# Subdivision schemes, network flows and linear optimization 

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#### Abstract

We link regularity and smoothness analysis of multivariate vector subdivision schemes with network flow theory and with special linear optimization problems. This connection allows us to prove the existence of what we call optimal difference masks that posses crucial properties unifying the regularity analysis of univariate and multivariate subdivision schemes. We also provide efficient optimization algorithms for construction of such optimal masks. Integrality of the corresponding optimal values leads to purely analytic proofs of $C^{k}$-regularity of subdivision.


## 1 Introduction

There is a large variety of results in the literature that study Hölder and Sobolev regularity and other important properties of scalar and vector, univariate and multivariate subdivision schemes, see [4, 14, 22] and references therein.

Such schemes are recursive algorithms for mesh refinement and, in the regular case, are based on the repeated application of the so-called subdivision operator

$$
S_{\boldsymbol{A}}: \ell^{n}\left(\mathbb{Z}^{s}\right) \rightarrow \ell^{n}\left(\mathbb{Z}^{s}\right), \quad S_{\boldsymbol{A}} \boldsymbol{c}(\alpha)=\sum_{\beta \in \mathbb{Z}^{s}} A(\alpha-M \beta) c(\beta), \quad \alpha \in \mathbb{Z}^{s} .
$$

The efficiency of such algorithms is guaranteed by their locality, indeed the so-called subdivision mask $\boldsymbol{A}=\left\{A(\alpha), \alpha \in \mathbb{Z}^{s}\right\}$ is usually finitely supported. The topology of the mesh is encoded in the dilation matrix $M \in \mathbb{Z}^{s \times s}$. For details on various applications of subdivision schemes see e.g. [3, 9, 10].

The methods for regularity analysis of such schemes are either based on the so-called joint spectral radius approach [8, 12, 20] or on the restricted radius approach [4, 6]. The results of [5] unify these approaches and show that both characterize the regularity of subdivision in terms of the same quantity: either called the joint spectral radius (JSR) of a certain family of square matrices derived from the subdivision mask $\boldsymbol{A}$, or the restricted spectral radius (RSR) of an associated linear operator of the difference subdivision scheme also derived from $\boldsymbol{A}$. Hölder or Sobolev regularity of subdivision is characterized in terms of $\infty-\mathrm{JSR}$ or $2-\mathrm{JSR}$, respectively. The question of exact computation on $2-\mathrm{JSR}$ in the subdivision context has been extensively studied in [18, 21]. The numerical methods for estimation of the $\infty$-JSR differ and its computation, in general, is an NP-hard problem [2]. Recent theoretical results and numerical tests in [17] lead to exact computation of $\infty-$ JSR for a wide class of families of matrices and rely on the special choice of the so-called extremal matrix norm. There are also various results on numerical methods for computation of $\infty-\mathrm{JSR}$ for particular subdivision schemes e.g. [16, 19].

We are interested in pursuing further the idea in [5, 6] of using optimization methods for estimation of $\infty-$ JSR. In sections 3.1.1 and 5, we show that these optimization problems are of a very special type, namely, they are network flow problems, or, in general, equivalent to special linear optimization problems. The properties of network flow problems allow for exact computation of what we call the optimal first difference subdivision mask, see section 4. The spectral properties of the corresponding optimal scheme characterize the convergence of $S_{\boldsymbol{A}}$. For such optimal masks the sufficient condition derived in [13] for the convergence of $S_{\boldsymbol{A}}$ also becomes necessary and, thus, coincides with the characterization of convergence given in [5, 6]. The advantage of working with optimal difference masks is twofold. Firstly, the
computation of the norm of the associated subdivision operator is straightforward, see [13], and exact, if all entries $A(\alpha)$ of the subdivision mask are rational. Thus, it allows for analytic arguments in the convergence proofs. Secondly, the proof of the existence of such optimal masks bridges the gap between the convergence analysis of univariate and multivariate schemes $S_{\boldsymbol{A}}$, i.e., the non-restricted and restricted norms of the corresponding difference operator coincide even in the multivariate case. In section 5, we also prove the existence of optimal masks for higher order difference schemes and provide an algorithm for their construction. In section 6, we illustrate our results with several examples.

## 2 Background and notation

In this paper we make use of the following notation.

- By $\epsilon_{\ell}, \ell=1, \ldots, s$, we denote the standard unit vectors of $\mathbb{R}^{s}$.
- For $\alpha \in \mathbb{N}_{0}^{s}$ we define $|\alpha|=\alpha_{1}+\ldots+\alpha_{s}$.
- In the multi-index notation we have $z^{\alpha}=z_{1}^{\alpha_{1}} \cdots z_{s}^{\alpha_{s}}, z \in \mathbb{C}^{s}$ and $\alpha \in \mathbb{N}_{0}^{s}$.
- The eigenvalues of the dilation matrix $M \in \mathbb{Z}^{s \times s}$ are all greater than 1 in absolute value.
- Vector sequences $\boldsymbol{c} \in \ell^{n}\left(\mathbb{Z}^{s}\right)$ indexed by $\mathbb{Z}^{s}$, i.e. functions from $\mathbb{Z}^{s}$ into $\mathbb{R}^{n}$, are denoted by boldface letters. Matrix sequences $\boldsymbol{A} \in \ell^{n \times m}\left(\mathbb{Z}^{s}\right)$ indexed by $\mathbb{Z}^{s}$, i.e. functions from $\mathbb{Z}^{s}$ into $\mathbb{R}^{n \times m}$, are denoted by boldface capital letters. The space of such sequences with finitely many non-zero elements is denoted by $\ell_{0}^{n}\left(\mathbb{Z}^{s}\right)$ or $\ell_{0}^{n \times m}\left(\mathbb{Z}^{s}\right)$, respectively.
- The finitely supported matrix sequence $\boldsymbol{\delta}_{I_{n}} \in \ell_{0}^{n \times n}\left(\mathbb{Z}^{s}\right)$ is defined by

$$
\delta_{I_{n}}(\alpha):=\left\{\begin{aligned}
I_{n}, & \alpha=0, \\
0, & \alpha \in \mathbb{Z}^{s} \backslash\{0\} .
\end{aligned}\right.
$$

- The norm on the Banach space $\ell_{\infty}^{n}\left(\mathbb{Z}^{s}\right)$ is given by

$$
\|\boldsymbol{c}\|_{\infty}=\sup _{\alpha \in \mathbb{Z}^{s}}\|c(\alpha)\|_{\infty}
$$

- For $C=\left(C_{i, j}\right) \in \mathbb{R}^{n \times n}$ we define $|C|=\left(\left|C_{i, j}\right|\right) \in \mathbb{R}^{n \times n}$.
- For a real number $a$ we define $a^{+}=\max \{a, 0\}$ and $a^{-}=-\min \{a, 0\}$.


### 2.1 Subdivision schemes

In this subsection we recall some basic facts about multivariate subdivision schemes.

Let $\boldsymbol{A} \in \ell_{0}^{n \times n}\left(\mathbb{Z}^{s}\right)$, the so-called subdivision mask, be given and be supported on $\{0, \ldots, N\}^{s}, N \in \mathbb{N}$. A subdivision scheme

$$
\begin{equation*}
\boldsymbol{c}^{[k+1]}(\alpha)=S_{\boldsymbol{A}} \boldsymbol{c}^{[k]}(\alpha), \quad k \in \mathbb{N}_{0}, \quad \boldsymbol{c}^{[0]} \in \ell^{n}\left(\mathbb{Z}^{s}\right), \quad \alpha \in \mathbb{Z}^{s} \tag{1}
\end{equation*}
$$

is a repeated application of the so-called subdivision operator

$$
\begin{equation*}
S_{\boldsymbol{A}}: \ell^{n}\left(\mathbb{Z}^{s}\right) \rightarrow \ell^{n}\left(\mathbb{Z}^{s}\right), \quad\left(S_{\boldsymbol{A}} \boldsymbol{c}\right)(\alpha)=\sum_{\beta \in \mathbb{Z}^{s}} A(\alpha-M \beta) c(\beta) \tag{2}
\end{equation*}
$$

Equivalently, the recursion in (1) can be written as

$$
\boldsymbol{c}^{[k+1]}=S_{\boldsymbol{A}}^{k+1} \boldsymbol{c}^{[0]}, \quad k \in \mathbb{N}_{0}, \quad \boldsymbol{c}^{[0]} \in \ell^{n}\left(\mathbb{Z}^{s}\right)
$$

where the iterated operator $S_{\boldsymbol{A}}^{k+1}$ is defined similarly as in (2) by replacing $\boldsymbol{A}$ by the so-called iterated mask $\boldsymbol{A}^{[k+1]}$ given by

$$
\begin{equation*}
A^{[k+1]}(\alpha)=\sum_{\beta \in \mathbb{Z}^{s}} A^{[k]}(\beta) A(\alpha-M \beta), \quad k \in \mathbb{N}_{0}, \quad \alpha \in \mathbb{Z}^{s}, \quad \boldsymbol{A}^{[0]}=\boldsymbol{\delta}_{I_{n}} . \tag{3}
\end{equation*}
$$

The definition of $S_{\boldsymbol{A}}$ in (2) implies that we have $|\operatorname{det}(M)|$ different subdivision rules, due to $\alpha=\varepsilon+M \beta, \varepsilon \in \Xi \simeq\left(\mathbb{Z}^{s} / M \mathbb{Z}^{s}\right)$ and $\beta \in \mathbb{Z}^{s}$. The set $\Xi$ is usually called the set of representatives of the equivalence classes $\mathbb{Z}^{s} / M \mathbb{Z}^{s}$. In the simplest case, $s=1, n=1$ and $M=2$, we have $\Xi \simeq\{0,1\}$. Thus, we have different subdivision rules for odd and even $\alpha \in \mathbb{Z}$. With a slight abuse of notation we denote the subdivision scheme also by $S_{\boldsymbol{A}}$.

We say that the subdivision scheme $S_{\boldsymbol{A}}$ is convergent, if for any starting sequence $\boldsymbol{c} \in \ell_{\infty}^{n}\left(\mathbb{Z}^{s}\right)$, there exists a uniformly continuous vector-valued function $f_{c} \in\left(C^{0}\left(\mathbb{R}^{s}\right)\right)^{n}$ such that

$$
\lim _{r \rightarrow \infty} \sup _{\alpha \in \mathbb{Z}^{s}}\left\|f_{\boldsymbol{c}}\left(M^{-r} \alpha\right)-S_{\boldsymbol{A}}^{r} \boldsymbol{c}(\alpha)\right\|_{\infty}=0
$$

To distinguish between scalar and vector subdivision schemes we denote their masks by $\boldsymbol{a}$ and $\boldsymbol{A}$, respectively.

### 2.2 Linear optimization and network flows

To familiarize the reader with terminology used in this paper, in this subsection, we introduce some basic notions from the theory of network flows and linear optimization.

A linear optimization (linear programming, LP) problem ([26]) is to maximize, or minimize, a linear function of $n$ variables subject to a finite number of linear constraints, each being a linear equation or a linear inequality. Since each equation can be written as two inequalities, every LP problem may be written as

$$
\begin{equation*}
\text { maximize } d^{T} x \text { subject to } C x \leq b \tag{4}
\end{equation*}
$$

where $C$ is an $m \times n$ matrix, $b$ and $d$ are (column) vectors of suitable dimensions, and $x \in \mathbb{R}^{n}$ is the vector containing the $n$ optimization variables $x_{1} x_{2}, \ldots, x_{n}$. A feasible solution of (4) is a vector $x \in \mathbb{R}^{n}$ satisfying all the constraints. A feasible solution is called optimal, if no other feasible solution attains a larger value on the objective function $d^{T} x$. There are practical, efficient algorithms for solving LP problems, see [26]. For LP problems there exist a powerful duality theory, which associates to every LP problem another LP problem, called the dual problem, and shows close connections between these two problems. In particular, the two problems have the same optimal value (under a weak assumption). If we take the dual twice, we are back to the original problem. We discuss duality in more detail for the special case of network flow problems.

A special class of LP problems, arising in several applications and useful in the context of subdivision, consists of network flow problems. Given a directed graph $G=(V, E)$ with vertex set $V$ and edge set $E$ (where each edge is an ordered pair of vertices), real numbers $b_{v}$ for each $v \in V$ and real numbers (costs) $c_{u v}$ for each $(u, v) \in E$; we write $u v$ sometimes instead of $(u, v)$. The minimum cost network flow problem is the following LP problem
minimize

$$
\sum_{(u, v) \in E} c_{u v} f_{u v}
$$

subject to

$$
\begin{array}{cl}
\sum_{u:(v, u) \in E} f_{v u}-\sum_{u:(u, v) \in E} f_{u v}=b_{v} & (v \in V)  \tag{5}\\
f_{u v} \geq 0 & ((u, v) \in E)
\end{array}
$$

The interpretation of (5) is as follows: The variable $f_{u v}$ represents the flow from $u$ to $v$ along the edge $(u, v)$, and $c_{u v}$ is the unit cost of sending this flow, so the objective function represents the total cost. The constraints represent flow balance at every vertex: the total flow leaving a vertex $v$ minus the total flow into the same vertex equals the given number $b_{v}$, which is called the supply at $v$. Finally, flows $f_{u v}$ are required to be nonnegative at each edge. Often one also has upper bounds (capacities) on flows, but we do not need this here. We assume that problem (5) has a feasible solution (conditions that guarantee this are known, and require that $\sum_{v \in V} b_{v}=0$, see [1]). This condition on $b_{v}$ has a natural interpretation in the subdivision context, see Remark 2, The dual of the minimum cost network flow problem (5) is

$$
\begin{array}{ll}
\text { maximize } & \sum_{v \in V} b_{v} x_{v} \\
\text { subject to } & x_{u}-x_{v} \leq c_{u v} \quad((u, v) \in E) . \tag{6}
\end{array}
$$

In this problem the variables $x_{v}$ are associated with vertices and may be interpreted as a kind of potential. Now, assume $f_{u v}((u, v) \in E)$ and $x_{v}$ $(v \in V)$ are feasible solutions of (5) and (6), respectively. Then

$$
\sum_{(u, v)} c_{u v} f_{u v} \geq \sum_{(u, v)}\left(x_{u}-x_{v}\right) f_{u v}=\sum_{v}\left(\sum_{u} f_{v u}-\sum_{u} f_{u v}\right) x_{u}=\sum_{v} b_{v} x_{v}
$$

This shows weak duality which says that the optimal value of (5) is not smaller than the optimal value of (6). Actually, a stronger result, the duality theorem, says that these two optimal values are equal. This fact is exploited in very efficient algorithms for solving the minimum cost network flow problem, see [1]. An important result is the integrality theorem which says that, if each $b_{v}$ is an integer, then problem (5) has an optimal solution where each $f_{u v}$ is an integer. We then say that the optimal solution is integral. This integrality property is due to special properties of the coefficient matrix of the flow balance equations, and for other classes of LP problems it may happen that no optimal solution is integral (i.e., in every optimal solution, at least one variable is not an integer).

## 3 First difference subdivision schemes and network flow problems

In this section we show that optimization problems considered in [6] for convergence analysis of subdivision are of a very special type, namely, they are network flow problems.

### 3.1 Scalar case

For simplicity of presentation we start with the scalar multivariate case, i.e. $n=1$. The subdivision mask $\boldsymbol{a} \in \ell_{0}\left(\mathbb{Z}^{s}\right)$ is a finitely supported sequence of real numbers $a(\alpha), \alpha \in \mathbb{Z}^{s}$. The convergence analysis of such subdivision schemes relies on what we call the backward difference operator $\nabla: \ell\left(\mathbb{Z}^{s}\right) \rightarrow$ $\ell^{s}\left(\mathbb{Z}^{s}\right)$ given by

$$
\nabla=\left(\begin{array}{c}
\nabla_{1}  \tag{7}\\
\vdots \\
\nabla_{s}
\end{array}\right), \quad\left(\nabla_{\ell} \boldsymbol{c}\right)(\alpha)=c(\alpha)-c\left(\alpha-e_{\ell}\right), \quad \alpha \in \mathbb{Z}^{s}, \quad \boldsymbol{c} \in \ell\left(\mathbb{Z}^{s}\right)
$$

A matrix sequence $\boldsymbol{B} \in \ell^{s \times s}\left(\mathbb{Z}^{s}\right)$ that satisfies

$$
\begin{equation*}
\nabla S_{a}^{r}=S_{B}^{r} \nabla, \quad r \in \mathbb{N} \tag{8}
\end{equation*}
$$

defines the so-called difference subdivision operator $S_{B}: \ell^{s \times s}\left(\mathbb{Z}^{s}\right) \rightarrow \ell^{s \times s}\left(\mathbb{Z}^{s}\right)$ by

$$
\begin{equation*}
S_{\boldsymbol{B}} \boldsymbol{d}(\alpha)=\sum_{\beta \in \mathbb{Z}^{s}} B(\alpha-M \beta) d(\beta), \quad \boldsymbol{d} \in \ell^{s}\left(\mathbb{Z}^{s}\right) \tag{9}
\end{equation*}
$$

The existence of $\boldsymbol{B}$ is equivalent to the fact that $\boldsymbol{a}$ satisfies sum rules of order 1, see [23] for details. The assumption that $\boldsymbol{a}$ satisfies sum rules of order 1 is by no means restrictive, as it is also a necessary condition for convergence of $S_{\boldsymbol{a}}$, see e.g. [4, 20].

We index the entries $B_{j, \ell}(\alpha)$ of the matrices $B(\alpha)$ by $j=1, \ldots, s$ and by $\ell=1, \ldots, s$ to match the indexing of the entries $\nabla_{\ell}$ of the difference operator $\nabla$. One of the approaches for characterizing convergence of subdivision schemes studies the spectral properties of the operator $S_{\boldsymbol{B}}$. Let $r \in \mathbb{N}$. The results of [13] use the non-restricted norm

$$
\left\|S_{B}^{r}\right\|_{\infty}=\max _{\varepsilon \in \Xi_{r}}\left\|\sum_{\beta \in \mathbb{Z}^{s}}\left|B^{[r]}\left(\varepsilon-M^{r} \beta\right)\right|\right\|_{\infty}, \quad \Xi_{r} \simeq\left(\mathbb{Z}^{s} / M^{r} \mathbb{Z}^{s}\right)
$$

to derive sufficient conditions for convergence of the subdivision scheme $S_{\boldsymbol{a}}$. In [6] the authors use the restricted norm

$$
\begin{equation*}
\left\|\left.S_{\boldsymbol{B}}^{r}\right|_{\nabla}\right\|_{\infty}=\max _{\|\nabla c\|_{\infty}=1}\left\|S_{\boldsymbol{B}}^{r} \nabla \boldsymbol{c}\right\|_{\infty} \tag{10}
\end{equation*}
$$

to characterize the convergence of $S_{\boldsymbol{a}}$. Due to (8), the operator $S_{\boldsymbol{B}}$ maps the difference subspace $\nabla \ell\left(\mathbb{Z}^{s}\right)$ into itself and, thus, its restriction $\left.S_{B}^{r}\right|_{\nabla}$ to $\nabla \ell\left(\mathbb{Z}^{s}\right)$ is well-defined.

Define $K=\{-N-1, \ldots, 0\}^{s}$. Due to $\boldsymbol{B} \in \ell_{0}^{s \times s}\left(\mathbb{Z}^{s}\right)$ and by the periodicity of the operator $S_{B}$, we have

$$
\begin{equation*}
\left\|\left.S_{\boldsymbol{B}}^{r}\right|_{\nabla}\right\|_{\infty}=\max _{\left\|\left.\nabla c\right|_{K}\right\|_{\infty}=1} \max _{\varepsilon \in \Xi_{r}}\left\|\sum_{\beta \in K} B^{[r]}\left(\varepsilon-M^{r} \beta\right)(\nabla \boldsymbol{c})(\beta)\right\|_{\infty} \tag{11}
\end{equation*}
$$

The problem of computing of $\left\|\left.S_{B}^{r}\right|_{\nabla}\right\|_{\infty}$ in (11) consists of several linear optimization problems for the finitely many unknowns $c(\beta), \beta \in\{-N-$ $2, \ldots, 0\}^{s}$. Indeed, to compute the maximum in (11), it suffices, for each pair $(\varepsilon, j) \in \Xi_{r} \times\{1, \ldots, s\}$, to solve the linear optimization problem
$\max$

$$
\begin{equation*}
\sum_{\beta \in K} \sum_{\ell=1}^{s} B_{j, \ell}^{[r]}\left(\varepsilon-M^{r} \beta\right)\left(\nabla_{\ell} \boldsymbol{c}\right)(\beta) \tag{12}
\end{equation*}
$$

subject to

$$
-1 \leq c(\beta)-c\left(\beta-\epsilon_{\ell}\right) \leq 1, \quad \beta \in K, \quad \ell=1, \ldots, s
$$

and, then, determine, over all $(\varepsilon, j) \in \Xi_{r} \times\{1, \ldots, s\}$, the maximum of the corresponding optimal values in (12). See [6] for details. In the following two subsections, we show that the problem in (12) can be interpreted as a network flow problem.

### 3.1.1 Dual of a minimum cost problem and its properties

In this subsection we show that the problem in (12) is the dual of a minimum cost network flow problem introduced in subsection [2.2, To arrive at this conclusion we need to introduce some additional notation.

We define a directed graph $G=(V, E)$ with the vertex set $V=\{-N-$ $2, \ldots, 0\}^{s}$ and the edge set

$$
E=\left\{(u, v)=\left(\beta, \beta-\epsilon_{\ell}\right): \beta \in\{-N-1, \ldots, 0\}^{s}, \ell=1, \ldots, s\right\}
$$

Note that the undirected graph corresponding to $G$ is connected. Moreover, $G$ is acyclic, i.e. $G$ does not contain a directed cycle. For a fixed $r \in \mathbb{N}$ and for each pair $(\varepsilon, j) \in \Xi_{r} \times\{1, \ldots, s\}$, we also define a function $d: E \rightarrow \mathbb{R}$ by

$$
d(e)=d_{u v}=B_{j, \ell}^{[r]}\left(\varepsilon-M^{r} \beta\right), \quad e=(u, v)=\left(\beta, \beta-\epsilon_{\ell}\right) \in E .
$$

Definition $1 A$ function $x: V \rightarrow \mathbb{R}$, where $x(v)=x_{v}$ denotes the function value at a vertex $v \in V$, is called $C-$ smooth if

$$
\begin{equation*}
-1 \leq x_{u}-x_{v} \leq 1 \quad \text { for all }(u, v) \in E \tag{13}
\end{equation*}
$$

We call such a function $C$-smooth to emphasize that the constraints in (13) appear in convergence analysis of $S_{\boldsymbol{a}}$.

Note that solving (12) for the unknown sequence $\boldsymbol{c}$ amounts to solving for $x$ the problems

$$
\begin{equation*}
z_{C}^{*}=\max \left\{\sum_{(u, v) \in E} d_{u v}\left(x_{u}-x_{v}\right): x \text { is } C \text {-smooth }\right\} \tag{14}
\end{equation*}
$$

and then finding the maximum of these values over $\varepsilon \in \Xi_{r}$ and $j=1, \ldots, s$.
We compare next the properties of the optimization problem (14) with the properties of the difference subdivision operator $S_{\boldsymbol{B}}$.

Remark 2 Define the weights

$$
b_{v}^{d}=\sum_{u:(v, u) \in E} d_{v u}-\sum_{u:(u, v) \in E} d_{u v} \quad \text { for } v \in V
$$

The identity

$$
\begin{equation*}
\sum_{v \in V} b_{v}^{d}=\sum_{v \in V}\left(\sum_{u:(v, u) \in E} d_{v u}-\sum_{u:(u, v) \in E} d_{u v}\right)=0 \tag{15}
\end{equation*}
$$

is due to the simple fact that each of the terms $d_{u v}$ appears in the above identity twice with the opposite signs. Note that the identity (15) is equivalent to

$$
\left\|S_{\boldsymbol{B}} \nabla \boldsymbol{c}\right\|_{\infty}=0 \quad \text { for a constant sequence } c(\alpha)=c(\beta), \quad \alpha, \beta \in \mathbb{Z}^{s}
$$

The property $z_{C}^{*}=\left\|\left.S_{B}^{r}\right|_{\nabla}\right\|_{\infty} \leq\left\|S_{\boldsymbol{B}}^{r}\right\|_{\infty}=\|d\|_{1}=\sum_{(u, v) \in E}\left|d_{u v}\right|$ is reflected in the following lemma.

Lemma 3 The problem (14) has an optimal solution, so it is feasible and not unbounded. Moreover, its optimal value $z_{C}^{*}$ satisfies

$$
0 \leq z_{C}^{*} \leq\|d\|_{1}=\sum_{(u, v) \in E}\left|d_{u v}\right|
$$

Proof. The constant function $x=0$ is $C$-smooth, so problem (14) is feasible, i.e., has feasible solutions, and $z_{C}^{*} \geq 0$. If $x$ is $C$-smooth, then the objective function

$$
f^{d}(x)=\sum_{(u, v) \in E} d_{u v}\left(x_{u}-x_{v}\right)
$$

satisfies

$$
\begin{aligned}
f^{d}(x) & \leq\left|f^{d}(x)\right| \leq \sum_{(u, v) \in E}\left|d_{u v}\left(x_{u}-x_{v}\right)\right| \\
& =\sum_{(u, v) \in E}\left|d_{u v}\right|\left|x_{u}-x_{v}\right| \leq \sum_{(u, v) \in E}\left|d_{u v}\right|=\|d\|_{1}
\end{aligned}
$$

Moreover, we associate to $G$ symmetric directed graph $\bar{G}=(V, \bar{E})$ with the edge set

$$
\bar{E}=E \cup\{(v, u):(u, v) \in E\}
$$

It is easy to see that the linear optimization problem in (14) is equivalent to
(DualFlow( $d$ ))

$$
\max \quad \sum_{v \in V} b_{v}^{d} x_{v}
$$

subject to

$$
x_{v}-x_{u} \leq 1 \quad((u, v) \in \bar{E})
$$

### 3.1.2 Minimum cost network flow problem

In this subsection we cast a more detailed look at network flow problems introduced in subsection 2.2. The problem in DualFlow $(d)$ is the standard
form of a dual of the following minimum cost network flow problem

$$
\min
$$

$$
\sum_{(u, v) \in \bar{E}} f_{u v}
$$

subject to
(Flow(d))

$$
\begin{array}{cl}
\sum_{u:(v, u) \in \bar{E}} f_{v u}-\sum_{u:(u, v) \in \bar{E}} f_{u v}=b_{v}^{d} & (v \in V) \\
f_{u v} \geq 0 & ((u, v) \in \bar{E}) .
\end{array}
$$

The flow variable $f_{u v}$ in $\operatorname{Flow}(d)$ represents the flow from $u$ to $v$ along the edge $(u, v)$. The linear constraints (equations) are flow balance constraints, see (5). For each vertex $v$, these constraints imply that the difference between total flow out of vertex $v$ and the total flow into the same vertex is equal to $b_{v}^{d}$, which may be considered as the divergence (net supply) at $v$. A feasible flow $f$ is a function $f: \bar{E} \rightarrow \mathbb{R}$ whose function values $f((u, v))=f_{u v}$ satisfy the constraints of $\operatorname{Flow}(d)$. An optimal flow $f^{*}$ is a feasible flow that minimizes the objective function $\sum_{(u, v) \in \bar{E}} f_{u v}$ in $\operatorname{Flow}(d)$.

Remark 4 If $f^{*}$ is an optimal flow, then, for each edge $(u, v) \in E$, either $f_{u v}^{*}$ or $f_{v u}^{*}$ is zero. Otherwise, one could reduce both $f_{u v}^{*}$ and $f_{v u}^{*}$ by the same small positive quantity, which would contradict the optimality of $f^{*}$.

The objective function $\sum_{(u, v) \in \bar{E}} f_{u v}$ represents the total flow cost. The problem $\operatorname{Flow}(d)$ is a quite special network flow problem: the costs, i.e. the coefficients of $f_{u v}$ in the objective function, on the edges are all 1 ; and there is no upper bound on the flow in each edge. These properties together with $\sum_{v \in V} b_{v}^{d}=0$ and the fact that the undirected graph associated with $G$ is connected, imply that the problem $\operatorname{Flow}(d)$ is feasible and has an optimal solution, see [26]. The existence of an optimal solution of $\operatorname{Flow}(d)$ also follows from Lemma 3 as the dual problem DualFlow ( $d$ ) has an optimal solution. The following result is one of the consequences of this duality relationship.

Theorem 5 The optimal value $z_{C}^{*}$ in (14) equals the optimal value in the network flow problem Flow(d).

Proof. This follows from the network flow duality theory, see e.g. [1].

### 3.2 Vector case

In the vector case, i.e. $n>1$, the mask $\boldsymbol{A} \in \ell^{n \times n}\left(\mathbb{Z}^{s}\right)$ has matrix entries $A(\alpha), \alpha \in \mathbb{Z}^{s}$. The associated first difference scheme is given by repeated applications of the operator $S_{B}: \ell^{n s}\left(\mathbb{Z}^{s}\right) \rightarrow \ell^{n s}\left(\mathbb{Z}^{s}\right)$. There is no conceptual change in the structure of the linear optimization problems in (12), see [6] for details. Therefore, even in the vector case, the convergence analysis of subdivision schemes profits from the theory of network flows. We omit the formulations of the corresponding results to avoid repetitions.

## 4 Optimal first difference masks

In this section we show that there exists an optimal difference mask $\boldsymbol{B}^{*} \in$ $\ell_{0}^{s \times s}\left(\mathbb{Z}^{s}\right)$, possibly different for each $r \in \mathbb{N}$, such that the corresponding operator $S_{B^{*}}$ in (9) satisfies $\nabla S_{a}^{r}=S_{B^{*}} \nabla$ and

$$
\begin{equation*}
\left\|S_{B^{*}}\right\|_{\infty}=\left\|\left.S_{B^{*}}\right|_{\nabla}\right\|_{\infty}=\left\|\left.S_{\boldsymbol{B}}^{r}\right|_{\nabla}\right\|_{\infty} \tag{16}
\end{equation*}
$$

for any other $S_{\boldsymbol{B}}$ satisfying $\nabla S_{\boldsymbol{a}}^{r}=S_{\boldsymbol{B}}^{r} \nabla$. The algorithm for construction of $\boldsymbol{B}^{*}$ in section 4.3 is such that for a given difference mask $\boldsymbol{B}$ with rational entries the optimal mask is also rational. Thus, the norm $\left\|S_{B^{*}}\right\|_{\infty}$ is rational, which allows for analytic arguments in convergence proofs for $S_{a}$. In the multivariate case, such masks $\boldsymbol{B}^{*}$ possibly differ for each $r \in \mathbb{N}$, see Example 12.

### 4.1 Univariate case

In the univariate case, it is well known that the operator $S_{B}$ is unique and the maximizing sequence in (12) is determined uniquely, up to a constant sequence, by

$$
c(\beta)-c(\beta-1)=\operatorname{sgn} B^{[r]}\left(\varepsilon-M^{r} \beta\right), \quad \beta \in\{-N-1, \ldots, 0\},
$$

i.e. $z_{C}^{*}=\left\|\left.S_{B}^{r}\right|_{\nabla}\right\|_{\infty}=\left\|S_{B}^{r}\right\|_{\infty}=\|d\|_{1}$ for any $r \in \mathbb{N}$. The same holds for higher order difference schemes. This property of $z_{C}^{*}$ also follows directly from Theorem 6 in section 4.2 and Theorem 15 in section 5.

### 4.2 Multivariate case

In the multivariate case, the difference subdivision operator $S_{B}$ in (8) is not unique, see [6].

Fix $r \in \mathbb{N}, \varepsilon \in \Xi_{r}$ and $j=1, \ldots, s$. The next result shows that there exists $\boldsymbol{B}^{*} \in \ell_{0}^{s \times s}\left(\mathbb{Z}^{s}\right)$ such that

$$
z_{C}^{*}=\sum_{\beta \in K} \sum_{\ell=1}^{s}\left|B_{j, \ell}^{*}\left(\varepsilon-M^{r} \beta\right)\right|=\sum_{(u, v) \in \bar{E}} f_{u v}^{*}
$$

i.e., the restricted and non-restricted norms of $S_{B^{*}}$ coincide.

Theorem 6 Let $d: E \rightarrow \mathbb{R}$ be given. Let $f^{*}$ be an optimal flow in $\operatorname{Flow}(d)$ and let $x^{*}$ be an optimal solution of DualFlow $(d)$. Define the function $d^{*}$ : $E \rightarrow \mathbb{R}$ by $d_{u v}^{*}=f_{u v}^{*}-f_{v u}^{*}$ for each $(u, v) \in E$.

Then $b_{v}^{d}=b_{v}^{d^{*}}$ for all $v \in V$, i.e. $\operatorname{Flow}(d)$ and $\operatorname{Flow}\left(d^{*}\right)$ coincide, and so do DualFlow (d) and DualFlow $\left(d^{*}\right)$. Moreover, the common optimal value of these problems is equal to $\left\|d^{*}\right\|_{1}$, i.e.

$$
\sum_{(u, v) \in \bar{E}} f_{u v}^{*}=\sum_{v \in V} b_{v}^{d^{*}} x_{v}^{*}=\left\|d^{*}\right\|_{1} .
$$

Proof. Let $x^{*}$ be an optimal solution of Dualflow $(d)$. Define $d^{*}$ as stated in the theorem. Then, as $f^{*}$ is a feasible solution of $\operatorname{Flow}(d)$, for each $v \in V$, we have

$$
\begin{aligned}
b_{v}^{d} & =\sum_{u:(v, u) \in \bar{E}} f_{v u}^{*}-\sum_{u:(u, v) \in \bar{E}} f_{u v}^{*} \\
& =\sum_{u:(v, u) \in E}\left(f_{v u}^{*}-f_{u v}^{*}\right)-\sum_{u:(u, v) \in E}\left(f_{u v}^{*}-f_{v u}^{*}\right) \\
& =\sum_{u:(v, u) \in E} d_{v u}^{*}-\sum_{u:(u, v) \in E} d_{u v}^{*}=b_{v}^{d^{*}}
\end{aligned}
$$

which proves the first statement.
Next, as $f^{*}$ is optimal, by Remark [4, for each $(u, v) \in E$ at most one of the two variables $f_{u v}^{*}$ and $f_{v u}^{*}$ can be positive. So for each $(u, v) \in E$

$$
f_{u v}^{*}+f_{v u}^{*}=\left|f_{u v}^{*}-f_{v u}^{*}\right|=\left|d_{u v}^{*}\right|
$$

and therefore, since $\operatorname{Flow}(d)$ and Dualflow $(d)$ have the same optimal value

$$
\sum_{v \in V} b_{v}^{d^{*}} x_{v}^{*}=\sum_{(u, v) \in \bar{E}} f_{u v}^{*}=\sum_{(u, v) \in E}\left(f_{u v}^{*}+f_{v u}^{*}\right)=\sum_{(u, v) \in E}\left|d_{u v}^{*}\right|=\left\|d^{*}\right\|_{1}
$$

as desired.
We are guaranteed to have an integral optimal $d^{*}$, i.e. a rational optimal $\boldsymbol{B}^{*}$, under the additional assumption that $d$ is integral.

Corollary 7 Let $d: E \rightarrow \mathbb{Z}$ and $f^{*}$ be an integral optimal flow in $\operatorname{Flow}(d)$. Then $d^{*}: E \rightarrow \mathbb{Z}$ defined by $d_{u v}^{*}=f_{u v}^{*}-f_{v u}^{*},(u, v) \in E$, satisfies

$$
\sum_{(u, v) \in \bar{E}} f_{u v}^{*}=\left\|d^{*}\right\|_{1} .
$$

Proof. The network flow theory (see Subsection 2.2 or [1]) guarantees the existence of an integral optimal solution $f^{*}$ of $\operatorname{Flow}(d)$, if $d$ is integral. The claim follows then from Theorem 6.

Remark 8 It can happen that $\left\|d^{*}\right\|_{1}<\|d\|_{1}$. For example, consider a directed graph with vertices $V=\{u=(0,0), v=(0,1), w=(1,1), z=(1,0)\}$ and let $d_{u v}=d_{v w}=1$ and $d_{z w}=-1$. Then $b_{u}^{d}=1, b_{z}^{d}=-1$ and $b_{v}^{d}=b_{w}^{d}=0$. An optimal flow $f^{*}$ is given by $f_{u z}^{*}=1$ and zero otherwise, so $d_{u z}^{*}=1$ and zero otherwise. One of the optimal dual solutions is $x_{u}^{*}=0, x_{v}^{*}=x_{z}^{*}=-1$ and $x_{w}^{*}=-2$. The common optimal value is 1 (recall that we have taken the negative in the dual), and so $\left\|d^{*}\right\|_{1}=1$ while $\|d\|_{1}=3$.

We state next some necessary and sufficient conditions for $d=d^{*}$. These sufficient conditions are easy to check and, if satisfied, yield $d^{*}$ without solving DualFlow ( $d$ ).

Recall that an edge $e \in \bar{E}$ has the form $e=(u, v)$ and that, for $u=$ $\left(u_{1}, u_{2}, \ldots, u_{s}\right), v=\left(v_{1}, v_{2}, \ldots, v_{s}\right)$, we have $\left|v_{k}-u_{k}\right|=1$ for some $k$ and $v_{j}=u_{j}$ for all $j \neq k$. We say that $e$ is a $k$-positive edge, if $v_{k}=u_{k}+1$, while if $v_{k}=u_{k}-1$ we call $e$ a $k$-negative edge. A path $P$ in $\bar{G}$ is called monotone if, for a fixed $k$, it either contains only $k$-positive edges or only $k$-negative edges. The path $P$ in the support of $f^{*}$ consists of the edge set $\left\{(u, v) \in P: f_{u v}^{*}>0\right\}$.

Theorem 9 Let $f^{*}$ be an optimal solution in $\operatorname{Flow}(d)$ for a graph $\bar{G}$. Then each path $P$ in the support of $f^{*}$ is monotone.

Proof. Assume that the support of $f^{*}$ contains a path $P$ which is not monotone. Say that $P$ has $m$ edges and that its vertices (in that order) are $u^{0}, u^{1}, \ldots, u^{m} \in V$. We represent $P$ by a $(0,1,-1)$-matrix $A$ of size $m \times s$ whose $i-$ th row is $u^{i}-u^{i-1}$. The definition of $E$ assures that each entry in $A$ is either 0,1 or -1 . Since $P$ is not monotone, $A$ contains a column with both 1 and -1 . We choose such a column $k$ for which the rows $i_{1}$ and $i_{2}$ containing 1 and -1 are such that $\left|i_{1}-i_{2}\right|$ is minimal; we may assume $i_{1}<i_{2}$. Note that in these rows $i_{1}$ and $i_{2}$ the only non-zero entries are in the $k-$ th column.

Let the matrix $A^{\prime}$ be obtained from $A$ by deleting rows $i_{1}$ and $i_{2}$. Then $A^{\prime}$ corresponds to a new path $P^{\prime}$ having the same end vertices as $P$. We may define a new flow $f^{\prime}$ accordingly by replacing a flow of one unit along $P$ by a flow of one unit along $P^{\prime}$. Then, since the symmetric difference between $P$ and $P^{\prime}$ is a cycle, $f^{\prime}$ satisfies the flow balance constraints. Moreover, $\sum f_{u v}^{\prime}=\sum f_{u v}^{*}-2$, contradicting that $f^{*}$ was optimal. Thus we have shown that the support of $f^{*}$ only contains paths that are monotone.

For $i=1, \ldots, s$ define $E_{i}=\left\{\left(u, u-\epsilon_{i}\right) \in E\right\}$. Then $\left\{E_{i}: i=1, \ldots, s\right\}$ is a partition of the edge set $E$, i.e. the sets $E_{i}$ are disjoint and $E$ is the union of $E_{i}$.

Theorem 10 Consider a graph $\bar{G}$. Let $\kappa_{i} \in\{-1,1\}$ for $1 \leq i \leq s$, and assume that $\operatorname{sign}\left(d_{u v}\right) \in\left\{0, \kappa_{i}\right\}$ for all $(u, v) \in E_{i}, 1 \leq i \leq s$.
(i) The flow $f^{*}$ with

$$
\left(f_{u v}^{*}, f_{v u}^{*}\right)=\left\{\begin{array}{ll}
\left(d_{u v}, 0\right), & \text { if } \kappa_{i}=1, \\
\left(0,-d_{u v}\right), & \text { if } \kappa_{i}=-1
\end{array} \quad \text { for all }(u, v) \in E_{i}, i=1, \ldots, s,\right.
$$

is optimal for $\operatorname{Flow}(d)$ with optimal value $\|d\|_{1}$.
(ii) The function $x^{*}: V \rightarrow \mathbb{R}$ with $x^{*}(v)=-\sum_{i=1}^{s} \kappa_{i} v_{i}$ for $v=\left(v_{1}, v_{2}, \ldots, v_{s}\right) \in$ $V$ is optimal in DualFlow $(d)$ with optimal value $\|d\|_{1}$.

Proof. To prove (i) and (ii) we use the standard technique, based on weak duality (see Subsection 2.2 and [26]). It suffices to show that $f^{*}$ and $x^{*}$
defined in $(i)$ and (ii), respectively, are feasible and

$$
\sum_{v \in V} b_{v}^{d} x^{*}(v)=\sum_{(u, v) \in \bar{E}} f_{u v}^{*} .
$$

Let $i=1, \ldots, s$ and $e=(u, v) \in E_{i}$. Then

$$
x^{*}(u)-x^{*}(v)=x^{*}(u)-x^{*}\left(u-\epsilon_{i}\right)=\kappa_{i} u_{i}-\kappa_{i}\left(u_{i}-1\right)=\kappa_{i} .
$$

This shows that $\left|x^{*}(u)-x^{*}(v)\right|=1$ for each edge $(u, v) \in \bar{E}$, so $x^{*}$ is feasible in DualFlow $(d)$. Clearly $f^{*}$ is feasible in $\operatorname{Flow}(d)$ as $f_{u v}^{*} \geq 0$ and its divergence in a vertex $v \in V$ equals the divergence of $d$ in $v$ which is $b_{v}^{d}$. Moreover, we have

$$
\begin{aligned}
\sum_{v \in V} b_{v}^{d} x^{*}(v) & =\sum_{v \in V}\left(\sum_{u:(v, u) \in E} d_{v u}-\sum_{u:(u, v) \in E} d_{u v}\right) x^{*}(v) \\
& =\sum_{(u, v) \in E} d_{u v}\left(x^{*}(u)-x^{*}(v)\right) \\
& =\sum_{i=1}^{s} \sum_{(u, v) \in E_{i}} d_{u v}\left(x^{*}(u)-x^{*}(v)\right) \\
& =\sum_{i=1}^{s} \sum_{(u, v) \in E_{i}} d_{u v} \kappa_{i}=\sum_{i=1}^{s} \sum_{(u, v) \in E_{i}}\left|d_{u v}\right| \\
& =\sum_{i=1}^{s} \sum_{(u, v) \in E_{i}}\left(f_{u v}^{*}+f_{v u}^{*}\right)=\sum_{(u, v) \in \bar{E}} f_{u v}^{*}
\end{aligned}
$$

Therefore, by duality (Theorem (5) it follows that $f^{*}$ is optimal in $\operatorname{Flow}(d)$, $x^{*}$ is optimal in DualFlow $(d)$ and, finally, that the optimal value equals $\|d\|_{1}$.

Corollary 11 Consider a graph $\bar{G}$, and let d be nonnegative. Define $f_{u v}^{*}=$ $d_{u v}$ and $f_{v u}^{*}=0$ for each $(u, v) \in E$ and $x^{*}(v)=-\sum_{i=1}^{s} v_{i}$ for $v=$ $\left(v_{1}, v_{2}, \ldots, v_{s}\right) \in V$. Then $f^{*}$ is optimal in $\operatorname{Flow}(d), x^{*}$ is optimal in DualFlow $(d)$ and the optimal value equals $\|d\|_{1}$.

Proof. Let $\kappa_{i}=1$ for each $i \leq s$ and apply Theorem 10 ,
We conclude this subsection with an example of an optimal first difference mask $\boldsymbol{B}^{*}$ that satisfies

$$
\left\|S_{B^{*}}\right\|_{\infty}=\left\|\left.S_{B^{*}}\right|_{\nabla}\right\|_{\infty},
$$

but does not satisfy

$$
\left\|S_{B^{*}}^{2}\right\|_{\infty} \neq\left\|\left.S_{B^{*}}^{2}\right|_{\nabla}\right\|_{\infty}
$$

Example 12 Let $M=2 I$. Consider a bivariate scalar subdivision scheme with the mask

$$
\begin{array}{ccccccc} 
& \vdots & \vdots & \vdots & \vdots & \vdots & \\
\ldots & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \ldots \\
& \ldots & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0
\end{array} \ldots
$$

supported on $\{0, \ldots, 3\}^{2}$. The bold entry corresponds to the index $(0,0)$. In this case, the directed graph $G=(V, E)$ has vertices $V=\{-5, \ldots, 0\}^{2}$ and the edge set $E=\left\{\left(\beta, \beta-\epsilon_{\ell}\right): \beta \in\{-4, \ldots, 0\}^{2}, \ell=1,2\right\}$. The nonzero part of the optimal mask $\boldsymbol{B}^{*} \in \ell_{0}^{2 \times 2}\left(\mathbb{Z}^{2}\right)$ for the first difference scheme is

The bold entry again corresponds to the index $(0,0)$. The nonzero entries of its second iterated mask $\left(B^{*}\right)^{[2]}$ for the coset $\varepsilon=(2,1)$ are

$$
\begin{aligned}
& \left(B^{*}\right)^{[2]}(2,5)=\left(\begin{array}{cc}
0 & 0 \\
\frac{1}{8} & 0
\end{array}\right), \quad\left(B^{*}\right)^{[2]}(6,5)=\left(\begin{array}{cc}
\frac{1}{8} & 0 \\
-\frac{1}{16} & 0
\end{array}\right) \\
& \left(B^{*}\right)^{[2]}(2,1)=\left(\begin{array}{cc}
\frac{1}{8} & 0 \\
-\frac{1}{16} & \frac{1}{8}
\end{array}\right) \quad \text { and } \quad\left(B^{*}\right)^{[2]}(6,1)=\left(\begin{array}{cc}
0 & 0 \\
0 & \frac{1}{8}
\end{array}\right) .
\end{aligned}
$$

On the contrary, the corresponding linear optimization problem for $\varepsilon=(2,1)$ and $j=2$ in (14) and with nonzero values

$$
\begin{aligned}
& d_{(0,0)(-1,0)}=d_{(-1,-1)(-2,-1)}=-\frac{1}{16}, \\
& d_{(0,0)(0,-1)}=d_{(-1,0)(-1,-1)}=d_{(0,-1)(-1,-1)}=\frac{1}{8}
\end{aligned}
$$

yields $z_{C}^{*}=\frac{6}{16}$. And, indeed, the corresponding nonzero entries of the optimal second iterated mask constructed from $d^{*}$ for this coset are

$$
\begin{array}{ccc} 
& & \vdots \\
\ldots & \left(\begin{array}{cc}
0 & 0 \\
\frac{1}{16} & 0
\end{array}\right) & \ldots \\
\vdots & \left(\begin{array}{cc}
\frac{1}{8} & 0 \\
-\frac{1}{16} & 0
\end{array}\right) & \ldots \\
\ldots & \left(\begin{array}{cc}
\frac{1}{8} & 0 \\
0 & \frac{1}{16}
\end{array}\right) & \ldots
\end{array}\left(\begin{array}{cc}
0 & 0 \\
0 & \frac{3}{16}
\end{array}\right) \quad \ldots .
$$

### 4.3 Flow algorithm

In this section we present the successive shortest path algorithm ([1]) for our minimum cost network flow problems. This algorithm determines the optimal flow $f^{*}$ in $\operatorname{Flow}(d)$, which defines an optimal mask of the first difference schemes as stated in Theorem 6. An advantage of using this particular algorithm is that it guarantees an integral solution, if the input is integral. For a given flow $f$, the algorithm defines the so-called residual network $G(f)$ which consists of

- the given vertices $V$ of the graph $\bar{G}$,
- all edges $e=(u, v) \in \bar{E}$ and their copies, called "backward" edges $e^{-1}$ whose direction is opposite to $e$. These edges are created only for the edges $e \in \bar{E}$ that consitute the shortests paths determined by Dijkstra's algorithm in step 2.2. of the Flow algorith given below.

In the residual network $G(f)$ each edge is assigned a capacity $r_{(u, v)}=f_{u v}$ for $e=(u, v) \in \bar{E}$ and for backward edges

$$
r_{(u, v)^{-1}}= \begin{cases}f_{u v} & \text { if, } f_{u v}>0 \\ 0 & \text { otherwise }\end{cases}
$$

For a given function $\pi: V \rightarrow \mathbb{Z}$, each edge in $G(f)$ is also assigned the so-called reduced cost

$$
c_{(u, v)}^{\pi}=1-\pi(u)+\pi(v), \quad(u, v) \in \bar{E},
$$

and for the backwards edges $(u, v)^{-1},(u, v) \in \bar{E}$,

$$
c_{(u, v)^{-1}}^{\pi}=-1-\pi(u)+\pi(v)
$$

The purpose of introducing the backward edges is that they allow us to decrease the flow through the original edges in $\bar{E}$ by sending it along the corresponding backward edges.

The algorithm starts with the zero flow $f=0$ and performs a finite number of iterations consisting of adding the flow along the shortest path between the end vertices of an edge $(u, v)$ where $d_{u v}$ is nonzero. The shortest path is computed in the residual network $G(f)$ using the reduced costs $c^{\pi}$. The function $\pi$ is introduced to ensure that the reduced costs stay nonnegative, which makes the shortest path calculation more efficient, as we can use Dijkstra's algorithm. For further explanation and details on the successive shortest path algorithm, see [1].

## Flow algorithm:

Input: a function $d: E \rightarrow \mathbb{Z}$.

1. Initial step: Compute $b_{v}^{d}$ for $v \in V$. Set $\epsilon(v):=b_{v}^{d}$ for $v \in V$. Define $\mathcal{E}_{+}=\{v \in V: \epsilon(v)>0\}$ and $\mathcal{E}_{-}=\{v \in V: \epsilon(v)<0\}$. Initialize the flow $f:=0$, define the residual network $G(f)$, and set $\pi(v)=0$ for each $v \in V$.
2. While $\mathcal{E}_{+} \neq \emptyset$ do

$$
\begin{aligned}
& \text { 2.1 Choose any } v_{+} \in \mathcal{E}_{+} \text {and } v_{-} \in \mathcal{E}_{-} . \\
& 2.2 \text { Dijkstra's algorithm: uses as edge lengths the reduced costs } c_{(u, v)}^{\pi}=1-\pi(u)+\pi(v) \text { to } \\
& \text { compute the shortest path distances } \delta(v) \text { from } v_{+} \text {to each other vertex } v \in V \text {; determines } \\
& \text { the shortest path } P \text { from } v_{+} \text {to } v_{-} . \\
& 2.3 \text { Update } \pi \text { by } \pi:=\pi-\delta \text {, compute } \gamma:=\min \left\{\epsilon\left(v_{+}\right),-\epsilon\left(v_{-}\right), \min \left\{r_{i j}:(i, j) \in P\right\}\right\} \text { and } \\
& \text { augment } f \text { by adding a flow of value } \gamma \text { along the path } P \text {. Update the residual network } \\
& G(f), \text { i.e. update the reduced costs; set } \epsilon\left(v_{+}\right)=\epsilon\left(v_{+}\right)-\gamma, \epsilon\left(v_{-}\right)=\epsilon\left(v_{-}\right)+\gamma ; \text { update } \mathcal{E}_{+}, \\
& \mathcal{E}_{-} .
\end{aligned}
$$

Output: optimal flow $f^{*}$ and optimal dual variable $x:=-\pi$.
Theorem 13 The flow algorithm solves both $\operatorname{Flow}(d)$ and the dual problem Dualflow $(d)$. Its complexity is $O\left(B n^{2}\right)$ where $B=(1 / 2) \sum_{v}\left|b_{v}^{d}\right|$ is an upper bound on the number of iterations, and $O\left(n^{2}\right)$ is the complexity of Dijkstra's algorithm for solving the shortest path problem with nonnegative costs in a graph with $n$ vertices.

Proof. The correctness of the general algorithm is shown in [1]. The complexity statement follows from the fact that in each iteration, by integrality of $d$, the flow is augmented by a positive integer.

The next simple example illustrates the flow algorithm. It also stresses the necessity of using the algorithm for finding the optimal flow $f^{*}$ even for seemingly simple examples of our very special network flow problems.

Example 14 Consider the graph $\bar{G}$ with $s=2$ and assume $d$ is such that $b_{v}^{d}=1$ when $v \in\{(0,0),(-2,-2)\}, b_{v}^{d}=-1$ when $v \in\{(-1,-1),(-3,-3)\}$, and $b_{v}^{d}=0$ otherwise. Initially, in the flow algorithm,

$$
f=0, \quad \mathcal{E}_{+}=\{(0,0),(-2,-2)\}, \quad \mathcal{E}_{-}=\{(-1,-1),(-3,-3)\}
$$

and we (may) choose $v_{+}=(-2,-2)$ and $v_{-}=(-1,-1)$. A shortest path $P$ from $v_{+}$to $v_{-}$consists of the nodes $(-2,-2),(-1,-2),(-1,-1)$ and it has cost 2. Note that the updated edge cost for each edge in $P$ is -1 . In the next, and final, iteration, we (must) choose $v_{+}=(0,0)$ and $v_{-}=(-3,-3)$ the shortest path $P^{\prime}$ consists of the nodes $(0,0),(0,-1),(-1,-1),(-1,-2)$, $(-2,-2),(-2,-3),(-3,-3)$. As a result, the flow cancels out on $P$, and we have an optimal flow with $f=1$ on four edges so the optimal value is 4 .

This example shows that the obvious heuristic method of successively adding a shortest path, while maintaining previous paths, may go wrong. Doing so we would get a solution with two paths, one of length 2 and the other of length 6, so a total cost of 8, while the optimal value is 4 .

## 5 Optimal higher order difference schemes

In this section we investigate the existence of optimal masks for higher order difference schemes used for studying the regularity of subdivision in the scalar case, i.e. $n=1$. The vector case is more technical, but the computation of the restricted norms we consider here and the derivation of the optimal difference schemes are conceptually very similar to what we do in the scalar case. The unconvinced reader is referred to [5] and Example 17 .

The smoothness analysis of $S_{a}$ is based on the spectral properties of the higher order difference schemes $S_{\boldsymbol{B}_{k}}, k \geq 2$, derived from $S_{\boldsymbol{a}}$. In our notation, we have $\boldsymbol{B}_{1}=\boldsymbol{B}$, where $\boldsymbol{B}$ is the first difference scheme from section 3.1.

The $k-t h$ order backward difference operator $\nabla^{k}: \ell\left(\mathbb{Z}^{s}\right) \rightarrow \ell^{N_{s, k}}\left(\mathbb{Z}^{s}\right), N_{s, k}=$ $\binom{s+k-1}{s-1}$, is defined by

$$
\begin{equation*}
\nabla^{k}=\left(\nabla_{1}^{\mu_{1}} \ldots \nabla_{s}^{\mu_{s}}\right)_{\substack{\mu=\left(\mu_{1}, \ldots, \mu_{s}\right) \in \mathbb{N}_{0}^{s} \\|\mu|=k}} \tag{17}
\end{equation*}
$$

where, for $\ell=1, \ldots, s$,

$$
\nabla_{\ell}^{\mu_{\ell}}=\nabla_{\ell} \nabla_{\ell}^{\mu_{\ell}-1}, \quad \mu_{\ell} \in \mathbb{N}, \quad \nabla_{\ell}^{0}=\mathrm{id}
$$

The $k-t h$ order difference schemes $S_{\boldsymbol{B}_{k}}$ satisfy

$$
\begin{equation*}
\nabla^{k} S_{\boldsymbol{a}}^{r}=S_{\boldsymbol{B}_{k}}^{r} \nabla^{k}, \quad r \in \mathbb{N} \tag{18}
\end{equation*}
$$

We denote the entries of the matrices $B_{k}^{[r]}(\alpha)$ by $B_{k, j, \mu}^{[r]}(\alpha), j=1, \ldots, N_{s, k}$, and $\mu=\left(\mu_{1}, \ldots, \mu_{s}\right)$ matches the ordering of $\nabla_{1}^{\mu_{1}} \ldots \nabla_{s}^{\mu_{s}}$ in $\nabla^{k}$. By [5], the study of the spectral properties of $\left.S_{\boldsymbol{B}_{k}}\right|_{\nabla^{k}}$ leads to computation of the restricted norms

$$
\left\|\left.S_{\boldsymbol{B}_{k}}^{r}\right|_{\nabla^{k}}\right\|_{\infty}=\max _{\left\|\left.\nabla^{k} \boldsymbol{c}\right|_{K}\right\|_{\infty}=1} \max _{\varepsilon \in \Xi_{r}}\left\|\sum_{\beta \in K} B_{k}^{[r]}\left(\varepsilon-M^{r} \beta\right)\left(\nabla^{k} \boldsymbol{c}\right)(\beta)\right\|_{\infty}
$$

where $K=\{-N-k, \ldots, 0\}^{s}$. Let $r \in \mathbb{N}, j=1, \ldots, N_{s, k}$ and $\varepsilon \in \Xi$. See 5] for details. The linear constraints $\left\|\left(\nabla^{k} \boldsymbol{c}\right)(\beta)\right\|_{\infty} \leq 1$ for $\beta \in K$ do not allow us to interpret the linear optimization problem
$\max$

$$
\sum_{\substack{\beta \in K}} \sum_{\substack{\mu \in \mathbb{N}_{0}^{s} \\|\mu|=k}} B_{k, j, \mu}^{[r]}\left(\varepsilon-M^{r} \beta\right)\left(\nabla_{1}^{\mu_{1}} \ldots \nabla_{s}^{\mu_{s}} \boldsymbol{c}\right)(\beta)
$$

subject to

$$
\begin{equation*}
-1 \leq\left(\nabla_{1}^{\mu_{1}} \ldots \nabla_{s}^{\mu_{s}} \boldsymbol{c}\right)(\beta) \leq 1, \quad \beta \in K, \quad \mu \in \mathbb{N}_{0}^{s}, \quad|\mu|=k \tag{19}
\end{equation*}
$$

as a network flow problem, compare with (5). However, we can still show that for each $r \in \mathbb{N}$ there exists an optimal mask $\boldsymbol{B}^{*}$ such that

$$
\left\|\left.S_{B_{k}}^{r}\right|_{\nabla^{k}}\right\|_{\infty}=\left\|\left.S_{B^{*}}\right|_{\nabla^{k}}\right\|_{\infty}=\left\|S_{B^{*}}\right\|_{\infty}
$$

and

$$
\nabla^{k} S_{\boldsymbol{A}}^{r}=S_{\boldsymbol{B}_{k}}^{r} \nabla^{k}=S_{\boldsymbol{B}^{*}} \nabla^{k}
$$

Denote by $\mathbf{1}$ and by $\mathbf{0}$ vectors of all ones and all zeros, respectively. The problem in (19) is equivalent to

$$
\begin{equation*}
z^{*}=\max \left\{d^{T}(\Delta x):-\mathbf{1} \leq \Delta x \leq \mathbf{1}\right\} \tag{20}
\end{equation*}
$$

with appropriately defined vector $d \in \mathbb{R}^{|N+k|^{s}}$ of the corresponding entries of $B_{k, j, \mu}^{[r]}\left(\varepsilon-M^{r} \beta\right)$ in the objective function and the matrix $\Delta$ reflecting the linear constraints in (19).

Theorem 15 There exists a vector $d^{*} \in \mathbb{R}^{N_{s, k}(N+k+1)^{s}}$ such that the solution of (20) satisfies

$$
z^{*}=\left\|d^{*}\right\|_{1} .
$$

Proof. Note that

$$
z^{*}=\max \left\{\left(d^{T} \Delta\right) x:\binom{\Delta}{-\Delta} x \leq\binom{\mathbf{1}}{-\mathbf{1}}\right\} .
$$

By duality [24], we get

$$
\begin{aligned}
z^{*} & =\min \left\{\mathbf{1}^{T}(w+y):\left(\begin{array}{ll}
w^{T} & y^{T}
\end{array}\right)\binom{\Delta}{-\Delta}=d^{T} \Delta, w, y \geq \mathbf{0}\right\} \\
& =\min \left\{\mathbf{1}^{T}(w+y):(d+y-w)^{T} \Delta=\mathbf{0}, w, y \geq \mathbf{0}\right\}
\end{aligned}
$$

Moreover, due to the fact that the supports of the optimal $w$ and $y$ are disjoint, we obtain

$$
\begin{align*}
z^{*} & =\min \left\{\mathbf{1}^{T}|g|:(d+g)^{T} \Delta=\mathbf{0}, g \in \mathbb{R}^{m}\right\} \\
& =\min \left\{\|d-g\|_{1}: g^{T} \Delta=\mathbf{0}, g \in \mathbb{R}^{m}\right\} \tag{21}
\end{align*}
$$

Define $d^{*}=d-g$.
Note that the entries of $d^{*}$ define the elements of the optimal mask $\boldsymbol{B}^{*}$ for the corresponding $r \in \mathbb{N}, j=1, \ldots, N_{s, k}$ and $\varepsilon \in \Xi$.

We would like to emphasize that the value $z^{*}$ coincides with the one determined by solving (19), which was already studied in [5] for $k \geq 1$ and in [6] for $k=1$. The equivalent formulation of (19) in (21) allows us not only to show the existence of optimal masks, but also yields its new, very intuitive interpretation: geometrically, $z^{*}$ is the distance from $d$ to the nullspace of $\Delta^{T}$ in the $\|\cdot\|_{1}$ norm.

## 6 Examples

In this section we illustrate our results with several examples. We give optimal masks for first and second difference schemes only for simplicity of presentation. The higher order difference masks can be determined analogously.

Example 16 Let $M=\left(\begin{array}{ll}2 & 1 \\ 0 & 2\end{array}\right)$. In this case $\Xi \simeq\{(0,0),(1,0),(1,1),(1,2)\}$.
Consider a scalar bivariate subdivision mask a given in terms of the associated symbol

$$
a(z)=\sum_{\alpha \in \mathbb{Z}^{2}} a(\alpha) z^{\alpha}=\frac{1}{4} b^{2}(z), \quad b(z)=\sum_{\varepsilon \in \Xi} z^{\varepsilon}, \quad z \in(\mathbb{C} \backslash\{0\})^{2} .
$$

The optimization problem in (21) implemented in Matlab yields an optimal mask $\boldsymbol{B}^{*} \in \ell_{0}^{2 \times 2}\left(\mathbb{Z}^{2}\right)$ given in terms of the associated matrix symbol

$$
B^{*}(z)=\frac{1}{4}\left(\begin{array}{ll}
b_{11}(z) & b_{12}(z) \\
b_{21}(z) & b_{22}(z)
\end{array}\right)
$$

with
$b_{11}(z)=\left(1+z_{1}\right)\left(1+2 z_{1} z_{2}+z_{1}^{2} z_{2}^{2}\right), \quad b_{12}(z)=0$,
$b_{21}(z)=-591 / 1739-z_{2}-709 / 1074 z_{1} z_{2}-709 / 1074 z_{1} z_{2}^{2}-z_{1}^{2} z_{2}^{2}-591 / 1739 z_{1}^{2} z_{2}^{3}$, $b_{22}(z)=1439 / 1074+2 z_{1}+709 / 1074 z_{1}^{2}+709 / 1074 z_{1} z_{2}+2 z_{1}^{2} z_{2}+1439 / 1074 z_{1}^{3} z_{2}$.

Note that there is also an optimal mask with integral $b_{i j}(z)$ given by
$b_{11}(z)=\left(1+z_{1}\right)\left(1+2 z_{1} z_{2}+z_{1}^{2} z_{2}^{2}\right), \quad b_{12}(z)=0$,
$b_{21}(z)=-z_{2}-z_{1} z_{2}^{2}-z_{1}^{2} z_{2}^{2}-z_{1}^{2} z_{2}^{3} \quad$ and $\quad b_{22}(z)=1+2 z_{1}+z_{1}^{2}+2 z_{1}^{2} z_{2}+2 z_{1}^{3} z_{2}$.
If we start the flow algorithm from section 4.3 with d derived from this optimal $\boldsymbol{B}^{*}$, we get $d^{*}=d$ as an output. For optimal masks we get

$$
\left\|S_{B^{*}}\right\|_{\infty}=\left\|\left.S_{B^{*}}\right|_{\nabla}\right\|_{\infty}=\frac{3}{4},
$$

which implies convergence of $S_{\boldsymbol{a}}$, i.e. continuity of its limits.
The next example is of a vector bivariate subdivision scheme introduced in [7. The corresponding dilation matrix is $M=2 I$ and $\Xi \simeq\{0,1\}^{2}$.

Example 17 We transform the mask in [7] following the steps in [5, Example 5.2] and obtain

$$
\begin{array}{cc}
\left(\begin{array}{ll}
1 & 1 \\
0 & 0
\end{array}\right) & \left(\begin{array}{ll}
2 & 1 \\
0 & 1
\end{array}\right)
\end{array} \begin{aligned}
& \left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right) \\
& \boldsymbol{A}=\frac{1}{8}\left(\begin{array}{ll}
1 & 1 \\
0 & 0
\end{array}\right) \\
& \left(\begin{array}{ll}
4 & 2 \\
0 & 2
\end{array}\right)
\end{aligned} \begin{array}{ll}
\left(\begin{array}{ll}
5 & 1 \\
0 & 3
\end{array}\right) & \left(\begin{array}{ll}
2 & 0 \\
0 & 1
\end{array}\right) \\
\left(\begin{array}{ll}
2 & 1 \\
0 & 1
\end{array}\right) & \left(\begin{array}{ll}
5 & 1 \\
0 & 3
\end{array}\right)
\end{array}\left(\begin{array}{ll}
4 & 0 \\
0 & 2
\end{array}\right)\left(\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right) .
$$

The optimal difference mask is given by its symbol $B^{*}(z)=\frac{1}{8}\left(b_{i j}(z)\right)_{1 \leq i, j \leq 4}$ with integral entries
$b_{11}(z)=\left(1+z_{2}\right)^{2}\left(1+z_{1}\right)\left(z_{2} z_{1}+1\right), \quad b_{12}(z)=\left(1-z_{1}\right) a_{11}(z)$,
$b_{13}(z)=b_{14}(z)=0, \quad b_{21}(z)=0, \quad b_{22}(z)=a_{22}(z), \quad b_{23}(z)=b_{24}(z)=0$,
$b_{31}(z)=0, \quad b_{12}(z)=\left(1-z_{2}\right) a_{11}(z), \quad b_{33}(z)=\left(1+z_{2}\right)\left(1+z_{1}\right)^{2}\left(z_{2} z_{1}+1\right)$,
$b_{34}(z)=0, \quad b_{41}(z)=0, \quad b_{42}(z)=a_{22}(z), \quad b_{43}(z)=b_{44}(z)=0$,

The entries $b_{11}, b_{13}, b_{31}$ and $b_{33}$ are computed using the optimization problem in (21) for the scalar mask given by $a_{11}(z)$, which is defined by the $A_{11}(\alpha)$ elements of $\boldsymbol{A}$. If we start the flow algorithm from section 4.3 with $d$ derived from this optimal $b_{11}, b_{13}, b_{31}$ and $b_{33}$, we get $d^{*}=d$ as an output. The rest of the entries in $B^{*}(z)$ are defined so that the associated operator $S_{B^{*}}$ satisfies

$$
\left(\begin{array}{cc}
\nabla_{1} & 0 \\
0 & 1 \\
\nabla_{2} & 0 \\
0 & 1
\end{array}\right) S_{\boldsymbol{A}}=S_{\boldsymbol{B}^{*}}\left(\begin{array}{cc}
\nabla_{1} & 0 \\
0 & 1 \\
\nabla_{2} & 0 \\
0 & 1
\end{array}\right)
$$

with $\nabla_{\ell}, \ell=1,2$, defined in (17), see [4, (5] for more details on the structure of the difference operator $\nabla$ and difference masks in the vector case. For the optimal mask $\boldsymbol{B}^{*}$ we have

$$
\left\|S_{B^{*}} \mid \nabla\right\|_{\infty}=\left\|S_{B^{*}}\right\|_{\infty}=\frac{3}{4}
$$

In the next example we determine an optimal second difference mask for the so-called butterfly scheme studied in e.g. [15].

Example 18 The dilation matrix is $M=2 I$ and the mask is given by its symbol

$$
a(z)=\frac{1}{2}\left(z_{1}+1\right)\left(z_{2}+1\right)\left(z_{1} z_{2}+1\right)\left(z_{1}^{2} z_{2}^{2}-\frac{1}{16} c(z)\right), \quad z \in(\mathbb{C} \backslash 0)^{2},
$$

where

$$
\begin{gathered}
c(z)=2 z_{2}+2 z_{1}-4 z_{1} z_{2}-4 z_{1} z_{2}^{2}-4 z_{1}^{2} z_{2}+2 z_{1} z_{2}^{3}+2 z_{1}^{3} z_{2}+12 z_{1}^{2} z_{2}^{2} \\
-4 z_{1}^{3} z_{2}^{2}-4 z_{1}^{2} z_{2}^{3}-4 z_{1}^{3} z_{2}^{3}+2 z_{1}^{4} z_{2}^{3}+2 z_{1}^{3} z_{2}^{4} .
\end{gathered}
$$

To show the $C^{1}$-regularity of the butterfly scheme by solving the optimization problem in (21), we have to determine an optimal mask for the third iteration of the second difference operator. An optimal mask for the second difference scheme is determined easily, if for its derivation, instead of $\nabla^{2}$ in (17), we make use of all three factors $\left(z_{1}+1\right)\left(z_{2}+1\right)\left(z_{1} z_{2}+1\right)$ as it is done in [15]. The diagonal structure of the symbol $B^{*}(z)=\frac{1}{16}\left(b_{i j}(z)\right)_{1 \leq i, j \leq 3}$ with non-zero elements
$b_{11}(z)=\left(1+z_{1}\right)^{-1}\left(1+z_{1} z_{2}\right)^{-1} A(z), \quad b_{22}(z)=\left(1+z_{1}\right)^{-1}\left(1+z_{2}\right)^{-1} A(z)$,
$b_{33}(z)=\left(1+z_{2}\right)^{-1}\left(1+z_{1} z_{2}\right)^{-1} A(z)$,
implies that the corresponding mask is optimal. This special structure of the symbol allows us to use the univariate strategy in section 4.1 to show the optimality of the mask. The iterates of $S_{B^{*}}$ are also optimal and $\left\|S_{B^{*}}^{2}\right\|_{\infty}<$ $1 / 2$ implies that the scheme is $C^{1}$.

The last example shows that although the optimal mask determined by solving the optimization problem in Theorem 15 can be non-integral, the optimal value $z^{*}$ still is. We were not able to find subdivision schemes with integral masks (after an appropriate normalization), which did not possess either integral optimal masks for higher order difference schemes or for which $z^{*}$ were not integral.

Example 19 Let $M=2 I$ and

$$
a(z)=4\left(\frac{1+z_{1}}{2}\right)\left(\frac{1+z_{2}}{2}\right)^{3}\left(\frac{1+z_{1} z_{2}}{2}\right)^{3}
$$

The associated bivariate scheme generates a three-directional box spline. Matlab yields $\left\|S_{B_{2}^{*}}\right\|_{\infty}=z^{*}=\frac{3}{8}<\frac{1}{2}$ implying the $C^{1}$-regularity of the scheme. The optimal second difference mask $2^{5} \boldsymbol{B}_{2}^{*} \in \ell_{0}^{3 \times 3}\left(\mathbb{Z}^{2}\right)$ satisfying (18) is not integral and we think it will serve no purpose to present it here. However, it allows us to derive another optimal second difference mask given by $B_{2}^{*}(z)=2^{-5}\left(b_{i j}(z)\right)_{i, j=1, \ldots, 3}$ with integral
$b_{11}(z)=z_{2}\left(z_{2}+1\right)\left(z_{1}^{2} z_{2}^{4}+2 z_{1}^{2} z_{2}^{3}+5 z_{1} z_{2}^{2}+z_{1}^{2} z_{2}^{2}+z_{2}+3 z_{1} z_{2}+3\right)$,
$b_{12}(z)=\left(1-z_{1}\right)\left(z_{1} z_{2}^{2}+1\right)^{2}, \quad b_{13}(z)=0, \quad b_{21}(z)=0$
$b_{22}(z)=\left(z_{2}+1\right)^{2}\left(z_{1} z_{2}+1\right)^{3}, \quad b_{31}(z)=z_{2}^{4}-z_{2}^{6}$,
$b_{32}(z)=z_{2}^{2}\left(-1+z_{1}^{2} z_{2}^{4}-z_{1}+z_{2}^{4} z_{1}-4 z_{1} z_{2}+4 z_{1} z_{2}^{3}\right)$,
$b_{33}(z)=(1+z 1)\left(z_{1}^{3} z_{2}^{3}+\left(2 z_{2}^{2}-z_{2}^{3}\right) z_{1}^{2}+\left(3 z_{2}^{2}+4 z_{2}^{3}+z_{2}^{4}+3 z_{2}\right) z_{1}+1+z_{2}^{2}+z_{2}\right)$.

## 7 Summary

In this paper we establish a link between convergence analysis of multivariate subdivision schemes and network flows as well as between the regularity analysis of subdivision and linear optimization. Advances in network flow theory and linear optimization provide efficient algorithms for determining what we call optimal difference masks. The regularity of the underlying subdivision scheme can be easily read off the corresponding optimal values, which determine the norm of the difference operators defined by such optimal difference masks. We would like to emphasize that we only prove the existence of the optimal masks, which are by no means unique. The existence of the optimal masks shows that there are no conceptual differences in the analysis of multivariate and univariate subdivision schemes. Moreover, if the subdivision mask has only rational entries, then so does the first difference optimal mask. There are no theoretical results that guarantee the same in the case of higher order difference masks, but we were not able to find an example of a subdivision scheme with rational entries whose higher order difference schemes would be irrational.

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