# Random sampling of signals concentrated on compact set in localized reproducing kernel subspace of $L^p(\mathbb{R}^n)$

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#### Abstract

The paper is devoted to studying the stability of random sampling in a localized reproducing kernel space. We show that if the sampling set on  $\Omega$  (compact) discretizes the integral norm of simple functions up to a given error, then the sampling set is stable for the set of functions concentrated on  $\Omega$ . Moreover, we prove with an overwhelming probability that  $\mathcal{O}(\mu(\Omega)(\log \mu(\Omega))^3)$  many random points uniformly distributed over  $\Omega$  yield a stable set of sampling for functions concentrated on  $\Omega$ .

**Keywords:** Random sampling; Reproducing kernel space; Sampling inequality; Covering number; Discretization.

## 1 Introduction

The sampling problem is one of the most active research area in the field of signal processing, image processing, and digital communication. The problem is related to find a discrete sample set such that a function f can be uniquely determined and reconstructed by its discrete sample values. However, this problem is not well defined unless we assume some additional information on the function space. In this paper, we focus on the space of localized reproducing kernel subspace of  $L^p(\mathbb{R}^n) := L^p(\mathbb{R}^n, \mu)$ , where  $\mu$  is a Lebesgue measure on  $\mathbb{R}^n$ .

A countable set  $X = \{x_j \in \mathbb{R}^n : j \in J\}$  is said to be a stable sample or stable set of sampling for the function space  $V \subseteq L^p(\mathbb{R}^n)$  if there exist positive constants A and B such that

$$A\|f\|_{L^{p}(\mathbb{R}^{n})}^{p} \leq \sum_{j \in J} |f(x_{j})|^{p} \leq B\|f\|_{L^{p}(\mathbb{R}^{n})}^{p} \quad \forall f \in V.$$
(1.1)

In case of Paley-Wiener space  $PW_{[a,b]}(\mathbb{R})$ , the stability of a sampling set is completely characterized by Beurling density condition. However, a similar result is not valid for  $PW_S(\mathbb{R}^n)$ , where S is a convex subset of  $\mathbb{R}^n$ , see [26, Section 5.7]. Hence, to overcome these difficulties in studying non-uniform sampling in higher-dimension, we consider a set of random points and check the probability of a stable sampling set. At the same time, the problem of finding a stable random sampling on  $\mathbb{R}^n$  is not feasible in general. Bass and Gröchenig [3] observed that for each random sample identically and uniformly distributed over each cube  $k + [0, 1]^n$ in  $\mathbb{R}^n$ , the sampling inequality (1.1) fails almost surely for Paley-Wiener space. Moreover, for

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smooth function f in  $L^p(\mathbb{R}^n)$ , the sample value  $f(x_j)$  may not assist in sampling inequality for large values of  $x_j$ . To resolve these problems, we consider random sample points are drawn uniformly and identically from a compact set  $\Omega$ , and a class of functions concentrated on  $\Omega$ . Such functions are useful in many application of engineering fields such as information and communication theory [12], signal detection and estimation [19], neuroscience [20], optics [16], and many more.

Random sampling problem is closely related to learning theory [8, 28, 30], compressed sensing [13], and widely applied in information recovery [22]. Trigonometric polynomials are effectively used in practical applications such as computer tomography [1], geophysics [29], image processing [31], and cardiology [32]. Bass and Gröchenig studied random sampling for multivariate trigonometric polynomial [2]; Candés, Romberg, and Tao reconstructed sparse trigonometric polynomial from a random sample set [6]. In the last decades, random sampling studied for Paley-Wiener space [3, 4]; shift-invariant space [15, 33, 35]; continuous function space with bounded derivative [34]; function space with finite rate of innovation [24]; reproducing kernel subspace of  $L^p(\mathbb{R}^n)$  which is an image of an idempotent integral operator [23, 27].

In this paper, we consider localized reproducing kernel subspace V of  $L^p(\mathbb{R}^n)$  (defined in Section 2), which takes into consideration of existing model spaces. A subset  $V^*(\Omega, \delta)$  of V, is a set of  $\delta$ -concentrated functions, define as follows

$$V^*(\Omega, \delta) := \Big\{ f \in V : \int_{\Omega} |f(x)|^p \, dx \ge (1 - \delta) \|f\|_{L^p(\mathbb{R}^n)}^p \Big\}, \quad 0 < \delta < 1$$

We are interested in finding probability bound for random sample  $\{\xi_{\nu} : \nu = 1, \ldots, r\}$  uniformly and identically drawn from  $\Omega$  to be a stable sample set for  $V^*(\Omega, \delta)$  and satisfy the sampling inequality

$$A\|f\|_{L^{p}(\mathbb{R}^{n})}^{p} \leq \frac{1}{r} \sum_{\nu=1}^{r} |f(\xi_{\nu})|^{p} \leq B\|f\|_{L^{p}(\mathbb{R}^{n})}^{p} \quad \forall f \in V^{*}(\Omega, \delta).$$
(1.2)

Of course, the sampling inequality can be achievable with high probability for large-scale sample size. However, the problem seems interesting if one can find a minimal the sample size required to satisfy the sampling inequality (1.2). It was proved for  $\delta$ -concentrated functions on the cube  $C_R := [-R/2, R/2]^n$  that sample size required to be of  $\mathcal{O}(R^{2n})$  in case of Paley-Wiener space [3], shift-invariant space [33], and image of an idempotent integral operator [27]. However, in recent years, it was shown that effective number of sample is indeed of order  $\mathcal{O}(R^n \log R^n)$  for Paley-Wiener space [4] and shift-invariant subspace of  $L^2(\mathbb{R}^n)$  [15]. The recent article [23] by Li et al. proved that for the space of image of an idempotent integral operator,  $\delta$ -concentrated functions on Corkscrew domain  $\Omega$  satisfy (1.2) with sample size of order  $\mathcal{O}(\nu(\Omega) \log \nu(\Omega))$ , where  $\nu$  denotes the metric measure on  $\Omega$ .

The main features of this paper are summarized as follow:

(i) In the recent articles [23, 27], the random sampling problem studied for an image of an idempotent integral operator T, with an additional assumption that the integral kernel K satisfies the regularity condition

$$\lim_{\epsilon \to 0} \left\| \sup_{z \in \mathbb{R}^n} |w_{\epsilon}(K)(\cdot + z, z)| \right\|_{L^1(\mathbb{R}^n)} = 0,$$

where  $w_{\epsilon}(K)(x,y) = \sup_{x',y'\in[-\epsilon,\epsilon]^n} |K(x+x',y+y') - K(x,y)|$ . In this paper, we drop

this assumption and study sampling inequality (1.2) for localizable reproducing kernel space. Further, instead of considering signals concentrated on cube  $C_R = [-R/2, R/2]^n$  in *n*-dimensional Euclidean space, we study the random sampling problem for signals concentrated on a compact subset  $\Omega$  of  $\mathbb{R}^n$ .

(ii) We show that any element in V can be approximated by an element in a finitedimensional subspace of V. As a consequence, we can prove the random sampling inequality (1.2) using the same line of proof in [3, 27, 35]. However, it does not lead us to better sample size estimation. In this paper, we apply the idea of Marcinkiewicz type discretization result introduced in [10] to solve random sampling problem in localizable reproducing kernel space. We show that if a sampling set is "good" discretization to the integral norm on  $\Omega$  for the class of simple functions, then it is a stable sampling set for  $\delta$ -concentrated functions on  $\Omega$ . In addition, we prove that the sampling inequality (1.2) can be achievable with high probability if the sample size of order  $\mathcal{O}(\mu(\Omega)(\log \mu(\Omega))^3)$ .

We pursue the approach of [10, 33] with a mild condition on generators. Note that the result in [33] based on strong decay condition of generators

$$\phi(x) \le \frac{C}{(1+|x|)^m}, \ x \in \mathbb{R}^n,$$

and in [10] relied on the assumption of boundedness of entropy number.

This paper is organized as follows. In Section 2, we give basic definitions, notations and preliminary results. In Section 3, we show that functions in a given compact set are bounded by simple functions. Moreover, under some condition on simple functions, we show that the sampling inequality hold for functions concentrated in a compact set. The main result of this paper is provided in Section 4.

## 2 Preliminaries

In this section, we define the localized reproducing kernel subspace V of  $L^p(\mathbb{R}^n)$ , and discuss some interesting examples.

A sample set  $U = \{u : u \in \mathbb{R}^n\}$  is relatively separated with positive gap  $\beta$  if

$$\beta = \inf_{\substack{u,u' \in U\\ u \neq u'}} \|u - u'\|_{\infty} > 0.$$

The Wiener amalgam space  $W(L^1)(\mathbb{R}^n)$  consist of all functions  $f \in L^{\infty}(\mathbb{R}^n)$  such that

$$||f||_{W(L^1)(\mathbb{R}^n)} := \sum_{k \in \mathbb{Z}^n} \sup_{x \in [0,1]^n} |f(x+k)| < \infty.$$

A family  $\{x_i : i \in I\}$  of elements of a Hilbert space H is called a *frame* for H if there exist constants  $c_1, c_2 > 0$  such that

$$c_1 \|x\|^2 \le \sum_{i \in I} |\langle x, x_i \rangle| \le c_2 \|x\|^2 \quad \forall x \in H.$$

**Definition 2.1.** We say that a closed subspace  $W \subseteq L^2(\mathbb{R}^n)$  is *localizable reproducing kernel Hilbert space*, if there exist:

- (a) a relatively separated set  $\Gamma \subseteq \mathbb{R}^n$  (nodes);
- (b) a function  $\Theta \in W(L^1)(\mathbb{R}^n)$  (envelope);
- (c) a collection of continuous functions  $\{F_{\gamma} : \gamma \in \Gamma\}$  is a frame for W, and satisfy the localization estimate

$$|F_{\gamma}(x)| \le \Theta(x - \gamma), \quad \forall \gamma \in \Gamma.$$
 (2.1)

The coefficient map  $f \mapsto Cf := (\langle f, F_{\gamma} \rangle)_{\gamma \in \Gamma}$  is bounded from  $L^2(\mathbb{R}^n) \to \ell^2(\Gamma)$ , and can be extended to the bounded operator  $C : L^p(\mathbb{R}^n) \to \ell^p(\Gamma)$ . Likewise, the adjoint operator  $C^* : \ell^p(\Gamma) \to L^p(\mathbb{R}^n), \quad 1 \le p < \infty$  defined by  $c \mapsto C^*c := \sum_{\gamma \in \Gamma} c_{\gamma}F_{\gamma}$  is bounded. Using the collection  $\{F_{\gamma} : \gamma \in \Gamma\}$  is a frame, we get the range space

$$V := C^*(\ell^p(\Gamma)) = \left\{ \sum_{\gamma \in \Gamma} c_{\gamma} F_{\gamma} : c \in \ell^p(\Gamma) \right\}$$

is a well-defined closed subspace of  $L^p(\mathbb{R}^n)$ , for  $1 \le p < \infty$ , and there exist constants A, B > 0such that

$$A\|f\|_{L^p(\mathbb{R}^n)}^p \le \sum_{\gamma \in \Gamma} |c_\gamma|^p \le B\|f\|_{L^p(\mathbb{R}^n)}^p,$$

$$(2.2)$$

for every  $f = \sum_{\gamma \in \Gamma} c_{\gamma} F_{\gamma} \in V$ , (see [17]). Moreover, it is easy to show that V is a reproducing kernel Banach space. First, let us recall a basic definition.

**Definition 2.2.** A Banach space  $\Sigma$  of functions on a set X is a *reproducing kernel Banach* space if the point evaluation functional  $f \mapsto f(x)$  is continuous for each  $x \in X$ , i.e., for every  $x \in X$ , there exists  $C_x > 0$ , such that  $|f(x)| \leq C_x ||f||$ , for all  $f \in \Sigma$ .

Lemma 2.3. The space V is a reproducing kernel Banach space.

**Proof.** Let  $x \in \mathbb{R}^n$  be fixed and  $f \in V$  be arbitrary.

$$\begin{split} f(x) &= \sum_{\gamma \in \Gamma} c_{\gamma} F_{\gamma}(x), \\ |f(x)| &\leq \left(\sum_{\gamma \in \Gamma} |c_{\gamma}|^{p}\right)^{\frac{1}{p}} \left(\sum_{\gamma \in \Gamma} |F_{\gamma}(x)|^{p'}\right)^{\frac{1}{p'}} \\ &\leq B^{\frac{1}{p}} \|f\|_{L^{p}(\mathbb{R}^{n})} \left(\sum_{\gamma \in \Gamma} |\Theta(x-\gamma)|^{p'}\right)^{\frac{1}{p'}} \\ &\leq B^{\frac{1}{p}} \|f\|_{L^{p}(\mathbb{R}^{n})} \left(\sum_{\gamma \in \Gamma} |\Theta(x-\gamma)|\right)^{\frac{1}{p'}} \\ &= B^{\frac{1}{p}} \|f\|_{L^{p}(\mathbb{R}^{n})} \left(\sum_{k \in \mathbb{Z}^{n}} \sum_{\gamma \in \Gamma \cap [k,k+1]^{n}} |\Theta(x-\gamma)|\right)^{\frac{1}{p'}} \\ &\leq B^{\frac{1}{p}} \|f\|_{L^{p}(\mathbb{R}^{n})} \left(\sum_{k \in \mathbb{Z}^{n}} N(\Gamma) \sup_{x \in [0,1]^{n}} |\Theta(x-k)|\right)^{\frac{1}{p'}} \\ &|f(x)| \leq B^{\frac{1}{p}} N(\Gamma)^{\frac{1}{p'}} \|\Theta\|_{W(L^{1})(\mathbb{R}^{n})}^{\frac{1}{p'}} \|f\|_{L^{p}(\mathbb{R}^{n})}. \end{split}$$

Therefore, the point evaluation functional is continuous for each  $x \in \mathbb{R}^n$ .

From the proof of the above lemma, we choose a constant D > 1 such that

$$\|f\|_{L^{\infty}(\mathbb{R}^n)} \le D\|f\|_{L^p(\mathbb{R}^n)}, \quad \forall f \in V.$$

$$(2.3)$$

**Example 2.1.** In the following, we give some well known examples of localizable reproducing kernel spaces.

1. Let  $\phi \in W(L^1)(\mathbb{R}^n)$  be such that for some positive constants  $A_1$  and  $B_1$ ,

$$A_1 \le \sum_{k \in \mathbb{Z}^n} |\hat{\phi}(\xi + k)|^2 \le B_1.$$

Then  $\{\phi(\cdot-k): k \in \mathbb{Z}^n\}$  is a frame for the shift-invariant space  $V(\phi) = \left\{\sum_{k \in \mathbb{Z}^n} c_k \phi(\cdot-k): (c_k) \in \ell^2(\mathbb{Z}^n)\right\}$ , see [7, Theorem 9.2.5]. In this case, we have  $F_{\gamma} = \phi(\cdot-\gamma), \ \gamma \in \Gamma = \mathbb{Z}^n$ .

2. Let T be an idempotent integral operator  $(T^2 = T)$  on  $L^p(\mathbb{R}^n)$  with the integral kernel K satisfies the off-diagonal decay condition

$$||K||_{S} := \max\left(\sup_{x \in \mathbb{R}^{n}} ||K(x, \cdot)||_{L^{1}(\mathbb{R}^{n})}, \sup_{y \in \mathbb{R}^{n}} ||K(\cdot, y)||_{L^{1}(\mathbb{R}^{n})}\right) < \infty,$$

and the regularity condition

$$\lim_{\epsilon \to 0} \|w_{\epsilon}(K)\|_{S} = 0, \tag{2.4}$$

where  $w_{\epsilon}(K)(x,y) = \sup_{x',y'\in[-\epsilon,\epsilon]^n} |K(x+x',y+y')-K(x,y)|$ . Then the image space Vof T is a reproducing kernel subspace of  $L^p(\mathbb{R}^n)$ , and there exist relatively separated set  $\Gamma \subseteq \mathbb{R}^n$  and the collection  $\{\phi_{\gamma} : \gamma \in \Gamma\}$  forms p-frame for V. Apart from that there exists  $h \in W(L^1)(\mathbb{R}^n)$  such that  $|\phi_{\gamma}(x)| \leq h(x-\gamma)$ , see [25].

So far, the sampling problem had been studied for the image of an idempotent integral operator in [25, 23, 27] and the regularity condition (2.4) was the key assumption. However, in practice the condition is not feasible. In the following, we give examples of localizable reproducing kernel space, which is the image of an idempotent integral operator but the regularity condition is either hard to verify or does not satisfy.

**Example 2.2.** Let  $X = \{x_k : k \in \mathbb{Z}\}$  be a relatively separated set of strictly increasing sequence in  $\mathbb{R}$  and  $\phi \in W(L^1)(\mathbb{R})$ . A quasi-shift invariant space is defined by

$$V_X(\phi) = \Big\{ f = \sum_{k \in \mathbb{Z}} c_k \phi(\cdot - x_k) : c = (c_k) \in \ell^2(\mathbb{Z}) \Big\}.$$

For more literature on quasi-shift invariant space, we refer the interested reader to [14, 21]. If the collection  $\{\phi(\cdot - x_k) : k \in \mathbb{Z}\}$  is a frame for  $V_X(\phi)$ , then the space  $V_X(\phi)$  is a localizable reproducing kernel Hilbert space. Furthermore,  $V_X(\phi)$  can be written as the image of an idempotent integral operator with the integral kernel

$$K(x,y) = \sum_{k \in \mathbb{Z}} \overline{\phi(x-x_k)} \widetilde{\phi_k}(y),$$

where  $\{\tilde{\phi}_k : k \in \mathbb{Z}\}$  is the canonical dual frame of  $\{\phi(\cdot - x_k) : k \in \mathbb{Z}\}$ . In case of irregular sample set X, there is no constructive method to compute  $\tilde{\phi}_k$ . As a consequence, the condition on K is not easily verifiable. Even for the simplest case, the kernel assumption (2.4) was not satisfied. For example, let  $\phi : \mathbb{R} \to \mathbb{R}$  be defined by

$$\phi(x) = \begin{cases} 1, & \text{if } 0 \le x \le \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

Then  $\phi \in W(L^1)(\mathbb{R})$  and  $\|\phi\|_{W(L^1)(\mathbb{R})} = 1$ . The collection  $\{\sqrt{2}\phi(\cdot - k) : k \in \mathbb{Z}\}$  is an orthonormal basis of the shift-invariant space  $V(\phi) \subset L^2(\mathbb{R})$  and the reproducing kernel of  $V(\phi)$  is defined by  $K(x,y) = \sum_{k \in \mathbb{Z}} 2\phi(x-k)\phi(y-k)$ .

For  $\epsilon < \frac{1}{2}$ ,

$$\sup_{x'\in[-\epsilon,\epsilon]} |K(x+x',y) - K(x,y)| = \begin{cases} 2\phi(y-k), & \text{if } x \in (k-\epsilon,k+\epsilon) \cup (k+\frac{1}{2}-\epsilon,k+\frac{1}{2}+\epsilon), \\ 0, & \text{otherwise.} \end{cases}$$

Hence,

$$\left\|\sup_{x'\in[-\epsilon,\epsilon]}|K(x+x',\cdot)-K(x,\cdot)|\right\|_{L^1(\mathbb{R})} = \begin{cases} 1, & \text{if } x\in(k-\epsilon,k+\epsilon)\cup(k+\frac{1}{2}-\epsilon,k+\frac{1}{2}+\epsilon), \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$1 = \sup_{x \in \mathbb{R}} \left\| \sup_{x' \in [-\epsilon,\epsilon]} |K(x+x',\cdot) - K(x,\cdot)| \right\|_{L^1(\mathbb{R})}$$
  
$$\leq \max\left( \sup_{x \in \mathbb{R}} \|w_\epsilon(K)(x,\cdot)\|_{L^1(\mathbb{R})}, \sup_{y \in \mathbb{R}} \|w_\epsilon(K)(\cdot,y)\|_{L^1(\mathbb{R})} \right) = \|w_\epsilon(K)\|_S.$$

We collect our assumption on compact domain  $\Omega$  and the function  $\Theta$  in the following list:

**Assumption 0.** Without loss of generality assume that  $\mu(\Omega) \geq 1$ , and denote d as the number of unit cube of the form  $m + [0,1]^n$  covers the boundary of  $\Omega$ , where  $m \in \mathbb{Z}^n$ . Let the number of  $\Gamma$  in  $\Omega$  is bounded by  $C(\Gamma)\mu(\Omega)$ , where  $C(\Gamma)$  is some positive constant.

Assumption 1. For every  $C_N = [-N/2, N/2]^n$ ,

$$\sum_{k \in \mathbb{Z}^n \setminus C_N} \sup_{x \in [0,1]^n} |\Theta(x-k)| < \frac{C}{N^{n\alpha}},\tag{2.5}$$

where C are positive constant, and  $\alpha \geq \frac{p}{p-1}$  for  $1 and <math>\alpha \geq 1$  for p = 1.

In the following, we provide an example of localizable reproducing kernel space satisfying Assumption 1.

**Example 2.3.** Let  $\phi : \mathbb{C}^n \to \mathbb{R}$  be a plurisubharmonic function and assume that there exist m, M > 0 such that

$$im\partial\bar{\partial}|z|^2 \le i\partial\bar{\partial}\phi \le iM\partial\bar{\partial}|z|^2.$$
(2.6)

Let  $A^2_{\phi}$  be the space of entire function on  $\mathbb{C}^n$  equipped with the norm

$$||f||_{\phi,2}^2 := \int_{\mathbb{C}^n} |f(z)|^2 e^{-2\phi(z)} \, dz.$$

Then  $A_{\phi}^2$  is a reproducing kernel Hilbert space and the reproducing kernel is denoted by  $K_{\phi}(z, w)$ . Moreover, the kernel  $K_{\phi}$  satisfy the following off-diagonal decay estimate [11]

$$|K_{\phi}(z,w)|e^{-\phi(z)-\phi(w)} \le Ce^{-c|z-w|} \quad \forall z,w \in \mathbb{C}^n$$

We define the weighted Fock space  $A_{\phi}^2$  as

$$V_{\phi}^{2} = \{ f = g e^{-\phi} : g \in A_{\phi}^{2} \}.$$

The space  $V_{\phi}^2$  is a localizable reproducing kernel Hilbert space in  $L^2(\mathbb{R}^{2n})$ , see [18]. If  $\beta \in (0, \sqrt{2/n})$ , then  $\Gamma = \beta \mathbb{Z}^{2n}$  is a relatively separated set and there exists  $\{F_{\gamma} : \gamma \in \Gamma\}$  frame for  $V_{\phi}^2$  with frame bounds  $A = \frac{1}{4\beta^{2n}}$  and  $B = \frac{3}{2}$ . In addition, the frame  $\{F_{\gamma} : \gamma \in \Gamma\}$  satisfy the localized estimate (2.1), where  $\Theta(z) = C_{\beta}e^{-c|z|}$ . It is easy to verify that  $\Theta \in W(L^1)(\mathbb{R}^{2n})$  and satisfy Assumption 1.

In the following lemma, we show that for any f in V there exists a function in finitedimensional subspace of V which is close to f. In order to proceed the lemma, let M be a compact set and we define finite-dimensional subspace  $V_M$  of V as

$$V_M = \Big\{ \sum_{\gamma \in \Gamma \cap M} c_{\gamma} F_{\gamma} : c_{\gamma} \in \mathbb{R} \Big\}.$$

**Lemma 2.4.** For a given  $\epsilon > 0$  and  $f \in V$ , there exist compact set M (depending on  $\epsilon$  and  $\Omega$ ) and  $\tilde{f} \in V_M$  such that

$$\|f-f\|_{L^p(\Omega)} < \epsilon \|f\|_{L^p(\mathbb{R}^n)}.$$

**Proof.** Let  $f \in V$ . Then there exists  $(c_{\gamma}) \in \ell^p(\Gamma)$  such that  $f = \sum_{\gamma \in \Gamma} c_{\gamma} F_{\gamma}$ . We consider  $\tilde{f} = \sum_{\gamma \in \Gamma \cap M} c_{\gamma} F_{\gamma}$ , where M is a compact subset of  $\mathbb{R}^n$  (chosen later).

Now, for  $x \in \Omega$ ,

$$\begin{aligned} f(x) - \tilde{f}(x)| &\leq \sum_{\gamma \in \Gamma \setminus M} |c_{\gamma}| |F_{\gamma}(x)| \\ &\leq \Big(\sum_{\gamma \in \Gamma \setminus M} |c_{\gamma}|^{p}\Big)^{\frac{1}{p}} \Big(\sum_{\gamma \in \Gamma \setminus C_{N}} |F_{\gamma}(x)|^{p'}\Big)^{\frac{1}{p'}} \\ &\leq B^{\frac{1}{p}} \|f\|_{L^{p}(\mathbb{R}^{n})} \Big(\sum_{\gamma \in \Gamma \setminus M} |\Theta(x - \gamma)|^{p'}\Big)^{\frac{1}{p'}} \\ &\leq B^{\frac{1}{p}} \Big(\sum_{\gamma \in \Gamma \setminus M} |\Theta(x - \gamma)|\Big)^{\frac{1}{p'}} \|f\|_{L^{p}(\mathbb{R}^{n})}, \end{aligned}$$

Since  $\Theta \in W(L^1)(\mathbb{R}^n)$ , for each  $\epsilon > 0$  there exists  $N = \left(\frac{CN(\Gamma)B^{\frac{p'}{p}}}{\epsilon^{p'}}\right)^{\frac{1}{n\alpha}} \mu(\Omega)^{\frac{1}{n}}$  such that

$$\sum_{k \in \mathbb{Z}^n \setminus C_N} \sup_{x \in [0,1]^n} |\Theta(x-k)| \le \frac{C}{N^{n\alpha}} < \frac{\epsilon^{p'}}{N(\Gamma)\mu(\Omega)^{p'} B^{\frac{p'}{p}}}$$

For  $x \in \partial \Omega$ , there exists  $m \in \mathbb{Z}^n$  such that  $x \in m + [0, 1]^n$ . Consider

$$\sum_{\gamma \in \Gamma \setminus \{m+C_N\}} |\Theta(x-\gamma)| \le N(\Gamma) \sum_{k \in \mathbb{Z}^n \setminus C_N} \sup_{y \in [0,1]^n} |\Theta(y-k)| < \frac{\epsilon^{p'}}{\mu(\Omega)^{p'} B^{\frac{p'}{p}}}.$$

The same is true for every  $x \in \partial \Omega \cap \{m_i + [0,1]^n\}$ , where  $m_i \in \mathbb{Z}^n$  and the collection  $\{m_i + [0,1]^n : i = 1, \ldots, d\}$  cover  $\partial \Omega$ . Choose  $M = \left(\bigcup_{i=1}^d \{m_i + C_N\}\right) \cup \Omega$ . Then for each  $x \in \Omega$ ,

$$\sum_{\gamma \in \Gamma \setminus M} |\Theta(x - \gamma)| < \frac{\epsilon^{p'}}{\mu(\Omega)^{p'} B^{\frac{p'}{p}}},$$

and hence

$$|f(x) - \tilde{f}(x)| < \frac{\epsilon}{\mu(\Omega)} ||f||_{L^p(\mathbb{R}^n)}, \quad \forall x \in \Omega.$$
(2.7)

This completes the proof.

*Remark* 2.5. The space  $V_M$  is generated by  $\{F_{\gamma} : \gamma \in \Gamma \cap M\}$  and the dimension  $d_M$  of  $V_M$  is at most the number of  $\gamma$  in M. Therefore,

$$d_{M} \leq |\Gamma \cap M| \leq \left(C(\Gamma)\mu(\Omega) + dN(\Gamma)N^{n}\right)$$
$$\leq \mu(\Omega) \left(dN(\Gamma)\left(\frac{CN(\Gamma)B^{\frac{p'}{p}}}{\epsilon^{p'}}\right)^{\frac{1}{\alpha}} + C(\Gamma)\right)$$
$$= C(\epsilon)\mu(\Omega),$$
where  $C(\epsilon) = \left(dN(\Gamma)\left(\frac{CN(\Gamma)B^{\frac{p'}{p}}}{\epsilon^{p'}}\right)^{\frac{1}{\alpha}} + C(\Gamma)\right)$  and  $N(\Gamma) = \sup_{k \in \mathbb{Z}^{n}} |\Gamma \cap (k + [0, 1]^{n})|.$ 

Remark 2.6. In the proof of the above Lemma, if we do not choose M carefully, the bound of  $d_M$  may not be effective estimation. For example, if d is the number of unit cubes that cover  $\Omega$ , then  $d > \mu(\Omega)$ . On the other hand, if we consider a cube Q in  $\mathbb{R}^n$  which contains  $\Omega$ , then  $\mu(Q) \ge \mu(\Omega)$ . In both cases, we get a larger bound for  $d_M$  as  $C\mu(\Omega)^2$  for some C > 0.

### **3** Discretization of functions

Let  $(X, \|\cdot\|)$  be a Banach space and B(f, r) denotes a ball of radius r center at f. For a compact set A and a positive number  $\epsilon$ , we define the covering number  $N_{\epsilon}(A)$  as follows

$$N_{\epsilon}(A) := N_{\epsilon}(A, X) := \min\left\{k : \exists f_1, f_2, \dots, f_k \in A, A \subseteq \bigcup_{j=1}^{\kappa} B(f_j, \epsilon)\right\}.$$

The corresponding minimal  $\epsilon$ -net is denoted by  $\mathcal{N}_{\epsilon}(A, X)$ , and  $N_{\epsilon}(A, X) = |\mathcal{N}_{\epsilon}(A, X)|$ . The following lemma is a well-known estimation of covering number of a closed ball in finite-dimensional space.

**Lemma 3.1** ([9]). Let X be a Banach space of dimension s. Then the number of open balls of radius  $\omega$  to cover  $\overline{B(0;r)}$  is bounded by  $\left(\frac{2r}{\omega}+1\right)^s$ .

Let M be a compact subset of  $\mathbb{R}^n$ . We consider the set

$$V_{M,\Omega} = \left\{ f \in V_M : \|f\|_{L^p(\mathbb{R}^n)}^p = \mu(\Omega) \right\}.$$

From (2.3), we see that  $V_{M,\Omega}$  is a compact subset of  $L^{\infty}(\mathbb{R}^n)$  and bounded by  $D\mu(\Omega)^{\frac{1}{p}}$ . Lemma 3.1 gives a bound of covering number of  $V_{M,\Omega} \subseteq L^{\infty}(\mathbb{R}^n)$ , i.e.

$$N_{\epsilon}(V_{M,\Omega}, L^{\infty}(\mathbb{R}^n)) \le \left(1 + \frac{2D\mu(\Omega)^{\frac{1}{p}}}{\epsilon}\right)^{d_M} \le \left(\frac{4D\mu(\Omega)^{\frac{1}{p}}}{\epsilon}\right)^{d_M}.$$
(3.1)

In the following, we now discuss on discretization of functions in a compact set  $V_{M,\Omega}$ . Let  $a \in (0, \frac{1}{2}]$  be a fixed small number (chosen later) and denote the set  $\mathcal{A}_j = \mathcal{N}_{a(1+a)^j}(V_{M,\Omega}, L^{\infty}(\mathbb{R}^n))$  for  $j \in \mathbb{Z}$ . Let  $j_0 \in \mathbb{Z}$  be a fixed integer (specified later), and for each  $j(\geq j_0) \in \mathbb{Z}$ , consider the map  $A_j : V_{M,\Omega} \to \mathcal{A}_j$  such that  $A_j(f)$  is a function in  $\mathcal{A}_j$  closest to f with respect to  $\| \cdot \|_{L^{\infty}(\mathbb{R}^n)}$ . Then

$$\|f - A_j(f)\|_{L^{\infty}(\mathbb{R}^n)} \le a(1+a)^j.$$
(3.2)

If  $f \in V_{M,\Omega}$  and  $j \in \mathbb{Z} \cap (j_0, \infty)$ , we construct a set of collection of points as follow.

$$U_j(f) := \left\{ x \in \mathbb{R}^n : |A_j(f)(x)| \ge (1+a)^j \right\},$$
  
$$D_j(f) := U_j(f) \setminus \bigcup_{k>j} U_k(f), \qquad D_{j_0}(f) := \mathbb{R}^n \setminus \bigcup_{k>j_0} U_k(f)$$

For each  $f \in V_{M,\Omega}$ , we define piecewise constant function h(f) by

$$h(f) = \sum_{j>j_0} (1+a)^j \chi_{D_j(f)}$$

where  $\chi_E$  is the characteristic function on E.

The following lemma gives some properties of the above *h*-mapping. We show that the absolute value of the function f in  $V_{M,\Omega}$  is bounded above and below by constants times of h(f).

**Lemma 3.2.** For each  $f \in V_{M,\Omega}$ ,

$$C_1(a)h(f)(x) \le |f(x)| \le C_2(a)h(f)(x) \quad \forall x \in \mathbb{R}^n \setminus D_{j_0}(f),$$
(3.3)

and

$$|f(x)| \le C_2(a)(1+a)^{j_0} \quad \forall x \in D_{j_0}(f),$$
(3.4)

where  $C_1(a) = (1-a)$  and  $C_2(a) = (1+a)^2$ .

**Proof.** Let  $x \in D_j(f)$  with  $j > j_0$ . Then  $x \in U_j(f)$  and  $x \notin U_k(f)$  for k > j. Using (3.2) and from the definition of  $U_j(f)$ , we get

$$|f(x)| \ge |A_j(f)(x)| - a(1+a)^j \ge (1+a)^j - a(1+a)^j = C_1(a)(1+a)^j$$

and

$$|f(x)| \le |A_{j+1}(f)(x)| + a(1+a)^{j+1} \le (1+a)^{j+1} + a(1+a)^{j+1} = C_2(a)(1+a)^j,$$

where  $C_1(a) = (1-a)$  and  $C_2(a) = (1+a)^2$ . Therefore, for all  $x \in \mathbb{R}^n \setminus D_{j_0}(f) = \bigcup_{j>j_0} D_j(f)$ we have,

$$C_1(a)h(f)(x) \le |f(x)| \le C_2(a)h(f)(x).$$

Similarly, the other inequality (3.4) can be derived.

*Remark* 3.3. For all  $a \in (0, \frac{1}{2}], C_1(a) \le 1 \le C_2(a)$ , and

$$\lim_{a \to 0} C_1(a) = \lim_{a \to 0} C_2(a) = 1.$$

For our results, we choose  $a \in (0, \frac{1}{2}]$  such that  $\left(\frac{C_2(a)}{C_1(a)}\right)^p \leq \frac{5}{4}$ .

We list some notations which will be used in the rest of the following sections.

Symbol	Remark
X  = The number of element in $X$	X is a finite set.
$N(\Gamma) = \sup_{k \in \mathbb{Z}^n}  \Gamma \cap (k + [0, 1]^n) $	Intuitively, the maximum number of elements of $\Gamma$ in a unit cube of $\mathbb{R}^n$ .
$p' = \frac{p}{p-1}$	$p' = \infty$ for $p = 1$ .
$C_1(a) = (1-a)$	with a satisfying $\left(\frac{C_2(a)}{a}\right)^p < \frac{5}{a}$
$C_2(a) = (1+a)^2$	$(C_1(a)) - 4$

In the following lemma, we give a condition on *h*-mapping such that for functions in  $V_{M,\Omega}$  the sample set discretize the integral norm on  $\Omega$ . For a sample set  $\xi = \{\xi_{\nu}\}_{\nu=1}^{r}$  and a function  $f \in V$ , denote

$$S(f,\xi) := (f(\xi_1), \dots, f(\xi_r)) \in \mathbb{R}^r, \quad \|S(f,\xi)\|_p^p := \frac{1}{r} \sum_{\nu=1}^r |f(\xi_\nu)|^p.$$

**Lemma 3.4.** Let  $\xi$  be a sample set in  $\Omega$  and  $f \in V_{M,\Omega}$ . Assume that the function h(f) satisfy the following inequality

$$\frac{1}{\mu(\Omega)} \|h(f)\|_{L^{p}(\Omega)}^{p} - \sigma \le \|S(h(f),\xi)\|_{p}^{p} \le \frac{1}{\mu(\Omega)} \|h(f)\|_{L^{p}(\Omega)}^{p} + \sigma,$$
(3.5)

for some  $\sigma > 0$ . Then

$$C_{1}(a)^{p} \Big( \frac{C_{2}(a)^{-p}}{\mu(\Omega)} \|f\|_{L^{p}(\Omega)}^{p} - (1+a)^{pj_{0}} - \sigma \Big) \\ \leq \|S(f,\xi)\|_{p}^{p} \leq C_{2}(a)^{p} \Big( \frac{C_{1}(a)^{-p}}{\mu(\Omega)} \|f\|_{L^{p}(\Omega)}^{p} + (1+a)^{pj_{0}} + \sigma \Big).$$
(3.6)

**Proof.** For points on the set  $D_{j_0}(f)$ , the inequality (3.4) implies

$$\int_{D_{j_0}(f)\cap\Omega} |f(x)|^p dx \le C_2(a)^p (1+a)^{pj_0} \mu(\Omega)$$

and

$$\frac{1}{r} \sum_{\nu:\xi_{\nu} \in D_{j_0}(f)} |f(\xi_{\nu})|^p \le C_2(a)^p (1+a)^{pj_0}.$$

By (3.3) we have

$$\begin{split} \|S(f,\xi)\|_{p}^{p} &\leq C_{2}(a)^{p}(1+a)^{pj_{0}} + C_{2}(a)^{p}\|S(h(f),\xi)\|_{p}^{p} \\ &\leq C_{2}(a)^{p}(1+a)^{pj_{0}} + \frac{C_{2}(a)^{p}}{\mu(\Omega)}\|h(f)\|_{L^{p}(\Omega)}^{p} + C_{2}(a)^{p}\sigma \\ &\leq C_{2}(a)^{p} \Big(\frac{C_{1}(a)^{-p}}{\mu(\Omega)}\|f\|_{L^{p}(\Omega)}^{p} + (1+a)^{pj_{0}} + \sigma\Big). \end{split}$$

On the other hand, we have

$$\begin{split} \|S(f,\xi)\|_{p}^{p} &\geq C_{1}(a)^{p} \|S(h(f),\xi)\|_{p}^{p} \\ &\geq C_{1}(a)^{p} \left(\frac{1}{\mu(\Omega)} \|h(f)\|_{L^{p}(\Omega)}^{p} - \sigma\right) \\ &\geq C_{1}(a)^{p} \left(\frac{C_{2}(a)^{-p}}{\mu(\Omega)} \int_{\Omega \setminus D_{j_{0}}(f)} |f(x)|^{p} dx - \sigma\right) \\ &\geq C_{1}(a)^{p} \left(\frac{C_{2}(a)^{-p}}{\mu(\Omega)} \|f\|_{L^{p}(\Omega)}^{p} - \frac{C_{2}(a)^{-p}}{\mu(\Omega)} \int_{D_{j_{0}}(f) \cap \Omega} |f(x)|^{p} dx - \sigma\right) \\ &\geq C_{1}(a)^{p} \left(\frac{C_{2}(a)^{-p}}{\mu(\Omega)} \|f\|_{L^{p}(\Omega)}^{p} - (1 + a)^{pj_{0}} - \sigma\right). \end{split}$$

The lemma implies that the discretization of the integral norm of  $f \in V_M$  and corresponding simple function h(f) are related. We aim to find the probability for which the random sample set  $\xi$  satisfy the condition (3.5) on h(f) for all  $f \in V_{M,\Omega}$ .

## 4 Random Sampling

In this section we discuss the main result of this paper. In order to derive the probabilistic estimates, we make use of the following lemma [5, Lemma 2.1].

**Lemma 4.1.** Let  $\{g_{\nu}\}_{\nu=1}^{r}$  be independent random variables with zero mean on probability space  $(X, \rho)$  such that

$$||g_{\nu}||_{L^{1}(X,\rho)} \leq 2, ||g_{\nu}||_{L^{\infty}(X,\rho)} \leq L, \quad 1 \leq \nu \leq r.$$

Then for any  $\eta \in (0,1)$  we have the following probability bound

$$\mathbb{P}\left\{\left|\sum_{\nu=1}^{r} g_{\nu}\right| \ge r\eta\right\} < 2\exp\left(-\frac{r\eta^2}{8L}\right).$$

It is easy to verify that the above lemma implies the following result.

**Corollary 4.2.** [5, Corollary 2.2] Let  $\xi = \{\xi_{\nu}\}_{\nu=1}^{r}$  be a random points drawn from probability space  $(X, \rho)$  and  $\{\mathcal{F}_j\}_{j \in G}$  be a finite collection of finite set of functions in  $L^1(X, \rho)$ . Assume that for each  $j \in G$  and  $f \in \mathcal{F}_j$ , we have

$$||f||_{L^1(X,\rho)} \le 1, \quad ||f||_{L^\infty(X,\rho)} \le L_j$$

Then for each  $j \in G$ , for any  $\eta_j \in (0,1)$  and for all  $f \in \mathcal{F}_j$ , we have

$$\left| \|f\|_{L^{1}(X,\rho)} - \frac{1}{r} \sum_{\nu=1}^{r} |f(\xi_{\nu})| \right| \leq \eta_{j},$$

with probability at least  $1 - 2\sum_{j \in G} |\mathcal{F}_j| \exp\left(-\frac{r\eta_j^2}{8L_j}\right)$ .

Let  $p \in [1, \infty)$ . For  $j > j_0$ , define

$$\mathcal{F}_j := \Big\{ \frac{4}{5} (1+a)^{pj} \chi_{D_j(f)} : f \in V_{M,\Omega} \Big\}.$$

By definition of  $D_j(f)$ , we consider those  $j > j_0$  such that  $C_1(a)(1+a)^j \leq D\mu(\Omega)^{\frac{1}{p}}$ . Otherwise  $D_j(f)$  is empty. Choose  $J \in \mathbb{Z}$  such that  $C_1(a)(1+a)^J \leq D\mu(\Omega)^{\frac{1}{p}}$ , i.e.,  $J \leq D_j(\Omega)^{\frac{1}{p}}$  $\frac{\log(D\mu(\Omega)^{\frac{1}{p}}/C_1(a))}{\log(1+a)}.$ 

Now, consider the index set  $G = [j_0, J] \cap \mathbb{Z}$ . Then, we apply Corollary 4.2 for the collection of sets  $\{\mathcal{F}_i\}_{i \in G}$  and prove the following theorem.

**Theorem 4.3.** Let  $\xi = \{\xi_{\nu}\}_{\nu=1}^{r}$  be a sequence of *i.i.d.* random points that are uniformly distributed over the compact set  $\Omega$ . If the sample size satisfy

$$r \ge \frac{10}{\sigma^2} d_M |G|^2,$$

then for every  $f \in V_M$  the following inequality

$$\frac{C_1(a)^p}{\mu(\Omega)} \Big( C_2(a)^{-p} \|f\|_{L^p(\Omega)}^p - (1+a)^{pj_0} \|f\|_{L^p(\mathbb{R}^n)}^p - \sigma \|f\|_{L^p(\mathbb{R}^n)}^p \Big) \le \|S(f,\xi)\|_p^p \\
\le \frac{C_2(a)^p}{\mu(\Omega)} \Big( C_1(a)^{-p} \|f\|_{L^p(\Omega)}^p + (1+a)^{pj_0} \|f\|_{L^p(\mathbb{R}^n)}^p + \sigma \|f\|_{L^p(\mathbb{R}^n)}^p \Big)$$

holds with probability at least  $1 - 2A_1|G| \exp\left(-\left(\frac{r\sigma^2}{|G|^2} - d_M\right)(1+a)^{-pj_0}\right)$ , where  $A_1 = C_1$  $\mathcal{O}(\exp(d_M \log \mu(\Omega))).$ 

**Proof.** Let  $\xi = \{\xi_{\nu}\}_{\nu=1}^{r}$  be the random points uniformly, identically and independently distributed over  $\Omega$  with probability measure  $d\rho = \frac{1}{\mu(\Omega)} dx$ . Hence for each  $j \in G$  and  $\phi_j =$  $\frac{4}{5}(1+a)^{pj}\chi_{D_{i}(f)}$ , we have

$$\|\phi_j\|_{L^{\infty}(\Omega,\rho)} \le \frac{4}{5}(1+a)^{pj} := L_j,$$

and

$$\begin{aligned} \|\phi_j\|_{L^1(\Omega,\rho)} &= \frac{1}{\mu(\Omega)} \int_{\Omega} \frac{4}{5} (1+a)^{pj} \chi_{D_j(f)} \, dx \\ &= \frac{4}{5\mu(\Omega)} \|(1+a)^{pj} \chi_{D_j(f)}\|_{L^1(\Omega)} \\ &= \frac{4C_1(a)^{-p}}{5\mu(\Omega)} \|f\|_{L^p(\Omega)}^p \\ &\leq C_2(a)^{-p} \leq 1. \end{aligned}$$

Denote  $\eta_j = \frac{4\sigma}{5|G|}$ , then from Corollary 4.2 we have

$$\left| \frac{1}{r} \sum_{\nu=1}^{r} |\phi_j(\xi_{\nu})| - \frac{1}{\mu(\Omega)} \|\phi_j\|_{L^1(\Omega)} \right| \le \eta_j$$
$$\frac{1}{\mu(\Omega)} \|\phi_j\|_{L^1(\Omega)} - \eta_j \le \frac{1}{r} \sum_{\nu=1}^{r} |\phi_j(\xi_{\nu})| \le \frac{1}{\mu(\Omega)} \|\phi_j\|_{L^1(\Omega)} + \eta_j, \tag{4.1}$$

holds for each  $j \in G$  and  $\phi_j \in \mathcal{F}_j$  with minimum probability of  $1 - 2\sum_{j \in G} |\mathcal{F}_j| \exp\left(-\frac{r\eta_j^2}{8L_j}\right)$ .

Since the sets  $D_j(f)$  are pairwise disjoint, we get

$$\sum_{j \in G} \frac{1}{r} \sum_{\nu=1}^{r} |\phi_j(\xi_\nu)| = \frac{4}{5} \frac{1}{r} \sum_{\nu=1}^{r} \sum_{j \in G} (1+a)^{pj} \chi_{D_j(f)}(\xi_\nu)$$
$$= \frac{4}{5} \|S(h(f),\xi)\|_p^p,$$

and

$$\sum_{j \in G} \|\phi_j\|_{L^1(C_R)} = \frac{4}{5} \int_{\Omega} \sum_{j \in G} (1+a)^{pj} \chi_{D_j(f)}(x) \, dx$$
$$= \frac{4}{5} \|h(f)\|_{L^p(\Omega)}^p.$$

Hence, for all  $f \in V_{M,\Omega}$ , (4.1) implies

$$\frac{1}{\mu(\Omega)} \sum_{j \in G} \|\phi_j\|_{L^1(\Omega)} - \sum_{j \in G} \eta_j \le \sum_{j \in G} \frac{1}{r} \sum_{\nu=1}^r |\phi_j(\xi_\nu)| \le \frac{1}{\mu(\Omega)} \sum_{j \in G} \|\phi_j\|_{L^1(\Omega)} + \sum_{j \in G} \eta_j$$
$$\frac{4}{5\mu(\Omega)} \|h(f)\|_{L^p(\Omega)}^p - \frac{4\sigma}{5} \le \frac{4}{5} \|S(h(f),\xi)\|_p^p \le \frac{4}{5\mu(\Omega)} \|h(f)\|_{L^p(\Omega)}^p + \frac{4\sigma}{5}$$
$$\frac{1}{\mu(\Omega)} \|h(f)\|_{L^p(\Omega)}^p - \sigma \le \|S(h(f),\xi)\|_p^p \le \frac{1}{\mu(\Omega)} \|h(f)\|_{L^p(\Omega)}^p + \sigma.$$

Therefore, Lemma 3.4 implies that (3.6) hold for every  $f \in V_{M,\Omega}$  with probability at least  $1 - 2 \sum_{j \in G} |\mathcal{F}_j| \exp\left(-\frac{r\eta_j^2}{8L_j}\right).$ 

In order to calculate the bound of  $\sum_{j \in G} |\mathcal{F}_j| \exp\left(-\frac{r\eta_j^2}{8L_j}\right)$ , we first evaluate the bound of  $|\mathcal{F}_j|$ . By the definition of  $D_j(f)$  and construction of  $\mathcal{F}_j$  we conclude that

$$|\mathcal{F}_j| \le |\mathcal{A}_j|.$$

Therefore, from (3.1) we have

$$\log |\mathcal{F}_j| \le \log |\mathcal{A}_j| \le d_M \log \left(\frac{4D\mu(\Omega)^{\frac{1}{p}}}{a(1+a)^j}\right)$$
$$\le d_M \log \left(\frac{4D\mu(\Omega)^{\frac{1}{p}}}{a}\right) + d_M (1+a)^{-pj}$$
$$\le \frac{d_M \log(C_4\mu(\Omega))}{p} + d_M (1+a)^{-pj},$$

where  $C_4 = (\frac{4D}{a})^p$ , and  $A_1 = \exp(p^{-1}d_M \log(C_4 \mu(\Omega)))$ .

Now,

$$\begin{split} \sum_{j \in G} |\mathcal{F}_j| \exp\left(-\frac{r\eta_j^2}{8L_j}\right) &\leq \sum_{j \in G} A_1 \exp\left(d_M (1+a)^{-pj} - \frac{r\eta_j^2}{8L_j}\right) \\ &= \sum_{j \in G} A_1 \exp\left(d_M (1+a)^{-pj} - \frac{r\sigma^2}{10|G|^2} (1+a)^{-pj}\right) \\ &= \sum_{j \in G} A_1 \exp\left(-R(1+a)^{-pj}\right) \\ &\leq A_1|G| \exp\left(-R(1+a)^{-pj_0}\right), \end{split}$$

where  $R = \frac{r\sigma^2}{10|G|^2} - d_M$ , for sufficiently large sample size r, we can chose R > 0.

Let  $f \in V_M \setminus \{0\}$  be arbitrary. Then  $g = \frac{f\mu(\Omega)^{\frac{1}{p}}}{\|f\|_{L^p(\mathbb{R}^n)}} \in V_{M,\Omega}$  and satisfy (3.6). Therefore,

$$\frac{C_1(a)^p}{\mu(\Omega)} \Big( C_2(a)^{-p} \|f\|_{L^p(\Omega)}^p - (1+a)^{pj_0} \|f\|_{L^p(\mathbb{R}^n)}^p - \sigma \|f\|_{L^p(\mathbb{R}^n)}^p \Big) \le \|S(f,\xi)\|_p^p \\
\le \frac{C_2(a)^p}{\mu(\Omega)} \Big( C_1(a)^{-p} \|f\|_{L^p(\Omega)}^p + (1+a)^{pj_0} \|f\|_{L^p(\mathbb{R}^n)}^p + \sigma \|f\|_{L^p(\mathbb{R}^n)}^p \Big),$$

holds with probability at least  $1-2A_1|G|\exp\left(-R(1+a)^{-pj_0}\right)$  with an additional assumption R > 0.

**Theorem 4.4.** Let  $\xi = \{\xi_{\nu} : \nu = 1, 2, ..., r\}$  be a sequence of independent random variable that are drawn uniformly from  $\Omega$ . Suppose that  $0 < \tau < 1$  is small enough and if the number of sample size r satisfies

$$r \ge \frac{10}{\sigma^2} d_M |G|^2 = \mathcal{O}(\mu(\Omega)(\log \mu(\Omega))^2),$$

then for every  $f \in V^*(\Omega, \delta)$  the sampling inequality

$$\frac{r}{\mu(\Omega)} \left(\frac{1}{5} - \frac{p2^{p+1}\tau}{5} - pD^{p-1}\tau\right) \|f\|_{L^p(\mathbb{R}^n)}^p \le \sum_{\nu=1}^r |f(\xi_\nu)|^p \le \frac{r}{\mu(\Omega)} \left(\frac{2B}{A} + pD^{p-1}\right) \|f\|_{L^p(\mathbb{R}^n)}^p, \quad (4.2)$$

holds with probability at least  $1 - 2A_1|G| \exp\left(-R(1+a)^{-pj_0}\right)$ .

**Proof.** Without loss of generality, let  $f \in V^*(\Omega, \delta)$  with  $||f||_{L^p(\mathbb{R}^n)} = 1$ .

From Lemma 2.4, for  $\tau > 0$  there exist compact set M and  $\tilde{f} \in V_M$  such that

$$\|f - \tilde{f}\|_{L^{\infty}(\Omega)} < \frac{\tau}{\mu(\Omega)},$$
$$\|f - \tilde{f}\|_{L^{p}(\Omega)} < \tau,$$

and

$$\begin{split} \tilde{f}(x) &= \sum_{\gamma \in \Gamma \cap M} c_{\gamma} F_{\gamma}(x), \\ |\tilde{f}(x)| &\leq \Big(\sum_{\gamma \in \Gamma \cap M} |c_{\gamma}|^{p}\Big)^{\frac{1}{p}} \Big(\sum_{\gamma \in \Gamma \cap M} |F_{\gamma}(x)|^{p'}\Big)^{\frac{1}{p'}} \\ &\leq B^{\frac{1}{p}} \|f\|_{L^{p}(\mathbb{R}^{n})} \Big(\sum_{\gamma \in \Gamma} |\Theta(x-\gamma)|^{p'}\Big)^{\frac{1}{p'}} \\ &\leq D \|f\|_{L^{p}(\mathbb{R}^{n})}. \end{split}$$

Next, we get

$$\left| \|f\|_{L^{p}(\Omega)}^{p} - \|\tilde{f}\|_{L^{p}(\Omega)}^{p} \right| \leq p(1+\tau)^{p-1}\tau$$
$$\leq p2^{p-1}\tau$$
$$\|f\|_{L^{p}(\Omega)}^{p} - p2^{p-1}\tau \leq \|\tilde{f}\|_{L^{p}(\Omega)}^{p} \leq \|f\|_{L^{p}(\Omega)}^{p} + p2^{p-1}\tau,$$
(4.3)

and

$$\left| |f(\xi_{\nu})|^{p} - |\tilde{f}(\xi_{\nu})|^{p} \right| \leq p \Big( \max\{|f(\xi_{\nu})|, |\tilde{f}(\xi_{\nu})|\} \Big)^{p-1} |f(\xi_{\nu}) - \tilde{f}(\xi_{\nu})| \\ \leq \frac{p D^{p-1} \tau}{\mu(\Omega)}.$$

Therefore,

$$\sum_{\nu=1}^{r} |\tilde{f}(\xi_{\nu})|^{p} - \frac{rpD^{p-1}\tau}{\mu(\Omega)} \le \sum_{\nu=1}^{r} |f(\xi_{\nu})|^{p} \le \sum_{\nu=1}^{r} |\tilde{f}(\xi_{\nu})|^{p} + \frac{rpD^{p-1}\tau}{\mu(\Omega)}.$$
(4.4)

Since  $V_M \subseteq V$  and using (2.2), we get

$$\|\tilde{f}\|_{L^p(\mathbb{R}^n)}^p \leq \frac{1}{A} \sum_{\gamma \in \Gamma \cap M} |c_\gamma|^p \leq \frac{1}{A} \sum_{\gamma \in \Gamma} |c_\gamma|^p \leq \frac{B}{A} \|f\|_{L^p(\mathbb{R}^n)}^p.$$

Hence, it follows from Theorem 4.3, and equations (4.4) and (4.3) that

$$\frac{r}{\mu(\Omega)}C_{1}(a)^{p}\left(C_{2}(a)^{-p}\|\tilde{f}\|_{L^{p}(\Omega)}^{p}-(1+a)^{pj_{0}}\|\tilde{f}\|_{L^{p}(\mathbb{R}^{n})}^{p}-\sigma\|\tilde{f}\|_{L^{p}(\mathbb{R}^{n})}^{p}\right) \leq \sum_{\nu=1}^{r}|\tilde{f}(\xi_{\nu})|^{p} \\
\leq \frac{r}{\mu(\Omega)}C_{2}(a)^{p}\left(C_{1}(a)^{-p}\|\tilde{f}\|_{L^{p}(\Omega)}^{p}+(1+a)^{pj_{0}}\|\tilde{f}\|_{L^{p}(\mathbb{R}^{n})}^{p}+\sigma\|\tilde{f}\|_{L^{p}(\mathbb{R}^{n})}^{p}\right) \\
\frac{r}{\mu(\Omega)}C_{1}(a)^{p}\left(C_{2}(a)^{-p}\|f\|_{L^{p}(\Omega)}^{p}-C_{2}(a)^{-p}p2^{p-1}\tau-(1+a)^{pj_{0}}\frac{B}{A}-\sigma\frac{B}{A}\right) \\
\leq \sum_{\nu=1}^{r}|\tilde{f}(\xi_{\nu})|^{p} \leq \frac{r}{\mu(\Omega)}C_{2}(a)^{p}\left(C_{1}(a)^{-p}+(1+a)^{pj_{0}}+\sigma\right)\frac{B}{A} \\
\frac{r}{\mu(\Omega)}\left[C_{1}(a)^{p}\left(C_{2}(a)^{-p}(1-\delta)-(1+a)^{pj_{0}}\frac{B}{A}-\sigma\frac{B}{A}\right)-\frac{4}{5}p2^{p-1}\tau-pD^{p-1}\tau\right] \\
\leq \sum_{\nu=1}^{r}|f(\xi_{\nu})|^{p} \leq \frac{r}{\mu(\Omega)}\left[C_{2}(a)^{p}\left(C_{1}(a)^{-p}+(1+a)^{pj_{0}}+\sigma\right)\frac{B}{A}+pD^{p-1}\right]. \tag{4.5}$$

Now, we choose  $j_0$  and  $\sigma$  such that

$$\frac{\sigma B}{A} = \frac{C_2(a)^{-p}(1-\delta)}{2},$$
$$\frac{B(1+a)^{pj_0}}{A} = \frac{C_2(a)^{-p}(1-\delta)}{4}.$$

This implies,

$$|j_0| = \log\left(\frac{4BC_2(a)^p}{A(1-\delta)}\right)/p\log(1+a).$$

and

$$|G| \le J + |j_0| \le \log(D^p C_1(a)^{-p} \mu(\Omega)) / p \log(1+a) + \log\left(\frac{4BC_2(a)^p}{A(1-\delta)}\right) / p \log(1+a)$$
  
$$\le \frac{\log(C_5 \mu(\Omega))}{p \log(1+a)},$$

where  $C_5 = \frac{5BD^p}{A(1-\delta)}$ . Also,  $\frac{C_2(a)^p}{C_1(a)^p} \leq \frac{5}{4}$ .

Hence, bounds of the sampling inequality (4.5) can be revised. We get the lower estimate as

$$\frac{r}{\mu(\Omega)} \Big( \frac{1}{5} - \frac{4}{5} p 2^{p-1} \tau - p D^{p-1} \tau \Big) \le \frac{r}{\mu(\Omega)} \Big( \frac{4C_1(a)^p}{5C_2(a)^p} - \frac{4}{5} p 2^{p-1} \tau - p D^{p-1} \tau \Big),$$

and upper bound as

$$\frac{r}{\mu(\Omega)} \Big( \frac{5B}{4A} + \frac{1-\delta}{4} + \frac{1-\delta}{2} + pD^{p-1} \Big) \le \frac{r}{\mu(\Omega)} \Big( \frac{2B}{A} + pD^{p-1} \Big).$$

This implies for every  $f \in V^*(\Omega, \delta)$ 

$$\frac{r}{\mu(\Omega)} \Big(\frac{1}{5} - \frac{p2^{p+1}\tau}{5} - pD^{p-1}\tau\Big) \|f\|_{L^p(\mathbb{R}^n)}^p \le \sum_{\nu=1}^r |f(\xi_\nu)|^p \le \frac{r}{\mu(\Omega)} \Big(\frac{2B}{A} + pD^{p-1}\Big) \|f\|_{L^p(\mathbb{R}^n)}^p,$$

holds with probability at least  $1 - 2A_1|G| \exp\left(-R(1+a)^{-pj_0}\right)$ .

Remark 4.5. Let  $\{\xi_{\nu}\}$  be random sample drawn uniformly from  $\Omega$ . The sampling inequality (4.2) hold with probability at least  $1 - \epsilon$  if

$$2A_{1}|G|\exp\left(-R(1+a)^{-pj_{0}}\right)$$

$$\leq 2|G|\exp\left(d_{M}p^{-1}\log(C_{4}\mu(\Omega)) - (1+a)^{-pj_{0}}\left(\frac{r\sigma^{2}}{10|G|^{2}} - d_{M}\right)\right)$$

$$\leq 2\frac{\log(C_{5}\mu(\Omega))}{p\log(1+a)}\exp\left(C(\omega)\mu(\Omega)p^{-1}\log(C_{4}\mu(\Omega)) - (1+a)^{-pj_{0}}\left(\frac{r\sigma^{2}p^{2}(\log(1+a))^{2}}{(\log(C_{5}\mu(\Omega)))^{2}} - C(\omega)\mu(\Omega)\right)\right)$$

$$\leq \epsilon$$

Hence, if the sample size  $r \geq \mathcal{O}(\mu(\Omega)(\log \mu(\Omega))^3)$ , then sampling inequality (4.2) holds with probability at least  $1 - \epsilon$ .

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