult to see that for each $\nu=1, \ldots, m-1$ we have

$$
\begin{align*}
& \left.D^{\beta}\left(c_{j}(x) q_{\beta j}(x)\right)\right|_{x=s_{\nu}^{*}}=D^{\beta} a_{l k}\left(s_{\nu}^{*}\right)  \tag{14}\\
& \left.D^{\delta}\left(c_{j}(x) q_{\beta j}(x)\right)\right|_{x=s_{\nu}^{*}}=0 \quad \forall \delta \in \mathbb{Z}_{+}^{d},[\delta] \leq n, \delta \neq \beta  \tag{15}\\
& \left.D^{\delta}\left(c_{i}(x) q_{\beta j}(x)\right)\right|_{x=s_{\nu}^{*}}=0 \quad \forall \delta \in \mathbb{Z}_{+}^{d},[\delta] \leq n-1, \forall i=1, \ldots, d . \tag{16}
\end{align*}
$$

Set

$$
\tilde{a}_{l k}(x):=a_{l k}(x)-\sum_{j=l+1}^{d} \sum_{\substack{[\beta]=n, \beta_{j}>0 \\ \beta_{l+1}=\ldots=\beta_{j-1}=0}} c_{j}(x) q_{\beta j}(x),
$$

$$
\tilde{a}_{j k}(x):=a_{j k}(x)+\sum_{\substack{[\beta]=n, \beta_{j}>0 \\ \beta_{l+1}=\ldots=\beta_{j-1}=0}} c_{l}(x) q_{\beta j}(x), \quad j=l+1, \ldots, d
$$

Because of construction, $\sum_{j=l}^{d} \tilde{a}_{j k} c_{j}=\sum_{j=l}^{d} a_{j k} c_{j}$, and, taking into account (14), (15), for each $\nu=1, \ldots, m-1$ we have $D^{\beta} \tilde{a}_{l k}\left(s_{\nu}^{*}\right)=0$ whenever $[\beta]=n$, $\sum_{i=l+1}^{d} \beta_{i}>0 ; D^{\beta} \tilde{a}_{l k}\left(s_{\nu}^{*}\right)=D^{\beta} a_{l k}\left(s_{\nu}^{*}\right)=0$ whenever $[\beta]=n, \sum_{i=l+1}^{d} \beta_{i}=0$. At last, due to $(16), D^{\delta} \tilde{a}_{j k}\left(s_{\nu}^{*}\right)=0, j=l, \ldots, d$ for all $\delta \in \mathbb{Z}_{+}^{d},[\delta] \leq n-1$. To complete the proof of the statement it remains to put $\tilde{t}_{j k}=\tilde{a}_{j k}\left(M^{*} x\right)$ for $j=l, \ldots, d$ and $\tilde{t}_{j k}=t_{j k}$ for $j=1, \ldots, l-1$.

So, we described one step for improvement of decomposition (7). Starting with $l=1$, after $(d-1)$ steps we will obtain required polynomials. $\diamond$

Analyzing the proof of Theorem 9 it is not difficult to describe an algorithm for finding polynomials $t_{j k} \in \mathcal{Z}^{n-1}, j, k=1, \ldots, d$, in decomposition (7). To realize this algorithm we will need the functions $g_{N \delta}$. Explicit recursive formulas for these functions are presented in [10].

## ALGORITHM 2

Input: $\quad t \in \mathcal{Z}^{N}, N>1$.
Output: $\quad t_{j k} \in \mathcal{Z}^{N-1}, j, k=1, \ldots, d$.
Step 1. Using Algorithm 1, find $t_{j k}, j, k=1, \ldots, d$, satisfying (7).
Step 2. For $n=1, \ldots, N-1$ do
Set $t_{j k}^{(0)}:=t_{j k}, j, k=1, \ldots, d ;$

$$
\text { For each } l=1, \ldots, d-1 \text { do }
$$

For each $k=1, \ldots, d$ do
$a_{l k}(x)=t_{l k}^{(l-1)}\left(M^{*-1} x\right) ;$
$t_{l k}^{(l)}(x):=t_{l k}^{(l-1)}(x)-\sum_{j=l+1}^{d}\left(1-e^{2 \pi i\left(M^{*} x, \mathbf{e}_{j}\right)}\right) \sum_{\substack{[\beta]=n, \beta_{j}>0 \\ \beta_{l+1}=\ldots=\beta_{j-1}=0}} q_{\beta j}\left(M^{*} x\right) ;$
For each $j=1, \ldots, l$ do
$t_{j k}^{(l)}(x):=t_{j k}^{(l-1)}(x) ;$
For each $j=l+1, \ldots, d$ do

$$
t_{j k}^{(l)}(x):=t_{j k}^{(l-1)}(x)+\left(1-e^{2 \pi i\left(M^{*} x, \mathbf{e}_{l}\right)}\right) \sum_{\substack{[\beta]=n, \beta_{j}>0 \\ \beta_{l+1}=\ldots=\beta_{j-1}=0}} q_{\beta j}\left(M^{*} x\right) . \diamond
$$

In the case $d=2$, Step 2 of Algorithm 2 does not look so frightening, it is reduced to the following.

For $n=1, \ldots, N-1$ do
Set $l=1, t_{j k}^{(0)}:=t_{j k}, j, k=1,2$;
For each $k=1,2$ do

$$
\begin{aligned}
a_{1 k}(x) & =t_{1 k}^{(0)}\left(M^{*-1} x\right) ; \\
t_{1 k}(x) & :=t_{1 k}^{(0)}(x)-\left(1-e^{2 \pi i\left(M^{*} x, \mathbf{e}_{2}\right)}\right) \sum_{[\beta]=n, \beta_{2}>0} q_{\beta j}\left(M^{*} x\right) ; \\
t_{2 k}(x) & :=t_{2 k}^{(0)}(x)+\left(1-e^{2 \pi i\left(M^{*} x, \mathbf{e}_{1}\right)}\right) \sum_{[\beta]=n, \beta_{2}>0} q_{\beta j}\left(M^{*} x\right) .
\end{aligned}
$$

In particular, for $d=2, N=2$, Step 2 may be realized as follows.
Set $t_{j k}^{(0)}:=t_{j k}, j, k=1,2$;
For each $k=1,2$ do

$$
\begin{aligned}
t_{1 k}(x) & :=t_{l k}^{(0)}(x)+\left.\frac{1}{2 \pi i}\left(1-e^{2 \pi i\left(M^{*} x\right)_{2}}\right) \sum_{\nu=1}^{m-1} h_{\nu}\left(M^{*} x\right) \frac{\partial}{\partial x_{2}} t_{1 k}^{(0)}\left(M^{*-1} u\right)\right|_{u=s_{\nu}^{*}} \\
t_{2 k}(x) & :=t_{2 k}^{(0)}(x)-\left.\frac{1}{2 \pi i}\left(1-e^{2 \pi i\left(M^{*} x\right)_{1}}\right) \sum_{\nu=1}^{m-1} h_{\nu}\left(M^{*} x\right) \frac{\partial}{\partial x_{2}} t_{1 k}^{(0)}\left(M^{*-1} u\right)\right|_{u=s_{\nu}^{*}}
\end{aligned}
$$

For each $n \in \mathbb{N}$ we introduce the set

$$
\Gamma^{n}:=\left\{k \in \mathbb{R}^{n}: \quad k_{l} \in\{1, \ldots, d\}, l=1, \ldots, n\right\} .
$$

Theorem 10 Let $n, n_{0} \in \mathbb{N}, n \leq n_{0}, t \in \mathcal{Z}^{n_{0}-1}$, then there exist trigonometric polynomials $t_{j k} \in \mathcal{Z}^{n_{0}-n-1}, k, j \in \Gamma^{n}$, such that

$$
\begin{align*}
\prod_{l=1}^{n}\left(1-e^{2 \pi i\left(x, \mathbf{e}_{k_{l}}\right)}\right) t(x) & =\sum_{j_{1}=1}^{d} \ldots \sum_{j_{n}=1}^{d} t_{j k}(x) \prod_{l=1}^{n}\left(1-e^{2 \pi i\left(M^{*} x, \mathbf{e}_{j_{l}}\right)}\right)  \tag{17}\\
t_{j k}(\mathbf{0}) & =\prod_{l=1}^{n}\left(M^{-1}\right)_{j_{l} k_{l}} t(\mathbf{0}) \tag{18}
\end{align*}
$$

Proof. We will proof by induction on $n$.
Base: $n=1$. Let $k \in \Gamma^{1}$. Due to Proposition 7 and Theorem 9, there exist trigonometric polynomials $t_{j k} \in \mathcal{Z}^{n_{0}-2}, j \in \Gamma^{1}$, such that (7) holds. So, we have (17) for $n=1$, (18) also follows from Proposition 7.

Inductive step: $n-1 \rightarrow n$. Let $1<n \leq n_{0}, k \in \Gamma^{n}, k^{\prime}:=\left(k_{1}, \ldots, k_{n-1}\right)$. By the inductive hypotheses there exist $t_{j^{\prime} k^{\prime}} \in \mathcal{Z}^{n_{0}-n}, j^{\prime} \in \Gamma^{n-1}$, such that

$$
\begin{gather*}
\prod_{l=1}^{n-1}\left(1-e^{2 \pi i\left(x, \mathbf{e}_{k_{l}}\right)}\right) t(x)=\sum_{j_{1}=1}^{d} \ldots \sum_{j_{n-1}=1}^{d} t_{j^{\prime} k^{\prime}}(x) \prod_{l=1}^{n-1}\left(1-e^{2 \pi i\left(M^{*} x, \mathbf{e}_{j_{l}}\right)}\right),  \tag{19}\\
t_{j k}(\mathbf{0})=\prod_{l=1}^{n-1}\left(M^{-1}\right)_{j_{l} k_{l}} t(\mathbf{0}) \tag{20}
\end{gather*}
$$

Since $n_{0}-n \geq 0$ and the theorem is proved already for $n=1$, for each $t_{j^{\prime} k^{\prime}}$, there exist trigonometric polynomials $t_{j_{n} k_{n}} \in \mathcal{Z}^{n_{0}-n-1}, j \in \Gamma^{1}$, such that

$$
\begin{align*}
\left(1-e^{2 \pi i\left(x, \mathbf{e}_{k_{n}}\right)}\right) t_{j^{\prime} k^{\prime}}(x) & =\sum_{j_{n}=1}^{d} t_{j_{n} k_{n}}(x)\left(1-e^{2 \pi i\left(M^{*} x, \mathbf{e}_{j_{n}}\right)}\right),  \tag{21}\\
t_{j_{n} k_{n}}(\mathbf{0}) & =\left(M^{-1}\right)_{j_{n} k_{n}} t_{j^{\prime} k^{\prime}}(\mathbf{0}) . \tag{22}
\end{align*}
$$

Combining (19), (20) with (21), (22) we comlete the proof. $\diamond$
To impart a more compact form to (17), (18), we introduce the following notations. Set

$$
\begin{aligned}
& \Delta_{k}(x)=\Delta_{k}=\left(1-e^{2 \pi i\left(\mathbf{e}_{k}, x\right)}\right), \quad \Delta(x)=\Delta=\left(\Delta_{1}, \ldots, \Delta_{d}\right)^{T}, \\
& \delta_{k}(x)=\delta_{k}=\left(1-e^{2 \pi i\left(\mathbf{e}_{k}, M^{*} x\right)}\right), \quad \delta(x)=\delta=\left(\delta_{1}, \ldots, \delta_{d}\right)^{T} .
\end{aligned}
$$

Now Theorem 10 can be rewritten as
Theorem $10^{\prime}$ Let $n, n_{0} \in \mathbb{N}, n \leq n_{0}, t \in \mathcal{Z}^{n_{0}-1}$, then there exists a $d^{n} \times d^{n}$ matrix $T$ whose entries are trigonometric polynomials $T_{k j} \in \mathcal{Z}^{n_{0}-n-1}$, for $k, j \in \Gamma^{n}$, such that

$$
(\Delta(x))^{[n]} t(x)=T(x)(\delta(x))^{[n]}, \quad T(\mathbf{0})=t(\mathbf{0})\left(M^{*-1}\right)^{[n]} .
$$

## 5. Construction of masks in $\mathcal{Z}^{n}$

A simple description of the classes $\mathcal{Z}^{n}$ is well known in the one-dimensional dyadic case. A general form is given by the formula $t(x)=\left(1+e^{2 \pi i x}\right)^{n} T(x)$,
where $T$ is an arbitrary trigonometric polynomial. In the multidimensional case $\mathcal{Z}^{n}$ can not be described in a similar way. We will give a characterization of the class $\mathcal{Z}^{n}$ for arbitrary dilation matrix which allows to construct its elements in practice.

Theorem 11 A trigonometric polynomial $t$ belongs to $\mathcal{Z}^{n}$ if and only if the derivatives of its polyphase function $\tau_{k}, k=0, \ldots, m-1$, up to order $n$ are given by

$$
\begin{equation*}
D^{\alpha} \tau_{k}(\mathbf{0})=\frac{1}{m} \sum_{\mathbf{0} \leq \beta \leq \alpha} \lambda_{\beta}\binom{\alpha}{\beta}\left(-2 \pi i r_{k}\right)^{\alpha-\beta}, \quad \alpha \in \mathbb{Z}_{+}^{d},[\alpha] \leq n \tag{23}
\end{equation*}
$$

where $\lambda_{\alpha}=\left.D^{\alpha} t\left(M^{*-1} x\right)\right|_{x=0}$.
Remark. For polynomials $t$ whose polyphase functions $\tau_{0}, \ldots, \tau_{m-1}$ form a unimodular row (i.e. there exists a dual row of trigonometric polynomials $\tilde{\tau}_{0}, \ldots, \tilde{\tau}_{m-1}$ such that $\sum_{k=0}^{m-1} \tau_{k} \tilde{\tau}_{k} \equiv 1$ ), the statement of Theorem 11 follows from combining the results of [2] and [10]. It was proved in these papers that both conditions are equivalent to vanishing moments of the corresponding wavelet system.

Proof. Assume that (23) holds with some complex numbers $\lambda_{\alpha}$. Let $s \in D\left(M^{*}\right)$. By Leibniz formula,

$$
\begin{aligned}
&\left.D^{\alpha}\left(e^{2 \pi i\left(r_{k}, x\right)} \tau_{k}(x)\right)\right|_{x=s}=\left.\sum_{0 \leq \beta \leq \alpha}\binom{\alpha}{\beta} D^{\beta}\left(e^{2 \pi i\left(r_{k}, x\right)}\right)\right|_{x=s} D^{\alpha-\beta} \tau_{k}(\mathbf{0})= \\
& \sum_{\mathbf{0} \leq \beta \leq \alpha}\binom{\alpha}{\beta} e^{2 \pi i\left(r_{k}, s\right)}\left(2 \pi i r_{k}\right)^{\beta} D^{\alpha-\beta} \tau_{k}(\mathbf{0})= \\
& \frac{1}{m} e^{2 \pi i\left(r_{k}, s\right)} \sum_{\mathbf{0} \leq \beta \leq \alpha}\binom{\alpha}{\beta}\left(2 \pi i r_{k}\right)^{\beta} \sum_{0 \leq \gamma \leq \alpha-\beta} \lambda_{\gamma}\binom{\alpha-\beta}{\gamma}\left(-2 \pi i r_{k}\right)^{\alpha-\beta-\gamma}= \\
& \frac{1}{m} e^{2 \pi i\left(r_{k}, s\right)} \sum_{\mathbf{0} \leq \beta \leq \alpha} \sum_{\mathbf{0} \leq \gamma \leq \alpha-\beta} \lambda_{\gamma}\binom{\alpha-\beta}{\gamma}\binom{\alpha}{\beta}\left(-2 \pi i r_{k}\right)^{\alpha-\gamma} \prod_{j=1}^{d}(-1)^{-\beta_{j}}= \\
& \frac{1}{m} e^{2 \pi i\left(r_{k}, s\right)} \sum_{\mathbf{0} \leq \gamma \leq \alpha} \lambda_{\gamma}\left(-2 \pi i r_{k}\right)^{\alpha-\gamma}\binom{\alpha}{\gamma} \sum_{\mathbf{0} \leq \beta \leq \alpha-\gamma}\binom{\alpha-\gamma}{\beta} \prod_{j=1}^{d}(-1)^{-\beta_{j}} .
\end{aligned}
$$

Since

$$
\begin{array}{r}
\sum_{\mathbf{0} \leq \beta \leq \alpha-\gamma}\binom{\alpha-\gamma}{\beta} \prod_{j=1}^{d}(-1)^{-\beta_{j}}=\prod_{j=1}^{d} \sum_{0 \leq \beta_{j} \leq \alpha_{j}-\gamma_{j}}\binom{\alpha_{j}-\gamma_{j}}{\beta_{j}}(-1)^{-\beta_{j}}= \\
\prod_{j=1}^{d}(1-1)^{\alpha_{j}-\gamma_{j}}= \begin{cases}0, & \alpha \neq \gamma \\
1, & \alpha=\gamma\end{cases}
\end{array}
$$

we have

$$
\left.D^{\alpha}\left(e^{2 \pi i\left(r_{k}, x\right)} \tau_{k}(x)\right)\right|_{x=s}=\frac{\lambda_{\alpha}}{m} e^{2 \pi i\left(r_{k}, s\right)}, \quad k=0, \ldots, m-1
$$

It follows from (2) and Proposition A that

$$
\begin{array}{r}
D^{\alpha}\left(\left.t\left(M^{*-1} x\right)\right|_{x=s}=\left.\sum_{k=0}^{m-1} D^{\alpha}\left(e^{2 \pi i\left(r_{k}, x\right)} \tau_{k}(x)\right)\right|_{x=s}=\right. \\
\frac{\lambda_{\alpha}}{m} \sum_{k=0}^{m-1} e^{2 \pi i\left(r_{k}, s\right)}= \begin{cases}\lambda_{\alpha}, & \text { if } s=\mathbf{0} \\
0, & \text { if } s \neq \mathbf{0}\end{cases}
\end{array}
$$

Now let us check that (23) follows from the relation $t \in \mathcal{Z}^{n}$. We will prove by induction on $n$. The base for $n=0$ was established in Proposition 1. To prove the inductive step $n \rightarrow n+1$, we assume that $t \in \mathcal{Z}^{n+1}$ and (23) holds. Let $\alpha \in \mathbb{Z}^{d},[\alpha]=n+1, s \in D\left(M^{*}\right)$. By (2) and Leibniz formula,

$$
\begin{align*}
& D^{\alpha}\left(\left.t\left(M^{*-1} x\right)\right|_{x=s}=\left.\sum_{k=0}^{m-1} \sum_{\mathbf{0} \leq \beta \leq \alpha}\binom{\alpha}{\beta} D^{\alpha-\beta}\left(e^{2 \pi i\left(r_{k}, x\right)}\right)\right|_{x=s} D^{\beta} \tau_{k}(\mathbf{0})=\right. \\
& \sum_{k=0}^{m-1}\left(D^{\alpha} \tau_{k}(0)+\sum_{\mathbf{0} \leq \beta<\alpha}\binom{\alpha}{\beta} D^{\alpha-\beta}\left(2 \pi i r_{k}\right)^{\alpha-\beta} D^{\beta} \tau_{k}(\mathbf{0})\right) e^{2 \pi i\left(r_{k}, s\right)} \tag{24}
\end{align*}
$$

Because of the inductive hypotheses, we have

$$
\begin{aligned}
\sum_{\mathbf{0} \leq \beta<\alpha}\binom{\alpha}{\beta}\left(2 \pi i r_{k}\right)^{\alpha-\beta} D^{\beta} \tau_{k}(\mathbf{0}) & = \\
\frac{1}{m} \sum_{\mathbf{0} \leq \beta<\alpha} \sum_{\mathbf{0} \leq \gamma \leq \beta} \lambda_{\gamma}\binom{\alpha}{\beta}\binom{\beta}{\gamma}\left(2 \pi i r_{k}\right)^{\alpha-\beta}\left(-2 \pi i r_{k}\right)^{\beta-\gamma} & =
\end{aligned}
$$

$$
\begin{array}{r}
\frac{1}{m} \sum_{\mathbf{0} \leq \gamma<\alpha} \lambda_{\gamma}\left(2 \pi i r_{k}\right)^{\alpha-\gamma} \sum_{\gamma \leq \beta<\alpha}\binom{\alpha}{\beta}\binom{\beta}{\gamma} \prod_{j=1}^{d}(-1)^{\beta_{j}-\gamma_{j}}= \\
\frac{1}{m} \sum_{\mathbf{0} \leq \gamma<\alpha}\binom{\alpha}{\gamma} \lambda_{\gamma}\left(2 \pi i r_{k}\right)^{\alpha-\gamma} \sum_{\mathbf{0} \leq \delta<\alpha-\gamma}\binom{\alpha-\gamma}{\delta} \prod_{j=1}^{d}(-1)^{\delta_{j}}= \\
\frac{1}{m} \sum_{\mathbf{0} \leq \gamma<\alpha}\binom{\alpha}{\gamma} \lambda_{\gamma}\left(2 \pi i r_{k}\right)^{\alpha-\gamma}\left(\prod_{j=1}^{d}(1-1)^{\alpha_{j}-\gamma_{j}}-\prod_{j=1}^{d}(-1)^{\alpha_{j}}\right)= \\
-\frac{1}{m} \sum_{\mathbf{0} \leq \gamma<\alpha}\binom{\alpha}{\gamma} \lambda_{\gamma}\left(-2 \pi i r_{k}\right)^{\alpha-\gamma} .
\end{array}
$$

Combining this with (24), we obtain

$$
\sum_{k=0}^{m-1} e^{2 \pi i\left(r_{k}, s\right)}\left(D^{\alpha} \tau_{k}(0)-\frac{1}{m} \sum_{0 \leq \gamma<\alpha}\binom{\alpha}{\gamma} \lambda_{\gamma}\left(-2 \pi i r_{k}\right)^{\alpha-\gamma}\right)=\left.D^{\alpha}\left(t\left(M^{*-1} x\right)\right)\right|_{x=s}
$$

Set $\lambda_{\alpha}=\left.D^{\alpha}\left(t\left(M^{*-1} x\right)\right)\right|_{x=0}$. Due to Proposition A, the linear system

$$
\sum_{k=0}^{m-1} e^{2 \pi i\left(r_{k}, s_{l}^{*}\right)} y_{k}=\lambda_{\alpha} \delta_{0 l}, \quad l=0, \ldots, m-1
$$

has a unique solution $y_{k}=\frac{\lambda_{\alpha}}{m}, k=0, \ldots, m-1$. It follows that

$$
D^{\alpha} \tau_{k}(0)-\frac{1}{m} \sum_{\mathbf{0} \leq \gamma<\alpha}\binom{\alpha}{\gamma} \lambda_{\gamma}\left(-2 \pi i r_{k}\right)^{\alpha-\gamma}=\frac{\lambda_{\alpha}}{m},
$$

which was to be proved. $\diamond$
So, if we want a polynomial $t$ to belong to $\mathcal{Z}^{n}$, its polyphase functions should have derivatives at the origin given by (23). This can be easily realized for an arbitrary set of parameters $\lambda_{\beta},[\beta] \leq n$. General forms for all such polynomials $t$ are presented in [10].

## 6. Subdivision schemes

In this section we apply the decompositions of Section 4 to the analysis of convergence and smoothness of multivariate subdivision schemes associated with general dilation matrices. We first present basic definitions, and prove an important observation about the matrices in a decomposition of $t \in \mathcal{Z}^{1}$.

### 6.1 Preliminaries

Let $A_{\alpha}, \alpha \in \mathbb{Z}^{d}$, be $N \times N^{\prime}$ matrices such that $A_{\alpha}$ is non-zero only for a finite number of $\alpha$, then $T(x):=\sum_{\alpha \in \mathbb{Z}^{d}} A_{\alpha} e^{-2 \pi i(\alpha, x)}$ is a $N \times N^{\prime}$ matrix whose entries are trigonometric polynomials $\left(N \times N^{\prime}\right.$ trigonometric matrix). The subdivision operator $S_{T}=S_{T, M}$ associated with $T$ and with a $d \times d$ dilation matrix $M$ is defined on $\ell_{\infty}^{N}=\ell_{\infty}^{N}\left(\mathbb{Z}^{d}\right)$ by

$$
\left(S_{T} f\right)_{\alpha}=\sum_{\beta \in \mathbb{Z}^{d}} A_{\alpha-M \beta} f_{\beta}, \quad f \in \ell_{\infty}^{N}
$$

It is clear that $S_{T}$ is a linear bounded operator taking $\ell_{\infty}^{N}$ into $\ell_{\infty}^{N^{\prime}}$. If $N=N^{\prime}$, the operators $S_{T}^{n}, n=1,2, \ldots$, are well defined, and the sequence $\left\{S_{T}^{n}\right\}_{n=1}^{\infty}$ is a $N \times N$ matrix subdivision scheme (scalar subdivision scheme if $N=1$ ).

Let $t$ be a trigonometric polynomial, $T$ be a $d \times d$ trigonometric matrix, set $L(x)=\left(L_{1}(x), \ldots, L_{d}(x)\right)^{T}$, where $L_{k}(x)=\left(1-e^{2 \pi i\left(x, \mathbf{e}_{k}\right)}\right) t(x), R(x)=$ $\left(R_{1}(x), \ldots, R_{d}(x)\right)^{T}$, where $R_{k}(x)=\sum_{j=1}^{d} T_{k j}(x)\left(1-e^{2 \pi i\left(M^{*} x, \mathbf{e}_{j}\right)}\right)$. To each $f \in$ $\ell_{\infty}$ assign the vector-valued sequence $\nabla f \in \ell_{\infty}^{d}$ defined by $(\nabla f)_{\alpha}=\left(f_{\alpha}-\right.$ $\left.f_{\alpha-\mathbf{e}_{1}}, \ldots, f_{\alpha}-f_{\alpha-\mathbf{e}_{d}}\right)^{T}$. It is clear that $S_{L} f=\nabla S_{t}(f), S_{R} f=S_{T}(\nabla f)$ for any $f \in \ell_{\infty}$. Hence equality (7), with $t_{j k}=T_{k j}$, may be rewritten in the form

$$
\begin{equation*}
\nabla S_{t}(f)=S_{T}(\nabla f), \quad \forall f \in \ell_{\infty} \tag{25}
\end{equation*}
$$

Similarly, to each $f \in \ell_{\infty}^{N}$ we assign $\nabla f \in \ell_{\infty}^{N d}$ defined by

$$
\left.(\nabla f)_{\alpha}=\left(\left(\nabla\left(f, \mathbf{e}_{1}\right)\right)_{\alpha}^{T}, \ldots, \nabla\left(f, \mathbf{e}_{N}\right)\right)_{\alpha}^{T}\right)^{T}
$$

If all trigonometric polynomials $t_{j k}$ in the right hand side of (7) are in $\mathcal{Z}^{0}$, we can decompose them (see Theorem 10). This second step of the decomposition may be rewritten as

$$
\begin{equation*}
\nabla S_{T}(f)=S_{Q}(\nabla f), \quad \forall f \in \ell_{\infty}^{d} \tag{26}
\end{equation*}
$$

where $Q$ is a $d^{2} \times d^{2}$ trigonometric matrix.
Proposition 12 Let $t$ be a trigonometric polynomial, $t \in \mathcal{Z}^{1}, t(\mathbf{0})=m$, and let $T$ be a $d \times d$ trigonometric matrix satisfying (25), $T(x):=\sum_{\alpha \in \mathbb{Z}^{d}} A_{\alpha} e^{-2 \pi i(\alpha, x)}$. Then

$$
\sum_{\beta \in \mathbb{Z}^{d}} A_{\alpha-M \beta}=M^{*-1} \quad \forall \alpha \in \mathbb{Z}^{d}
$$

Proof. First of all, we note that $\sum_{\beta \in \mathbb{Z}^{d}} A_{\alpha-M \beta}=\sum_{\beta \in \mathbb{Z}^{d}} A_{\alpha^{\prime}-M \beta}$ whenever $\alpha \equiv \alpha^{\prime} \quad(\bmod M)$. So, it suffices to check that

$$
X_{\nu}:=\sum_{\beta \in \mathbb{Z}^{d}} A_{s_{\nu}-M \beta}=M^{*-1}, \quad \nu=0, \ldots, m-1 .
$$

Substituting $x=M^{*-1} s_{k}^{*}$ into the equality

$$
T(x)=\sum_{\nu=0}^{m-1} e^{-2 \pi i\left(s_{\nu}, x\right)} \sum_{\beta \in \mathbb{Z}^{d}} A_{s_{\nu}-M \beta} e^{-2 \pi i(M \beta, x)},
$$

we have $T\left(M^{*-1} s_{k}^{*}\right)=\sum_{\nu=0}^{m-1} e^{2 \pi i\left(s_{\nu}, M^{*-1} s_{k}^{*}\right)} X_{\nu}$ It follows from Proposition 7 and Theorem 8 that

$$
\sum_{\nu=0}^{m-1} e^{-2 \pi i\left(s_{\nu}, M^{*-1} s_{k}^{*}\right)} X_{\nu}=m \delta_{k 0} M^{*-1}, \quad k=0, \ldots, m-1
$$

This linear system has a unique solution $X_{\nu}=M^{*-1}, \nu=0, \ldots, m-1$, because of Proposition A. $\diamond$

### 6.2 Convergence

A scalar subdivision scheme $S_{t}=S_{t, M}$, associated with a dilation matrix $M$, is called uniformly convergent, if for any $f \in \ell_{\infty}$, there exists a continuous function $S_{t}^{\infty} f$ such that

$$
\begin{equation*}
\lim _{k \rightarrow \infty}\left\|S_{t}^{k} f-S_{t}^{\infty} f\left(M^{-k} \cdot\right)\right\|_{\infty}=0 \tag{27}
\end{equation*}
$$

and if for at least one $f \in \ell_{\infty}$, the limit function $S_{t}^{\infty} f$ is not identically zero.
The issue of the convergence of a multivariate scalar subdivision scheme associated with a dilation matrix $M$ is studied in [4]. The convergence result there is limited to dilation matrices having a self-similar tile. A $d \times d$ dilation matrix $M$ is said to have a self-similar tile if there exist a set of digits $D(M)$ and a bounded set $E \subset \mathbb{R}^{d}$ whose integer translates form a disjoint decomposition of $\mathbb{R}^{d}$ such that

$$
\begin{equation*}
M E=\underset{s \in D(M)}{\cup}(E+s) \tag{28}
\end{equation*}
$$

Theorem 13 [4] Let $S_{t}=S_{t, M}$ be a scalar subdivision scheme and let there exists a self-similar tile related to $M$. Then $S_{t}$ is uniformly convergent if and only if: a) $t(0)=m$; b) there exists a $d \times d$ trigonometric matrix $T$ such that (25) holds, c) $\lim _{k \rightarrow \infty} \sup _{\substack{f \in \in \infty \\\|\nabla f\| \infty=1}}\left\|S_{T}^{k} \nabla f\right\|_{\ell_{\infty}^{d}}=0$.

Here we present sufficient conditions for convergence of a scalar multivariate subdivision scheme associated with any dilation matrix $M$, and then give an example of a subdivision scheme satisfying these conditions, where the verefication of the conditions is done with Algorithm 1.

Theorem $13^{\prime}$ Let $t \in \mathcal{Z}^{0}$ and $M$ be a $d \times d$ dilation matrix. Then there exists a trigonometric matrix $T$ of order $d \times d$ satisfying (25). Moreover, if $t(0)=m$ and

$$
\begin{equation*}
\lim _{k \rightarrow+\infty}\left\|S_{T}^{k}\right\|_{\infty}=0 \tag{29}
\end{equation*}
$$

then the subdivision scheme $S_{t}$ is uniformly convergent.
First we prove a simple lemma.
Lemma 14 Let $n \in \mathbb{Z}_{+}$, $M$ be a $d \times d$ dilation matrix, and let $\varphi$ be a compactly supported function satisfying the refinement equation

$$
\begin{equation*}
\varphi(x)=\sum_{\alpha \in \mathbb{Z}^{d}} a_{\alpha} \varphi(M x-\alpha), \quad \forall x \in \mathbb{R}^{d} \tag{30}
\end{equation*}
$$

Then

$$
\begin{equation*}
\sum_{\beta \in \mathbb{Z}^{d}} f_{\beta} \varphi(x-\beta)=\sum_{\alpha \in \mathbb{Z}^{d}}\left(S_{\tilde{t} I_{n}} f\right)_{\alpha} \varphi(M x-\alpha), \quad \forall x \in \mathbb{R}^{d} \tag{31}
\end{equation*}
$$

for any $f \in \ell_{\infty}^{n}$, where

$$
\tilde{t}(x)=\sum_{\alpha \in \mathbb{Z}^{d}} a_{\alpha} e^{-2 \pi i(x, \alpha)}
$$

Proof. Using (30), we have

$$
\begin{array}{r}
\sum_{\alpha \in \mathbb{Z}^{d}}\left(S_{\tilde{t} I_{n}} f\right)_{\alpha} \varphi(M x-\alpha)=\sum_{\alpha \in \mathbb{Z}^{d}} \sum_{\beta \in \mathbb{Z}^{d}} a_{\alpha-M \beta} f_{\beta} \varphi(M x-\alpha)= \\
\sum_{\beta \in \mathbb{Z}^{d}} f_{\beta} \sum_{\alpha \in \mathbb{Z}^{d}} a_{\alpha} \varphi(M(x-\beta)-\alpha)=\sum_{\beta \in \mathbb{Z}^{d}} f_{\beta} \varphi(x-\beta) . \diamond
\end{array}
$$

Proof of Theorem $13^{\prime}$. The existence of $T$ satisfying (25) follows from Proposition 7, as indicated in Subsection 6.1.

Let $\varphi$ be a continuous compactly supported function satisfying (30) and the interpolatory conditions

$$
\begin{equation*}
\varphi(\alpha)=\delta_{\alpha \mathbf{0}}, \quad \forall \alpha \in \mathbb{Z}^{d} \tag{32}
\end{equation*}
$$

The existence of such a function for an arbitrary dilation matrix $M$ is proved in [3, Proposition 4.1]. It is proved in [3] that the corresponding mask $\tilde{t}$ is in $\mathcal{Z}^{0}$ and that $\tilde{t}(0)=m$. To prove the second part of the claim, we will show that the sequence of functions

$$
F_{k}(x)=\sum_{\alpha \in \mathbb{Z}^{d}} \varphi\left(M^{k} x-\alpha\right)\left(S_{t}^{k} f\right)_{\alpha}, \quad k=1,2, \ldots
$$

is a Cauchy sequence in $C\left(\mathbb{R}^{d}\right)$ for any $f \in \ell_{\infty}$.
By Lemma 14 with $n=1$, we get for $f \in \ell_{\infty}$

$$
F_{k}(x)=\sum_{\alpha \in \mathbb{Z}^{d}} \varphi\left(M^{k} x-\alpha\right)\left(S_{t}^{k} f\right)_{\alpha}=\sum_{\alpha \in \mathbb{Z}^{d}} \varphi\left(M^{k+1} x-\alpha\right)\left(S_{\tilde{t}} S_{t}^{k} f\right)_{\alpha}
$$

and therefore

$$
\begin{equation*}
F_{k+1}(x)-F_{k}(x)=\sum_{\alpha \in \mathbb{Z}^{d}} \varphi\left(M^{k+1} x-\alpha\right)\left(\left(S_{t}-S_{\tilde{t}}\right) S_{t}^{k} f\right)_{\alpha} \tag{33}
\end{equation*}
$$

Since $t$ and $\tilde{t}$ are in $\mathcal{Z}^{0}$ and $t(0)=\tilde{t}(0)$, we get from Lemma 3 that

$$
t(x)-\tilde{t}(x)=\sum_{k=1}^{d} q_{k}(x)\left(1-e^{2 \pi i\left(M^{*} x, \mathbf{e}_{k}\right)}\right) .
$$

This leads, by similar derivations to those leading to (25), to the existence of a vector of subdivision schemes $\left(S_{q_{1}}, \ldots, S_{q_{d}}\right)$, such that for any $f \in \ell_{\infty}$,

$$
\left(S_{t}-S_{\tilde{t}}\right) f=S_{t-\tilde{t}} f=\left(S_{q_{1}}, \ldots, S_{q_{d}}\right) \nabla f
$$

Hence, (33) can be rewritten as

$$
F_{k+1}(x)-F_{k}(x)=\sum_{\alpha \in \mathbb{Z}^{d}}\left(\left(S_{q_{1}}, \ldots, S_{q_{d}}\right) \nabla S_{t}^{k} f\right)_{\alpha} \varphi\left(M^{k+1} x-\alpha\right),
$$

Using relation (25), we obtain

$$
\begin{array}{r}
\left|F_{k+1}(x)-F_{k}(x)\right|=\left|\sum_{\alpha \in \mathbb{Z}^{d}}\left(\left(S_{q_{1}}, \ldots, S_{q_{d}}\right) S_{T}^{k} \nabla f\right)_{\alpha} \varphi\left(M^{k+1} x-\alpha\right)\right| \leq \\
C\left\|S_{T}^{k} \nabla f\right\|_{\infty} \leq C\left\|S_{T}^{k}\right\|_{\infty}|\nabla f|,
\end{array}
$$

where $C$ depends on $\varphi, q_{1}, \ldots, q_{d}$.
Condition (29) is equivalent to the existence of a positive integer $L$ such that $\left\|S_{T}^{L}\right\|_{\infty}=\mu<1$. Thus

$$
\left|F_{k+1}(x)-F_{k}(x)\right| \leq C|\nabla f| \mu^{\frac{k}{L}}
$$

This yields that for all $k, n \in \mathbb{N}$

$$
\begin{equation*}
\left|F_{k+n}(x)-F_{k}(x)\right| \leq C|\nabla f| \sum_{j=0}^{n-1} \mu^{\frac{k+j}{L}} \leq \frac{C}{1-\mu}|\nabla f| \mu^{\frac{k}{L}}, \tag{34}
\end{equation*}
$$

which implies that $\left\{F_{k}\right\}$ is a Cauchy sequence. Denote the limit function by $F$ which is, evidently, a continuous function. Passing to the limit in (34) as $n \rightarrow \infty$, we have

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \sup _{x \in \mathbb{R}^{d}}\left|F(x)-F_{k}(x)\right|=0 \tag{35}
\end{equation*}
$$

Due to (35),

$$
\lim _{k \rightarrow \infty} \sup _{x \in \mathbb{R}^{d}}\left|F(x)-\sum_{\alpha \in \mathbb{Z}^{d}}\left(S_{t}^{k} f\right)_{\alpha} \varphi\left(M^{k} x-\alpha\right)\right|=0
$$

Substituting $x=M^{-k} \beta, \beta \in \mathbb{Z}^{d}$, we have

$$
\lim _{k \rightarrow \infty} \sup _{\beta \in \mathbb{Z}^{d}}\left|F\left(M^{-k} \beta\right)-\sum_{\alpha \in \mathbb{Z}^{d}}\left(S_{t}^{k} f\right)_{\alpha} \varphi(\beta-\alpha)\right|=0
$$

The claim of the theorem follows now from (32). $\diamond$
Example. Let $M=\left(\begin{array}{rr}0 & 2 \\ 2 & -1\end{array}\right)$,

$$
s_{0}=\binom{0}{0}, s_{1}=\binom{1}{0}, s_{2}=\binom{0}{1}, s_{3}=\binom{1}{1} .
$$

Define the polyphase functions of a polynomial $t$ by

$$
\begin{aligned}
& \tau_{0}(x)=\frac{1}{16}\left(4+4 z_{1}+4 z_{2}+4 z_{1} z_{2}\right) \\
& \tau_{1}(x)=\frac{1}{16}\left(5+4 z_{1}+z_{1}^{-1}+2 z_{2}+3 z_{2}^{-1}+z_{1} z_{2}\right) \\
& \tau_{2}(x)=\frac{1}{16}\left(4+z_{1}+2 z_{1}^{-1}+5 z_{2}+z_{2}^{-1}+3 z_{1} z_{2}\right) \\
& \tau_{3}(x)=\frac{1}{16}\left(5+z_{1}+4 z_{1}^{-1}+z_{2}+3 z_{2}^{-1}+z_{1} z_{2}+z_{1}^{-1} z_{2}\right)
\end{aligned}
$$

It is not difficult to see that $t \in \mathcal{Z}^{0}$, and that $t(0)=m$. Using Algorithm 1, we find

$$
\begin{array}{ll}
\tau_{110}(x)=\frac{1}{16}\left(-1+2 z_{2}+z_{1} z_{2}+2 z_{2}^{2}+z_{1} z_{2}^{2}\right), & \tau_{210}(x)=\frac{1}{16}\left(5+3 z_{2}\right), \\
\tau_{111}(x)=\frac{1}{16}\left(z_{1}^{-1}+3 z_{2}\right), & \tau_{211}(x)=\frac{1}{16}\left(5+3 z_{2}^{-1}\right), \\
\tau_{112}(x)=\frac{1}{16}\left(-1+2 z_{1}^{-1}+z_{2}+z_{2}^{2}-z_{1}^{-1} z_{2}\right), & \tau_{212}(x)=\frac{1}{16}\left(5+3 z_{2}+z_{2}^{-1}\right), \\
\tau_{113}(x)=\frac{1}{16}\left(2 z_{1}^{-1}+2 z_{2}+z_{1}^{-1} z_{2}\right), & \tau_{213}(x)=\frac{1}{16}\left(5+2 z_{2}^{-1}\right), \\
\tau_{120}(x)=\frac{1}{16}\left(1+z_{1}+4 z_{2}+z_{2}^{-1}+3 z_{1} z_{2}\right), & \tau_{220}(x)=\frac{1}{16} z_{2}^{-1}, \\
\tau_{121}(x)=\frac{1}{16}\left(2+z_{1}+z_{1}^{-1}+z_{2}+3 z_{2}^{-1}+z_{1} z_{2}\right), & \tau_{221}(x)=0, \\
\tau_{122}(x)=\frac{1}{16}\left(3+z_{2}+2 z_{1}^{-1}\right), & \tau_{222}(x)=\frac{1}{16} z_{2}^{-1}, \\
\tau_{123}(x)=\frac{1}{16}\left(3+3 z_{1}^{-1}+z_{1}^{-1} z_{2}\right), & \tau_{223}(x)=0 .
\end{array}
$$

By Theorem $13^{\prime}$, to prove that the subdivision scheme $S_{t}$ is convergent, it remains to check that the norm of the matrix subdivision operator $S_{T}$ is strictly less than 1 . This requirement is fulfilled because

$$
\left\|S_{T}\right\|_{\infty} \leq \max _{(\nu, k)}\left(\left\|\tau_{1 k \nu}\right\|_{\ell_{1}}+\left\|\tau_{2 k \nu}\right\|_{\ell_{1}}\right) \leq \frac{15}{16} .
$$

(Here we identify a trigonometric polynomial with its sequence of Fourier coefficients).

### 6.3 Smoothness

Now we discuss how to study the smoothness of the limit function of a uniformly convergent scalar subdivision scheme.

Let $t=\sum_{\alpha \in \mathbb{Z}^{d}} a_{\alpha} e^{-2 \pi i(\alpha, \cdot)}$ be a trigonometric polynomial, and let $\tau_{\nu}, \nu=$ $0, \ldots, m-1$, be its polyphase functions. It is well known (see, e.g., [4]) that

$$
\tau_{\nu}(0)=\sum_{\beta \in \mathbb{Z}^{d}} a_{s_{\nu}-M \beta}=1, \nu=0, \ldots, m-1
$$

whenever (27) is fulfilled for at least one $f \in \ell_{\infty}$ for which $S_{t}^{\infty} f \not \equiv 0$. So, if $S_{t}$ uniformly converges then, due to Propositions 1 and 7, there exists a $d \times d$ trigonometric matrix $T$ such that (25) holds.

Following [9], we introduce the notions of a normalized subdivision scheme and its subconvergence.

For $X$ a $N \times N$ matrix, and $T$ a $N \times N$ trigonometric matrix, the sequence $\left\{X^{* k} S_{T}^{k}\right\}_{k=1}^{\infty}$ is called the normalized (by $X$ ) subdivision scheme $S_{T}$.

We say that a subdivision scheme $S_{T}=S_{T, M}$ normalized by $X$ is uniformly convergent on a subspace $L$ of $\ell_{\infty}^{N}, L \neq\{0\}$, if for any $F \in L$, there exists a continuous vector-valued function $S_{T / X}^{\infty} F\left(S_{T / X}^{\infty} F(x) \in \mathbb{R}^{N}\right)$ such that

$$
\begin{equation*}
\lim _{k \rightarrow \infty}\left\|X^{* k} S_{T}^{k} F-S_{T / X}^{\infty} F\left(M^{-k} \cdot\right)\right\|_{\infty}=0 \tag{36}
\end{equation*}
$$

and if for at least one $F \in L$ the limit $S_{T / X} F$ is not identically zero.
We say that a subdivision scheme $S_{T}$ normalized by $X$ is uniformly subconvergent on a subspace $L, L \subseteq \ell_{\infty}^{d}, L \neq\{0\}$, if for some infinite set $\mathfrak{N} \subset \mathbb{N}$ and for any $F \in L$ there exists a continuous vector-valued function $S_{T / X}^{\infty} F$ $\left(S_{T / X}^{\infty} F(x) \in \mathbb{R}^{d}\right)$ such that

$$
\begin{equation*}
\lim _{\substack{k \in \mathfrak{n} \\ k \rightarrow \infty}}\left\|M^{* k} S_{T}^{k} F-S_{T / X}^{\infty} F\left(M^{-k} \cdot\right)\right\|_{\infty}=0 \tag{37}
\end{equation*}
$$

and if for at least one $F \in L$, the limit $S_{T / X} F$ is not identically zero.
Next, following [8], we consider the class of isotropic dilation matrices. A matrix $M$ is called isotropic if there exists a constant $C$ such that $\left\|M^{k}\right\|_{\infty}\left\|M^{-k}\right\|_{\infty} \leq C$ for all $k \in \mathbb{N}$.

Here we give a detailed proof of the sufficiency of a necessary and sufficirnt condition for the limits of a scalar subdivision scheme to be in $C^{1}$. This condition is stated in [9] with a sketch of a proof, which is not applicable
for general isotropic dilation matrices. Indeed, by Proposition 12 here, $S_{T}$ cannot be of full rank, as required by the sketch of the proof, whenever $M$ is not a multiple of the identity matrix.

Theorem 15 Let $M$ be isotropic, let a scalar subdivision scheme $S_{t}=S_{t, M}$ be uniformly convergent, and let $S_{T}$ be a $d \times d$ matrix subdivision scheme satisfying (25). If the subdivision scheme $S_{T}$ normalized by $M$ uniformly subconverges on $\nabla \ell_{\infty}$, then the function $S_{t}^{\infty} f$ is in $C^{1}\left(\mathbb{R}^{d}\right)$ for all $f \in \ell_{\infty}$.

First we prove a lemma.
Lemma 16 Let $\mathfrak{M}$ be an infinite subset of $\mathbb{N}, g_{k} \in \ell_{\infty}^{n}, k \in \mathfrak{M}, g$ be a continuous vector-valued function $\left(g(x) \in \mathbb{R}^{n}\right)$, and let $\Psi_{k}, k \in \mathfrak{M}$, be $n \times n$ matrix-valued functions which are uniformly bounded, uniformly compactly supported and such that

$$
\begin{equation*}
\sum_{\alpha \in \mathbb{Z}^{n}} \Psi_{k}(\cdot-\alpha) \equiv I_{n}, \quad \forall k \in \mathfrak{M} . \tag{38}
\end{equation*}
$$

If

$$
\begin{equation*}
\lim _{\substack{k \in \mathfrak{M} \\ k \rightarrow \infty}}\left\|g_{k}-g\left(M^{-k} \cdot\right)\right\|_{\infty}=0, \tag{39}
\end{equation*}
$$

then the sequence of vector-valued functions $\sum_{\alpha \in \mathbb{Z}^{d}} \Psi_{k}\left(M^{k} \cdot-\alpha\right)\left(g_{k}\right)_{\alpha}, k \in \mathfrak{M}$, uniformly converges to $g$ on any compact set $K \subset \mathbb{R}^{d}$.

Proof. Let $K \in \mathbb{R}^{d}$ be a compact set, $x \in K, k \in \mathfrak{M}$. Because of (38),

$$
\begin{align*}
& \sum_{\alpha \in \mathbb{Z}^{n}} \Psi_{k}\left(M^{k} x-\alpha\right)\left(g_{k}\right)_{\alpha}-g(x)=\sum_{\alpha \in \mathbb{Z}^{n}} \Psi_{k}\left(M^{k} x-\alpha\right)\left(\left(g_{k}\right)_{\alpha}-g(x)\right)= \\
& \sum_{\alpha \in \Omega\left(M^{k} x\right)} \Psi_{k}\left(M^{k} x-\alpha\right)\left(\left(g_{k}\right)_{\alpha}-g\left(M^{-k} \alpha\right)\right)+ \\
& \sum_{\alpha \in \Omega\left(M^{k} x\right)} \Psi_{k}\left(M^{k} x-\alpha\right)\left(g\left(M^{-k} \alpha\right)-g(x)\right), \tag{40}
\end{align*}
$$

where $\Omega(t)=\underset{k \in \mathfrak{M}}{\cup} \Omega_{k}(t)$, with $\Omega_{k}(t)=\left\{\alpha \in \mathbb{Z}^{d}: \quad \Psi_{k}(t-\alpha) \neq \mathbb{O}_{n}\right\}$.
It is clear, that $\sharp \Omega(t) \leq C$, where $C$ is a constant depending only on the joint support of the functions $\Psi_{k}$. It follows from (39) that

$$
\begin{equation*}
\lim _{\substack{k \in \mathfrak{M} \\ k \rightarrow \infty}} \sup _{x \in K}\left|\sum_{\alpha \in \Omega\left(M^{k} x\right)} \Psi_{k}\left(M^{k} x-\alpha\right)\left(\left(g_{k}\right)_{\alpha}-g\left(M^{-k} \alpha\right)\right)\right|=0 . \tag{41}
\end{equation*}
$$

If $\Psi_{k}\left(M^{k} x-\alpha\right) \neq \mathbb{O}_{n}$, then $\left|M^{-k} \alpha-x\right| \leq\left\|M^{-k}\right\| R$, where $R$ is the radius of a ball containing the supports of all $\Psi_{k}$. Due to the uniform continuity of $g$ on a compact set, this yields

$$
\begin{equation*}
\lim _{\substack{k \in \mathcal{M} \\ k \rightarrow \infty}} \sup _{x \in K}\left|\sum_{\alpha \in \Omega\left(M^{k} x\right)} \Psi_{k}\left(M^{k} x-\alpha\right)\left(g\left(M^{-k} \alpha\right)-g(x)\right)\right|=0 . \tag{42}
\end{equation*}
$$

To complete the proof it remains to combine (41) and (42) with (40). $\checkmark$
Proof of Theorem 15. For $\sigma \in \mathbb{N}^{d}$, the function

$$
b_{\sigma}(x)=\prod_{j=1}^{d}\left(\chi_{[0,1]} * \ldots * \chi_{[0,1]}\right)\left(x_{j}\right)
$$

is the tensor product B -spline of order $\sigma$, when the number of convolutions in the above product are $\sigma_{j}+1$. In the following we take $\sigma_{j}>1, j=1, \ldots, d$. Thus, $b_{\sigma}$ has the following properties:

$$
\begin{gather*}
\sum_{\alpha \in \mathbb{Z}^{d}} b_{\sigma}(x-\alpha) \equiv 1,  \tag{43}\\
\operatorname{grad} b_{\sigma}=\left(b_{\sigma-\mathbf{e}_{1}}\left(\cdot-\mathbf{e}_{1}\right)-b_{\sigma-\mathbf{e}_{1}}, \ldots, b_{\sigma-\mathbf{e}_{d}}\left(\cdot-\mathbf{e}_{d}\right)-b_{\sigma-\mathbf{e}_{d}}\right)^{T} . \tag{44}
\end{gather*}
$$

Given $f \in L$, using (43) and Lemma 16 with $\mathfrak{M}=\mathbb{N}, n=1, g_{k}=S_{t}^{k} f$, $g=S_{t}^{\infty} f$, we obtain that the sequence $F^{k}:=\sum_{\alpha \in \mathbb{Z}^{n}} b_{\sigma}\left(M^{k} \cdot-\alpha\right)\left(S_{t}^{k} f\right)_{\alpha}$ uniformly converges to $S_{t}^{\infty} f$ on any compact set $K \subset \mathbb{R}^{d}$. These functions are in $C^{1}\left(\mathbb{R}^{d}\right)$ by the choice of $\sigma$. To prove the theorem it remains to check that the sequence of vector-valued functions grad $F^{k}$ uniformly converges to a continuous vector-valued function on any compact set $K \subset \mathbb{R}^{d}$.

Set

$$
B_{\sigma}=\left(\begin{array}{rlr}
b_{\sigma-\mathbf{e}_{1}} & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & b_{\sigma-\mathbf{e}_{d}}
\end{array}\right)
$$

Using (25), we have

$$
\begin{array}{r}
\operatorname{grad} \sum_{\alpha \in \mathbb{Z}^{d}} b_{\sigma}\left(M^{k} x-\alpha\right)\left(S_{t}^{k} f\right)_{\alpha}=M^{* k} \sum_{\alpha \in \mathbb{Z}^{d}} \operatorname{grad} b_{\sigma}\left(M^{k} x-\alpha\right)\left(S_{t}^{k} f\right)_{\alpha}= \\
M^{* k} \sum_{\alpha \in \mathbb{Z}^{d}} B_{\sigma}\left(M^{k} x-\alpha\right)\left(\nabla S_{t}^{k} f\right)_{\alpha}=\sum_{\alpha \in \mathbb{Z}^{d}} M^{* k} B_{\sigma}\left(M^{k} x-\alpha\right) M^{*-k}\left(M^{* k} S_{T}^{k} \nabla f\right)_{\alpha} .
\end{array}
$$

Due to the normalized uniform subconvergence of $S_{T}$ on $\nabla L$, there exists an infinite set $\mathfrak{N} \subset \mathbb{N}$ such that for any $F \in \nabla L$ there exists a continuous vector-valued function $S_{T / M}^{\infty} F$ for which (37) holds. Let $f \in L$. Since $\sum_{\alpha \in \mathbb{Z}^{d}} M^{* k} B_{\sigma}(x-\alpha) M^{*-k} \equiv I_{d}$, by Lemma 16 with $\mathfrak{M}=\mathfrak{N}, n=d$, $g_{k}=M^{* k} S_{T}^{k} \nabla f, g=S_{T / M}^{\infty} \nabla f$ and $\Psi_{k}=M^{* k} B_{\sigma} M^{*-k}$ (the functions $\Psi_{k}$ are uniformly bounded because $M$ is isotropic), we can state that the sequence $\operatorname{grad} F^{k}, k \in \mathfrak{N}$, uniformly converges to $S_{T / M}^{\infty} \nabla f$ on any compact set $K \subset \mathbb{R}^{d} . \diamond$

A direct conclusion from Theorem 15 in case equation (25) holds for a trigonometric polynomial $t$, is that the limit function of the subdivision scheme $S_{t}$ is in $C^{1}$ if $S_{t}$ is uniformly convergent and if $S_{T}$ normalized by $M$ is uniformly subconvergent on $\nabla \ell_{\infty}$.

How to check the uniform convergence of a scalar subdivision scheme is discussed in Subsection 6.2. The method is based on Theorem 13 from [4]. In the following we prove a sufficient condition for the uniform convergence of $S_{T}$ normalized by $M$ on $\nabla \ell_{\infty}$, because to the best of our knowledge, such a theorem has not been published yet. Here we formulate and prove such a theorem for our specific case, namely for checking the smoothness of limits of a convergent scalar multivariate subdivision scheme, corresponding to a trigonometric polynomial in $\mathcal{Z}^{1}$.

Theorem 17 Let $t \in \mathcal{Z}^{1}, M$ be a $d \times d$ dilation matrix. Then there exist trigonometric matrices $T, Q$ of orders $d \times d$ and $d^{2} \times d^{2}$ respectively, satisfying (25) and (26). Moreover, if $t(0)=m$ and

$$
\begin{equation*}
\left\|M^{* L}\right\|_{\infty}\left\|S_{Q}^{L}\right\|_{\infty} \leq \mu<1 \tag{45}
\end{equation*}
$$

for some positive integer $L$, then the subdivision scheme $S_{T}$ normalized by $M$ uniformly converges on $\nabla \ell_{\infty}$.

Proof. The existence of $T, Q$, satisfying (25) and (26), follows from Theorem 10, as indicated in Subsection 6.1.

To prove the second part of the claim, we will show that the sequence of vector-valued functions

$$
F_{k}(x)=\sum_{\alpha \in \mathbb{Z}^{d}} \varphi\left(M^{k} x-\alpha\right)\left(M^{* k} S_{T}^{k} \nabla f\right)_{\alpha}, \quad k=1,2, \ldots
$$

with $\varphi$ as in the proof of Theorem $13^{\prime}$, is a Cauchy sequence in $\left(C\left(\mathbb{R}^{d}\right)\right)^{d}$ for any $f \in \ell_{\infty}$.

Recall that $\varphi$ is a continuous compactly supported function satisfying (30) and the interpolatory conditions (32), and that $\tilde{t}$, defined as in Lemma 14 , is in $\mathcal{Z}^{0}$ and satisfies $\tilde{t}(0)=m$.

By Lemma 14 with $n=d$,

$$
F_{k}(x)=M^{* k} \sum_{\alpha \in \mathbb{Z}^{d}} \varphi\left(M^{k} x-\alpha\right)\left(S_{T}^{k} \nabla f\right)_{\alpha}=M^{* k} \sum_{\alpha \in \mathbb{Z}^{d}} \varphi\left(M^{k+1} x-\alpha\right)\left(S_{\tilde{t} I_{d}} S_{T}^{k} \nabla f\right)_{\alpha}
$$

This yields that

$$
\begin{array}{r}
F_{k+1}(x)-F_{k}(x)=M^{* k} \sum_{\alpha \in \mathbb{Z}^{d}} \varphi\left(M^{k+1} x-\alpha\right)\left(\left(M^{*} S_{T}-S_{\tilde{t}_{d}}\right) S_{T}^{k} \nabla f\right)_{\alpha}= \\
M^{* k} \sum_{\alpha \in \mathbb{Z}^{d}} \varphi\left(M^{k+1} x-\alpha\right)\left(S_{\widetilde{T}} S_{T}^{k} \nabla f\right)_{\alpha}, \tag{46}
\end{array}
$$

where $\widetilde{T}=M^{*} T-\tilde{t} I_{d}$. The entries of the matrix $\widetilde{T}$ are trigonometric polynomials in $\mathcal{Z}^{0}$ because $\tilde{t} \in \mathcal{Z}^{0}$ as was mentioned above, and the entries of $T$ are in $\mathcal{Z}^{0}$ due to Theorem 10. Moreover, by Proposition 12,

$$
\widetilde{T}(\mathbf{0})=M^{*} T(\mathbf{0})-\tilde{t}(\mathbf{0}) I_{d}=M^{*} m M^{*-1}-m I_{d}=\mathbb{O}_{d}
$$

Due to Lemma 3, it follows that for any $g \in \ell_{\infty}^{d}$

$$
\begin{equation*}
S_{\widetilde{T}} g=S_{\widetilde{Q}} \nabla g \tag{47}
\end{equation*}
$$

where $\widetilde{Q}$ is a $d \times d^{2}$ trigonometric matrix.
So, (46) may be rewritten as

$$
F_{k+1}(x)-F_{k}(x)=M^{* k} \sum_{\alpha \in \mathbb{Z}^{d}} \varphi\left(M^{k+1} x-\alpha\right)\left(S_{\widetilde{Q}} \nabla S_{T}^{k} \nabla f\right)_{\alpha}
$$

and

$$
\begin{equation*}
\left|F_{k+1}(x)-F_{k}(x)\right| \leq C\left\|M^{* k}\right\|_{\infty}\left\|S_{\widetilde{Q}} \nabla S_{T}^{k} \nabla f\right\|_{\infty}, \tag{48}
\end{equation*}
$$

where $C$ depends only on $\varphi$.
It follows from (26) that $\nabla S_{T}^{k} g=S_{Q}^{k} \nabla g$ for any $g \in \ell_{\infty}^{d}$, and hence

$$
\begin{equation*}
\nabla S_{T}^{k} \nabla f=S_{Q}^{k} \nabla^{2} f \tag{49}
\end{equation*}
$$

Thus we get from (48), in view of (45),

$$
\begin{aligned}
\left|F_{k+1}(x)-F_{k}(x)\right| \leq & C\left\|M^{* k}\right\|_{\infty}\left\|S_{\widetilde{Q}}\right\|_{\infty}\left\|S_{Q}^{k} \nabla^{2} f\right\|_{\infty} \leq \\
& C_{1}\left\|M^{* L}\right\|_{\infty}^{\frac{k}{L}}\left\|S_{Q}^{L}\right\|_{\infty}^{\frac{k}{L}}\left|\nabla^{2} f\right| \leq C_{1}\left|\nabla^{2} f\right| \mu^{\frac{k}{L}} .
\end{aligned}
$$

This yields that for all $k, n \in \mathbb{N}$

$$
\begin{equation*}
\left|F_{k+n}(x)-F_{k}(x)\right| \leq C_{1}\left\|\nabla^{2} f\right\|_{\infty} \sum_{j=0}^{n-1} \mu^{\frac{k+j}{L}} \leq C_{2}\left\|\nabla^{2} f\right\|_{\infty} \mu^{\frac{k}{L}}, \tag{50}
\end{equation*}
$$

which implies that $\left\{F_{k}\right\}$ is a Cauchy sequence. Denote the limit vector-valued function by $F$ which is, evidently continuous. Passing to the limit in (50) as $n \rightarrow \infty$, we have

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \sup _{x \in \mathbb{R}^{d}}\left|F(x)-F_{k}(x)\right|=0 \tag{51}
\end{equation*}
$$

Due to (51),

$$
\lim _{k \rightarrow \infty} \sup _{x \in \mathbb{R}^{d}}\left|F(x)-\sum_{\alpha \in \mathbb{Z}^{d}}\left(M^{* k} S_{T}^{k} \nabla f\right)_{\alpha} \varphi\left(M^{k} x-\alpha\right)\right|=0
$$

Substituting $x=M^{-k} \beta, \beta \in \mathbb{Z}^{d}$, we have

$$
\lim _{k \rightarrow \infty} \sup _{\beta \in \mathbb{Z}^{d}}\left|F\left(M^{-k} \beta\right)-\sum_{\alpha \in \mathbb{Z}^{d}}\left(M^{* k} S_{T}^{k} \nabla f\right)_{\alpha} \varphi(\beta-\alpha)\right|=0
$$

The claim of the theorem follows now from (32). $\diamond$

## References

[1] Dyn N. Subdivision schemes in Computer Aided Geometric Design, In "Advances in Numerical Analysis-Vol. II, Wavelets, Subdivision Algorithms and Radial Basis Functions", W. Light (ed.), Clarendon Press, Oxford, (1992), 36-104.
[2] Jia R.Q. Approximation properties of multivariate wavelets, Math. Comp. 67 (1998), 647-655.
[3] Han B., Compactly supported tight wavelet frames and orthonormal wavelets of exponential decay with a general dilation matrix, J. Comput. Appl. Math. 155 (2003) 43-67.
[4] Latour V., Müller J., Nickel W. Stationry subdivision for general scaling matrices, Math. Zeitschrift 227 (1998), 645-661.
[5] Möller H.M. and Sauer T. Multivariate refinable functions of high approximation order via quotient ideals of Laurent polynomials, Adv. Comput. Math. 20 (2004), No.1-3, 205-228.
[6] I. Novikov, V. Protassov, and M. Skopina, Wavelet Theory, Moscow: Fizmatlit, 2005 (in Russian).
[7] Sauer T. Polynomial interpolation, ideals and approximation order of multivariate refinable functions Proceedings of the American Mathematical Society 130 (2002), 11, 3335-3347.
[8] Sauer T. How to generate smoother refinable functions from given one In: W. Haussmann, K. Jetter, M. Reimer and J. Stökler (eds): Modern Developments in Multivariate Approximation, Vol. 145 of International Series of Numerical Mathematics (2003), 279-294.
[9] Sauer T. Differentiability of multivariate refinable functions and factorization Adv. Comput. Math., 26 (2006), No.1-3, 211-235.
[10] Skopina M., On Construction of Multivariate Wavelets with Vanishing Moments ACHA 20 (2006), 3, 375-390.
[11] Wojtaszczyk P. A mathematical introduction to wavelets, London Math. Soc. Student texts 37, 1997.


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