On the parameterized complexity of manipulating Top Trading Cycles

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Abstract

We study the problem of exchange when 1) agents are endowed with heterogeneous indivisible objects, and 2) there is no money. In general, no rule satisfies the three central properties Pareto-efficiency, individual rationality, and strategy-proofness [62]. Recently, it was shown that Top Trading Cycles is **NP**-hard to manipulate [32], a relaxation of strategy-proofness. However, parameterized complexity is a more appropriate framework for this and other economic settings. Certain aspects of the problem - number of objects each agent brings to the table, goods up for auction, candidates in an election [25], legislative figures to influence [24] - may face natural bounds or are fixed as the problem grows. We take a parameterized complexity approach to indivisible goods exchange for the first time. Our results represent good and bad news for TTC. When the size of the endowments k is a fixed constant, we show that the computational task of manipulating TTC can be performed in polynomial time. On the other hand, we show that this parameterized problem is **W**[1]-hard, and therefore unlikely to be *fixed parameter tractable*.

1 Introduction

In many economic environments, agents are endowed with heterogeneous indivisible objects, exchange is desirable, and there is no money. For example, workers trading shifts/tasks/assignments, users sharing time blocks on a supercomputer, etc. A rule or mechanism recommends for each possible profile of preferences and endowments a re-allocation of the objects. The general program is to define desirable properties (axioms) and design rules that satisfy as many of them as possible. Three central and well-studied properties are *Pareto-efficiency* (no rearrangement could make all agents at least as well off, and some better off), *individual rationality* (no agent is worse off than they started), and *strategy-proofness* (no agent is better off reporting a lie than their true preference). Unfortunately, in this environment and many others, there is no rule satisfying all three [62]. This motivates the study of properties which are relaxations of strategy-proofness.

We focus on the Top Trading Cycles (TTC) mechanism due to Gale. TTC is strategyproof when the endowments are of size 1, but when the endowments are multiple, manipulation is possible. It was recently shown that TTC is **NP**-hard to manipulate. This result suggests that there may not be an incentive to manipulate TTC, as agents have bounded computational resources. However, the result could be very misleading to policy makers. As we will see, manipulating TTC can be done in time approximately n^k , where n is the number of goods and k is the size of the endowments. This does not contradict [32] because the size of the endowment is part of the input to the problem in that paper. Indeed, the hardness reduction makes the implicit assumption that an agent may have a number of goods that grows with the number of agents.

Still, when k is large, the algorithm is not practical. If TTC could be manipulated in time $f(k)n^c$ for some function f and constant c (i.e. if the problem were *fixed parameter tractable*) Then TTC would cease to be an attractive rule in the general case. The parameterized complexity of manipulating TTC is therefore an important and interesting question.

Our previously mentioned algorithm is bad news for TTC, but our main result represents good news. We show that manipulating TTC is W[1]-hard (which we define properly later) and therefore very unlikely to be fixed parameter tractable.

Related Literature

We discuss two bodies of related work: the progression of the study of indivisible objects exchange in the economics literature, and recent work in computational social choice.

In 1974, Shapley & Scarf introduced the problem of exchanging indivisible objects without money, also known as the Housing Market [60]. Each agent is endowed one object, may consume one object, and has strict preferences over all objects. They showed that the Top Trading Cycles algorithm (attributed to David Gale) could be used to compute a core allocation. It turns out that the core is unique, and the rule derived from recommending the core for each preference profile is the only *Pareto-efficient*, *individually rational*, and *strategy-proof* rule [5, 55, 42, 44, 59, 62, 65].

Subsequently, the literature considered various generalizations of the model and/or applications of TTC: the case of no ownership [36, 67, 65], the case where some agents may own nothing (generalizing the two previous cases) [2, 63, 65, 49, 53], fairer probabilistic rules [1, 8, 6, 11, 15, 16, 22, 21, 23, 35, 41] allowing for indifferences in preferences [4, 9, 14, 37, 52, 54, 56, 64], School Choice [3, 29, 48, 28, 47, 46], and dynamic environments [40, 58]. Several authors considered manipulation not by preference misreport but by the merging/splitting/withholding of endowments [7, 18].

Our paper considers the case when each agent may be endowed with multiple objects [50, 51, 64, 66]. As mentioned, there is no rule satisfying all three properties [62]. In response to this, [51] weakens the *Pareto-efficiency* requirement to *range-efficiency* and characterizes the resulting family of rules on a large preference domain. In a complementary manner, [33] shows that in the Lexicographic domain of preferences ATTC satisfies the properties when *strategy-proofness* is weakened to *NP-hard to Manipulate*. An immediate corollary of their result is the extension of the statement to larger domains. Other authors consider an environment where objects have types [38, 39, 45, 43, 61], or where there is no ownership [19, 20].

The idea of studying the complexity of manipulation was proposed by [10] in response to [34, 57]—the latter showing that, in the environment of voting, requiring *strategy-proofness* leads to dictatorship. We refer the reader to surveys in the subsequent computational social choice literature [26, 30, 31], and highlight works that take the parameterized complexity approach [12, 13].

2 Preliminaries

Let N be a set of agents and let \mathcal{O} be a set of objects. Let $\omega = {\{\omega_i\}_{i \in N}}$ be a set of subsets of \mathcal{O} such that $\omega_i \cap \omega_j = \emptyset$ for all $i \neq j$ and $\cup \omega_i = \mathcal{O}$; we call ω_i the endowment of agent i. If a good α is in ω_i then we say that agent *i* is the owner of α and that $a(\alpha) = i$. Let \mathcal{R} be the set of all relations over $2^{\mathcal{O}}$ that are complete, transitive and anti-symmetric. Let $R = \{R_i\}_{i \in N}$ be an element of $\mathcal{R}^{|N|}$; we call R_i the preference relation of agent *i*, and R the preference profile. We denote the strict component of R_i by P_i , i.e. XP_iY if and only if XR_iY and $\neg YR_iX$. We say that $\mathcal{E} = (N, \mathcal{O}, \omega, R)$ is an economy. If $|\omega_i| = 1$ for all *i* we say E is a housing market and otherwise a generalised housing market. If $z = \{z_i\}_{i \in N}$ is a set of disjoint subsets of \mathcal{O} such $\cup z_i = \mathcal{O}$ we say that z is an allocation for the economy \mathcal{E} . Note that the endowment is an allocation. A rule $\phi : \mathcal{R}^{|N|} \to Z$ recommends an allocation given a particular preference profile. We denote by $\phi_i(R)$ the allocation of agent *i* under ϕ at R; if $\phi(R) = z$ then $\phi_i(R) = z_i$.

Properties of rules Following standard notation we write (R'_i, R_{-i}) to be the preference profile obtained from R by replacing R_i with R'_i . We say that R'_i is a misreport for agent i. A misreport R'_i is beneficial under ϕ if $\phi_i(R'_i, R_{-i})P_i\phi_i(R)$. A rule ϕ is strategy-proof if for all economies, no agent has a beneficial misreport under ϕ . We emphasise that this property implies that no agent can lie even if they have full information about the preferences of the other agents. One trivial example of a strategy-proof rule is the "no deal" rule $\phi(R) = \omega$, but this rule is clearly sub optimal. We say that an allocation z is Pareto-optimal for R if for any z' we have that $z_i R_i z'_i$ and for at least one agent j we do not have that $z'_i R_i z_i$. We say that a rule is Pareto-efficient (PE) if it always recommends a Pareto-optimal allocation. If an agent might be worse off after the trade according to their own preference relation, there is no incentive to take part. A rule is said to be individually rational (IR) if $\phi_i(R)R_i\omega_i$ for each agent i.

Graph theory In order to describe the rule that is the focus on this paper, and our results, we require some definitions from graph theory. We follow the definitions in [17], but we now recall some important notions. A (directed) walk in a graph is an ordered multiset $(v_1, e_1, v_2, e_2, \ldots, e_k, v_{k+1})$ where v_i is a vertex and e_i is a (directed) edge from v_i to v_{i+1} for $1 \leq i < k$. A path is a walk where no vertex is repeated. A cycle is a path plus an edge from v_k to v_1 . A *clique* in a graph G is a set of vertices C such that there is an edge between every pair of vertices in C. A proper colouring (or simply a colouring) of a graph G is an assignment of colours to its vertices such that no edge joins two vertices of the same colour.

Top Trading Cycles For housing markets, there is exactly one rule that is simultaneously SP, PE and IR [62]. The allocation that the rule recommends can be obtained by following the *Top Trading Cycles* procedure which we define below (see Figure 1 for an example).



Figure 1: The first step of the TTC procedure (dotted edges denote second preferences)

In a housing market, the endowments are singletons and so any IR rule must also produce an assignment whose elements are singletons. We can assume that each agent has a strict preference relation over \mathcal{O} . We introduce the following useful notation: if $\alpha P_i\beta$ for all β in some subset \mathcal{O}' of the goods, we say that agent *i* topranks α in \mathcal{O}' (if $\mathcal{O}' = \mathcal{O}$ we say simply that *i* topranks α).

Top Trading Cycles

Input: An economy $\mathcal{E} = (N, \mathcal{O}, \omega, \mathcal{R})$. **Output:** An assignment z.

- 1. Create a directed graph H_1 whose vertex set is $V_1 = \mathcal{O}$ with an edge (α, β) in E_t if and only if $a(\alpha)$ topranks β .
- 2. For $t = 1, 2, \ldots$:
 - (a) If V_t is empty, stop.
 - (b) Otherwise, select an arbitrary cycle $(\gamma_1, \gamma_2, \ldots, \gamma_j)$ in H_t .
 - i. Add γ_1 to $z_{a(\gamma_i)}$, and add γ_{i+1} to $z_{a(\gamma_i)}$ for $1 \leq i < j$.
 - ii. Let $V_{t+1} = V_t \setminus \{\gamma_1, \ldots, \gamma_j\}.$
 - iii. Let H_{t+1} be the directed graph on V_{t+1} with an edge (α, β) in E_{t+1} if and only if $a(\alpha)$ topranks β in V_{t+1} .

In Step 2b, an arbitrary cycle was selected. Indeed, the order that cycles are removed from the graph in TTC does not matter. However, it will be useful to refer to the time at which goods are traded under TTC. In order to make this notion well-defined, we insist that an economy \mathcal{E} is equipped with a total ordering over \mathcal{O} ; we can refer to the *first* good in \mathcal{O} . Observe that for each good α in V_t there is a unique directed walk with no repeated edges starting at α ; we call this the *trading walk* starting at α . Since every element of V_t has outdegree 1, this walk must contain a cycle. We define the cycle in Step 2b to be the one contained in the trading walk starting at the first good in V_t . We can now define the trading time $tt_{\mathcal{E}}(\alpha)$ of a good α in a run of TTC on \mathcal{E} to be the least integer t such that $\alpha \in V_t \setminus V_{t+1}$. When the economy is unambiguous, we write $tt(\alpha) = tt_{\mathcal{E}}(\alpha)$. The following simple observations will be very useful later.

Observation 1. Suppose α and β are goods in V_t during a run of TTC. If the owner of α topranks β in V_t , then $tt(\alpha) \ge tt(\beta)$.

Observation 2. Suppose $\alpha_i \in \omega_i$ and $\alpha_j \in \omega_j$. If $\alpha_i R_i \alpha_j$ and $\alpha_j R_j \alpha_i$. Then $tt(\alpha_i) \neq tt(\alpha_j)$.

Observation 3. Suppose α is a good in V_t , and the trading walk W starting at α in H_t is not a trading cycle. Let β be a good on the trading cycle in W. Then $tt(\beta) < tt(\alpha)$.

Observation 4. Suppose $\beta \in TTC_i(R)$ and $\alpha P_i\beta$. Then $tt(\alpha) < tt(\beta)$.

Observe that TTC (as we have described it above) does not require the endowments to be singletons. In other words, it can be applied in the setting of generalised housing markets. However, we know that in this case TTC is not strategy-proof in general. For example, let $\mathcal{E} = (N, \mathcal{O}, \omega, R)$ be the economy shown in Figure 1. In this economy, $a(\delta)$ and $a(\gamma)$ have the same preferences. Suppose that $a(\delta)$ and $a(\gamma)$ are the same; let $\omega_1 = \{\gamma, \delta\}, \omega_2 = \{\alpha\}, \omega_3 = \{\beta\}$ for instance. If agent 1 prefers the bundle $\{\alpha, \beta\}$ to its assignment $\{\alpha, \delta\}$ there is a possibility for agent 1 to benefit by misreporting their preferences. Agent 1 can



Figure 2: Agent 1 (who owns e_0, e_α, e_β) can get α and β by trading e_0 for x.

report a preference relation R'_1 such that $\beta P'_1 \alpha P'_1 \gamma P'_1 \delta$. It is easy to verify that the allocation $\text{TTC}_1(R'_1, R_{-1})$ when agent 1 reports R'_1 is $\{\alpha, \beta\}$.

In Figure 2 we see a more complicated example. We adopt the convention throughout the paper that \longrightarrow denotes a first preference, \ldots denotes second preference, \ldots denotes third preference, \ldots denotes fourth preference, and thereafter a dashed line with *i* dots denotes the (3 + i)th preference. We set $\omega_1 = \{e_0, e_\alpha, e_\beta\}$. The preferences of agent 1 are such that any bundle including both α and β is preferable to any bundle including one or the other or neither. Informally, agent 1 wants to get α and β . However, the order of preference of the individual goods, according to R_1 is $\alpha, \beta, e_\alpha, e_\beta, e_0, \gamma, x, y$. In the first round of TTC, the goods α, γ, e_0 form a trading cycle. It can be seen that after these goods are removed, first x, y form a trading cycle, and then β forms a trading cycle. Thus the assignment to agent 1 is $\{\alpha, e_\alpha, e_\beta\}$. Agent 1 has an incentive to lie; even though x is not preferable to any individual good in ω_1 , obtaining it prevents x, y from forming a trading cycle.

Since manipulation of TTC is clearly possible with multiple endowments, it is necessary to consider relaxing the strict condition that a rule is strategy-proof. Instead, we consider requiring that computing a beneficial misreport is computationally intractable.

Computational complexity We are interested in the complexity of the following problem:

BENEFICIAL MISREPORT(ϕ)

INPUT: A generalised housing market economy \mathcal{E} **QUESTION**: Does agent 1 have a beneficial misreport under ϕ ?

For simplicity's sake, we always assume that agent 1 is the would-be liar. Since we are mainly interested in proving (conditional) lower bounds on the complexity of manipulating TTC, we focus on the decision version of the problem. Fujita *et al.* [32] showed that BM(TTC) is **NP**-complete in general (they refer to *Augmented* Top Trading Cycles, but the description is equivalent). This result suggests that TTC might yet be of practical use despite not being SP; an agent with limited computational resources would have no incentive to lie.

However, the hardness established by this result seems to depend heavily on the size of the endowments. Indeed, the proof makes the implicit assumption that one agent may have a number of goods that grows with the number of agents. This strongly suggests that a parameterized approach is more appropriate. In fact, the **NP**-completeness of the problem could be very misleading; as we shall see later, there is a polynomial time solution to the problem when the size of the endowment is a fixed constant.

Parameterized complexity For a full treatment of the topic of parameterized complexity, we refer the reader to the textbook by Downey and Fellows [27]. We give a brief overview aimed at non-specialists. Consider the following decision problems.

CLIQUE	VERTEX COVER
INPUT : A graph G and an integer	INPUT : A graph G and an integer
K	K
QUESTION : Is there a set C of K	QUESTION : Is there a set C of K
vertices of G every pair of which is	vertices of G such that every edge of
adjacent?	G contains a vertex of C ?

Both of these problems are **NP**-hard, which means that if they can be solved in an amount of time that is polynomial in the total size of the input (G, K) then **P** = **NP**. On the other hand, when K is fixed, and not part of the input of the problem, both can be solved in polynomial time. Indeed, CLIQUE can be solved in $|G|^{O(K)}$ time, and VERTEX COVER can be solved in $O(2^{K}|G|)$ time. It should be clear that there is a big difference in these run times: $2^{20} \times 1000$ operations will take a modern computer mere seconds whereas 1000^{20} is larger than the number of atoms in the observable universe. This 2-dimensional approach shows us that the complexity of these problems is very sensitive to the size of the solution sought, and in general a problem's complexity may depend heavily on the size of some *parameter* in a way that classical complexity ignores.

We define a parameterized language to be a subset of $\Sigma^* \times \mathbb{N}$ for some alphabet Σ . If (x, k) is a member of a parameterized language L we say that k is the *parameter*. If there exists an algorithm which can decide whether (x, k) belongs to L in $|x|^{f(k)}$ for some computable function f, then L is in the complexity class **XP**. If, additionally, there exists an algorithm that decides membership of L in time $f(k) \cdot |x|$ for an arbitrary function f, then L is in the complexity class **FPT**. The above discussion shows that VERTEX COVER is in **FPT**, but CLIQUE is thought not to be.

The fact that CLIQUE is **NP**-hard is a *conditional lower bound* for the run time of an algorithm that solves CLIQUE. Since every **NP** problem reduces to CLIQUE, a polynomial time solution to this problem implies $\mathbf{P} = \mathbf{NP}$. An analogous conditional lower bound exists in the parameterized setting. We define the class of $\mathbf{W}[1]$ parameterized languages to be those that reduce to the following.

SHORT NONDETERMINISTIC TURING MACHINE HALTING

INPUT: A nondeterministic Turing machine M**PARAMETER**: k

QUESTION: Is it possible for M to reach a halting state in at most k steps?

It is considered extremely unlikely that $\mathbf{FPT} = \mathbf{W}[1]$. We remark that CLIQUE happens to be $\mathbf{W}[1]$ -hard (in fact $\mathbf{W}[1]$ -complete) and thus unlikely to be in \mathbf{FPT} . There is a whole hiearchy of classes $\mathbf{FPT} \subseteq \mathbf{W}[1] \subseteq \mathbf{W}[2] \subseteq \ldots \subseteq \mathbf{XP}$ (deciding the existence of a dominating set of size k is a $\mathbf{W}[2]$ -complete problem for instance) and the inequality $\mathbf{FPT} \subset \mathbf{XP}$ is known to be strict. For our purposes, it is enough to consider $\mathbf{W}[1]$ -hardness as a conditional lower bound on the complexity of a decision problem.



Figure 3: A directed 4-coloured graph G (snake in red).

Our main result is that BM(TTC), parameterized by the size of the endowments, is W[1]-hard. For the rest of the paper, we refer only to the parameterized version of BM(TTC) In fact, BM(TTC) remains hard under a strong restriction on the preference relations.

Preference domains A preference relation R_i is **lexicographic** if for each $X, Y \subseteq \mathcal{O}$, $X P_i Y$ iff there is $b \in \mathcal{O}$ such that 1) $b \in X \setminus Y$, 2) for each $a \in \mathcal{O}$ with $a P_i b$, $a \in X \cap Y$. The lexicographic, additive, responsive, and monotonic domains are ordered by inclusion ¹. Our result holds even on the lexicographic domain; it immediately holds for the more general domains.

For the rest of this paper, we assume that all agents have lexicographic preference relations unless stated otherwise. This allows us to write the preference relations in a compressed format. We may write the preference relation of an agent i as a list of the singletons ordered by R_i , up to and including the least preferred element of ω_i .

3 The Main Result

In order to prove that BM(TTC) is W[1]-hard, we introduce an auxiliary problem; our reduction is ultimately from the following problem.

MULTICOLOUR CLIQUE

INPUT: A graph G with a proper vertex colouring ϕ **PARAMETER**: The number of colours k **QUESTION**: Does there exists a clique of size k in G?

In order to simplify our exposition, we introduce an intermediate problem which we will prove is $\mathbf{W}[1]$ -hard and reduce to BM(TTC). In a directed graph G with a proper vertex colouring $\phi: V(G) \to [k]$, an edge (u, v) with $\phi(v) = \phi(u) + 1$ will be called a *rung*. On the other hand if $\phi(v) < \phi(u)$, we say that (u, v) is a *snake*. A *partial ladder* in such a graph is a set of k vertices $\{v_1, \ldots, v_j\}$ such that (v_i, v_{i+1}) is a rung for all $1 \le i \le j - 1$. A (partial) ladder is *snakeless* if there are no snakes between its vertices. We define the problem of deciding the existence of a snakeless ladder in a k-coloured graph as follows:

SNAKELESS LADDER

INPUT: A directed graph G with a proper vertex colouring ϕ

¹A preference relation R_i is **monotonic** if for each $X, Y \subseteq O$ such that $Y \subseteq X, X R_i Y$. A preference relation R_i is **responsive** if for each $X \subset O$, and each $a, b \in O \setminus X, X \cup \{a\} R_i X \cup \{b\}$ iff $a R_i b$. A preference relation R_i is **additive** if there is $u_i : O \to \mathbb{R}$ such that for each $X, Y \subseteq O, X R_i Y$ iff $\sum_{a \in X} u_i(a) \geq \sum_{a \in Y} u_i(a)$.



Figure 4: Three vertex gadgets.

PARAMETER: The number of colours k**QUESTION**: Does G contain a snakeless ladder?

Lemma 5. SNAKELESS LADDER is $\mathbf{W}[1]$ -hard.

Proof. The proof is a simple reduction from MULTICOLOUR CLIQUE. From a k-coloured graph G we obtain a directed graph G' by first complementing the set of edges between non-adjacent colour classes. In other words if $|\phi(u) - \phi(v)| \neq 1$, then uv is an edge in G' if and only if it was not an edge in G. We then direct the edge uv from u to v if $\phi(v) = \phi(u) + 1$ or if $\phi(v) < \phi(u)$. Now consider a set of vertices $\{v_1, \ldots, v_k\}$ in G'. By definition $(v_1, v_2), (v_2, v_3) \ldots, (v_{k+1}, v_k)$ are rungs in G' if and only if $v_1v_2, v_2v_3, \ldots, v_{k+1}v_k$ are edges in G. Similarly, for i < j there is no snake (v_j, v_i) in G' if and only if v_iv_j is an edge of G. Therefore, $\{v_1, \ldots, v_k\}$ form a snakeless ladder in G' if and only if they form a clique in G.

Theorem 6. BM(TTC) is W[1]-hard, even when the preference domain is lexicographic

Proof. We reduce from SNAKELESS LADDER. Let G be a graph with a proper k-colouring for some integer k. We will obtain an economy \mathcal{E}_G and an integer k' such that the endowment of player 1 in \mathcal{E}_G has size k' and furthermore player 1 has a beneficial misreport R'_1 if and only if there is a snakeless ladder in G. We will then show that \mathcal{E}_G, k' can be constructed in time $f(k)|G|^c$ for some function f and constant c.

We begin the construction of $\mathcal{E}_G = (N, \mathcal{O}, \omega, R)$ by setting $\omega_1 = e_\alpha, e_\beta \cup \{e_j : 1 \leq j \leq k\}$. We can assume that all other agents have a singleton endowment. For each vertex v_i in G, we add to \mathcal{E}_G a vertex gadget. A vertex gadget is a pair of goods x_i, y_i and their respective owners, who have particular preferences depending on the colour and neighbourhood of v_i . The agent $a(y_i)$ always topranks x_i , and $a(x_i)$ topranks e_j where j is the colour of v_i . The full preference relation of $a(x_i)$ is (e_j, y_i, x_i) . The preferences of $a(y_i)$ are as follows. Let v_{s_1}, v_{s_2}, \ldots be the endpoints of the snakes which start at v_i and let v_{r_1}, v_{r_2}, \ldots be the endpoints of the rungs which start at v_i . For a vertex v_i of colour k, the full preference relation of $a(y_i)$ is $(x_i, y_{s_1}, y_{s_2}, \ldots, e_\beta, y_i)$. For a vertex v_i not of colour k, the full preference relation of $a(y_i)$ is $(x_i, y_{s_1}, y_{s_2}, \ldots, y_{r_1}, y_{r_2}, \ldots, y_i)$. Note that $a(y_i)$ prefers each of the goods representing the snakes of v_i to each of the goods representing the rungs. In Figure 4 we see three vertex gadgets. For i = 1, 2, 3 we have that v_i has colour i and we see that (v_2, v_1) is a snake and (v_2, v_3) is a rung.

1	α	β	γ	x_i	y_i
α	γ	γ	e_1	e_j	x_i
β	α	$\{y_i \text{ of colour } 1\}$	e_2	y_i	$\{\text{snakes}\}$
e_{α}		β	•••	x_i	$\{rungs\}$
e_{β}			e_k		$(e_{\beta} \text{ if } j = k)$
e_k			e_{α}		y_i
• • •			γ		
e_1					

Table 1: The preferences of some agents in \mathcal{E}_G (v_i has colour j)

We continue our construction by the addition of three goods (and their respective owners) α, β, γ . The full preference relation of $a(\alpha)$ is (γ, α) . The full preference relation of $a(\gamma)$ is $(e_1, e_2, \ldots, e_k, e_\alpha, \gamma)$. The preference relation of $a(\beta)$ is as follows. Suppose v_1, v_2, \ldots, v_j are the vertices of colour 1 in G in an arbitrary order. The full preference relation of $a(\beta)$ is $(\gamma, y_1, y_2, \ldots, y_j, \beta)$. Note that $a(\beta)$ prefers all goods representing vertices of colour 1 to β and that β is preferred to all other goods representing vertices.

The construction is completed by revealing the true preference relation of agent 1; namely, $(\alpha, \beta, e_{\alpha}, e_{\beta}, e_k, e_{k-1}, \ldots, e_1)$. For clarity, we have provided Table 1 which shows the preferences of the agents described above. In Figure 5 we see an example of an economy constructed from the graph G in Figure 3. The preferences of agent 1 are omitted. When agent 1 reports the truth, the assignment received is $\{\alpha, e_{\alpha}, e_{\beta}, e_k, \ldots, e_2\}$.

Claim 7. If there is a beneficial misreport R'_1 for agent 1, then $\alpha, \beta \in TTC_1(R'_1, R_{-1})$

Proof. The two best goods obtained by agent 1 by reporting the truth are α and e_{α} . By the lexicographic property of R_1 , any bundle preferred by agent 1 to its true assignment must include α since agent 1 topranks α . Similarly, a preferred bundle must include a good that is preferred by agent 1 to e_{α} . The only such good is β .

The only bundles agent 1 prefers to their true assignment include both α and β . The reader may wish to verify that there is no beneficial misreport available to agent 1 in the economy in Figure 3. This is in contrast with Figure 2, where agent 1 was able to prevent x, y forming a trading cycle by obtaining x, and therefore obtain both α and β . If agent 1 tries the misreport $(x_1, x_2, x_3, x_4, \alpha, \beta)$ it is easy to see that y_1, y_2, y_3 will at some point form a trading cycle. Thus β will form a trading cycle on its own and not be included in the assignment to agent 1. This is because v_1, v_2, v_3, v_4 is not a snakeless ladder in G. If the snake (v_3, v_1) was omitted from G, then G would have a snakeless ladder and y_3 would no longer prefer y_1 to y_4 .

We now formalise this intuition and argue that agent 1 has a beneficial misreport in \mathcal{E}_G if and only if G has a snakeless ladder. Suppose that $L = (v_1, v_2, \ldots, v_k)$ is a snakeless ladder in G (so v_i has colour i in G). We claim that $R'_1 = (x_1, x_2, \ldots, x_k, \alpha, \beta, e_1, \ldots, e_k, e_\alpha, e_\beta)$ is a beneficial misreport. We abuse our terminology slightly and say that if v_i is of colour j, then x_i, y_i are goods of colour j. Let \mathcal{E}'_G be the economy obtained from \mathcal{E}_G by replacing R_1 by R'_1 , and consider a run of TTC on \mathcal{E}'_G . For the rest of this proof, we will write $tt(\delta) = tt_{\mathcal{E}'}(\delta)$ for the trade time of a good δ during a run of TTC on \mathcal{E}' . It is clear that in H_1 , there is a trading cycle e_1, x_1 . After this is removed, there will be a trading cycle e_2, x_2 . Observe that of all the vertices of colour 1 and 2 in G, only v_1 and v_2 are represented by goods in $H_{tt(e_2)+1}$. Furthermore, $a(y_1)$ topranks y_2 in $V_{tt(e_2)+1}$. Now e_3, x_3 form a trading cycle, and after this is removed, only y_3 remains among the goods of colour 3. Since L is a snakeless ladder,



Figure 5: The economy \mathcal{E}_G associated with the graph in Figure 3

 $a(y_3)$ prefers each good of colour 4 to y_1 . Similarly, we have that $a(y_i)$ topranks $a(y_{i+1})$ in $H_{tt(e_k)+1}$ for $1 \leq i < k$. Since v_k is of colour k, y_k topranks e_β . The goods $x_i, 1 \leq i \leq k$ are not in $V_{tt(e_k)+1}$, and so agent 1 topranks α in $H_{tt(e_k)+1}$ (according to the false preference relation R'_1), and $a(\gamma)$ topranks e_α . Thus, e_α, α, γ form a trading cycle. The assignment of agent 1 under R'_1 includes α as required. Since γ is not in $V_{tt(\alpha)+1}$, so $a(\beta)$ topranks y_1 , the only remaining good of colour 1. Furthermore, α is not in $V_{tt(\alpha)+1}$, so agent 1 topranks β in $H_{tt(\alpha)+1}$. Finally, $e_\beta, \beta, y_1, \ldots, y_k$ form a trading cycle, and β is included in the assignment to agent 1.

Suppose instead that there exists a beneficial misreport R'_1 for agent 1. We show that the trading cycle in $H_{tt(\beta)}$ that contains β is of the form $(\beta, y_1, y_2, \ldots, y_k, e_\beta)$ where v_1, v_2, \ldots, v_k is a snakeless ladder in G. We now demonstrate that $\alpha P'_1\beta$ must hold. By Observation 1, we have that $tt(\gamma) \leq tt(\alpha)$. On the other hand, if $tt(\gamma) < tt(\alpha)$, then $a(\alpha)$ topranks α in $V_{tt(\gamma)+1}$; thus $a(\alpha)$ keeps α , a contradiction. This shows that $tt(\alpha) = tt(\gamma)$. Observation 1 also gives us that $tt(\gamma) \leq tt(\beta)$, and by Observation 2 this inequality is strict. This shows that $tt(\alpha) < tt(\beta)$. Since α and β are both in the assignment to agent 1 by assumption, we must have $\alpha P'_1\beta$.

Observation 1 also tells us that $tt(e_1) \leq tt(\gamma)$. We show that this inequality is strict If $tt(e_1) = tt(\gamma)$, then e_1, α, γ form a trading cycle in $H_{tt(e_1)}$. Then in $H_{tt(e_1)+1}$, each pair of colour 1 forms a trading cycle. By Observation 3, no pair of colour 1 is in $H_{tt(\beta)}$, and β forms a trading cycle with itself, contradicting the definition of R'_1 .

In $H_{tt(e_1)+1}$, the agent $a(\gamma)$ topranks e_2 . Again, by Observation 1 we have that $tt(e_2) \leq tt(\gamma)$. A very similar argument to the above shows that this inequality is strict. Indeed, since β does not form a trading cycle on its own by assumption, there must be at least one good y_i of colour 1 in $H_{tt(\beta)}$. Since there must be a trading cycle including β in $H_{tt(\beta)}$, the good x_i cannot be in $H_{tt(\beta)}$. If $tt(e_2) = tt(\gamma)$, every pair of colour 2 forms a trading cycle in $H_{tt(e_2)+1}$. Thus no pair of colour 2 is in $H_{tt(\beta)}$, so y_i forms a trading cycle with itself a contradiction.

Proceeding by induction, we see that $tt(e_i) < tt(\alpha)$ for $1 \le i \le k$. Consider $H_{tt(\alpha)}$. For

 $1 \leq i \leq k$, the goods e_i are not in $V_{tt(\alpha)}$. The only other good that $a(\gamma)$ ranks above γ is e_{α} . Thus the trading cycle in $H_{tt(\alpha)}$ containing α must be $\alpha, \gamma, e_{\alpha}$.

Now consider $H_{tt(\beta)}$. Suppose y_i is a good of colour j in $V_{tt(\beta)}$. Then, without loss of generality, x_i is not in $V_{tt(\beta)}$, since we have shown e_j is not, and the order in which cycles are removed is arbitrary. Moreover, suppose there are distinct goods $y_i, y_{i'}$ of colour j in $V_{tt(\beta)}$. By Observation 1, $tt(e_j) \leq tt(x_i)$ and $tt(e_j) \leq tt(x_{i'})$. By Observation 2, $tt(x_i) \neq tt(x_{i'})$. Thus e_j must have been in a trading cycle with at most one of $x_i, x_{i'}$. Without loss of generality, $tt(e_j)$ is strictly less than $tt(x_i)$, and y_i, x_i form a trading cycle in $H_{tt(e_j)+1}$, a contradiction.

The trading walk in $H_{tt(\beta)}$ starting at β must be a trading cycle. Since γ is not in $V_{tt(\beta)}$, there must be exactly one good y_i of colour 1 in $V_{tt(\beta)}$. Without loss of generality, that good is y_1 . The agent $a(y_1)$ only ranks goods of the form y_j of colour 2 above y_1 in $V_{tt(\beta)}$. As we have discussed, there must be exactly one such good; without loss of generality, that good is y_2 . We proceed by induction. Suppose there is a path P in $H_{tt(\beta)}$ of the form $(\beta, y_1, \ldots, y_i)$ where y_i is of colour i. Observe that v_1, \ldots, v_{i-1} must be a snakeless partial ladder, though there may yet be a snake $(v_i, v_{i'})$ with i' < i. However, if $a(y_i)$ topranks some good $y_{i'} \in P$ with i' < i then the trading walk starting at β is not a cycle, which is a contradiction. All other goods of colour i + 1 in $V_{tt(\beta)}$. Without loss of generality, that good is y_{i+1} . Observe that $v_1, ldots, v_i$ is a snakeless partial ladder, and that there is a path in $H_{tt(\beta)}$ of the form $(\beta, y_1, \ldots, y_{i+1})$. We conclude that there is a path of the form $(\beta, y_1, \ldots, y_k)$ in $H_{tt(\beta)}$, and by the same argument, y_k must toprank e_β . In other words, we have that v_1, \ldots, v_k is a snakeless ladder as required.

4 An Upper Bound

We leave the possibility of a matching upper bound on the complexity of BM(TTC) (i.e. a proof of membership in W[1]) as an interesting open problem. We conclude the paper with an upper bound that nevertheless represents a negative result for TTC. Informally, if the size of the endowments is a fixed constant, BM(TTC) can be decided in polynomial time. In fact, our result is slightly stronger, in that we present an explicit constructive algorithm that can produce a beneficial misreport. We also highlight that this result holds regardless of the preference domain.

Proposition 8. BM(TTC) is in **XP**.

Proof. We will show that the following algorithm computes a beneficial misreport, if one exists, for agent 1 in time at most $k!n^{k+c}$ where n is the number of goods, k is the size of the endowment and c is a constant associated with the runtime of TTC.

Algorithm \mathcal{A}

Input: An economy $\mathcal{E} = (N, \mathcal{O}, \omega, R)$. **Output:** An beneficial misreport for agent 1.

- 1. Let $\omega_1 = \{e_1, \ldots, e_k\}$ be the endowment of agent 1.
- 2. Let X_1, X_2, \ldots, X_m be the bundles of size k such that $X_i R_1 \omega_1$ for each i (ordered according to R_1).
- 3. For $i = 1, 2, \dots, m$:

- (a) Let $X_i = \{\gamma_1, \ldots, \gamma_k\}$
- (b) Let $\alpha_1, \ldots, \alpha_{n-k}$ be the goods not in X_i .
- (c) For each permutation $\{i_1, \ldots, i_k\}$ of $\{1, \ldots, k\}$:
 - i. Let R'_1 induce the following ordering over the singletons: $(\gamma_{i_1}, \ldots, \gamma_{i_k}, \alpha_1, \ldots, \alpha_{n-k})$ (the order of $\alpha_1, \ldots, \alpha_{n-k}$ is arbitrary).
 - ii. Let z be the output of $TTC(R'_1, R_{-1})$
 - iii. If $z_1 \in z$ is the bundle X_i , then return R'_1

4. Return 0

The correctness of our algorithm is a corollary of the following claim.

Claim 9. Suppose the assignment to agent 1 under TTC is $z_1 = \{\gamma_1, \ldots, \gamma_k\}$ if they report R'_1 , with $\gamma_i R'_1 \gamma_{i+1}$ for $1 \leq i < k$. Let R''_1 be a misreport such that $\gamma_i R''_1 \gamma_{i+1}$ for $1 \leq i < k$ and $\gamma_k R''_1 \alpha$ for every good $\alpha \notin z_1$. Then the assignment to agent 1 if they report R''_1 is also z_1

Proof. Let \mathcal{E}' and \mathcal{E}'' be the (otherwise identical) economies in which agent 1 reports R'_1 and R''_1 respectively. For t = 1, 2, ... let H'_t and H''_t be the graphs generated by running TTC on \mathcal{E}' and \mathcal{E}'' respectively. Let $tt'(\alpha)$ and $tt''(\alpha)$ be the trade times of α during a run of TTC on \mathcal{E}' and \mathcal{E}'' respectively. By Observation 4, $tt'(\gamma_1) < tt'(\gamma_i)$ and $tt''(\gamma_1) < tt''(\gamma_i)$ for $1 < i \leq k$. Consider $H'_{tt'(\gamma_1)}$. Clearly, none of the cycles that have been removed from $H'_1, H'_2, \ldots, H'_{tt'(\gamma_1)-1}$ have included any of $\gamma_1, \ldots, \gamma_k$. Therefore we may assume without loss of generality that the same cycles are removed from $H''_1, \ldots, H''_{tt''(\gamma_1)-1}$ and thus $H''_{tt''(\gamma_1)}$ and $H'_{tt'(\gamma_1)}$ are identical. Since γ_1 is assigned to agent 1 in \mathcal{E}' , it must also be assigned to agent 1 in \mathcal{E}'' . The claim follows by induction.

Thus if there is a beneficial misreport such that the assignment to agent 1 is $X = \{\gamma_1, \ldots, \gamma_k\}$, then it is enough to check only those misreports that rank the goods in X above any other goods.

It remains for us to analyse the runtime of Algorithm \mathcal{A} . There are at most $\binom{n}{k} \leq n^k$ bundles that agent 1 can prefer above the endowment. There are k! different permutations of a bundle of size k. So Step 3(c)ii is performed at most $k!n^k$ times. In this step, TTC is called. Since TTC takes polynomial time to perform (it takes at most n steps to find a cycle, and at least one good is removed for each time step), there exists a constant c such that this step takes at most n^c time. The overall run time is therefore at most $k!n^{k+c}$ as required.

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