Minkowski products of unit quaternion sets

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Abstract

The *Minkowski product* of unit quaternion sets is introduced and analyzed, motivated by the desire to characterize the overall variation of compounded spatial rotations that result from individual rotations subject to known uncertainties in their rotation axes and angles. For a special type of unit quaternion set, the spherical caps of the 3–sphere S^3 in $\mathbb{R}^4$, closure under the Minkowski product is achieved. Products of sets characterized by fixing either the rotation axis or rotation angle, and allowing the other to vary over a given domain, are also analyzed. Two methods for visualizing unit quaternion sets and their Minkowski products in $\mathbb{R}^3$ are also discussed, based on stereographic projection and the Lie algebra formulation. Finally, some general principles for identifying Minkowski product boundary points are discussed in the case of full–dimension set operands.

Keywords: Minkowski products, unit quaternions, spatial rotations 3–sphere, stereographic projection, Lie algebra, boundary evaluation.

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1 Introduction

The Minkowski sum $A \oplus B$ of two point sets $A, B \in \mathbb{R}^n$ is the set of all points generated [16] by the vector sums of points chosen independently from those sets, i.e.,

$$A \oplus B := \{ \mathbf{a} + \mathbf{b} \, : \, \mathbf{a} \in A \text{ and } \mathbf{b} \in B \} \,. \tag{1}$$

The Minkowski sum has applications in computer graphics, geometric design, image processing, and related fields [9, 11, 12, 13, 14, 15, 20]. The validity of the definition (1) in $\mathbb{R}^n$ for all $n \geq 1$ stems from the straightforward extension of the vector sum $\mathbf{a} + \mathbf{b}$ to higher–dimensional Euclidean spaces. However, to define a Minkowski *product* set

$$A \otimes B := \{ \mathbf{a}\,\mathbf{b} \, : \, \mathbf{a} \in A \text{ and } \mathbf{b} \in B \} \,, \tag{2}$$

it is necessary to specify *products* of points in $\mathbb{R}^n$. In the case $n = 1$, this is simply the real–number product — the resulting algebra of point sets in $\mathbb{R}^1$ is called *interval arithmetic* [17, 18] and is used to monitor the propagation of uncertainty through computations in which the initial operands (and possibly also the arithmetic operations) are not precisely determined.

A natural realization of the Minkowski product (2) in $\mathbb{R}^2$ may be achieved [7] by interpreting the points $\mathbf{a}$ and $\mathbf{b}$ as *complex numbers*, with $\mathbf{a}\,\mathbf{b}$ being the usual complex–number product. Algorithms to compute Minkowski products of complex–number sets have been formulated [6], and extended to determine Minkowski roots and powers [3, 8] of complex sets; to evaluate polynomials specified by complex–set coefficients and arguments [4]; and to solve simple equations expressed in terms of complex–set coefficients and unknowns [5]. The Minkowski algebra of complex sets introduces rich geometrical structures and has useful applications to mathematical morphology, geometrical optics, and the stability analysis of linear dynamic systems.

In proceeding to higher dimensions, it is natural to consider next the case of $\mathbb{R}^4$, in which a (non–commutative) "product of points" may be specified by invoking the quaternion algebra. In this context, the study of the Minkowski sum has no obvious and intuitive motivation, but the use of unit quaternions to describe spatial rotations provides a compelling case for the investigation of Minkowski products in $\mathbb{R}^4$. Applications in computer animation, robot path planning, 5–axis CNC machining, and related fields frequently involve compounded sequences of spatial rotations, that are individually subject to certain indeterminacies. The set of all possible outcomes of such compounded sequences of indeterminate spatial rotations possesses a natural description as the (ordered) Minkowski product of unit quaternion sets.

The remainder of this paper is organized as follows. Section 2 provides a brief review of the key properties of unit quaternions and their interpretation as rotation operators, while Section 3 discusses the visualization of the set of all unit quaternions (namely, the 3–sphere S^3 in $\mathbb{R}^4$) through its stereographic projection to $\mathbb{R}^3$. The concept of a Minkowski product of two unit quaternion sets is then introduced in Section 4. In Sections 5 and Section 6, we focus on a specific type of unit quaternion set, the *spherical caps* on S^3, and show that they exhibit closure under the Minkowski product. Section 7 then considers products of sets more closely related to applications, specified by fixing either the rotation axis or the angle, and varying the other over a given domain. As an alternative to stereographic projection, Section 8 describes a Lie algebra approach to visualizing unit quaternion sets and their Minkowski products, which has the virtue of generating bounded images in $\mathbb{R}^3$. Finally, Section 9 discusses general principles for identifying boundary points of Minkowski products, giving some necessary conditions and a sufficient condition for a product of two points to lie on the boundary. Section 10 summarizes the key results of this study and suggests further lines of investigation.

2 Quaternions and spatial rotations

Quaternions are "four–dimensional numbers" of the form

$$\mathcal{A} \,=\, a + a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \qquad \text{and} \qquad \mathcal{B} \,=\, b + b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}\,.$$

where the elements $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ of the quaternion algebra $\mathbb{H}$ obey the multiplication rules $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and $\mathbf{i}\,\mathbf{j} = \mathbf{k}$, $\mathbf{j}\,\mathbf{k} = \mathbf{i}$, $\mathbf{k}\,\mathbf{i} = \mathbf{j}$. A quaternion $\mathcal{A}$ may be regarded as comprising scalar (real) and vector (imaginary) parts $a = \mathrm{scal}(\mathcal{A})$ and $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} = \mathrm{vect}(\mathcal{A})$, and we write $\mathcal{A} = a + \mathbf{a}$. Real numbers and 3–vectors are subsumed as "pure scalar" and "pure vector" quaternions.

The sum and (non–commutative) product of $\mathcal{A} = a + \mathbf{a}$ and $\mathcal{B} = b + \mathbf{b}$ can be expressed using scalar and cross products of vectors as

$$\mathcal{A} + \mathcal{B} \,=\, a + b + \mathbf{a} + \mathbf{b}\,, \quad \mathcal{A}\,\mathcal{B} \,=\, ab - \langle \mathbf{a}, \mathbf{b} \rangle + a\,\mathbf{b} + b\,\mathbf{a} + \mathbf{a} \times \mathbf{b}\,.$$

Every quaternion $\mathcal{A} = a + \mathbf{a}$ has a *conjugate* $\mathcal{A}^* = a - \mathbf{a}$, and a non–negative *magnitude* $|\mathcal{A}|$ defined by $|\mathcal{A}|^2 = \mathcal{A}^*\mathcal{A} = \mathcal{A}\mathcal{A}^* = a^2 + |\mathbf{a}|^2$, and one can verify that $(\mathcal{A}\,\mathcal{B})^* = \mathcal{B}^*\mathcal{A}^*$ and $|\mathcal{A}\,\mathcal{B}| = |\mathcal{A}|\,|\mathcal{B}|$. If $|\mathcal{A}| \neq 0$, the quaternion $\mathcal{A}$ has an *inverse* $\mathcal{A}^{-1} = \mathcal{A}^*/|\mathcal{A}|^2$ satisfying $\mathcal{A}^{-1}\mathcal{A} = \mathcal{A}\,\mathcal{A}^{-1} = 1$, and $\mathcal{A}^{-1}\mathcal{B}$ and $\mathcal{B}\mathcal{A}^{-1}$ specify the *left* and *right* division of $\mathcal{B}$ by $\mathcal{A}$. We also define an *inner product* $\langle \mathcal{A}, \mathcal{B} \,\rangle$ of $\mathcal{A}$ and $\mathcal{B}$ (regarded as vectors in $\mathbb{R}^4$) by

$$\langle \mathcal{A}, \mathcal{B} \,\rangle \,:=\, a\,b + \langle \mathbf{a}, \mathbf{b} \rangle \,=\, \mathrm{scal}(\mathcal{A}\,\mathcal{B}^*)\,. \tag{3}$$

A *unit* quaternion $\mathcal{U} = u + u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$ satisfies $|\mathcal{U}| = 1$, and it may be identified with a point on the unit 3–sphere S^3 in $\mathbb{R}^4$ defined by the equation $u^2 + u_x^2 + u_y^2 + u_z^2 = 1$. Since a product $\mathcal{U}_1\,\mathcal{U}_2$ of two unit quaternions is also a unit quaternion, the points of S^3 have the structure of a (non–commutative) *group* with respect to the quaternion product. Note that the inner product (3) is invariant under multiplication of the operands by a unit quaternion,

$$\langle \mathcal{A}, \mathcal{B} \,\rangle \,=\, \langle\, \mathcal{U}\mathcal{A}, \mathcal{U}\mathcal{B} \,\rangle \,=\, \langle\, \mathcal{A}\mathcal{U}, \mathcal{B}\mathcal{U} \,\rangle\,, \tag{4}$$

since such multiplications correspond [2] to rotations in $\mathbb{R}^4$.

Any unit quaternion $\mathcal{U}$ may be expressed in the form

$$\mathcal{U} \,=\, \cos\tfrac{1}{2}\theta + \sin\tfrac{1}{2}\theta\,\mathbf{n} \tag{5}$$

for some angle $\theta \in [\,-\pi, \pi\,]$ and unit vector $\mathbf{n}$. This defines a *rotation operator* in $\mathbb{R}^3$ — for any pure vector $\mathbf{v}$, the product $\mathcal{U}\,\mathbf{v}\,\mathcal{U}^*$ also defines a pure vector, corresponding to a rotation of $\mathbf{v}$ through angle θ about an axis defined by $\mathbf{n}$. Note that $\mathcal{U}$ and $-\mathcal{U}$ define equivalent rotations.

Successive spatial rotations can be replaced by a "compounded" rotation — the result of consecutively applying rotations $\mathcal{U}_2 = \cos\frac{1}{2}\theta_2 + \sin\frac{1}{2}\theta_2\,\mathbf{n}_2$ and $\mathcal{U}_1 = \cos\frac{1}{2}\theta_1 + \sin\frac{1}{2}\theta_1\,\mathbf{n}_1$ to $\mathbf{v}$ is $\mathcal{U}_1\,(\,\mathcal{U}_2\mathbf{v}\,\mathcal{U}_2^*\,)\,\mathcal{U}_1^*$, which can be expressed as $\mathcal{U}\,\mathbf{v}\,\mathcal{U}^*$ with $\mathcal{U} = \mathcal{U}_1\,\mathcal{U}_2$. The non–commutative product captures the fact that the result of a sequence of rotations depends upon the *order* of their application. The rotation angle θ and axis $\mathbf{n}$ of $\mathcal{U}$ are given by

$$\cos\tfrac{1}{2}\theta \,=\, \cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2 - \sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\,\langle \mathbf{n}_1, \mathbf{n}_2 \rangle\,, \tag{6}$$

$$\mathbf{n} \,=\, \frac{\sin\frac{1}{2}\theta_1\cos\frac{1}{2}\theta_2\,\mathbf{n}_1 \,+\, \cos\frac{1}{2}\theta_1\sin\frac{1}{2}\theta_2\,\mathbf{n}_2 \,+\, \sin\frac{1}{2}\theta_1\sin\frac{1}{2}\theta_2\,\mathbf{n}_1 \times \mathbf{n}_2}{\sin\frac{1}{2}\theta}\,. \tag{7}$$

3 Stereographic projection to $\mathbb{R}^3$

The set of all unit quaternions occupies the 3–sphere S^3 in $\mathbb{R}^4$. To visualize S^3 a *stereographic projection* can be used to map it into $\mathbb{R}^3$, just as points on the 2–sphere can be imaged onto $\mathbb{R}^2$ to generate a map of the earth's surface. We recall the following definition.

Definition 1 *Consider the conformal map* $\Psi : \mathbb{H} \setminus \{1\} \to \mathbb{H} \setminus \{-1\}$ *defined by*

$$\Psi(\mathcal{Q}) := (1 - \mathcal{Q})^{-1}(1 + \mathcal{Q}) = (1 + \mathcal{Q})(1 - \mathcal{Q})^{-1}\,,$$

and its inverse

$$\Phi(\mathcal{Q}) := (\mathcal{Q} + 1)^{-1}(\mathcal{Q} - 1) = (\mathcal{Q} - 1)(\mathcal{Q} + 1)^{-1}\,.$$

The maps Ψ *and* Φ*, or rather their continuous extensions to the Alexandroff compactification* $\widehat{\mathbb{H}} := \mathbb{H} \cup \{\infty\}$*, are called* quaternionic Cayley transformations*.*

As conformal maps, Ψ and Φ map n–spheres and n–spaces to n–spheres and n–spaces for $0 \leq n \leq 3$. In particular, the next result concerns the unit 3-sphere in $\mathbb{H}$, denoted S^3, and the 3–space of purely imaginary quaternions, denoted $\mathbb{R}^3$.

Proposition 1 *The stereographic projection from the point* -1 *of* S^3 *to* $\mathbb{R}^3$*, defined by*

$$u + \mathbf{u} \mapsto \frac{\mathbf{u}}{1 + u}\,, \tag{8}$$

is the restriction of Φ *to* S^3*.*

Proof : By direct computation, for all $\mathcal{Q} = q + \mathbf{q} \in \mathbb{H} \setminus \{-1\}$ we have

$$\begin{aligned} \Phi(\mathcal{Q}) &:= (|\mathcal{Q}|^2 + 2q + 1)^{-1}(\mathcal{Q}^* + 1)(\mathcal{Q} - 1) \\ &= (|\mathcal{Q}|^2 + 2q + 1)^{-1}(|\mathcal{Q}|^2 + 2\,\mathbf{q} - 1)\,. \end{aligned} \tag{9}$$

Hence, writing $\mathcal{Q} = \mathcal{U} = u + \mathbf{u}$ when $|\mathcal{Q}| = 1$, we obtain

$$\Phi(\mathcal{U}) = \frac{2\,\mathbf{u}}{2 + 2u} = \frac{\mathbf{u}}{1 + u}\,. \quad \blacksquare$$

The set of unit quaternions $\mathcal{U} = u + u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k} \in S^3$ can be parameterized in terms of *hyperspherical coordinates* (α, β, γ) through the expression

$$(u, u_x, u_y, u_z) \,=\, (\cos\alpha, \sin\alpha\cos\beta, \sin\alpha\sin\beta\cos\gamma, \sin\alpha\sin\beta\sin\gamma)\,, \quad (10)$$

where $\alpha, \beta \in [\,0, \pi\,]$ and $\gamma \in [\,0, 2\pi\,]$. In terms of the scalar–vector form (5) with $\mathbf{n} = n_x\mathbf{i} + n_y\mathbf{j} + n_z\mathbf{k}$ we have

$$\theta \,=\, 2\alpha\,, \quad n_x \,=\, \cos\beta\,, \quad n_y \,=\, \sin\beta\cos\gamma\,, \quad n_z \,=\, \sin\beta\sin\gamma\,,$$

and conversely

$$\alpha \,=\, \tfrac{1}{2}\theta\,, \quad \beta \,=\, \arccos n_x\,, \quad \gamma \,=\, \arctan(n_y, n_z)\,,$$

where $\arctan(a, b)$ is the angle with cosine $a/\sqrt{a^2+b^2}$ and sine $b/\sqrt{a^2+b^2}$.

For each unit quaternion $\mathcal{U} \in S^3$, the point $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k} = \Phi(\mathcal{U}) \in \mathbb{R}^3$ defined by the stereographic projection (8) may be identified as the intersection with $\mathbb{R}^3$ of the line in $\mathbb{R}^4$ that passes through -1 and $\mathcal{U}$. In terms of the hyperspherical coordinates (10) on S^3, we have

$$(v_x, v_y, v_z) \,=\, \tan\tfrac{1}{2}\alpha\,(\cos\beta, \sin\beta\cos\gamma, \sin\beta\sin\gamma)\,. \quad (11)$$

Thus $\mathbf{v}$ may be interpreted as the point with "ordinary" spherical coordinates (β, γ) on the 2–sphere in $\mathbb{R}^3$ with radius $r = \tan\frac{1}{2}\alpha$. The points $\mathcal{U} = 1$ and $\mathcal{U} = -1$, corresponding to $\alpha = 0$ and $\alpha = \pi$, are mapped to the origin of $\mathbb{R}^3$ and to infinity, respectively. In terms of the scalar–vector form (5) of $\mathcal{U}$, the stereographic projection to $\mathbb{R}^3$ becomes

$$\mathbf{v} \,=\, \tan\tfrac{1}{4}\theta\,\mathbf{n}\,, \quad (12)$$

i.e., $\mathcal{U}$ is mapped to the point identified by the unit vector $\mathbf{n}$ on the 2–sphere of radius $r = \tan\frac{1}{4}\theta$ in $\mathbb{R}^3$. Note that, although the unit quaternion $-\mathcal{U} = -\cos\frac{1}{2}\theta - \sin\frac{1}{2}\theta\,\mathbf{n}$ specifies a rotation by angle $-\theta$ about $-\mathbf{n}$, equivalent to that specified by $\mathcal{U}$, it is mapped to the *distinct* point

$$\mathbf{v}' \,=\, -\cot\tfrac{1}{4}\theta\,\mathbf{n}\,.$$

4 Quaternionic Minkowski product

Definition 2 *The* Minkowski product *of two subsets U, V of $\mathbb{H}$ is defined by*

$$U \otimes V \,=\, \{\, \mathcal{U}\mathcal{V} \,:\, \mathcal{U} \in U\,,\, \mathcal{V} \in V\,\}\,.$$

If $U, V \subseteq S^3$, then the elements of $U \otimes V$ describe all possible compounded rotations generated by a rotation $\mathcal{V} \in V$ followed by a rotation $\mathcal{U} \in U$.

Remark 1 *As a consequence of the properties of quaternionic multiplication, $\otimes$ is an associative but noncommutative operation on the power set of $\mathbb{H}$.*

The following Lemma can be verified by direct computation.

Lemma 1 *Let*

$$\begin{aligned} T : \mathbb{H} \times \mathbb{H} &\to \mathbb{H} \\ (\mathcal{P}, \mathcal{Q}) &\mapsto \mathcal{P}\mathcal{Q} \,. \end{aligned}$$

Then for all $\mathcal{V} \in \mathbb{H}$ the directional derivatives of T in the directions $(\mathcal{V}, 0)$ and $(0, \mathcal{V})$ are

$$\frac{\partial T}{\partial(\mathcal{V}, 0)}(\mathcal{P}, \mathcal{Q}) = \mathcal{V}\mathcal{Q} \quad \textit{and} \quad \frac{\partial T}{\partial(0, \mathcal{V})}(\mathcal{P}, \mathcal{Q}) = \mathcal{P}\mathcal{V} \,.$$

Remark 2 *Consider two unit quaternions*

$$\mathcal{U}_1 = \cos\tfrac{1}{2}\theta_1 + \sin\tfrac{1}{2}\theta_1 \, \mathbf{n}_1, \quad \mathcal{U}_2 = \cos\tfrac{1}{2}\theta_2 + \sin\tfrac{1}{2}\theta_2 \, \mathbf{n}_2$$

and the unit quaternions $\mathcal{U}_1 + \delta\,\mathcal{U}_1, \mathcal{U}_2 + \delta\,\mathcal{U}_2$ that result from perturbations $\delta\theta_1, \delta\mathbf{n}_1, \delta\theta_2, \delta\mathbf{n}_2$ to the rotation angles and axes of $\mathcal{U}_1, \mathcal{U}_2$. By Lemma 1, the following equality holds to first order:

$$(\mathcal{U}_1 + \delta\,\mathcal{U}_1)(\mathcal{U}_2 + \delta\,\mathcal{U}_2) = \mathcal{U}_1\,\mathcal{U}_2 + \delta\mathcal{U}_1\,\mathcal{U}_2 + \mathcal{U}_1\,\delta\mathcal{U}_2 \,.$$

Thus, to first order in $\delta\theta_1, \delta\mathbf{n}_1$ and $\delta\theta_2, \delta\mathbf{n}_2$ the product $(\mathcal{U}_1 + \delta\,\mathcal{U}_1)(\mathcal{U}_2 + \delta\,\mathcal{U}_2)$ is always distinct from $\mathcal{U}_1\,\mathcal{U}_2$ — i.e., the unit quaternion product map $S^3 \times S^3 \to S^3$ has no stationary points.

We address the problem of determining Minkowski products of different types of subsets of S^3. For subsets U, V of S^3 of full dimension, we would ideally like to determine either (i) a “faithful” (one–to-one) parameterization, over a suitable domain in three parameters, of the product set $U \otimes V \subset S^3$; or (ii) a characterization of its boundary $\partial(U \otimes V)$ in S^3. Problem (ii) is, in general, more tractable. We show in Section 9 that $\partial(U \otimes V) \subseteq \partial U \otimes \partial V$, so (ii) amounts to identifying corresponding points $\mathcal{U} \in \partial U$ and $\mathcal{V} \in \partial V$ that generate (potential) points on the Minkowski product boundary $\partial(U \otimes V)$.

5 Unit quaternion spherical caps

In the Minkowski algebra of complex sets [7], emphasis was placed on circular disks as set operands, and it seems natural to extend this to the context of unit quaternion sets. For this reason, the first class of subsets of S^3 that we will study is that of *spherical caps*, namely the subsets

$$\{\,\mathcal{U}\in S^3 : |\,\mathcal{U}-\mathcal{U}_0| \le \rho\,\} \tag{13}$$

defined by the intersection of S^3 with a 4–ball that has a prescribed radius ρ and unit quaternion $\mathcal{U}_0$ as center. The set (13) includes all unit quaternions $\mathcal{U}$ whose distance (measured on S^3) from $\mathcal{U}_0$ does not exceed ρ. In the case of deviations $\delta\mathcal{U}$ resulting from small perturbations $\delta\theta$ and $\delta\mathbf{n}$ to the rotation angle θ_0 and axis $\mathbf{n}_0$ of $\mathcal{U}_0$, it includes all unit quaternions satisfying

$$|\,\delta\mathcal{U}\,| \,=\, \sqrt{\tfrac{1}{4}(\delta\theta)^2+\sin^2\tfrac{1}{2}\theta_0\,|\delta\mathbf{n}|^2} \,<\, \rho\,.$$

Remark 3 *For $\mathcal{U}_0\in S^3$ the intersection of S^3 with the ball of radius ρ and center $\mathcal{U}_0$ in $\mathbb{H}$ is identical to its intersection with a half–space orthogonal to $\mathcal{U}_0$, namely*

$$\{\,\mathcal{U}\in S^3 : |\,\mathcal{U}-\mathcal{U}_0| \le \rho\,\} = \{\,\mathcal{U}\in S^3 : \langle\,\mathcal{U},\mathcal{U}_0\rangle \ge 1-\tfrac{1}{2}\rho^2\,\}\,,$$

since $|\,\mathcal{U}-\mathcal{U}_0|^2 = 2\,(1-\langle\,\mathcal{U},\mathcal{U}_o\rangle)$. We distinguish three cases:

- *for $\rho\ge 2$, this set coincides with S^3;*
- *for $0<\rho<2$ it is a proper subset of S^3 — a* spherical cap *— and its boundary in the topology of S^3 induced by $\mathbb{R}^4$ is a 2–sphere;*
- *for $\rho=0$, it is the singleton set $\{\,\mathcal{U}_0\}$.*

Based on the preceding remark, we set $\rho = 2\sin\frac{1}{2}t$ with $t\in[\,0,\pi\,]$ so that $1-\frac{1}{2}\rho^2=\cos t$, and formulate unit quaternion spherical caps as follows.

Definition 3 *For $\mathcal{U}_0\in S^3$ and $t\in[\,0,\pi\,]$, we define*

$$U(\mathcal{U}_0,t) := \{\,\mathcal{U}\in S^3 : \langle\,\mathcal{U},\mathcal{U}_0\rangle \ge \cos t\,\}\,.$$

and denote its boundary in the topology of S^3 as $\partial U(\mathcal{U}_0,t)$.

Regarding quaternions $\mathcal{U} \in S^3$ as unit vectors in $\mathbb{R}^4$, we can also interpret $U(\mathcal{U}_0, t)$ as the intersection of S^3 with the cone of vectors $\mathcal{U}$ whose inclinations with $\mathcal{U}_0$ do not exceed $t = 2 \arcsin \frac{1}{2}\rho$. We note that, if $\mathcal{U}_0 = 1$, then $\langle \mathcal{U}, \mathcal{U}_0 \rangle$ is just the scalar part of $\mathcal{U}$. For the unit quaternion $\mathcal{U} = \cos \frac{1}{2}\theta + \sin \frac{1}{2}\theta \, \mathbf{n}$ corresponding to a rotation through angle θ about the axis $\mathbf{n}$, the condition $|\mathcal{U} - 1| \leq \rho = 2 \sin \frac{1}{2} t$ reduces to

$$\cos \tfrac{1}{2}\theta \;\geq\; 1 - \tfrac{1}{2}\rho^2 \;=\; \cos t \,, \tag{14}$$

i.e., $|\theta| \leq 2\,t = 4 \arcsin \frac{1}{2}\rho$. The boundary $\partial U(1,t)$ corresponds to satisfaction of (14) with equality, i.e., $|\theta| = 2\,t = 4 \arcsin \frac{1}{2}\rho$. As observed in Remark 3, $\mathcal{U}(1,t) = \{1\}$ if $t = 0$ ($\rho = 0$), and $\mathcal{U}(1,t) = S^3$ if $t = \pi$ ($\rho = 2$). In the case $t = \frac{1}{2}\pi$ ($\rho = \sqrt{2}$), it is the set of all unit quaternions with any rotation axis $\mathbf{n}$ and rotation angles $\theta \in [\,-\pi, \pi\,]$.

Remark 4 *Setting* $\exp(s\,\mathbf{n}) = \cos s + \sin s\,\mathbf{n}$*, we have*

$$U(1,t) = \{\, \exp(s\,\mathbf{n}) : 0 \leq s \leq t, |\mathbf{n}| = 1 \,\} \,.$$

5.1 Visualization by stereographic projection

We now consider the stereographic projection of the set $U(\mathcal{U}_0, t)$ onto $\mathbb{R}^3$.

Proposition 2 *Let* $\mathcal{U}_0 \in S^3$ *and* $t \in (0, \pi)$*.*

1. *If* $-1 \notin U(\mathcal{U}_0, t)$ *then* $\Phi(U(\mathcal{U}_0, t))$ *is a closed* 3*–ball in* $\mathbb{R}^3$*.*
2. *If* $-1 \in \partial U(\mathcal{U}_0, t)$ *then* $\Phi(U(\mathcal{U}_0, t) \setminus \{-1\})$ *is a closed half–space in* $\mathbb{R}^3$*.*
3. *If* $-1 \in U(\mathcal{U}_0, t) \setminus \partial U(\mathcal{U}_0, t)$ *then* $\Phi(U(\mathcal{U}_0, t) \setminus \{-1\})$ *is* $\mathbb{R}^3$ *minus an open 3–ball.*

Finally, for $t = 0$ *the image of* $U(\mathcal{U}_0, t)$ *through* Φ *is* $\{\Phi(\mathcal{U}_0)\}$*, and for* $t = \pi$ *it is* $\mathbb{R}^3$*.*

Proof : Since the statements for the cases $t = 0$ and π are trivial, we focus on the case $t \in (0, \pi)$. The set $U(\mathcal{U}_0, t)$ is the intersection of S^3 with a 4–ball B centered at $\mathcal{U}_0$ in $\mathbb{H}$. Thus,

$$\Phi(U(\mathcal{U}_0, t)) = \Phi(S^3) \cap \Phi(B) = \mathbb{R}^3 \cap \Phi(B) \,.$$

where $\Phi(B)$ is a closed subset of $\widehat{\mathbb{H}}$ that includes $\Phi(\mathcal{U}_0) \in \mathbb{R}^3 \cup \{\infty\}$ as an interior point. The proposition then follows from the following observations:

1. If $-1 \notin U(\mathcal{U}_0, t)$ then $-1 \notin B$ and $\Phi(B)$ is a closed 4–ball in $\mathbb{H}$.
2. If $-1 \in \partial U(\mathcal{U}_0, t)$ then $-1 \in \partial B$ and $\Phi(B \setminus \{-1\})$ is a closed half–space in $\mathbb{H}$.
3. If $-1 \in U(\mathcal{U}_0, t) \setminus \partial U(\mathcal{U}_0, t)$ then -1 is an interior point of B and $\Phi(B \setminus \{-1\})$ is $\mathbb{H}$ minus an open 4–ball. ∎

6 Products of unit quaternion spherical caps

We now compute the product $U(\mathcal{U}_0, s) \otimes U(\mathcal{V}_0, t)$. To this end, the following remark will be useful.

Remark 5 *For all $\mathcal{U}_0, \mathcal{V}_0 \in S^3$ and for all $t \in [\,0, \pi\,]$, we have*

$$U(\mathcal{U}_0, t) \otimes \{\mathcal{V}_0\} = U(\mathcal{U}_0 \mathcal{V}_0, t) = \{\mathcal{U}_0\} \otimes U(\mathcal{V}_0, t),$$

since from (4) we note that $\langle \mathcal{U}\mathcal{V}_0^*, \mathcal{U}_0 \rangle = \langle \mathcal{U}, \mathcal{U}_0 \mathcal{V}_0 \rangle = \langle \mathcal{U}_0^* \mathcal{U}, \mathcal{V}_0 \rangle$.

As a first consequence, it is possible to visualize $U(\mathcal{U}_0, t) = U(1, t) \otimes \{\mathcal{U}_0\}$ and its boundary $\partial U(\mathcal{U}_0, t) = \partial U(1, t) \otimes \{\mathcal{U}_0\}$ as copies of $U(1, t)$ and $\partial U(1, t)$, rotated within S^3 so as to have center $\mathcal{U}_0$ instead of 1. We recall that $U(1, t)$ and $\partial U(1, t)$ have been described in detail in the first part of Section 5.

Before considering the general product $U(\mathcal{U}_0, s) \otimes U(\mathcal{V}_0, t)$, it is instructive to examine a special case.

Lemma 2 *Let $s, t \in [\,0, \pi\,]$. Then*

$$U(1, s) \otimes U(1, t) = \begin{cases} U(1, s + t) & \textit{if } s + t \in [\,0, \pi\,], \\ S^3 & \textit{if } s + t \in [\,\pi, 2\pi\,]. \end{cases}$$

Proof : By Remark 4, $U(1, s) \otimes U(1, t)$ is the set of all products of the form $\exp(a\,\mathbf{m})\,\exp(b\,\mathbf{n})$ with $|\mathbf{m}| = |\mathbf{n}| = 1$, $0 \le a \le s$, $0 \le b \le t$. Now, the scalar part of $\exp(a\,\mathbf{m})\,\exp(b\,\mathbf{n})$ is equal to

$$\cos a \cos b - \sin a \sin b \,\langle \mathbf{m}, \mathbf{n} \rangle,$$

which is greater than or equal to $\cos a \cos b - \sin a \sin b = \cos(a + b)$. If $s + t \in [\,0, \pi\,]$ this bound implies that $U(1, s) \otimes U(1, t) \subseteq U(1, s+t)$. If $s+t \in [\,\pi, 2\pi\,]$, the bound only implies the trivial inclusion $U(1, s) \otimes U(1, t) \subseteq S^3$.

On the other hand, let $\mathcal{U} = u + \mathbf{u} \in S^3$. If we can identify real numbers a, b with $0 \leq a \leq s$ and $0 \leq b \leq t$ such that $u = \cos(a+b)$, then $\mathcal{U} = \exp((a+b)\,\mathbf{p})$ for a suitably chosen $\mathbf{p}$ with $|\mathbf{p}| = 1$, and we conclude that $\mathcal{U} = \exp(a\,\mathbf{p})\exp(b\,\mathbf{p}) \in U(1,s) \otimes U(1,t)$. If $s+t \in [\,0,\pi\,]$, then such a and b exist when $u \geq \cos(s+t)$. If $s+t \in [\,\pi, 2\pi\,]$, then they exist for all $u \geq -1$. This proves that $U(1,s) \otimes U(1,t) \supseteq U(1,s+t)$ in the former case, and that $U(1,s) \otimes U(1,t) \supseteq S^3$ in the latter case. ∎

We are now ready to present the general result for the Minkowski products of unit quaternion spherical caps.

Theorem 1 *Let $\mathcal{U}_0, \mathcal{V}_0 \in S^3$ and $s,t \in [\,0,\pi\,]$. Then*

$$U(\mathcal{U}_0, s) \otimes U(\mathcal{V}_0, t) = \begin{cases} U(\mathcal{U}_0\mathcal{V}_0, s+t) & \text{if } s+t \in [\,0,\pi\,], \\ S^3 & \text{if } s+t \in [\,\pi, 2\pi\,]. \end{cases}$$

Proof : From Remark 5, we have $U(\mathcal{U}_0, s) = \{\mathcal{U}_0\} \otimes U(1,s)$ and $U(\mathcal{V}_0, t) = U(1,t) \otimes \{\mathcal{V}_0\}$. Taking into account Remark 1, we write

$$U(\mathcal{U}_0, s) \otimes U(\mathcal{V}_0, t) = \{\mathcal{U}_0\} \otimes U(1,s) \otimes U(1,t) \otimes \{\mathcal{V}_0\}\,.$$

We now apply Lemma 2. If $s+t \in [\,0,\pi\,]$ then

$$U(\mathcal{U}_0, s) \otimes U(\mathcal{V}_0, t) = \{\mathcal{U}_0\} \otimes U(1,s+t) \otimes \{\mathcal{V}_0\}\,,$$

whence

$$U(\mathcal{U}_0, s) \otimes U(\mathcal{V}_0, t) = U(\mathcal{U}_0\mathcal{V}_0, s+t)$$

by two further applications of Remark 5. However, if $s+t \in [\,\pi, 2\pi\,]$, then

$$U(\mathcal{U}_0, s) \otimes U(\mathcal{V}_0, t) = \{\mathcal{U}_0\} \otimes S^3 \otimes \{\mathcal{V}_0\} = S^3\,. \quad ∎$$

7 Bounded rotation angles and axes

Although unit quaternion spherical caps admit a simple and elegant theory for their Minkowski products, they correspond to somewhat complicated and non–intuitive relations among the feasible rotation axes and angles. We now consider different types of sets, of greater relevance to the physical actuators used in robot manipulators, 5–axis milling machines, and related contexts.

Specifically, we analyze below the Minkowski products of sets defined by (1) fixed rotation axes, and rotation angles that vary over prescribed subsets of $[-\pi, \pi]$; and (2) fixed rotation angles, and rotation axes that deviate from prescribed directions by no more than a given angle. In case (1) the operand sets are curves and their product is a 2–surface in S^3, while in case (2) the operand sets are 2–surfaces and their product is of full dimension in S^3.

7.1 Fixed rotation axis, bounded angle

Let us consider the following sets.

Definition 4 *For each quaternionic imaginary unit* $\mathbf{c}$*, and for all* $\phi \in \mathbb{R}$ *and all* $\delta \in [0, \pi]$*,*

$$C(\mathbf{c}, \phi, \delta) := \{ \exp(s\,\mathbf{c}) : |s - \phi| \leq \delta \} .$$

$C(\mathbf{c}, \phi, \pi)$ is the great circle in S^3 that passes through 1 and $\exp(\phi\,\mathbf{c})$. For all $\delta \in (0, \pi)$ the set $C(\mathbf{c}, \phi, \delta)$ is an arc of this great circle. Finally, $C(\mathbf{c}, \phi, 0)$ is the singleton $\{\exp(\phi\,\mathbf{c})\}$.

Proposition 3 *For a fixed quaternionic imaginary unit* $\mathbf{c}$*, angles* $\phi_1, \phi_2 \in \mathbb{R}$*, and ranges* $\delta_1, \delta_2 \in [0, \pi]$*, the product* $C(\mathbf{c}, \phi_1, \delta_1) \otimes C(\mathbf{c}, \phi_2, \delta_2)$ *is a circle, a circular arc, or a singleton:*

$$C(\mathbf{c}, \phi_1, \delta_1) \otimes C(\mathbf{c}, \phi_2, \delta_2) = \begin{cases} C(\mathbf{c}, \phi_1 + \phi_2, \delta_1 + \delta_2) & \text{if } \delta_1 + \delta_2 \in [0, \pi], \\ C(\mathbf{c}, \phi_1 + \phi_2, \pi) & \text{if } \delta_1 + \delta_2 \in [\pi, 2\pi]. \end{cases}$$

Proof : The statement is an immediate consequence of the fact that

$$\exp(s\,\mathbf{c}) \exp(t\,\mathbf{c}) = \exp((s + t)\mathbf{c})$$

for all $s, t \in \mathbb{R}$. ∎

We now study the nature of the product $C(\mathbf{c}_1, \phi_1, \delta_1) \otimes C(\mathbf{c}_2, \phi_2, \delta_2)$ in greater detail. To this end, the following remark will prove useful.

Remark 6 *For each quaternionic imaginary unit* $\mathbf{c}$*, and for all* $\phi \in \mathbb{R}$ *and all* $\delta \in [0, \pi]$*,*

$$C(\mathbf{c}, 0, \delta) \otimes \{\exp(\phi\,\mathbf{c})\} = C(\mathbf{c}, \phi, \delta) = \{\exp(\phi\,\mathbf{c})\} \otimes C(\mathbf{c}, 0, \delta) .$$

Theorem 2 *For fixed quaternionic imaginary units $\mathbf{c}_1, \mathbf{c}_2$ with $\mathbf{c}_2 \neq \pm\,\mathbf{c}_1$, the product*

$$C(\mathbf{c}_1, \phi_1, \delta_1) \otimes C(\mathbf{c}_2, \phi_2, \delta_2) \subset S^3$$

is an immersed 2–surface in $\mathbb{H}$, possibly with boundary. The smallest unit quaternion spherical cap $U(\exp(\phi_1 \mathbf{c}_1)\exp(\phi_2 \mathbf{c}_2), \eta)$ that includes this product has $\eta := \arccos(r)$, where

$$r := \min_{|s|\le\delta_1, |t|\le\delta_2} (\cos s \cos t - \sin s \sin t \, \langle \mathbf{c}_1, \mathbf{c}_2 \rangle) \,. \tag{15}$$

Moreover, if neither δ_1 nor δ_2 is equal to π, and at least one of them is less than $\frac{1}{2}\pi$, then $C(\mathbf{c}_1, \phi_1, \delta_1) \otimes C(\mathbf{c}_2, \phi_2, \delta_2) \subset S^3$ is an embedded 2–surface in $\mathbb{H}$, with boundary. Its boundary consists of four circular arcs which have pairwise intersections at the four points

$$\exp((\phi_1 \pm \delta_1)\mathbf{c}_1) \exp((\phi_2 \pm \delta_2)\mathbf{c}_2) \,, \quad \exp((\phi_1 \pm \delta_1)\mathbf{c}_1) \exp((\phi_2 \mp \delta_2)\mathbf{c}_2) \,.$$

Proof : By Remarks 5 and 6, it suffices to consider the case $\phi_1 = \phi_2 = 0$. Consider the surjective map

$$\begin{aligned} P : [-\delta_1, \delta_1] \times [-\delta_2, \delta_2] &\to C(\mathbf{c}_1, 0, \delta_1) \otimes C(\mathbf{c}_2, 0, \delta_2) \\ (s,t) &\mapsto \exp(s\,\mathbf{c}_1) \exp(t\,\mathbf{c}_2) \,. \end{aligned}$$

Note that P is non–singular, since the s–derivative $\exp(s\,\mathbf{c}_1)\,\mathbf{c}_1 \exp(t\,\mathbf{c}_2)$ and t–derivative $\exp(s\,\mathbf{c}_1)\exp(t\,\mathbf{c}_2)\,\mathbf{c}_2 = \exp(s\,\mathbf{c}_1)\,\mathbf{c}_2 \exp(t\,\mathbf{c}_2)$ cannot be linearly dependent over $\mathbb{R}$: if they were, then $\mathbf{c}_1, \mathbf{c}_2$ would also be linearly dependent, contradicting the hypothesis $\mathbf{c}_2 \neq \pm\mathbf{c}_1$.

Now let us determine for which η the inclusion

$$C(\mathbf{c}_1, 0, \delta_1) \otimes C(\mathbf{c}_2, 0, \delta_2) \subseteq U(1, \eta)$$

holds. The scalar part of the product $\exp(s\,\mathbf{c}_1)\exp(t\,\mathbf{c}_2)$ is

$$\cos s \cos t - \sin s \sin t \, \langle \mathbf{c}_1, \mathbf{c}_2 \rangle \,.$$

This quantity spans the whole interval $[\,r, 1\,]$ with

$$r := \min_{|s|\le\delta_1, |t|\le\delta_2} (\cos s \cos t - \sin s \sin t \, \langle \mathbf{c}_1, \mathbf{c}_2 \rangle) \,.$$

Moreover, P is an embedding if and only if P is injective. The equality $\exp(p\,\mathbf{c}_1)\exp(r\,\mathbf{c}_2) = \exp(s\,\mathbf{c}_1)\exp(t\,\mathbf{c}_2)$ holds if, and only if, $\exp((p-s)\mathbf{c}_1) =$

$\exp((t-r)\mathbf{c}_2)$, i.e., $p-s, t-r \in \{-2\pi, 0, 2\pi\}$ or $p-s, t-r \in \{\pm\pi\}$. Thus, P is injective if and only if neither δ_1 nor δ_2 equals π and at least one of them is less than $\frac{1}{2}\pi$. When P is an embedding, the boundary of $C(\mathbf{c}_1, 0, \delta_1) \otimes C(\mathbf{c}_2, 0, \delta_2)$ consists of the four circular arcs

$$[-\delta_1, \delta_1] \to C(\mathbf{c}_1, 0, \delta_1) \otimes C(\mathbf{c}_2, 0, \delta_2) \quad s \mapsto \exp(s\,\mathbf{c}_1)\exp(\pm\delta_2\mathbf{c}_2)\,,$$

$$[-\delta_2, \delta_2] \to C(\mathbf{c}_1, 0, \delta_1) \otimes C(\mathbf{c}_2, 0, \delta_2) \quad t \mapsto \exp(\pm\delta_1\mathbf{c}_1)\exp(t\,\mathbf{c}_2)\,,$$

as desired. ∎

Proposition 4 *If $\delta_1, \delta_2 \in [\,0, \frac{1}{2}\pi\,]$ then the smallest unit quaternion spherical cap $U(\exp(\phi_1\mathbf{c}_1)\exp(\phi_2\mathbf{c}_2), \eta)$ that includes $C(\mathbf{c}_1, \phi_1, \delta_1) \otimes C(\mathbf{c}_2, \phi_2, \delta_2)$ has*

$$\eta = \arccos\left(\cos\delta_1\cos\delta_2 - \sin\delta_1\sin\delta_2|\langle\mathbf{c}_1, \mathbf{c}_2\rangle|\right) < \delta_1 + \delta_2\,.$$

Furthermore, the boundary of the embedded surface $C(\mathbf{c}_1, \phi_1, \delta_1) \otimes C(\mathbf{c}_2, \phi_2, \delta_2)$ intersects the boundary of $U(\exp(\phi_1\mathbf{c}_1)\exp(\phi_2\mathbf{c}_2), \eta)$ either at the two corners $\exp((\phi_1 \pm \delta_1)\mathbf{c}_1)\exp((\phi_2 \pm \delta_2)\mathbf{c}_2)$ or at $\exp((\phi_1 \pm \delta_1)\mathbf{c}_1)\exp((\phi_2 \mp \delta_2)\mathbf{c}_2)$.

Proof : Let $F(s,t) := \cos s \cos t - \sin s \sin t\,\langle\mathbf{c}_1, \mathbf{c}_2\rangle$. By direct computation, the only critical point of F in the interior of $[-\delta_1, \delta_1] \times [-\delta_2, \delta_2]$ is its maximum point $(0,0)$. Moreover, the restrictions to $(-\delta_1, \delta_1) \times \{\pm\delta_2\}$, $\{\pm\delta_1\} \times (-\delta_2, \delta_2)$ are concave. Hence, the minimum of F in $[-\delta_1, \delta_1] \times [-\delta_2, \delta_2]$ is attained at one of the four corner points $(\pm\delta_1, \pm\delta_2), (\pm\delta_1, \mp\delta_2)$. Now

$$F(\pm\delta_1, \pm\delta_2) = \cos\delta_1\cos\delta_2 - \sin\delta_1\sin\delta_2\,\langle\mathbf{c}_1, \mathbf{c}_2\rangle,$$

$$F(\pm\delta_1, \mp\delta_2) = \cos\delta_1\cos\delta_2 + \sin\delta_1\sin\delta_2\,\langle\mathbf{c}_1, \mathbf{c}_2\rangle,$$

and consequently

$$\eta = \arccos\left(\cos\delta_1\cos\delta_2 - \sin\delta_1\sin\delta_2|\langle\mathbf{c}_1, \mathbf{c}_2\rangle|\right) < \delta_1 + \delta_2\,,$$

since $\cos(\delta_1 + \delta_2) = \cos\delta_1\cos\delta_2 - \sin\delta_1\sin\delta_2$. ∎

7.2 Bounded rotation axis, fixed angle

We now consider the following sets.

Definition 5 *For each quaternionic imaginary unit* $\mathbf{c}$ *and for all* $\phi \in (0, \pi)$ *and all* $\xi \in [\,0, \pi\,]$*, we define*

$$S(\mathbf{c}, \phi, \xi) := \{\, \exp(\phi\, \mathbf{m}) = \cos\phi + \sin\phi\, \mathbf{m} : \langle \mathbf{m}, \mathbf{c} \rangle \geq \cos\xi \,\}\,.$$

$S(\mathbf{c}, \phi, \pi)$ is the 2–sphere obtained by intersecting S^3 with the 3–space of quaternions whose scalar part is equal to $\cos\phi$. For all $\xi \in (0, \pi)$, the set $S(\mathbf{c}, \phi, \xi)$ is a spherical cap of that 2–sphere, whose boundary in the topology of the 2–sphere is a circle:

$$bS(\mathbf{c}, \phi, \xi) = \{\, \exp(\phi\, \mathbf{m}) = \cos\phi + \sin\phi\, \mathbf{m} : \langle \mathbf{m}, \mathbf{c} \rangle = \cos\xi \,\}\,.$$

Here we introduce the symbol b to distinguish this type of boundary from the boundary ∂ of the same set in the topology of S^3. Finally, we note that $S(\mathbf{c}, \phi, 0) = \{\, \exp(\phi\, \mathbf{c}) \,\}$.

Proposition 5 *Choose quaternionic imaginary units* $\mathbf{c}_1, \mathbf{c}_2$ *and let* $\phi_1, \phi_2 \in (0, \pi)$ *and* $\xi_1, \xi_2 \in (0, \pi\,]$*. Then the rank of the real differential of the map*

$$\begin{aligned} &S(\mathbf{c}_1, \phi_1, \xi_1) \times S(\mathbf{c}_2, \phi_2, \xi_2) \to S^3 \\ &\qquad (\exp(\phi_1 \mathbf{m}), \exp(\phi_2 \mathbf{n})) \mapsto \exp(\phi_1 \mathbf{m}) \exp(\phi_2 \mathbf{n}) \end{aligned}$$

at a point $(\exp(\phi_1 \mathbf{m}), \exp(\phi_2 \mathbf{n}))$ *is less than* 3 *if and only if* $\mathbf{m} = \pm\mathbf{n}$.

Proof : We denote the map by σ and fix a point $(\exp(\phi_1 \mathbf{m}), \exp(\phi_2 \mathbf{n})) \in S(\mathbf{c}_1, \phi_1, \xi_1) \times S(\mathbf{c}_2, \phi_2, \xi_2)$. By Lemma 1, we have

$$\frac{\partial \sigma}{\partial(\mathbf{v}, 0)}(\exp(\phi_1 \mathbf{m}), \exp(\phi_2 \mathbf{n})) = \mathbf{v} \exp(\phi_2 \mathbf{n})\,,$$

$$\frac{\partial \sigma}{\partial(0, \mathbf{w})}(\exp(\phi_1 \mathbf{m}), \exp(\phi_2 \mathbf{n})) = \exp(\phi_1 \mathbf{m})\, \mathbf{w}\,,$$

for all $(\mathbf{v}, 0)$ and $(0, \mathbf{w})$ in the tangent space to $S(\mathbf{c}_1, \phi_1, \xi_1) \times S(\mathbf{c}_2, \phi_2, \xi_2)$ at the point $(\exp(\phi_1 \mathbf{m}), \exp(\phi_2 \mathbf{n}))$, i.e., for all

$$\mathbf{v} \in \Pi_{\mathbf{m}} := \{\mathbf{v} : \mathbf{v} \perp \mathbf{m}\}, \quad \mathbf{w} \in \Pi_{\mathbf{n}} := \{\mathbf{w} : \mathbf{w} \perp \mathbf{n}\}.$$

The image of the differential of σ is the sum $\Pi_1 + \Pi_2$ of the 2–plane $\Pi_1 := \Pi_{\mathbf{m}} \exp(\phi_2 \mathbf{n})$ through the origin and the 2–plane $\Pi_2 := \exp(\phi_1 \mathbf{m}) \, \Pi_{\mathbf{n}}$ through the origin. Now, $\Pi_1 + \Pi_2$ has dimension less than 3 if and only if $\Pi_1 = \Pi_2$, which is equivalent to

$$\exp(-\phi_1 \mathbf{m}) \, \Pi_{\mathbf{m}} = \Pi_{\mathbf{n}} \exp(-\phi_2 \mathbf{n}) \,,$$

and this in turn is equivalent to $\Pi_{\mathbf{m}} = \Pi_{\mathbf{n}}$, i.e., $\mathbf{m} = \pm \mathbf{n}$. ∎

The previous result immediately implies the following description of the product $S(\mathbf{c}_1, \phi_1, \xi_1) \otimes S(\mathbf{c}_2, \phi_2, \xi_2)$ for small ξ_1, ξ_2.

Corollary 1 *Choose quaternionic imaginary units $\mathbf{c}_1, \mathbf{c}_2$ with $\mathbf{c}_1 \neq \pm \mathbf{c}_2$ and let $\phi_1, \phi_2 \in (0, \pi)$. Then if $\xi_1, \xi_2 \in (0, \pi)$ are sufficiently small, we have:*

1. *the map*

$$\begin{aligned} \sigma : S(\mathbf{c}_1, \phi_1, \xi_1) \times S(\mathbf{c}_2, \phi_2, \xi_2) &\to S^3 \\ (\exp(\phi_1 \mathbf{m}), \exp(\phi_2 \mathbf{n})) &\mapsto \exp(\phi_1 \mathbf{m}) \exp(\phi_2 \mathbf{n}) \end{aligned}$$

   is a submersion;

2. *its image $S(\mathbf{c}_1, \phi_1, \xi_1) \otimes S(\mathbf{c}_2, \phi_2, \xi_2)$ is (the closure in S^3 of) an open subset of S^3;*

3. *the boundary $\partial(S(\mathbf{c}_1, \phi_1, \xi_1) \otimes S(\mathbf{c}_2, \phi_2, \xi_2))$ is included in*

$$(bS(\mathbf{c}_1, \phi_1, \xi_1) \otimes S(\mathbf{c}_2, \phi_2, \xi_2)) \cup (S(\mathbf{c}_1, \phi_1, \xi_1) \otimes bS(\mathbf{c}_2, \phi_2, \xi_2)) \,,$$

   where the two members of the union intersect in the set $bS(\mathbf{c}_1, \phi_1, \xi_1) \otimes bS(\mathbf{c}_2, \phi_2, \xi_2)$.

The next result determines which spherical caps $U(\exp(\phi \, \mathbf{c}), t)$ contain $S(\mathbf{c}, \phi, \xi)$ and it provides a rough estimate of which $U(\exp(\phi_1 \mathbf{c}_1) \exp(\phi_2 \mathbf{c}_2), t)$ contain $S(\mathbf{c}_1, \phi_1, \xi_1) \otimes S(\mathbf{c}_2, \phi_2, \xi_2)$.

Proposition 6 *For each quaternionic imaginary unit $\mathbf{c}$ and for all $\phi \in (0, \pi)$ and all $\xi \in [\,0, \pi\,]$, the smallest unit quaternion spherical cap $U(\exp(\phi \, \mathbf{c}), t)$ that includes $S(\mathbf{c}, \phi, \xi)$ has t equal to*

$$t(\phi, \xi) := \arccos(\cos^2 \phi + \sin^2 \phi \cos \xi) \leq \xi \,.$$

As a consequence, for all $\mathbf{c}_1, \mathbf{c}_2$*, all* $\phi_1, \phi_2 \in (0, \pi)$*, and all* $\xi_1, \xi_2 \in [\,0, \pi\,]$*, if* $T := t(\phi_1, \xi_1) + t(\phi_2, \xi_2) \in [\,0, \pi\,]$ *then the unit quaternion spherical cap*

$$U(\exp(\phi_1 \mathbf{c}_1)\exp(\phi_2 \mathbf{c}_2), T)$$

includes $S(\mathbf{c}_1, \phi_1, \xi_1) \otimes S(\mathbf{c}_2, \phi_2, \xi_2)$*.*

Proof : The first statement is a consequence of the fact that, for $\exp(\phi\, \mathbf{m}) \in S(\mathbf{c}, \phi, \xi)$, the expression

$$\langle \exp(\phi\, \mathbf{m}), \exp(\phi\, \mathbf{c}) \rangle = \cos^2 \phi + \sin^2 \phi\, \langle \mathbf{m}, \mathbf{c} \rangle$$

attains its minimum when $\langle \mathbf{m}, \mathbf{c} \rangle = \cos \xi$. The second statement follows from the inclusions

$$S(\mathbf{c}_i, \phi_i, \xi_i) \subset U(\exp(\phi_i\, \mathbf{c}_i), t(\phi_i, \xi_i))\,, \quad i = 1, 2$$

and from Theorem 1. ■

Remark 7 *The inequality* $t(\phi, \xi) \leq \xi$ *is strict if and only if* $\phi \neq \frac{1}{2}\pi$ *and* $\xi \neq 0$*.*

In some instances, the estimate for the product $S(\mathbf{c}_1, \phi_1, \xi_1) \otimes S(\mathbf{c}_2, \phi_2, \xi_2)$ presented in Proposition 6 is sharp.

Example 1 For all quaternionic imaginary units $\mathbf{c}$ and all $\xi \in [\,0, \frac{1}{2}\pi\,]$,

$$S\left(\mathbf{c}, \tfrac{1}{2}\pi, \xi\right) \otimes S\left(\mathbf{c}, \tfrac{1}{2}\pi, \xi\right) \subseteq U(-1, 2\xi)$$

by Proposition 6 and Remark 7. For a fixed unitary $\mathbf{v}$ orthogonal to $\mathbf{c}$, the product of the two elements

$$\cos \xi\, \mathbf{c} \pm \sin \xi\, \mathbf{v}$$

of $S\left(\mathbf{c}, \frac{1}{2}\pi, \xi\right)$ has scalar product with -1 equal to $\cos^2 \xi - \sin^2 \xi = \cos 2\xi$. Hence, the product of these elements does not belong to $U(-1, t)$ if $t < 2\xi$.

8 Lie algebra representation

As an alternative to stereographic projection, the Lie algebra $\mathfrak{so}(3)$ associated with the Lie group $SO(3)$ provides a more intuitive visualization in $\mathbb{R}^3$ of the Minkowski products of unit quaternion sets. In this algebra, spatial rotations are represented by *Euler vectors* of the form $\theta\,\mathbf{n}$, where θ is the rotation angle and the unit vector $\mathbf{n}$ defines the rotation axis. With $\theta \in [\,-\pi, \pi\,]$, any set of spatial rotations lies inside the sphere with center at the origin and radius π in $\mathbb{R}^3$. This approach avoids mapping finite points to infinity, which was the case with stereographic projection in Section 5.

We recall that the elements of the Lie algebra $\mathfrak{so}(3)$ are exactly all skew-symmetric 3×3 real matrices. Thus,

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \;\mapsto\; \mathbf{A} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

defines an isomorphism between the vector spaces $\mathbb{R}^3$ and $\mathfrak{so}(3)$. It maps any vector $\mathbf{a}$ to the matrix $\mathbf{A}$ associated with the linear map $\mathbf{v} \mapsto \mathbf{a} \times \mathbf{v}$. If the inverse isomorphism is denoted by

$$\mathscr{I} : \mathfrak{so}(3) \to \mathbb{R}^3\,,$$

then by direct computation one can verify that for all $\mathbf{A}, \mathbf{B} \in \mathfrak{so}(3)$,

$$\mathscr{I}([\mathbf{A}, \mathbf{B}]) = \mathscr{I}(\mathbf{A}) \times \mathscr{I}(\mathbf{B})\,,$$

where $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$ is the commutator of $\mathbf{A}$ and $\mathbf{B}$.

Each element of $\mathfrak{so}(3)$ has the form $\theta\,\mathbf{N}$ with $\theta \in \mathbb{R}$ and

$$\mathbf{N} \,=\, \begin{bmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{bmatrix},$$

where $\mathbf{n} = n_x\mathbf{i} + n_y\mathbf{j} + n_z\mathbf{k} = \mathscr{I}(\mathbf{N})$ is a unitary element of $\mathbb{R}^3$.

Remark 8 *If $\mathbf{n} = \mathscr{I}(\mathbf{N})$ is unitary, then by direct computation*[1] *we have*

$$\mathbf{N}^2 = \begin{bmatrix} n_x^2 - 1 & n_x n_y & n_x n_z \\ n_x n_y & n_y^2 - 1 & n_y n_z \\ n_x n_z & n_y n_z & n_z^2 - 1 \end{bmatrix} = \mathbf{n}\,\mathbf{n}^T - \mathbf{I}\,,$$

[1]Here the vector $\mathbf{n} \in \mathbb{R}^3$ is regarded as a column matrix, subject to the matrix product.

and

$$\mathbf{N}^3 = \mathbf{N}\,\mathbf{n}\,\mathbf{n}^T - \mathbf{N} = (\mathbf{n}\times\mathbf{n})\,\mathbf{n}^T - \mathbf{N} = -\,\mathbf{N}\,.$$

Consequently, the exponential map

$$\exp : \mathfrak{so}(3) \to SO(3)$$

acts on $\theta\,\mathbf{N}$ *as follows:*

$$\begin{aligned}\exp(\theta\,\mathbf{N}) &= \mathbf{I} + \theta\,\mathbf{N} + \frac{\theta^2}{2!}\mathbf{N}^2 + \frac{\theta^3}{3!}\mathbf{N}^3 + \cdots \\ &= \mathbf{I} + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)\mathbf{N} + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right)\mathbf{N}^2 \\ &= \mathbf{I} + \sin\theta\,\mathbf{N} + (1-\cos\theta)\,\mathbf{N}^2\,.\end{aligned}$$

The explicit form of the matrix $\mathbf{M} \in SO(3)$ defining a rotation by angle θ about a unit axis vector $\mathbf{n} = n_x\mathbf{i} + n_y\mathbf{j} + n_z\mathbf{k}$ is well known, e.g., [1, p. 75].

Remark 9 *If* $\mathbf{M} = \exp(\theta\,\mathbf{N})$*, then* $\frac{1}{2}(\mathbf{M} - \mathbf{M}^T) = \sin\theta\,\mathbf{N}$*. Consequently, the logarithmic map* $\log : SO(3) \to \mathfrak{so}(3)$ *acts on* $\mathbf{M}$ *as follows:*

$$\log(\mathbf{M}) = \frac{\arcsin(\|\mathscr{I}(\mathbf{A})\|)}{\|\mathscr{I}(\mathbf{A})\|}\,\mathbf{A}, \quad \mathbf{A} = \tfrac{1}{2}(\mathbf{M} - \mathbf{M}^T)\,, \tag{16}$$

where the range of $\arcsin$ *is* $[\,-\frac{1}{2}\pi, \frac{1}{2}\pi\,]$*.*

Multiplication on $SO(3)$ induces, through the logarithmic map (16), an operation on $\mathfrak{so}(3)$ defined by the Baker–Campbell–Hausdorff (BCH) formula [10, 19]. We give a brief introduction to this approach, and illustrate its use in visualizing Minkowski products of unit quaternion sets by some examples.

Definition 6 (Baker–Campbell–Hausdorff formula) *We define* $\mathrm{BCH} : \mathfrak{so}(3) \times \mathfrak{so}(3) \to \mathfrak{so}(3)$ *as the unique function such that*

$$\log(\mathbf{M}_1\mathbf{M}_2) = \mathrm{BCH}(\log(\mathbf{M}_1), \log(\mathbf{M}_2))$$

for all $\mathbf{M}_1, \mathbf{M}_2 \in SO(3)$*. The symbol* BCH *is also used to denote the function* $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ *mapping any pair* $(\mathscr{I}(\mathbf{A}_1), \mathscr{I}(\mathbf{A}_2))$ *to* $\mathscr{I}(\mathrm{BCH}(\mathbf{A}_1, \mathbf{A}_2))$*.*

Note that expressions (6) and (7) suffice to determine the rotation angle θ and axis $\mathbf{n}$ in the relation

$$\theta\,\mathbf{n} \;=\; \mathrm{BCH}(\theta_1\mathbf{n}_1,\theta_2\mathbf{n}_2)\,.$$

We now illustrate the Lie algebra approach by several computed examples.

Example 2 We consider (see Section 7) the great circles

$$U = C(\mathbf{j},0,\pi) \quad \text{and} \quad V = C(\mathbf{k},0,\pi)$$

associated with the subsets $\mathbf{U},\mathbf{V}$ of $SO(3)$ that comprise rotations about the y and z axes, respectively. We can visualize the Minkowski product $U \otimes V$ as $\mathscr{I}(\log \mathbf{UV}) = \mathrm{BCH}(\mathscr{I}(\log \mathbf{U}),\mathscr{I}(\log \mathbf{V})) \subset \mathbb{R}^3$. The BCH operands can be parameterized as

$$\mathscr{I}(\log \mathbf{U}) = \{t\,\mathbf{j} :\; -2\pi \le t \le 2\pi\},$$
$$\mathscr{I}(\log \mathbf{V}) = \{s\,\mathbf{k} :\; -2\pi \le s \le 2\pi\}.$$

The set $\mathscr{I}(\log \mathbf{UV})$ is a surface lying inside the sphere with center 0 and radius π, as illustrated in Figure 1 (for reference, the spheres of radius 1 and π are also shown). The hue value of the plot corresponds to the parameter t.

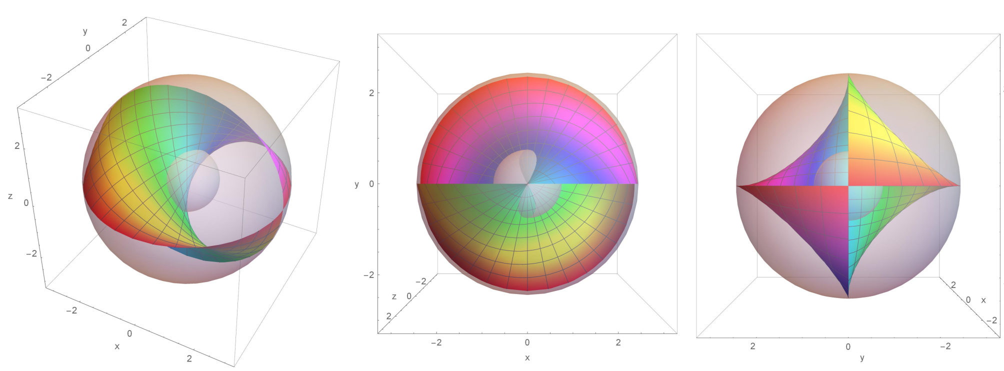

Figure 1: A general view (left), top view (center), and left view (right) of the Minkowski product of the two great circles U and V specified in Example 2.

Example 3 Consider (see Section 7) the Minkowski product

$$C(\mathbf{j}, 0, \tfrac{1}{4}\pi) \otimes S(\mathbf{k}, \tfrac{1}{8}\pi, \tfrac{1}{8}\pi)\,.$$

The former set is defined by a fixed rotation axis and variable rotation angle, and the latter corresponds to a fixed rotation angle $\theta = \frac{1}{4}\pi$ with rotation axes varying in a neighborhood of the z axis. The BCH operands are

$$\mathscr{I}(\log \mathbf{U}) = \{t\,\mathbf{j} :\, -\tfrac{1}{2}\pi \le t \le \tfrac{1}{2}\pi\},$$
$$\mathscr{I}(\log \mathbf{V}) = \left\{\tfrac{1}{4}\pi(\cos u \sin v\,\mathbf{i} + \sin u \sin v\,\mathbf{j} + \cos v\,\mathbf{k}) :\, -\pi \le u \le \pi,\, 0 \le v \le \tfrac{1}{8}\pi\right\}.$$

The 3-dimensional set $\mathscr{I}(\log \mathbf{UV})$ is the union of a one–parameter family of surfaces.

Example 4 Consider now the Minkowski product

$$S(\mathbf{j}, \tfrac{1}{8}\pi, \tfrac{1}{8}\pi) \otimes S(\mathbf{k}, \tfrac{1}{8}\pi, \tfrac{1}{8}\pi)\,.$$

These sets have a fixed rotation angle $\theta = \frac{1}{4}\pi$, and rotation axes varying in neighborhoods of the y and z axes, respectively. The BCH operands are

$$\mathscr{I}(\log \mathbf{U}) = \left\{\tfrac{1}{4}\pi(\cos s \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \sin s \sin t\,\mathbf{k}) :\, -\pi \le s \le \pi,\, 0 \le t \le \tfrac{1}{8}\pi\right\},$$
$$\mathscr{I}(\log \mathbf{V}) = \left\{\tfrac{1}{4}\pi(\cos u \sin v\,\mathbf{i} + \sin u \sin v\,\mathbf{j} + \cos v\,\mathbf{k}) :\, -\pi \le u \le \pi,\, 0 \le v \le \tfrac{1}{8}\pi\right\}.$$

The 3-dimensional set $\mathscr{I}(\log \mathbf{UV})$ can be regarded as the union of a two–parameter family of surfaces.

9 Minkowski product boundaries

Thus far, we have investigated Minkowski products for specific 1–dimensional, 2–dimensional, and 3–dimensional subsets of S^3. We now consider properties of the Minkowski product $U \otimes V$ valid for any subsets U, V of S^3 that have interior points $\mathcal{U}, \mathcal{V}$ in the topology of S^3 induced by $\mathbb{R}^4$. We note that, in this setting, $\mathcal{U}$ and $\mathcal{V}$ are interior points if and only if U and V include unit quaternion spherical caps $U(\mathcal{U}, \delta)$ and $V(\mathcal{V}, \epsilon)$ with $\delta > 0$ and $\epsilon > 0$.

Theorem 3 *Let $U, V \subseteq S^3$ and let $\mathcal{U} \in U$, $\mathcal{V} \in V$. Then $\mathcal{U}\mathcal{V}$ is an interior point of $U \otimes V$ if $\mathcal{U}$ is an interior point of U, or $\mathcal{V}$ is an interior point of V.*

Proof : When $\mathcal{U}$ is an interior point of U, there exists a $\delta > 0$ such that $U(\mathcal{U}, \delta) \subseteq U$. Then by Remark 5,

$$U(\mathcal{U}\mathcal{V}, \delta) = U(\mathcal{U}, \delta) \otimes \{\mathcal{V}\} \,,$$

whence $U(\mathcal{U}\mathcal{V}, \delta)$ is included in $U \otimes V$, so $\mathcal{U}\mathcal{V}$ is an interior point of $U \otimes V$. The case when $\mathcal{V}$ is an interior point of V is treated in the same fashion. ∎

Corollary 2 *Let U, V be subsets of S^3 and let $\partial U, \partial V$ be their boundaries in the topology of S^3. Then the boundary $\partial(U \otimes V)$ of $U \otimes V$ in the topology of S^3 is included in $\partial U \otimes \partial V$.*

The requirement that $\mathcal{U} \in \partial U$, $\mathcal{V} \in \partial V$ is a *necessary* but not *sufficient* condition for $\mathcal{U}\,\mathcal{V} \in \partial(U \otimes V)$. In general, products of points on ∂U and ∂V may generate interior points of $U \otimes V$, and it can be difficult to identify only those pairs of boundary points such that $\mathcal{U}\,\mathcal{V} \in \partial(U \otimes V)$:

- $U \otimes V$ may cover all of S^3 (and thus have no boundary) even in cases where U and V are proper subsets of S^3;

- it may happen for $\mathcal{U} \in \partial U, \mathcal{V} \in \partial V$ that

$$\mathcal{U}\,\mathcal{V} = \mathcal{U}'\,\mathcal{V}' \,,$$

  where $\mathcal{U}'$ is an interior point of U or $\mathcal{V}'$ is an interior point of V;

- if $\partial U, \partial V, \partial(U \otimes V)$ are 2-surfaces in S^3 then we cannot expect $\partial U \otimes \partial V$ to coincide with $\partial(U \otimes V)$ by dimensional considerations.

All three phenomena can be observed in the next example:

Example 5 By Theorem 1, the equality $U(1, s) \otimes U(1, s) = U(1, 2s)$ holds for all $s \in (0, \frac{1}{2}\pi]$. Now:

- if $s = \frac{1}{2}\pi$, then $U(1, 2s) = S^3$ has no boundary;

- the product 1 of the boundary points $\exp(s\mathbf{i})$ and $\exp(-s\mathbf{i})$ equals the product of the interior points $\exp(\frac{1}{2}s\mathbf{i})$ and $\exp(-\frac{1}{2}s\mathbf{i})$;

- if $s < \frac{1}{2}\pi$ then, by direct computation, the product of two boundary points $\exp(s\mathbf{m}), \exp(s\mathbf{n})$ of $U(1,s)$ belongs to $\partial U(1,2s)$ if, and only if, $\mathbf{m} = \mathbf{n}$; therefore,

$$(\partial U \otimes \partial U) \setminus \{\mathcal{U}^2 \, : \, \mathcal{U} \in \partial U\}$$

is included in the interior of $U(1,2s)$.

For the case of Minkowski sums $S_1 \oplus S_2$ of point sets S_1, S_2 in $\mathbb{R}^2$ or $\mathbb{R}^3$, a necessary condition for the sum of boundary points $\mathbf{p}_1 \in \partial S_1$ and $\mathbf{p}_2 \in \partial S_2$ to belong to $\partial(S_1 \oplus S_2)$ is well–known: namely, the normals $\mathbf{n}_1, \mathbf{n}_2$ to $\partial S_1, \partial S_2$ at $\mathbf{p}_1, \mathbf{p}_2$ must be linearly dependent — or, equivalently, the tangent spaces to ∂S_1 and ∂S_2 at $\mathbf{p}_1$ and $\mathbf{p}_2$ must be identical. This principle was extended to Minkowksi products of point sets in $\mathbb{R}^2$ by interpreting points as complex numbers $\mathbf{p}_1 = x_1 + \mathrm{i}\, y_1$, $\mathbf{p}_2 = x_2 + \mathrm{i}\, y_2$ and invoking the complex logarithm to transform Minkowski products into Minkowski sums [6, 7].

The situation with the Minkowski products of unit quaternion sets is more subtle, because the map $\log : SO(3) \to \mathfrak{so}(3)$ is not a homomorphism from the multiplicative group $SO(3)$ to the additive group $\mathfrak{so}(3)$, i.e., in general

$$\log(\mathbf{M}_1\mathbf{M}_2) = \mathrm{BCH}(\log(\mathbf{M}_1), \log(\mathbf{M}_2)) \neq \log(\mathbf{M}_1) + \log(\mathbf{M}_2)\,.$$

Nevertheless, we can find a necessary condition using a different strategy. We begin by considering the case of spherical caps, before proceeding to state and prove a general result.

Lemma 3 *Let $\mathcal{U}_0, \mathcal{V}_0 \in S^3$ and let s_0, t_0 be such that $0 < s_0 \leq t_0 < \frac{1}{2}\pi$. Suppose that the boundaries in S^3 of the spherical caps $U(\mathcal{U}_0, s_0)$ and $U(\mathcal{V}_0, t_0)$ intersect at 1. Then 1 belongs to the boundary of the Minkowski product $U(\mathcal{U}_0, s_0) \otimes U(\mathcal{V}_0, t_0) = U(\mathcal{U}_0\mathcal{V}_0, s_0 + t_0)$ in S^3 if, and only if, there exists an imaginary unit $\mathbf{c}$ such that*

$$\mathcal{U}_0 = \exp(s_0\mathbf{c}), \quad \mathcal{V}_0 = \exp(t_0\mathbf{c})\,.$$

This happens if, and only if, $U(\mathcal{U}_0, s_0)$ is included in $U(\mathcal{V}_0, t_0)$ and the boundaries $\partial U(\mathcal{U}_0, s_0), \partial U(\mathcal{V}_0, t_0)$ are tangent at 1.

Proof : By hypothesis, the scalar part of $\mathcal{U}_0$ equals $\cos s_0$ and the scalar part of $\mathcal{V}_0$ equals $\cos t_0$. In this case, the scalar part of $\mathcal{U}_0\mathcal{V}_0$ equals $\cos(s_0 + t_0) = \cos s_0 \cos t_0 - \sin s_0 \sin t_0$ if, and only if, there exists an imaginary unit $\mathbf{c}$ such that $\mathcal{U}_0 = \exp(s_0\mathbf{c})$ and $\mathcal{V}_0 = \exp(t_0\mathbf{c})$. This happens if, and only if, $\mathcal{U}_0$ is included in the (smaller) arc of the great circle in S^3 with endpoints 1 and $\mathcal{V}_0$. This is the same as asking for $U(\mathcal{U}_0, s_0)$ to be included in $U(\mathcal{V}_0, t_0)$ and for $\partial U(\mathcal{U}_0, s_0), \partial U(\mathcal{V}_0, t_0)$ to be tangent at 1. ∎

Definition 7 *Let $U \subset S^3$. Then the boundary of U in S^3 is* tame *if, for every point $\mathcal{U}_0 \in \partial U$, the following conditions hold:*

(1) U includes a spherical cap $U(\mathcal{U}, s)$ with $s > 0$, such that $U(\mathcal{U}, s) \cap \partial U = \{\mathcal{U}_0\}$;

(2) at $\mathcal{U}_0$, the set of all tangent vectors to curves in U through $\mathcal{U}_0$ spans a 2–plane, which we denote by $T_{\mathcal{U}_0}\partial U$ and call the tangent plane to ∂U at $\mathcal{U}_0$, as usual.

In the situation described in the previous definition, $T_{\mathcal{U}_0}\partial U$ coincides with the tangent plane to the 2–sphere $\partial U(\mathcal{U}, s)$ at $\mathcal{U}_0$.

Remark 10 *If U is the closure in S^3 of an open connected subset of S^3 and its boundary ∂U is a smooth surface, then ∂U is tame.*

Theorem 4 *Let U, V be proper subsets of S^3 and assume that ∂U, ∂V are both tame. Let $\mathcal{U} \in \partial U$ and $\mathcal{V} \in \partial V$. If $\mathcal{U}\mathcal{V}$ belongs to $\partial(U \otimes V)$, then*

$$\mathcal{U}^* \left(T_{\mathcal{U}}(\partial U)\right) = \left(T_{\mathcal{V}}(\partial V)\right) \mathcal{V}^*.$$

Furthermore, for each spherical cap $U(\mathcal{U}_0, s) \subseteq U$ with $s \in (0, \frac{1}{2}\pi)$ whose boundary is tangent to ∂U at $\mathcal{U}$, and each spherical cap $U(\mathcal{V}_0, t) \subseteq V$ with $t \in (0, \frac{1}{2}\pi)$ whose boundary is tangent to ∂V at $\mathcal{V}$, we have

$$\mathcal{U}^* \, U(\mathcal{U}_0, s) \subseteq U(\mathcal{V}_0, t) \, \mathcal{V}^* \quad \text{or} \quad \mathcal{U}^* \, U(\mathcal{U}_0, s) \supseteq U(\mathcal{V}_0, t) \, \mathcal{V}^* \, .$$

Proof : Fix any spherical cap $U(\mathcal{U}_0, s) \subseteq U$ whose boundary is tangent to ∂U at $\mathcal{U}$, and any spherical cap $U(\mathcal{V}_0, t) \subseteq V$ whose boundary is tangent to ∂V at $\mathcal{V}$. We rotate U to $U_1 := \mathcal{U}^* U$, and V to $V_1 := V\mathcal{V}^*$ in S^3. After these rotations:

- the point 1 belongs to both ∂U_1 and ∂V_1;
- $\mathcal{U}^* \, U(\mathcal{U}_0, s)$ is a spherical cap $U(\mathcal{P}, s) \subseteq U_1$ tangent to ∂U_1 at 1;

- $U(\mathcal{V}_0, t)\,\mathcal{V}^*$ is a spherical cap $U(\mathcal{Q}, t) \subseteq V_1$ tangent to ∂V_1 at 1;

- $T_1\partial U_1 = \mathcal{U}^*\,(T_{\mathcal{U}}(\partial U)), \quad T_1\partial V_1 = (T_{\mathcal{V}}(\partial V))\,\mathcal{V}^*$.

Finally, since $U_1 \otimes V_1$ is obtained from $U \otimes V$ by means of the rotation $\mathcal{W} \mapsto \mathcal{U}^*\mathcal{W}\mathcal{V}^*$, we conclude that $\mathcal{U}\mathcal{V} \in \partial(U \otimes V)$ is equivalent to $1 \in \partial(U_1 \otimes V_1)$.

We now show that $1 \in \partial(U_1 \otimes V_1)$ implies both the equality $T_1\partial U_1 = T_1\partial V_1$ and one of the inclusions $U(\mathcal{P}, s) \subseteq U(\mathcal{Q}, t), U(\mathcal{P}, s) \supseteq U(\mathcal{Q}, t)$. We claim that

$$1 \in \partial(U(\mathcal{P}, s) \otimes U(\mathcal{Q}, t)) = \partial U(\mathcal{P}\mathcal{Q}, s + t)\,.$$

Indeed, if 1 were an interior point of $U(\mathcal{P}\mathcal{Q}, s+t) \subseteq U_1 \otimes V_1$ then it would be an interior point of $U_1 \otimes V_1$, contradicting our hypothesis. Lemma 3 allows us to deduce that one of $U(\mathcal{P}, s)$ and $U(\mathcal{Q}, t)$ is included in the other, and that their boundaries are tangent at 1. As a consequence,

$$T_1\partial U_1 = T_1\partial U(\mathcal{P}, s) = T_1\partial U(\mathcal{Q}, t) = T_1\partial V_1\,,$$

as desired. ∎

The previous theorem immediately implies the next corollary: a criterion to identify parts of $\partial U \otimes \partial V$ that are *not* included in $\partial(U \otimes V)$.

Corollary 3 *Let U, V be proper subsets of S^3 with tame boundaries, and let $\mathcal{U} \in \partial U$, $\mathcal{V} \in \partial V$. Suppose two spherical caps $U(\mathcal{U}_0, s)$, $U(\mathcal{U}_0', s')$ exist, such that*

1. *neither cap includes the other;*

2. *both caps are included in U;*

3. *the boundaries of both caps are tangent to ∂U at $\mathcal{U}$.*

Then

$$\mathcal{U}\mathcal{V} \notin \partial(U \otimes V) \quad \textit{and} \quad \mathcal{V}\mathcal{U} \notin \partial(V \otimes U)\,.$$

In other words, $\partial(U \otimes V)$ does not include any point of $\{\mathcal{U}\} \otimes V$, and $\partial(V \otimes U)$ does not include any point of $V \otimes \{\mathcal{U}\}$.

Example 6 Let $s \in (0, \frac{1}{2}\pi)$ and set

$$U := U(\exp(s\,\mathbf{i}), s) \cup U(\exp(-s\,\mathbf{i}), s)\,.$$

Consider the boundary point 1 of U. Then as a consequence of the previous corollary, for each proper subset V of S^3 with tame boundary, the boundary $\partial(U \otimes V)$ does not include any point of V.

Theorem 4 inspires our last result, which identifies a sufficient condition for a product $\mathcal{U}\mathcal{V}$ of points $\mathcal{U} \in \partial U$ and $\mathcal{V} \in \partial V$ to belong to $\partial(U \otimes V)$.

Theorem 5 *Let $\mathcal{U} \in \partial U$, $\mathcal{V} \in \partial V$ for proper subsets U, V of S^3. Assume that there exist $s \in (0, \frac{1}{2}\pi)$ and a spherical cap $U(\mathcal{P}, s)$ with $1 \in \partial U(\mathcal{P}, s)$ and with*

$$\mathcal{U}^* U \subseteq U(\mathcal{P}, s) \supseteq V\mathcal{V}^* .$$

Then

$$\mathcal{U}\mathcal{V} \in \partial(U \otimes V) .$$

Proof : Suppose, by contradiction, that $\mathcal{U}\mathcal{V}$ is an interior point of

$$U \otimes V \subseteq (\mathcal{U}\, U(\mathcal{P}, s)) \otimes (U(\mathcal{P}, s)\, \mathcal{V}) = U(\mathcal{U}\mathcal{P}^2\mathcal{V}, 2s) .$$

Then 1 would be an interior point of

$$U(\mathcal{P}^2, 2s) = U(\mathcal{P}, s) \otimes U(\mathcal{P}, s) ,$$

and since $1 \in \partial U(\mathcal{P}, s)$, this would contradict Lemma 3. ∎

We mention that, in order to apply this sufficient condition, it is not necessary to assume that the tangent planes $T_{\mathcal{U}}(\partial U)$, $T_{\mathcal{V}}(\partial V)$ are well–defined. However, if they are well-defined, then

$$\mathcal{U}^* T_{\mathcal{U}}(\partial U) = T_1 U(\mathcal{P}, s) = T_{\mathcal{V}}(\partial V)\mathcal{V}^* .$$

We also point out that, in the statement, the assumption $s < \frac{1}{2}\pi$ is essential. This fact is illustrated in the next examples.

Example 7 If $\mathcal{P} = \mathbf{i}$ and $s = \frac{1}{2}\pi$ then, although 1 is a boundary point of $U(\mathcal{P}, s)$, the Minkowski product

$$U(\mathcal{P}, s) \otimes U(\mathcal{P}, s) = U(-1, \pi) = S^3$$

admits no boundary points.

Example 8 Fix $s \in (\frac{1}{2}\pi, \pi)$ and let $\mathcal{P} := \exp(s\mathbf{i})$, so that $1 \in \partial U(\mathcal{P}, s)$. Set

$$U := U(\mathcal{P}, s) \cap U(1, \tfrac{1}{8}\pi)$$

and consider the boundary point 1 of U. Although $U \subseteq U(\mathcal{P}, s)$, the point 1 is an interior point of $U \otimes U$. Indeed, it can be obtained not only as the product of 1 with itself, but also as the product of the two points $\exp(\frac{1}{16}\pi\mathbf{j})$ and $\exp(-\frac{1}{16}\pi\mathbf{j})$. These two points are interior points of U because they are interior points of $U(1, \frac{1}{8}\pi)$ and because

$$\langle \exp(\pm\tfrac{1}{16}\pi\mathbf{j}), \exp(s\mathbf{i}) \rangle = \cos(\tfrac{1}{16}\pi) \cos(s) > \cos(s) .$$

10 Closure

The characterization of the uncertainties in a compounded spatial rotation, arising from the ordered product of a sequence of individual rotations subject to prescribed uncertainties in their rotation angles and axes, is of fundamental interest in diverse contexts. Unit quaternion sets offer compact and intuitive representations for families of spatial rotations, and their Minkowski products describe the outcomes of all combinations of the individual rotations.

Whereas Minkowski sums have been extensively studied, and methods for computing them readily generalize to higher dimensions, Minkowski products have received less attention. Using complex–number multiplication to define products of points in $\mathbb{R}^2$, algorithms for the Minkowski products of planar sets are based upon invoking the complex logarithm to transform Minkowski products into Minkowski sums. However, the situation with unit quaternion sets is much more challenging, since (i) they reside in a non–Euclidean space, the 3–sphere S^3; and (ii) their product is non–commutative.

This preliminary study of the Minkowski products of unit quaternion sets describes some basic results and guiding principles for their systematic study. A family of unit quaternion sets that is closed under the Minkowski product — the spherical caps of S^3 — was identified, and some key results concerning sets defined by fixing either the rotation axis or rotation angle, and allowing the other to vary over a given domain, were also developed. To help visualize unit quaternion sets and their Minkowski products, mappings to $\mathbb{R}^3$ based upon stereographic projection and the associated Lie algebra were proposed. Finally, general principles to identify boundary points of Minkowski products, for full–dimension operand sets with smooth boundaries, were analyzed.

The results presented herein serve to introduce the problem of computing Minkowksi products of unit quaternions, to establish some basic foundations, and to identify some possibilities and difficulties it entails. Through its depth and practical importance, this problem offers scope for much further study.

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References

[1] S. L. Altmann (1986), *Rotations, Quaternions, and Double Groups*, Dover Publications (reprint), Mineola, NY.

[2] P. Du Val (1964), *Homographies, Quaternions, and Rotations*, Clarendon Press, Oxford.

[3] R. T. Farouki, W. Gu, and H. P. Moon (2000), Minkowski roots of complex sets, *Geometric Modeling and Processing 2000*, IEEE Computer Society Press, 287–300.

[4] R. T. Farouki and C. Y. Han (2004), Computation of Minkowski values of polynomials over complex sets, *Numerical Algorithms* **36**, 13–29.

[5] R. T. Farouki and C. Y. Han (2005), Solution of elementary equations in the Minkowski geometric algebra of complex sets, *Advances in Computational Mathematics* **22**, 301–323.

[6] R. T. Farouki, H. P. Moon, and B. Ravani (2000), Algorithms for Minkowski products and implicitly–defined complex sets, *Advances in Computational Mathematics* **13**, 199–229.

[7] R. T. Farouki, H. P. Moon, and B. Ravani (2001), Minkowski geometric algebra of complex sets, *Geometriae Dedicata* **85**, 283–315.

[8] R. T. Farouki and H. Pottmann (2002), Exact Minkowski products of N complex disks, *Reliable Computing* **8**, 43–66.

[9] P. K. Ghosh (1988), A mathematical model for shape description using Minkowski operators, *Computer Vision, Graphics, and Image Processing* **44**, 239–269.

[10] R. Gilmore (2008), *Lie Groups, Physics, and Geometry: An Introduction for Physicists, Engineers, and Chemists*, Cambridge University Press, Cambridge.

[11] E. E. Hartquist, J. P. Menon, K. Suresh, H. B. Voelcker, and J. Zagajac (1999), A computing strategy for applications involving offsets, sweeps, and Minkowski operators, *Comput. Aided Design* **31**, 175–183.

[12] A. Kaul (1993), Computing Minkowski sums, PhD Thesis, Columbia University.

[13] A. Kaul and R. T. Farouki (1995), Computing Minkowski sums of plane curves, *International Journal of Computational Geometry and Applications* **5**, 413–432.

[14] A. Kaul and J. R. Rossignac (1992), Solid interpolating deformations: Construction and animation of PIP, *Computers and Graphics* **16**, 107–115.

[15] A. E. Middleditch (1988), Applications of a vector sum operator, *Comput. Aided Design* **20**, 183–188.

[16] H. Minkowski (1903), Volumen und Oberfläche, *Mathematische Annalen* **57**, 447–495.

[17] R. E. Moore (1966), *Interval Analysis*, Prentice–Hall, Englewood Cliffs, NJ.

[18] R. E. Moore (1979), *Methods and Applications of Interval Analysis*, SIAM, Philadelphia.

[19] J. M. Selig (1996), *Geometrical Methods in Robotics*, Springer, New York.

[20] J. Serra (1982), *Image Analysis and Mathematical Morphology*, Academic Press, London.