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Published in:
Advances in Computational Mathematics

Link to article, DOI:
10.1007/s10444-016-9492-x

Publication date:
2016

Document Version
Peer reviewed version

Explicit constructions and properties of generalized shift-invariant systems in $L^2(\mathbb{R})$

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October 19, 2016

Abstract: Generalized shift-invariant (GSI) systems, originally introduced by Hernández, Labate & Weiss and Ron & Shen, provide a common frame work for analysis of Gabor systems, wavelet systems, wave packet systems, and other types of structured function systems. In this paper we analyze three important aspects of such systems. First, in contrast to the known cases of Gabor frames and wavelet frames, we show that for a GSI system forming a frame, the Calderón sum is not necessarily bounded by the lower frame bound. We identify a technical condition implying that the Calderón sum is bounded by the lower frame bound and show that under a weak assumption the condition is equivalent with the local integrability condition introduced by Hernández et al. Second, we provide explicit and general constructions of frames and dual pairs of frames having the GSI-structure. In particular, the setup applies to wave packet systems and in contrast to the constructions in the literature, these constructions are not based on characteristic functions in the Fourier domain. Third, our results provide insight into the local integrability condition (LIC).

1 Introduction

Generalized shift-invariant systems provide a common framework for analysis of a large class of function systems in $L^2(\mathbb{R})$. Defining the translation operators $T_c, c \in \mathbb{R}$, by $T_c f(x) = f(x-c)$, a *generalized shift-invariant (GSI) system* has the form $\{T_{c_j k} g_j\}_{k\in\mathbb{Z}, j\in J}$, where $\{c_j\}_{j\in J}$ is a countable set in $\mathbb{R}_+$ and $g_j \in L^2(\mathbb{R})$. GSI systems were introduced by Hernández, Labate & Weiss [14], and Ron & Shen [21].

In the analysis of a GSI system, the function $\sum_{j\in J} c_j^{-1} |\hat{g}_j(\cdot)|^2$, which we will call the Calderón sum in analogue with the standard terminology used in the special case of a wavelet system, plays an important role. Intuitively, the Calderón sum measures the total energy concentration of the generators g_j in the frequency domain. Hence, whenever a GSI system has the frame property, one would expect the Calderón sum to be bounded from below since the GSI frame can reproduce all frequencies in a stable way. Indeed, whenever a Gabor frame or a wavelet frame is considered as a GSI system in the natural way (see the details below), it is known that the Calderón sum is bounded above and below by the upper and lower frame

2010 *Mathematics Subject Classification.* Primary 42C15, 43A70, Secondary: 43A32.

Key words and phrases. Dual frames, frame, Gabor system, Generalized shift invariant system, LIC, local integrability condition, wave packets, wavelets

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bounds, respectively. In the general case of a GSI system the Calderón sum is known to be bounded above by the upper frame bound. In this paper we prove by an example that the Calderón sum is not always bounded below by the lower frame bound. On the other hand, we identify a technical condition implying that the Calderón sum is bounded by the lower frame bound. Under a weak assumption, this condition is proved to be equivalent with the local integrability condition introduced by Hernández et. al. [14].

Our second main contribution is to provide constructions of pairs of dual frames having the GSI structure. The construction procedure allows for smooth and well-localized generators, and it unifies several known constructions of dual frames with Gabor, wavelet, and so-called Fourier-like structure [6,7,9,19,20]. Due to its generality the setup is technical, but nevertheless it is possible to extract attractive new constructions, as we will explain below.

We will apply our results for GSI systems on the important special case of wave packet systems. We consider necessary and sufficient conditions for frame properties of wave packet systems. In particular, by the just mentioned construction procedure, we obtain dual pairs of wave packet frames that are not based on characteristic functions in the Fourier domain. Recall that a wave packet system is a the collection of functions that arises by letting a class of translation, modulation, and scaling operators act on a fixed function $\psi \in L^2(\mathbb{R})$. The precise setup is as follows. Given $a \in \mathbb{R}$, we define the modulation operator $(E_a f)(x) = \mathrm{e}^{2\pi i a x} f(x)$, and (for $a > 0$) the scaling operator $(D_a f)(x) = a^{-1/2} f(x/a)$; these operators are unitary on $L^2(\mathbb{R})$. Let $b > 0$ and $\{(a_j, d_j)\}_{j\in J}$ be a countable set in "scale/frequency" space $\mathbb{R}^+ \times \mathbb{R}$. The *wave packet system* generated by a function $\psi \in L^2(\mathbb{R})$ is the collection of functions $\{D_{a_j} T_{bk} E_{d_j} \psi\}_{k\in\mathbb{Z}, j\in J}$.

The key feature of wave packet system is that it allows us to *combine* the Gabor structure and the wavelet structure into one system that yields a very flexible analysis of signals. For the particular parameter choice $(a_j, d_j) := (a^j, 1)$ for $j \in J = \mathbb{Z}$ and some $a \neq 0$, the wave packet system $\{D_{a_j} T_{bk} E_{d_j} \psi\}_{k\in\mathbb{Z}, j\in J}$ simply becomes the wavelet system $\{D_{a^j} T_{kb} \psi\}_{j,k\in\mathbb{Z}}$ generated by the function $\psi \in L^2(\mathbb{R})$. On the other hand, for the choice $(a_j, d_j) := (1, aj)$ for $j \in J = \mathbb{Z}$ and some $a > 0$, we recover the system $\{T_{bk} E_{aj} \psi\}_{j,k\in\mathbb{Z}}$ which is unitarily equivalent with the Gabor system $\{E_{aj} T_{bk} \psi\}_{j,k\in\mathbb{Z}}$. Hence, we can consider both wavelet and Gabor systems as special cases of wave packet systems. Furthermore, other choices of the parameters $\{(a_j, d_j)\}_{j\in J}$, which intuitively control how the scale/frequency information of a signal is analyzed, combine Gabor and wavelet structure. Finally, the translations by $b\mathbb{Z}$ allow for time localization of the wave packet atom.

The generality of GSI systems is known to lead to some technical issues. Indeed, local integrability conditions play an important role in the theory of GSI systems as a mean to control the interplay between the translation lattices $c_j\mathbb{Z}$ and the generators g_j, $j \in J$. Our third main contribution is new insight into the role of local integrability conditions. In particular, we will see that local integrability conditions also play an important role for wave packet systems, and that it is important to distinguish between the so-called local integrability condition (LIC) and the weaker α-LIC. This is in sharp contrast to the case of Gabor and wavelet systems in $L^2(\mathbb{R})$, where one largely can ignore local integrability conditions.

The paper is organized as follows. In Section 2, we introduce the theory of GSI systems and extend the well-known duality conditions to certain subspaces of $L^2(\mathbb{R})$. In Section 3 we discuss various technical conditions under which the Calderón sum for a GSI frame is bounded below by the lower frame bound; applications to wavelet systems and Gabor systems are considered in Section 4. In Section 5 we provide explicit constructions of dual GSI frames for certain subspaces of $L^2(\mathbb{R})$. The general version of the result is technical, but we are nevertheless able to provide concrete realizations of the results. Finally, Section 6 applies the key results of the paper to wave packet systems. In particular we show that a successful analysis of such systems must be based on the α-LIC rather than the LIC. Furthermore, we provide explicit constructions of dual pairs of wave packet frames.

We end this introduction by putting our work in a perspective with other known results. Córdoba and Fefferman [11] considered continuous wave packet transforms in $L^2(\mathbb{R}^n)$ generated by the gaussian which is well localized in time and frequency. The results in [11] yield *approximate* reproducing formulas. In [15,18] the authors constructs frequency localized wave packet systems associated with exact reproducing formulas in terms of Parseval frames. However, these generators are poorly localized in time as the generators are characteristic functions in the frequency space. In this work we construct wave packet dual frames well localized in time and frequency.

For an introduction to frame theory we refer to the books [4,12,13].

2 Preliminary results on GSI systems

To set the stage, we will recall and extend some of the most important results on GSI systems. Let J be a finite or a countable index set. As already mentioned in the introduction, analysis of GSI systems $\{T_{c_jk}g_j\}_{k\in\mathbb{Z},j\in J}$ cover several of the cases considered in the literature. In case $c_j = c > 0$ for all $j \in J$, the system $\{T_{ck}g_j\}_{k\in\mathbb{Z},j\in J}$ is a shift invariant (SI) system; if one further takes $g_j = E_{aj}g, j \in J := \mathbb{Z}$ for some $a > 0$ and $g \in L^2(\mathbb{R})$, we recover the Gabor case. The wavelet system $\{D_{a^j}T_{bk}\psi\}_{j,k\in\mathbb{Z}}$ with $a > 0$ and $b > 0$ can naturally be represented as a GSI system via

$$\{D_{a^j}T_{bk}\psi\}_{j,k\in\mathbb{Z}} = \{T_{c_jk}g_j\}_{k\in\mathbb{Z},j\in J} \quad \text{with } c_j = a^jb,\ g_j = D_{a^j}\psi,\ \text{for } j \in J = \mathbb{Z}. \tag{2.1}$$

Note that this representation is non-unique, hence unless it is clear from the context, we will always specify the choice of c_j and g_j, $j \in J$.

The upper bound of the Calderón sum for GSI systems obtained by Hernández, Labate, and Weiss [14] only relies on the Bessel property. The precise statement is as follows.

Theorem 2.1 ([14])**.** *Suppose the GSI system $\{T_{c_jk}g_j\}_{k\in\mathbb{Z},j\in J}$ is a Bessel sequence with bound B. Then*

$$\sum_{j\in J}\frac{1}{c_j}|\hat{g}_j(\gamma)|^2 \le B \qquad \textit{for a.e. } \gamma \in \mathbb{R}. \tag{2.2}$$

Here, for $f \in L^1(\mathbb{R})$, the Fourier transform is defined as

$$\hat{f}(\gamma) = \int_{\mathbb{R}} f(x)\, \mathrm{e}^{-2\pi i\gamma x}\, \mathrm{d}x$$

with the usual extension to $L^2(\mathbb{R})$.

In the special cases where $\{T_{c_jk}g_j\}_{k\in\mathbb{Z},j\in J}$ is a Gabor frame or a wavelet frame with lower frame bound A, it is known that A is also a lower bound on the sum in (2.2). For instance, for a wavelet frame $\{D_{a^j}T_{bk}\psi\}_{j,k\in\mathbb{Z}} = \{T_{a^jbk}D_{a^j}\psi\}_{j,k\in\mathbb{Z}}$ with bounds A and B, Chui and Shi [10] proved that

$$A \le \sum_{j\in J}\frac{1}{b}|\hat{\psi}(a^j\gamma)|^2 \le B \qquad \text{for a.e. } \gamma \in \mathbb{R}. \tag{2.3}$$

2.1 Frame theory for GSI systems

We will consider GSI frames $\{T_{c_j k} g_j\}_{k\in\mathbb{Z},j\in J}$ for certain closed subspaces of $L^2(\mathbb{R})$. To this end, for a measurable subset S of $\mathbb{R}$ we define

$$\check{L}^2(S) := \left\{ f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \subset S \right\}.$$

The set S is usually some collection of frequency bands that are of interest. In case one is not interested in subspaces of $L^2(\mathbb{R})$, simply set $S=\mathbb{R}$. If S is chosen to be a finite, symmetric interval around the origin, we obtain the important special case of Paley-Wiener spaces. We will always assume that the generators g_j satisfy that $\operatorname{supp} \hat{g}_j \subset S$ for every $j \in J$. Note that this guarantees that the GSI system $\{T_{c_j k} g_j\}_{k\in\mathbb{Z},j\in J}$ belongs to $\check{L}^2(S)$.

In order to check that a GSI system is a frame for $\check{L}^2(S)$ it is enough to check the frame condition on a dense set in $\check{L}^2(S)$. Depending on the given GSI system, we will fix a measurable set $E \subset S$ whose closure has measure zero and define the subspace $\mathcal{D}_E$ by

$$\mathcal{D}_E := \left\{ f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \subset S \setminus E \text{ is compact and } \hat{f} \in L^\infty(\mathbb{R}) \right\}.$$

For example, for a Gabor system $\{E_{mb} T_{na} g\}_{m,n\in\mathbb{Z}}$ we can take E to be the empty set, and for a wavelet system $\{D_{a^j} T_{bk} \psi\}_{j,k\in\mathbb{Z}}$ we take $E=\{0\}$.

In order to consider frame properties for GSI systems we will need a local integrability condition, introduced in [14] and generalized in [16].

Definition 2.2. Consider a GSI system $\{T_{c_j k} g_j\}_{k\in\mathbb{Z},j\in J}$ and let $E \in \mathcal{E}$, where $\mathcal{E}$ denotes the set of measurable subsets of $S \subset \mathbb{R}$ whose closure has measure zero.

(i) If

$$L(f) := \sum_{j\in J} \sum_{m\in\mathbb{Z}} \frac{1}{c_j} \int_{\operatorname{supp} \hat{f}} |\hat{f}(\gamma + c_j^{-1} m) \hat{g}_j(\gamma)|^2 \, \mathrm{d}\gamma < \infty \tag{2.4}$$

for all $f \in \mathcal{D}_E$, we say that $\{T_{c_j k} g_j\}_{k\in\mathbb{Z},j\in J}$ satisfies the local integrability condition (LIC) with respect to the set E.

(ii) $\{T_{c_j k} g_j\}_{k\in\mathbb{Z},j\in J}$ and $\{T_{c_j k} h_j\}_{k\in\mathbb{Z},j\in J}$ satisfy the *dual* α*-LIC* with respect to E if

$$L'(f) := \sum_{j\in J} \sum_{m\in\mathbb{Z}} \frac{1}{c_j} \int_{\mathbb{R}} |\hat{f}(\gamma) \overline{\hat{f}(\gamma + c_j^{-1} m) \hat{g}_j(\gamma)} \hat{h}_j(\gamma + c_j^{-1} m)| \, \mathrm{d}\gamma < \infty \tag{2.5}$$

for all $f \in \mathcal{D}_E$. We say that $\{T_{c_j k} g_j\}_{k\in\mathbb{Z},j\in J}$ satisfies the α-LIC with respect to E, if (2.5) holds with $g_j = h_j$, $j \in J$.

By an application of the Cauchy-Schwarz inequality, we see that if the local integrability condition holds, then the α-local integrability condition also holds. Clearly, if a local integrability condition holds with respect to $E=\emptyset$, it holds with respect to any $E \in \mathcal{E}$.

In [14] it is shown that any Gabor system satisfies the LIC for $E=\emptyset$. To arrive at the same conclusion for SI systems, it suffices to assume that the system is a Bessel sequence, see [16]. In fact, it is not difficult to show the following more general result.

Lemma 2.3. *Consider the SI system* $\{T_{ck} g_j\}_{k\in\mathbb{Z},j\in J}$*, and let* $E \in \mathcal{E}$*. Then* $\{T_{ck} g_j\}_{k\in\mathbb{Z},j\in J}$ *satisfies the LIC with respect to E if and only if the Calderón sum for* $\{T_{ck} g_j\}_{k\in\mathbb{Z},j\in J}$ *is locally integrable on* $\mathbb{R} \setminus E$*, i.e.,*

$$\sum_{j\in J} \frac{1}{c} |\hat{g}_j(\cdot)|^2 \in L^1_{\mathrm{loc}}(\mathbb{R} \setminus E). \tag{2.6}$$

Of course, one can leave out the factor $\frac{1}{c}$ in the Calderón sum in (2.6). Note that if $\{T_{ck}g_j\}_{k\in\mathbb{Z},j\in J}$ is a Bessel sequence, then, by Theorem 2.1, the Calderón sum satisfies (2.6) for any $E \in \mathcal{E}$. Similarly, it was shown in [2] that a wavelet system $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ satisfies the LIC with respect to $E = \{0\}$ if and only if

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi}(a^j\cdot)|^2 \in L^1_{\mathrm{loc}}(\mathbb{R}\setminus\{0\}).$$

Hernández, Labate and Weiss [14] characterized duality for two GSI systems satisfying the LIC. In [16] Jakobsen and Lemvig generalized this to not necessarily discrete GSI systems defined on a locally compact abelian group and satisfying the weaker α-LIC. The following generalization to discrete GSI system in $\check{L}^2(S)$ follows the original proofs closely, so we only sketch the proof.

Theorem 2.4. *Let $S \subset \mathbb{R}$. Suppose that $\{T_{c_jk}g_j\}_{k\in\mathbb{Z},j\in J}$ and $\{T_{c_jk}h_j\}_{k\in\mathbb{Z},j\in J}$ are Bessel sequences in $\check{L}^2(S)$ satisfying the dual α-LIC for some $E \in \mathcal{E}$. Then $\{T_{c_jk}g_j\}_{k\in\mathbb{Z},j\in J}$ and $\{T_{c_jk}h_j\}_{k\in\mathbb{Z},j\in J}$ are dual frames for $\check{L}^2(S)$ if and only if*

$$\sum_{j\in J:\alpha\in c_j^{-1}\mathbb{Z}} \frac{1}{c_j}\,\overline{\hat{g}_j(\gamma)}\,\hat{h}_j(\gamma+\alpha) = \delta_{\alpha,0}\,\chi_S(\gamma) \qquad a.e.\, \gamma\in\mathbb{R} \tag{2.7}$$

for all $\alpha \in \bigcup_{j\in J} c_j^{-1}\mathbb{Z}$.

Proof. For simplicity assume that $E = \emptyset$; the case of general E only requires few modifications of the following proof. For $f \in \mathcal{D}_E$ define the function $w_f : \mathbb{R} \to \mathbb{C}$ by

$$w_f(x) = \sum_{j\in J}\sum_{k\in\mathbb{Z}} \langle T_x f, T_{c_jk}g_j\rangle \langle T_{c_jk}h_j, T_x f\rangle. \tag{2.8}$$

By [14, Proposition 9.4] (the given proof also hold with the α-LIC replacing the LIC) we know that w_f is a continuous, almost periodic function that coincides pointwise with its absolutely convergent Fourier series

$$w_f(x) = \sum_{\alpha\in \bigcup_{j\in J} c_j^{-1}\mathbb{Z}} d_\alpha\, \mathrm{e}^{2\pi i\alpha x}, \tag{2.9}$$

where

$$d_\alpha = \int_{\mathbb{R}} \hat{f}(\gamma)\overline{\hat{f}(\gamma+\alpha)}\, t_\alpha(\gamma) d\gamma. \tag{2.10}$$

and $t_\alpha(\gamma)$ denotes the left hand side of (2.7).

Assume that (2.7) holds. Inserting $t_\alpha(\gamma) = \delta_{\alpha,0}\,\chi_S(\gamma)$ into (2.9) for $x = 0$ yields

$$\sum_{j\in J}\sum_{k\in\mathbb{Z}} \langle f, T_{c_jk}g_j\rangle \langle T_{c_jk}h_j, f\rangle = w_f(0) = \int_S \left|\hat{f}(\gamma)\right|^2 d\gamma = \|f\|^2.$$

By a standard density argument, this completes the proof of the "if"-direction.

Assume now that $\{T_{c_jk}g_j\}_{k\in\mathbb{Z},j\in J}$ and $\{T_{c_jk}h_j\}_{k\in\mathbb{Z},j\in J}$ are dual frames for $\check{L}^2(S)$. Then $w_f(x) = \|f\|^2$ for all $f \in \mathcal{D}_E$ and all $x \in \mathbb{R}$. By uniqueness of Fourier coefficients of almost periodic functions, this only happens if, for $\alpha \in \bigcup_{j\in J} c_j^{-1}\mathbb{Z}$,

$$d_0 = \|f\|^2 \qquad \text{and} \qquad d_\alpha = 0 \quad \text{for } \alpha \neq 0. \tag{2.11}$$

Since $\mathcal{D}_E$ is dense in $\check{L}^2(S)$, it follows from $d_0 = \|f\|^2$ that $t_0(\gamma) = 1$ for a.e. $\gamma \in S$.

Assume that $\alpha = c_j^{-1}k$ for some $j \in J$ and $k \in \mathbb{Z} \setminus \{0\}$. For each $\ell \in \mathbb{Z}$, take

$$\hat{f}(\gamma) = \begin{cases} 1 & \text{for } \gamma \in \left[c_j^{-1}\ell, c_j^{-1}(\ell+1)\right] \cap S, \\ t_\alpha(\gamma) & \text{for } \gamma \in \left[c_j^{-1}\ell - \alpha, c_j^{-1}(\ell+1) - \alpha\right] \cap S, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f \in \mathcal{D}_E$ and

$$0 = \int_{\mathbb{R}} \hat{f}(\gamma)\overline{\hat{f}(\gamma+\alpha)}\, t_\alpha(\gamma)\, d\gamma = \int_{[c_j^{-1}\ell, c_j^{-1}(\ell+1)]\cap S} |t_\alpha(\gamma)|^2\, d\gamma.$$

Since $\ell \in \mathbb{Z}$ was arbitrarily chosen, we deduce that $t_\alpha(\gamma)$ vanishes almost everywhere for $\gamma \in S$.

For a.e. $\gamma \notin S$ the assumption $\operatorname{supp} \hat{g}_j \subset S$ implies that $t_\alpha(\gamma) = 0$ for any α. Summarizing, we have shown that $t_\alpha(\gamma) = \delta_{\alpha,0}\, \chi_S(\gamma)$ for a.e. $\gamma \in \mathbb{R}$. □

In the characterization of tight frames, we can leave out the Bessel condition.

Theorem 2.5. *Let $S \subset \mathbb{R}$ and $A > 0$. Suppose that the GSI system $\{T_{c_jk}g_j\}_{k\in\mathbb{Z},j\in J}$ satisfies the α-LIC condition for some $E \in \mathcal{E}$. Then $\{T_{c_jk}g_j\}_{k\in\mathbb{Z},j\in J}$ is a tight frame for $\check{L}^2(S)$ with frame bound A if and only if*

$$\sum_{j\in J:\alpha\in c_j^{-1}\mathbb{Z}} \frac{1}{c_j} \overline{\hat{g}_j(\gamma)}\, \hat{g}_j(\gamma+\alpha) = A\, \delta_{\alpha,0}\, \chi_S(\gamma) \qquad a.e.\ \gamma \in \mathbb{R}$$

for all $\alpha \in \bigcup_{j\in J} c_j^{-1}\mathbb{Z}$.

For tight frames, Theorem 2.5 gives information about the Calderón sum. Indeed, it follows immediately from Theorem 2.5 that if $\{T_{c_jk}g_j\}_{k\in\mathbb{Z},j\in J}$ is a tight frame with constant A satisfying the α-local integrability condition, then

$$\sum_{j\in J} \frac{1}{c_j} |\hat{g}_j(\gamma)|^2 = A \qquad \text{a.e. } \gamma \in S.$$

Finally, the following result allows us to construct frames without worrying about technical local integrability conditions. In fact, the condition (2.12) below implies that the α-LIC with respect to any set $E \in \mathcal{E}$ is satisfied.

Theorem 2.6. *Consider the generalized shift invariant system $\{T_{c_jk}g_j\}_{k\in\mathbb{Z},j\in J}$.*

(i) If

$$B := \operatorname*{ess\,sup}_{\gamma\in S} \sum_{j\in J}\sum_{m\in\mathbb{Z}} \frac{1}{c_j} \left|\hat{g}_j(\gamma)\hat{g}_j(\gamma + c_j^{-1}m)\right| < \infty, \tag{2.12}$$

then $\{T_{c_jk}g_j\}_{k\in\mathbb{Z},j\in J}$ is a Bessel sequence in $\check{L}^2(S)$ with bound B.

(ii) Furthermore, if also

$$A := \operatorname*{ess\,inf}_{\gamma\in S} \Big(\sum_{j\in J} \frac{1}{c_j} |\hat{g}_j(\gamma)|^2 - \sum_{j\in J}\sum_{0\neq m\in\mathbb{Z}} \frac{1}{c_j} \left|\hat{g}_j(\gamma)\hat{g}_j(\gamma + c_j^{-1}m)\right| \Big) > 0,$$

then $\{T_{c_jk}g_j\}_{k\in\mathbb{Z},j\in J}$ is a frame for $\check{L}^2(S)$ with bounds A and B.

The proof of Theorem 2.6 is a straightforward modification of the standard proof for $L^2(\mathbb{R})$ (see, e.g., [5, 8, 16]).

3 A lower bound of the Calderón sum for GSI systems

Following a construction by Bownik and Rzeszotnik [3] and Kutyniok and Labate [17], we first show that the Calderón sum for arbitrary GSI frames is not necessarily bounded from below by the lower frame bound.

Example 1. Consider the orthonormal basis $\{E_k T_m \chi_{[0,1[}\}_{k,m\in\mathbb{Z}}$, an integer $N \geq 3$, and the lattices $\Gamma_j = N^j\mathbb{Z}$, $j \in \mathbb{N}$. There exists a sequence $\{t_i\}_{i=1}^{\infty}$ such that

$$\bigcup_{i=1}^{\infty}(t_i+\Gamma_i)=\mathbb{Z}, \quad (t_i+\Gamma_i)\cap(t_j+\Gamma_j)=\emptyset \quad \text{for } i\neq j,$$

i.e., $\mathbb{Z}$ can be decomposed into translates of the sparser lattices Γ_j. It follows that

$$\{\mathcal{F}^{-1}E_kT_m\chi_{[0,1[}\}_{k,m\in\mathbb{Z}} = \{T_kE_m\mathcal{F}^{-1}\chi_{[0,1[}\}_{k,m\in\mathbb{Z}} = \{T_{N^jk}T_{t_j}E_m\mathcal{F}^{-1}\chi_{[0,1[}\}_{k,m\in\mathbb{Z},j\in\mathbb{N}}.$$

Hence, the GSI system defined by

$$c_{(j,m)} = N^j \text{ and } g_{(j,m)} = T_{t_j}E_m\mathcal{F}^{-1}\chi_{[0,1[}, \quad \text{for } (j,m)\in J=\mathbb{N}\times\mathbb{Z},$$

is an orthonormal basis and therefore, in particular, a Parseval frame. Since

$$\begin{aligned}\sum_{j=1}^{\infty}\sum_{m\in\mathbb{Z}}\frac{1}{c_{(j,m)}}\left|\hat{g}_{(j,m)}(\gamma)\right|^2 &= \sum_{j=1}^{\infty}\sum_{m\in\mathbb{Z}}\frac{1}{N^j}|\mathcal{F}T_{t_j}E_m\mathcal{F}^{-1}\chi_{[0,1[}(\gamma)|^2\\ &= \sum_{j=1}^{\infty}\sum_{m\in\mathbb{Z}}\frac{1}{N^j}|E_{t_j}\chi_{[m,m+1)}(\gamma)|^2 = \sum_{j=1}^{\infty}\frac{1}{N^j}|\chi_{\mathbb{R}(\gamma)}|^2 = \frac{1}{N-1},\end{aligned}$$

we conclude that the Calderón sum (2.2) is not bounded below by the lower frame bound $A = 1$ whenever $N \geq 3$. On the other hand, we see that the Calderón sum is indeed bounded from above by the (upper) frame bound 1 as guaranteed by Theorem 2.1. ■

We will now provide a technical condition on a frame $\{T_{c_jk}g_j\}_{k\in\mathbb{Z},j\in J}$ that implies that the Calderón sum in (2.2) is bounded by the lower frame bound. The proof generalizes the argument in [10].

Theorem 3.1. *Assume that*

$$\sum_{j\in J}|\hat{g}_j(\cdot)|^2 \in L^1_{\mathrm{loc}}(\mathbb{R}\setminus E). \tag{3.1}$$

If the GSI system $\{T_{c_jk}g_j\}_{k\in\mathbb{Z},j\in J}$ is a frame for $L^2(\mathbb{R})$ with lower bound A, then

$$A \leq \sum_{j\in J}\frac{1}{c_j}|\hat{g}_j(\gamma)|^2 \qquad a.e.\, \gamma\in\mathbb{R}. \tag{3.2}$$

Proof. The assumption that the function $\sum_{j\in J}|\hat{g}_j(\cdot)|^2$ is locally integrable in $\mathbb{R}\setminus E$ implies, by the Lebesgue differentiation theorem, that the set of its Lebesgue point is dense in $\mathbb{R}\setminus E$. Let $\omega_0 \in \mathbb{R}\setminus\overline{E}$ be a Lebesgue point. Choose $\varepsilon' > 0$ such that $[\omega_0-\varepsilon', \omega_0+\varepsilon'] \subseteq \mathbb{R}\setminus\overline{E}$. By assumption, we have

$$\int_{\omega_0-\varepsilon'}^{\omega_0+\varepsilon'}\sum_{j\in J}|\hat{g}_j(\gamma)|^2\,\mathrm{d}\gamma < \infty.$$

This means that for every $\eta > 0$, there exists a finite set $J' \subset J$ such that

$$\sum_{j \in J \setminus J'} \int_{\omega_0 - \varepsilon'}^{\omega_0 + \varepsilon'} |\hat{g}_j(\gamma)|^2 \, \mathrm{d}\gamma < \eta. \tag{3.3}$$

Now define $M := \max_{j \in J'} c_j$. Let $0 < \varepsilon < \min\{\varepsilon', \frac{M^{-1}}{2}\}$. It is clear that (3.3) also holds for every $\varepsilon > 0$ with $\varepsilon < \varepsilon'$.

Using Lemma 20.2.3 of [4], we have, for $f \in \mathcal{D}_E$,

$$A\|f\|^2 \leq \sum_{j \in J} \sum_{k \in \mathbb{Z}} \frac{1}{c_j} \int_{\mathbb{R}} \hat{f}(\gamma) \overline{\hat{f}(\gamma + c_j^{-1}k) \hat{g}_j(\gamma)} \hat{g}_j(\gamma + c_j^{-1}k) \, \mathrm{d}\gamma. \tag{3.4}$$

Consider $\hat{f} = \frac{1}{\sqrt{2\varepsilon}} \chi_K$, where $K = [\omega_0 - \varepsilon, \omega_0 + \varepsilon]$. Since $f \in \mathcal{D}_E$, by inequality (3.4), we have

$$\begin{aligned}
A &\leq \sum_{j \in J} \sum_{k \in \mathbb{Z}} \frac{1}{2\varepsilon c_j} \int_{K \cap (K - c_j^{-1}k)} \overline{\hat{g}_j(\gamma)} \hat{g}_j(\gamma + c_j^{-1}k) \, \mathrm{d}\gamma \\
&= \sum_{j \in J'} \sum_{k \in \mathbb{Z}} \frac{1}{2\varepsilon c_j} \int_{K \cap (K - c_j^{-1}k)} \overline{\hat{g}_j(\gamma)} \hat{g}_j(\gamma + c_j^{-1}k) d\gamma \\
&\quad + \sum_{j \in J \setminus J'} \sum_{k \in \mathbb{Z}} \frac{1}{2\varepsilon c_j} \int_{K \cap (K - c_j^{-1}k)} \overline{\hat{g}_j(\gamma)} \hat{g}_j(\gamma + c_j^{-1}k) \, \mathrm{d}\gamma \\
&=: S_1 + S_2.
\end{aligned}$$

For $j \in J'$, we have $2\varepsilon < M^{-1} \leq c_j^{-1}|k|$ for all $k \in \mathbb{Z} \setminus \{0\}$. Hence

$$K \cap (K - c_j^{-1}k) = \emptyset, \quad k \in \mathbb{Z} \setminus \{0\}, j \in J'.$$

Therefore

$$S_1 = \sum_{j \in J'} \frac{1}{2\varepsilon c_j} \int_K |\hat{g}_j(\gamma)|^2 \, \mathrm{d}\gamma. \tag{3.5}$$

In particular, S_1 is a non-negative number. For $j \in J \setminus J'$, $K \cap (K - c_j^{-1}k) = \emptyset$ if $|k| > 2\varepsilon c_j$. Therefore,

$$S_2 \leq \sum_{j \in J \setminus J'} \sum_{|k| \leq 2\varepsilon c_j} \frac{1}{2\varepsilon c_j} \int_{K \cap (K - c_j^{-1}k)} |\overline{\hat{g}_j(\gamma)} \hat{g}_j(\gamma + c_j^{-1}k)| \, \mathrm{d}\gamma.$$

Now by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
S_2 &\leq \sum_{j \in J \setminus J'} \sum_{|k| \leq 2\varepsilon c_j} \frac{1}{2\varepsilon c_j} \Big(\int_{K \cap (K - c_j^{-1}k)} |\hat{g}_j(\gamma)|^2 \, \mathrm{d}\gamma \Big)^{\frac{1}{2}} \Big(\int_{(K + c_j^{-1}k) \cap K} |\hat{g}_j(\gamma)|^2 \, \mathrm{d}\gamma \Big)^{\frac{1}{2}} \\
&\leq \sum_{j \in J \setminus J'} (4\varepsilon c_j + 1) \frac{1}{2\varepsilon c_j} \int_K |\hat{g}_j(\gamma)|^2 \, \mathrm{d}\gamma \\
&= 2 \sum_{j \in J \setminus J'} \int_K |\hat{g}_j(\gamma)|^2 \, \mathrm{d}\gamma + \sum_{j \in J \setminus J'} \frac{1}{2\varepsilon c_j} \int_K |\hat{g}_j(\gamma)|^2 \, \mathrm{d}\gamma \\
&\leq 2\eta + \sum_{j \in J \setminus J'} \frac{1}{2\varepsilon c_j} \int_K |\hat{g}_j(\gamma)|^2 \, \mathrm{d}\gamma.
\end{aligned} \tag{3.6}$$

Since $A \le S_1 + S_2$, it follows from (3.5) and (3.6) that

$$\begin{aligned} A &\le \sum_{j\in J'} \frac{1}{2\varepsilon c_j} \int_K |\hat{g}_j(\gamma)|^2 \,\mathrm{d}\gamma + 2\eta + \sum_{j\in J\setminus J'} \frac{1}{2\varepsilon c_j} \int_K |\hat{g}_j(\gamma)|^2 \,\mathrm{d}\gamma \\ &= \sum_{j\in J} \frac{1}{2\varepsilon c_j} \int_K |\hat{g}_j(\gamma)|^2 \,\mathrm{d}\gamma + 2\eta \\ &= \frac{1}{2\varepsilon} \int_{\omega_0-\varepsilon}^{\omega_0+\varepsilon} \sum_{j\in J} \frac{1}{c_j} |\hat{g}_j(\gamma)|^2 \,\mathrm{d}\gamma + 2\eta. \end{aligned}$$

Letting $\varepsilon \to 0$, we arrive at

$$A \le \sum_{j\in J} \frac{1}{c_j} |\hat{g}_j(\omega_0)|^2 + 2\eta.$$

Since $\eta > 0$ is arbitrary, the proof is complete. □

In order to check the condition (3.1), it is enough to consider the partial sum over the $j \in J$ for which $c_j > M$ for some $M > 0$:

Proposition 3.2. *Suppose that $\{T_{c_j k} g_j\}_{k\in\mathbb{Z}, j\in J}$ is a Bessel sequence with bound B. Then* (3.1) *holds if and only if there exist some $M > 0$ such that*

$$\sum_{\{j:c_j>M\}} \left|\hat{g}_j(\cdot)\right|^2 \in L^1_{\mathrm{loc}}(\mathbb{R}\setminus E). \tag{3.7}$$

Proof. To see this, consider any $M > 0$. Then

$$\begin{aligned} \sum_{\{j:c_j>M\}} |\hat{g}_j(\gamma)|^2 \le \sum_{j\in J} |\hat{g}_j(\gamma)|^2 &= \sum_{\{j:c_j\le M\}} |\hat{g}_j(\gamma)|^2 + \sum_{\{j:c_j>M\}} |\hat{g}_j(\gamma)|^2 \\ &\le M \sum_{\{j:c_j\le M\}} \frac{1}{c_j} |\hat{g}_j(\gamma)|^2 + \sum_{\{j:c_j>M\}} |\hat{g}_j(\gamma)|^2 \\ &\le MB + \sum_{\{j:c_j>M\}} |\hat{g}_j(\gamma)|^2. \end{aligned}$$

□

Let us take a closer look at the essential condition (3.1). First we note that in Example 1 this condition is indeed violated: in fact, a slight modification of the calculation in Example 1 shows that the infinite series in (3.1) is divergent for all $\gamma \in \mathbb{R}$. The next result shows that under mild regularity conditions on c_j and g_j, condition (3.1) is equivalent to the LIC. However, in practice, condition (3.1), or rather condition (3.7), is often much easier to work with than the LIC.

Proposition 3.3. *Let $E \in \mathcal{E}$. Suppose that the Calderón sum for $\{T_{c_j k} g_j\}_{k\in\mathbb{Z}, j\in J}$ is locally integrable on $\mathbb{R}\setminus E$, i.e.,*

$$\sum_{j\in J} \frac{1}{c_j} |\hat{g}_j(\cdot)|^2 \in L^1_{\mathrm{loc}}(\mathbb{R}\setminus E),$$

Then the GSI system $\{T_{c_j k} g_j\}_{k\in\mathbb{Z}, j\in J}$ satisfies the LIC with respect to the set E if and only if (3.1) *holds.*

Proof. Assume that

$$L(f) = \sum_{j\in J}\sum_{m\in\mathbb{Z}} \frac{1}{c_j}\int_{\operatorname{supp}\hat{f}} |\hat{f}(\gamma + c_j^{-1}m)\hat{g}_j(\gamma)|^2 \,\mathrm{d}\gamma < \infty \tag{3.8}$$

for every $f \in \mathcal{D}_E$. Let K be a compact set in $\mathbb{R}\setminus E$, i.e., $K \subset [c,d]\setminus E$, and let $\hat{f} = \chi_K$ in (3.8). Then for each $j \in J$, the set $K\cap(K - c_j^{-1}m)$ can only be nonempty if $|m| \le c_j(d-c)$. Hence for the inner sum in (3.8), we have

$$\begin{aligned}\sum_{m\in\mathbb{Z}} \frac{1}{c_j}\int_{K\cap(K-c_j^{-1}m)} |\hat{g}_j(\gamma)|^2 \,\mathrm{d}\gamma &= \sum_{|m|\le c_j(d-c)} \frac{1}{c_j}\int_{K\cap(K-c_j^{-1}m)} |\hat{g}_j(\gamma)|^2 \,\mathrm{d}\gamma \\ &\ge \frac{[c_j(d-c)]}{c_j}\int_K |\hat{g}_j(\gamma)|^2\,\mathrm{d}\gamma \\ &\ge (d-c)\int_K |\hat{g}_j(\gamma)|^2 d\gamma - \frac{1}{c_j}\int_K |\hat{g}_j(\gamma)|^2\,\mathrm{d}\gamma,\end{aligned}$$

where we for the first inequality use that the union of the sets $K\cap(K-c_j^{-1}m)$, $|m| \le [c_j(d-c)]$, contains $[c_j(d-c)]$ copies of K. Hence,

$$(d-c)\sum_{j\in J}\int_K |\hat{g}_j(\gamma)|^2\,\mathrm{d}\gamma \le L(f) + \sum_{j\in J}\frac{1}{c_j}\int_K |\hat{g}_j(\gamma)|^2\,\mathrm{d}\gamma < \infty.$$

Thus (3.1) holds. Conversely, assume that (3.1) holds. For $f \in \mathcal{D}_E$, let $R > 0$ such that $\operatorname{supp}\hat{f} \subseteq \{\gamma : |\gamma| \le R\}$. Therefore the inner sum in (3.8) has at most $4Rc_j + 1$ terms. By splitting the sum into two terms, we have

$$\begin{aligned}L(f) &\le \|\hat{f}\|_\infty^2 \sum_{j\in J}\sum_{|m|\le 2Rc_j} \frac{1}{c_j}\int_{\operatorname{supp}\hat{f}} |\hat{g}_j(\gamma)|^2\,\mathrm{d}\gamma \\ &\le 4R\|\hat{f}\|_\infty^2 \sum_{j\in J}\int_{\operatorname{supp}\hat{f}} |\hat{g}_j(\gamma)|^2\,\mathrm{d}\gamma + \|\hat{f}\|_\infty^2\sum_{j\in J}\frac{1}{c_j}\int_{\operatorname{supp}\hat{f}} |\hat{g}_j(\gamma)|^2\,\mathrm{d}\gamma < \infty.\end{aligned}$$

□

Using Proposition 3.3, the following result is now just a reformulation of Theorem 3.1.

Corollary 3.4. *Assume that the LIC with respect to some set $E \in \mathcal{E}$ holds for the GSI system $\{T_{c_jk}g_j\}_{k\in\mathbb{Z},j\in J}$. If $\{T_{c_jk}g_j\}_{k\in\mathbb{Z},j\in J}$ is a frame for $L^2(\mathbb{R})$ with lower bound A, then* (3.2) *holds.*

Remark 1. Corollary 3.4 also holds for GSI frames for $\check{L}^2(S)$ in which case the Calderón sum is bounded from below by A on S and zero otherwise.

The next example shows that the Calderón sum might be bounded from below by the lower frame bound, even if neither the technical condition (3.1) nor the LIC is satisfied.

Example 2. Consider the GSI system $\{T_{N^jk}g_{j,p,\ell}\}_{k,\ell\in\mathbb{Z},j\in\mathbb{N},p\in\mathcal{P}_j}$, where

$$\mathcal{P}_j = \{1,\dots,N^j-1\},\ \hat{g}_{j,p,\ell} = (N-1)^{1/2}T_\ell\chi_{[\frac{p}{N^j},\frac{p+1}{N^j}]},\ c_{j,p,\ell} = N^j.$$

This system satisfies the α-LIC. To see this, fix $f \in \mathcal{D}_\emptyset$. We then have

$$L'(f) = \sum_{j\in\mathbb{N}}\sum_{p\in\mathcal{P}_j}\sum_{\ell\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \frac{1}{N^j}\int_{\mathbb{R}} |\hat{f}(\gamma)\overline{\hat{f}(\gamma + \frac{m}{N^j})\hat{g}_{j,p,\ell}(\gamma)}\hat{g}_{j,p,\ell}(\gamma - \frac{m}{N^j})|\,\mathrm{d}\gamma$$

$$
\begin{aligned}
&\le (N-1)\|\hat{f}\|_\infty^2 \sum_{j\in\mathbb{N}} \sum_{p\in\mathcal{P}_j} \sum_{\ell\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \frac{1}{N^j} \int_{\operatorname{supp}\hat{f}} \chi_{[\frac{p}{N^j},\frac{p+1}{N^j}]}(\gamma-\ell)\chi_{[\frac{p}{N^j},\frac{p+1}{N^j}]}(\gamma-\ell-\frac{m}{N^j})\,\mathrm{d}\gamma\\
&= (N-1)\|\hat{f}\|_\infty^2 \sum_{j\in\mathbb{N}} \sum_{p\in\mathcal{P}_j} \sum_{\ell\in\mathbb{Z}} \frac{1}{N^j} \int_{\operatorname{supp}\hat{f}} \chi_{[\frac{p}{N^j},\frac{p+1}{N^j}]}(\gamma-\ell)\,\mathrm{d}\gamma\\
&= (N-1)\|\hat{f}\|_\infty^2 \sum_{j\in\mathbb{N}} \frac{1}{N^j} \int_{\operatorname{supp}\hat{f}} \chi_{\mathbb{R}}(\gamma)\,\mathrm{d}\gamma\\
&= \|\hat{f}\|_\infty^2 \int_{\operatorname{supp}\hat{f}} \chi_{\mathbb{R}}(\gamma)\,\mathrm{d}\gamma < \infty.
\end{aligned}
$$

The considered GSI system is also a Parseval frame. To see this, using Proposition 2.5, it is sufficient to show that

$$
\sum_{j\in J_\alpha} \sum_{p\in\mathcal{P}_j} \sum_{\ell\in\mathbb{Z}} \frac{1}{N^j} \overline{\hat{g}_{j,p,\ell}(\gamma)}\hat{g}_{j,p,\ell}(\gamma+\alpha) = \delta_{\alpha,0} \qquad \text{a.e. } \gamma\in\mathbb{R}. \tag{3.9}
$$

Let $\Lambda = \{N^{-j}n \ : \ j\in J, n\in\mathbb{Z}\}$; and, for $\alpha\in\Lambda$, let

$$
J_\alpha = \{j\in J \ : \ \exists n\in\mathbb{Z} \text{ such that } \alpha = N^{-j}n\}.
$$

Assume that $0\neq\alpha=\frac{n}{N^{j_0}}$ and $n\le N^{j_0}$. Then $J_\alpha\subseteq\{j : j\ge j_0\}$. But for each $j\ge j_0$ and $\gamma\in\mathbb{R}$ we have

$$
\chi_{[\frac{p}{N^j},\frac{p+1}{N^j}]}(\gamma-\ell)\chi_{[\frac{p}{N^j},\frac{p+1}{N^j}]}(\gamma-\ell-\frac{n}{N^{j_0}}) = 0.
$$

Hence, (3.9) is satisfied for $\alpha\neq 0$. Now, consider (3.9) with $\alpha=0$:

$$
\begin{aligned}
\sum_{j\in\mathbb{N}} \sum_{p\in\mathcal{P}_j} \sum_{\ell\in\mathbb{Z}} \frac{1}{N^j} |\hat{g}_{j,p,\ell}(\gamma)|^2 &= (N-1) \sum_{j\in\mathbb{N}} \sum_{p\in\mathcal{P}_j} \sum_{\ell\in\mathbb{Z}} \frac{1}{N^j} \chi_{[\frac{p}{N^j},\frac{p+1}{N^j}]}(\gamma-\ell)\\
&= (N-1)\sum_{j\in\mathbb{N}} \frac{1}{N^j}\chi_{\mathbb{R}}(\gamma) = 1.
\end{aligned}
$$

One can easily show that this Parseval frame does not satisfies condition (3.1), and consequently by Proposition 3.3, it cannot satisfy the LIC for any $E\in\mathcal{E}$, while the inequality (3.2) holds with $A=1$. ■

4 Special cases of GSI systems

We will now show that Theorem 3.1 indeed generalizes the known results for wavelet and Gabor systems. First, for regular wavelet systems the condition (3.7) is always satisfied for $E=\{0\}$:

Lemma 4.1. *Let $\psi\in L^2(\mathbb{R})$ and $a>1, b>0$. Consider the wavelet system $\{D_{a^j}T_{bk}\psi\}$, written on the form (2.1). Then there exists $M\in\mathbb{R}$ such that (3.7) holds for $E=\{0\}$, i.e.,*

$$
\sum_{\{j:c_j>M\}} |\hat{g}_j(\cdot)|^2 \in L^1_{\mathrm{loc}}(\mathbb{R}\setminus\{0\}). \tag{4.1}
$$

Proof. Assume that K is a compact subset of $\mathbb{R}\setminus\{0\}$. Let $M=ab$; then $a^jb>M$ if and only if $j>0$. Now,

$$
\sum_{j>0}\int_K |\hat{g}_j(\gamma)|^2\,\mathrm{d}\gamma = \sum_{j>0}\int_K |D_{a^{-j}}\hat{\psi}(\gamma)|^2\,\mathrm{d}\gamma = \sum_{j>0}\int_K a^j|\hat{\psi}(a^j\gamma)|^2\,\mathrm{d}\gamma = \sum_{j>0}\int_{a^jK} |\hat{\psi}(\gamma)|^2\,\mathrm{d}\gamma.
$$

Since K is a compact subset of $\mathbb{R}\setminus\{0\}$, one can find $L, R > 0$ such that $K \subset \{\gamma : \frac{1}{R} < |\gamma| < R\}$ and $a^L > R^2$. Hence if $j - j_0 \geq L$, then $a^{j_0}K \cap a^j K = \emptyset$. Thus the family of subsets $\{a^j K\}_{j>0}$ can be considered as a finite union of mutually disjoint sets,

$$\{a^j K\}_{j>0} = \bigcup_{i=1}^{L} \{a^{i+kL} K\}_{k=0}^{\infty},$$

Therefore,

$$\sum_{j>0} \int_K |\hat{g}_j(\gamma)|^2 \, \mathrm{d}\gamma = \sum_{j>0} \int_{a^j K} |\hat{\psi}(\gamma)|^2 \, \mathrm{d}\gamma \leq L \int_{\mathbb{R}} |\hat{\psi}(\gamma)|^2 \, \mathrm{d}\gamma < \infty,$$

which implies that (4.1) holds. □

From Theorem 3.1 we can now recover the lower bound in (2.3) for wavelet frames. Assume that $\{D_{a^j} T_{bk} \psi\}_{j,k\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with bounds A and B. By Lemma 4.1 and Proposition 3.2, the wavelet system satisfies (3.1); thus (2.3) holds by Theorem 3.1.

To establish the lower bound of the Calderón sum for irregular wavelet systems $\{D_{a_j} T_{bk} \psi\}_{j,k\in\mathbb{Z}}$, first obtained by Yang and Zhou [22], we have to work a little harder. We mention that the result by Yang and Zhou covers the more general setting of irregular dilations *and* translations. Recall that a sequence $\{a_j\}_{j\in\mathbb{Z}}$ of positive numbers is said to be logarithmically separated by $\lambda > 0$ if $\frac{a_{j+1}}{a_j} \geq \lambda$ for each $j \in \mathbb{Z}$.

Lemma 4.2. *If a sequence $\{a_j\}_{j\in\mathbb{Z}}$ of positive numbers is logarithmically separated by $\lambda > 0$, then for each $\psi \in L^2(\mathbb{R})$ and every compact subset $K \subset \mathbb{R} \setminus \{0\}$, we have*

$$\sum_{j\in\mathbb{Z}} \int_{a_j K} |\widehat{\psi}(\gamma)|^2 \, d\gamma < \infty.$$

Proof. Without loss of generality, assume that $\{a_j\}_{j\in\mathbb{Z}}$ is an increasing sequence and $K \subset \mathbb{R}_+ \setminus \{0\}$. There exist positive number c, d such that $K \subset [c, d]$. Take $r \in \mathbb{N}$ such that $\lambda^{r-1} > \frac{d}{c}$; then

$$\frac{a_{j+r}}{a_j} = \frac{a_{j+r}}{a_{j+r-1}} \frac{a_{j+r-1}}{a_{j+r-2}} \cdots \frac{a_{j+1}}{a_j} \geq \lambda^{r-1} > \frac{d}{c}.$$

Hence $a_{j+r} c > a_j d$. This shows that $a_j K \cap a_{j+r} K = \emptyset$ for all $j \in \mathbb{Z}$. By a similar argument, $a_j K \cap a_{j-r} K = \emptyset$. Therefore

$$\sum_{j\in\mathbb{Z}} \int_{a_j K} |\widehat{\psi}(\gamma)|^2 \, d \leq 2r \int_{\mathbb{R}} |\widehat{\psi}(\gamma)|^2 \, d\gamma < \infty.$$

□

We will now show that any wavelet system (with regular translates) that form a Bessel sequence automatically satisfies the LIC:

Proposition 4.3. *Let $\psi \in L^2(\mathbb{R})$ and $\{a_j\}_{j\in\mathbb{Z}}$ be a sequence in $\mathbb{R}_+$. If $\{D_{a_j} T_{bk} \psi\}_{j,k\in\mathbb{Z}}$ is a Bessel sequence, then it satisfies the LIC with respect to $E = \{0\}$.*

Proof. Define $I_n = (2^{n-\frac{1}{2}}, 2^{n+\frac{1}{2}}]$, for $n \in \mathbb{Z}$. By Lemma 1 in [22], there exists $M > 0$ such that for each $n \in \mathbb{Z}$, the number of j which a_j belongs to I_n is less than M. On the other hand, each point of I_{2n} is logarithmically separated with points from the interval I_{2m} for $m, n \in \mathbb{Z}$ and $m \neq n$. Similarly a point of I_{2n+1} is logarithmically separated with points from the interval

I_{2m+1} for $m, n \in \mathbb{Z}$ and $m \neq n$. Hence we can consider $\{a_j\}_{j\in\mathbb{Z}}$ as a disjoint union of finitely many logarithmically separated subsets $\{a_j\}_{j\in J_i}$, $i = 1, \cdots, N$. Let $K \subset \mathbb{R}\setminus\{0\}$ be a compact set. By Lemma 4.2, we know that $\sum_{j\in J_i} \int_{a_j K} |\widehat{\psi}(\gamma)|^2 < \infty$ for each $i = 1, \cdots, N$; it follows that

$$\sum_{j\in\mathbb{Z}} \int_K |a_j^{\frac{1}{2}} \widehat{\psi}(a_j\gamma)|^2 d\gamma = \sum_{j\in\mathbb{Z}} \int_{a_j K} |\widehat{\psi}(\gamma)|^2 d\gamma = \sum_{i=1}^{N} \sum_{j\in J_i} \int_{a_j K} |\widehat{\psi}(\gamma)|^2 d\gamma < \infty,$$

as desired. □

Corollary 4.4. *Suppose that $\{D_{a_j} T_{bk}\psi\}_{j,k\in\mathbb{Z}}$ is a frame with lower bounds $A > 0$. Then $\{D_{a_j} T_{bk}\psi\}_{j,k\in\mathbb{Z}}$ satisfies the LIC with respect to $E = \{0\}$ and*

$$A \leq \sum_{j\in\mathbb{Z}} \frac{1}{b} |\widehat{\psi}(a_j\gamma)|^2 \qquad \text{for a.e. } \gamma \in \mathbb{R}.$$

For other variants of GSI systems, covering the Gabor case, we also have the desired bounds of the Calderón sum immediately from the frame property.

Corollary 4.5. *Assume that $\{T_{c_j k} g_j\}_{k\in\mathbb{Z}, j\in J}$ is a frame with lower bound $A > 0$. If the sequence $\{c_j\}_{j\in\mathbb{Z}}$ is bounded above, then*

$$A \leq \sum_{j\in J} \frac{1}{c_j} |\hat{g}_j(\gamma)|^2 \qquad \text{for a.e. } \gamma \in \mathbb{R}.$$

Proof. Assume that there exists $M > 0$ such that $0 < c_j \leq M$, for all $j \in \mathbb{Z}$. Using the proof of Theorem 3.1, we have $I_1 = \mathbb{Z}$ and $I_2 = \emptyset$. Hence by letting $\varepsilon \to 0$ in (3.5), we have the result. □

5 Constructing dual GSI frames

We now turn to the question of how to obtain dual pairs of frames. Indeed, we present a flexible construction procedure that yields dual GSI frames for $\check{L}^2(S)$, where $S \subset \mathbb{R}$ is any countable collection of frequency bands. The precise choice of S depends on the application; we refer to [1] for an implementation and applications of GSI systems within audio signal processing. Our construction relies on a certain partition of unity, closely related to the Calderón sum, and unifies similar constructions of dual frames with Gabor, wavelet, and Fourier-like structure in [6, 7, 9, 19, 20]. Due to its generality the method will be technically involved; however, we will show that we nevertheless are able to extract the interesting cases from the general setup.

Methods in wavelet theory and Gabor analysis, respectively, share many common features. However, the decomposition into frequency bands is very different for the two approaches. To handle these differences in one unified construction procedure, we need a very flexible setup. In order to motivate the setup, we first consider the Gabor case and the wavelet case more closely in the following example.

Example 3. For a GSI system $\{T_{c_j k} g_j\}_{k\in\mathbb{Z}, j\in J}$ to be a frame for $\check{L}^2(S)$ it is necessary that the union of the sets $\operatorname{supp} \hat{g}_j$, $j \in J$, covers the frequency domain S. This is an easy consequence of Remark 1 and the fact that the Fourier transform of $\{T_{c_j k} g_j\}_{k\in\mathbb{Z}, j\in J}$ is $\{E_{c_j k} \hat{g}_j\}_{k\in\mathbb{Z}, j\in J}$. For simplicity we here consider only $S = \mathbb{R}$.

As we later want to apply the construction procedure for wave packet systems, it is necessary that the setup covers as well constructions of bandlimited dual wavelet as dual Gabor frames. For wavelet systems take the dyadic Shannon wavelet for $L^2(\mathbb{R})$ as an example. In this case, we split the frequency domain $S = \mathbb{R}$ in two sets $S_0 = (-\infty, 0]$ and $S_1 = (0, \infty)$. The support of the dilates of the Shannon wavelet is $\operatorname{supp} \hat{g}_j = \operatorname{supp} D_{2^{-j}} \hat{\psi} = \left[-2^{-j}, -2^{-j-1}\right] \cup \left[2^{-j-1}, 2^{-j}\right]$. To control the support of $\hat{g}_j$ we will define certain knots at the dyadic fractions $\left\{\pm 2^j\right\}_{j\in\mathbb{Z}}$. Observe that $\{\operatorname{supp} \hat{g}_j : j \in J\}$ covers the frequency line $\mathbb{R}$. However, in addition, we also need to control in which order this covering is done. On S_0 the support of $\hat{g}_j$ is moved to the left with increasing $j \in \mathbb{Z}$, while $\operatorname{supp} \hat{g}_j$ on S_1 is moved to the right. To handle S_0 and S_1 in the same setup, we introduce auxiliary functions φ_i, $i = 0, 1$, to allow for a change of how we cover the frequency set S_i with $\operatorname{supp} \hat{g}_j$. If we consider knots $\xi^{(0)}_j = -2^{-j}$ and $\xi^{(1)}_j = 2^{j-1}$ for $j \in \mathbb{Z}$ and define two bijective functions on $\mathbb{Z}$, $\varphi_0 = \mathrm{id}$ and $\varphi_1 = -\mathrm{id}$, then we have

$$\operatorname{supp} \hat{g}_j \subset \left[\xi^{(0)}_{\varphi_0(j)}, \xi^{(0)}_{\varphi_0(j)+1}\right] \cup \left[\xi^{(1)}_{\varphi_1(j)}, \xi^{(1)}_{\varphi_1(j)+1}\right] \tag{5.1}$$

The key point is that, for both $i = 0$ and $i = 1$, the contribution of $\operatorname{supp} \hat{g}_j$ in S_i can be written uniformly as $[\xi^{(i)}_{\varphi_i(j)}, \xi^{(i)}_{\varphi_i(j)+1}]$.

For Gabor systems the situation is simpler. Consider the Gabor-like orthonormal basis $\{T_k g_j\}_{j,k\in\mathbb{Z}}$, where $g_j = E_j g$, $j \in \mathbb{Z}$, with $g \in L^2(\mathbb{R})$ defined by $\hat{g} = \chi_{[0,1]}$. Hence, we only need one set $S_0 = \mathbb{R}$ with knots $\mathbb{Z}$. Here, φ_0 is the identity on $\mathbb{Z}$.

We remark that, in most applications, each φ_i will be an affine map of the form $z \mapsto az+b$, $a, b \in \mathbb{Z}$. The choice of the knots $\xi^{(i)}_k$ is usually not unique, but is simply chosen to match the support of $\hat{g}_j$. ■

Motivated by the concrete cases in Example 3 we now formulate the general setup as follows:

I) Let $S \subset \mathbb{R}$ be an at most countable collection of disjoint intervals

$$S = \bigcup_{i\in I} S_i,$$

where $I \subset \mathbb{Z}$. We write $S_i = (\alpha_i, \beta_i]$ with the convention that $\alpha_i = -\infty$ if $S_i = (-\infty, \beta_i]$, $\beta_i = \infty$ if $S_i = (\alpha_i, \infty]$, and $\alpha_i = -\infty$ and $\beta_i = \infty$ if $S_i = \mathbb{R}$. We assume an ordering of $\{S_i\}_{i\in I}$ so that $\beta_i \le \alpha_j$ whenever $i < j$. For each $i \in I$ we consider a sequence of knots

$$\{\xi^{(i)}_k\}_{k\in\mathbb{Z}} \subset S_i, \tag{5.2}$$

such that

$$\lim_{k\to-\infty} \xi^{(i)}_k = \alpha_i, \quad \lim_{k\to\infty} \xi^{(i)}_k = \beta_i, \quad \xi^{(i)}_k \le \xi^{(i)}_{k+1},\ k \in \mathbb{Z}.$$

II) Let $c_j > 0$. For each $j \in J$ we take $g_j \in L^2(\mathbb{R})$ such that $\hat{g}_j$ is a bounded, real function with compact support in a finite union of the sets S_i, $i \in I$. We further assume that $\{c_j^{-1/2}\hat{g}_j\}_{j\in J}$ are uniformly bounded, i.e., $\sup_j \|c_j^{-1/2}\hat{g}_j\|_\infty < \infty$.

III) For each $j \in J$, define the index set I_j by

$$I_j = \{i \in I : \hat{g}_j \neq 0 \text{ on } S_i\}.$$

On the other hand, for each $i \in I$, we fix an index set $J_i \subset J$ such that

$$\{j \in J : \hat{g}_j \neq 0 \text{ on } S_i\} \subset J_i.$$

Note that $i \in I_j$ implies that $j \in J_i$. Assume further that there is a bijective mapping $\varphi_i : J_i \to \mathbb{Z}$ such that, for each $j \in J$,

$$\operatorname{supp} \hat{g}_j \subset \bigcup_{i \in I_j} \left[\xi^{(i)}_{\varphi_i(j)}, \xi^{(i)}_{\varphi_i(j)+N}\right] \tag{5.3}$$

for some $N \in \mathbb{N}$. Often, we take J_i to be equal to the index set of the "active" generators g_j on the interval S_i, that is, $J_i = \{j \in J : \hat{g}_j \neq 0 \text{ on } S_i\}$, but this need not be the case, e.g., if $\{j \in J : \hat{g}_j \neq 0 \text{ on } S_i\}$ is a finite set.

In the final step of our setup we have only left to define the dual generators.

IV) Let $h_j \in \check{L}^2(S)$ be given by

$$\hat{h}_j(\gamma) = \begin{cases} \sum_{n=-N+1}^{N-1} a^{(i)}_{\varphi_i(j),n}\, \hat{g}_{\varphi_i^{-1}(\varphi_i(j)+n)}(\gamma) & \text{for } \gamma \in S_i, i \in I_j \\ 0 & \text{for } \gamma \in S_i, i \notin I_j \end{cases} \tag{5.4}$$

where $\{a^{(i)}_{k,n}\}_{i \in I, k \in \mathbb{Z}, n \in \{-N+1,\ldots,N-1\}}$ will be specified later.

Example 3 (continuation)**.** For a rigorous introduction of φ_i in Example 3 above, we need to specify the sets J_i. In the wavelet case, as all dilates $D_{2^{-j}}\hat{\psi}$ have support intersecting *both* S_0 and S_1, the active sets J_i, $i = 0, 1$, correspond to all dilations, that is, $\mathbb{Z}$. In the Gabor case, one easily also verifies that the active set J_0 is $\mathbb{Z}$. ■

We are now ready to present the construction of dual GSI frames. Recall from (5.2) that the superscript (i) on the points $\xi^{(i)}_k$ refer to the set S_i.

Theorem 5.1. *Assume the general setup I–IV. Suppose that*

$$\sum_{j \in J} c_j^{-1/2} \hat{g}_j(\gamma) = \chi_S(\gamma) \qquad \textit{for a.e. } \gamma \in \mathbb{R}. \tag{5.5}$$

Suppose further that $c_j \leq 1/M_j$, where

$$M_j = \max\left\{\xi^{(\max I_j)}_{\varphi_{\max I_j}(j)+2N} - \xi^{(\min I_j)}_{\varphi_{\min I_j}(j)}, \xi^{(\max I_j)}_{\varphi_{\max I_j}(j)+N} - \xi^{(\min I_j)}_{\varphi_{\min I_j}(j)-N}\right\}, \tag{5.6}$$

and that $\{a^{(i)}_{k,n}\}_{k \in \mathbb{Z}, n \in \{-N+1,\ldots,N-1\}, i \in I}$ is a bounded sequence satisfying

$$a^{(i)}_{\varphi_i(j),0} = 1 \quad \textit{and} \quad \left(\frac{c_{\varphi_i^{-1}(\varphi_i(j)+n)}}{c_j}\right)^{1/2} a^{(i)}_{\varphi_i(j),n} + \left(\frac{c_j}{c_{\varphi_i^{-1}(\varphi_i(j)+n)}}\right)^{1/2} a^{(i)}_{\varphi_i(j)+n,-n} = 2, \tag{5.7}$$

for $n \in \{1, \ldots, N-1\}$, $j \in J_i$, and $i \in I$. Then $\{T_{c_j k} g_j\}_{k \in \mathbb{Z}, j \in \mathbb{Z}}$ and $\{T_{c_j k} h_j\}_{k \in \mathbb{Z}, j \in \mathbb{Z}}$ are a pair of dual frames for $\check{L}^2(S)$.

Proof. First, note that for each $j \in J$, we have $j \in J_{\max I_j} \cap J_{\min I_j}$ and therefore that M_j in (5.6) is well-defined.

Now, note that by assumption (5.3) and the definition in (5.4),

$$\operatorname{supp} \hat{g}_j \subset \left[\xi^{(\min I_j)}_{\varphi_{\min I_j}(j)}, \xi^{(\max I_j)}_{\varphi_{\max I_j}(j)+N}\right]$$

and

$$\operatorname{supp} \hat{h}_j \subset \left[\xi^{(\min I_j)}_{\varphi_{\min I_j}(j)-N}, \xi^{(\max I_j)}_{\varphi_{\max I_j}(j)+2N}\right],$$

where the constant N is given by assumption III. Thus, if $j \in J$ and $0 \neq m \in \mathbb{Z}$, then $\hat{g}_j(\gamma)\hat{h}_j(\gamma+c_j^{-1}m) = 0$ for a.e. $\gamma \in \mathbb{R}$ since $c_j^{-1} \geq M_j$ for each $j \in J$. Therefore, by Theorem 2.4, we only need to show that $\{T_{c_jk}g_j\}_{k\in\mathbb{Z},j\in J}$ and $\{T_{c_jk}h_j\}_{k\in\mathbb{Z},j\in J}$ are Bessel sequences, satisfy the dual α-LIC and that

$$\sum_{j\in J} c_j^{-1}\overline{\hat{g}_j(\gamma)}\hat{h}_j(\gamma) = \chi_S(\gamma) \qquad \text{for a.e. } \gamma \in \mathbb{R}. \tag{5.8}$$

holds.

Choose $K > 0$ so that $\frac{1}{\sqrt{c_j}}\,|\hat{g}_j(\gamma)| \leq K$ for all $j \in J$ and a.e. $\gamma \in S$. For a.e. $\gamma \in S$ we have

$$\sum_{j\in J}\sum_{m\in\mathbb{Z}} \frac{1}{c_j}\left|\hat{g}_j(\gamma)\hat{g}_j(\gamma + c_j^{-1}m)\right| = \sum_{j\in J}\frac{1}{c_j}\left|\hat{g}_j(\gamma)\right|^2 \leq K\sum_{j\in J}\frac{1}{c_j^{1/2}}\,|\hat{g}_j(\gamma)| \leq NK^2,$$

where the last inequality follows from the fact that at most N functions from $\{\hat{g}_j : j \in J\}$ can be nonzero on a given interval $[\xi^{(i)}_k, \xi^{(i)}_{k+1}]$. It now follows from Theorem 2.6(i) that $\{T_{c_jk}g_j\}_{k\in\mathbb{Z},j\in J}$ is a Bessel sequence in $\check{L}^2(S)$ with bound NK^2.

A simple argument shows that if $\{a^{(i)}_{k,n}\}_{k\in\mathbb{Z},n\in\{-N+1,\dots,N-1\},i\in I}$ is a bounded sequence, then each term on the left-hand side of (5.7) must be bounded with respect to $i \in I, j \in J_i$ and $n = -N+1,\cdots,N-1$. Let $M > 0$ be such a bound. Then, for each $i \in I$ and $\gamma \in S_i$, we have

$$\begin{aligned}
c_j^{-1/2}|\hat{h}_j(\gamma)| &\leq \sum_{n=-N+1}^{N-1} |c_j^{-1/2} a^{(i)}_{\varphi_i(j),n}\; \hat{g}_{\varphi_i^{-1}(\varphi_i(j)+n)}(\gamma)| \\
&\leq K\sum_{n=-N+1}^{N-1}\left|\frac{c_{\varphi_i^{-1}(\varphi_i(j)+n)}}{c_j}\right|^{1/2}\left|a^{(i)}_{\varphi_i(j),n}\right| \leq KM(2N-1) =: L.
\end{aligned}$$

Hence, the sequence of functions $\{c_j^{-1/2}\hat{h}_j\}_{j\in J}$ is uniformly bounded by L.

For the remainder of the proof, we let $i \in I$ and $j \in J_i$ be fixed, but arbitrary. Note that the functions $\hat{g}_{\varphi_i^{-1}(\varphi_i(j)+n)}$, $n = 0, 1, \dots, N-1$, are the only nonzero generators $\{\hat{g}_k\}_{k\in J}$ on $[\xi^{(i)}_{\varphi_i(j)+N-1}, \xi^{(i)}_{\varphi_i(j)+N}]$. Hence, for $\hat{h}_{\varphi_i^{-1}(\varphi_i(j)+l)}$ only $l = -N+1,\cdots,2N-2$ can be nonzero on $[\xi^{(i)}_{\varphi_i(j)+N-1}, \xi^{(i)}_{\varphi_i(j)+N}]$. Thus, for $\gamma \in [\xi^{(i)}_{\varphi_i(j)+N-1}, \xi^{(i)}_{\varphi_i(j)+N}]$, we have

$$\begin{aligned}
\sum_{j\in J}\sum_{m\in\mathbb{Z}} c_k^{-1}\left|\hat{h}_j(\gamma)\hat{h}_j(\gamma + c_j^{-1}m)\right| &= \sum_{j\in J}\sum_{m=-1}^{1} c_j^{-1}\left|\hat{h}_j(\gamma)\hat{h}_j(\gamma + c_j^{-1}m)\right| \\
&\leq 3L\sum_{j\in J} c_j^{-1/2}\left|\hat{h}_j(\gamma)\right|
\end{aligned}$$

$$
\begin{aligned}
&= 3L \sum_{\ell=-N+1}^{2N-2} c^{-1/2}_{\varphi_i^{-1}(\varphi_i(j)+\ell)} \left| \hat{h}_{\varphi_i^{-1}(\varphi_i(j)+\ell)}(\gamma) \right| \\
&\le 3L^2(3N-2).
\end{aligned}
$$

It follows from Theorem 2.6(i) that $\{T_{c_j k} h_j\}_{k\in\mathbb{Z}, j\in J}$ is also a Bessel sequence. Similar computations show that the α-LIC holds:

$$
\begin{aligned}
\sum_{j\in J}\sum_{m\in\mathbb{Z}} \frac{1}{c_j} \left|\hat{g}_j(\gamma)\hat{h}_j(\gamma + c_j^{-1}m)\right| &= \sum_{j\in J} \frac{1}{c_j}\left|\hat{g}_j(\gamma)\hat{h}_j(\gamma)\right| \\
&\le \left(\sum_{j\in J} \frac{1}{c_j} \left|\hat{g}_j(\gamma)\right|^2 \right)^{1/2} \left(\sum_{j\in J} \frac{1}{c_j} \left|\hat{h}_j(\gamma)\right|^2 \right)^{1/2} \\
&\le N^{1/2} K B^{1/2},
\end{aligned}
$$

where B is a Bessel bound for $\{T_{c_j k} h_j\}_{k\in\mathbb{Z}, j\in J}$.

Finally, we need to show that (5.8) holds. Set $\tilde{g}_k = c_k^{-1/2} g_k$, $k \in J$, and $\ell_n = \varphi_i^{-1}(\varphi_i(j)+n)$. Then, for a.e. $\gamma \in [\xi^{(i)}_{\varphi_i(j)}, \xi^{(i)}_{\varphi_i(j)+1}]$.

$$
\begin{aligned}
1 &= \left(\sum_{k\in J} \hat{\tilde{g}}_k(\gamma) \right)^2 = \left(\sum_{n=0}^{N-1} \hat{\tilde{g}}_{\ell_n}(\gamma) \right)^2 \\
&= \hat{\tilde{g}}_{\ell_0}(\gamma) \left[\hat{\tilde{g}}_{\ell_0}(\gamma) + 2\hat{\tilde{g}}_{\ell_1}(\gamma) + 2\hat{\tilde{g}}_{\ell_2}(\gamma) + \cdots + 2\hat{\tilde{g}}_{\ell_{N-1}}(\gamma) \right] \\
&+ \hat{\tilde{g}}_{\ell_1}(\gamma) \left[\qquad \hat{\tilde{g}}_{\ell_1}(\gamma) + 2\hat{\tilde{g}}_{\ell_2}(\gamma) + \cdots + 2\hat{\tilde{g}}_{\ell_{N-1}}(\gamma) \right] \\
&+ \cdots \\
&+ \hat{\tilde{g}}_{\ell_{N-1}}(\gamma) \left[\qquad\qquad\qquad \hat{\tilde{g}}_{\ell_{N-1}}(\gamma) \right].
\end{aligned}
$$

Clearly, each mixed term in this sum has coefficient 2, e.g., $2\hat{\tilde{g}}_{\ell_n}(\gamma)\hat{\tilde{g}}_{\ell_m}(\gamma)$ whenever $n \neq m$. Replacing $\hat{\tilde{g}}_{\ell_n}$ with $c_{\ell_n}^{-1/2}\hat{g}_{\ell_n}$ yields a mixed term with coefficient $2c_{\ell_n}^{-1/2}c_{\ell_m}^{-1/2}\hat{g}_{\ell_n}(\gamma)\hat{g}_{\ell_m}(\gamma)$. By (5.7), we have

$$
a_{\varphi_i(j),0} = 1 \quad \text{and} \quad c_j^{-1} a^{(i)}_{\varphi_i(j),n} + c^{-1}_{\varphi_i^{-1}(\varphi_i(j)+n)} a^{(i)}_{\varphi_i(j)+n,-n} = 2c_j^{-1/2} c^{-1/2}_{\varphi_i^{-1}(\varphi_i(j)+n)},
$$

for $\ell = 1, \dots, N-1, j \in J$. Hence, we can factor the sum in the following way:

$$
\begin{aligned}
1 &= c_{\ell_0}^{-1}\hat{g}_{\ell_0}(\gamma) \left[a_{\ell_0,0}\, \hat{g}_{\ell_0}(\gamma) + a_{\ell_0,1}\, \hat{g}_{\ell_1}(\gamma) + \cdots + a_{\ell_0,N-1}\, \hat{g}_{\ell_{N-1}}(\gamma) \right] \\
&+ c_{\ell_1}^{-1}\hat{g}_{\ell_1}(\gamma) \left[a_{\ell_1,-1}\, \hat{g}_{\ell_0}(\gamma) + a_{\ell_1,0}\, \hat{g}_{\ell_1}(\gamma) + \cdots + a_{\ell_1,N-2}\, \hat{g}_{\ell_{N-1}}(\gamma) \right] \\
&+ \cdots \\
&+ c_{\ell_{N-1}}^{-1}\hat{g}_{\ell_{N-1}}(\gamma) \left[a_{\ell_{N-1},-N+1}\, \hat{g}_{\ell_0}(\gamma) + \cdots + a_{\ell_{N-1},0}\, \hat{g}_{\ell_{N-1}}(\gamma) \right] \\
&= \sum_{n=0}^{N-1} c^{-1}_{\varphi_i^{-1}(\varphi_i(j)+n)} \hat{g}_{\varphi_i^{-1}(\varphi_i(j)+n)}(\gamma) \hat{h}_{\varphi_i^{-1}(\varphi_i(j)+n)}(\gamma) = \sum_{k\in J} c_k^{-1} \hat{g}_k(\gamma)\hat{h}_k(\gamma)
\end{aligned}
$$

for a.e. $\gamma \in [\xi^{(i)}_j, \xi^{(i)}_{j+1}]$. Since $i \in I$ and $j \in J_i$ were arbitrary, the proof is complete. □

Remark 2. A few comments on the definition of M_j are in place. Firstly, the only feature of M_j is to guarantee that $\hat{g}_j$ and $\hat{h}_j(\cdot + c_j^{-1}k)$ has non-overlapping supports for all $k \in \mathbb{Z} \setminus \{0\}$.

Secondly, it is possible to make a sharper choice of M_j in Theorem 5.1. Indeed, let $n_{j,\max} = \max\{n : a^{(\max I_j)}_{\varphi_{\max I_j}(j),n} \neq 0\}$ and let $n_{j,\min} = \min\{n : a^{(\min I_j)}_{\varphi_{\min I_j}(j),n} \neq 0\}$. We then take:

$$M_j = \max\left\{\xi^{(\max I_j)}_{\varphi_{\max I_j}(j)+n_{j,\max}+N} - \xi^{(\min I_j)}_{\varphi_{\min I_j}(j)}, \xi^{(\max I_j)}_{\varphi_{\max I_j}(j)+N} - \xi^{(\min I_j)}_{\varphi_{\min I_j}(j)+n_{j,\min}}\right\}. \tag{5.9}$$

For wavelet systems Theorem 5.1 reduces to the construction from [19,20] of dual wavelet frames in $L^2(\mathbb{R})$.

Corollary 5.2. *Let $a > 1$ and $\psi \in L^2(\mathbb{R})$. Suppose that $\hat{\psi}$ is real-valued, that $\operatorname{supp}\hat{\psi} \subseteq \left[-a^M, -a^{M-N}\right] \cup \left[a^{M-N}, a^M\right]$ for some $M \in \mathbb{Z}, N \in \mathbb{N}$ and that*

$$\sum_{j\in\mathbb{Z}} \hat{\psi}(a^j\gamma) = 1 \qquad \text{for a.e. } \gamma \in \mathbb{R}. \tag{5.10}$$

Take $b_n \in \mathbb{C}$, $n = -N+1, \dots, N-1$, satisfying

$$b_0 = 1, \quad b_{-n} + b_n = 2,\ n = 1, 2, \dots, N-1,$$

and set $\ell := \max\{n : b_n \neq 0\}$. Let $b \in \left(0, a^{-M}(1+a^\ell)^{-1}\right]$. Then the function ψ and the function $\tilde{\psi} \in L^2(\mathbb{R})$ defined by

$$\hat{\tilde{\psi}}(\gamma) = b \sum_{n=-N+1}^{N-1} b_n\, \hat{\psi}(a^{-n}\gamma) \qquad \text{for a.e. } \gamma \in \mathbb{R} \tag{5.11}$$

generate dual frames $\{D_{a^j}T_{bk}\psi\}_{j,k\in\mathbb{Z}}$ and $\{D_{a^j}T_{bk}\tilde{\psi}\}_{j,k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$.

Proof. Let $\phi = \sqrt{b}\psi$. We consider the wavelet system $\{D_{a^j}T_{bk}\phi\}_{j,k\in\mathbb{Z}}$ as a GSI system with $c_j = a^j b$ and $g_j = D_{a^j}\phi$ for $j \in J = \mathbb{Z}$. We apply Theorem 5.1 with $S_0 = (-\infty, 0]$ and $S_1 = (0, +\infty)$, $\xi_j^{(0)} = -a^{M-j}$ and $\xi_j^{(1)} = a^{M-N+j}$ for $j \in \mathbb{Z}$. The assumption (5.5) corresponds to

$$\sum_{j\in\mathbb{Z}} \hat{\phi}(a^j\gamma) = \sqrt{b} \qquad \text{for } \gamma \in \mathbb{R}$$

which is satisfied by (5.10). Let $a^{(0)}_{j,n} = a^{-n/2}b_n$ and $a^{(1)}_{j,n} = a^{n/2}b_{-n}$ for $n = -N+1, \dots, N-1$ and $j \in \mathbb{Z}$ and define $\varphi_0, \varphi_1 : \mathbb{Z} \to \mathbb{Z}$ with $\varphi_0(j) = j$ and $\varphi_1(j) = -j$. The definition of h_j in (5.4) reads

$$\hat{h}_j(\gamma) = \begin{cases} \sum_{n=-N+1}^{N-1} a^{(0)}_{j,n}\, a^{(j+n)/2}\hat{\phi}(a^{(j+n)}\gamma) & \text{for } \gamma \in S_0, \\ \sum_{n=-N+1}^{N-1} a^{(1)}_{-j,-n}\, a^{(j+n)/2}\hat{\phi}(a^{(j+n)}\gamma) & \text{for } \gamma \in S_1. \end{cases}$$

Setting $\hat{\tilde{\phi}} = D_{a^{-j}}h_j$ yields

$$\hat{\tilde{\phi}}(\gamma) = \sum_{n=-N+1}^{N-1} b_n\, \hat{\phi}(a^n\gamma) \qquad \gamma \in \mathbb{R}.$$

For all $j \in \mathbb{Z}$, we have $I_j = \{0, 1\}$ and thus $\min I_j = 0$ and $\max I_j = 1$. Note also that $\ell = n_{j,\max} = -n_{j,\min}$ for all $j \in \mathbb{Z}$. Hence, condition (5.9) reads

$$M_j = \max\left\{\xi^{(1)}_{-j+n_{j,\max}+N} - \xi^{(0)}_j, \xi^{(1)}_{-j+N} - \xi^{(0)}_{j+n_{j,\min}}\right\}$$

$$= \max \left\{ a^{M-j+n_{j,\max}} + a^{M-j}, a^{M-j} + a^{M-j-n_{\mathrm{j,min}}} \right\}$$
$$= a^{M-j}(a^{\ell}+1).$$

From Theorem 5.1 we have that $\{D_{a^j}T_{bk}\phi\}_{j,k\in\mathbb{Z}}$ and $\{D_{a^j}T_{bk}\tilde{\phi}\}_{j,k\in\mathbb{Z}}$ are dual frames for $b \in \big(0, a^{-M}(1+a^{\ell})^{-1}\big]$. By setting $\tilde{\psi} = \sqrt{b}\tilde{\phi}$, we therefore have that $\{D_{a^j}T_{bk}\psi\}_{j,k\in\mathbb{Z}}$ and $\{D_{a^j}T_{bk}\tilde{\psi}\}_{j,k\in\mathbb{Z}}$ are dual frames; it is clear that $\hat{\tilde{\psi}}$ is given by the formula in (5.11). □

Using the Fourier transform we can move the construction in Theorem 5.1 to the time domain. In this setting we obtain dual frames $\left\{E_{b_p m} g_p\right\}_{m\in\mathbb{Z},p\in\mathbb{Z}}$ and $\left\{E_{b_p m} h_p\right\}_{m\in\mathbb{Z},p\in\mathbb{Z}}$ with compactly supported generators g_p and h_p. A simplified version of this result, useful for application in Gabor analysis, is as follows.

Theorem 5.3. *Let $\{x_p : p \in \mathbb{Z}\} \subset \mathbb{R}$ be a sequence such that*

$$\lim_{p\to\pm\infty} x_p = \pm\infty, \quad x_{p-1} \le x_p, \quad \text{and } x_{p+2N-1} - x_p \le M, \quad p \in \mathbb{Z},$$

for some constants $N \in \mathbb{N}$ and $M > 0$. Let $g_p \in L^2(\mathbb{R})$, $p \in \mathbb{Z}$, be real-valued functions with such that $\{b_p^{-1/2} g_p\}_{p\in\mathbb{Z}}$ are uniformly bounded functions with $\operatorname{supp} g_p \subset [x_p, x_{p+N}]$. Assume that $\sum_{p\in\mathbb{Z}} b_p^{-1/2} g_p(x) = 1$ for a.e. $x \in \mathbb{R}$. Let

$$h_p(x) = \sum_{n=-N+1}^{N-1} a_{p,n}\, g_{p+n}(x), \quad \text{for } x \in \mathbb{R},$$

where $\{a_{p,n}\}_{p\in\mathbb{Z},n\in\{-N+1,\dots,N-1\}}$ is a bounded sequence in $\mathbb{C}$. Suppose that $0 < b_p < 1/M$ and that

$$a_{p,0} = 1 \quad \text{and} \quad \left(\frac{b_{p+n}}{b_p}\right)^{1/2} a_{p,n} + \left(\frac{b_p}{b_{p+n}}\right)^{1/2} a_{p+n,-n} = 2, \tag{5.12}$$

for $p \in \mathbb{Z}$ and $n = 1, \dots, N-1$. Then $\left\{E_{b_p m} g_p\right\}_{m\in\mathbb{Z},p\in\mathbb{Z}}$ and $\left\{E_{b_p m} h_p\right\}_{m\in\mathbb{Z},p\in\mathbb{Z}}$ are a pair of dual frames for $L^2(\mathbb{R})$.

Theorem 5.3 generalizes results on SI systems by Christensen and Sun [9] and Christensen and Kim [6] in the following way: Taking $b_p = b$ for all $p \in \mathbb{Z}$, Theorem 5.3 reduces to [7, Theorem 2.2] and to [9, Theorem 2.5] when further choosing $a_{p,n} = 0$ for $n = 1, \dots, N-1$ and $p \in \mathbb{Z}$ and to Theorem 3.1 in [6] when choosing $g_p = T_p g$ for some $g \in L^2(\mathbb{R})$.

6 Wave packet systems

Let $b > 1$, and let $\{(a_j, d_j)\}_{j\in J}$ be a countable set in $\mathbb{R}^+ \times \mathbb{R}$. The wave packet system $\{D_{a_j}T_{bk}E_{d_j}\psi\}_{k\in\mathbb{Z},j\in J}$ is a GSI system with

$$g_j = D_{a_j}E_{d_j}\psi \text{ and } c_j = a_j b, \quad j \in J.$$

It is of course possible to apply the dilation, modulation and translation operator in a different order than in $\{D_{a_j}T_{bk}E_{d_j}\psi\}_{k\in\mathbb{Z},j\in J}$. Indeed, we will also consider the collection of functions $\{T_{bk}D_{a_j}E_{d_j}\psi\}_{k\in\mathbb{Z},j\in J}$, which is a shift-invariant version of the wave packet system. It takes the form of a GSI system with

$$g_j = D_{a_j}E_{d_j}\psi \text{ and } c_j = b, \quad j \in J,$$

and it contains the Gabor-like system $\{T_{bk}E_{aj}\psi\}_{j,k\in\mathbb{Z}}$, but not the wavelet system, as a special case.

There are four more ordering of the dilation, modulation and translation operators; however, the study of these systems reduces to either the study of $\{D_{a_j}T_{bk}E_{d_j}\psi\}_{k\in\mathbb{Z},j\in J}$ or $\{T_{bk}D_{a_j}E_{d_j}\psi\}_{k\in\mathbb{Z},j\in J}$. This is clear from the commutator relations:

$$D_{a_j}T_{bk}E_{d_j} = \mathrm{e}^{-2\pi i d_j bk}\, D_{a_j}E_{d_j}T_{bk} = \mathrm{e}^{-2\pi i d_j bk}\, E_{a_j^{-1}d_j}D_{a_j}T_{bk}$$

and

$$T_{bk}D_{a_j}E_{d_j} = T_{bk}E_{a_j^{-1}d_j}D_{a_j} = \mathrm{e}^{-2\pi i a_j^{-1}d_j bk}\, E_{a_j^{-1}d_j}T_{bk}D_{a_j}.$$

The following result is a special case of Theorem 2.6 for the case of wave packet systems of the form $\{D_{a_j}T_{bk}E_{d_j}\psi\}_{k\in\mathbb{Z},j\in J}$; a direct proof was given in [8].

Theorem 6.1. *Let $b > 1$, and let $\{(a_j,d_j)\}_{j\in J}$ be a countable set in $\mathbb{R}\setminus\{0\}\times\mathbb{R}$. Assume that*

$$B := \frac{1}{b}\sup_{\gamma\in\mathbb{R}}\sum_{j\in J}\sum_{k\in\mathbb{Z}}|\hat{\psi}(a_j\gamma-d_j)\hat{\psi}(a_j\gamma-d_j-k/b)| < \infty. \tag{6.1}$$

Then $\{D_{a_j}T_{bk}E_{d_j}\psi\}_{k\in\mathbb{Z},j\in J}$ is a Bessel sequence with bound B. Further, if also

$$A := \frac{1}{b}\inf_{\gamma\in\mathbb{R}}\left(\sum_{j\in J}|\hat{\psi}(a_j\gamma-d_j)|^2 - \sum_{j\in J}\sum_{0\neq k\in\mathbb{Z}}|\hat{\psi}(a_j\gamma-d_j)\hat{\psi}(a_j\gamma-d_j-k/b)|\right) > 0,$$

then $\{D_{a_j}T_{bk}E_{d_j}\psi\}_{k\in\mathbb{Z},j\in J}$ is a frame for $L^2(\mathbb{R})$ with bounds A and B.

A similar result holds for the shift-invariant wave packet system $\{T_{bk}D_{a_j}E_{d_j}\psi\}_{k\in\mathbb{Z},j\in J}$.

6.1 Local integrability conditions and a lower bound for the Calderón sum

Theorem 6.1 allows us to construct frames, even tight frames, without worrying about technical local integrability conditions; in fact, the condition (6.1) implies that the α-LIC is satisfied.

Lemma 6.2 ([16])**.** *If* (6.1) *holds, then the α-LIC for wave packet systems holds with respect to some set $E\in\mathcal{E}$, i.e.,*

$$L'(f) = \sum_{j\in J}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}|\hat{f}(\gamma)\hat{f}(\gamma-k/(a_jb))\hat{\psi}(a_j\gamma-d_j)\hat{\psi}(a_j\gamma-d_j-k/b)|\,\mathrm{d}\gamma < \infty$$

for all $f\in\mathcal{D}_E$.

On the other hand, as we will see in Example 4, condition (6.1) does not imply the LIC. Indeed, for wave packet system, if possible, one should work with α-LIC instead of the LIC. We will see further results supporting this claim in the following. We continue our study of local integrability conditions with a special case of the wave packet system $\{D_{a_j}T_{bk}E_{d_j}\psi\}_{k\in\mathbb{Z},j\in J}$ that is highly redundant. Given $a>1$ and a sequence $\{d_m\}_{m\in\mathbb{Z}}$ in $\mathbb{R}$, we consider the collection of functions

$$\{D_{a^j}T_{bk}E_{d_m}\psi\}_{j,m,k\in\mathbb{Z}}. \tag{6.2}$$

which can be considered as GSI systems with $c_{(j,m)} = a^jb$ and $g_{j,m} = D_{a^j}E_{d_m}\psi$. For wave packet systems of this form the point set $\{(a_j,d_j)\}_{j\in J}\subset\mathbb{R}^+\times\mathbb{R}$ is a separable set of the form $\{(a^j,d_m)\}_{j,m\in Z}$. For each fixed $m\in\mathbb{Z}$ in (6.2) the system $\{D_{a^j}T_{bk}E_{d_m}\psi\}_{j,k\in\mathbb{Z}}$ is a wavelet system (with generator $E_{d_m}\psi$).

The first obvious constraint is that if the system $\{D_{a^j}T_{bk}E_{d_m}\psi\}_{j\in J,m,k\in\mathbb{Z}}$ is a frame, then the sequence $\{d_m\}_{m\in\mathbb{Z}}$ cannot be bounded. In fact, in that case the sequence $\{d_m\}_{m\in\mathbb{Z}}$ has an accumulation point and thus $\{D_{a^j}T_{bk}E_{d_m}\psi\}_{j\in J,m,k\in\mathbb{Z}}$ cannot be a Bessel sequence, see [8, Lemma 2.3]. We will now show that a wave packet system on the form (6.2) cannot satisfy condition (3.1).

Lemma 6.3. *Assume that $\{d_m\}_{m\in\mathbb{Z}}$ is unbounded and $\psi \neq 0$. Then*

$$\sum_{\{j:a^j b>M\}}\sum_{m\in\mathbb{Z}} |a^{\frac{j}{2}}\hat{\psi}(a^j\cdot - d_m)|^2 \notin L^1_{\text{loc}}(\mathbb{R}\setminus E), \tag{6.3}$$

for all $M>0$ and all $E\in\mathcal{E}$.

Proof. Let $M>0$ and let $E\in\mathcal{E}$. Assume that $\{d_m\}_{m\in\mathbb{Z}}$ is not bounded below. Note that $\{j : a^j b > M\} = \{j : j \geq M'\}$ where $M' = [\frac{\ln M - \ln b}{\ln a}] + 1$. Taking $K = [1,a]\setminus E$, we have

$$\begin{aligned} I_M &:= \sum_{m\in\mathbb{Z}}\sum_{j=M'}^{\infty}\int_K a^j|\hat{\psi}(a^j\gamma + d_m)|^2 d\gamma = \sum_{m\in\mathbb{Z}}\sum_{j=M'}^{\infty}\int_{a^jK}|\hat{\psi}(\gamma+d_m)|^2 d\gamma \\ &= \sum_{m\in\mathbb{Z}}\sum_{j=M'}^{\infty}\int_{a^j[1,a]}|\hat{\psi}(\gamma+d_m)|^2 d\gamma \geq \sum_{m\in\mathbb{Z}}\int_{a^{M'}}^{\infty}|\hat{\psi}(\gamma+d_m)|^2 d\gamma. \end{aligned} \tag{6.4}$$

Since $\{d_m\}_{m\in\mathbb{Z}}$ is not bounded below, there exists a subsequence $\{d_{m_l}\}_{l=1}^{\infty}$ of $\{d_m\}_{m\in\mathbb{Z}}$ such that $d_{m_l}\to -\infty$. Hence, for each $N\in\mathbb{R}$ there exists $L\in\mathbb{N}$ such that $d_{m_l} < N - a^{M'}$ for $l\geq L$. Thus

$$\int_{a^{M'}+d_{m_l}}^{\infty}|\hat{\psi}(\gamma)|^2 > \int_N^{\infty}|\hat{\psi}(\gamma)|^2\,\mathrm{d}\gamma,$$

for all $l\geq L$. But in this case, using inequality (6.4) and choosing N small enough, we have

$$I_M \geq \sum_{l\geq L}\int_N^{\infty}|\hat{\psi}(\gamma)|^2\,\mathrm{d}\gamma = \infty.$$

A similar argument shows that if $\{d_m\}_{m\in\mathbb{Z}}$ is not bounded above, then $I_M=\infty$. Therefore (6.3) holds. □

Thus, for the case where $\{d_m\}_{m\in\mathbb{Z}}$ is unbounded, it is impossible for a wave packet system $\{D_{a^j}T_{bk}E_{d_m}\psi\}_{j\in J,m,k\in\mathbb{Z}}$ to satisfy the LIC-condition. On the other hand, if $\{d_m\}_{m\in\mathbb{Z}}$ is bounded, we know that it is impossible for such a system to form a Bessel sequence. Hence, the wave packet system $\{D_{a^j}T_{bk}E_{d_m}\psi\}_{j\in J,m,k\in\mathbb{Z}}$ cannot simultaneously be a Bessel sequence and satisfy the LIC.

Corollary 6.4. *If the system $\{D_{a^j}T_{bk}E_{d_m}\psi\}_{j\in J,m,k\in\mathbb{Z}}$ is a Bessel sequence, then this system does not satisfies the LIC.*

The following example introduces a family of wave packet tight frames of the form $\{D_{a^j}T_{bk}E_{d_m}\psi\}_{j\in\mathbb{Z},k\in\mathbb{Z},m\in\mathbb{Z}}$ that satisfies (6.1) and thus the α-LIC by Lemma 6.2, but not the LIC.

Example 4. Let $\psi\in L^2(\mathbb{R})$ be a Shannon-type scaling function defined by $\hat{\psi}=\chi_{[-1/4,1/4]}$. Let $a=2$, $b=1$, and define $d_m = \operatorname{sgn}(m)(2^{|m|}-\frac{3}{4})$ for $m\in\mathbb{Z}\setminus\{0\}$. We will first argue that the wave packet system $\{D_{a^j}T_{bk}E_{d_m}\psi\}_{j\in\mathbb{Z},k\in\mathbb{Z},m\in\mathbb{Z}\setminus\{0\}}$ is a tight frame for $L^2(\mathbb{R})$ with bound

1. To see this, we will simply verify the conditions in Theorem 6.1. All we need to do is to prove that

$$\sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}\setminus\{0\}} |\hat{\psi}(2^j\gamma - d_m)|^2 = 1 \quad \text{for a.e. } \gamma \in \mathbb{R}. \tag{6.5}$$

With our definition of ψ, this amounts to verifying that the sets

$$I_{m,j} := 2^{-j}\left[-\tfrac{1}{4} + d_m, \tfrac{1}{4} + d_m\right),\ j \in \mathbb{Z}, m \in \mathbb{Z}\setminus\{0\}, \tag{6.6}$$

form a disjoint covering of $\mathbb{R}$. To see this, let $m \in \mathbb{N}$ and $k \in \mathbb{Z}$, then

$$I_{m,m-k} = 2^{-m+k}\left[-1 + 2^m, -\tfrac{1}{2} + 2^m\right) = 2^k\Big(1 + 2^{-m}\left[-1, -\tfrac{1}{2}\right)\Big).$$

The sets $1 + 2^{-m}\left[-1, -\frac{1}{2}\right)$, $m \in \mathbb{N}$, form a disjoint covering of $\left[\frac{1}{2}, 1\right)$. Hence the sets $2^k(1 + 2^{-m}\left[-1, -\frac{1}{2}\right))$, $k \in \mathbb{Z}, m \in \mathbb{N}$, form a disjoint covering of $(0, \infty)$.

A similar argument for $m < 0$ shows that $\{I_{m,j}\}_{j\in\mathbb{Z}, m\in-\mathbb{N}}$ is a disjoint covering of $(-\infty, 0)$. We conclude that (6.5) holds. Thus, the system $\{D_{2^j}T_kE_{d_m}\psi\}_{j\in\mathbb{Z},k\in\mathbb{Z},m\in\mathbb{Z}\setminus\{0\}}$ is a tight frame.

Since $\{D_{2^j}T_kE_{d_m}\psi\}_{j\in\mathbb{Z},k\in\mathbb{Z},m\in\mathbb{Z}\setminus\{0\}}$ satisfies (6.1), it satisfies the α-LIC by Lemma 6.2. On the other hand, by Corollary 6.4 the wave packet system does not satisfy the LIC. ■

We finally consider lower bounds of the Calderón sum for wave packet systems. From Corollary 3.4 we know that any GSI frame that satisfies the LIC will have a lower bounded Calderón sum. A first result utilizing this observation is the following result for the shift-invariant version of the wave packet systems.

Theorem 6.5. *Let $\psi \in L^2(\mathbb{R})$. If $\{T_{bk}D_{a_j}E_{d_j}\psi\}_{k\in\mathbb{Z},j\in J}$ is a frame with lower bound A, then*

$$A \le \sum_{j\in J} \frac{a_j}{b}|\hat{\psi}(a_j\gamma - d_j)|^2 \qquad a.e.\, \gamma \in \mathbb{R}. \tag{6.7}$$

Proof. Any SI system satisfies the LIC. Hence, in particular, the system $\{T_{bk}D_{a_j}E_{d_j}\psi\}_{k\in\mathbb{Z},j\in J}$ satisfies the LIC. The result is now immediate from Corollary 3.4. □

On the other hand, Example 4 shows that, in general, we cannot expect wave packet frames $\{D_{a_j}T_{bk}E_{d_j}\psi\}_{k\in\mathbb{Z},j\in J}$, even tight frames, to satisfy the LIC. Hence, in general, we can only say that if the wave packet system $\{D_{a_j}T_{bk}E_{d_j}\psi\}_{k\in\mathbb{Z},j\in J}$ is a frame with lower bound A *and* if it satisfies the LIC, then

$$A \le \sum_{j\in J} \frac{1}{b}|\hat{\psi}(a_j\gamma - d_j)|^2 \qquad \text{a.e.}\, \gamma \in \mathbb{R}. \tag{6.8}$$

6.2 Constructing dual wave packet frames

We now want to apply the general construction of dual GSI frames in Theorem 5.1 to the case of wave packet systems. We first consider how to construct suitable partitions of the unity.

Example 5. Let $f : [a^{j_0}, a^{j_0+1}] \to \mathbb{R}$ be a continuous function such that $f(a^{j_0}) = 0$, $f(a^{j_0+1}) = 1$ and $f(-\gamma + a^{j_0+1} + a^{j_0}) + f(\gamma) = 1$. Define

$$\hat{\psi}(\gamma) := \begin{cases} f(|\gamma|) & a.e.\ |\gamma| \in [a^{j_0}, a^{j_0+1}], \\ 1 - f(\frac{|\gamma|}{a}) & a.e.\ |\gamma| \in [a^{j_0+1}, a^{j_0+2}], \\ 0 & \text{otherwise.} \end{cases}$$

For almost every $\gamma \in [-a^{j_0+1}, a^{j_0+1}]$ there is exactly one $j \in J$ such that $a^j|\gamma| \in [a^{j_0}, a^{j_0+1}]$. Hence $\hat{\psi}(a^j\gamma) = f(|a^j\gamma|)$, $\hat{\psi}(a^{j+1}\gamma) = 1 - f(\frac{a^{j+1}|\gamma|}{a})$. Then for $J = \mathbb{N} \cup \{0\}$, we have

$$\sum_{j\in J} \hat{\psi}(a^j\gamma) = \begin{cases} 1 & a.e.\ \gamma \in [-a^{j_0+1}, a^{j_0+1}], \\ 1 - f(\frac{|\gamma|}{a}) & a.e.\ |\gamma| \in [a^{j_0+1}, a^{j_0+2}], \\ 0 & \text{otherwise.} \end{cases}$$

By shifting this function along $d\mathbb{Z}$, where $d = a^{j_0+1}(a+1)$, we have

$$\sum_{m\in\mathbb{Z}} \sum_{j\in J} \hat{\psi}(a^j(\gamma - dm)) = \sum_{m\in\mathbb{Z}} \sum_{j\in J} \hat{\psi}(a^j\gamma - a^j dm) = 1 \quad a.e.\ \gamma \in \mathbb{R}.$$

■

In the remainder of this section we let ψ be defined as in Example 5. Note that depending on the choice of f, we can make $\hat{\psi}$ as smooth as we like. Hence, we can construct generators ψ that are band-limited functions with arbitrarily fast decay in time domain and that have a partition of unity property.

Note that if $f(\gamma) \in [0,1]$ for all $\gamma \in \left[a^{j_0}, a^{j_0+1}\right]$, the function $\hat{\psi}$ is non-negative. In this case we can use the partition of unity to construct tight frames. Indeed, for the parameter choice $b < 2a^{j_0+2}$ the sums over $k \in \mathbb{Z}$ in Theorem 6.1 only have non-zero terms for $k = 0$. Define $\hat{\phi} = \hat{\psi}^{1/2}$. Since

$$\sum_{m\in\mathbb{Z}} \sum_{j\in J} |\hat{\phi}(a^j\gamma - a^j dm)|^2 = 1,$$

it follows from Theorem 6.1 that $\{D_{a^j} T_{bk} E_{a^j dm} \phi\}_{k,m\in\mathbb{Z}, j\in J}$ is a 1-tight wave packet frame for $L^2(\mathbb{R})$.

However, taking the square root of $\hat{\psi}$ might destroy desirable properties of the generator, e.g., if f is a polynomial, then $\hat{\psi}$ is piecewise polynomial, but this property is not necessarily inherited by $\hat{\phi} := \hat{\psi}^{1/2}$. By constructing dual frames from Theorem 5.1, we can circumvent this issue.

Example 6. In order to apply Theorem 5.1 we need to setup notation. Let $\tilde{\psi} = \sqrt{b}\psi$ and consider the wave packet system $\{D_{a^j} T_{bk} E_{a^j dm} \tilde{\psi}\}_{k,m\in\mathbb{Z}, j\in\mathbb{N}_0}$ as a GSI-system $\{T_{c_{(j,m)}k} g_{(j,m)}\}_{j\in\mathbb{N}_0, m,k\in\mathbb{Z}}$, where $g_{(j,m)} = D_{a^j} E_{a^j dm} \tilde{\psi}$ and $c_{(j,m)} = a^j b$ for all $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}$.

For $i \in I$, define $S_i = (di, d(i+1)]$. Since $\hat{g}_{(j,m)} = D_{a^{-j}} T_{a^j dm} \hat{\tilde{\psi}}$, we have

$$I_{(j,m)} = \{m, m-1\}, \quad J_i = \{(j,i), (j,i+1) : j \in \mathbb{N}_0\}.$$

Define the knots $\{\xi_k^{(i)}\}_{k,i\in\mathbb{Z}}$ by

$$\xi_k^{(i)} = \begin{cases} -a^{j_0+3-k} + d(i+1) & k > 0, \\ a^{j_0+k} + di & k \le 0. \end{cases}$$

Then the sequence $\{\xi_k^{(i)}\}_{k,i\in\mathbb{Z}}$ fulfills the properties in I in the setup from Section 5. For each $i \in \mathbb{Z}$, we define the bijective mapping $\varphi_i : J_i \mapsto \mathbb{Z}$ by $\varphi_i(j,i) = -j$ and $\varphi_i(j,i+1) = 1 + j$ for all $j \in \mathbb{N}_0$. Then

$$\operatorname{supp} \hat{g}_{(j,m)} \subset \bigcup_{i\in I_{(j,m)}} \left[\xi^{(i)}_{\varphi_i(j,m)}, \xi^{(i)}_{\varphi_i(j,m)+2}\right]$$

The definition of $h_{(j,m)}$ for $j \in \mathbb{N}$ and $m \in \mathbb{Z}$, is

$$\hat{h}_{(j,m)} = \begin{cases} \sum_{n=-1}^{1} a^m_{-j,n}\hat{g}_{(j-n,m)}(\gamma) & \gamma \in S_m, \\ \sum_{n=-1}^{1} a^{m-1}_{j+1,n}\hat{g}_{(j+n,m-1)}(\gamma) & \gamma \in S_{m-1}, \\ 0 & \gamma \in \mathbb{R} \setminus (S_m \cup S_{m-1}), \end{cases}$$

and, for $j = 0$, we have

$$\hat{h}_{(0,m)} = \begin{cases} a^m_{0,-1}\hat{g}_{(1,m)}(\gamma) + \hat{g}_{(0,m)}(\gamma) + a^m_{0,1}\hat{g}_{(0,m+1)}(\gamma) & \gamma \in S_m, \\ a^{m-1}_{1,-1}\hat{g}_{(0,m-1)}(\gamma) + \hat{g}_{(0,m)}(\gamma) + a^{m-1}_{1,1}\hat{g}_{(1,m)}(\gamma) & \gamma \in S_{m-1}, \\ 0 & \gamma \in \mathbb{R} \setminus (S_m \cup S_{m-1}). \end{cases}$$

Hence, if we define $\hat{\phi}_1 = T_{-a^j dm}D_{a^j}\hat{h}_{(j,m)}$ and $\hat{\phi}_0 = T_{-dm}\hat{h}_{(0,m)}$. Take any two numbers b_1, b_{-1} in $\mathbb{R}$ with $b_1 + b_{-1} = 2$ and set $a^m_{k+1,-n} = a^m_{-k,n} = a^{n/2}b_n$ for all $k \in \mathbb{N}$ and $m \in \mathbb{Z}$. We then have

$$\hat{\phi}_1(\gamma) = b_{-1}\hat{\hat{\psi}}(a\gamma) + \hat{\hat{\psi}}(\gamma) + b_1\hat{\hat{\psi}}(a^{-1}\gamma) \qquad \gamma \in \mathbb{R}. \tag{6.9}$$

For $j = 0$, based on Theorem 5.1, we should set $a^m_{0,-1} = a^m_{1,1} = a^{-1/2}b_{-1}$, also we set $c_{-1} = a^m_{1,-1}$ and $c_1 = a^m_{0,1}$ for all $m \in \mathbb{Z}$, where $c_{-1}, c_1 \in \mathbb{R}$ and $c_{-1} + c_1 = 2$. In this case, we have

$$\hat{\phi}_0 = b_{-1}\hat{\hat{\psi}}(a\gamma) + \hat{\hat{\psi}}(\gamma) + c_1\hat{\hat{\psi}}(\gamma - d) + c_{-1}\hat{\hat{\psi}}(\gamma + d) \qquad \gamma \in \mathbb{R}. \tag{6.10}$$

By Theorem 5.1, we conclude that the systems

$$\{T_{bk}E_{a^j dm}\psi\}_{k,m\in\mathbb{Z}} \cup \{D_{a^j}T_{bk}E_{a^j dm}\psi\}_{k,m\in\mathbb{Z},j\in\mathbb{N}}$$

and

$$\{b^{1/2}\, T_{bk}E_{dm}\phi_0\}_{k,m\in\mathbb{Z}} \cup \{b^{1/2}\, D_{a^j}T_{bk}E_{a^j dm}\phi_1\}_{k,m\in\mathbb{Z},j\in\mathbb{N}}$$

are dual wave packet frames for $L^2(\mathbb{R})$. Note that in the definitions of ϕ_0 and ϕ_1 in (6.10) and (6.9), respectively, we are free to choose any set of coefficients satisfying $b_1 + b_{-1} = 2$ and $c_{-1} + c_1 = 2$. ■

In Example 6 we constructed dual wave packet frames with *two* generators, akin to the case of scaling and wavelet functions for non-homogeneous wavelet systems. By a special choice of the coefficients b_1, b_{-1}, c_{-1}, c_1, we can reduce the number of generators to one.

Example 7. Take $b_1 = b_{-1} = c_{-1} = c_1 = 1$ in (6.9) and (6.10). Note that $\hat{\hat{\psi}}(a^{-1}\gamma)$ is equal to $\hat{\hat{\psi}}(\gamma - d) + \hat{\hat{\psi}}(\gamma + d)$ on the support of $\hat{\psi}$. Thus, $\hat{\phi}_0$ and $\hat{\phi}_1$ agree on $\operatorname{supp}\hat{\psi}$. Hence if we set

$$\hat{\phi}(\gamma) = b\hat{\psi}(a\gamma) + b\hat{\psi}(\gamma) + b\hat{\psi}(\gamma - d) + b\hat{\psi}(\gamma + d), \quad \gamma \in \mathbb{R},$$

then the wave packet systems $\{D_{a^j}T_{bk}E_{a^j dm}\psi\}_{k,m\in\mathbb{Z},j\in J}$ and $\{D_{a^j}T_{bk}E_{a^j dm}\phi\}_{k,m\in\mathbb{Z},j\in J}$ are dual frames for $L^2(\mathbb{R})$ for $b < a^{-j_0}(2a^2 + a - 1)^{-1}$. Alternatively, we can take

$$\hat{\phi}(\gamma) = b\hat{\psi}(a\gamma) + b\hat{\psi}(\gamma) + b\hat{\psi}(a^{-1}\gamma), \quad \gamma \in \mathbb{R},$$

in which case we need to take $b < a^{-j_0-2}(a + 1)^{-1}$ to obtain dual frames. ■

Acknowledgment: Marzieh Hasannasab would like to thank Kharazmi University for financial support during the preparation of the first draft of this manuscript. The three authors thank the reviewers for useful suggestions that improved the presentation.

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