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Jad Dakroub, Joanna Faddoul, Pascal Omnes, Toni Sayah

A POSTERIORI ERROR ESTIMATES FOR THE TIME DEPENDENT NAVIER-STOKES SYSTEM COUPLED WITH THE CONVECTION-DIFFUSION-REACTION EQUATION

JAD DAKROUB [1], JOANNA FADDOUL [2,3], PASCAL OMNES [3,4], TONI SAYAH [2].

Abstract. In this paper we study the *a posteriori* error estimates for the time dependent Navier-Stokes system coupled with the convection-diffusion-reaction equation. The problem is discretized in time using the implicit Euler method and in space using the finite element method. We establish *a posteriori* error estimates with two types of computable error indicators, the first one linked to the space discretization and the second one to the time discretization. Finally, numerical investigations are performed and presented.

Keywords: *A posteriori* error estimation, Navier-Stokes problem, convection-diffusion-reaction equation, finite element method, adaptive methods.

1. Introduction

The modeling of physical phenomena arising in engineering and sciences leads to partial differential equations in space and time, expressing the mathematical model of the problem to be solved. In general, analytical solutions of these equations do not exist, hence numerical methods such as the finite element method are employed. A major feature of numerical methods is that they involve different sources of numerical errors. The focus of this paper is on an *a posteriori* error analysis for the time dependent Navier-Stokes system coupled with the convection-diffusion-reaction equation .
The *a posteriori* analysis was first introduced by Babuška and Rheinboldt [6, 7], and developed, among other authors, by Verfürth [31] or Ainsworth and Oden [3]. This analysis controls the overall discretization error of a problem by providing error indicators that are easy to compute. Once these error indicators are constructed, their efficiency can be proven by bounding each indicator by the local error, a property also called optimality. A large amount of work has been made concerning the *a posteriori* errors. With no claim to exhaustivity, we can cite for example, Ladevèze [26] for constitutive relation error estimators for time-dependent non-linear Finite Element analysis, Verfürth [32] for the heat equation, Bernardi and Verfürth [14] for the time dependent Stokes equations, Bernardi and Süli [13] for the time and space adaptivity for the second-order wave equation, Bergam, Bernardi and Mghazli [8] for some parabolic equations, Ern and Vohralík [22] for estimations based on potential and flux reconstruction for the heat equation. A chronological perspective of *a posteriori* error estimation in various norms for parabolic problems is presented in [21].

As far as incompressible flow problems are concerned, various works deal with *a posteriori* error estimators for mixed finite element discretizations of the Navier-Stokes equations. We may cite Luo and Zhu [27], El Akkad, El Khalfi and Guessous [20], Bernardi *et al.* [10], Durango and Novo [19]. Bernardi and Sayah

November 29, 2022.
AUF CNRS USJ
[1] Faculté des ingénieurs, Université Saint-Joseph de Beyrouth, Liban.
E-mail: jad.dakroub@usj.edu.lb
[2] Laboratoire de Mathématiques et Applications, Unité de recherche Mathématiques et Modélisation, CAR, Faculté des Sciences, Université Saint-Joseph, B.P 11-514 Riad El Solh, Beyrouth 1107 2050, Liban.
E-mail: joanna.faddoul@net.usj.edu.lb, toni.sayah@usj.edu.lb.
[3] Université Sorbonne Paris Nord, LAGA, CNRS UMR 7539, Institut Galilée 99 Av. J.-B. Clément, 93430 Villetaneuse, France.
E-mail:joanna.faddoul@edu.univ-paris13.fr
[4] Université Paris-Saclay, CEA, Service de Thermo-hydraulique et de Mécanique des Fluides, 91191, Gif-sur-Yvette, France.
E-mail: pascal.omnes@cea.fr.

establish *a posteriori* error estimates for the time dependent Stokes [11] and then Navier-Stokes [12] systems with mixed boundary conditions; for the latter, Nassreddine and Sayah propose improved estimates in [29]. In the stationary case, Dakroub, Faddoul and Sayah [17] present an *a posteriori* analysis of the Newton method applied to the Navier-Stokes problem, while *a posteriori* error estimations of the Large Eddy Simulation methodology applied to the Navier-Stokes Problem are given by Nassreddine, Omnes and Sayah [28].

As far as the coupling of the convection-diffusion equation with other models is concerned, we may cite Chalhoub *et al.* [16] who establish optimal *a posteriori* error estimation of the time dependent convection-diffusion-reaction equation coupled with the Darcy equation and Agroum [1] for an *a posteriori* error analysis for solving the stationary coupled Navier-Stokes problem and convection-diffusion equation.

In this paper we consider the time dependent Navier-Stokes problem coupled with the convection-diffusion-reaction equation and a discrete formulation based on the Euler scheme in time and on a finite element scheme for the space discretization, for which Aldbaissy *et al.* [4] establish an optimal *a priori* error estimate. The coupling of both equations is due to the fact that both the viscosity coefficient and the forcing term in the Navier-Stokes equations depend on the concentration, and to the fact that the convective velocity in the transport equation is the velocity involved in the Navier-Stokes system. Here, we establish *a posteriori* error estimates based on two types of computable error indicators, the first one being linked to the time discretization and the second one to the space discretization. We also show corresponding numerical investigations.

Let Ω be a connected bounded open domain in $\mathbb{R}^d$, $d = 2, 3$, with a Lipschitz continuous boundary $\partial\Omega$ and let $[0, T]$ be an interval of $\mathbb{R}$. We consider the following system:

$$\left\{\begin{array}{ll}
\dfrac{\partial \mathbf{u}}{\partial t}(x,t) - \mathrm{div}(2\nu(C(x,t))\mathbb{D}(\mathbf{u})(x,t)) + (\mathbf{u}(x,t)\cdot\nabla)\mathbf{u}(x,t) + \nabla p(x,t) & = \mathbf{f}(x,t,C(x,t)) \text{ in } \Omega\times]0,T[,\\
\dfrac{\partial C}{\partial t}(x,t) + (\mathbf{u}(x,t)\cdot\nabla)C(x,t) - \alpha\Delta C(x,t) + r_0 C(x,t) & = g(x,t) \text{ in } \Omega\times]0,T[,\\
\mathrm{div}\ \mathbf{u}(x,t) & = 0 \quad \text{in } \Omega\times]0,T[,\\
\mathbf{u}(x,t) & = 0 \text{ on } \partial\Omega\times]0,T[,\\
C(x,t) & = 0 \text{ on } \partial\Omega\times]0,T[,\\
\mathbf{u}(x,0) & = \mathbf{u}_0 \text{ in } \Omega,\\
C(x,0) & = C_0 \text{ in } \Omega,
\end{array}\right. \tag{1.1}$$

where the unknowns are the velocity $\mathbf{u}$, the pressure p and the concentration C in the fluid. Classically, we have set $\mathbb{D}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$. The function $\mathbf{f}$ represents an external force that depends on the concentration C and the function g represents an external concentration source. The viscosity ν is the sum of a constant viscosity ν_0 and of an additional term $\nu_c(C)$ that accounts for the variation of the fluid viscosity as a function of the concentration. In this work, the diffusion coefficient α and the parameter r_0 are positive constants. To simplify, a homogeneous Dirichlet boundary condition is prescribed on the concentration C.

The outline of the paper is as follows:

- In Section 2, we introduce some notations and functional spaces that are useful for the study of the problem.
- In section 3, we introduce the variational formulation.
- In section 4, we introduce the discrete problem and we recall its main properties.
- In section 5, we study the *a posteriori* error estimation.
- Section 6 is devoted to the numerical experiments.
- In section 7, we give some conclusions about this work.

2. Preliminaries

In this section, we recall the main notations and results which we will use later on.

We denote by $L^p(\Omega)^d$ the space of measurable functions $\mathbf{v}$ such that $|\mathbf{v}|^p$ is integrable. For $\mathbf{v} \in L^p(\Omega)^d$, the norm is defined by
$$\| \mathbf{v} \|_{L^p(\Omega)^d} = \left(\int_\Omega |\mathbf{v}(x)|^p d\mathbf{x} \right)^{1/p}.$$
We introduce the Sobolev space
$$W^{m,r}(\Omega)^d = \left\{ \mathbf{v} \in [L^r(\Omega)]^d; \partial^k \mathbf{v} \in [L^r(\Omega)]^d, \forall |k| \le m \right\},$$
where $k = (k_1, \cdots, k_d)$ is a vector of non negative integers, such that $|k| = k_1 + \cdots + k_d$ and
$$\partial^k \mathbf{v} = \frac{\partial^{|k|}\mathbf{v}}{\partial x_1^{k_1} \cdots \partial x_d^{k_d}}.$$
This space is equipped with the semi-norm
$$|\mathbf{v}|_{m,r,\Omega} = \left(\sum_{|k|=m} \int_\Omega |\partial^k \mathbf{v}|^r d\mathbf{x} \right)^{1/r},$$
and is a Banach space for the norm
$$\| \mathbf{v} \|_{m,r,\Omega} = \left(\sum_{\ell=0}^{m} |\mathbf{v}|^r_{\ell,r,\Omega} d\mathbf{x} \right)^{1/r}.$$
When $r = 2$, this space is the Hilbert space $H^m(\Omega)^d$. In particular, we consider the following spaces
$$H_0^1(\Omega)^d = \{ \mathbf{v} \in H^1(\Omega)^d, \mathbf{v}_{|\partial\Omega} = 0 \},$$
equipped with the norm
$$|\mathbf{v}|_{H_0^1(\Omega)^d} = |\mathbf{v}|_{1,\Omega} = \left(\int_\Omega |\nabla \mathbf{v}|^2 d\mathbf{x} \right)^{1/2}.$$
The dual of $H_0^1(\Omega)^d$ is denoted by $H^{-1}(\Omega)^d$.
We also introduce
$$L_0^2(\Omega) = \{ q \in L^2(\Omega); \int_\Omega q(\mathbf{x}) d\mathbf{x} = 0 \}$$
and we define the following scalar product in $L^2(\Omega)^d$
$$(\mathbf{v}, \mathbf{w}) = \int_\Omega \mathbf{v}(\mathbf{x}) \cdot \mathbf{w}(\mathbf{x}) d\mathbf{x}, \quad \forall \mathbf{v}, \mathbf{w} \in L^2(\Omega)^d.$$
Moreover, we shall use the same notation as soon as the integral on the right-hand side has a meaning, even if $\mathbf{v}$ and $\mathbf{w}$ are not in $L^2(\Omega)^d$. Similar notations are also used for scalar functions instead of vector valued functions.

Lemma 2.1. *For any $p \ge 1$, when $d = 1$ or 2, or $1 \le p \le \dfrac{2d}{d-2}$ when $d \ge 3$, there exist two positive constants S_p and S_p^0 such that*
$$\forall \mathbf{v} \in H_0^1(\Omega)^d, \quad \| \mathbf{v} \|_{L^p(\Omega)^d} \le S_p^0 |\mathbf{v}|_{1,\Omega}, \tag{2.1}$$
and
$$\forall \mathbf{v} \in H^1(\Omega)^d, \quad \| \mathbf{v} \|_{L^p(\Omega)^d} \le S_p \| \mathbf{v} \|_{1,\Omega}. \tag{2.2}$$

Lemma 2.2. *When $d = 2$, for all $\mathbf{v} \in H_0^1(\Omega)^d$, we have*
$$\| \mathbf{v} \|_{L^4(\Omega)^2} \le 2^{\frac{1}{4}} \| \mathbf{v} \|^{1/2}_{L^2(\Omega)^2} \ |\mathbf{v}|^{1/2}_{1,\Omega}. \tag{2.3}$$

As usual, for handling time-dependent problems, it is convenient to consider functions defined on a time interval $[a, b]$ with values in a separable functional space W equipped with a norm $\| \, . \, \|_W$. For all $r \ge 1$ we introduce the space
$$L^r(a, b; W) = \left\{ \mathbf{f} \text{ is measurable on }]a, b[\text{ and } \int_a^b \| \mathbf{f}(t) \|_W^r \, dt < \infty \right\},$$

equipped with the norm

$$\| \mathbf{f} \|_{L^r(a,b;W)} = \left(\int_a^b \| \mathbf{f}(t) \|_W^r \, dt \right)^{1/r}.$$

If $r = \infty$, then

$$L^\infty(a,b;W) = \left\{ \mathbf{f} \text{ is measurable on }]a,b[\text{ and } \sup_{t\in[a,b]} \| \mathbf{f}(t) \|_W < \infty \right\},$$

equipped with the norm

$$\| \mathbf{f} \|_{L^\infty(a,b;W)} = \sup_{t\in[a,b]} \| \mathbf{f}(t) \|_W .$$

Remark 2.3. *$L^r(a,b;W)$ is Banach space if W is a Banach space.*

In addition, we define $C^j(0,T;W)$ as the space of functions C^j in time with values in W.

We consider the following spaces:

$$X = H_0^1(\Omega)^d, M = L_0^2(\Omega) \text{ and } Y = H_0^1(\Omega),$$

and

$$V = \{\mathbf{v} \in X; \text{div } \mathbf{v} = 0 \text{ in } \Omega\}.$$

Henceforth, we suppose the following hypothesis:

Assumption 2.4. *We assume that the data $\mathbf{f}, g$ and ν verify:*

i) $\mathbf{f}$ *can be written as follows*

$$\mathbf{f}(x,t,C(x,t)) = \mathbf{f}_0(x,t) + \mathbf{f}_1(x,C(x,t)),$$

where $\mathbf{f}_0 \in C^0(0,T;L^2(\Omega)^d)$ and $\mathbf{f}_1$ is $c^*_{\mathbf{f}_1}$-lipschitz with respect to its second argument from $\mathbb{R}$ with value in $\mathbb{R}^d$. In addition, we suppose that

$$\forall x \in \Omega, \forall \xi \in \mathbb{R}, |\mathbf{f}_1(x,\xi)| \le c_{\mathbf{f}_1}|\xi|, \tag{2.4}$$

where $c_{\mathbf{f}_1}$ is a positive constant.

ii) $g \in C^0(0,T;L^2(\Omega))$,

iii) $\nu = \nu_0 + \nu_C$ *where $\nu_0 > 0$ is a given constant and $0 \le \nu_C \in L^\infty(\mathbb{R})$ and is Lipschitz-continuous, with Lipschitz constant c_ν. The upper bound of ν_C is denoted by $\hat{\nu}_2$: for any $\theta \in \mathbb{R}$ we have*

$$0 \le \nu_C(\theta) \le \hat{\nu}_2. \tag{2.5}$$

iv) $\mathbf{u}_0 \in L^2(\Omega)^d$, div $\mathbf{u}_0 = 0$, $\mathbf{u}_0 \cdot \mathbf{n}_{\partial\Omega} = 0$ *and* $C_0 \in L^2(\Omega)$.

As a consequence of this assumption, since ν_0 is a constant, and $\mathbb{D}(\mathbf{u})$ is symmetrical, for any functions $(\mathbf{u}, \mathbf{v}) \in [H_0^1(\Omega)^d]^2$ with div $\mathbf{u} = 0$, there holds

$$(2\nu\mathbb{D}(\mathbf{u}), \nabla\mathbf{v}) = \nu_0(\nabla\mathbf{u}, \nabla\mathbf{v}) + (2\nu_C\mathbb{D}(\mathbf{u}), \mathbb{D}(\mathbf{v})).$$

Moreover, for any function $\mathbf{u} \in [H_0^1(\Omega)^d]$ there holds

$$\nu_0|\mathbf{u}|^2_{H_0^1(\Omega)^d} \le \nu_0(\nabla\mathbf{u}, \nabla\mathbf{u}) + (2\nu_C\mathbb{D}(\mathbf{u}), \mathbb{D}(\mathbf{u})) \le (\nu_0 + 2\hat{\nu}_2)|\mathbf{u}|^2_{H_0^1(\Omega)^d}. \tag{2.6}$$

In the next lemma we recall the Gronwall-Bellman inequality shown in [33, p. 292] and in [18, p. 252]. We will use this Lemma in different proofs as in Theorem 5.10, Theorem 5.15 and Theorem 5.19.

Lemma 2.5. *(Gronwall lemma) Let*

(1) $\tilde{f}, \tilde{g}$ *and k, be integrable functions defined* $\mathbb{R}^+ \to \mathbb{R}$,
(2) $\tilde{g} \ge 0$, $k \ge 0$,
(3) $\tilde{g} \in L^\infty(\mathbb{R}^+)$,
(4) $\tilde{g}k$ *is an integrable function on* $\mathbb{R}^+$.

If $y : \mathbb{R}^+ \to \mathbb{R}$ satisfies

$$y(t) \leq \tilde{f}(t) + \tilde{g}(t) \int_0^t k(\tau) y(\tau) d\tau, \forall t \in \mathbb{R}^+ \tag{2.7}$$

then

$$y(t) \leq \tilde{f}(t) + \tilde{g}(t) \int_0^t k(\tau) \tilde{f}(\tau) \exp\big(\int_\tau^t k(s) \tilde{g}(s) ds \big) d\tau. \tag{2.8}$$

Lemma 2.6. *(Discrete Gronwall lemma)* [18, p. 254] *Let $(y_n)_n$, $(\tilde{f}_n)_n$ and $(\tilde{g}_n)_n$ three positive sequences that verify:*

$$y_n \leq \tilde{f}_n + \sum_{k=0}^{n-1} \tilde{g}_k y_k, \quad \forall n \geq 0.$$

Then we have:

$$y_n \leq \tilde{f}_n + \sum_{k=0}^{n-1} \tilde{f}_k \tilde{g}_k \prod_{j=k+1}^{n-1} (1 + \tilde{g}_j), \quad \forall n \geq 0,$$

and

$$y_n \leq \tilde{f}_n + \sum_{k=0}^{n-1} \tilde{f}_k \tilde{g}_k \exp\big(\sum_{j=k}^{n-1} \tilde{g}_j \big), \quad \forall n \geq 0.$$

For the sake of simplicity in notations, we set

$$\mathbf{u}(t) = \mathbf{u}(., t), \ C(t) = C(., t) \text{ and } p(t) = p(., t).$$

3. Variational formulation

The variational formulation corresponding to problem (1.1) in the sense of distributions on $[0, T]$ is the following:

$$(E) \left\{ \begin{array}{lcll} \text{Find } t \mapsto (\mathbf{u}(t), p(t), C(t)) \in X \times M \times Y \text{such that} & & & \\ \dfrac{d}{dt}(\mathbf{u}(t), \mathbf{v}) + \nu_0(\nabla \mathbf{u}(t), \nabla \mathbf{v}) + (2\nu_C(C(t))\mathbb{D}(\mathbf{u}(t)), \mathbb{D}(\mathbf{v})) & & & \\ \qquad -(p(t), \operatorname{div} \mathbf{v}) + c_{\mathbf{u}}(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v}) & = & (\mathbf{f}(t, C(t)), \mathbf{v}) & \forall \mathbf{v} \in X, \\ \dfrac{d}{dt}(C(t), r) + c_C(\mathbf{u}(t), C(t), r) + \alpha(\nabla C(t), \nabla r) + r_0(C(t), r) & = & (g(t), r) & \forall r \in Y, \\ (\operatorname{div} \mathbf{u}(t), q) & = & 0 & \forall q \in M, \\ \mathbf{u}(0) & = & \mathbf{u}_0 & \text{in } \Omega, \\ C(0) & = & C_0 & \text{in } \Omega. \end{array} \right. \tag{3.1}$$

with

$$c_{\mathbf{u}}(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v}) = ((\mathbf{u}(t) \cdot \nabla)\mathbf{u}(t), \mathbf{v}) \qquad \text{and} \qquad c_C(\mathbf{u}(t), C(t), r) = ((\mathbf{u}(t) \cdot \nabla)C(t), r).$$

The existence and conditional uniqueness of the solution of (E) are treated and studied in [2, 4] for a slightly different problem and we list here the corresponding theorems:
The existence and conditional uniqueness of the solution of (E) are treated for a slightly different problem in [2, 4] where the Navier-Stokes equation is coupled with the heat equation without the reaction term $r_0(C(t), r)$ and in which the diffusion term is $(\nu(C(t))\nabla \mathbf{u}(t), \nabla \mathbf{v})$. The reaction term adds coercivity in the concentration equation since $r_0 \geq 0$, and thus does not affect the energy estimates which are at the basis of the results below. The difference in the diffusion term does not affect the results either, because we still have the key condition (2.6). We list here the corresponding theorems that are still valid in our case: (Theorem 2.3 in [2] and Theorems 3.1 and 3.2 in [4]).

Theorem 3.1. *Under Assumption 2.4, Problem (E) admits at least one solution*

$$(\mathbf{u}, p, C) \in L^2(0, T; H_0^1(\Omega)^d) \cap L^\infty(0, T; L^2(\Omega)^d) \times L^2(0, T; L^2(\Omega)) \times L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)).$$

Theorem 3.2. *Under Assumption 2.4, every solution of (E) satisfies the bound*

$$\begin{aligned}&\| \mathbf{u} \|_{L^\infty(0,T;L^2(\Omega)^d)} + \| \mathbf{u} \|_{L^2(0,T;H_0^1(\Omega)^d)} + \| C \|_{L^\infty(0,T;L^2(\Omega))} + \| C \|_{L^2(0,T;H_0^1(\Omega))}\\ &\le \hat{C}\Big(\| g \|_{L^2(0,T;L^2(\Omega))} + \| \mathbf{f}_0 \|_{L^2(0,T;L^2(\Omega)^d)} + \| \mathbf{u}_0 \|_{L^2(\Omega)^d} + \| C_0 \|_{L^2(\Omega)} \Big)\end{aligned} \tag{3.2}$$

where $\hat{C}$ is a positive constant which depends of $S_2^0, \nu_0, \alpha, r_0$ and $c_{\mathbf{f}_1}$.

Theorem 3.3. *Let $d = 2$ and assume that ν_C is Lipschitz-continuous, with Lipschitz constant c_ν. If Problem (E) admits a solution $(\mathbf{u}, p, C)$ which verifies*

$$\mathbf{u} \in L^p(0,T;W^{1,r}(\Omega)^d), \text{ where } p \ge 4 \text{ and } r \ge 4,$$

then this solution is unique.

4. Discrete problem

In this section, we use the semi-implicit Euler method (cf. [23]) for the time discretization and the finite element method for the space discretization.
In order to describe the time discretization with an adaptive choice of local time steps, we introduce a partition of the interval $[0,T]$ into sub-intervals $[t_{n-1}, t_n]$, $1 \le n \le N$, with $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$. For all $n \in [1, N]$, we denote by τ_n the length of the interval $[t_{n-1}, t_n]$. For later use, we set for convenience $\tau_0 = \tau_1$. And finally we denote by σ_τ the regularity parameter

$$\sigma_\tau = \max_{1 \le n \le N} \max\left(\frac{\tau_n}{\tau_{n-1}}, \frac{\tau_{n-1}}{\tau_n} \right).$$

We introduce the following operator π_τ (resp. $\pi_{l,\tau}$): let Y be a Banach space and g a continuous function from $]0,T]$ (resp. $[0,T[$) into Y, $\pi_\tau g$ (resp. $\pi_{l,\tau} g$) denotes the piecewise constant function which is equal to $g(t_n)$ (resp. $g(t_{n-1})$) on each interval $]t_{n-1}, t_n]$, $1 \le n \le N$.
In the same way, for $(\phi_n)_{0 \le n \le N}$ in Y^{N+1} we associate the piecewise constant function $\pi_\tau \phi_\tau$ (resp. $\pi_{l,\tau}\phi_\tau$) which is equal to ϕ_n (resp. ϕ_{n-1}) on each interval $]t_{n-1}, t_n]$, $1 \le n \le N$. Furthermore, for any $(C^n)_{0 \le n \le N}$ in Y^{N+1}, we associate the function C_τ on $[0,T]$ which is globally continuous and affine on each interval $[t_{n-1}, t_n]$, $1 \le n \le N$, and equals to C^n in t_n, for $1 \le n \le N$.
More precisely, on the interval $[t_{n-1}, t_n]$, C_τ is defined by:

$$C_\tau = \frac{t - t_{n-1}}{\tau_n}(C^n - C^{n-1}) + C^{n-1} = \frac{t - t_n}{\tau_n}(C^n - C^{n-1}) + C^n. \tag{4.1}$$

Moreover, the same type of definition will be used component by component for any family $(\mathbf{v}^n)_{0 \le n \le N}$ in $(Y^d)^{N+1}$ and the corresponding piecewise affine function is denoted by $\mathbf{v}_\tau$.

We assume that Ω is a polygon ($d = 2$) or a polyhedron ($d = 3$), so it can be completely meshed. Now, we describe the space discretization. For each step n such that $1 \le n \le N$, let $(\mathcal{T}_{nh})$ be a regular family of triangles ($d = 2$) or tetrahedra ($d = 3$) of Ω. The intersection of two different elements of $(\mathcal{T}_{nh})$, if not empty, is a vertex or a whole edge or a whole face ($d = 3$) of both of them. We also have:

$$\overline{\Omega} = \bigcup_{\kappa \in \mathcal{T}_{nh}} \kappa.$$

For each element κ of $\mathcal{T}_{nh}$, let h_κ be its diameter. For each $n \in \{1, \cdots, N\}$, h_n denotes the maximum diameter of $\mathcal{T}_{nh}$ and h denotes the maximum of h_n, for $n \in \{1, \cdots, N\}$.

Let X_{nh}, M_{nh} and Y_{nh} such that $X_{nh} \subset X, M_{nh} \subset M$ and $Y_{nh} \subset Y$, and for each $n \in \{1, \cdots, N\}$,

$$\begin{aligned}
Z_{nh} &= \{q_h \in C^0(\bar{\Omega})\ \forall \kappa \in \mathcal{T}_{nh},\ q_{h|\kappa} \in P_1\},\\
X_{nh} &= \big\{\mathbf{v}_h \in C^0(\bar{\Omega})^d\ ; \forall \kappa \in \mathcal{T}_{nh},\ \mathbf{v}_{h|\kappa} \in P_{1b}, \mathbf{v}_{h|\partial\Omega=0}\big\},\\
Y_{nh} &= \big\{r_h \in Z_{nh};\ r_{h|\partial\Omega=0}\big\},\\
M_{nh} &= \big\{q_h \in Z_{nh};\ \int_\Omega q_h\, \mathbf{dx} = 0\big\}.
\end{aligned} \tag{4.2}$$

Here $P_1(\kappa)$ is the space of restrictions to κ of affine functions, $P_{1b}(\kappa)$ the sum of a polynomial of $P_1(\kappa)$ and a "bubble" function ψ_κ, $P_{1b}(\kappa) = P_1(\kappa) + \text{vect}(\psi_\kappa)$. Denoting by $a_i, 1 \le i \le d+1$, the vertices of κ and by λ_i its barycentric coordinates, the basic bubble function ψ_κ is the polynomial of degree $d+1$ defined by

$$\psi_\kappa(x) = \lambda_1(x) \cdots \lambda_{d+1}(x).$$

We observe that $\psi_\kappa(x) = 0$ on $\partial\kappa$ and that $\psi_\kappa(x) > 0$ on κ. The graph of ψ_κ looks like a bubble attached to the boundary of κ, hence its name.

We introduce the discrete space:

$$V_{nh} = \big\{ \mathbf{v}_h \in X_{nh} \,; \forall q_h \in M_{nh},\ \int_\Omega q_h(x) \text{div } \mathbf{v}_h(x) = 0 \big\}. \tag{4.3}$$

Remark 4.1. *(cf. [5]) The spaces X_{nh} and M_{nh} satisfy the following discrete inf-sup condition:*

$$\inf_{p_h \in M_{nh}} \sup_{\mathbf{v}_h \in X_{nh}} \frac{\displaystyle\int_\Omega \text{div } \mathbf{v}_h(x) p_h(x)\, \mathbf{dx}}{|\mathbf{v}_h|_{H_0^1} \parallel p_h \parallel_{L^2(\Omega)}} \ge \beta, \tag{4.4}$$

where β is positive constant independent of h.

In order to introduce the discrete scheme, we define the following forms: for all $\mathbf{u}_h^n$, $\mathbf{v}_h \in X_{nh}$, and for all C_h^n, $r_h \in Y_{nh}$,

$$d_{\mathbf{u}}(\mathbf{u}_h^{n-1}, \mathbf{u}_h^n, \mathbf{v}_h) = ((\mathbf{u}_h^{n-1} \cdot \nabla)\mathbf{u}_h^n, \mathbf{v}_h) + \frac{1}{2}(\text{div}(\mathbf{u}_h^{n-1})\mathbf{u}_h^n, \mathbf{v}_h)$$

and

$$d_C(\mathbf{u}_h^n, C_h^n, r_h) = ((\mathbf{u}_h^n \cdot \nabla)C_h^n, r_h) + \frac{1}{2}(\text{div}(\mathbf{u}_h^n)C_h^n, r_h).$$

Proposition 4.2. *For all $\mathbf{u}_h, \mathbf{v}_h \in X_{nh}$ and $r_h \in Y_{nh}$, we have*

$$d_{\mathbf{u}}(\mathbf{u}_h, \mathbf{v}_h, \mathbf{v}_h) = 0 \quad \textit{and} \quad d_C(\mathbf{u}_h, r_h, r_h) = 0.$$

Moreover, the definitions of $d_{\mathbf{u}}$ and d_C can be extended to $\mathbf{u}, \mathbf{v} \in X$ and $r \in Y$ and we also have

$$d_{\mathbf{u}}(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0 \quad \textit{and} \quad d_C(\mathbf{u}, r, r) = 0.$$

Let us now introduce the fully discrete scheme associated to Problem (P): For every $n \in \{1, \cdots, N\}$, having $\mathbf{u}_h^{n-1} \in X_{(n-1)h}$ and $C_h^{n-1} \in Y_{(n-1)h}$, Find $(\mathbf{u}_h^n, p_h^n) \in X_{nh} \times M_{nh}, C_h^n \in Y_{nh}$ such that,

$$(\text{Eds1}) \begin{cases} \dfrac{1}{\tau_n}(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{v}_h) + \nu_0(\nabla \mathbf{u}_h^n, \nabla \mathbf{v}_h) + (2\nu_C(C_h^{n-1})\mathbb{D}(\mathbf{u}_h^n), \mathbb{D}(\mathbf{v}_h)) & \\ \qquad + d_{\mathbf{u}}(\mathbf{u}_h^{n-1}, \mathbf{u}_h^n, \mathbf{v}_h) - (p_h^n, \text{div } \mathbf{v}_h) = (\mathbf{f}^n(C_h^{n-1}), \mathbf{v}_h) & \forall \mathbf{v}_h \in X_{nh}, \\ \dfrac{1}{\tau_n}(C_h^n - C_h^{n-1}, r_h) + d_C(\mathbf{u}_h^n, C_h^n, r_h) + \alpha(\nabla C_h^n, \nabla r_h) + r_0(C_h^n, r_h) = (g^n, r_h) & \forall r_h \in Y_{nh}, \\ (q_h, \text{div } \mathbf{u}_h^n) = 0 & \forall q_h \in M_{nh}, \end{cases}$$

where $\mathbf{u}_h^0$ and C_h^0 are given approximations of $\mathbf{u}_0$ and C_0, and g^n and $\mathbf{f}^n$ are defined as follows:

$$g^n = g(t_n)$$

and

$$\mathbf{f}^n(C_h^{n-1}) = \mathbf{f}_0^n + \mathbf{f}_1(C_h^{n-1}) \quad \text{ with } \quad \mathbf{f}_0^n = \mathbf{f}_0(t_n).$$

The following theorem states existence and uniqueness of the solution to Problem (Eds1); the proof is similar to that given in [4], taking into account that $\big(\nu_C(C_h^{n-1})\mathbb{D}(\mathbf{u}_h^n), \mathbb{D}(\mathbf{u}_h^n)\big) \ge 0$:

Theorem 4.3. *At each time step n, for a given $\mathbf{u}_h^{n-1} \in X_{(n-1)h}, C_h^{n-1} \in Y_{(n-1)h}$ and under Assumption 2.4, problem (Eds1) admits a unique solution $(\mathbf{u}_h^n, p_h^n, C_h^n) \in X_{nh} \times M_{nh} \times Y_{nh}$. Furthermore we have, for $m = 1, \cdots, N$, the following bounds*

$$\begin{aligned} &\frac{1}{2} \parallel \mathbf{u}_h^m \parallel^2_{L^2(\Omega)^d} + \frac{\nu_0}{2} \sum_{n=0}^m \tau_n |\mathbf{u}_h^n|^2_{H_0^1(\Omega)^d} \\ &\quad \le \tilde{C}_d \Big(\sum_{n=0}^m \tau_n \parallel g^n \parallel^2_{L^2(\Omega)} + \sum_{n=0}^m \tau_n \parallel \mathbf{f}_0^n \parallel^2_{L^2(\Omega)} + \parallel C_h^0 \parallel^2_{L^2(\Omega)} + \parallel \mathbf{u}_h^0 \parallel^2_{L^2(\Omega)^d} \Big), \end{aligned} \tag{4.5}$$

$$\begin{aligned} \frac{1}{2} \parallel C_h^m \parallel^2_{L^2(\Omega)} + \frac{\alpha}{2} \sum_{n=0}^{m} \tau_n |C_h^n|^2_{H_0^1(\Omega)} + r_0 \sum_{n=0}^{m} \tau_n \parallel C_h^n \parallel^2_{L^2(\Omega)} & \\ \leq \tilde{C'}_d \Big(\sum_{n=0}^{m} \tau_n \parallel g^n \parallel^2_{L^2(\Omega)} + \parallel C_h^0 \parallel^2_{L^2(\Omega)} \Big). & \end{aligned} \tag{4.6}$$

where $\tilde{C}_d$ et $\tilde{C'}_d$ are positive constants independent of h and m.

5. A posterior error analysis

In this section, we shall prove *a posteriori* error estimates between the exact solution of problem (1.1) and the numerical solution of Problem (Eds1) for $d = 2$. We assume that the solution of Problem (E) is unique and sufficiently regular, in particular $\mathbf{u} \in C^0(0, T; L^2(\Omega)^2)$ and $C \in C^0(0, T; L^2(\Omega))$. We begin by constructing the indicators, then we derive the *a posteriori* error estimates and finally establish their quasi-optimality, in the sense that we need to require higher regularity of the exact solution to be able to bound the indicators by the local errors.

5.1. Construction of the error indicators.

We first introduce the space

$$Z^l_{nh} = \{g_h \in L^2(\Omega); \forall \kappa_n \in \mathcal{T}_{nh}, g_h|_{\kappa_n} \in P_l(\kappa_n)\},$$

where $l \leq 1$. For $1 \leq n \leq N$, we fix an approximation $\mathbf{f}_h^n$ of the data $\mathbf{f}^n$ in $Z_{nh} \times Z_{nh}$ and g_h^n of the data g^n in Z_{nh}, we fixe $l = 1$. we denote by

- Γ_h^i the set of edges of the mesh that are not contained in $\partial\Omega$,
- Γ_h^b the set of edges of the mesh which are contained in $\partial\Omega$.

Next, for all $\kappa_n \in \mathcal{T}_{nh}$, we denote by:

- ξ_{κ_n} the set of edges of κ_n that are not contained in $\partial\Omega$,
- Δ_{κ_n} the union of elements of $\mathcal{T}_{nh}$ that share at least one common vertex with κ_n,
- h_{κ_n} the diameter of κ_n and h_{e_n} the diameter of e_n,
- $[.]_{e_n}$ the jump through the edge e_n in ξ_{κ_n},
- n_{κ_n} stands for the unit outward normal vector to κ_n on $\partial\kappa_n$.

In order to establish the *a posteriori* error estimates, we shall use for each element κ_n of $\mathcal{T}_{nh}$ the bubble function ψ_{κ_n} defined below (4.2) and for each edge $e_n \in \kappa_n$ the function ψ_{e_n} which is equal to the product of the d barycentric coordinates associated with the vertices of e_n. We also consider a lifting operator $\mathcal{L}_{e_n}$ defined on polynomials on e_n vanishing on ∂e_n into polynomials on at most the two elements κ_n and κ'_n containing e_n and vanishing on $\partial(\kappa_n \cup \kappa'_n)\backslash e_n$, which is constructed by affine transformation from a fixed operator on the reference element into κ_n. For the details of the construction of the lifting operator, we refer to [31, Page 65].

We recall the following results from ([31], Lemma 3.3):

Property 5.1. *Denoting by $P_r(\kappa_n)$ the space of polynomials of degree smaller than r on κ_n, we have*

$$\forall v \in P_r(\kappa_n), \left\{ \begin{array}{c} \parallel v \parallel_{0,\kappa_n} \leq \parallel v\psi_{\kappa_n}^{1/2} \parallel_{0,\kappa_n} \leq c' \parallel v \parallel_{0,\kappa_n}, \\ |v|_{1,\kappa_n} \leq c h_{\kappa_n}^{-1} \parallel v \parallel_{0,\kappa_n}, \end{array} \right. \tag{5.1}$$

where c and c' are constants independent of mesh steps and of v.

Property 5.2. *Denoting by $P_r(e_n)$ the space of polynomials of degree smaller than r on e_n, we have*

$$\forall v \in P_r(e_n), \quad c \parallel v \parallel_{0,e_n} \leq \parallel v\psi_{e_n}^{1/2} \parallel_{0,e_n} \leq c' \parallel v \parallel_{0,e_n}, \tag{5.2}$$

and for all polynomials v in $P_r(e_n)$ vanishing on $\partial(e_n)$, if κ_n is an element which contains e_n,

$$\parallel \mathcal{L}_{e_n} v \parallel_{0,\kappa_n} + h_{e_n} |\mathcal{L}_{e_n} v|_{1,\kappa_n} \leq c h_{e_n}^{1/2} \parallel v \parallel_{0,e_n}, \tag{5.3}$$

where c and c' are constants independent of mesh steps and of v.

We also introduce a Clément type regularization operator $\mathcal{C}_{hn}$ [15] which has the following properties, see [9, section IX.3]: for all $\mathbf{w}$ in $H_0^1(\Omega)^2$, $\mathcal{C}_{nh}\mathbf{w}$ belongs to the continuous affine finite element space, preserves homogeneous Dirichlet boundary conditions and satisfies for any κ_n in $\mathcal{T}_{nh}$ and e_n in ξ_{κ_n}:

$$\| \mathbf{w} - \mathcal{C}_{nh}\mathbf{w} \|_{0,\kappa_n} \le c h_{\kappa_n} |\mathbf{w}|_{1,\Delta_{\kappa_n}} \quad \text{and} \quad \| \mathbf{w} - \mathcal{C}_{nh}\mathbf{w} \|_{0,e_n} \le c h_{e_n}^{1/2} |\mathbf{w}|_{1,\Delta_{\kappa_n}}, \tag{5.4}$$

where c and c' are constants independent of the mesh steps and of w.
Of course, the same type of properties remain true when applying this regularization operator to a scalar field C in $H_0^1(\Omega)$.

For the *a posteriori* error studies, we consider the piecewise affine functions $\mathbf{u}_h$ and C_h which take in the interval $[t_{n-1}, t_n]$, the values

$$\mathbf{u}_h(t) = \frac{t - t_{n-1}}{\tau_n}(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}) + \mathbf{u}_h^{n-1} = \frac{t - t_n}{\tau_n}(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}) + \mathbf{u}_h^n \tag{5.5}$$

and

$$C_h(t) = \frac{t - t_{n-1}}{\tau_n}(C_h^n - C_h^{n-1}) + C_h^{n-1} = \frac{t - t_n}{\tau_n}(C_h^n - C_h^{n-1}) + C_h^n. \tag{5.6}$$

The piecewise constant function p_h is equal to p_h^n on the interval $]t_{n-1}, t_n]$. We prove quasi-optimal *a posteriori* error estimates by using the norms

$$\begin{aligned}
[[\mathbf{u} - \mathbf{u}_h]](t_m) = \Bigg(& \| \mathbf{u}(t_m) - \mathbf{u}_h(t_m) \|^2_{L^2(\Omega)^2} \\
& + \nu_0 \ \max \Big(\int_0^{t_m} \| \mathbf{u}(t) - \mathbf{u}_h(t) \|_X^2 \, dt, \sum_{n=1}^m \int_{t_{n-1}}^{t_n} \| \mathbf{u}(t) - \pi_\tau \mathbf{u}_h(t) \|_X^2 \, dt \Big) \Bigg)^{1/2}
\end{aligned} \tag{5.7}$$

and

$$\begin{aligned}
[[C - C_h]](t_m) = \Bigg(& \| C(t_m) - C_h(t_m) \|^2_{L^2(\Omega)^2} \\
& + \alpha \ \max \Big(\int_0^{t_m} \| C(t) - C_h(t) \|_Y^2 \, dt, \sum_{n=1}^m \int_{t_{n-1}}^{t_n} \| C(t) - \pi_\tau C_h(t) \|_Y^2 \, dt \Big) \Bigg)^{1/2}.
\end{aligned} \tag{5.8}$$

Remark 5.3. *In definitions 5.7 and 5.8, the quantities* $\int_0^{t_m} \| \mathbf{u}(t) - \mathbf{u}_h(t) \|_X^2 \, dt$ *and* $\int_0^{t_m} \| C(t) - C_h(t) \|_Y^2 \, dt$ *are closely related to* $\sum_{n=1}^m \int_{t_{n-1}}^{t_n} \| \mathbf{u}(t) - \pi_\tau \mathbf{u}_h(t) \|_X^2 \, dt$ *and* $\sum_{n=1}^m \int_{t_{n-1}}^{t_n} \| C(t) - \pi_\tau C_h(t) \|_Y^2 \, dt$*, respectively. This distinction would not be necessary to derive upper bounds of the errors (Subsection 5.2); however it is useful as far as the lower bounds are concerned (Subsection 5.3).*

In the following lemma we calculate the residuals which will allow us to define the error indicators.

Lemma 5.4. *A standard calculation shows that the solution* $(\mathbf{u}, p, C)$ *of problem (1.1) verifies the following equalities for all* $(\mathbf{v}, q, r) \in (X, Y, M)$ *and all* $(\mathbf{u}_h, p_h, r_h)$*, for* $1 \le n \le N$ *and* t *in* $]t_{n-1}, t_n]$

$$\left\{ \begin{aligned}
& (\frac{\partial}{\partial t}(\mathbf{u} - \mathbf{u}_h)(t), \mathbf{v}) + \nu_0(\nabla(\mathbf{u}(t) - \pi_\tau \mathbf{u}_h(t)), \nabla \mathbf{v}) + (2\nu_C(C(t))\mathbb{D}(\mathbf{u}(t)) - 2\nu_C(\pi_{l,\tau} C_h)\mathbb{D}(\pi_\tau \mathbf{u}_h(t)), \mathbb{D}(\mathbf{v})) \\
& + (\mathbf{u}(t) \cdot \nabla \mathbf{u}(t) - \frac{1}{2}(\mathrm{div}\ (\pi_{l,\tau} \mathbf{u}_h(t))\pi_\tau \mathbf{u}_h(t), \mathbf{v}) - (\pi_{l,\tau} \mathbf{u}_h(t) \cdot \nabla \pi_\tau \mathbf{u}_h(t), \mathbf{v}) - (\mathrm{div}\ \mathbf{v}(t), p(t) - p_h(t)) \\
& \qquad = (\mathbf{f}(t, C(t)) - \mathbf{f}^n(C_h^{n-1}), \mathbf{v}) + \langle R_u^h(t), \mathbf{v} \rangle, \\
& (\frac{\partial}{\partial t}(C - C_h)(t), r) + \alpha(\nabla(C - C_h)(t), \nabla r) + (\mathbf{u}(t) \cdot \nabla C(t) - \pi_\tau \mathbf{u}_h(t) \cdot \nabla \pi_\tau C_h(t), r) \\
& + r_0(C(t) - C_h(t), r) - \frac{1}{2}(\mathrm{div}\ (\pi_\tau \mathbf{u}_h(t))\pi_\tau C_h(t), r) \\
& \qquad = (g(t) - g^n, r) + \langle R_c(t), r \rangle, \\
& \int_\Omega q(t,x)\mathrm{div}(\mathbf{u}(t,x) - \mathbf{u}_h(t,x)) = - \int_\Omega q(t,x)\mathrm{div}(\mathbf{u}_h(t,x)),
\end{aligned} \right. \tag{5.9}$$

with

$$\begin{aligned}\langle R_u^h(t),\mathbf{v}\rangle \quad &= \sum_{\kappa_n\in\mathcal{T}_{nh}}\left\{\int_{\kappa_n}\Big((\mathbf{f}^n(C_h^{n-1})-\frac{1}{\tau_n}(\mathbf{u}_h^n-\mathbf{u}_h^{n-1})+\nu_0\Delta\mathbf{u}_h^n+2\nabla\cdot(\nu_C(C_h^{n-1})\mathbb{D}(\mathbf{u}_h^n))\right.\\
&-\mathbf{u}_h^{n-1}\cdot\nabla\mathbf{u}_h^n-\frac{1}{2}\mathrm{div}\ (\mathbf{u}_h^{n-1})\mathbf{u}_h^n-\nabla p_h^n)(x)\cdot\mathbf{v}(x)\Big)\,\mathbf{dx}\\
&\left.-\frac{1}{2}\sum_{e_n\in\varepsilon_{\kappa n}}\int_{e_n}[(\nu_0\nabla\mathbf{u}_h^n+2\nu_C(C_h^{n-1})\mathbb{D}(\mathbf{u}_h^n)-p_h^n\mathbb{I})(\sigma)]_{e_n}\mathbf{n}\cdot\mathbf{v}(\sigma)\mathbf{d}\sigma\right\},\end{aligned}\tag{5.10}$$

and

$$\langle R_c,r\rangle=\langle R_c^\tau,r\rangle+\langle R_c^h,r\rangle\tag{5.11}$$

where

$$\begin{aligned}\langle R_c^\tau(t),r\rangle \quad &= \quad \frac{t_n-t}{\tau_n}\sum_{\kappa_n\in\mathcal{T}_{nh}}\left\{\alpha\int_{\kappa_n}\nabla(C_h^n-C_h^{n-1})(x)\cdot\nabla r(x)\,\mathbf{dx}\right.\\
&\left.\qquad +r_0\int_{\kappa_n}(C_h^n-C_h^{n-1})(x)r(x)\,\mathbf{dx}\right\}\end{aligned}\tag{5.12}$$

and

$$\begin{aligned}\langle R_c^h(t),r\rangle \quad &= \quad \sum_{\kappa_n\in\mathcal{T}_{nh}}\left\{\int_{\kappa_n}(g^n-\frac{1}{\tau_n}(C_h^n-C_h^{n-1})-\frac{1}{2}\mathrm{div}\ (\mathbf{u}_h^n)\ C_h^n\right.\\
&+\alpha\Delta C_h^n-\mathbf{u}_h^n\cdot\nabla C_h^n-r_0C_h^n)r(x)\,\mathbf{dx}\\
&\left.-\frac{1}{2}\sum_{e_n\in\varepsilon_{\kappa n}}\alpha\int_{e_n}[\nabla C_h^n(\sigma)]_{e_n}\cdot\mathbf{n}\,r(\sigma)\mathbf{d}\sigma\right\}.\end{aligned}\tag{5.13}$$

Moreover, if $(\mathbf{u}_h,p_h,C_h)$ *is solution of problem (Eds1), then it holds that*

$$\langle R_u^h,\mathbf{v}\rangle=\langle R_u^h,\mathbf{v}-\mathbf{v}_h\rangle \text{ for all } \mathbf{v}_h\in X_{nh}\tag{5.14}$$

and

$$\langle R_c^h,r\rangle=\langle R_c^h,r-r_h\rangle \text{ for all } r_h\in Y_{nh}.\tag{5.15}$$

Definition 5.5. *For each* $\kappa_n\in\mathcal{T}_{nh}$*, we introduce the following indicators:*

$$(\eta_{u,n,\kappa_n}^\tau)^2=\tau_n\parallel\mathbf{u}_h^n-\mathbf{u}_h^{n-1}\parallel_{H^1(\kappa_n)}^2,\tag{5.16}$$

$$(\eta_{c,n,\kappa_n}^\tau)^2=\tau_n\parallel C_h^n-C_h^{n-1}\parallel_{H^1(\kappa_n)}^2,\tag{5.17}$$

$$\begin{aligned}(\eta_{u,n,\kappa_n}^h)^2 \quad &= \quad h_{\kappa_n}^2\parallel\mathbf{f}^n(C_h^{n-1})-\frac{1}{\tau_n}(\mathbf{u}_h^n-\mathbf{u}_h^{n-1})+\nu_0\Delta\mathbf{u}_h^n+\nabla\cdot(2\nu_C(C_h^{n-1})\mathbb{D}(\mathbf{u}_h^n))\\
&-\mathbf{u}_h^{n-1}\cdot\nabla\mathbf{u}_h^n-\frac{1}{2}\mathrm{div}\ (\mathbf{u}_h^{n-1})\mathbf{u}_h^n-\nabla p_h^n)(x)\parallel_{0,\kappa_n}^2\\
&+\frac{1}{2}\sum_{e_n\in\varepsilon_{\kappa n}}h_{e_n}\parallel[(\nu_0\nabla\mathbf{u}_h^n+2\nu_C(C_h^{n-1})\mathbb{D}(\mathbf{u}_h^n)-p_h^n\mathbb{I})(\sigma)]_{e_n}\mathbf{n}\parallel_{0,e_n}^2\\
&+\parallel\mathrm{div}\ \mathbf{u}_h^n\parallel_{0,\kappa_n}^2,\end{aligned}\tag{5.18}$$

$$\begin{aligned}(\eta_{c,n,\kappa_n}^h)^2= \quad & h_{\kappa_n}^2\parallel g^n-\frac{1}{\tau_n}(C_h^n-C_h^{n-1})+\alpha\Delta C_h^n-\mathbf{u}_h^n\cdot\nabla C_h^n-\frac{1}{2}\mathrm{div}\ (\mathbf{u}_h^n)C_h^n-r_0C_h^n\parallel_{0,\kappa_n}^2\\
&+\frac{1}{2}\sum_{e_n\in\varepsilon_{\kappa n}}h_{e_n}\parallel[\alpha\nabla C_h^n(\sigma)]_{e_n}\cdot\mathbf{n}\parallel_{0,e_n}^2.\end{aligned}\tag{5.19}$$

These indicators are easy to compute since they only depend on the discrete solution and they involve polynomials.

Lemma 5.6. *There exist constants* c_{ru} *and* c_{rc} *independent of the discretization parameters such that the following estimates hold for* $1\le n\le N$, *(1) For all* $\mathbf{v} \in X$, *let* $\mathbf{v}_h = \mathcal{C}_{nh}\mathbf{v}$, *we have*

$$|\langle R_u^h, \mathbf{v} - \mathbf{v}_h \rangle| \leq c_{ru} \bigg(\sum_{\kappa_n \in \mathcal{T}_{nh}} (\eta^h_{u,n,\kappa_n})^2 \bigg)^{1/2} \parallel \mathbf{v} \parallel_{1,\Omega} . \tag{5.20}$$

(2) For all $r \in Y$, *let* $r_h = \mathcal{C}_{nh} r$, *we have*

$$|\langle R_c^h, r - r_h \rangle| \leq c_{rc} \bigg(\sum_{\kappa_n \in \mathcal{T}_{nh}} (\eta^h_{c,n,\kappa_n})^2 \bigg)^{1/2} \parallel r \parallel_{1,\Omega} . \tag{5.21}$$

In Table 2, we can also see the value of the efficiency index for different values of space-time unknowns STU but for the uniform mesh. We can notice that the efficiency index varies around 5.

Proof. To derive inequality (5.20), we use Formula (5.10) applied to $(\mathbf{v} - \mathcal{C}_{nh}\mathbf{v})$ with the Cauchy-Schwarz inequality on each κ_n and each e_n, then relations (5.4) and the discrete Cauchy-Schwarz inequality, together with definition (5.18). In order to prove inequality (5.21), we follow the same steps using (5.13) applied to $(r - \mathcal{C}_{nh} r)$ and definition (5.19). ☐

5.2. **Upper bounds of the error.** In this section, we establish the *a posteriori* error estimates where we bound the error between the exact solution $(\mathbf{u}, p, C)$ of problem (1.1) and the numerical solution $(\mathbf{u}_h, p_h, C_h)$ of problem (Eds1) using the indicators given in Definition 5.5. For this, several steps are needed. First, Theorem 5.7 establishes an upper bound of the concentration error in the energy norm. Then, in order to obtain an upper bound of the velocity error in the energy norm, we need to introduce an auxiliary time semi-discrete problem and we perform intermediary estimations given by Theorems 5.10 and Corollary 5.17. The resulting upper bound for the velocity error is given in Corollary 5.18. Next, combining the results on the concentration and velocity errors lead to Theorem 5.19, which provides a reliable a posteriori error estimate for the total error; this requires mainly the application of Gronwall's Lemma 2.5. Moreover, additional bounds are dealt with respectively in Theorem 5.20 and Theorem 5.21; they are needed to obtain the efficiency of the estimates which is proven in Subsection 5.3.

First, we establish the upper bound on the concentration error, that depends on the velocity error.

Theorem 5.7. *Let* $\mathbf{u}$ *and* C *be the velocity and concentration solutions of problem* (1.1). *Supposing that* $\mathbf{u} \in L^\infty(0,T; L^3(\Omega)^2)$, $C \in L^\infty(0,T; L^3(\Omega))$ *and* $\nabla C \in L^\infty(0,T; L^2(\Omega)^2)$, *the following a posteriori error estimate holds between* C *and the solution* C_h *associated to* $(C_h^n)_{0 \leq n \leq N}$, *solution of problem (Eds1): for* $1 \leq m \leq N$,

$$\begin{aligned} &\parallel C(t_m) - C_h(t_m) \parallel^2_{L^2(\Omega)} + \alpha \int_0^{t_m} \parallel C(s) - C_h(s) \parallel^2_Y ds + 2r_0 \int_0^{t_m} \parallel C(s) - C_h(s) \parallel^2_{L^2(\Omega)} ds \\ &\qquad \leq c \bigg(\sum_{n=1}^m \parallel g - g^n \parallel^2_{L^2(t_{n-1},t_n;Y')} + \parallel C_0 - C_h^0 \parallel^2_{L^2(\Omega)} + \parallel \mathbf{u} - \mathbf{u}_h \parallel^2_{L^2(0,t_m;X)} \\ &\qquad\quad + \sum_{n=1}^m \sum_{\kappa_n \in \tau_{nh}} \big((\eta^\tau_{c,n,\kappa_n})^2 + (\eta^\tau_{u,n,\kappa_n})^2 + \tau_n (\eta^h_{c,n,\kappa_n})^2 \big) \bigg), \end{aligned} \tag{5.22}$$

where c *is a constant independent of the time and mesh steps.*

Proof. Let $t \in]t_{n-1}, t_n]$. We first insert $(\mathbf{u}(t) \cdot \nabla C_h^n, r)$ in the second equality of (5.9), take $r = C - C_h$, use (5.11) and (5.15) and use the incompressibility relation div $\mathbf{u} = 0$ to get the equality, valid for any $r_h \in Y_{nh}$:

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \parallel r(t) \parallel^2_{L^2(\Omega)} + \alpha \parallel r(t) \parallel^2_Y + r_0 \parallel r(t) \parallel^2_{L^2(\Omega)} \\ &\quad = (g(t) - g^n, r(t)) + \langle R_c^\tau(t), r(t) \rangle + \langle R_c^h(t), r(t) - r_h(t) \rangle \\ &\quad - (\mathbf{u}(t) \cdot \nabla(C(t) - C_h^n), r(t)) + ((\mathbf{u}_h^n - \mathbf{u}(t)) \cdot \nabla C_h^n), r(t)) + \frac{1}{2} (\text{div } (\mathbf{u}_h^n - \mathbf{u}(t)) C_h^n, r(t)). \end{aligned} \tag{5.23}$$

We start by bounding the term $(g(t)-g^n, r(t))$, by using the relation $ab \le \frac{1}{\alpha}a^2 + \frac{\alpha}{4}b^2$; this leads to:

$$\begin{aligned}(g(t)-g^n, r(t)) &\le \| g(t)-g^n \|_{Y'} \| r(t) \|_Y\\ &\le \frac{1}{\alpha} \| g(t)-g^n \|^2_{Y'} + \frac{\alpha}{4} \| r(t) \|^2_Y .\end{aligned}$$

Let us now bound the last three terms in (5.23). We first insert $C_h(t)$ and $C(t)$, use Proposition 4.2 and get

$$\begin{aligned}T_a &:= ((\mathbf{u}_h^n - \mathbf{u}(t))\cdot\nabla C_h^n, r(t)) - (\mathbf{u}(t)\cdot\nabla(C(t)-C_h^n), r(t)) + \frac{1}{2}(\text{div } (\mathbf{u}_h^n - \mathbf{u}(t))C_h^n, r(t))\\ &= ((\mathbf{u}_h^n - \mathbf{u}(t))\cdot\nabla(C_h^n - C_h(t)), r(t)) + (\mathbf{u}(t)\cdot\nabla(C_h^n - C(t)), r(t)) + \frac{1}{2}(\text{div } (\mathbf{u}_h^n - \mathbf{u}(t))(C_h^n - C_h(t)), r(t))\\ &\quad + ((\mathbf{u}_h^n - \mathbf{u}(t))\cdot\nabla C(t), r(t)) + \frac{1}{2}(\text{div } (\mathbf{u}_h^n - \mathbf{u}(t))\, C(t), r(t)).\end{aligned}$$

By integrating by parts the third and fifth terms in the right-hand side of the previous equality, we get

$$\begin{aligned}T_a &= (\mathbf{u}(t)\cdot\nabla(C_h^n - C(t)), r) + \frac{1}{2}((\mathbf{u}_h^n - \mathbf{u}(t))\cdot\nabla(C_h^n - C_h(t)), r(t)) + \frac{1}{2}((\mathbf{u}_h^n - \mathbf{u}(t))\cdot\nabla C(t), r(t))\\ &\quad - \frac{1}{2}((\mathbf{u}_h^n - \mathbf{u}(t))\cdot\nabla r(t), (C_h^n - C_h(t))) - \frac{1}{2}((\mathbf{u}_h^n - \mathbf{u}(t))\cdot\nabla r(t), C(t))\\ &=: T_1 + T_2 + T_3 + T_4 + T_5.\end{aligned}$$

Now we will bound the terms $T_i, i = 1, \dots, 5$.
We insert $C_h(t)$, recall that $r = (C - C_h)$ and use the relation $(\mathbf{u}(t)\cdot\nabla r, r) = 0$ to obtain

$$T_1 = (\mathbf{u}(t)\cdot\nabla(C_h^n - C(t)), r(t)) = (\mathbf{u}(t)\cdot\nabla(C_h^n - C_h(t)), r(t)).$$

Let $\mathbf{u} \in L^\infty(0, T, L^3(\Omega)^2)$. Using (5.6) with $|t - t_n| \le \tau_n$, the $L^2 - L^3 - L^6$ generalized Cauchy-Schwarz inequality, Lemma 2.1 and the inequality $ab \le \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2$ we get for any $\varepsilon_1 > 0$

$$\begin{aligned}|T_1| &\le \| \mathbf{u} \|_{L^3(\Omega)^2} \| C_h^n - C_h^{n-1} \|_Y \| r(t) \|_{L^6(\Omega)}\\ &\le c_1 \| \mathbf{u} \|^2_{L^\infty(0,T;L^3(\Omega)^2)} \frac{\varepsilon_1}{2} \| C_h^n - C_h^{n-1} \|^2_Y + \frac{1}{2\varepsilon_1} \| r(t) \|^2_Y .\end{aligned}$$

We consider the term T_2, insert $\mathbf{u}_h(t)$, use (5.6), the $L^2 - L^4 - L^4$ generalized Cauchy-Schwarz inequality and $ab \le 2a^2 + \frac{1}{8}b^2$ with $b = \sqrt{2} \| C_h^n - C_h^{n-1} \|_Y$ to get

$$\begin{aligned}|T_2| &= \frac{1}{2}\big|((\mathbf{u}_h^n - \mathbf{u}_h(t))\cdot\nabla(C_h^n - C_h(t)), r(t)) + ((\mathbf{u}_h(t) - \mathbf{u}(t))\cdot\nabla(C_h^n - C_h), r(t))\big|\\ &\le \frac{1}{2} \| r(t) \|_{L^4(\Omega)} \| C_h^n - C_h^{n-1} \|_Y \big(\| \mathbf{u}_h^n - \mathbf{u}_h(t) \|_{L^4(\Omega)^2} + \| \mathbf{u}_h(t) - \mathbf{u}(t) \|_{L^4(\Omega)^2} \big)\\ &\le \frac{1}{2}\big(\| r(t) \|^2_{L^4(\Omega)} \| \mathbf{u}_h^n - \mathbf{u}_h(t) \|^2_{L^4(\Omega)^2} + \| r(t) \|^2_{L^4(\Omega)} \| \mathbf{u}_h(t) - \mathbf{u}(t) \|^2_{L^4(\Omega)^2} \big)\\ &\quad + \frac{1}{4} \| C_h^n - C_h^{n-1} \|^2_Y .\end{aligned}$$

By using Lemma 2.2, the fact that the exact and numerical velocities and concentrations are bounded in $L^\infty(0, T; L^2(\Omega))$ (cf. Theorem 3.2 and Theorem 4.3), and relation (5.5), we get for any $\varepsilon_2 > 0$ and $\varepsilon_3 > 0$

$$\begin{aligned}|T_2| &\le c_2\big(\| r(t) \|_Y \| \mathbf{u}_h^n - \mathbf{u}_h(t) \|_X + \| r(t) \|_Y \| \mathbf{u}_h(t) - \mathbf{u}(t) \|_X \big) + \frac{1}{4} \| C_h^n - C_h^{n-1} \|^2_Y\\ &\le \left(\frac{1}{2\varepsilon_2} + \frac{1}{2\varepsilon_3}\right) \| r(t) \|^2_Y + c_2^2\left(\frac{\varepsilon_2}{2} \| \mathbf{u}_h^n - \mathbf{u}_h^{n-1} \|^2_Y + \frac{\varepsilon_3}{2} \| \mathbf{u}_h(t) - \mathbf{u}(t) \|^2_Y \right) + \frac{1}{4} \| C_h^n - C_h^{n-1} \|^2_Y .\end{aligned}$$

In order to bound T_3, we insert $\mathbf{u}_h(t)$ use relation (5.5) and the fact that $\nabla C \in L^\infty(0, T; L^2(\Omega)^2)$. We also use Lemma 2.1 and we get for any $\varepsilon_4 > 0$ and $\varepsilon_5 > 0$

$$\begin{aligned}|T_3| &= \frac{1}{2}\big|\big((\mathbf{u}_h^n - \mathbf{u}_h(t))\cdot\nabla C(t), r\big) + (\mathbf{u}_h(t) - \mathbf{u}(t))\cdot\nabla C(t), r(t))\big|\\ &\le \frac{1}{2}\Big(\| \mathbf{u}_h^n - \mathbf{u}_h^{n-1} \|_{L^6(\Omega)^2} + \| \mathbf{u}_h(t) - \mathbf{u}(t) \|_{L^6(\Omega)^2} \Big) \| \nabla C(t) \|_{L^2(\Omega)} \| r(t) \|_{L^3(\Omega)}\\ &\le c_3\Big(\frac{\varepsilon_4}{2} \| \mathbf{u}_h^n - \mathbf{u}_h^{n-1} \|^2_X + \frac{\varepsilon_5}{2} \| \mathbf{u}(t) - \mathbf{u}_h(t) \|^2_X \Big) + \Big(\frac{1}{2\varepsilon_4} + \frac{1}{2\varepsilon_5}\Big) \| r(t) \|^2_Y .\end{aligned}$$

The term T_4 is treated by following the same steps as in the bound for T_2 and we get for any $\varepsilon_6 > 0$ and $\varepsilon_7 > 0$

$$|T_4| = \frac{1}{2}\Big|((\mathbf{u}_h^n - \mathbf{u}(t)) \cdot \nabla r(t), (C_h^n - C_h(t))\Big|$$
$$\leq \left(\frac{1}{2\varepsilon_6} + \frac{1}{2\varepsilon_7}\right) \| r(t) \|_Y^2 + c_4\Big(\frac{\varepsilon_6}{2} \| \mathbf{u}_h^n - \mathbf{u}_h^{n-1} \|_X^2 + \frac{\varepsilon_7}{2} \| \mathbf{u}_h(t) - \mathbf{u}(t) \|_X^2 \Big) + \frac{1}{4} \| C_h^n - C_h^{n-1} \|_Y^2 .$$

The term T_5 can be treated like T_3 and we get for any $\varepsilon_8 > 0$ and $\varepsilon_9 > 0$ by using the fact that $C \in L^\infty(0, T; L^3(\Omega))$

$$|T_5| \leq c_5\Big(\frac{\varepsilon_8}{2} \| \mathbf{u}_h^n - \mathbf{u}_h^{n-1} \|_X^2 + \frac{\varepsilon_9}{2} \| \mathbf{u}(t) - \mathbf{u}_h(t) \|_X^2 \Big) + \left(\frac{1}{2\varepsilon_8} + \frac{1}{2\varepsilon_9}\right) \| r(t) \|_Y^2 .$$

To end the proof, we need to bound the third and fourth terms of (5.23). According to (5.12), we have

$$\langle R_c^\tau(t), r\rangle = \frac{t_n - t}{\tau_n}\Big(\alpha \int_\Omega \nabla(C_h^n - C_h^{n-1})(x) \cdot \nabla r(x)\, \mathbf{dx} + r_0 \int_\Omega (C_h^n - C_h^{n-1})(x) r(x)\, \mathbf{dx}\Big). \tag{5.24}$$

Then we easily obtain for any $\varepsilon_{10} > 0$

$$|\langle R_c^\tau(t), r(t)\rangle| \leq c_6 \frac{\varepsilon_{10}}{2} \| C_h^n - C_h^{n-1} \|_Y^2 + \frac{1}{2\varepsilon_{10}} \| r(t) \|_Y^2 .$$

The final term that we need to bound is $\langle R_c^h, r - r_h\rangle$. Choosing $r_h = \mathcal{C}_{nh} r$, and applying (5.21) we get for any $\varepsilon_{11} > 0$:

$$|\langle R_c^h, r - r_h\rangle| \leq c_7 \frac{\varepsilon_{11}}{2} \sum_{\kappa_n \in \mathcal{T}_{nh}} (\eta_{c,n,\kappa_n}^h)^2 + \frac{1}{2\varepsilon_{11}} \| r(t) \|_Y^2 . \tag{5.25}$$

Finally, by regrouping all the above bounds, plugging them into (5.23) and choosing $\varepsilon_i, i = \{1, \cdots, 11\}$ such that $\frac{1}{2}\sum_{i=1}^{11} \frac{1}{\varepsilon_i}$ equals to $\frac{\alpha}{4}$, we get

$$\begin{aligned} \frac{d}{dt} \| r(t) \|_{L^2(\Omega)}^2 + \alpha \| r(t) \|_Y^2 + 2r_0 \| r(t) \|_{L^2(\Omega)}^2 \leq c\Big(\| g(t) - g^n \|_{Y'}^2 \\ + \| \mathbf{u}(t) - \mathbf{u}_h(t) \|_X^2 + \| C_h^n - C_h^{n-1} \|_Y^2 + \| \mathbf{u}_h^n - \mathbf{u}_h^{n-1} \|_X^2 + \sum_{\kappa_n \in \mathcal{T}_{nh}} (\eta_{c,n,\kappa_n}^h)^2\Big). \end{aligned} \tag{5.26}$$

We integrate (5.26) between t_{n-1} and t_n, use Definition 5.5 and then we sum over $n = 1, \ldots m$ to obtain

$$\begin{aligned} \| r(t_m) \|_{L^2(\Omega)}^2 + \alpha \int_0^{t_m} \| r(s) \|_Y^2\, ds + 2r_0 \int_0^{t_m} \| r(s) \|_{L^2(\Omega)}^2\, ds \\ \leq c\Big(\| g - g^n \|_{L^2(0,t_m;Y')}^2 + \| C_0 - C_h^0 \|_{L^2(\Omega)}^2 + \| \mathbf{u} - \mathbf{u}_h \|_{L^2(0,t_m;X)}^2 \\ + \sum_{n=1}^m \sum_{\kappa_n \in \tau_{nh}} \big((\eta_{c,n,\kappa_n}^\tau)^2 + (\eta_{u,n,\kappa_n}^\tau)^2 + \tau_n (\eta_{c,n,\kappa_n}^h)^2\big)\Big). \end{aligned} \tag{5.27}$$

Hence we get the desired result. □

To prove the upper bound of the velocity error, we refer to the idea of Bernardi and Verfürth in [14] or Bernardi and Sayah in [11] in order to uncouple time and space errors. In order to achieve our objective, we introduce an auxiliary problem denoted by (P_{aux}) and we first compute an upper bound of the error between the time-affine function $\mathbf{u}_\tau$ constructed from its solution by a formula like (4.1) and the exact solution $\mathbf{u}$ of Problem (1.1) and then an upper bound of the error between $\mathbf{u}_\tau$ and the discrete solution $\mathbf{u}_h$. Finally, we combine these two error bounds to obtain the desired *a posteriori* error estimation.

We introduce the following time semi-discrete problem and prove an energy estimate on its solution.

$$(P_{aux})\begin{cases} \text{Let } C_h^{n-1} \text{ be the concentration component of the finite element solution of (Eds1), then} \\ \text{knowing } \mathbf{u}^{n-1},\ \text{find}(\mathbf{u}^n,p^n)\in X\times M \text{ such that} \\ \dfrac{1}{\tau_n}(\mathbf{u}^n-\mathbf{u}^{n-1},\mathbf{v})+\nu_0(\nabla\mathbf{u}^n,\nabla\mathbf{v})+(2\nu_C(C_h^{n-1})\mathbb{D}(\mathbf{u}^n),\mathbb{D}(\mathbf{v})) \\ \qquad +(\mathbf{u}^{n-1}\cdot\nabla\mathbf{u}^n,\mathbf{v})-(\text{div }\mathbf{v},p^n) = (\mathbf{f}^n(C_h^{n-1}),\mathbf{v}) \qquad \forall\mathbf{v}\in X, \\ \qquad (\text{div }\mathbf{u}^n,q) = 0 \qquad \forall q\in M, \end{cases}$$

with $\mathbf{u}^0=\mathbf{u}_0$ and $\mathbf{f}^n(C_h^{n-1})=\mathbf{f}_0^n+\mathbf{f}_1(C_h^{n-1})$ with $\mathbf{f}_0^n=\mathbf{f}_0(t_n)$.

Proposition 5.8. *Under assumption 2.4, for each iteration n, the solution $(\mathbf{u}^n,p^n)\in X\times M$ of problem (P_{aux}) satisfies for any $m=1,\cdots,N$:*

$$\begin{aligned} &\frac{1}{2}\parallel \mathbf{u}^m\parallel^2_{L^2(\Omega)^d}+\frac{\nu_0}{2}\sum_{n=0}^m\tau_n|\mathbf{u}^n|^2_{H^1_0(\Omega)^d} \\ &\quad\le \tilde{C}_d\Big(\sum_{n=0}^m\tau_n\parallel g^n\parallel^2_{L^2(\Omega)}+\sum_{n=0}^m\tau_n\parallel \mathbf{f}_0^n\parallel^2_{L^2(\Omega)}+\parallel C_h^0\parallel^2_{L^2(\Omega)}+\parallel\mathbf{u}_0\parallel^2_{L^2(\Omega)^d}\Big), \end{aligned} \tag{5.28}$$

where $\tilde{C}_d$ is a positive constant independent of m.

Proof. The proof is obtained by energy estimates and by the concentration bound of Theorem 4.3. □

In order to derive an estimate between the solution of this auxiliary problem and the exact solution, we first define the corresponding residual.

Lemma 5.9. *By combining (1.1) and (P_{aux}), we get*

$$(\mathbf{u}-\mathbf{u}_\tau)(0)=\mathbf{0}$$

and, for $1\le n\le N$, for t in $]t_{n-1},t_n]$, for any $\mathbf{v}$ in X and q in M we have

$$\begin{cases} (\dfrac{\partial}{\partial t}(\mathbf{u}-\mathbf{u}_\tau)(t),\mathbf{v})+\nu_0(\nabla(\mathbf{u}(t)-\mathbf{u}_\tau(t)),\nabla\mathbf{v})+(2\nu_C(C(t))\mathbb{D}(\mathbf{u}(t))-2\nu_C(\pi_{l,\tau}C_h)\mathbb{D}(\mathbf{u}_\tau(t)),\mathbb{D}(\mathbf{v})) \\ \qquad +(\mathbf{u}(t)\cdot\nabla\mathbf{u}(t)-\mathbf{u}_\tau(t)\cdot\nabla\mathbf{u}_\tau(t),\mathbf{v})-(\text{div }\mathbf{v}(t),p(t)-\pi_\tau p_\tau(t)) \\ \qquad\qquad =(\mathbf{f}(t,C(t))-\mathbf{f}^n(C_h^{n-1}),\mathbf{v})+\langle R_u^{\tau,2}(t),\mathbf{v}\rangle, \\ \\ (\text{div }(\mathbf{u}-\mathbf{u}_\tau),q)=0, \end{cases} \tag{5.29}$$

where $R_u^{\tau,2}$ is defined by

$$\begin{aligned} \langle R_u^{\tau,2}(t),\mathbf{v}\rangle &= \frac{t_n-t}{\tau_n}\int_\Omega \nu_0\nabla(\mathbf{u}^n-\mathbf{u}^{n-1})(x):\nabla\mathbf{v}(x)+2\nu_C(C_h^{n-1})\mathbb{D}(\mathbf{u}^n-\mathbf{u}^{n-1})(x):\mathbb{D}(\mathbf{v}(x))\,\mathbf{dx} \\ &+\frac{t_n-t}{\tau_n}\int_\Omega(\mathbf{u}^{n-1}\cdot\nabla(\mathbf{u}^n-\mathbf{u}^{n-1}))(x)\cdot\mathbf{v}(x)\,\mathbf{dx}-\frac{t-t_{n-1}}{\tau_n}\int_\Omega((\mathbf{u}^n-\mathbf{u}^{n-1})\cdot\nabla\mathbf{u}_\tau(t))(x)\cdot\mathbf{v}(x)\,\mathbf{dx}. \end{aligned} \tag{5.30}$$

Theorem 5.10. *Let $\mathbf{u}$ be the velocity of problem (1.1) and $\mathbf{u}_\tau$ the velocity associated to $(\mathbf{u}^n)_{0\le n\le N}$ solution of the auxiliary problem (P_{aux}). Suppose that $\nabla\mathbf{u}\in L^\infty(0,T;L^4(\Omega)^{2\times2})$ and $\mathbf{u}\in L^\infty(0,T;L^3(\Omega)^2)$. Then, the following a posteriori error estimate holds*

$$\begin{aligned} &\parallel\mathbf{u}(t)-\mathbf{u}_\tau(t)\parallel^2_{L^2(\Omega)^2}+\nu_0\int_0^t\parallel\mathbf{u}(\tau)-\mathbf{u}_\tau(\tau)\parallel^2_X\,d\tau \\ &\quad\le\ c\Big(\parallel\mathbf{f}_0-\pi_\tau\mathbf{f}_0\parallel^2_{L^2(0,t;L^2(\Omega)^2)}+\parallel C-\pi_{l,\tau}C_h\parallel^2_{L^2(0,t;L^4(\Omega))}+\parallel\pi_\tau\mathbf{u}-\pi_{l,\tau}\mathbf{u}\parallel^2_{L^2(0,t;X)}\Big). \end{aligned} \tag{5.31}$$

where c is a positive constant independent of the time and mesh steps.

***Proof*.** We first insert $(\nu_C(C_h^{n-1})\mathbb{D}(\mathbf{u}(t)), \mathbb{D}(\mathbf{v}))$ and $(\mathbf{u}_\tau(t)\cdot\nabla\mathbf{u}(t), \mathbf{v})$ in the first equality of (5.29) where $t \in]t_{n-1}, t_n]$, we get

$$\begin{aligned}
&(\frac{\partial}{\partial t}(\mathbf{u}(t)-\mathbf{u}_\tau(t)),\mathbf{v}) + (2(\nu_C(C(t))-\nu_C(C_h^{n-1}))\mathbb{D}(\mathbf{u}(t)),\mathbb{D}(\mathbf{v})) + 2(\nu_C(C_h^{n-1})\mathbb{D}(\mathbf{u}(t)-\mathbf{u}_\tau(t)),\mathbb{D}(\mathbf{v}))\\
&\quad +(\nu_0\nabla(\mathbf{u}(t)-\mathbf{u}_\tau(t)),\nabla\mathbf{v}) + ((\mathbf{u}(t)-\mathbf{u}_\tau(t))\cdot\nabla\mathbf{u}(t),\mathbf{v}) - (\mathbf{u}_\tau(t)\cdot\nabla(\mathbf{u}_\tau(t)-\mathbf{u}(t)),\mathbf{v})\\
&\quad -(\mathrm{div}\ \mathbf{v}(t), p(t)-\pi_\tau p_\tau(t)) = (\mathbf{f}(t,C(t))-\mathbf{f}^n(C_h^{n-1}),\mathbf{v}) + \langle R_u^{\tau,2}(t),\mathbf{v}\rangle.
\end{aligned}$$

Let $\mathbf{v}=\mathbf{u}-\mathbf{u}_\tau$. By using that div $\mathbf{u}_\tau = 0$ and $(\mathbf{u}_\tau(t)\cdot\nabla\mathbf{v}(t),\mathbf{v}(t)) = 0$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\parallel \mathbf{v}(t)\parallel^2_{L^2(\Omega)} + \nu_0 \parallel \mathbf{v}\parallel_X^2 + 2(\nu_C(C_h^{n-1})\mathbb{D}(\mathbf{v}(t)),\mathbb{D}(\mathbf{v}(t))) =\\
&\quad (\mathbf{f}(t,C(t))-\mathbf{f}^n(C_h^{n-1}),\mathbf{v}(t)) + \langle R_u^{\tau,2}(t),\mathbf{v}(t)\rangle\\
&\quad -(\mathbf{v}(t)\cdot\nabla\mathbf{u}(t),\mathbf{v}(t)) + (2(\nu_C(C_h^{n-1})-\nu_C(C(t)))\mathbb{D}(\mathbf{u}(t)),\mathbb{D}(\mathbf{v}(t))).
\end{aligned} \tag{5.32}$$

We need to bound the terms in the right-hand side of (5.32). Let us start with the third one. Using the $L^2 - L^4 - L^4$ generalized Cauchy-Schwarz inequality as well as (2.3) we obtain

$$\begin{aligned}
|(\mathbf{v}(t)\cdot\nabla\mathbf{u}(t),\mathbf{v}(t))| &\le \parallel \mathbf{u}(t)\parallel_X \parallel \mathbf{v}(t)\parallel^2_{L^4(\Omega)^2}\\
&\le \sqrt{2}\parallel \mathbf{u}(t)\parallel_X\parallel \mathbf{v}(t)\parallel_{L^2(\Omega)^2}\parallel \mathbf{v}(t)\parallel_X\\
&\le \varepsilon_1 \parallel \mathbf{u}(t)\parallel_X^2\parallel \mathbf{v}(t)\parallel^2_{L^2(\Omega)^2} + \frac{1}{2\varepsilon_1}\parallel \mathbf{v}(t)\parallel_X^2 .
\end{aligned} \tag{5.33}$$

In the same way, we bound the first term in the right-hand side of (5.32) as follows:

$$\begin{aligned}
|(\mathbf{f}(t,C(t))-\mathbf{f}^n(C_h^{n-1}),\mathbf{v}(t))| &\le \parallel \mathbf{f}(t,C(t))-\mathbf{f}^n(C_h^{n-1})\parallel_{L^2(\Omega)^2}\parallel \mathbf{v}(t)\parallel_{L^2(\Omega)^2}\\
&\le \frac{(S_0^2)^2\varepsilon_2}{2}\parallel \mathbf{f}(t,C(t))-\mathbf{f}^n(C_h^{n-1})\parallel^2_{L^2(\Omega)^2} + \frac{1}{2\varepsilon_2}\parallel \mathbf{v}(t)\parallel_X^2 .
\end{aligned}$$

Using the definition of the function $\mathbf{f}$ and the fact that $\mathbf{f}_1$ is $c^*_{\mathbf{f}_1}$-lipschitz, we get

$$|(\mathbf{f}(t,C(t))-\mathbf{f}^n(C_h^{n-1}),\mathbf{v})| \le (S_0^2)^2\varepsilon_2\big(\parallel \mathbf{f}_0(t)-\mathbf{f}_0^n\parallel^2_{L^2(\Omega)^2} + c^*_{\mathbf{f}_1}\parallel C(t)-C_h^{n-1}\parallel^2_{L^2(\Omega)}\big) + \frac{1}{2\varepsilon_2}\parallel \mathbf{v}(t)\parallel_X^2 . \tag{5.34}$$

For the fourth term in the right-hand side of (5.32), we use Assumption 2.4 and the assumption $\nabla\mathbf{u}\in L^\infty(0,T;L^4(\Omega)^{2\times2})$ to get

$$\begin{aligned}
2(\nu_C(C_h^{n-1})-\nu_C(C(t))\mathbb{D}(\mathbf{u}(t)),\mathbb{D}(\mathbf{v})) &\le 2\parallel \nu_C(C_h^{n-1})-\nu_C(C(t))\parallel_{L^4(\Omega)}\parallel \mathbb{D}(\mathbf{u}(t))\parallel_{L^4(\Omega)^{2\times2}}\parallel \mathbb{D}(\mathbf{v}(t))\parallel_{L^2(\Omega)^{2\times2}}\\
&\le c_0\frac{\varepsilon_3}{2}\parallel C_h^{n-1}-C(t)\parallel^2_{L^4(\Omega)} + \frac{1}{2\varepsilon_3}\parallel \mathbf{v}(t)\parallel_X^2 .
\end{aligned} \tag{5.35}$$

The second term in the right-hand side of (5.32) is given by (5.30) and denoted by $T_a =: T_1+T_2+T_3$. We will bound separately T_1, T_2 and T_3.
T_1 can be bounded as

$$\begin{aligned}
|T_1| &= \left|\frac{t_n-t}{\tau_n}\int_\Omega \nu_0\nabla(\mathbf{u}^n-\mathbf{u}^{n-1})(x):\nabla\mathbf{v}(x) + 2\nu_C(C_h^{n-1})\mathbb{D}(\mathbf{u}^n-\mathbf{u}^{n-1})(x):\mathbb{D}(\mathbf{v}(x))\,\mathbf{dx}\right|\\
&\le (\nu_0+2\hat{\nu}_2)\parallel \mathbf{u}^n-\mathbf{u}^{n-1}\parallel_X\parallel \mathbf{v}(t)\parallel_X\\
&\le (\nu_0+2\hat{\nu}_2)^2\frac{\varepsilon_4}{2}\parallel \mathbf{u}^n-\mathbf{u}^{n-1}\parallel_X^2 + \frac{1}{2\varepsilon_4}\parallel \mathbf{v}(t)\parallel_X^2 .
\end{aligned}$$

T_2 can be treated by inserting $\mathbf{u}(t)$ as follow:

$$\begin{aligned}
|T_2| &= \left|\frac{t_n-t}{\tau_n}\int_\Omega(\mathbf{u}^{n-1}\cdot\nabla(\mathbf{u}^n-\mathbf{u}^{n-1}))(x)\cdot\mathbf{v}(x)\,\mathbf{dx}\right|\\
&\le \parallel \mathbf{u}^{n-1}-\mathbf{u}(t)\parallel_{L^4(\Omega)^2}\parallel \mathbf{v}(t)\parallel_{L^4(\Omega)^2}\parallel \mathbf{u}^n-\mathbf{u}^{n-1}\parallel_X + \parallel \mathbf{u}(t)\parallel_{L^3(\Omega)^2}\parallel \mathbf{v}(t)\parallel_{L^6(\Omega)^2}\parallel \mathbf{u}^n-\mathbf{u}^{n-1}\parallel_X .
\end{aligned}$$

Using the fact that $\mathbf{u}\in L^\infty(0,T;L^3(\Omega)^2)$ we have the relation

$$\parallel \mathbf{u}(t)\parallel_{L^3(\Omega)^2}\parallel \mathbf{v}(t)\parallel_{L^6(\Omega)^2}\parallel \mathbf{u}^n-\mathbf{u}^{n-1}\parallel_X \le c_1\frac{\varepsilon_5}{2}\parallel \mathbf{u}^n-\mathbf{u}^{n-1}\parallel_X^2 + \frac{1}{2\varepsilon_5}\parallel \mathbf{v}(t)\parallel_X^2 .$$

We insert $\mathbf{u}_\tau$ and use its definition given by a formula similar to (4.1); using the fact that $\mathbf{v} = \mathbf{u} - \mathbf{u}_\tau$, we get

$$\begin{aligned}
&\| \mathbf{u}^{n-1} - \mathbf{u}(t) \|_{L^4(\Omega)^2} \| \mathbf{v} \|_{L^4(\Omega)^2} \| \mathbf{u}^n - \mathbf{u}^{n-1} \|_X \\
&\quad \le c_2^{'} (\| \mathbf{u}^{n-1} - \mathbf{u}^n \|_{L^4(\Omega)^2} \| \mathbf{v} \|_{L^4(\Omega)^2} \| \mathbf{u}^n - \mathbf{u}^{n-1} \|_X) + \| \mathbf{v} \|^2_{L^4(\Omega)^2} \| \mathbf{u}^n - \mathbf{u}^{n-1} \|_X \\
&\quad \le c_2 \big(\| \mathbf{u}^{n-1} - \mathbf{u}^n \|^2_{L^4(\Omega)^2} \| \mathbf{v} \|^2_{L^4(\Omega)^2} + \| \mathbf{u}^n - \mathbf{u}^{n-1} \|^2_X + \| \mathbf{v} \|^2_{L^4(\Omega)^2} \| \mathbf{u}^n - \mathbf{u}^{n-1} \|_X \big),
\end{aligned}$$

and by using relation (2.3) both on $\mathbf{v}$ and on $(\mathbf{u}^n - \mathbf{u}^{n-1})$ and the fact that the exact and semi-discrete velocities are bounded in $L^2(\Omega)^2$ (see Theorem 3.2 and Proposition 5.8), we get

$$\| \mathbf{u}^{n-1} - \mathbf{u}(t) \|_{L^4(\Omega)^2} \| \mathbf{v} \|_{L^4(\Omega)^2} \| \mathbf{u}^n - \mathbf{u}^{n-1} \|_X \le c_3 (1 + \frac{\varepsilon_6}{2}) \| \mathbf{u}^n - \mathbf{u}^{n-1} \|^2_X + \frac{1}{2\varepsilon_6} \| \mathbf{v} \|^2_X .$$

Thus, T_2 is bounded as follows:

$$|T_2| \le c_4 (1 + \frac{\varepsilon_5}{2} + \frac{\varepsilon_6}{2}) \| \mathbf{u}^n - \mathbf{u}^{n-1} \|^2_X + (\frac{1}{2\varepsilon_5} + \frac{1}{2\varepsilon_6}) \| \mathbf{v} \|^2_X .$$

Let us now bound the term T_3. By using $\mathbf{u}_\tau = \mathbf{u} - \mathbf{v}$ and remarking that $((\mathbf{u}^n - \mathbf{u}^{n-1}) \cdot \nabla \mathbf{v}, \mathbf{v}) = 0$ we get by remarking that $\nabla \mathbf{u} \in L^\infty(0, T; L^4(\Omega)^{2\times 2})$:

$$\begin{aligned}
|T_3| &= \left| \frac{t - t_{n-1}}{\tau_n} \int_\Omega \big((\mathbf{u}^n - \mathbf{u}^{n-1}) \cdot \nabla \mathbf{u}_\tau \cdot \mathbf{v} \big)(x) \, \mathbf{dx} \right| = \left| \frac{t - t_{n-1}}{\tau_n} \int_\Omega \big((\mathbf{u}^n - \mathbf{u}^{n-1}) \cdot \nabla \mathbf{u} \cdot \mathbf{v} \big)(x) \, \mathbf{dx} \right| \\
&\le \| \nabla \mathbf{u} \|_{L^4(\Omega)^{2\times 2}} \| \mathbf{v} \|_{L^2(\Omega)} \| \mathbf{u}^n - \mathbf{u}^{n-1} \|_{L^4(\Omega)^2} \\
&\le c_5 \frac{\varepsilon_7}{2} \| \mathbf{u}^n - \mathbf{u}^{n-1} \|^2_X + \frac{1}{2\varepsilon_7} \| \mathbf{v} \|^2_X .
\end{aligned}$$

Thereby, we get

$$|\langle R_u^{\tau,2}(t), \mathbf{v}(t) \rangle| \le c_6 (1 + \frac{\varepsilon_4}{2} + \frac{\varepsilon_5}{2} + \frac{\varepsilon_6}{2} + \frac{\varepsilon_7}{2}) \| \mathbf{u}^n - \mathbf{u}^{n-1} \|^2_X + (\frac{1}{2\varepsilon_4} + \frac{1}{2\varepsilon_5} + \frac{1}{2\varepsilon_6} + \frac{1}{2\varepsilon_7}) \| \mathbf{v} \|^2_X . \quad (5.36)$$

Next, we regroup (5.32), (5.33)–(5.34)–(5.35)–(5.36) and choose $\varepsilon_i, i = 1, \dots 7$ such that $\sum_{i=1}^{7} \frac{1}{2\varepsilon_i} \| \mathbf{v} \|^2_X$ is smaller than $\frac{\nu_0}{2} \| \mathbf{v} \|^2_X$ and we integrate between 0 and t to obtain:

$$\begin{aligned}
&\| \mathbf{v}(t) \|^2_{L^2(\Omega)^2} + \nu_0 \int_0^t \| \mathbf{v}(s) \|^2_X \, ds \\
&\qquad \le c_7 \left(\| \mathbf{f}_0 - \pi_\tau \mathbf{f}_0 \|^2_{L^2(0,t;L^2(\Omega)^2)} + \| C - \pi_{l,\tau} C_h \|^2_{L^2(0,t;L^4(\Omega))} + \| \pi_\tau \mathbf{u} - \pi_{l,\tau} \mathbf{u} \|^2_{L^2(0,t;X)} \right) \\
&\qquad\qquad + c_7^{'} \int_0^t \| \mathbf{u}(\tau) \|^2_X \| \mathbf{v}(\tau) \|^2_{L^2(\Omega)^2} \, d\tau.
\end{aligned}$$

We apply the Gronwall Lemma 2.5 with the following functions:

$y(t) = \| \mathbf{v}(t) \|^2_{L^2(\Omega)^2} + \nu_0 \int_0^t \| \mathbf{v}(s) \|^2_X \, ds$,

$\tilde{f}(t) = c_7 \left(\| \mathbf{f}_0 - \pi_\tau \mathbf{f}_0 \|^2_{L^2(0,t;L^2(\Omega)^2)} + \| C - \pi_{l,\tau} C_h \|^2_{L^2(0,t;L^4(\Omega))} + \| \pi_\tau \mathbf{u} - \pi_{l,\tau} \mathbf{u} \|^2_{L^2(0,t;X)} \right)$,

$\tilde{g}(t) = 1$,

$k(t) = c_7^{'} \| \mathbf{u}(t) \|^2_X$,

and we get

$$\begin{aligned}
&\| \mathbf{u}(t) - \mathbf{u}_\tau(t) \|^2_{L^2(\Omega)^2} + \nu_0 \int_0^t \| \mathbf{u}(s) - \mathbf{u}_\tau(s) \|^2_X \, ds \\
&\quad \le c_7 \left(\| \mathbf{f}_0 - \pi_\tau \mathbf{f}_0 \|^2_{L^2(0,t;L^2(\Omega)^2)} + \| C - \pi_{l,\tau} C_h \|^2_{L^2(0,t;L^4(\Omega))} + \| \pi_\tau \mathbf{u} - \pi_{l,\tau} \mathbf{u} \|^2_{L^2(0,t;X)} \right) \\
&\quad\quad + c_7^{'} \int_0^t \left(\tilde{f}(\tau) \| \mathbf{u}(\tau) \|^2_X \times \exp c_7^{'} \int_0^t \| \mathbf{u}(s) \|^2_X \, ds \right) d\tau.
\end{aligned}$$

Since $\tau \le t$, we have $\tilde{f}(\tau) \le \tilde{f}(t)$, so

$$\begin{aligned}
&\| \mathbf{u}(t) - \mathbf{u}_\tau(t) \|^2_{L^2(\Omega)^2} + \nu_0 \int_0^t \| \mathbf{u}(s) - \mathbf{u}_\tau(s) \|^2_X \, ds \\
&\quad \le \ c_8 \bigg(\| \mathbf{f}_0 - \pi_\tau \mathbf{f}_0 \|^2_{L^2(0,t;L^2(\Omega)^2)} + \| C - \pi_{l,\tau} C_h \|^2_{L^2(0,t;L^4(\Omega))} + \| \pi_\tau \mathbf{u} - \pi_{l,\tau} \mathbf{u} \|^2_{L^2(0,t;X)} \bigg) \\
&\qquad + c_9 \, \tilde{f}(t) \int_0^t \bigg(\| \mathbf{u}(\tau) \|^2_X \times \ \exp \int_0^t \| \mathbf{u}(s) \|^2_X \, ds \bigg) d\tau.
\end{aligned}$$

Based on (3.2), integrals of the type $\int_0^t \| \mathbf{u}(s) \|^2_X \, ds$ are bounded by constants and we get (5.31). □

After obtaining an *a posteriori* estimate between the exact solution $\mathbf{u}$ and the solution $\mathbf{u}_\tau$ of problem (P_{aux}), we establish an estimate between $\mathbf{u}_\tau$ and the discrete solution $\mathbf{u}_h$. For this, we shall prove the intermediary Lemmas 5.11, 5.13 and 5.14. They lead to Theorem 5.15 which, with Lemma 5.16, prove Corollary 5.17, the final estimate between $\mathbf{u}_\tau$ and $\mathbf{u}_h$.

Lemma 5.11. *For all $n \in \{1, \dots, N\}$, the solutions of Problem (P_{aux}) and (Eds1) verify for all $\mathbf{v} \in X$ and $\mathbf{v}_h \in X_{nh}$:*

$$\begin{aligned}
&\frac{1}{\tau_n}((\mathbf{u}^n - \mathbf{u}^{n-1}) - (\mathbf{u}^n_h - \mathbf{u}^{n-1}_h)), \mathbf{v}) + \nu_0(\nabla(\mathbf{u}^n - \mathbf{u}^n_h), \nabla \mathbf{v}) + 2(\nu_C(C^{n-1}_h)\mathbb{D}(\mathbf{u}^n - \mathbf{u}^n_h), \mathbb{D}\mathbf{v}) \\
&\quad + (\mathbf{u}^{n-1} \cdot \nabla \mathbf{u}^n - \mathbf{u}^{n-1}_h \cdot \nabla \mathbf{u}^n_h, \mathbf{v}) - \frac{1}{2}(\text{div } (\mathbf{u}^{n-1}_h)\mathbf{u}^n_h, \mathbf{v}) - (\text{div } \mathbf{v}, p^n - p^n_h) = (R^h_u, \mathbf{v} - \mathbf{v}_h),
\end{aligned} \tag{5.37}$$

$$(q, \text{div}(\mathbf{u}^n - \mathbf{u}^n_h)) = -(q, \text{div}(\mathbf{u}^n_h)),$$

where R^h_u defined in (5.10) *also verifies the following equality*

$$\begin{aligned}
\langle R^h_u(t), \mathbf{v} - \mathbf{v}_h \rangle = &(\mathbf{f}^n(C^{n-1}_h) - \frac{1}{\tau_n}(\mathbf{u}^n_h - \mathbf{u}^{n-1}_h), \mathbf{v} - \mathbf{v}_h) - (\nu_0 \nabla \mathbf{u}^n_h, \nabla(\mathbf{v} - \mathbf{v}_h)) - 2(\nu_C(C^{n-1}_h)\mathbb{D}(\mathbf{u}^n_h), \mathbb{D}(\mathbf{v} - \mathbf{v}_h)) \\
&- (\mathbf{u}^{n-1}_h \cdot \nabla \mathbf{u}^n_h, \mathbf{v} - \mathbf{v}_h) - \frac{1}{2}(\text{div } (\mathbf{u}^{n-1}_h) \, \mathbf{u}^n_h, \mathbf{v} - \mathbf{v}_h) + (\text{div}(\mathbf{v} - \mathbf{v}_h), p^n_h).
\end{aligned} \tag{5.38}$$

Proof. We start by subtracting the first equation of problem (Eds1) from the first equation of problem (P_{aux}) and we obtain

$$\begin{aligned}
&\langle \mathbf{f}^n(C^{n-1}_h), \mathbf{v} \rangle - \langle \mathbf{f}^n(C^{n-1}_h), \mathbf{v}_h \rangle = \langle \mathbf{f}^n(C^{n-1}_h), \mathbf{v} \rangle - \langle \mathbf{f}^n(C^{n-1}_h), \mathbf{v}_h - \mathbf{v} + \mathbf{v} \rangle \\
&\quad = \frac{1}{\tau_n}((\mathbf{u}^n - \mathbf{u}^{n-1}) - (\mathbf{u}^n_h - \mathbf{u}^{n-1}_h)), \mathbf{v}) - (\text{div } \mathbf{v}, p^n - p^n_h) + (\nu_0 \nabla(\mathbf{u}^n - \mathbf{u}^n_h), \nabla \mathbf{v}) \\
&\qquad + 2(\nu_C(C^{n-1}_h)\mathbb{D}(\mathbf{u}^n - \mathbf{u}^n_h), \mathbb{D}\mathbf{v}) + (\mathbf{u}^{n-1} \cdot \nabla \mathbf{u}^n - \mathbf{u}^{n-1}_h \cdot \nabla \mathbf{u}^n_h, \mathbf{v}) \\
&\qquad - \frac{1}{2}(\text{div } (\mathbf{u}^{n-1}_h)\mathbf{u}^n_h, \mathbf{v}) + (\frac{1}{\tau_n}(\mathbf{u}^n_h - \mathbf{u}^{n-1}_h, \mathbf{v} - \mathbf{v}_h) + (\nu_0 \nabla \mathbf{u}^n_h, \nabla(\mathbf{v} - \mathbf{v}_h)) \\
&\qquad + 2(\nu_C(C^{n-1}_h)\mathbb{D}(\mathbf{u}^n_h), \mathbb{D}(\mathbf{v} - \mathbf{v}_h)) + (\mathbf{u}^{n-1}_h \cdot \nabla \mathbf{u}^n_h, \mathbf{v} - \mathbf{v}_h) \\
&\qquad + \frac{1}{2}(\text{div } (\mathbf{u}^{n-1}_h)\mathbf{u}^n_h, \mathbf{v} - \mathbf{v}_h) - (\text{div}(\mathbf{v} - \mathbf{v}_h), p^n_h).
\end{aligned} \tag{5.39}$$

Using (5.38) and (5.39), we get the first equality in (5.37). For the second equality of (5.37), we just need to insert $\mathbf{u}^n_h$ in the last equation of (P_{aux}). □

Before introducing the second lemma, we will introduce the Stokes operator Π defined from X into itself as follows: For each $\mathbf{v}$ in X, $\Pi\mathbf{v}$ denote the velocity $\mathbf{w}$ of the unique solution $(\mathbf{w}, r)$ in $X \times M$ of the following Stokes problem:

$$\begin{cases} \forall \, \mathbf{t} \in X, (\nabla \mathbf{w}, \nabla \mathbf{t}) - (\text{div } \mathbf{t}, r) = 0, \\ \forall \, q \in M, (\text{div } \mathbf{w}, q) = (\text{div } \mathbf{v}, q). \end{cases} \tag{5.40}$$

The next lemma states some properties of the operator Π ([11], [14] or [28]).

Lemma 5.12. *The operator Π has the following properties:*

(1) $\forall \, \mathbf{v} \in V, \ \Pi\mathbf{v} = \mathbf{0}$.

(2) $\forall\, \mathbf{v} \in X$, *we have the following estimates:*

$$\| \mathbf{v} - \Pi\mathbf{v} \|_{H_0^1(\Omega)^2} \leq \| \mathbf{v} \|_X \qquad \text{and} \qquad \| \Pi\mathbf{v} \|_X \leq \frac{1}{\beta*} \| \operatorname{div} \mathbf{v} \|_{L^2(\Omega)} .$$

(3) $\forall\, \mathbf{v}_h \in V_{nh}$ *and* $1 \leq n \leq N$*:*

$$\| \Pi\mathbf{v}_h \|_{L^2(\Omega)^2} \leq c h_n^{1/2} \| \operatorname{div} \mathbf{v}_h \|_{L^2(\Omega)} .$$

Now, we state the second and third lemmas that help us prove an *a posteriori* error estimate between the solution $\mathbf{u}_\tau$ of problem (P_{aux}) and the solution $\mathbf{u}_h$ of problem (Eds1).

Lemma 5.13. *Let* $\mathbf{u}^m$ *and* $\mathbf{u}_h^m$ *be respectively the solutions of* (P_{aux}) *and (Eds1). We denote by* $\mathbf{e}^n = \mathbf{u}^n - \mathbf{u}_h^n$ *and* $\mathbf{v} = \mathbf{e}^n - \Pi\mathbf{e}^n$*. We have the following equality:*

$$\begin{aligned}\frac{1}{2} \| \mathbf{e}^n \|^2_{L^2(\Omega)^2} - \frac{1}{2} \| \mathbf{e}^{n-1} \|^2_{L^2(\Omega)^2} + \frac{1}{2} \| \mathbf{e}^n - \mathbf{e}^{n-1} \|^2_{L^2(\Omega)^2} + \nu_0 \tau_n \| \mathbf{e}^n \|^2_X + 2\tau_n(\nu_C(\pi_{l,\tau} C_h)\mathbb{D}(\mathbf{e}^n), \mathbb{D}(\mathbf{e}^n))\\ = (\mathbf{e}^n - \mathbf{e}^{n-1}, \Pi\mathbf{e}^n) + \nu_0\tau_n(\nabla\mathbf{e}^n, \nabla(\Pi\mathbf{e}^n)) + 2\tau_n(\nu_C(\pi_{l,\tau} C_h)\mathbb{D}(\mathbf{e}^n), \mathbb{D}(\Pi\mathbf{e}^n)) + \tau_n\langle R_u^h, \mathbf{v} - \mathbf{v}_h\rangle\\ - \frac{\tau_n}{2}(\mathbf{e}^{n-1}\cdot\nabla\mathbf{u}^n, \mathbf{v}) + \frac{\tau_n}{2}(\mathbf{e}^{n-1}\cdot\nabla\mathbf{v}, \mathbf{u}^n) - \tau_n(\mathbf{u}_h^{n-1}\cdot\nabla\Pi\mathbf{e}^n, \mathbf{e}^n) - \frac{\tau_n}{2}(\operatorname{div}\,(\mathbf{u}_h^{n-1})\Pi\mathbf{e}^n, \mathbf{e}^n).\end{aligned} \tag{5.41}$$

Proof. First of all, we denote for all $n \in \{1, \ldots, N\}$, $\mathbf{e}^n = \mathbf{u}^n - \mathbf{u}_h^n$. We have

$$\begin{aligned}\frac{1}{2} \| \mathbf{e}^n \|^2_{L^2(\Omega)^2} - \frac{1}{2} \| \mathbf{e}^{n-1} \|^2_{L^2(\Omega)^2} + \frac{1}{2} \| \mathbf{e}^n - \mathbf{e}^{n-1} \|^2_{L^2(\Omega)^2} + \nu_0 \tau_n \| \mathbf{e}^n \|^2_X + 2\tau_n(\nu_C(\pi_{l,\tau} C_h)\mathbb{D}(\mathbf{e}^n), \mathbb{D}(\mathbf{e}^n)) =\\ (\mathbf{e}^n - \mathbf{e}^{n-1}, \mathbf{e}^n) + \nu_0\tau_n(\nabla\mathbf{e}^n, \nabla\mathbf{e}^n) + 2\tau_n(\nu_C(\pi_{l,\tau} C_h)\mathbb{D}(\mathbf{e}^n), \mathbb{D}(\mathbf{e}^n)).\end{aligned} \tag{5.42}$$

Let $\mathbf{v} = \mathbf{e}^n - \Pi\mathbf{e}^n$ in the first equality in Lemma 5.11. Thereby, knowing that div $(\mathbf{e}^n - \Pi\mathbf{e}^n) = 0$, the pressure term vanishes (this is the reason for the introduction of $\Pi\mathbf{e}^n$) and we get for all $\mathbf{v}_h \in X_{nh}$

$$\begin{aligned}(\mathbf{e}^n - \mathbf{e}^{n-1}, \mathbf{e}^n) + \nu_0\tau_n(\nabla\mathbf{e}^n, \nabla\mathbf{e}^n) + 2\tau_n(\nu_C(\pi_{l,\tau} C_h)\mathbb{D}(\mathbf{e}^n), \mathbb{D}(\mathbf{e}^n)) =\\ (\mathbf{e}^n - \mathbf{e}^{n-1}, \Pi\mathbf{e}^n) + \nu_0\tau_n(\nabla\mathbf{e}^n, \nabla(\Pi\mathbf{e}^n)) + 2\tau_n(\nu_C(\pi_{l,\tau} C_h)\mathbb{D}(\mathbf{e}^n), \mathbb{D}(\Pi\mathbf{e}^n))\\ + \tau_n\langle R_u^h, \mathbf{v} - \mathbf{v}_h\rangle - \tau_n(\mathbf{u}^{n-1}\cdot\nabla\mathbf{u}^n - \mathbf{u}_h^{n-1}\cdot\nabla\mathbf{u}_h^n, \mathbf{v}) + \frac{1}{2}\tau_n(\operatorname{div}\,(\mathbf{u}_h^{n-1})\mathbf{u}_h^n, \mathbf{v}).\end{aligned} \tag{5.43}$$

In the last two terms of (5.43), we insert $\tau_n(\mathbf{u}_h^{n-1}\cdot\nabla\mathbf{u}^n, \mathbf{v})$, $\frac{1}{2}\tau_n(\operatorname{div}\,(\mathbf{u}_h^{n-1})\mathbf{u}^n, \mathbf{v})$ and we add the vanishing term $-\frac{1}{2}\tau_n(\operatorname{div}\,(\mathbf{u}^{n-1})\mathbf{u}^n, \mathbf{v})$, to obtain :

$$\begin{aligned}-\tau_n(\mathbf{u}^{n-1}\cdot\nabla\mathbf{u}^n - \mathbf{u}_h^{n-1}\cdot\nabla\mathbf{u}_h^n, \mathbf{v}) + \frac{\tau_n}{2}(\operatorname{div}\,(\mathbf{u}_h^{n-1})\mathbf{u}_h^n, \mathbf{v}) =\\ -\tau_n(\mathbf{e}^{n-1}\cdot\nabla\mathbf{u}^n, \mathbf{v}) - \tau_n(\mathbf{u}_h^{n-1}\cdot\nabla\mathbf{e}^n, \mathbf{v}) - \frac{\tau_n}{2}(\operatorname{div}\,(\mathbf{u}_h^{n-1})\mathbf{e}^n, \mathbf{v}) - \frac{\tau_n}{2}(\operatorname{div}\,(\mathbf{e}^{n-1})\mathbf{u}^n, \mathbf{v}).\end{aligned}$$

We insert $\tau_n(\mathbf{u}_h^{n-1}\cdot\nabla\Pi\mathbf{e}^n, \mathbf{v})$ and $\frac{\tau_n}{2}(\operatorname{div}\,(\mathbf{u}_h^{n-1})\Pi\mathbf{e}^n, \mathbf{v})$, then use that $\mathbf{v} = \mathbf{e}^n - \Pi\mathbf{e}^n$ and apply Proposition 4.2; we get

$$\begin{aligned}-\tau_n(\mathbf{u}^{n-1}\cdot\nabla\mathbf{u}^n - \mathbf{u}_h^{n-1}\cdot\nabla\mathbf{u}_h^n, \mathbf{v}) + \frac{\tau_n}{2}(\operatorname{div}\,(\mathbf{u}_h^{n-1})\mathbf{u}_h^n, \mathbf{v}) =\\ -\tau_n(\mathbf{e}^{n-1}\cdot\nabla\mathbf{u}^n, \mathbf{v}) - \frac{\tau_n}{2}(\operatorname{div}\,(\mathbf{e}^{n-1})\mathbf{u}^n, \mathbf{v}) - \tau_n(\mathbf{u}_h^{n-1}\cdot\nabla\Pi\mathbf{e}^n, \mathbf{e}^n) - \frac{\tau_n}{2}(\operatorname{div}\,(\mathbf{u}_h^{n-1})\Pi\mathbf{e}^n, \mathbf{e}^n).\end{aligned} \tag{5.44}$$

By integrating by parts the second term in the right-hand side of (5.44), plugging it into the right-hand side of (5.43) and using (5.42), we get the desired result. □

Now we bound the very last line of (5.41).

Lemma 5.14. *Let us suppose that $h_n \le c_s\tau_n, \forall n \in \{1,\dots,N\}$, where c_s is a positive constant independent of n. Let also $\mathbf{u}^m$ and $\mathbf{u}_h^m$ be respectively the solutions of (P_{aux}) and (Eds1). We denote by $\mathbf{e}^n = \mathbf{u}^n - \mathbf{u}_h^n$ and $\mathbf{v} = \mathbf{e}^n - \Pi\mathbf{e}^n$. We have the following bound:*

$$
\begin{aligned}
&|\frac{\tau_n}{2}(\mathbf{e}^{n-1}\cdot\nabla\mathbf{u}^n,\mathbf{v}) + \frac{\tau_n}{2}(\mathbf{e}^{n-1}\cdot\nabla\mathbf{v},\mathbf{u}^n) + \tau_n(\mathbf{u}_h^{n-1}\cdot\nabla\Pi\mathbf{e}^n,\mathbf{e}^n) + \frac{\tau_n}{2}(\mathrm{div}\ (\mathbf{u}_h^{n-1})\Pi\mathbf{e}^n,\mathbf{e}^n)| \\
&\quad\le c\tau_n\Big(\parallel \mathrm{div}\ \mathbf{u}_h^n\parallel^2_{L^2(\Omega)} + \parallel \mathrm{div}\ \mathbf{u}_h^{n-1}\parallel^2_{L^2(\Omega)} + \big(\parallel \mathbf{u}_h^{n-1}\parallel_X^2 + \parallel \mathbf{u}_h^n\parallel_X^2 + \parallel \mathbf{u}^n\parallel_X^2\big)\parallel \mathbf{e}^{n-1}\parallel^2_{L^2(\Omega)^2}\Big) \\
&\quad + \frac{\nu_0}{20}\tau_n\parallel \mathbf{e}^n\parallel_X^2 + \frac{\nu_0}{25}\tau_{n-1}\parallel \mathbf{e}^{n-1}\parallel_X^2 + \frac{3}{100}\parallel \mathbf{e}^n - \mathbf{e}^{n-1}\parallel^2_{L^2(\Omega)^2},
\end{aligned}
\tag{5.45}
$$

where c is a positive constant independent of the time and mesh steps.

Proof. Let us denote:

$$
\begin{aligned}
A &:= \frac{\tau_n}{2}(\mathbf{e}^{n-1}\cdot\nabla\mathbf{u}^n,\mathbf{v}) + \frac{\tau_n}{2}(\mathbf{e}^{n-1}\cdot\nabla\mathbf{v},\mathbf{u}^n) =: A_1 + A_2\,, \\
B &:= \tau_n(\mathbf{u}_h^{n-1}\cdot\nabla\Pi\mathbf{e}^n,\mathbf{e}^n) + \frac{\tau_n}{2}(\mathrm{div}\ (\mathbf{u}_h^{n-1})\Pi\mathbf{e}^n,\mathbf{e}^n) =: B_1 + B_2\,.
\end{aligned}
$$

By using the $L^2 - L^4 - L^4$ generalized Cauchy-Schwarz inequality, then inequality (2.3) on $\mathbf{e}^{n-1}$, $\mathbf{e}^n$ and $\Pi\mathbf{e}^n$, Lemma 5.12 and the fact that $\Pi\mathbf{e}^n = -\Pi\mathbf{u}_h^n$, we have:

$$
\begin{aligned}
|A_1| &\le \frac{\tau_n}{2}\parallel \mathbf{e}^{n-1}\parallel_{L^4(\Omega)^2}\parallel \mathbf{u}^n\parallel_X\parallel \mathbf{v}\parallel_{L^4(\Omega)^2} \\
&\le c_5\tau_n\parallel \mathbf{e}^{n-1}\parallel^{1/2}_{L^2(\Omega)^2}\parallel \mathbf{e}^{n-1}\parallel_X^{1/2}\parallel \mathbf{u}^n\parallel_X\big(\parallel \mathbf{e}^n\parallel^{1/2}_{L^2(\Omega)^2}\parallel \mathbf{e}^n\parallel_X^{1/2} + \parallel \Pi\mathbf{e}^n\parallel^{1/2}_{L^2(\Omega)^2}\parallel \Pi\mathbf{e}^n\parallel_X^{1/2}\big) \\
&\le c_5\tau_n\big(\parallel \mathbf{e}^{n-1}\parallel^{1/2}_{L^2(\Omega)^2}\parallel \mathbf{e}^{n-1}\parallel_X^{1/2}\parallel \mathbf{u}^n\parallel_X\parallel \mathbf{e}^n\parallel^{1/2}_{L^2(\Omega)^2}\parallel \mathbf{e}^n\parallel_X^{1/2} \\
&\qquad + \parallel \mathbf{e}^{n-1}\parallel^{1/2}_{L^2(\Omega)^2}\parallel \mathbf{e}^{n-1}\parallel_X^{1/2}\parallel \mathbf{u}^n\parallel_X\,(h_n^{1/2}\parallel \mathrm{div}\ \mathbf{u}_h^n\parallel_{L^2(\Omega)})^{1/2}\parallel \mathbf{e}^n\parallel_X^{1/2}\big).
\end{aligned}
\tag{5.46}
$$

We denote by $A_{1,1}$ and $A_{1,2}$ the terms in the right-hand side of (5.46). Then we have by taking into consideration that $h_n \le c_s\tau_n$:

$$
\begin{aligned}
A_{1,1} &\le c_5\tau_n\parallel \mathbf{u}^n\parallel_X\big(\frac{\varepsilon_1}{2}\parallel \mathbf{e}^{n-1}\parallel_{L^2(\Omega)^2}\parallel \mathbf{e}^{n-1}\parallel_X + \frac{1}{2\varepsilon_1}\parallel \mathbf{e}^n\parallel_{L^2(\Omega)^2}\parallel \mathbf{e}^n\parallel_X\big) \\
&\le c_5\tau_n\frac{\varepsilon_1}{2}\big(\frac{1}{2\varepsilon_2}\parallel \mathbf{e}^{n-1}\parallel^2_{L^2(\Omega)^2}\parallel \mathbf{u}^n\parallel_X^2 + \frac{\varepsilon_2}{2}\parallel \mathbf{e}^{n-1}\parallel_X^2\big) \\
&\qquad + c_5\tau_n\frac{1}{2\varepsilon_1}\big(\frac{1}{2\varepsilon_3}\parallel \mathbf{e}^n\parallel^2_{L^2(\Omega)^2}\parallel \mathbf{u}^n\parallel_X^2 + \frac{\varepsilon_3}{2}\parallel \mathbf{e}^n\parallel_X^2\big).
\end{aligned}
$$

and

$$
\begin{aligned}
A_{1,2} &\le c_5\tau_n\big(\frac{\varepsilon_4}{2}h_n^{1/2}\parallel \mathbf{e}^{n-1}\parallel_{L^2(\Omega)^2}\parallel \mathbf{e}^{n-1}\parallel_X\parallel \mathbf{u}^n\parallel_X^2 + \frac{1}{2\varepsilon_4}\parallel \mathrm{div}\ \mathbf{u}_h^n\parallel_{L^2(\Omega)}\parallel \mathbf{e}^n\parallel_X\big) \\
&\le c_5\tau_n\frac{\varepsilon_4}{2}\big(\frac{\varepsilon_5}{2}h_n\parallel \mathbf{e}^{n-1}\parallel_X^2\parallel \mathbf{u}^n\parallel_X^2 + \frac{1}{2\varepsilon_5}\parallel \mathbf{e}^{n-1}\parallel^2_{L^2(\Omega)^2}\parallel \mathbf{u}^n\parallel_X^2\big) \\
&\qquad + c_5\tau_n\frac{1}{2\varepsilon_4}\parallel \mathrm{div}\ \mathbf{u}_h^n\parallel_{L^2(\Omega)}\parallel \mathbf{e}^n\parallel_X \\
&\le c_6\tau_n^2\frac{\varepsilon_4\varepsilon_5}{4}\parallel \mathbf{e}^{n-1}\parallel_X^2\parallel \mathbf{u}^n\parallel_X^2 + \tau_n\frac{\varepsilon_4}{2\varepsilon_5}\parallel \mathbf{e}^{n-1}\parallel^2_{L^2(\Omega)^2}\parallel \mathbf{u}^n\parallel_X^2 \\
&\qquad + c_6\tau_n\frac{1}{2\varepsilon_4}\big(\frac{1}{2\varepsilon_6}\parallel \mathrm{div}\ \mathbf{u}_h^n\parallel^2_{L^2(\Omega)} + \frac{\varepsilon_6}{2}\parallel \mathbf{e}^n\parallel_X^2\big).
\end{aligned}
$$

Let us now bound $A_2 = \frac{\tau_n}{2}(\mathbf{e}^{n-1}\cdot\nabla\mathbf{e}^n,\mathbf{u}^n) - \frac{\tau_n}{2}(\mathbf{e}^{n-1}\cdot\nabla\Pi\mathbf{e}^n,\mathbf{u}^n) =: A_{2,1} + A_{2,2}$.

We use the $L^2-L^4-L^4$ generalized Cauchy-Schwarz inequality, then (2.3) on $\mathbf{u}^n$ and $\mathbf{e}^{n-1}$ to get

$$\begin{aligned} A_{2,1} &\leq c_7\tau_n \parallel \mathbf{e}^{n-1}\parallel_{L^4(\Omega)^2}\parallel \mathbf{e}^n\parallel_X\parallel \mathbf{u}^n\parallel_{L^4(\Omega)^2}\\ &\leq c_8\big(\frac{\tau_n}{\varepsilon_7}\parallel \mathbf{e}^n\parallel_X^2+\tau_n\varepsilon_7\parallel \mathbf{e}^{n-1}\parallel_{L^2(\Omega)^2}\parallel \mathbf{e}^{n-1}\parallel_X\parallel \mathbf{u}^n\parallel_{L^2(\Omega)^2}\parallel \mathbf{u}^n\parallel_X\big)\\ &\leq c_9\big(\frac{\tau_n}{\varepsilon_7}\parallel \mathbf{e}^n\parallel_X^2+\tau_n\frac{\varepsilon_7}{\varepsilon_8}\parallel \mathbf{e}^{n-1}\parallel^2_{L^2(\Omega)^2}\parallel \mathbf{u}^n\parallel_X^2+\tau_n\varepsilon_7\varepsilon_8\parallel \mathbf{e}^{n-1}\parallel_X^2\parallel \mathbf{u}^n\parallel^2_{L^2(\Omega)^2}\big). \end{aligned}$$

We use Lemma 5.12 then we follow the same steps as $A_{2,1}$, to get

$$\begin{aligned} A_{2,2} &\leq c_{10}\tau_n \parallel \mathbf{e}^{n-1}\parallel_{L^4(\Omega)^2}\parallel \operatorname{div}\mathbf{u}_h^n\parallel_{L^2(\Omega)}\parallel \mathbf{u}^n\parallel_{L^4(\Omega)^2}\\ &\leq c_{11}\big(\frac{\tau_n}{\varepsilon_9}\parallel \operatorname{div}\mathbf{u}_h^n\parallel^2_{L^2(\Omega)}+\tau_n\varepsilon_9\parallel \mathbf{e}^{n-1}\parallel_{L^2(\Omega)^2}\parallel \mathbf{e}^{n-1}\parallel_X\parallel \mathbf{u}^n\parallel_{L^2(\Omega)^2}\parallel \mathbf{u}^n\parallel_X\big)\\ &\leq c_{12}\big(\frac{\tau_n}{\varepsilon_9}\parallel \operatorname{div}\mathbf{u}_h^n\parallel^2_{L^2(\Omega)}+\tau_n\frac{\varepsilon_9}{\varepsilon_{10}}\parallel \mathbf{e}^{n-1}\parallel^2_{L^2(\Omega)^2}\parallel \mathbf{u}^n\parallel_X^2+\tau_n\varepsilon_9\varepsilon_{10}\parallel \mathbf{e}^{n-1}\parallel_X^2\parallel \mathbf{u}^n\parallel^2_{L^2(\Omega)^2}\big). \end{aligned}$$

Finally, by gathering what precedes and using Proposition 5.8, we get:

$$\begin{aligned} |A| &\leq c_1\,\tau_n\Bigg(\big(\frac{1}{\varepsilon_4\varepsilon_6}+\frac{1}{\varepsilon_9}\big)\parallel \operatorname{div}\mathbf{u}_h^n\parallel^2_{L^2(\Omega)}+\big(\frac{\varepsilon_3}{\varepsilon_1}+\frac{\varepsilon_6}{\varepsilon_4}+\frac{1}{\varepsilon_7}\big)\parallel \mathbf{e}^n\parallel_X^2\\ &+\ (\varepsilon_1\varepsilon_2+\varepsilon_4\varepsilon_5\tau_n\parallel \mathbf{u}^n\parallel_X^2+\varepsilon_7\varepsilon_8+\varepsilon_9\varepsilon_{10})\parallel \mathbf{e}^{n-1}\parallel_X^2+\frac{1}{\varepsilon_1\varepsilon_3}\parallel \mathbf{e}^n-\mathbf{e}^{n-1}\parallel^2_{L^2(\Omega)^2}\parallel \mathbf{u}^n\parallel_X^2\\ &+\ \big(\frac{\varepsilon_1}{\varepsilon_2}+\frac{1}{\varepsilon_1\varepsilon_3}+\frac{\varepsilon_4}{\varepsilon_5}+\frac{\varepsilon_7}{\varepsilon_8}+\frac{\varepsilon_9}{\varepsilon_{10}}\big)\parallel \mathbf{u}^n\parallel_X^2\parallel \mathbf{e}^{n-1}\parallel^2_{L^2(\Omega)^2}\Bigg). \end{aligned} \tag{5.47}$$

By using the relation $\tau_n\parallel \mathbf{u}^n\parallel_X^2\leq \sum_{n=1}^{N}\tau_n\parallel \mathbf{u}^n\parallel_X^2$ and Proposition 5.8, we deduce that

$$\tau_n\parallel \mathbf{u}^n\parallel_X^2\leq \hat{C}, \tag{5.48}$$

where $\hat{C}$ is a given positive constant .

Thus, by choosing $\varepsilon_1=\dfrac{100c_1\sqrt{\hat{C}}}{\sqrt{\nu_0}}$, $\varepsilon_2=\dfrac{\nu_0}{100c_1\varepsilon_1\sigma_\tau}$, $\varepsilon_3=\sqrt{\hat{C}\nu_0}$, $\varepsilon_4=1$, $\varepsilon_5=\dfrac{\nu_0}{100c_1\hat{C}\sigma_\tau}$, $\varepsilon_6=\dfrac{\nu_0}{100c_1}$, $\varepsilon_7=\dfrac{100c_1}{\nu_0}$, $\varepsilon_8=\dfrac{1}{\varepsilon_7^2}\sigma_\tau$ and $\varepsilon_9=\varepsilon_{10}=\sqrt{\dfrac{\nu_0}{100c_1\sigma_\tau}}$, we obtain,

$$\begin{aligned} |A| &\leq \frac{3\nu_0}{100}\tau_n\parallel \mathbf{e}^n\parallel_X^2+\frac{\nu_0}{25}\tau_{n-1}\parallel \mathbf{e}^{n-1}\parallel_X^2+\frac{1}{100}\parallel \mathbf{e}^n-\mathbf{e}^{n-1}\parallel^2_{L^2(\Omega)^2}\\ &\quad+c_2\,\tau_n\Bigg(\parallel \operatorname{div}\mathbf{u}_h^n\parallel^2_{L^2(\Omega)}+\parallel \mathbf{u}^n\parallel_X^2\parallel \mathbf{e}^{n-1}\parallel^2_{L^2(\Omega)^2}\Bigg). \end{aligned} \tag{5.49}$$

We will now bound the term B. We use the $L^2-L^4-L^4$ generalized Cauchy-Schwarz inequality, then Lemma 5.12, the incompressibility condition $\operatorname{div}\mathbf{u}^n=\mathbf{0}$ and Lemma 2.2, we get

$$\begin{aligned} |B_1| &\leq c_{13}\tau_n\parallel \mathbf{u}_h^{n-1}\parallel_{L^4(\Omega)^2}\parallel \operatorname{div}\mathbf{u}_h^n\parallel_{L^2(\Omega)}\parallel \mathbf{e}^n\parallel_{L^4(\Omega)^2}\\ &\leq c_{14}\tau_n\big(\frac{1}{2\varepsilon_{11}}\parallel \operatorname{div}\mathbf{u}_h^n\parallel^2_{L^2(\Omega)}+\frac{\varepsilon_{11}}{2}\parallel \mathbf{u}_h^{n-1}\parallel_{L^2(\Omega)^2}\parallel \mathbf{u}_h^{n-1}\parallel_X\parallel \mathbf{e}^n\parallel_{L^2(\Omega)^2}\parallel \mathbf{e}^n\parallel_X\big). \end{aligned}$$

As $\parallel \mathbf{u}_h^n\parallel_{L^2(\Omega)^2}$ is bounded according to (4.5), we obtain

$$\begin{aligned} |B_1| &\leq c_{15}\tau_n\big(\frac{1}{2\varepsilon_{11}}\parallel \operatorname{div}\mathbf{u}_h^n\parallel^2_{L^2(\Omega)}+\frac{\varepsilon_{11}}{2}\big(\frac{\varepsilon_{12}}{2}\parallel \mathbf{e}^n\parallel_X^2+\frac{1}{2\varepsilon_{12}}\parallel \mathbf{u}_h^{n-1}\parallel_X^2\parallel \mathbf{e}^n\parallel^2_{L^2(\Omega)^2}\big)\big)\\ &\leq c_{16}\tau_n\Bigg(\frac{1}{2\varepsilon_{11}}\parallel \operatorname{div}\mathbf{u}_h^n\parallel^2_{L^2(\Omega)}+\frac{\varepsilon_{11}}{2}\big(\frac{\varepsilon_{12}}{2}\parallel \mathbf{e}^n\parallel_X^2+\frac{1}{2\varepsilon_{12}}\parallel \mathbf{u}_h^{n-1}\parallel_X^2\parallel \mathbf{e}^{n-1}\parallel^2_{L^2(\Omega)^2}\big)\\ &\qquad+\frac{\varepsilon_{11}}{4\varepsilon_{12}}\parallel \mathbf{u}_h^{n-1}\parallel_X^2\parallel \mathbf{e}^n-\mathbf{e}^{n-1}\parallel^2_{L^2(\Omega)^2}\Bigg). \end{aligned}$$

By following the same steps, remarking that $\Pi\mathbf{u}^n = \mathbf{0}$ and using Lemma 5.12, we get

$$\begin{aligned}
|B_2| &\leq \frac{\tau_n}{2}\big(\frac{1}{2\varepsilon_{13}} \parallel \text{div } \mathbf{u}_h^{n-1} \parallel^2_{L^2(\Omega)} + \frac{\varepsilon_{13}}{2} \parallel \Pi\mathbf{u}_h^n \parallel^2_{L^4(\Omega)^2} \parallel \mathbf{e}^n \parallel^2_{L^4(\Omega)^2}\big)\\
&\leq c_{17}\tau_n\Bigg(\frac{1}{\varepsilon_{13}} \parallel \text{div } \mathbf{u}_h^{n-1} \parallel^2_{L^2(\Omega)} + h_n^{1/2}\varepsilon_{13} \parallel \text{div } \mathbf{u}_h^n \parallel_{L^2(\Omega)} \parallel \Pi\mathbf{u}_h^n \parallel_X \parallel \mathbf{e}^n \parallel_{L^2(\Omega)^2} \parallel \mathbf{e}^n \parallel_X\Bigg)\\
&\leq c_{18}\tau_n\Bigg(\frac{1}{\varepsilon_{13}} \parallel \text{div } \mathbf{u}_h^{n-1} \parallel^2_{L^2(\Omega)} + \varepsilon_{13}\big(\frac{\varepsilon_{14}}{2} \parallel \mathbf{e}^n \parallel^2_X + \frac{h_n}{2\varepsilon_{14}} \parallel \text{div } \mathbf{u}_h^n \parallel^2_{L^2(\Omega)} \parallel \mathbf{u}_h^n \parallel^2_X \parallel \mathbf{e}^n \parallel^2_{L^2(\Omega)^2}\big)\Bigg)\\
&\leq c_{19}\tau_n\Bigg(\frac{1}{\varepsilon_{13}} \parallel \text{div } \mathbf{u}_h^{n-1} \parallel^2_{L^2(\Omega)} + \varepsilon_{13}\big(\frac{\varepsilon_{14}}{2} \parallel \mathbf{e}^n \parallel^2_X + \frac{\tau_n}{2\varepsilon_{14}} \parallel \mathbf{u}_h^n \parallel^4_X \parallel \mathbf{e}^{n-1} \parallel^2_{L^2(\Omega)^2}\big)\\
&\qquad + \frac{\tau_n\varepsilon_{13}}{2\varepsilon_{14}} \parallel \mathbf{u}_h^n \parallel^4_X \parallel \mathbf{e}^n - \mathbf{e}^{n-1} \parallel^2_{L^2(\Omega)^2}\Bigg).
\end{aligned}$$

Thus, B will be bounded as follows:

$$\begin{aligned}
|B| &\leq c_{20}\tau_n\Bigg(\frac{1}{\varepsilon_{11}} \parallel \text{div } \mathbf{u}_h^n \parallel^2_{L^2(\Omega)} + \frac{1}{\varepsilon_{13}} \parallel \text{div } \mathbf{u}_h^{n-1} \parallel^2_{L^2(\Omega)} + \big(\varepsilon_{11}\varepsilon_{12} + \varepsilon_{13}\varepsilon_{14}\big) \parallel \mathbf{e}^n \parallel^2_X\\
&\qquad + \frac{\varepsilon_{11}}{\varepsilon_{12}} \parallel \mathbf{u}_h^{n-1} \parallel^2_X \parallel \mathbf{e}^{n-1} \parallel^2_{L^2(\Omega)^2} + \tau_n\frac{\varepsilon_{13}}{\varepsilon_{14}} \parallel \mathbf{u}_h^n \parallel^4_X \parallel \mathbf{e}^{n-1} \parallel^2_{L^2(\Omega)^2}\\
&\qquad + \big(\frac{\varepsilon_{11}}{\varepsilon_{12}} \parallel \mathbf{u}_h^{n-1} \parallel^2_X + \frac{\varepsilon_{13}}{\varepsilon_{14}}\tau_n \parallel \mathbf{u}_h^n \parallel^4_X\big) \parallel \mathbf{e}^n - \mathbf{e}^{n-1} \parallel^2_{L^2(\Omega)^2}\Bigg)
\end{aligned} \tag{5.50}$$

By using the relation $\tau_n \parallel \mathbf{u}_h^n \parallel^2_X \leq \sum_{n=1}^{N} \tau_n \parallel \mathbf{u}_h^n \parallel^2_X$ and Theorem 4.3, we deduce that

$$\tau_n \parallel \mathbf{u}_h^n \parallel^2_X \leq \hat{C}, \tag{5.51}$$

where $\hat{C}$ is a given positive constant. We use this to obtain

$$\begin{aligned}
|B| &\leq c_{20}\tau_n\Bigg(\frac{1}{\varepsilon_{11}} \parallel \text{div } \mathbf{u}_h^n \parallel^2_{L^2(\Omega)} + \frac{1}{\varepsilon_{13}} \parallel \text{div } \mathbf{u}_h^{n-1} \parallel^2_{L^2(\Omega)} + \big(\varepsilon_{11}\varepsilon_{12} + \varepsilon_{13}\varepsilon_{14}\big) \parallel \mathbf{e}^n \parallel^2_X\\
&\qquad + \frac{\varepsilon_{11}}{\varepsilon_{12}} \parallel \mathbf{u}_h^{n-1} \parallel^2_X \parallel \mathbf{e}^{n-1} \parallel^2_{L^2(\Omega)^2} + \hat{C}\frac{\varepsilon_{13}}{\varepsilon_{14}} \parallel \mathbf{u}_h^n \parallel^2_X \parallel \mathbf{e}^{n-1} \parallel^2_{L^2(\Omega)^2}\Bigg)\\
&\quad + c_{20}\big(\hat{C}\frac{\varepsilon_{11}}{\varepsilon_{12}} + \hat{C}^2\frac{\varepsilon_{13}}{\varepsilon_{14}}\big) \parallel \mathbf{e}^n - \mathbf{e}^{n-1} \parallel^2_{L^2(\Omega)^2}.
\end{aligned} \tag{5.52}$$

We choose $\varepsilon_{11} = \dfrac{\sqrt{\nu_0}}{100c_{20}\sqrt{\hat{C}}}$, $\varepsilon_{12} = \sqrt{\hat{C}\nu_0}$, $\varepsilon_{13} = \dfrac{\sqrt{\nu_0}}{100c_{20}\hat{C}}$ and $\varepsilon_{14} = \hat{C}\sqrt{\nu_0}$, and we obtain

$$\begin{aligned}
|B| &\leq \frac{\nu_0}{50}\tau_n \parallel \mathbf{e}^n \parallel^2_X + \frac{1}{50} \parallel \mathbf{e}^n - \mathbf{e}^{n-1} \parallel^2_{L^2(\Omega)^2}\\
&\quad + c_3\tau_n\Bigg(\parallel \text{div } \mathbf{u}_h^n \parallel^2_{L^2(\Omega)} + \parallel \text{div } \mathbf{u}_h^{n-1} \parallel^2_{L^2(\Omega)}\\
&\qquad + \parallel \mathbf{u}_h^{n-1} \parallel^2_X \parallel \mathbf{e}^{n-1} \parallel^2_{L^2(\Omega)^2} + \parallel \mathbf{u}_h^n \parallel^2_X \parallel \mathbf{e}^{n-1} \parallel^2_{L^2(\Omega)^2}\Bigg).
\end{aligned} \tag{5.53}$$

Finally, relations (5.49) and (5.53) give the result. □

Now that we have established Lemmas 5.11, 5.13 and 5.14, we use them to get Theorem 5.15 which, with Lemma 5.16, prove Corollary 5.17, the final estimate between $\mathbf{u}_\tau$ and $\mathbf{u}_h$.

Theorem 5.15. *Let $h_n \leq c_s\tau_n, \forall n \in \{1, \dots, N\}$, where c_s is a positive constant independent of n. The following a posteriori error holds between $\mathbf{u}^m$ and $\mathbf{u}_h^m$:*

$$\parallel \mathbf{u}^m - \mathbf{u}_h^m \parallel^2_{L^2(\Omega)^2} + \sum_{n=1}^{m} \tau_n \parallel \mathbf{u}^n - \mathbf{u}_h^n \parallel^2_X \leq c\Bigg(\parallel \mathbf{u}_0 - \mathbf{u}_h^0 \parallel^2_{L^2(\Omega)^2} + \tau_0 \parallel \mathbf{u}_0 - \mathbf{u}_h^0 \parallel^2_X + \sum_{n=1}^{m} \sum_{\kappa_n \in \mathcal{T}_{nh}} \tau_n(\eta^h_{u,n,\kappa_n})^2\Bigg), \tag{5.54}$$

where C is a positive constant independent of the time and mesh steps.

Proof. First, we bound the right-hand side of (5.41); we shall make a frequent use of $ab \le \dfrac{\varepsilon}{2}a^2 + \dfrac{1}{2\varepsilon}b^2$ for all $\varepsilon > 0$, in particular for $\varepsilon = \frac{1}{2}$. Knowing that $\Pi \mathbf{e}^n = -\Pi \mathbf{u}_h^n$ (as $\Pi \mathbf{u}^n = 0$, see Lemma 5.12) and $h_n \le c_s \tau_n$, and using Lemma 5.12, the first and the second terms will be bounded as follows:

$$
\begin{aligned}
(\mathbf{e}^n - \mathbf{e}^{n-1}, \Pi \mathbf{e}^n) &\le \frac{1}{4} \| \mathbf{e}^n - \mathbf{e}^{n-1} \|^2_{L^2(\Omega)^2} + \| \Pi \mathbf{e}^n \|^2_{L^2(\Omega)^2} \\
&\le \frac{1}{4} \| \mathbf{e}^n - \mathbf{e}^{n-1} \|^2_{L^2(\Omega)^2} + c\, c_s\, \tau_n \| \operatorname{div} \mathbf{u}_h^n \|^2_{L^2(\Omega)^2},
\end{aligned}
\tag{5.55}
$$

and

$$
\begin{aligned}
\nu_0 \tau_n (\nabla \mathbf{e}^n, \nabla(\Pi \mathbf{e}^n)) + 2\tau_n (\nu_C(\pi_{l,\tau} C_h)\mathbb{D}(\mathbf{e}^n), \mathbb{D}(\Pi \mathbf{e}^n)) &\le \frac{\nu_0 \tau_n}{16} \| \mathbf{e}^n \|^2_X + 4\nu_0\, \tau_n \| \Pi \mathbf{u}_h^n \|^2_X \\
&\quad + \frac{\nu_0 \tau_n}{16} \| \mathbf{e}^n \|^2_X + \frac{2^2 \times 4 \times (\hat{\nu}_2)^2}{\nu_0} \tau_n \| \Pi \mathbf{u}_h^n \|^2_X \\
&\le \frac{\nu_0 \tau_n}{8} \| \mathbf{e}^n \|^2_X + c_2 \tau_n \| \operatorname{div} \mathbf{u}_h^n \|^2_{L^2(\Omega)^2} .
\end{aligned}
\tag{5.56}
$$

To treat the third term in the right-hand side of (5.41), we set $\mathbf{v}_h = C_{nh}\mathbf{v}$ and we use the definition R_u^h and Lemma 5.6. Then, since $\mathbf{v} = \mathbf{e}^n - \Pi \mathbf{e}^n$ and using lemma 5.12 we get:

$$
\begin{aligned}
\tau_n \langle R_u^h, \mathbf{v} - \mathbf{v}_h \rangle &\le c_3 \tau_n \Big(\sum_{\kappa_n \in \mathcal{T}_{nh}} (\eta^h_{u,n,\kappa_n})^2 \Big)^{1/2} \| \mathbf{v} \|_X \\
&\le \frac{c_4}{\nu_0} \sum_{\kappa_n \in \mathcal{T}_{nh}} \tau_n (\eta^h_{u,n,\kappa_n})^2 + \frac{\nu_0 \tau_n}{8} \| \mathbf{e}^n \|^2_X .
\end{aligned}
\tag{5.57}
$$

To bound the last terms of (5.41), we use Lemma 5.14. By Regrouping (5.45), (5.55), (5.56) and (5.57) summing over n from 1 to m and using $\tau_n \le \sigma_\tau \tau_{n-1}$ we get

$$
\begin{aligned}
\| \mathbf{e}^m \|^2_{L^2(\Omega)^2} + \sum_{n=1}^m \tau_n \| \mathbf{e}^n \|^2_X \le c_5 \Big(&\| \mathbf{e}^0 \|^2_{L^2(\Omega)^2} + \tau_0 \| \mathbf{e}^0 \|^2_X + \sum_{n=1}^m \sum_{\kappa_n \in \mathcal{T}_{nh}} \tau_n (\eta^h_{u,n,\kappa_n})^2 \\
&+ \sum_{n=0}^{m-1} \tau_n \big(\| \mathbf{u}^{n+1} \|^2_X + \| \mathbf{u}_h^n \|^2_X + \| \mathbf{u}_h^{n+1} \|^2_X \big) \| \mathbf{e}^n \|^2_{L^2(\Omega)^2} \Big).
\end{aligned}
$$

We apply Gronwall's lemma 2.6 with the following given functions:

$y_m = \| \mathbf{e}^m \|^2_{L^2(\Omega)^2} + \sum_{n=1}^m \tau_n \| \mathbf{e}^n \|^2_X$,

$\tilde{f}_m = c_5 \Big(\| \mathbf{e}^0 \|^2_{L^2(\Omega)^2} + \tau_0 \| \mathbf{e}^0 \|^2_X + \sum_{n=1}^m \sum_{\kappa_n \in \mathcal{T}_{nh}} \tau_n (\eta^h_{u,n,\kappa_n})^2 \Big)$,

and

$\tilde{g}_n = c_5 \tau_n \big(\| \mathbf{u}^{n+1} \|^2_X + \| \mathbf{u}_h^n \|^2_X + \| \mathbf{u}_h^{n+1} \|^2_X \big)$.

We get

$$
\begin{aligned}
\| \mathbf{e}^m \|^2_{L^2(\Omega)^2} + \sum_{n=1}^m \tau_n \| \mathbf{e}^n \|^2_X \le c_6 \Big(&\| \mathbf{e}^0 \|^2_{L^2(\Omega)^2} + \tau_0 \| \mathbf{e}^0 \|^2_X + \sum_{n=1}^m \sum_{\kappa_n \in \mathcal{T}_{nh}} \tau_n (\eta^h_{u,n,\kappa_n})^2 \\
&+ \sum_{n=0}^{m-1} \tau_n \tilde{f}_n \big(\| \mathbf{u}^{n+1} \|^2_X + \| \mathbf{u}_h^n \|^2_X + \| \mathbf{u}_h^{n+1} \|^2_X \big) \exp\big(\sum_{j=n}^{m-1} \tau_j (\| \mathbf{u}^{j+1} \|^2_X + \| \mathbf{u}_h^j \|^2_X + \| \mathbf{u}_h^{j+1} \|^2_X) \big) \Big).
\end{aligned}
$$

Knowing that for all $n \le m$, $\tilde{f}_n \le \tilde{f}_m$ we obtain

$$
\begin{aligned}
\| \mathbf{e}^m \|^2_{L^2(\Omega)^2} + \sum_{n=1}^m \tau_n \| \mathbf{e}^n \|^2_X \le c_6 \Big(&\| \mathbf{e}^0 \|^2_{L^2(\Omega)^2} + \tau_0 \| \mathbf{e}^0 \|^2_X + \sum_{n=1}^m \sum_{\kappa_n \in \mathcal{T}_{nh}} \tau_n (\eta^h_{u,n,\kappa_n})^2 \\
&+ \tilde{f}_m \sum_{n=0}^{m-1} \tau_n \big(\| \mathbf{u}^{n+1} \|^2_X + \| \mathbf{u}_h^n \|^2_X + \| \mathbf{u}_h^{n+1} \|^2_X \big) \exp\big(\sum_{j=n}^{m-1} \tau_j (\| \mathbf{u}^{j+1} \|^2_X + \| \mathbf{u}_h^j \|^2_X + \| \mathbf{u}_h^{j+1} \|^2_X) \big) \Big).
\end{aligned}
$$

Using Theorem 4.3, Proposition 5.8 and the definition of σ_τ we obtain

$$\| \mathbf{e}^m \|^2_{L^2(\Omega)^2} + \sum_{n=1}^{m} \tau_n \| \mathbf{e}^n \|^2_X \leq c\Big(\| \mathbf{e}^0 \|^2_{L^2(\Omega)^2} + \tau_0 \| \mathbf{e}^0 \|^2_X + \sum_{n=1}^{m} \sum_{\kappa_n \in \mathcal{T}_{nh}} \tau_n (\eta^h_{u,n,\kappa_n})^2 \Big).$$

This ends the proof of Theorem 5.15, which establishes an *a posteriori* estimation between $\mathbf{u}^m$ and $\mathbf{u}_h^m$. □

In order to relate the left-hand side of (5.54) to the augmented error norm defined by (5.7), we prove the following two inequalities

Lemma 5.16. *Let $\mathbf{u}^m$ and $\mathbf{u}_h^m$ be respectively the solutions of (P_{aux}) and (Eds1). It holds that*

$$\frac{1}{3} \sum_{n=1}^{m} \tau_n \| \mathbf{u}^n - \mathbf{u}_h^n \|^2_X \leq \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \| \mathbf{u}_\tau(s) - \mathbf{u}_h(s) \|^2_X \, ds \leq \frac{1+\sigma_\tau}{2} \sum_{n=0}^{m} \tau_n \| \mathbf{u}^n - \mathbf{u}_h^n \|^2_X . \tag{5.58}$$

***Proof*.** Since $\mathbf{u}_\tau - \mathbf{u}_h$ is affine with respect to time on $[t_{n-1}; t_n]$, its square is a second order polynomial in time and the Simpson formula allows us to compute the following integral exactly

$$\begin{aligned} \int_{t_{n-1}}^{t_n} \| \mathbf{u}_\tau(s) - \mathbf{u}_h(s) \|^2_X \, ds \;&=\; \frac{\tau_n}{3} \big(\| \mathbf{u}_\tau(t_n) - \mathbf{u}_h(t_n) \|^2_X + \| \mathbf{u}_\tau(t_{n-1}) - \mathbf{u}_h(t_{n-1}) \|^2_X \\ &\quad + \| \mathbf{u}_\tau(t_n) - \mathbf{u}_h(t_n) \|_X \| \mathbf{u}_\tau(t_{n-1}) - \mathbf{u}_h(t_{n-1}) \|_X \big), \end{aligned} \tag{5.59}$$

in which we used the fact that $\| \mathbf{u}_\tau - \mathbf{u}_h \|_X$ is affine to replace its value in $\frac{t_{n-1}+t_n}{2}$ by the half sum of its values in t_{n-1} and in t_n. Then we immediately get

$$\frac{\tau_n}{3} \| \mathbf{u}_\tau(t_n) - \mathbf{u}_h(t_n) \|^2_X \leq \int_{t_{n-1}}^{t_n} \| \mathbf{u}_\tau(s) - \mathbf{u}_h(s) \|^2_X \, ds. \tag{5.60}$$

On the other hand, using $ab \leq \frac{1}{2}(a^2+b^2)$ and $\tau_n \leq \sigma_\tau \tau_{n-1}$ in the right-hand side of (5.59), we obtain

$$\begin{aligned} \int_{t_{n-1}}^{t_n} \| \mathbf{u}_\tau(s) - \mathbf{u}_h(s) \|^2_X \, ds \;&\leq\; \frac{\tau_n}{2} \big(\| \mathbf{u}_\tau(t_n) - \mathbf{u}_h(t_n) \|^2_X + \| \mathbf{u}_\tau(t_{n-1}) - \mathbf{u}_h(t_{n-1}) \|^2_X \big) \\ &\leq\; \frac{\tau_n}{2} \| \mathbf{u}_\tau(t_n) - \mathbf{u}_h(t_n) \|^2_X + \sigma_\tau \frac{\tau_{n-1}}{2} \| \mathbf{u}_\tau(t_{n-1}) - \mathbf{u}_h(t_{n-1}) \|^2_X \end{aligned} \tag{5.61}$$

Summing (5.60) and (5.61) over n, we get the desired result (5.58). □

Corollary 5.17. *Let $t \in]t_{m-1}, t_m]$. Let $h_n \leq c_s \tau_n, \forall n \in \{1, \ldots, N\}$, where c_s is a positive constant independent of n. The following a posteriori error holds between $\mathbf{u}_\tau(t)$ and $\mathbf{u}_h(t)$:*

$$\begin{aligned} &\| (\mathbf{u}_\tau - \mathbf{u}_h)(t) \|^2_{L^2(\Omega)^2} + \int_0^t \| \mathbf{u}_\tau - \mathbf{u}_h \|^2_X (s) ds \\ &\quad \leq c\Big(\| \mathbf{u}_0 - \mathbf{u}_h^0 \|^2_{L^2(\Omega)^2} + \tau_0 \| \mathbf{u}_0 - \mathbf{u}_h^0 \|^2_X + \sum_{n=1}^{m} \sum_{\kappa_n \in \mathcal{T}_{nh}} \tau_n (\eta^h_{u,n,\kappa_n})^2 \Big), \end{aligned} \tag{5.62}$$

where c is a positive constant independent of the time and mesh steps.

***Proof*.** Let $t \in]t_{m-1}, t_m]$. Since $t \mapsto \| (\mathbf{u}_\tau - \mathbf{u}_h)(t) \|_{L^2(\Omega)^2}$ is an affine function of time on $[t_{m-1}, t_m]$, then

$$\| (\mathbf{u}_\tau - \mathbf{u}_h)(t) \|^2_{L^2(\Omega)^2} \leq \sup(\| \mathbf{u}^{m-1} - \mathbf{u}_h^{m-1} \|^2_{L^2(\Omega)^2}, \| \mathbf{u}^m - \mathbf{u}_h^m \|^2_{L^2(\Omega)^2}).$$

Now both terms in the right-hand side above are bounded by the right-hand side of (5.54) since it is a growing function of m. Thus

$$\| (\mathbf{u}_\tau - \mathbf{u}_h)(t) \|^2_{L^2(\Omega)^2} \leq c\Big(\| \mathbf{u}_0 - \mathbf{u}_h^0 \|^2_{L^2(\Omega)^2} + \tau_0 \| \mathbf{u}_0 - \mathbf{u}_h^0 \|^2_X + \sum_{n=1}^{m} \sum_{\kappa_n \in \mathcal{T}_{nh}} \tau_n (\eta^h_{u,n,\kappa_n})^2 \Big). \tag{5.63}$$

Next, we get from(5.58) that

$$\int_0^t \| \mathbf{u}_\tau - \mathbf{u}_h \|^2_X (s) ds \leq \int_0^{t_m} \| \mathbf{u}_\tau - \mathbf{u}_h \|^2_X (s) ds \leq \frac{1+\sigma_\tau}{2} \sum_{n=0}^{m} \tau_n \| \mathbf{u}^n - \mathbf{u}_h^n \|^2_X .$$

With (5.54) we obtain

$$\int_0^t \| \mathbf{u}_\tau - \mathbf{u}_h \|_X^2 (s)ds \le c\Big(\| \mathbf{u}_0 - \mathbf{u}_h^0 \|_{L^2(\Omega)^2}^2 + \tau_0 \| \mathbf{u}_0 - \mathbf{u}_h^0 \|_X^2 + \sum_{n=1}^m \sum_{\kappa_n \in \mathcal{T}_{nh}} \tau_n (\eta_{u,n,\kappa_n}^h)^2 \Big). \quad (5.64)$$

From (5.63) and (5.64) we obtain (5.62). □

Having bounded the error between $\mathbf{u}$ and $\mathbf{u}_\tau$ in Theorem 5.10 and then the error between $\mathbf{u}_\tau$ and $\mathbf{u}_h$ in Corollary 5.17, we can combine these two results to bound the error between $\mathbf{u}$ and $\mathbf{u}_h$.

Corollary 5.18. *Let $h_n \le c_s \tau_n, \forall n \in \{1, \dots, N\}$, where c_s is a positive constant independent of n. Under assumptions of Theorem 5.10, for $t \in]t_{m-1}, t_m]$, we have the following a posteriori estimation between the velocity $\mathbf{u}$ solution of problem (1.1) and $\mathbf{u}_h$ corresponding to $\mathbf{u}_h^n$ of problem (Eds1):*

$$\begin{aligned} &\| \mathbf{u}(t) - \mathbf{u}_h(t) \|_{L^2(\Omega)^2}^2 + \nu_0 \int_0^t \| \mathbf{u}(s) - \mathbf{u}_h(s) \|_X^2 \, ds \le c\Big(\| \mathbf{f}_0 - \pi_\tau \mathbf{f}_0 \|_{L^2(0,t;L^2(\Omega)^2)}^2 + \| \mathbf{u}_0 - \mathbf{u}_h^0 \|_{L^2(\Omega)^2}^2 \\ &+ \tau_0 \| \mathbf{u}_0 - \mathbf{u}_h^0 \|_X^2 + \| C - \pi_{l,\tau} C_h \|_{L^2(0,t,L^4(\Omega))}^2 + \| \pi_\tau \mathbf{u} - \pi_{l,\tau} \mathbf{u} \|_{L^2(0,t;X)}^2 + \sum_{n=1}^m \tau_n \sum_{\kappa_n \in \mathcal{T}_{nh}} (\eta_{u,n,\kappa_n}^h)^2 \Big), \end{aligned} \quad (5.65)$$

where c is a positive constant.

Proof. We start by applying the triangular inequality

$$\begin{aligned} \| \mathbf{u}(t) - \mathbf{u}_h \|_{L^2(\Omega)^2}^2 &+ \nu_0 \int_0^t \| \mathbf{u}(s) - \mathbf{u}_h(s) \|_X^2 \, ds \\ &\le 2 \| \mathbf{u}(t) - \mathbf{u}_\tau(t) \|_{L^2(\Omega)^2}^2 + 2\nu_0 \int_0^t \| \mathbf{u}(s) - \mathbf{u}_\tau(s) \|_X^2 \, ds \\ &\quad + 2 \| \mathbf{u}_\tau(t) - \mathbf{u}_h(t) \|_{L^2(\Omega)^2}^2 + 2\nu_0 \int_0^t \| \mathbf{u}_\tau(s) - \mathbf{u}_h(s) \|_X^2 \, ds. \end{aligned}$$

By using Theorem 5.10 and Corollary 5.17 we obtain (5.65). □

Till now, the upper bound (5.22) on the concentration error depends on the velocity error and, conversely, the upper bound (5.65) on the velocity error depends on the concentration error. Thus we have to combine these two inequalities to get the desired upper bound. This is done in the next Theorem, by using Corollary 5.18 and Theorem 5.7.

Theorem 5.19. *Under assumptions of Theorem 5.7 and Corollary 5.18, we have the following bound for each $m \in \{1, \dots, N\}$:*

$$\begin{aligned} &\| C(t_m) - C_h(t_m) \|_{L^2(\Omega)}^2 + \alpha \int_0^{t_m} \| C(s) - C_h(s) \|_Y^2 \, ds + r_0 \int_0^{t_m} \| C(s) - C_h(s) \|_{L^2(\Omega)}^2 \, ds \\ &\quad + \| \mathbf{u}(t_m) - \mathbf{u}_h(t_m) \|_{L^2(\Omega)^2}^2 + \nu_0 \int_0^{t_m} \| \mathbf{u}(s) - \mathbf{u}_h(s) \|_X^2 \, ds \\ &\le c\Big(\| g - \pi_\tau g \|_{L^2(0,t_m;Y')}^2 + \| C_0 - C_h^0 \|_{L^2(\Omega)}^2 + \| \mathbf{u}_0 - \mathbf{u}_h^0 \|_{L^2(\Omega)^2}^2 + \tau_0 \| \mathbf{u}_0 - \mathbf{u}_h^0 \|_X^2 \\ &\quad + \| \mathbf{f}_0 - \pi_\tau \mathbf{f}_0 \|_{L^2(0,t_m;L^2(\Omega)^2)}^2 + \sum_{n=1}^m \sum_{\kappa_n \in \tau_{nh}} \big((\eta_{c,n,\kappa_n}^\tau)^2 + (\eta_{u,n,\kappa_n}^\tau)^2 + \tau_n (\eta_{c,n,\kappa_n}^h)^2 + \tau_n (\eta_{u,n,\kappa_n}^h)^2 \big) \Big), \end{aligned}$$

where c is a positive constant.

Proof. We use Relation (2.3) and the inequality $ab \le \frac{1}{2\varepsilon} a^2 + \frac{\varepsilon}{2} b^2$ to get

$$\| C - \pi_{l,\tau} C_h \|_{L^2(0,t;L^4(\Omega))}^2 \le c\big(\frac{1}{2\varepsilon} \| C - \pi_{l,\tau} C_h \|_{L^2(0,t;L^2(\Omega))}^2 + \frac{\varepsilon}{2} \| C - \pi_{l,\tau} C_h \|_{L^2(0,t;Y)}^2 \big). \quad (5.66)$$

On the one hand, for $t \in]t_{m-1}, t_m]$ we consider Inequalities (5.65) and (5.66) to get the following bound

$$\begin{aligned} &\| \mathbf{u}(t) - \mathbf{u}_h(t) \|^2_{L^2(\Omega)^2} + \nu_0 \int_0^t \| \mathbf{u}(s) - \mathbf{u}_h(s) \|^2_X \, ds \\ &\quad \le c_1 \Big(\sum_{n=1}^m \sum_{\kappa_n \in \mathcal{T}_{nh}} \tau_n (\eta^h_{u,n,\kappa_n})^2 + \| \pi_\tau \mathbf{u} - \pi_{l,\tau} \mathbf{u} \|^2_{L^2(0,t;X)} + \| \mathbf{u}_0 - \mathbf{u}^0_h \|^2_{L^2(\Omega)^2} + \tau_0 \| \mathbf{u}_0 - \mathbf{u}^0_h \|^2_X \\ &+ \| \mathbf{f}_0 - \pi_\tau \mathbf{f}_0 \|^2_{L^2(0,t;L^2(\Omega)^2)} + \frac{1}{2\varepsilon} \| C - \pi_{l,\tau} C_h \|^2_{L^2(0,t;L^2(\Omega))} + \frac{\varepsilon}{2} \| C - \pi_{l,\tau} C_h \|^2_{L^2(0,t;Y)} \Big). \end{aligned} \tag{5.67}$$

On the other hand, for each $n \in [1, N]$, and for each $s \in]t_{n-1}, t_n]$, we have

$$\begin{aligned} \| C(s) - \pi_{l,\tau} C_h(s) \|^2_Y &\le \big(\| C(s) - C_h(s) \|_y + \| C_h(s) - C^{n-1}_h \|_Y \big)^2 \\ &\le \Big(\| C(s) - C_h(s) \|_Y + \big(\frac{s - t_{n-1}}{\tau_n}\big) \| C^{n-1}_h - C^n_h \|_Y \Big)^2 \\ &\le 2 \Big(\| C(s) - C_h(s) \|^2_Y + \| C^n_h - C^{n-1}_h \|^2_Y \Big). \end{aligned}$$

Let $t \in [t_{m-1}, t_m]$; in order to integrate the above inequality from 0 to t, we first integrate it between t_{n-1} and t_n, then sum over n from 1 to $m-1$ and we finally add the integration from t_{m-1} to t. We obtain

$$\int_0^t \| C(t) - \pi_{l,\tau} C_h(t) \|^2_Y \, dt \le c' \Big(\int_0^t \| C(s) - C_h(s) \|^2_Y \, ds + \sum_{n=1}^m \sum_{\kappa_n \in \mathcal{T}_{nh}} (\eta^\tau_{c,n,\kappa_n})^2 \Big), \tag{5.68}$$

where c' is a positive constant independent of the time and mesh steps. We also have in the same way

$$\| C - \pi_{l,\tau} C_h \|^2_{L^2(0,t;L^2(\Omega))} \le c' \Big(\| C - C_h \|^2_{L^2(0,t;L^2(\Omega))} + \sum_{n=1}^m \sum_{\kappa_n \in \mathcal{T}_{nh}} (\eta^\tau_{c,n,\kappa_n})^2 \Big). \tag{5.69}$$

Following the same reasoning we integrate (5.26) between 0 and t, we get

$$\begin{aligned} &\| C(t) - C_h(t) \|^2_{L^2(\Omega)} + \alpha \int_0^t \| C(s) - C_h(s) \|^2_Y \, ds + 2r_0 \int_0^t \| C(s) - C_h(s) \|^2_{L^2(\Omega)} \, ds \\ &\le c \Big(\| g - \pi_\tau g \|^2_{L^2(0,t;Y')} + \| C_0 - C^0_h \|^2_{L^2(\Omega)} + \| \mathbf{u} - \mathbf{u}_h \|^2_{L^2(0,t;X)} \\ &\quad + \sum_{n=1}^m \sum_{\kappa_n \in \tau_{nh}} \big((\eta^\tau_{c,n,\kappa_n})^2 + (\eta^\tau_{u,n,\kappa_n})^2 + \tau_n (\eta^h_{c,n,\kappa_n})^2 \big) \Big). \end{aligned} \tag{5.70}$$

To treat the term $\| \pi_\tau \mathbf{u} - \pi_{l,\tau} \mathbf{u} \|^2_X$ in (5.67) we use the triangle inequality for each $n \le m$, so we get

$$\| \mathbf{u}^n - \mathbf{u}^{n-1} \|^2_X \le 3 (\| \mathbf{u}^n - \mathbf{u}^n_h \|^2_X + \| \mathbf{u}^n_h - \mathbf{u}^{n-1}_h \|^2_X + \| \mathbf{u}^{n-1} - \mathbf{u}^{n-1}_h \|^2_X).$$

We multiply by τ_n, we use the error indicator definition (5.16) and the property $\tau_n \le \sigma_\tau \tau_{n-1}$, to get

$$\tau_n \| \mathbf{u}^n - \mathbf{u}^{n-1} \|^2_X \le 3 \big(\tau_n \| \mathbf{u}^n - \mathbf{u}^n_h \|^2_X + \sum_{\kappa_n \in \tau_{nh}} (\eta^\tau_{u,n,\kappa_n})^2 + \sigma_\tau \tau_{n-1} \| \mathbf{u}^{n-1} - \mathbf{u}^{n-1}_h \|^2_X \big).$$

Since $t \in [t_{m-1}, t_m]$, we get

$$\int_0^t \| \pi_\tau \mathbf{u} - \pi_{l,\tau} \mathbf{u} \|^2_X \le \sum_{n=1}^m \tau_n \| \mathbf{u}^n - \mathbf{u}^{n-1} \|^2_X \le c'' \Big(\sum_{n=0}^m \tau_n \| \mathbf{u}^n - \mathbf{u}^n_h \|^2_X + \sum_{n=1}^m \sum_{\kappa_n \in \tau_{nh}} (\eta^\tau_{u,n,\kappa_n})^2 \Big). \tag{5.71}$$

Summing (5.67) and (5.70) and using (5.68), (5.69), (5.71) and (5.54) we get

$$\begin{aligned}
&\| C(t)-C_h(t) \|^2_{L^2(\Omega)} + \alpha \int_0^t \| C(s)-C_h(s) \|^2_Y \, ds + 2r_0 \int_0^t \| C(s)-C_h(s) \|^2_{L^2(\Omega)} \, ds \\
&\qquad + \| \mathbf{u}(t)-\mathbf{u}_h(t) \|^2_{L^2(\Omega)^2} + \nu_0 \int_0^t \| \mathbf{u}(s)-\mathbf{u}_h(s) \|^2_X \\
&\leq c_2 \Big(\| g-\pi_\tau g \|^2_{L^2(0,t;Y')} + \| C_0 - C_h^0 \|^2_{L^2(\Omega)} + \| \mathbf{u}_0 - \mathbf{u}_h^0 \|^2_{L^2(\Omega)^2} + \tau_0 \| \mathbf{u}_0 - \mathbf{u}_h^0 \|^2_X \\
&\qquad + \| \mathbf{f}_0 - \pi_\tau \mathbf{f}_0 \|^2_{L^2(0,t;L^2(\Omega)^2)} + \frac{1}{2\varepsilon} \| C - C_h \|^2_{L^2(0,t;L^2(\Omega))} + \frac{\varepsilon}{2} \| C - C_h \|^2_{L^2(0,t;Y)} \\
&\qquad + \sum_{n=1}^m \sum_{\kappa_n \in \tau_{nh}} \big((\eta^\tau_{c,n,\kappa_n})^2 + (\eta^\tau_{u,n,\kappa_n})^2 + \tau_n (\eta^h_{c,n,\kappa_n})^2 \big) + \tau_n (\eta^h_{u,n,\kappa_n})^2) \Big).
\end{aligned}$$

We consider $\varepsilon = \dfrac{\alpha}{c_2}$ to get

$$\begin{aligned}
&\| C(t)-C_h(t) \|^2_{L^2(\Omega)} + \frac{\alpha}{2} \int_0^t \| C(s)-C_h(s) \|^2_Y \, ds + 2r_0 \int_0^t \| C(s)-C_h(s) \|^2_{L^2(\Omega)} \, ds \\
&\qquad + \| \mathbf{u}(t)-\mathbf{u}_h(t) \|^2_{L^2(\Omega)^2} + \nu_0 \int_0^t \| \mathbf{u}(s)-\mathbf{u}_h(s) \|^2_X \, ds \\
&\leq c_3 \Big(\| g-\pi_\tau g \|^2_{L^2(0,t;Y')} + \| C_0 - C_h^0 \|^2_{L^2(\Omega)} + \| \mathbf{u}_0 - \mathbf{u}_h^0 \|^2_{L^2(\Omega)^2} + \tau_0 \| \mathbf{u}_0 - \mathbf{u}_h^0 \|^2_X \\
&\qquad + \| \mathbf{f}_0 - \pi_\tau \mathbf{f}_0 \|^2_{L^2(0,t;L^2(\Omega)^2)} + \| C - C_h \|^2_{L^2(0,t;L^2(\Omega))} \\
&\qquad + \sum_{n=1}^m \sum_{\kappa_n \in \tau_{nh}} \big((\eta^\tau_{c,n,\kappa_n})^2 + (\eta^\tau_{u,n,\kappa_n})^2 + \tau_n (\eta^h_{c,n,\kappa_n})^2 \big) + \tau_n (\eta^h_{u,n,\kappa_n})^2) \Big).
\end{aligned}$$

Then we apply Gronwall's Theorem 2.5 by considering:

$$\begin{aligned}
\tilde{f}(t) &= \| g-\pi_\tau g \|^2_{L^2(0,t;Y')} + \| C_0 - C_h^0 \|^2_{L^2(\Omega)} + \| \mathbf{u}_0 - \mathbf{u}_h^0 \|^2_{L^2(\Omega)^2} + \tau_0 \| \mathbf{u}_0 - \mathbf{u}_h^0 \|^2_X \\
&\quad + \| \mathbf{f}_0 - \pi_\tau \mathbf{f}_0 \|^2_{L^2(0,t;L^2(\Omega)^2)} + \sum_{n=1}^m \sum_{\kappa_n \in \tau_{nh}} \big((\eta^\tau_{c,n,\kappa_n})^2 + (\eta^\tau_{u,n,\kappa_n})^2 + \tau_n (\eta^h_{c,n,\kappa_n})^2 \big) + \tau_n (\eta^h_{u,n,\kappa_n})^2), \\
y(t) &= \| C(t)-C_h(t) \|^2_{L^2(\Omega)} + \alpha \int_0^t \| C(s)-C_h(s) \|^2_Y \, ds + r_0 \int_0^t \| C(s)-C_h(s) \|^2_{L^2(\Omega)} \, ds \\
&\quad + \| \mathbf{u}(t)-\mathbf{u}_h^m \|^2_{L^2(\Omega)^2} + \nu_0 \int_0^t \| \mathbf{u}(s)-\mathbf{u}_h(s) \|^2_X \, ds,
\end{aligned}$$

$\tilde{g}(t) = 1$ and $k(\tau) = c_3$.

Since k and $\tilde{g}$ are constants, we have

$$\begin{aligned}
&\| C(t)-C_h(t) \|^2_{L^2(\Omega)} + \alpha \int_0^t \| C(s)-C_h(s) \|^2_Y \, ds + r_0 \int_0^t \| C(s)-C_h(s) \|^2_{L^2(\Omega)} \, ds \\
&\qquad + \| \mathbf{u}(t)-\mathbf{u}_h(t) \|^2_{L^2(\Omega)^2} + \nu_0 \int_0^t \| \mathbf{u}(s)-\mathbf{u}_h(s) \|^2_X \, ds \\
&\leq c \Big(\| g-\pi_\tau g \|^2_{L^2(0,t;Y')} + \| C_0 - C_h^0 \|^2_{L^2(\Omega)} + \| \mathbf{u}_0 - \mathbf{u}_h^0 \|^2_{L^2(\Omega)^2} + \tau_0 \| \mathbf{u}_0 - \mathbf{u}_h^0 \|^2_X \\
&\quad + \| \mathbf{f}_0 - \pi_\tau \mathbf{f}_0 \|^2_{L^2(0,t;L^2(\Omega)^2)} + \sum_{n=1}^m \sum_{\kappa_n \in \tau_{nh}} \big((\eta^\tau_{c,n,\kappa_n})^2 + (\eta^\tau_{u,n,\kappa_n})^2 + \tau_n (\eta^h_{c,n,\kappa_n})^2 \big) + \tau_n (\eta^h_{u,n,\kappa_n})^2) \Big).
\end{aligned}$$

For $t = t_m$ we get the result of Theorem 5.19. $\square$

Theorem 5.19 in itself provides a reliable a posteriori error estimator for the total error in the energy norm. However, as usual for time-dependent problems, we need to augment this norm by additional terms which are needed to obtain the efficiency of the error indicators. Additional norms related to the concentration and to the velocity are dealt with respectively in Theorem 5.20 and Theorem 5.21.

Theorem 5.20. *Under assumption of Theorem 5.7, the exact solution C of Problem* (1.1) *and the solution C_h of Problem (Eds1), for $1 \le n \le m$ with $m \in \{1, \cdots, N\}$, satisfy the following estimation:*

$$\begin{aligned} &\| \frac{\partial}{\partial t}(C - C_h) + \mathbf{u} \cdot \nabla C - \pi_\tau \mathbf{u}_h \cdot \nabla \pi_\tau C_h - \frac{1}{2} \operatorname{div} (\pi_\tau \mathbf{u}_h) \pi_\tau C_h \|_{L^2(0,t_m;Y')} \\ &\le c \Big(\sum_{n=1}^{m} \sum_{\kappa_n \in \mathcal{T}_{nh}} \big(\tau_n (\eta^h_{c,n,\kappa_n})^2 + (\eta^\tau_{c,n,\kappa_n})^2 + (\eta^\tau_{u,n,\kappa_n})^2 \big) \\ &\qquad + \| g - \pi_\tau g \|^2_{L^2(0,t_m;Y')} + \| C_0 - C_h^0 \|^2_{L^2(\Omega)} + \| \mathbf{u} - \mathbf{u}_h \|^2_{L^2(0,t_m;X)} \Big)^{1/2}, \end{aligned} \tag{5.72}$$

where c is a positive constant independent of the time and mesh steps.

Proof. We consider the second equality of system (5.9) of Lemma 5.4, with (5.11), (5.15):

$$\begin{aligned} &(\frac{\partial}{\partial t}(C - C_h)(t), r) + (\mathbf{u}(t) \cdot \nabla C(t) - \pi_\tau \mathbf{u}_h(t) \cdot \nabla \pi_\tau C_h(t), r) - \frac{1}{2}(\operatorname{div} (\pi_\tau \mathbf{u}_h(t)) \pi_\tau C_h(t), r) \\ &\quad = -\alpha(\nabla(C(t) - \pi_\tau C_h), \nabla r) - r_0(C(t) - \pi_\tau C_h, r) + \langle R^h_c(t), r - r_h \rangle \\ &\qquad + \langle R^\tau_c(t), r \rangle + (g(t) - g(t_n), r). \end{aligned}$$

However,

$$\begin{aligned} &\| \frac{\partial}{\partial t}(C - C_h)(t) + \mathbf{u}(t) \cdot \nabla C(t) - \pi_\tau \mathbf{u}_h(t) \cdot \nabla \pi_\tau C_h(t) - \frac{1}{2} \operatorname{div} (\pi_\tau \mathbf{u}_h(t)) \pi_\tau C_h(t) \|_{Y'} \\ &\quad = \sup_{r \in Y} \frac{1}{\| r \|_Y} \Big((\frac{\partial}{\partial t}(C - C_h)(t), r) + (\mathbf{u}(t) \cdot \nabla C(t) - \pi_\tau \mathbf{u}_h(t) \cdot \nabla \pi_\tau C_h(t), r) \\ &\qquad - \frac{1}{2}(\operatorname{div} (\pi_{l,\tau} \mathbf{u}_h(t)) \pi_\tau \mathbf{u}_h(t), r) \Big) \\ &\quad = \sup_{r \in Y} \frac{1}{\| r \|_Y} \big(-\alpha(\nabla(C(t) - \pi_\tau C_h), \nabla r) - r_0(C(t) - \pi_\tau C_h, r) + \langle R^h_c(t), r - r_h \rangle \\ &\qquad + \langle R^\tau_c(t), r \rangle + (g(t) - g(t_n), r) \big). \end{aligned}$$

All the terms in the right-hand side, except the third and the fourth ones, can be treated with the Cauchy-Schwarz inequality. For the third term, we take $r_h = C_{nh} r$ and use (5.21) and we obtain

$$|\langle R^h_c(t), r - r_h \rangle| \le c_1 \Big(\sum_{\kappa_n \in \mathcal{T}_{nh}} (\eta^h_{c,n,\kappa_n})^2 \Big)^{1/2} \| r \|_Y .$$

For the fourth term an easy calculation leads to

$$|\langle R^\tau_c(t), r \rangle| \le c_2 \Big(\sum_{\kappa_n \in \mathcal{T}_{nh}} \| C_h^n - C_h^{n-1} \|^2_{H^1(\kappa_n)} \Big)^{1/2} \| r \|_Y .$$

Whence

$$\begin{aligned} &\| \frac{\partial}{\partial t}(C - C_h)(t) + \mathbf{u}(t) \cdot \nabla C(t) - \pi_\tau \mathbf{u}_h(t) \cdot \nabla \pi_\tau C_h(t) - \frac{1}{2} \operatorname{div} (\pi_\tau \mathbf{u}_h(t)) \pi_\tau C_h(t) \|_{Y'} \\ &\quad \le c_3 \Big(\alpha \| C(t) - \pi_\tau C_h(t) \|^2_Y + r_0 \| C(t) - \pi_\tau C_h(t) \|^2_{L^2(\Omega)} + \| g(t) - g(t_n) \|^2_{Y'} \\ &+ \sum_{\kappa_n \in \mathcal{T}_{nh}} \| C_h^n - C_h^{n-1} \|^2_{H^1(\kappa_n)} + \sum_{\kappa_n \in \mathcal{T}_{nh}} (\eta^h_{c,n,\kappa_n})^2 \Big)^{1/2}. \end{aligned}$$

We obtain the desired result (5.72) by integrating over t between t_{n-1} and t_n, summing over n from 0 to m and by using inequalities similar to (5.68) and (5.69) applied to $(C - \pi_\tau C_h)$ and then (5.22). □

To get the final form of the *a posteriori* bound, we have to bound the following last term:

$$\frac{\partial}{\partial t}(\mathbf{u} - \mathbf{u}_h) + \mathbf{u} \cdot \nabla \mathbf{u} - \pi_{l,\tau} \mathbf{u}_h \cdot \nabla \pi_\tau \mathbf{u}_h - \frac{1}{2} \operatorname{div} (\pi_{l,\tau} \mathbf{u}_h) \pi_\tau \mathbf{u}_h + \nabla(p - p_h).$$

Theorem 5.21. *Let* $(\mathbf{u}, p, C)$ *be the solution of problem* (1.1) *and* $(\mathbf{u}_h, p_h, C_h)$ *the solution of problem (Eds1). For* $1 \leq n \leq m$ *with* $m \in \{1, \cdots, N\}$*, we have the following inequality*

$$\begin{aligned}
&\| \frac{\partial}{\partial t}(\mathbf{u}-\mathbf{u}_h) + \mathbf{u}\cdot\nabla\mathbf{u} - \pi_{l,\tau}\mathbf{u}_h\cdot\nabla\pi_\tau\mathbf{u}_h - \frac{1}{2}\,\mathrm{div}\,(\pi_{l,\tau}\mathbf{u}_h)\pi_\tau\mathbf{u}_h + \nabla(p-p_h) \|_{L^2(0,t_m;X')} \\
&\quad \leq c\Big(\sum_{n=1}^{m} \sum_{\kappa_n\in\mathcal{T}_{nh}} \big(\tau_n(\eta^h_{u,n,\kappa_n})^2 + (\eta^\tau_{c,n,\kappa_n})^2 + (\eta^\tau_{u,n,\kappa_n})^2\big) + \| \mathbf{f}_0 - \pi_\tau\mathbf{f}_0 \|^2_{L^2(0,t_m;L^2(\Omega)^2)} \\
&\qquad + \| C - C_h \|^2_{L^2(0,t_m;Y)} + \| \mathbf{u}-\mathbf{u}_h \|^2_{L^2(0,t_m;X)} \Big)^{1/2}.
\end{aligned} \tag{5.73}$$

Proof. The first equality of system (5.9) in Lemma 5.4 gives us:

$$\begin{aligned}
&(\frac{\partial}{\partial t}(\mathbf{u}-\mathbf{u}_h)(t),\mathbf{v}) + (\mathbf{u}(t)\cdot\nabla\mathbf{u}(t) - \pi_{l,\tau}\mathbf{u}_h(t)\cdot\nabla\pi_\tau\mathbf{u}_h(t),\mathbf{v}) - \frac{1}{2}(\mathrm{div}\,(\pi_{l,\tau}\mathbf{u}_h(t))\pi_\tau\mathbf{u}_h(t),\mathbf{v}) \\
&-(\mathrm{div}\,\mathbf{v}, p(t)-p_h(t)) \\
&\qquad = -\nu_0(\nabla\mathbf{u}(t) - \nabla\pi_\tau\mathbf{u}_h(t),\nabla\mathbf{v}) - 2(\nu_C(C(t))\mathbb{D}(\mathbf{u}(t)) - \nu_C(\pi_{l,\tau}C_h)\mathbb{D}(\pi_\tau\mathbf{u}_h(t)),\mathbb{D}(\mathbf{v})) \\
&\qquad +\langle R^h_u(t),\mathbf{v}\rangle + (\mathbf{f}(t,C(t)) - \mathbf{f}^n(C_h^{n-1}),\mathbf{v}).
\end{aligned}$$

However,

$$\begin{aligned}
&\| \frac{\partial}{\partial t}(\mathbf{u}-\mathbf{u}_h)(t) + \mathbf{u}(t)\cdot\nabla\mathbf{u}(t) - \pi_{l,\tau}\mathbf{u}_h(t)\cdot\nabla\pi_\tau\mathbf{u}_h(t) - \frac{1}{2}\mathrm{div}\,(\pi_{l,\tau}\mathbf{u}_h(t))\pi_\tau\mathbf{u}_h(t) + \nabla(p-p_h)(t) \|_{X'} \\
&\quad = \sup_{\mathbf{v}\in X}\frac{1}{\|\mathbf{v}\|_X}\Big((\frac{\partial}{\partial t}(\mathbf{u}-\mathbf{u}_h)(t),\mathbf{v}) + (\mathbf{u}(t)\cdot\nabla\mathbf{u}(t) - \pi_{l,\tau}\mathbf{u}_h(t)\cdot\nabla\pi_\tau\mathbf{u}_h(t),\mathbf{v}) \\
&\qquad\qquad -\frac{1}{2}(\mathrm{div}\,(\pi_{l,\tau}\mathbf{u}_h(t))\pi_\tau\mathbf{u}_h(t),\mathbf{v}) - (\mathrm{div}\,\mathbf{v}, p(t)-p_h(t))\Big) \\
&\quad = \sup_{\mathbf{v}\in X}\frac{1}{\|\mathbf{v}\|_X}\Big(-\nu_0(\nabla\mathbf{u}(t) - \nabla\pi_\tau\mathbf{u}_h(t),\nabla\mathbf{v}) - 2(\nu_C(C(t))\mathbb{D}(\mathbf{u}(t)) - \nu_C(\pi_{l,\tau}C_h)\mathbb{D}(\pi_\tau\mathbf{u}_h(t)),\mathbb{D}(\mathbf{v})) \\
&\qquad\qquad +\langle R^h_u(t),\mathbf{v}\rangle + (\mathbf{f}(t,C(t)) - \mathbf{f}^n(C_h^{n-1}),\mathbf{v})\Big).
\end{aligned} \tag{5.74}$$

We start by bounding the first two terms of (5.74). Foremost, we insert $2(\nu_C(\pi_{l,\tau}C_h)\mathbb{D}(\mathbf{u}(t)),\mathbb{D}(\mathbf{v}))$; we use (2.5) for the second term of (5.75) below. For the third one, we use that ν_C is lipschitz, then the $L^4-L^4-L^2$ generalized Cauchy-Schwarz inequality, the fact that $\nabla(\mathbf{u})\in L^\infty(0,T;L^4(\Omega)^{2\times2})$ and (2.2) with $p=4$; we get

$$\begin{aligned}
&\big| -\nu_0(\nabla\mathbf{u}(t) - \nabla\pi_\tau\mathbf{u}_h(t),\nabla\mathbf{v}) - 2(\nu_C(\pi_{l,\tau}C_h)\mathbb{D}(\mathbf{u}(t)-\pi_\tau\mathbf{u}_h(t)),\mathbb{D}(\mathbf{v})) \\
&\qquad -2((\nu_C(C(t)) - \nu_C(\pi_{l,\tau}C_h))\mathbb{D}(\mathbf{u}(t)),\mathbb{D}(\mathbf{v}))\big| \\
&\leq (\nu_0 + 2\hat{\nu}_2)\,\| \mathbf{u}(t) - \pi_\tau\mathbf{u}_h(t) \|_X \| \mathbf{v} \|_X + c_0 \| C(t) - C_h^{n-1} \|_Y \| \mathbf{v} \|_X .
\end{aligned} \tag{5.75}$$

For the other terms of (5.74), by using (5.14) and (5.20) we obtain

$$\begin{aligned}
&|\langle \mathbf{f}(t,C(t)) - \mathbf{f}^n(C_h^{n-1}) + R^h_u(t),\mathbf{v}\rangle| \\
&\quad \leq \sum_{\kappa_n\in\mathcal{T}_{nh}} c_1 \| \mathbf{f}(t,C(t)) - \mathbf{f}^n(C_h^{n-1}) \|_{L^2(\kappa_n)} \| \mathbf{v} \|_{H^1(\kappa_n)} + c_2\Big(\sum_{\kappa_n\in\mathcal{T}_{nh}} (\eta^h_{u,n,\kappa_n})^2 \Big)^{1/2} \| \mathbf{v} \|_X .
\end{aligned}$$

Whence, by using the definition and the properties of $\mathbf{f}$ we have,

$$\begin{aligned}
&\| \frac{\partial}{\partial t}(\mathbf{u}-\mathbf{u}_h)(t) + \mathbf{u}(t)\cdot\nabla\mathbf{u}(t) - \pi_{l,\tau}\mathbf{u}_h(t)\cdot\nabla\pi_\tau\mathbf{u}_h(t) - \frac{1}{2}\mathrm{div}\,(\pi_{l,\tau}\mathbf{u}_h(t))\pi_\tau\mathbf{u}_h(t) + \nabla(p-p_h)(t) \|_{X'} \\
&\quad \leq c_3\Big(\| \mathbf{u}(t) - \pi_\tau\mathbf{u}_h(t) \|^2_X + \sum_{\kappa_n\in\mathcal{T}_{nh}} \| \mathbf{f}_0(t) - \mathbf{f}_0^n \|^2_{L^2(\kappa_n)} \\
&\qquad + \sum_{\kappa_n\in\mathcal{T}_{nh}} \| C(t) - C_h^{n-1} \|^2_Y + \sum_{\kappa_n\in\mathcal{T}_{nh}} (\eta^h_{u,n,\kappa_n})^2 \Big)^{1/2}.
\end{aligned}$$

We insert $C_h(t)$ in the third term of the right-hand side of the above inequality, use (5.6) (the definition of $C_h(t)$), integrate over t between t_{n-1} and t_n, and sum over n from 0 to m to get:

$$\begin{aligned}&\| \frac{\partial}{\partial t}(\mathbf{u}-\mathbf{u}_h)+\mathbf{u}\cdot\nabla\mathbf{u}-\pi_{l,\tau}\mathbf{u}_h\cdot\nabla\pi_\tau\mathbf{u}_h-\frac{1}{2}\,\mathrm{div}\;(\pi_{l,\tau}\mathbf{u}_h)\pi_\tau\mathbf{u}_h+\nabla(p-p_h)\,\|_{L^2(0,t_m;X')}\\ &\quad\le\; c\Big(\sum_{n=1}^{m}\sum_{\kappa_n\in\mathcal{T}_{nh}}\big(\tau_n(\eta^h_{u,n,\kappa_n})^2+(\eta^\tau_{c,n,\kappa_n})^2\big)+\|\,\mathbf{f}_0-\pi_\tau\mathbf{f}_0\,\|^2_{L^2(0,t_m;L^2(\Omega)^2)}\\ &\qquad\qquad+\|\,C-C_h\,\|^2_{L^2(0,t_m;Y)}+\|\,\mathbf{u}-\pi_\tau\mathbf{u}_h\,\|^2_{L^2(0,t_m;X)}\Big)^{1/2}.\end{aligned}\tag{5.76}$$

In order to obtain the final result of this theorem, it remains to bound the last term in the right-hand side of equation (5.76). To do so, knowing that : $\|\,\mathbf{u}-\pi_\tau\mathbf{u}_h\,\|^2_{L^2(0,t_m;X)}=\displaystyle\sum_{n=1}^{m}\int_{t_{n-1}}^{t_n}\|\,\mathbf{u}(t)-\pi_\tau\mathbf{u}_h(t)\,\|^2_X\,dt$, we follow the same idea that led to (5.68) and we obtain

$$\sum_{n=1}^{m}\int_{t_{n-1}}^{t_n}\|\,\mathbf{u}(t)-\pi_\tau\mathbf{u}_h(t)\,\|^2_X\,dt\le c\Big(\sum_{n=1}^{m}\int_{t_{n-1}}^{t_n}\|\,\mathbf{u}(s)-\mathbf{u}_h(s)\,\|^2_X\,ds+\sum_{n=1}^{m}\sum_{\kappa_n\in\mathcal{T}_{nh}}(\eta^\tau_{u,n,\kappa_n})^2\Big).\tag{5.77}$$

By combining (5.76) and (5.77) we get the desired result. □

Having all we need, we can now introduce the following corollary which establishes the upper bound.

Corollary 5.22. *Let $h_n\le c_s\tau_n,\forall n\in\{1,\ldots,N\}$, where c_s is a positive constant independent of n. For each $m\in\{1,\cdots,N\}$, the solutions $(\mathbf{u},p,C)$ of* (1.1) *and $(\mathbf{u}_h,p_h,C_h)$ of (Eds1) satisfy the following a posteriori estimation:*

$$\begin{aligned}&[[\mathbf{u}-\mathbf{u}_h]]^2(t_m)+\|\,\frac{\partial}{\partial t}(\mathbf{u}-\mathbf{u}_h)+\mathbf{u}\cdot\nabla\mathbf{u}-\pi_{l,\tau}\mathbf{u}_h\cdot\nabla\pi_\tau\mathbf{u}_h-\frac{1}{2}\,\mathrm{div}\;(\pi_{l,\tau}\mathbf{u}_h)\pi_\tau\mathbf{u}_h+\nabla(p-p_h)\,\|_{L^2(0,t_m;X')}\\ &\quad+[[C-C_h]]^2(t_m)+\|\,\frac{\partial}{\partial t}(C-C_h)+\mathbf{u}\cdot\nabla C-\pi_\tau\mathbf{u}_h\cdot\nabla\pi_\tau C_h-\frac{1}{2}\,\mathrm{div}\;(\pi_\tau\mathbf{u}_h)\pi_\tau C_h\,\|_{L^2(0,t_m;Y')}\\ &\le c\Big(\,\|\,g-\pi_\tau g\,\|^2_{L^2(0,t_m;Y')}+\|\,C_0-C_h^0\,\|^2_{L^2(\Omega)}+\|\,\mathbf{u}_0-\mathbf{u}_h^0\,\|^2_{L^2(\Omega)}+\tau_0\,\|\,\mathbf{u}_0-\mathbf{u}_h^0\,\|^2_X\\ &\quad+\|\,\mathbf{f}_0-\pi_\tau\mathbf{f}_0\,\|^2_{L^2(0,t_m;L^2(\Omega)^2)}+\sum_{n=1}^{m}\sum_{\kappa_n\in\tau_{nh}}\big((\eta^\tau_{c,n,\kappa_n})^2+(\eta^\tau_{u,n,\kappa_n})^2+\tau_n(\eta^h_{c,n,\kappa_n})^2+\tau_n(\eta^h_{u,n,\kappa_n})^2\big)\Big).\end{aligned}\tag{5.78}$$

Proof. The result is a simple consequence of the definition of $[[\mathbf{u}-\mathbf{u}_h]]^2(t_m)$ and $[[C-C_h]]^2(t_m)$, Theorem 5.19 and inequalities (5.72) and (5.73), in which the error terms $\|\,C-C_h\,\|^2_{L^2(0,t_m;Y)}$ and $\|\,\mathbf{u}-\mathbf{u}_h\,\|^2_{L^2(0,t_m;X)}$ are themselves bounded by Theorem 5.19. □

Thereby, we bounded the error between the exact solution $(\mathbf{u},p,C)$ of problem (1.1) and the numerical solution $(\mathbf{u}_h,p_h,C_h)$ of problem (Eds1) using the indicators. Let us now move on to the last stage of the *a posteriori* error estimation, namely the lower bounds, which prove efficiency of the error indicators.

5.3. **Lower bounds of the error.** To complete the *a posteriori* estimation, we establish the efficiency of the error indicators, which consists in bounding each indicator η^τ_{u,n,κ_n}, η^h_{u,n,κ_n}, η^τ_{c,n,κ_n} and η^h_{c,n,κ_n} locally by the error between the exact and numerical solutions.

To accomplish the desired efficiency proof, we introduce approximations $\mathbf{f}_h^n$ of $\mathbf{f}^n$, ν_h of ν_C and g_h^n of g^n as follows: for any function $\xi\in L^p(\kappa_n)$ and for each element $\kappa_n\in\mathcal{T}_{nh}$ we set

$$\mathbf{f}_h^n(\xi)|_{\kappa_n}=\frac{1}{|\kappa_n|}\int_{\kappa_n}\mathbf{f}^n(\xi(x))\,\mathbf{dx},\tag{5.79}$$

$$\nu_h(\xi)|_{\kappa_n}=\frac{1}{|\kappa_n|}\int_{\kappa_n}\nu_C(\xi(x))\,\mathbf{dx}\tag{5.80}$$

and

$$g_h^n|_{\kappa_n} = \frac{1}{|\kappa_n|}\int_{\kappa_n} g^n(x)\,\mathbf{dx}. \tag{5.81}$$

A straightforward consequence of the properties of $\mathbf{f}^n$ and ν_C is that ν_h is bounded and $\mathbf{f}_h^n$ and ν_h are Lipschitz with respect to ξ.

Remark 5.23. *We introduce $\mathbf{f}_h^n, \nu_h$ and g_h^n, which are constant on each κ_n, so we shall be able to apply Property 5.1 which holds for polynomials only.*

Theorem 5.24. *For all $n \in \{1, \cdots, N\}$, we have the following estimation*

$$(\eta^{\tau}_{c,n,\kappa_n})^2 \leq c\ \big(\parallel C - C_h \parallel^2_{L^2(t_{n-1},t_n;H^1(\kappa_n))} + \parallel C - \pi_\tau C_h \parallel^2_{L^2(t_{n-1},t_n;H^1(\kappa_n))}\ \big)\,,$$

where c is a constant independent of the time and mesh steps.

Proof. Based on the definition of C_h and C_h^n, we have for all $n \in \{1, \cdots, N\}$ and all $t \in [t_{n-1} - t_n]$,

$$C_h(t) - \pi_\tau C_h(t) = \frac{t - t_n}{\tau_n}(C_h^n - C_h^{n-1}).$$

So, by inserting $C(t)$ in the above equality, we get

$$\left(\frac{t-t_n}{\tau_n}\right)^2 \parallel C_h^n - C_h^{n-1} \parallel^2_{H^1(\kappa_n)} \leq 2(\parallel C(t) - C_h(t) \parallel^2_{H^1(\kappa_n)} + \parallel (C(t) - \pi_\tau C_h(t) \parallel^2_{H^1(\kappa_n)}).$$

We integrate over t between t_{n-1} and t_n to get the result. $\square$

We will now bound the indicator η^h_{c,n,κ_n}.

Theorem 5.25. *For all $n \in \{1, \cdots, N\}$, we have*

$$\begin{aligned}
&\tau_n(\eta^h_{c,n,\kappa_n})^2 \leq \\
&c\Bigg(\parallel \frac{\partial}{\partial t}(C - C_h)(t) + \mathbf{u}\cdot\nabla C - \mathbf{u}_h^n \cdot \nabla C_h^n - \frac{1}{2}\text{div }(\mathbf{u}_h^n)C_h^n + r_0(C - C_h^n) \parallel^2_{L^2(t_{n-1},t_n;H^{-1}(\Delta\kappa_n))}\big) \\
&\quad + \parallel C - \pi_\tau C_h \parallel^2_{L^2(t_{n-1},t_n;H^1(\Delta\kappa_n))} + h^2_{\kappa_n} \parallel g - \pi_\tau g \parallel^2_{L^2(t_{n-1},t_n;L^2(\Delta\kappa_n))} + \tau_n h^2_{\kappa_n} \parallel g^n - g_h^n \parallel^2_{L^2(\Delta\kappa_n)}\Bigg),
\end{aligned} \tag{5.82}$$

where c is a positive constant independent of the mesh and time steps.

Proof. We consider the second equality of system (5.9) in Lemma 5.4 in which we simplify the left and right hand side by $\langle R_c^\tau(t), r\rangle$, so we have for $t =]t_{n-1}, t_n]$:

$$\begin{aligned}
&\int_\Omega \frac{\partial}{\partial t}(C - C_h)(t)\, r\,\mathbf{dx} + \alpha\int_\Omega \nabla(C(t) - C_h^n)\cdot\nabla r\,\mathbf{dx} + \int_\Omega \mathbf{u}(t)\cdot\nabla C(t)\, r\,\mathbf{dx} \\
&- \int_\Omega \mathbf{u}_h^n\cdot\nabla C_h^n\, r\,\mathbf{dx} + r_0\int_\Omega (C(t) - C_h^n)\, r\,\mathbf{dx} - \frac{1}{2}\int_\Omega \text{div }(\mathbf{u}_h^n)C_h^n\, r\,\mathbf{dx} \\
&\qquad = \sum_{\kappa_n\in\mathcal{T}_{nh}}\Bigg\{\int_{\kappa_n}(g(t) - g^n)\, r\,\mathbf{dx} + \int_{\kappa_n}(g^n - g_h^n)\, r\,\mathbf{dx} \\
&\qquad + \int_{\kappa_n}(g_h^n - \frac{1}{\tau_n}(C_h^n - C_h^{n-1}) + \alpha\Delta C_h^n - \mathbf{u}_h^n\cdot\nabla C_h^n - \frac{1}{2}\text{div }(\mathbf{u}_h^n)C_h^n - r_0 C_h^n)\, r\,\mathbf{dx} \\
&\qquad - \frac{\alpha}{2}\sum_{e_n\in\varepsilon_{\kappa_n}}\int_{e_n}[\nabla C_h^n(\sigma)\cdot\mathbf{n}]r(\sigma)\mathbf{d}\sigma\Bigg\}.
\end{aligned} \tag{5.83}$$

We take $r = r_{\kappa_n}$, where

$$r_{\kappa_n} = \begin{cases} (g_h^n - \frac{1}{\tau_n}(C_h^n - C_h^{n-1}) + \alpha\Delta C_h^n - \mathbf{u}_h^n\cdot\nabla C_h^n - \frac{1}{2}\text{div }(\mathbf{u}_h^n)C_h^n - r_0 C_h^n)\psi_{\kappa_n} & \text{on } \kappa_n, \\ 0 & \text{on } \Omega\backslash\kappa_n. \end{cases}$$

We remember that ψ_{κ_n} is the bubble function which is equal to the product of the barycentric coordinates associated with the vertices of κ_n.
So, we obtain

$$\begin{aligned}
&\int_{\kappa_n} (g_h^n - \frac{1}{\tau_n}(C_h^n - C_h^{n-1}) + \alpha\Delta C_h^n - \mathbf{u}_h^n \cdot \nabla C_h^n - \frac{1}{2}\text{div }(\mathbf{u}_h^n)C_h^n - r_0 C_h^n)^2\, \psi_{\kappa_n}\, \mathbf{dx} \\
&\quad = \int_{\kappa_n} \frac{\partial}{\partial t}(C - C_h)(t)\, r_{\kappa_n}\, \mathbf{dx} + \alpha \int_{\kappa_n} \nabla(C(t) - C_h^n)\cdot \nabla r_{\kappa_n}\, \mathbf{dx} + \int_{\kappa_n} \mathbf{u}(t)\cdot\nabla C(t)\, r_{\kappa_n}\, \mathbf{dx} \\
&\qquad - \int_{\kappa_n} \mathbf{u}_h^n \cdot \nabla C_h^n\, r_{\kappa_n}\, \mathbf{dx} + r_0 \int_{\kappa_n} (C(t) - C_h^n)\, r_{\kappa_n}\, \mathbf{dx} - \frac{1}{2}\int_{\kappa_n} \text{div }(\mathbf{u}_h^n) C_h^n\, r_{\kappa_n}\, \mathbf{dx} \\
&\qquad - \int_{\kappa_n} (g(t) - g^n)\, r_{\kappa_n}\, \mathbf{dx} - \int_{\kappa_n} (g^n - g_h^n)\, r_{\kappa_n}\, \mathbf{dx}.
\end{aligned}$$

Since $r_{\kappa_n} \in H_0^1(\kappa_n)$, we can apply the definition of the $H^{-1}(\kappa_n)$ norm. We apply the Cauchy-Schwarz inequality to get

$$\begin{aligned}
&\| \, (g_h^n - \frac{1}{\tau_n}(C_h^n - C_h^{n-1}) + \alpha\Delta C_h^n - \mathbf{u}_h^n \cdot \nabla C_h^n - \frac{1}{2}\text{div }(\mathbf{u}_h^n)C_h^n - r_0 C_h^n)\, \psi_{\kappa_n}^{1/2} \, \|_{0,\kappa_n}^2 \\
&\quad \le c_1 \Big(\| \, \frac{\partial}{\partial t}(C - C_h)(t) + \mathbf{u}\cdot\nabla C(t) - \mathbf{u}_h^n \cdot \nabla C_h^n - \frac{1}{2}\text{div }(\mathbf{u}_h^n)C_h^n \\
&\qquad + r_0(C(t) - C_h^n) \, \|_{H^{-1}(\kappa_n)} \| \, r_{\kappa_n} \, \|_{H_0^1(\kappa_n)} + \alpha \, \| \, \nabla(C(t) - C_h^n) \, \|_{0,\kappa_n} \| \, r_{\kappa_n} \, \|_{H_0^1(\kappa_n)} \\
&\qquad + \| \, g(t) - g^n \, \|_{H^{-1}(\kappa_n)} \| \, r_{\kappa_n} \, \|_{0,\kappa_n} + \| \, g^n - g_h^n \, \|_{0,\kappa_n} \| \, r_{\kappa_n} \, \|_{0,\kappa_n} \Big).
\end{aligned}$$

We multiply by h_{κ_n}, use Property 5.1 and the fact that $0 \le \psi_{\kappa_n} \le 1$; then we simplify and we get

$$\begin{aligned}
&h_{\kappa_n} \| \, (g_h^n - \frac{1}{\tau_n}(C_h^n - C_h^{n-1}) + \alpha\Delta C_h^n - \mathbf{u}_h^n \cdot \nabla C_h^n - \frac{1}{2}\text{div }(\mathbf{u}_h^n)C_h^n - r_0 C_h^n) \, \|_{0,\kappa_n} \\
&\quad \le c_2 \Big(\| \, \frac{\partial}{\partial t}(C - C_h)(t) + \mathbf{u}\cdot\nabla C(t) - \mathbf{u}_h^n \cdot \nabla C_h^n - \frac{1}{2}\text{div }(\mathbf{u}_h^n)C_h^n + r_0(C(t) - C_h^n) \, \|_{H^{-1}(\kappa_n)} \\
&\qquad + \| \, \nabla(C(t) - C_h^n) \, \|_{0,\kappa_n} + h_{\kappa_n} \| \, g(t) - g^n \, \|_{0,\kappa_n} + h_{\kappa_n} \| \, g^n - g_h^n \, \|_{0,\kappa_n} \Big).
\end{aligned} \tag{5.84}$$

First, we insert g^n and use the triangular inequality. Then by squaring the previous inequality and integrating over t between t_{n-1} and t_n, we can bound the first term of the indicator η_{c,n,κ_n}^h as follows:

$$\begin{aligned}
&\tau_n h_{\kappa_n}^2 \| \, g^n - \frac{1}{\tau_n}(C_h^n - C_h^{n-1}) + \alpha\Delta C_h^n - \mathbf{u}_h^n \cdot \nabla C_h^n - \frac{1}{2}\text{div }(\mathbf{u}_h^n)C_h^n - r_0 C_h^n \, \|_{0,\kappa_n}^2 \\
&\le c_3 \big(\| \, \frac{\partial}{\partial t}(C - C_h)(t) + \mathbf{u}\cdot\nabla C - \mathbf{u}_h^n \cdot \nabla C_h^n - \frac{1}{2}\text{div }(\mathbf{u}_h^n)C_h^n + r_0(C - C_h^n) \, \|_{L^2(t_{n-1},t_n;H^{-1}(\kappa_n))}^2 \\
&\quad + \| \, C - C_h^n \, \|_{L^2(t_{n-1},t_n;H^1(\kappa_n))}^2 + h_{\kappa_n}^2 \| \, g - g^n \, \|_{L^2(t_{n-1},t_n;L^2(\kappa_n))}^2 + \tau_n h_{\kappa_n}^2 \| \, g^n - g_h^n \, \|_{0,\kappa_n}^2 \, \big).
\end{aligned} \tag{5.85}$$

To obtain a bound for the second term of η_{c,n,κ_n}^h, we consider an element $\kappa_n \in \mathcal{T}_{nh}$ and $e_n \subset \partial\kappa_n \cap \Gamma_h^i$; we denote by κ_n' the other cell of the mesh that shares e_n and we consider Equation (5.83) with $r = r_{e_n}$ where:

$$r_{e_n} = \begin{cases} \mathcal{L}_{e_n}(\alpha[\nabla C_h^n \cdot \mathbf{n}]_{e_n} \psi_{e_n}) & \text{on } \kappa_n \cup \kappa_n', \\ 0 & \text{on } \Omega \backslash (\kappa_n \cup \kappa_n'). \end{cases}$$

We get:

$$\int_{e_n} [\alpha \nabla C_h^n \cdot \mathbf{n}]_{e_n}^2 \psi_{e_n} ds = \int_{\kappa_n \cup \kappa'_n} (g^n - \frac{1}{\tau_n}(C_h^n - C_h^{n-1}) + \alpha \Delta C_h^n - \mathbf{u}_h^n \cdot \nabla C_h^n - \frac{1}{2}\text{div } (\mathbf{u}_h^n) C_h^n + r_0 C_h^n) r_{e_n} \, \mathbf{dx}$$
$$- \int_{\kappa_n \cup \kappa'_n} (\frac{\partial}{\partial t}(C - C_h)(t) + \mathbf{u}(t) \cdot \nabla C(t) - \mathbf{u}_h^n \cdot \nabla C_h^n - \frac{1}{2}\text{div } (\mathbf{u}_h^n) C_h^n + r_0 (C(t) - C_h^n)) \, r_{e_n} \, \mathbf{dx}$$
$$+ \int_{\kappa_n \cup \kappa'_n} (g(t) - g^n) \, r_{e_n} \, \mathbf{dx} + \alpha \int_{\kappa_n \cup \kappa'_n} \nabla (C(t) - C_h^n) \, \nabla r_{e_n} \, \mathbf{dx}.$$

Since $r_{e_n} \in H_0^1(K_n \cup K'_n)$, we can apply the definition of the $H^{-1}(K_n \cup K'_n)$ norm; we apply the Cauchy-Schwarz inequality, we use the properties 5.1 and 5.2, we multiply by $h_{e_n}^{1/2}$ and simplify by $\| \alpha [\nabla C_h^n \cdot \mathbf{n}]_{e_n} \|_{0,e_n}$ to get

$$\begin{aligned}
& h_{e_n}^{1/2} \| \alpha [\nabla C_h^n \cdot \mathbf{n}]_{e_n} \|_{0,e_n} \\
& \leq c_4 \Big(h_{e_n} \| g^n - \frac{1}{\tau_n}(C_h^n - C_h^{n-1}) + \alpha \Delta C_h^n - \mathbf{u}_h^n \cdot \nabla C_h^n - \frac{1}{2}\text{div } (\mathbf{u}_h^n) C_h^n + r_0 C_h^n \|_{0,\kappa_n \cup \kappa'_n} \\
& \| \frac{\partial}{\partial t}(C(t) - C_h)(t) + \mathbf{u} \cdot \nabla C - \mathbf{u}_h^n \cdot \nabla C_h^n - \frac{1}{2}\text{div } (\mathbf{u}_h^n) C_h^n + r_0 (C - C_h^n) \|_{H^{-1}(\kappa_n \cup \kappa'_n)} \\
& \qquad + \| g(t) - g^n \|_{L^2(\kappa_n \cup \kappa'_n)} + \| C(t) - C_h^n \|_{H^1(\kappa_n \cup \kappa'_n)} \Big).
\end{aligned} \tag{5.86}$$

We can bound the first term of the right hand side of (5.86), using (5.84) and the fact that $\frac{h_{e_n}}{h_{\kappa_n}}$ and $\frac{h_{e_n}}{h_{\kappa'_n}}$ are bounded by 1. By squaring the previous inequality, integrating over t from between t_{n-1} and t_n, we can bound the second term of the indicator η_{c,n,κ_n}^h as follows:

$$\begin{aligned}
& h_{e_n} \tau_n \| \alpha [\nabla C_h^n \cdot \mathbf{n}]_{e_n} \|_{0,e_n}^2 \\
& \leq c_5 \Big(\| \frac{\partial}{\partial t}(C - C_h)(t) + \mathbf{u} \cdot \nabla C - \mathbf{u}_h^n \cdot \nabla C_h^n - \frac{1}{2}\text{div } (\mathbf{u}_h^n) C_h^n + r_0 (C - C_h^n) \|_{L^2(t_{n-1},t_n;H^{-1}(\kappa_n \cup \kappa'_n))}^2 \\
& \quad + h_{e_n}^2 \| g - g^n \|_{L^2(t_{n-1},t_n;L^2(\kappa_n \cup \kappa'_n))}^2 + \tau_n h_{e_n}^2 \| g^n - g_h^n \|_{0,\kappa_n \cup \kappa'_n}^2 + \| C - C_h^n \|_{L^2(t_{n-1},t_n;H^1(\kappa_n \cup \kappa'_n))}^2 \Big).
\end{aligned} \tag{5.87}$$

Regrouping (5.85) and (5.87), we get the desired result. □

We still have to bound the indicators of velocity η_{u,n,κ_n}^τ and η_{u,n,κ_n}^h.

Theorem 5.26. *For all $n \in \{1, \cdots, N\}$, we have the following estimation:*

$$(\eta_{u,n,\kappa_n}^\tau)^2 \leq c \left(\| \mathbf{u} - \mathbf{u}_h \|_{L^2(t_{n-1},t_n;H^1(\kappa_n)^2)}^2 + \| \mathbf{u} - \pi_\tau \mathbf{u}_h \|_{L^2(t_{n-1},t_n;H^1(\kappa_n)^2)}^2 \right),$$

where c is a positive constant independent of the time and space steps.
For the proof, we just have to follow the proof of theorem 5.24.

Theorem 5.27. *Let $\nabla \mathbf{u} \in L^\infty(0,T;L^\infty(\Omega)^{2\times 2})$, we have for all $n \in \{1, \cdots, N\}$:*

$$\begin{aligned}
& \tau_n (\eta_{u,n,\kappa_n}^h)^2 \leq \\
& c \Big(\| \frac{\partial}{\partial t}(\mathbf{u} - \mathbf{u}_h) + \mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{u}_h^{n-1} \cdot \nabla \mathbf{u}_h^n - \frac{1}{2}\text{div } (\mathbf{u}_h^{n-1}) \mathbf{u}_h^n + \nabla (p - p_h) \|_{L^2(t_{n-1},t_n;H^{-1}(\Delta\kappa_n)^2)}^2 \\
& \quad + \| C - C_h \|_{L^2(t_{n-1},t_n;H^1(\Delta\kappa_n))} + \tau_n \| C_h^n - C_h^{n-1} \|_{H^1(\Delta\kappa_n)}^2 + \| \mathbf{u} - \mathbf{u}_h \|_{L^2(t_{n-1},t_n;H^1(\Delta\kappa_n))}^2 \\
& \quad + \| \mathbf{u}_h^n - \mathbf{u} \|_{L^2(t_{n-1},t_n;H^1(\Delta\kappa_n))}^2 + \| \nu_h(C) - \nu_C(C) \|_{L^2(t_{n-1},t_n;L^2(\Delta\kappa_n))}^2 \\
& \quad + h_{\kappa_n}^2 \| \mathbf{f}_0 - \pi_\tau \mathbf{f}_0 \|_{L^2(t_{n-1},t_n;L^2(\Delta\kappa_n))}^2 + h_{\kappa_n}^2 \| \mathbf{f}^n(C) - \mathbf{f}_h^n(C) \|_{L^2(t_{n-1},t_n;L^2(\Delta\kappa_n))^2}^2 \Big).
\end{aligned} \tag{5.88}$$

where c is a positive constant independent of the time and mesh steps.

Proof. Based on the first equality of system (5.9) in Lemma 5.4, we use 5.10 and we insert the terms $(\mathbf{f}_h^n(C_h^{n-1}),\mathbf{v})$, $2(\nu_h(C(t))\mathbb{D}(\mathbf{u}),\mathbb{D}(\mathbf{v}))$, and $2(\nu_h(C_h^{n-1})\mathbb{D}(\mathbf{u}),\mathbb{D}(\mathbf{v}))$ to get:

$$\begin{aligned}
&\int_\Omega \big(\frac{\partial}{\partial t}(\mathbf{u}-\mathbf{u}_h)(t)+\mathbf{u}(t)\cdot\nabla\mathbf{u}(t)-\frac{1}{2}\text{div }(\mathbf{u}_h^{n-1})\mathbf{u}_h^n-\mathbf{u}_h^{n-1}\cdot\nabla\mathbf{u}_h^n\big)\ \cdot\mathbf{v}\,\mathbf{dx}\\
&-\int_\Omega \text{div }\mathbf{v}(t)(p(t)-p_h(t))\,\mathbf{dx}+\nu_0\int_\Omega\nabla(\mathbf{u}(t)-\mathbf{u}_h^n):\nabla\mathbf{v}\,\mathbf{dx}+2\int_\Omega \nu_h(C_h^{n-1})\mathbb{D}(\mathbf{u}(t)-\mathbf{u}_h^n):\mathbb{D}(\mathbf{v})\,\mathbf{dx}\\
&+2\int_\Omega(\nu_C(C(t))-\nu_h(C(t))\mathbb{D}(\mathbf{u}):\mathbb{D}(\mathbf{v})\,\mathbf{dx}+2\int_\Omega(\nu_h(C(t))-\nu_h(C_h^{n-1}))\mathbb{D}(\mathbf{u}):\mathbb{D}(\mathbf{v})\,\mathbf{dx}\\
&-\int_\Omega(\mathbf{f}(t,C(t))-\mathbf{f}^n(C_h^{n-1}))\ \cdot\mathbf{v}\,\mathbf{dx}-\int_\Omega(\mathbf{f}^n(C_h^{n-1})-\mathbf{f}_h^n(C_h^{n-1}))\ \cdot\mathbf{v}\,\mathbf{dx}\\
&=\sum_{\kappa_n\in\mathcal{T}_{nh}}\Big\{\int_{\kappa_n}(\mathbf{f}_h^n(C_h^{n-1})-\frac{1}{\tau_n}(\mathbf{u}_h^n-\mathbf{u}_h^{n-1})+\nu_0\Delta\mathbf{u}_h^n+\nabla\cdot(2\nu_h(C_h^{n-1})\mathbb{D}(\mathbf{u}_h^n))-\mathbf{u}_h^{n-1}\cdot\nabla\mathbf{u}_h^n\\
&\qquad-\frac{1}{2}\text{div }(\mathbf{u}_h^{n-1})\mathbf{u}_h^n-\nabla p_h^n))\cdot\mathbf{v}\,\mathbf{dx}-\frac{1}{2}\sum_{e_n\in\varepsilon_{\kappa n}}\int_{e_n}[(\nu_0\nabla\mathbf{u}_h^n+2\nu_h(C_h^{n-1})\mathbb{D}(\mathbf{u}_h^n)-p_h^n)(\sigma)\mathbf{n}]\cdot\mathbf{v}(\sigma)\mathbf{d}\sigma\Big\}.
\end{aligned}\tag{5.89}$$

We take $\mathbf{v}=\mathbf{v}_{\kappa_n}$ such that

$$\mathbf{v}_{\kappa_n}=\begin{cases}((\mathbf{f}_h^n(C_h^{n-1})-\frac{1}{\tau_n}(\mathbf{u}_h^n-\mathbf{u}_h^{n-1})+\nu_0\Delta\mathbf{u}_h^n+\nabla\cdot(2\nu_h(C_h^{n-1})\mathbb{D}(\mathbf{u}_h^n)) & \\ \qquad -\mathbf{u}_h^{n-1}\cdot\nabla\mathbf{u}_h^n-\frac{1}{2}\text{div }(\mathbf{u}_h^{n-1})\mathbf{u}_h^n-\nabla p_h^n)(x))\psi_{\kappa_n} & \text{on } \kappa_n,\\ 0 & \text{on } \Omega\backslash\kappa_n.\end{cases}$$

Then we get:

$$\begin{aligned}
&\int_{\kappa_n}\big(\mathbf{f}_h^n(C_h^{n-1})-\frac{1}{\tau_n}(\mathbf{u}_h^n-\mathbf{u}_h^{n-1})+\nu_0\Delta\mathbf{u}_h^n+\nabla\cdot(2\nu_h(C_h^{n-1})\mathbb{D}(\mathbf{u}_h^n))\\
&\qquad\qquad-\mathbf{u}_h^{n-1}\cdot\nabla\mathbf{u}_h^n-\frac{1}{2}\text{div }(\mathbf{u}_h^{n-1})\mathbf{u}_h^n-\nabla p_h^n\big)^2\psi_{\kappa_n}\,\mathbf{dx}\\
&=\int_{\kappa_n}\big(\frac{\partial}{\partial t}(\mathbf{u}-\mathbf{u}_h)(t)+\mathbf{u}(t)\cdot\nabla\mathbf{u}(t)-\mathbf{u}_h^{n-1}.\nabla\mathbf{u}_h^n-\frac{1}{2}\text{div }(\mathbf{u}_h^{n-1})\mathbf{u}_h^n+\nabla(p(t)-p_h(t))\ \big)\ \cdot\mathbf{v}_{\kappa_n}\,\mathbf{dx}\\
&\quad+\nu_0\int_{\kappa_n}\nabla(\mathbf{u}(t)-\mathbf{u}_h^n):\nabla\mathbf{v}_{\kappa_n}\,\mathbf{dx}+\int_{\kappa_n}2\nu_h(C_h^{n-1})\mathbb{D}(\mathbf{u}(t)-\mathbf{u}_h^n):\mathbb{D}(\mathbf{v}_{\kappa_n})\,\mathbf{dx}\\
&\quad+\int_{\kappa_n}2(\nu_C(C(t))-\nu_h(C(t)))\mathbb{D}(\mathbf{u}):\mathbb{D}(\mathbf{v}_{\kappa_n})\,\mathbf{dx}+\int_{\kappa_n}2(\nu_h(C(t))-\nu_h(C_h^{n-1}))\mathbb{D}(\mathbf{u}):\mathbb{D}(\mathbf{v}_{\kappa_n})\,\mathbf{dx}\\
&\quad-\int_{\kappa_n}(\mathbf{f}(t,C(t))-\mathbf{f}^n(C_h^{n-1}))\ \cdot\mathbf{v}_{\kappa_n}\,\mathbf{dx}-\int_{\kappa_n}(\mathbf{f}^n(C_h^{n-1})-\mathbf{f}_h^n(C_h^{n-1}))\ \cdot\mathbf{v}_{\kappa_n}\,\mathbf{dx}.
\end{aligned}\tag{5.90}$$

Since $\mathbf{v}_{\kappa_n}\in H_0^1(K_n)^2$, we can apply the definition of the $H^{-1}(\kappa_n)^2$ norm. We denote by $I_i, i=0,\dots,6$ the terms in the right-hand side of equality (5.90). The first three terms can easily be bounded by using the properties of ν_h and the Cauchy-Schwarz inequality and we get

$$\begin{aligned}|I_0+I_1+I_2|\le c_1\big(\ \|\ \frac{\partial}{\partial t}(\mathbf{u}-\mathbf{u}_h)(t)+\mathbf{u}(t)\cdot\nabla\mathbf{u}(t)-\mathbf{u}_h^{n-1}\cdot\nabla\mathbf{u}_h^n-\frac{1}{2}\text{div }(\mathbf{u}_h^{n-1})\mathbf{u}_h^n+\nabla(p(t)-p_h(t))\ \|_{H^{-1}(\kappa_n)^2}\\
+\|\ \mathbf{u}-\mathbf{u}_h^n\ \|_{H^1(\kappa_n)^2}\ \big)\ \|\ \mathbf{v}_{\kappa_n}\ \|_{H^1(\kappa_n)^2}\end{aligned}$$

Let us now bound I_3. We apply the Cauchy-Schwarz inequality by considering $\nabla\mathbf{u}\in L^\infty(0,T;L^\infty(\Omega)^{2\times2})$. Thus we get:

$$\begin{aligned}|I_3|&\le 2\ \|\ \nu_C(C(t))-\nu_h(C(t))\ \|_{L^2(\kappa_n)}\|\ \mathbb{D}(\mathbf{u}(t))\ \|_{L^\infty(0,T;L^\infty(\Omega)^{2\times2})}\|\ \mathbf{v}_{\kappa_n}\ \|_{H^1(\kappa_n)^2}\\
&\le c_2\ \|\ \nu_C(C(t))-\nu_h(C(t))\ \|_{L^2(\kappa_n)}\|\ \mathbf{v}_{\kappa_n}\ \|_{H^1(\kappa_n)^2}\ .\end{aligned}$$

As ν_h is Lipschitz, and $\nabla\mathbf{u}\in L^\infty(0,T;L^\infty(\Omega)^{2\times2})$, we apply the Cauchy-Schwarz inequality and we insert $C_h(t)$ to get :

$$\begin{aligned}|I_4|&\le c_5\ \|\ C(t)-C_h^{n-1}\ \|_{L^2(\kappa_n)}\|\ \mathbb{D}(\mathbf{u})\ \|_{L^\infty(0,T;L^\infty(\Omega)^{2\times2})}\|\ \mathbf{v}_{\kappa_n}\ \|_{H^1(\kappa_n)^2}\\
&\le c_6\big(\ \|\ C(t)-C_h(t)\ \|_{H^1(\kappa_n)}+\|\ C_h^n-C_h^{n-1}\ \|_{H^1(\kappa_n)}\ \big)\ \|\ \mathbf{v}_{\kappa_n}\ \|_{H^1(\kappa_n)^2}\ .\end{aligned}$$

For the Term I_5, we insert $\mathbf{f}(C_h(t))$, use the definition and the properties of $\mathbf{f}$ (assumption 2.4) and apply the Cauchy-Schwarz inequality to obtain

$$|I_5| \quad \leq \quad c_7\bigg(\parallel \mathbf{f}_0(t) - \mathbf{f}_0(t_n) \parallel_{L^2(\kappa_n)^2} + \parallel C(t) - C_h(t) \parallel_{L^2(\kappa_n)} + \parallel C_h^n - C_h^{n-1} \parallel_{L^2(\kappa_n)} \bigg) \parallel \mathbf{v}_{\kappa_n} \parallel_{L^2(\kappa_n)^2} .$$

We still have to bound the term I_6. We insert $\mathbf{f}^n(C(t))$ and $\mathbf{f}_h^n(C(t))$ and we use the Cauchy-Schwarz inequality to get:

$$\begin{aligned} |I_6| \quad \leq \quad c_8\bigg(& \parallel \mathbf{f}_h^n(C_h^{n-1}) - \mathbf{f}_h^n(C(t)) \parallel_{L^2(\kappa_n)^2} + \parallel \mathbf{f}_h^n(C(t)) - \mathbf{f}^n(C(t)) \parallel_{L^2(\kappa_n)^2} \\ & + \parallel \mathbf{f}^n(C(t)) - \mathbf{f}^n(C_h^{n-1}) \parallel_{L^2(\kappa_n)^2} \bigg) \parallel \mathbf{v}_{\kappa_n} \parallel_{L^2(\kappa_n)^2} . \end{aligned}$$

By using the properties of $\mathbf{f}^n$ and $\mathbf{f}_h^n$ and then inserting $C_h(t)$ in the first and third term in the right-hand side of the above bound, we obtain:

$$\begin{aligned} |I_6| \quad \leq \quad c_9\bigg(& \parallel C_h^n - C_h^{n-1} \parallel_{L^2(\kappa_n)^2} + \parallel \mathbf{f}_h^n(C(t)) - \mathbf{f}^n(C(t)) \parallel_{L^2(\kappa_n)^2} \\ & + \parallel C(t) - C_h(t) \parallel_{L^2(\kappa_n)^2} \bigg) \parallel \mathbf{v}_{\kappa_n} \parallel_{L^2(\kappa_n)^2} . \end{aligned}$$

Thus, we consider Equation (5.90), gather the above bounds corresponding to $I_i, i = 0, \ldots, 6$, use the Cauchy-Schwarz inequality and properties 5.1 and 5.2, multiply by $h_{\kappa_n}^2$ and integrate between t_{n-1} and t_n to have the following inequality:

$$\begin{aligned} & \tau_n h_{\kappa_n}^2 \parallel \mathbf{f}^n(C_h^{n-1}) - \frac{1}{\tau_n}(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}) + \nu_0 \Delta \mathbf{u}_h^n + \nabla \cdot (2\nu_h(C_h^{n-1})\mathbb{D}(\mathbf{u}_h^n)) \\ & \qquad\qquad\qquad\qquad - \mathbf{u}_h^{n-1} \cdot \nabla \mathbf{u}_h^n - \frac{1}{2}\text{div } (\mathbf{u}_h^{n-1})\mathbf{u}_h^n - \nabla p_h^n \parallel_{0,\kappa_n}^2 \\ & \leq c_{10}\bigg(\parallel \frac{\partial}{\partial t}(\mathbf{u} - \mathbf{u}_h) + (\mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{u}_h^{n-1} \cdot \nabla \mathbf{u}_h^n - \frac{1}{2}\text{div } (\mathbf{u}_h^{n-1})\mathbf{u}_h^n + \nabla(p - p_h) \parallel_{L^2(t_{n-1},t_n;H^{-1}(\kappa_n))}^2 \\ & \quad + \parallel C - C_h \parallel_{L^2(t_{n-1},t_n;H^1(\kappa_n))}^2 + \tau_n \parallel C_h^n - C_h^{n-1} \parallel_{H^1(\kappa_n)}^2 \\ & \quad + \parallel \mathbf{u}_h^n - \mathbf{u} \parallel_{L^2(t_{n-1},t_n;H^1(\kappa_n))}^2 + \parallel \nu_h(C) - \nu_C(C) \parallel_{L^2(t_{n-1},t_n;L^2(\kappa_n))}^2 \\ & \quad + h_{\kappa_n}^2 \parallel \mathbf{f}_0 - \pi_\tau \mathbf{f}_0 \parallel_{L^2(t_{n-1},t_n;L^2(\kappa_n))}^2 + h_{\kappa_n}^2 \parallel \mathbf{f}_h^n(C) - \mathbf{f}^n(C) \parallel_{L^2(t_{n-1},t_n;L^2(\kappa_n))}^2 \bigg), \end{aligned} \tag{5.91}$$

This last inequality constitutes the bound of the first term of the indicator η_{u,n,κ_n}^h. Let us now bound the second one.
We consider an element $\kappa_n \in \mathcal{T}_{nh}$ and $e_n \subset \partial\kappa_n \cap \Gamma_h^i$; we denote by κ_n' the other cell of the mesh that shares e_n and we consider Equation (5.89) with $\mathbf{v} = \mathbf{v}_{e_n}$ such that

$$\mathbf{v}_{e_n} = \begin{cases} \mathcal{L}_{e_n}([(\nu_0 \nabla \mathbf{u}_h^n + 2\nu_h(C_h^{n-1})\mathbb{D}(\mathbf{u}_h^n) - p_h^n)\,\mathbf{n}]_{e_n}\psi_{e_n}) & \text{on } \kappa_n \cup \kappa_n', \\ 0 & \text{on } \Omega \backslash (\kappa_n \cup \kappa_n'). \end{cases}$$

We obtain,

$$
\begin{aligned}
&-\int_{e_n}[(\nu_0\nabla\mathbf{u}_h^n+2\nu_h(C_h^{n-1})\mathbb{D}(\mathbf{u}_h^n)-p_h^n)\mathbf{n}]_{e_n}^2\,\psi_{e_n}ds\\
&=\int_{\kappa_n\cup\kappa_n'}\big(\frac{\partial}{\partial t}(\mathbf{u}-\mathbf{u}_h)(t)+\mathbf{u}(t)\cdot\nabla\mathbf{u}(t)-\frac{1}{2}\text{div }(\mathbf{u}_h^{n-1})\mathbf{u}_h^n-\mathbf{u}_h^{n-1}\cdot\nabla\mathbf{u}_h^n+\nabla(p-p_h)(t)\big)\cdot\mathbf{v}_{e_n}\,\mathbf{dx}\\
&+\int_{\kappa_n\cup\kappa_n'}\nu_0\nabla(\mathbf{u}(t)-\mathbf{u}_h^n):\nabla\mathbf{v}_{e_n}\,\mathbf{dx}+\int_{\kappa_n\cup\kappa_n'}2(\nu_h(C_h^{n-1})\mathbb{D}(\mathbf{u}(t)-\mathbf{u}_h^n)):\mathbb{D}(\mathbf{v}_{e_n})\,\mathbf{dx}\\
&+\int_{\kappa_n\cup\kappa_n'}2(\nu_C(C(t))-\nu_h(C(t)))\mathbb{D}(\mathbf{u}):\mathbb{D}(\mathbf{v}_{e_n})\,\mathbf{dx}+\int_{\kappa_n\cup\kappa_n'}2(\nu_h(C_h(t))-\nu_h(C_h^{n-1}))\mathbb{D}(\mathbf{u}):\mathbb{D}(\mathbf{v}_{e_n})\,\mathbf{dx}\\
&-\int_{\kappa_n\cup\kappa_n'}(\mathbf{f}(t,C(t))-\mathbf{f}^n(C_h^{n-1}))\cdot\mathbf{v}_{e_n}\,\mathbf{dx}-\int_{\kappa_n\cup\kappa_n'}(\mathbf{f}^n(C_h^{n-1})-\mathbf{f}_h^n(C_h^{n-1}))\cdot\mathbf{v}_{e_n}\,\mathbf{dx}\\
&-\int_{\kappa_n\cup\kappa_n'}(\mathbf{f}_h^n(C_h^{n-1})-\frac{1}{\tau_n}(\mathbf{u}_h^n-\mathbf{u}_h^{n-1})+\nu_0\Delta\mathbf{u}_h^n\\
&\qquad+\nabla\cdot(2\nu_h(C_h^{n-1})\mathbb{D}(\mathbf{u}_h^n))-\mathbf{u}_h^{n-1}\cdot\nabla\mathbf{u}_h^n-\frac{1}{2}\text{div }(\mathbf{u}_h^{n-1})\mathbf{u}_h^n-\nabla p_h^n)\cdot\mathbf{v}_{e_n}\,\mathbf{dx}.
\end{aligned}
\tag{5.92}
$$

We follow the same steps as in the case of the first term of $(\eta_{u,n,\kappa_n}^h)^2$. Using (5.91) and the fact that $\frac{h_{e_n}}{h_{\kappa_n}}$ and $\frac{h_{e_n}}{h_{\kappa_n'}}$ are bounded by 1, we get:

$$
\begin{aligned}
&\tau_n h_{e_n}\parallel[(\nu_0\nabla\mathbf{u}_h^n+2\nu_h(C_h^{n-1})\mathbb{D}(\mathbf{u}_h^n)-p_h^n)\mathbf{n}]_{e_n}\parallel_{0,e_n}^2\\
&\le c_{11}\Big(\parallel\frac{\partial}{\partial t}(\mathbf{u}-\mathbf{u}_h)+\mathbf{u}\cdot\nabla\mathbf{u}-\mathbf{u}_h^{n-1}\cdot\nabla\mathbf{u}_h^n-\frac{1}{2}\text{div }(\mathbf{u}_h^{n-1})\mathbf{u}_h^n+\nabla(p-p_h)\parallel_{L^2(t_{n-1},t_n;H^{-1}(\kappa_n\cup\kappa_n'))}^2\\
&\quad+\quad\parallel C-C_h\parallel_{L^2(t_{n-1},t_n;H^1(\kappa_n\cup\kappa_n'))}^2+\tau_n\parallel C_h^n-C_h^{n-1}\parallel_{H^1(\kappa_n\cup\kappa_n')}^2\\
&\quad+\quad\parallel\mathbf{u}_h^n-\mathbf{u}\parallel_{L^2(t_{n-1},t_n;H^1(\kappa_n\cup\kappa_n'))}^2+\parallel\nu_h(C)-\nu_C(C)\parallel_{L^2(t_{n-1},t_n;L^2(\kappa_n\cup\kappa_n'))}^2\\
&\quad+\quad h_{e_n}^2\parallel\mathbf{f}_0-\mathbf{f}_0(t_n)\parallel_{L^2(t_{n-1},t_n;L^2(\kappa_n\cup\kappa_n'))}^2+\quad h_{e_n}^2\parallel\mathbf{f}^n(C)-\mathbf{f}_h^n(C)\parallel_{L^2(t_{n-1},t_n;L^2(\kappa_n\cup\kappa_n'))}^2\Big).
\end{aligned}
\tag{5.93}
$$

We finish this proof by bounding the last term of $(\eta_{u,n,\kappa_n}^h)^2$.
By using the impressibility condition div $(\mathbf{u})=0$ we get

$$\parallel\text{div }\mathbf{u}_h(t)\parallel_{L^2(\kappa_n)}\le\parallel\text{div }(\mathbf{u}(t)-\mathbf{u}_h(t))\parallel_{L^2(\kappa_n)}.$$

We integrate between t_{n-1} and t_n, and we use Relation (5.60) with the term (div $(\mathbf{u}_h(t_n))$) instead of the term $(\mathbf{u}_\tau(t_n)-\mathbf{u}_h(t_n))$ to get:

$$
\begin{aligned}
\frac{\tau_n}{3}\parallel\text{div }\mathbf{u}_h^n\parallel_{0,\kappa_n}&\le\parallel\text{div }\mathbf{u}_h\parallel_{L^2(t_{n-1},t_n;L^2(\kappa_n))}\\
&\le\parallel\text{div }(\mathbf{u}-\mathbf{u}_h)\parallel_{L^2(t_{n-1},t_n;L^2(\kappa_n))}\\
&\le\parallel\mathbf{u}-\mathbf{u}_h\parallel_{L^2(t_{n-1},t_n;H^1(\kappa_n))}.
\end{aligned}
\tag{5.94}
$$

Gathering (5.91), (5.93) and (5.94), and the fact that $h_{e_n}\le h_{\kappa_n}$, we obtain the desired result. □

Each indicator is now bounded by the error between the exact and the numerical solutions; the lower bound of the error is thus obtained done. This establishes the complete equivalence between the *a posteriori* error estimators and the error.

6. Numerical Results

In this last section, to validate the results proved in this paper, we use numerical simulations performed with Freefem++[25]. We treat two different cases. We first study an academic case where the exact solution is known; in a second step, we move on to a more realistic case.

6.1. **The Academic case.** We consider a square domain Ω, where each edge is initially divided into N segments of equal length.

Firstly, we define the following relative total error between the exact and numerical solutions:

$$err = \left(\frac{\sum_{n=1}^{N} \tau_n |\mathbf{u}_h^n - \mathbf{u}(t_n)|^2_{H^1(\Omega)^2} + \sum_{n=1}^{N} \tau_n ||p_h^n - p(t_n)||^2_{L^2(\Omega)} + \sum_{n=1}^{N} \tau_n |C_h^n - C(t_n)|^2_{H^1(\Omega)}}{\sum_{n=1}^{N} \tau_n \big(|\mathbf{u}(t_n)|^2_{H^1(\Omega)^2} + ||p(t_n)||^2_{L^2(\Omega)} + |C(t_n)|^2_{H^1(\Omega)} \big)} \right)^{1/2},$$

where $(\mathbf{u}, p, C)$ is the solution of Problem (1.1) and $(\mathbf{u}_h^n, p_h^n, C_h^n)$ is the solution of Problem (Eds1). As far as the velocity and concentration variables are concerned, this is a (relative) discrete version of the $L^2(0, T, H_0^1(\Omega))$ norms that are involved in definitions 5.7 and 5.8. As far as the pressure error is concerned, the discrete inf-sup condition 4.4 very classically allows to bound the L^2 pressure error by the velocity H_0^1 error. So, the denominator of this error is contained in the left-hand side of Corollary 5.22. We use the *a posteriori* error estimates given by this corollary to show numerical results based on mesh adaptation. For this objective, we introduce the relative time indicator, defined by

$$E_{\tau_u} = \left(\frac{\sum_{n=0}^{N} (\eta_{u,n}^{\tau})^2}{D} \right)^{1/2} \qquad \text{and} \qquad E_{\tau_c} = \left(\frac{\sum_{n=0}^{N} (\eta_{c,n}^{\tau})^2}{D} \right)^{1/2},$$

with

$$(\eta_{u,n}^{\tau})^2 = \sum_{\kappa_n \in \mathcal{T}_{nh}} \left(\eta_{u,n,\kappa_n}^{\tau} \right)^2 \quad , \quad (\eta_{c,n}^{\tau})^2 = \sum_{\kappa_n \in \mathcal{T}_{nh}} \left(\eta_{c,n,\kappa_n}^{\tau} \right)^2,$$

and

$$D = \sum_{n=1}^{N} \tau_n \big(|\mathbf{u}_h^n|^2_{H^1(\Omega)^2} + ||p_h^n||^2_{L^2(\Omega)} + |C_h^n|^2_{H^1(\Omega)} \big).$$

We alse introduce the relative space indicators, defined by

$$E_{h_u} = \left(\frac{\sum_{n=0}^{N} \tau_n (\eta_{u,n}^{h})^2}{D} \right)^{1/2} \qquad \text{and} \qquad E_{h_c} = \left(\frac{\sum_{n=0}^{N} \tau_n (\eta_{c,n}^{h})^2}{D} \right)^{1/2},$$

with

$$(\eta_{u,n}^{h})^2 = \sum_{\kappa_n \in \mathcal{T}_{nh}} \left(\eta_{u,n,\kappa_n}^{h} \right)^2 \quad \text{and} \quad (\eta_{c,n}^{h})^2 = \sum_{\kappa_n \in \mathcal{T}_{nh}} \left(\eta_{c,n,\kappa_n}^{h} \right)^2,$$

and we denote

$$E_{tot} = E_{\tau_u} + E_{\tau_c} + E_{h_u} + E_{h_c}.$$

We consider the case where the exact solution of (1.1) is given by $(\mathbf{u}_{ex}, p_{ex}, C_{ex}) = (\mathrm{rot}\,\psi, p_{ex}, C_{ex})$, where

$$\begin{aligned}
&\psi(x, y, t) = x^2 (x-1)^2\, y^2 (y-1)^2 \, \sin(t),\\
&p_{ex}(x, y, t) = (t+1)\, \cos(\pi x)\, \cos(\pi y),\\
&C_{ex}(x, y, t) = -t\, e^{-100((x-0.3-0.3t)^2 + (y-0.3)^2)}.
\end{aligned}$$

Thereby, we can compare the exact and numerical solutions by computing the corresponding error.
We consider $\Omega = [0, 1]^2$, $T = 1$, $\alpha = 1$, $r_0 = 1$, $\nu_0 = 0.5$ and $\nu_C(C) = 0.2 \sin(C) + 0.5$. First, we consider the adaptive algorithm with an initial time step $\tau = \dfrac{1}{N}$ and an initial mesh corresponding to $N = 20$.
Starting by $\mathbf{u}_h^0 = \mathbf{u}_0 = 0$ and $C_h^0 = C_0 = 0$, we used the algorithm below.

Algorithm:

(1) Having $\mathbf{u}_h^n$ and C_h^n, we calculate $\mathbf{u}_h^{n+1}$, C_h^{n+1}, p_h^{n+1} and the indicators $(\eta_{u,n}^{\tau})^2$, $(\eta_{c,n}^{\tau})^2$, $(\eta_{u,n}^{h})^2$, $(\eta_{c,n}^{h})^2$.

(2) If the error E_{tot} is less than a fixed tolerance, we move on to the next time step $(n \to n+1)$ and we go to step (1) with a larger time step when E_{tot} is lower than 90% of the fixed tolerance. Otherwise we need to adapt either the mesh or the time step and therefore we move to step (3).

(3) We are in one of these two situations:

(a) if the error in space $E_{h_u} + E_{h_c}$ is less than the error in time $E_{\tau_u} + E_{\tau_c}$, we adapt the time step by replacing τ_n by $\dfrac{\tau_n}{2}$ and start over from step (1) without moving to the next time step.
(b) if the error in time is smaller than the error in space, we adapt the mesh according to the local error indicators and repeat from step (1) without moving to the next time step.

More precisely, concerning the mesh refinement, we use the "ReMeshIndicator" macro provided by FreeFem++ [24, Section 5.1.9]; this typically provides the meshing tool with an indication that, in the new adapted mesh, the mesh step of the current mesh should be locally (around a given cell κ) divided by the ratio $\frac{\eta^h_{c,\kappa}+\eta^h_{u,\kappa}}{\langle \eta^h \rangle}$, where $\langle \eta^h \rangle$ is the mean value of the space error indicators over the mesh.

Moreover, when the mesh is modified, FreeFem++ interpolates the previous finite elements functions $\mathbf{u}_h^{n-1}$ and C_h^{n-1} on the quadrature points of the new mesh and use these values to evaluate integrals that are needed on the new mesh, both in the discrete system (Eds1) and in the evaluation of the error indicators (5.16)–(5.19). In practice, this results in the fact that in the time step that follows a mesh adaptation, the time estimators (5.16) and (5.17) are twice to three times larger than they usually are. According to our adaptation algorithm, this leads to a reduction of the time step after each mesh adaptation, by a factor 2 or 4. Note that solving (Eds1) with a small time step results in the definition of a new velocity field that is (up to terms of size τ_n) the L^2 projection on the new mesh of the previous velocity field, constrained by the weak divergence free condition on the new mesh.
Figures 1 to 4, show the evolution of the mesh during the time steps.

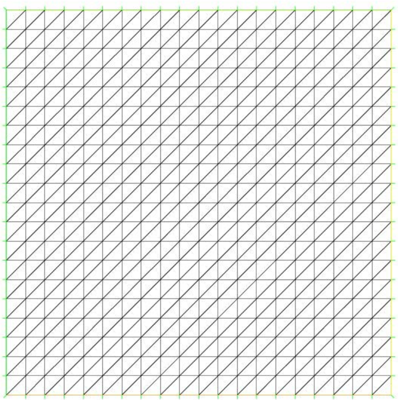


FIGURE 1. Initial mesh

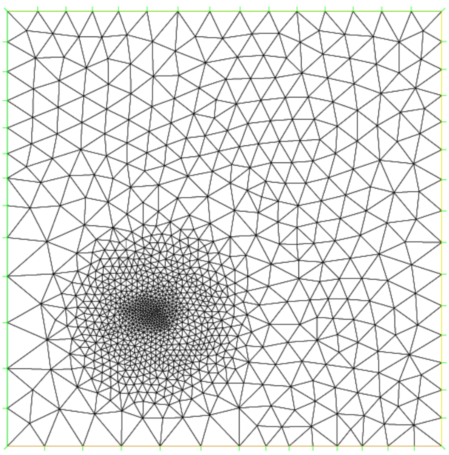


FIGURE 2. Mesh at $t = 0.241$

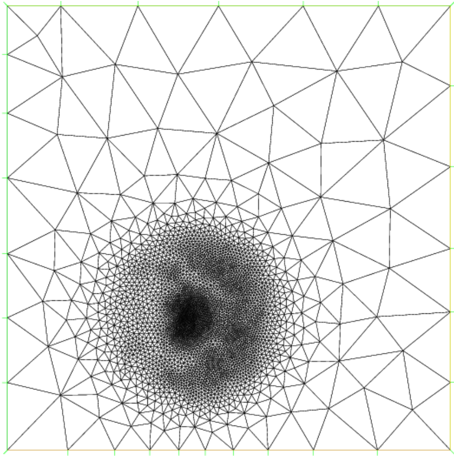


FIGURE 3. Mesh at $t = 0.496$

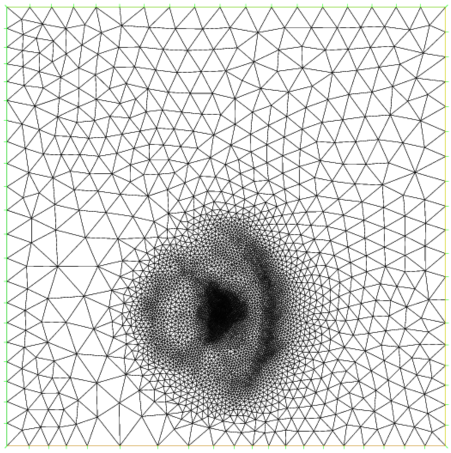


FIGURE 4. Mesh at $t = 0.897$

In figure 5 (respectively in figure 6), we compare the relative total error (respectively the global indicators E_{tot}) with respect to the total number of space-time unknowns in logarithmic scale, for both uniform and adaptive numerical algorithms. Both figures clearly show the advantage of the adaptive method versus the uniform method, since the errors corresponding to the adaptive method are several times smaller than those corresponding to the uniform method for a given number of space-time unknowns. We stress that these graphs can also be read horizontally rather than vertically: for example, in order to reach a total relative error of 10^{-1}, the adaptive strategy requires around 10^4 space-time unknowns, while around 3×10^5 are needed on uniform meshes. This factor 30 in the number of unknowns results in a huge speed-up in the adaptive simulations for a given targeted accuracy.

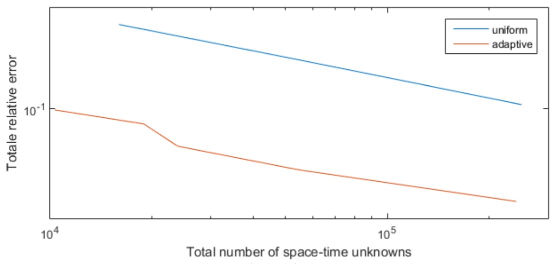


FIGURE 5. Total relative errors err.

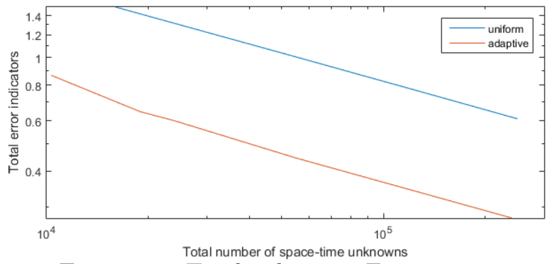


FIGURE 6. Total indicators E_{tot}.

We define the efficiency index as follows:

$$\mathrm{EI} = \left(\frac{\sum_{n=1}^{N} \sum_{\kappa_n \in \mathcal{T}_{nh}} \left((\eta^{\tau}_{u,n,\kappa_n})^2 + (\eta^{\tau}_{c,n,\kappa_n})^2 + (\eta^{h}_{u,n,\kappa_n})^2 + (\eta^{h}_{c,n,\kappa_n})^2 \right)}{\sum_{n=1}^{N} \tau_n \left(|\mathbf{u}_h^n - \mathbf{u}(t_n)|^2_{H^1(\Omega)^2} + ||p_h^n - p(t_n)||^2_{L^2(\Omega)} + |C_h^n - C(t_n)|^2_{H^1(\Omega)} \right)} \right)^{1/2}.$$

In Table 1, we can see the value of the efficiency index for different values of space-time unknowns (STU) for the adaptive mesh. We can notice that the efficiency index varies between 7.67 and 9.09.

STU	10 324	18 965	23 913	54 948	241 276
EI	8.80	7.67	9.09	8.74	7.60

TABLE 1. Efficiency index with respect to the total number of space-time unknowns.

In Table 2, we can also see the value of the efficiency index for different values of STU but for the uniform mesh. We can notice that the efficiency index varies around 5.

STU	16 000	54 000	128 000	250 000
EI	5.79	5.82	5.83	5.82

TABLE 2. Efficiency index with respect to the total number of space-time unknowns.

6.2. **A more realistic case.** Let us introduce a more realistic test case. Here the unknown $C(t)$ represents the variation of the temperature at a time t in the domain assumed to be here the L-shaped vertical section of a room with two different roof heights. We suppose that the initial temperature is uniformly equal to $C_0 = 290K$ (around 16°C). We also suppose that this room is heated by a heater $(BDEF)$ placed at the bottom of the domain as shown in Figure 7 where $AB = 1.6$, $BD = 0.4$, $BF = 0.8$, $FG = 1.6$, $GH = 2.5$, $HI = 2$, $IJ = 1$, $JK = 2$ and $KA = 3.5$.

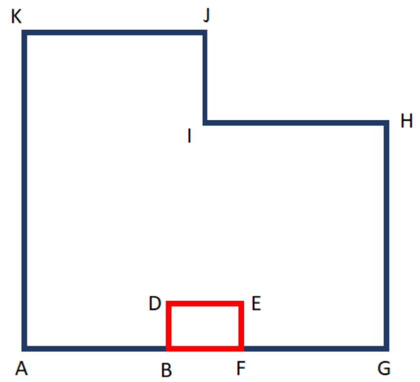


FIGURE 7. The domain

Therefore, the simulation describes the evolution of the temperature variation and of the velocity of the air in this room with respect to time. Physical parameters are chosen to represent typical values of air and are all given in SI units: $T = 1200$, $r_0 = 0$, $\alpha = 2 \times 10^{-5}$, and $\nu = 10^{-5}$. Moreover, the source term in the momentum equation is given by the buoyancy force $\mathbf{f}(C) = (0, \text{grav} \times C/C_0)$ where grav $= 10$. Furthermore, the heat source term is chosen low enough so that the flow remains below a turbulent regime; it is given by

$$g = \begin{cases} 10^{-3} & \text{in the heater } BDEF, \\ 0 & \text{elsewhere.} \end{cases}$$

Finally, on the boundary $\partial\Omega$, we use Robin boundary conditions with a Robin coefficient equal to 10^{-6}, in order to modelize low heat losses.

We start the simulation with a uniform mesh. The adaptive process generates meshes that follow well the velocity and the temperature variation over time. Figures 8, 9, 10 display the meshes, the velocity norms and the temperature variations at the different times $t = 50$, $t = 150$, $t = 200$; figures 11 and 12 for $t = 600$ and $t = 1200$ do not display the meshes because of the high density of cells.

In figure 13, we compare the global indicators E_{tot} with respect to the total number of space-time unknowns in logarithmic scale, for both uniform and adaptive numerical algorithms. The figures show the advantage of the adaptive method versus the uniform method, since the errors corresponding to the adaptive method are smaller than those corresponding to the uniform method. The adaptive strategy requires around 2 to 3 times fewer space-time unknowns than the uniform strategy in order to reach a given error indicator, which results in a significant speed-up in the simulations.

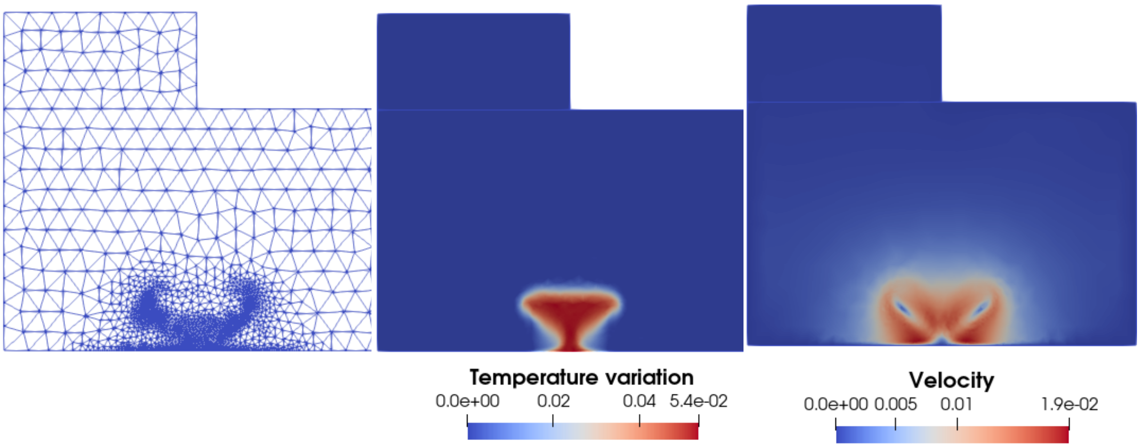


FIGURE 8. Mesh (left), numerical temperature (centre) and numerical velocity (right) at $t = 50$.

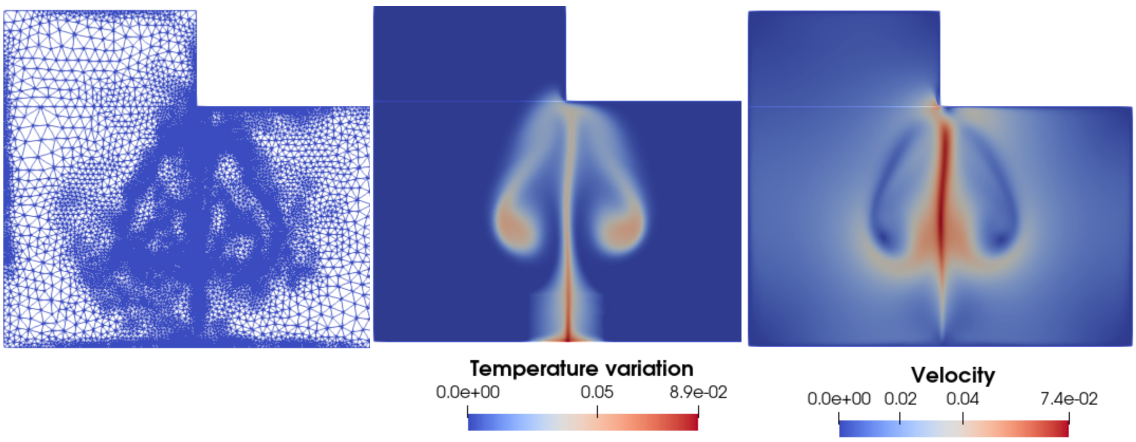


FIGURE 9. Mesh (left), numerical temperature (centre) and numerical velocity (right) at $t = 150$.

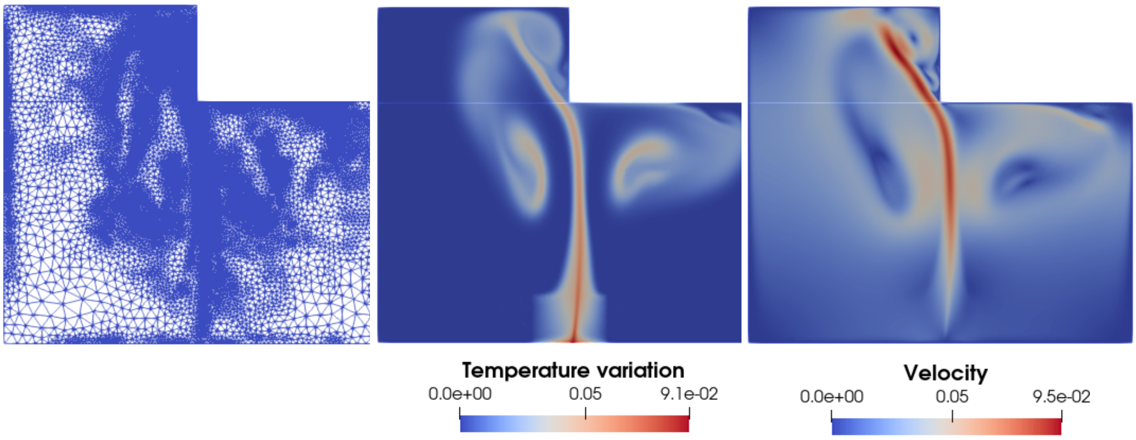
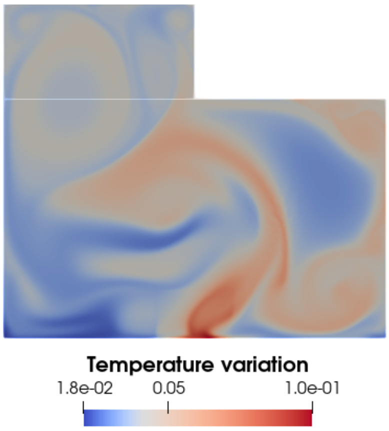


FIGURE 10. Mesh (left), numerical temperature (centre) and numerical velocity (right) at $t = 200$.

FIGURE 11. Numerical temperature (left) and numerical velocity (right) at $t = 600$.



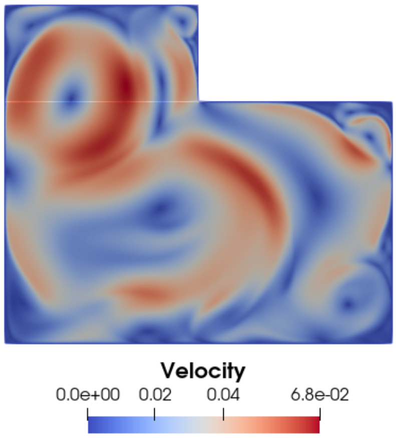


FIGURE 12. Numerical temperature (left) and numerical velocity (right) at $t = 1200$.

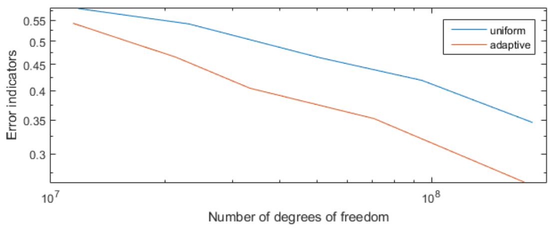


FIGURE 13. Total indicators E_{tot}.

7. Conclusion

In this work, we have derived *a posteriori* error estimates for the time dependent Navier-Stokes system coupled with the convection-diffusion-reaction equation. We started by introducing a variational formulation that we discretized in time using the semi-implicit Euler method and in space using the "P1 bubble / P1 / P1" finite element method. Then we established an *a posteriori* error estimate between the exact solution of our problem and the numerical solution, where we introduced an auxiliary problem to establish the velocity error estimate. In order to show the equivalence between the error and the indicators, we first bounded the velocity and the concentration errors by a sum, over the mesh and over the time steps, of indicators that are local in time and in space; then we bounded each of these indicators using the local error. Finally, based on our theoretical results and using the FreeFem++ software, we developed an adaptive algorithm monitors the mesh and time step refinement and leads to more accurate results than an algorithm based on uniform mesh and time step refinement.

A further step in our work would be to establish the same kind of error estimate for this same problem when the diffusion coefficient α in the convection diffusion reaction equation also depends on the concentration.

Conflict of Interest: The authors declared that they have no conflict of interest.

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