Artificial Intelligence Review (2023) 56:8955–9008
https://doi.org/10.1007/s10462-022-10376-1

Generalized q-rung orthopair fuzzy interactive Hamacher power average and Heronian means for MADM

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Published online: 18 January 2023

Abstract

In this paper, we establish a novel q-rung orthopair fuzzy (q-ROF) multi-attribute decision making (MADM) model on the basis of the proposed q-ROF interactive Hamacher weighted adjustable power average (q-ROFIHWAPA) and q-ROF interactive Hamacher weighted coordinated Heronian means (HMs), which (1) can reflect the correlations among multiple attributes; (2) weakens the impacts of the extreme evaluation values more reasonably; (3) considers the interactions between the membership degree (MD) and non-membership degree (N-MD) of different q-ROF numbers (q-ROFNs); (4) has the characteristic of generality (It can generate different methods by different operations). Firstly, the q-ROF interactive Hamacher operations, improved score function and new q-ROF entropy (q-ROFE) formula, which are the necessary raw materials for the implementation of MADM, are presented. Secondly, we introduce the adjustable power average (APA) and its weight form (WAPA) to remedy the deficiencies of the classical power averages (PAs). Afterwards we extend the WAPA to q-ROF circumstance and propose the q-ROF interactive Hamacher WAPA (q-ROFIHWAPA), and its basic properties are analyzed. Further, the entropy weight fitting method is presented to determine the parameter carried by the q-ROFIHWAPA. Thirdly, inspired by the evolutionary process of Bonferroni means (BMs), we define the weighted coordinated HM (WCHM) and weighted geometric coordinated HM (WGCHM) based on the traditional HMs, respectively, which eliminate the redundancy of the dual generalized weighted BM (DGWBM) and dual generalized weighted Bonferroni geometric mean (DGWBGM), i.e., the case of $\tau_1 > \tau_2 > \cdots > \tau_n$. Then we develop the q-ROF interactive Hamacher WCHM (q-ROFIHWCHM) and q-ROF interactive Hamacher WGCHM (q-ROFIHWGCHM) by combining them with the q-ROF interactive Hamacher operations, and the common properties and special cases are also investigated. Finally, we create a MADM algorithm relied on the q-ROFIHWAPA and q-ROFIHWCHM (resp. q-ROFIHWGCHM), and a practical example is introduced to illustrate the effectiveness and superiority of the proposed method.

Keywords MADM · q-ROF interactive Hamacher operations · Generalized PAs · Entropy weight fitting method · Generalized HMs

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1 Introduction

MADM is based on the available decision information to rank the limited alternatives in a certain way. For practical MADM problems, it is often difficult for decision makers (DMs) to quantify their views with crisp values. In 1965, Zadeh (1965) introduced the concept of fuzzy set (FS) on the basis of classical set, which vividly characterizes the fuzziness of attributes and opens a new era of fuzzy MADM. However, with regard to any fixed element in the universe of discourse, FS can only rely on the membership degree (MD) to indicate its certainty, which obviously does not give a complete picture of the fuzzy problem. In view of this, Atanassov and Yager improved the FS and proposed the intuitionistic fuzzy set (IFS) (Atanassov 1986), Pythagorean fuzzy set (PFS) (Yager and Abbasov 2013; Yager 2014) and q-ROF set (q-ROFS) (Yager 2017; Yager and Alajlan 2017) in succession by adding the parts of non-membership degree (N-MD) and hesitancy degree (HD). As a matter of fact, q-ROFS is a generalized concept, and IFS and PFS are its special cases (when $q = 1$ and $q = 2$). For a pair (u, v) separated from q-ROFS (i.e., q-ROFN (Liu and Wang 2018)), it satisfies the restrictions:$0 \leq u, v \leq 1$ and $u^q + v^q \leq 1 (q \geq 1)$, and thus the larger q, the wider the space of fuzzy information it delineates. Next, we give an overview on q-ROFS from these aspects: information measures, decision making techniques, the fundamentals of analysis and feasible extensions.

(1) *Information measures* Peng and Liu (2019) gave the axiomatic definitions of several information measures (distance measure, similarity measure, entropy measure and inclusion measure) for q-ROFSs, a series of corresponding calculation formulas and conversion relations among them, and then applied the proposed similarity measure to pattern recognition, clustering analysis and medical diagnosis. Tang et al. (2020) introduced the possibility degree measure for q-ROFNs, which is the basis of ranking q-ROFNs. Khan et al. (2021a, b) explored the knowledge measures for q-ROFS by utilizing inverse tangent function and inverse cosine function.

(2) *Decision making techniques* Usually, we can refine the decision making techniques into the following three divisions: classical methods, means and preference relations (PRs).

    (1) *Classical methods* Wang and Li (2018) presented the q-ROF TOmada de Decisao Interativa e Multicritevio (TODIM) method to rank the green suppliers. Based on the proposed distance measure, Pinar and Boran (2020) developed two independent group decision making (GDM) algorithms, i.e., q-ROF Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) and q-ROF elimination and choice translating reality (ELECTRE), and then applied them to selecting the best supplier for a construction company. Gong et al. (2020) explored the multi-attribute border approximation area comparison (MABAC) method to evaluate the teaching quality of universities. The q-ROF preference ranking organization method for enrichment of evaluations (PROMETHEE) method was successfully applied to selecting the suitable contractor for a construction company (Akram 2021). Appropriate region for carrying out mass vaccination campaigns against COVID-19 was chosen by q-ROF robust Vlsekriterijumska Optimizacija I Kompromisno Resenje (VIKOR) method (Khan et al. 2021c). Alkan and Kahraman (2021) also proposed two novel q-ROF TOPSIS methods to evaluate government strategies against COVID-19. Relied on the q-ROF environment, Arya and Kumar

(2021) consolidated the canonical TODIM and VIKOR methods into a q-ROF TODIM-VIKOR method and employed it to choose the most potential supplier of medical consumption products.

(2) *Means* Under the q-ROF context the weighted averages and weighted geometrics are established on different operations, such as algebraic operations (Liu and Wang 2018), Hamacher operations (Darko and Liang 2020), Dombi operations (Jana et al. 2019) and neutral operations (Garg and Chen 2020). Peng et al. (2018) proposed the q-ROF weighted exponential aggregation (q-ROFWEA) with the help of exponential operational law. Du (2019) presented two generalized weighted averages, which are called q-ROF weighted power means. By combining the sine trigonometric operational law (STOL) with a sequence of weighted averages and weighted geometrics, Garg (2020) derived the q-ROF sine trigonometric weighted averages and weighted geometrics. Zeng et al. (2021) improved the induced ordered weighted logarithmic averaging distance (IOWLAD) based on q-ROF environment. Furthermore, in order to reflect the correlations between attributes, the typical correlation means, such as BM (Bonferroni 1950), HMs (Yu and Wu 2012; Yu 2013) and Maclaurin symmetric mean (MSM) (Maclaurin 1729), have been applied to integrating q-ROF preference information with success (Darko and Liang 2020; Liu and Wang 2019; Wei et al. 2018; Yang et al. 2020; Liu et al. 2020).

(3) *PRs* Zhang et al. (2019) defined the q-ROF preference relation (q-ROFPR) and additive consistent q-ROFPR, and then constructed two goal programming models to obtain the priority weight vectors from individual q-ROFPR and group q-ROFPRs; soon afterwards, Zhang et al. (2020) defined the multiplicative consistent q-ROFPR, proposed two optimization models to attain the priority weight vectors from individual q-ROFPR and group q-ROFPRs, provided the methods for repairing the inconsistent q-ROFPRs and for generating the weight vector of DMs and finally established a GDM algorithm based on them. Li et al. (2019) developed a range of PRs relied on the q-ROF circumstance, including q-ROFPR, additive consistent q-ROFPR, multiplicative consistent q-ROFPR, incomplete q-ROFPR, additive consistent incomplete q-ROFPR, multiplicative consistent incomplete q-ROFPR and acceptable incomplete q-ROFPR. Zhang and Chen (2021a, b) presented quite a few optimization methods to dispose of incomplete and unacceptable (additive and multiplicative) consistent q-ROFPRs and then established the corresponding GDM algorithms.

(3) *The fundamentals of analysis* Gao et al. (2019, 2020) created the q-ROF calculus system, which contains derivative, differential, indefinite and additive definite integrals, etc. Shu et al. (2019) constructed the structure of q-ROF additive and multiplicative double integrals, including their definitions, integrable criteria and fundamental properties; subsequently, Ai et al. (2021) extended the q-ROF additive double integral to the framework of Archimedean t-norms and t-conorms (ATT). Inspired by the membership and non-membership functions of q-ROF function (q-ROFF) in Gao et al. (2019, 2020), Shu et al. (2019), Ai et al. (2021), Ye et al. (2019) proposed the concept of q-rung orthopair single variable fuzzy function (q-ROSVFF) and investigated the differential calculus established on it.

(4) *Feasible extensions*: Joshi et al. (2018) and Wang et al. (2019a) developed the interval valued q-ROFS (IVq-ROFS) or q-rung interval-valued orthopair fuzzy set (q-RIVOFS)

by extending the MD and N-MD into interval numbers within [0, 1]. To adapt the situation that DMs describe their preferences by linguistic terms, the different linguistic contexts were introduced, such as q-rung picture linguistic set (q-RPLS) (Li et al. 2018), linguistic q-ROFS (Lq-ROFS) (Liu and Liu 2019), q-ROF linguistic set (q-ROFLS) (Wang et al. 2019b) and probabilistic linguistic q-ROFS (PLq-ROFS) (Liu and Huang 2020). Xu et al. (2018) presented the q-rung dual hesitant fuzzy set (q-RDHFS) to indicate the DMs' hesitation.

T-norm (TN) and T-conorm (TC) are the necessary ingredients to derive the generalized union and intersection for q-ROFNs. As the generalization of algebraic and Einstein TN and TC, Hamacher TN and TC are more elastic. For this reason, Liu and Wang (2019) and Darko and Liang (2020) introduced the Hamacher sum $\oplus$ and product $\otimes$ for q-ROFNs with the help of them. However, it is easy to see that these two operations have the following defects: (1) If $Q_1 = \langle u_1, v_1 \rangle \left(v_1 \neq 0\right)$ and $Q_2 = \langle u_2, 0 \rangle$, then by addition operation, we have $v_{Q_1 \oplus Q_2} = 0$, which implies that v_1 doesn't work at all and that the operational regulation parameter γ has no effect on $v_{Q_1 \oplus Q_2}$; (2) If $Q_1 = \langle u_1, v_1 \rangle \left(u_1 \neq 0\right)$ and $Q_2 = \langle 0, v_2 \rangle$, then by multiplication operation, we get $u_{Q_1 \otimes Q_2} = 0$, which indicates that u_1 is invalid and the parameter γ is independent of $u_{Q_1 \otimes Q_2}$; (3) Neither of these two operations considers the situation where there exists a certain correction between the MD and N-MD of different q-ROFNs, which needs to be reflected by their interactions. In view of these shortcomings, we extend the interactive Hamacher operations for IFNs (Garg 2016) and PFNs (Wang et al. 2021) into the interactive Hamacher operations for q-ROFNs.

The PA was first introduced by Yager (2001), whose most striking feature is to endow each data with certain credibility by the support and strengthening among the input arguments, so as to highlight the role of the data close to the overall information and weaken the influence of the data deviating from the overall information. To be specific, if a_i is close to the overall information, its total support $T(a_i)$ from other arguments is large, and thus a_i obtain high credibility; otherwise, such data should be evaluated a small weight. After that, Zhou et al. (2012) put forward the generalized PA (GPA) by combining the PA with the generalized mean (Dyckhoff and Pedrycz 1984). It should be made clear that when given a set of input arguments, the GPA (Zhou et al. 2012) can only alter the aggregation result but not the nonlinear weight of each data, and thus it doesn't reflect the most essential characteristic of the PA. Furthermore, we note that the classical weighted PA (WPA) does not satisfy reducibility. For these reasons, we propose the APA and its weight form (WAPA). Subsequently, employing the q-ROF interactive Hamacher operations on the WAPA, we present the q-ROFIHWAPA. Besides, we also propose an approach to determine the parameter carried by the q-ROFIHWAPA, which is called entropy weight fitting method. Incidentally, by this means the weighted nonlinear weights derived from the q-ROFIHWAPA are more objective.

The BM (Bonferroni 1950), whose typical feature is to consider the correlations between any two arguments, was introduced by Bonferroni. Also, Xia et al. (2013) acquired its geometric form, i.e., geometric BM (GeoBM). Further, Beliakov et al. (2010) expanded the BM to the generalized BM (GBM), which reflects the correlations among any three arguments. Then Xia et al. (2012) pointed out that the GBM don't consider the case where $i = j$ or $j = k$ or $i = k$, and it doesn't stress the importance of each argument. To this end, Xia et al. (2012) proposed the generalized weighted BM (GWBM) to revise it, as well as introducing the generalized weighted Bonferroni geometric mean (GWBGM). Afterwards Zhang et al. (2017) developed the dual generalized weighted BM (DGWBM) and dual generalized weighted Bonferroni geometric mean (DGWBGM), which can reflect the correlations among different numbers of attributes by embedding different numbers of parameters to R. Stimulated by the above development process, in this paper, we propose the WCHM and WGCHM on the basis of the HM and geometric HM (GHM), respectively, which can eliminate the redundancy of the DGWBM and DGWBGM, i.e., the case of $\tau_1 > \tau_2 > \cdots > \tau_n$. Then we extend the WCHM and WGCHM to q-ROF environment and propose the q-ROFIHWCHM and q-ROFIHWGCHM based on the interactive Hamacher operation rules.

Our ultimate goal is to establish a MADM algorithm relied on the q-ROFIHWAPA and q-ROFIHWCHM (resp. q-ROFIHWGCHM). More precisely, before aggregating all the individual attribute values of the alternatives into the overall attribute values with the q-ROFIHWCHM or q-ROFIHWGCHM, the weight of each data has been replaced with the weighted nonlinear weight carried by the q-ROFIHWAPA.

The rest of this paper is arranged as follows. In Sect. 2, some basic definitions involved in q-ROF environment are improved, such as Hamacher operations (Darko and Liang 2020; Liu and Wang 2019), score functions (Liu and Wang 2018; Li et al. 2019) and entropy axiomatic definition (Peng and Liu 2019). In Sect. 3, the APA and WAPA are defined to remedy the deficiencies of the PA and WPA. Afterwards the q-ROFIHWAPA is introduced, and its basic properties are analyzed. Furthermore, a MADM model based on the q-ROFIHWAPA and its supporting application example are presented. Finally, the entropy weight fitting method is proposed to determine the parameter carried by the q-ROFIHWAPA. In Sect. 4, the WCHM and WGCHM are defined to eliminate the redundancy of the DGWBM and DGWBGM. Then the q-ROFIHWCHM and q-ROFIHWGCHM are developed, and their related properties and special cases are also explored. In Sects. 5 and 6, a novel q-ROF MADM algorithm is devised by using the q-ROFIHWAPA and q-ROFIHWCHM (resp. q-ROFIHWGCHM), and an application example is presented to illustrate the effectiveness and superiority of the introduced algorithm. Section 7 gives some conclusions.

2 Preliminaries

In this section, we focus mainly on improving some basic definitions involved in q-ROF setting, including Hamacher operations, score functions and entropy axiomatic definition.

2.1 Interactive Hamacher operation rules for q-ROFNs

Definition 2.1 Yager (2017), Yager and Alajlan (2017).

Let X is a finite universe of discourse, a q-ROFS A on X is characterized as:

$$A = \left\{ \langle x, u_A(x), v_A(x) \rangle | x \in X \right\}, \tag{1}$$

where $u_A : X \to [0,1]$ and $v_A : X \to [0,1]$, for any $x \in X$,$u_A(x)$ and $v_A(x)$ represent the MD and N-MD of the element x to A, respectively, with the condition $\left(u_A(x)\right)^q + \left(v_A(x)\right)^q \leq 1\,(q \geq 1)$. Besides,$\pi_A(x) = \left(1 - \left(u_A(x)\right)^q - \left(v_A(x)\right)^q\right)^{\frac{1}{q}}$ is called the HD of the element x to A.

For convenience,$Q = \langle u, v \rangle$ is called a q-ROFN (Liu and Wang 2018), with the conditions:$0 \leq u, v \leq 1$, $u^q + v^q \leq 1\,(q \geq 1)$. Meanwhile, we denote the set of all q-ROFNs as $\mathcal{Q}$.

Definition 2.2 Peng and Liu (2019), Liu et al. (2018).

Let $Q_1 = \langle u_1, v_1\rangle$ and $Q_2 = \langle u_2, v_2\rangle$ be two q-ROFNs, the normalized Hamming distance between Q_1 and Q_2 can be defined as follows:

$$d(Q_1, Q_2) = \frac{1}{2}\left(\left|u_1^q - u_2^q\right| + \left|v_1^q - v_2^q\right| + \left|\pi_1^q - \pi_2^q\right|\right), \tag{2}$$

where $\pi_1 = \left(1 - u_1^q - v_1^q\right)^{\frac{1}{q}}$ and $\pi_2 = \left(1 - u_2^q - v_2^q\right)^{\frac{1}{q}}$.

By the Hamacher TN and TC, Liu and Wang (2019) and Darko and Liang (2020) defined the generalized union and intersection for q-ROFNs, i.e., q-ROF Hamacher sum $\oplus$ and product $\otimes$.

Definition 2.3 Darko and Liang (2020), Liu and Wang (2019).

Let $Q = \langle u, v\rangle$, $Q_1 = \langle u_1, v_1\rangle$ and $Q_2 = \langle u_2, v_2\rangle$ be three q-ROFNs, and $\gamma > 0$, the Hamacher operations for q-ROFNs are shown as follows:

(1) $Q_1 \oplus Q_2 = \left\langle \left(\frac{u_1^q+u_2^q-u_1^q u_2^q-(1-\gamma)u_1^q u_2^q}{1-(1-\gamma)u_1^q u_2^q}\right)^{\frac{1}{q}}, \left(\frac{v_1^q v_2^q}{\gamma+(1-\gamma)\left(v_1^q+v_2^q-v_1^q v_2^q\right)}\right)^{\frac{1}{q}} \right\rangle$;

(2) $Q_1 \otimes Q_2 = \left\langle \left(\frac{u_1^q u_2^q}{\gamma+(1-\gamma)\left(u_1^q+u_2^q-u_1^q u_2^q\right)}\right)^{\frac{1}{q}}, \left(\frac{v_1^q+v_2^q-v_1^q v_2^q-(1-\gamma)v_1^q v_2^q}{1-(1-\gamma)v_1^q v_2^q}\right)^{\frac{1}{q}} \right\rangle$;

(3) $\lambda Q = \left\langle \left(\frac{(1+(\gamma-1)u^q)^\lambda-(1-u^q)^\lambda}{(1+(\gamma-1)u^q)^\lambda+(\gamma-1)(1-u^q)^\lambda}\right)^{\frac{1}{q}}, \left(\frac{\gamma v^{q\lambda}}{(\gamma-1)v^{q\lambda}+(1+(\gamma-1)(1-v^q))^\lambda}\right)^{\frac{1}{q}} \right\rangle$, $\lambda > 0$;

(4) $Q^\lambda = \left\langle \left(\frac{\gamma u^{q\lambda}}{(\gamma-1)u^{q\lambda}+(1+(\gamma-1)(1-u^q))^\lambda}\right)^{\frac{1}{q}}, \left(\frac{(1+(\gamma-1)v^q)^\lambda-(1-v^q)^\lambda}{(1+(\gamma-1)v^q)^\lambda+(\gamma-1)(1-v^q)^\lambda}\right)^{\frac{1}{q}} \right\rangle$, $\lambda > 0$.

It is worth pointing out that the above operations have several drawbacks:

(1) If $Q_1 = \langle u_1, v_1\rangle$ $\left(v_1 \neq 0\right)$ and $Q_2 = \langle u_2, 0\rangle$, then for any $\gamma > 0$, according to addition operation rule, we have

$$Q_1 \oplus Q_2 = \left\langle \left(\frac{u_1^q + u_2^q - u_1^q u_2^q - (1-\gamma)u_1^q u_2^q}{1-(1-\gamma)u_1^q u_2^q}\right)^{\frac{1}{q}}, 0 \right\rangle. \tag{3}$$

It is clear that v_1 doesn't work at all in this case. That is, if $v_2 = 0$, no matter what value v_1 takes, the $v_{Q_1 \oplus Q_2}$ is always 0. Besides, the parameter γ is independent of $v_{Q_1 \oplus Q_2}$. In other words, the parameter γ has no effect on the $v_{Q_1 \oplus Q_2}$ in this case.

(2) If $Q_1 = \langle u_1, v_1\rangle$ $\left(u_1 \neq 0\right)$ and $Q_2 = \langle 0, v_2\rangle$, then for any $\gamma > 0$, by multiplication operation rule, we get

$$Q_1 \otimes Q_2 = \left\langle 0, \left(\frac{v_1^q + v_2^q - v_1^q v_2^q - (1-\gamma)v_1^q v_2^q}{1-(1-\gamma)v_1^q v_2^q}\right)^{\frac{1}{q}} \right\rangle. \tag{4}$$

In this case,u_1 is invalid and the parameter γ is independent of $u_{Q_1 \otimes Q_2}$.

(3) The Hamacher operations for q-ROFNs in Definition 2.3 don't consider the situation where there exists a certain correction between the MD and N-MD of different q-ROFNs, which needs to be reflected by their interactions.

Given the above shortcomings, we now propose novel operation rules.

Definition 2.4 Let $Q = \langle u, v \rangle$, $Q_1 = \langle u_1, v_1 \rangle$ and $Q_2 = \langle u_2, v_2 \rangle$ be three q-ROFNs, and $\gamma > 0$, the interactive Hamacher operations for q-ROFNs are defined as follows:

(1) $Q_1 \oplus Q_2 = \left\langle \left(\frac{\prod_{i=1}^2 (1+(\gamma-1)u_i^q) - \prod_{i=1}^2 (1-u_i^q)}{\prod_{i=1}^2 (1+(\gamma-1)u_i^q) + (\gamma-1)\prod_{i=1}^2 (1-u_i^q)} \right)^{\frac{1}{q}}, \left(\frac{\gamma \prod_{i=1}^2 (1-u_i^q) - \gamma \prod_{i=1}^2 (1-u_i^q-v_i^q)}{\prod_{i=1}^2 (1+(\gamma-1)u_i^q) + (\gamma-1)\prod_{i=1}^2 (1-u_i^q)} \right)^{\frac{1}{q}} \right\rangle$;

(2) $Q_1 \otimes Q_2 = \left\langle \left(\frac{\gamma \prod_{i=1}^2 (1-v_i^q) - \gamma \prod_{i=1}^2 (1-u_i^q-v_i^q)}{\prod_{i=1}^2 (1+(\gamma-1)v_i^q) + (\gamma-1)\prod_{i=1}^2 (1-v_i^q)} \right)^{\frac{1}{q}}, \left(\frac{\prod_{i=1}^2 (1+(\gamma-1)v_i^q) - \prod_{i=1}^2 (1-v_i^q)}{\prod_{i=1}^2 (1+(\gamma-1)v_i^q) + (\gamma-1)\prod_{i=1}^2 (1-v_i^q)} \right)^{\frac{1}{q}} \right\rangle$;

(3) $\lambda Q = \left\langle \left(\frac{(1+(\gamma-1)u^q)^\lambda - (1-u^q)^\lambda}{(1+(\gamma-1)u^q)^\lambda + (\gamma-1)(1-u^q)^\lambda} \right)^{\frac{1}{q}}, \left(\frac{\gamma(1-u^q)^\lambda - \gamma(1-u^q-v^q)^\lambda}{(1+(\gamma-1)u^q)^\lambda + (\gamma-1)(1-u^q)^\lambda} \right)^{\frac{1}{q}} \right\rangle, \lambda > 0$;

(4) $Q^\lambda = \left\langle \left(\frac{\gamma(1-v^q)^\lambda - \gamma(1-u^q-v^q)^\lambda}{(1+(\gamma-1)v^q)^\lambda + (\gamma-1)(1-v^q)^\lambda} \right)^{\frac{1}{q}}, \left(\frac{(1+(\gamma-1)v^q)^\lambda - (1-v^q)^\lambda}{(1+(\gamma-1)v^q)^\lambda + (\gamma-1)(1-v^q)^\lambda} \right)^{\frac{1}{q}} \right\rangle, \lambda > 0$.

Notably, If $\gamma = 1$, the interactive Hamacher operations degenerate into the traditional interactive operations for q-ROFNs presented by Yang et al. (2020), and if $\gamma = 2$, the interactive Hamacher operations degenerate into the interactive Einstein operations for q-ROFNs.

Theorem 2.1. *The results obtained by Definition 2.4 are still q-ROFNs.*

In mathematics, Theorem 2.1 is stated as Q *is closed under the addition, multiplication, scalar multiplication and power.*

Theorem 2.2 *Let* $Q = \langle u, v \rangle$, $Q_1 = \langle u_1, v_1 \rangle$ *and* $Q_2 = \langle u_2, v_2 \rangle$ *be three q-ROFNs, and* $\lambda, \lambda_1, \lambda_2 > 0$, *we have*

(1) $Q_1 \oplus Q_2 = Q_2 \oplus Q_1$; (2) $(Q_1 \oplus Q_2) \oplus Q_3 = Q_1 \oplus (Q_2 \oplus Q_3)$;
(3) $\lambda(Q_1 \oplus Q_2) = \lambda Q_1 \oplus \lambda Q_2$; (4) $(\lambda_1 + \lambda_2)Q = \lambda_1 Q \oplus \lambda_2 Q$;
(5) $\lambda_1(\lambda_2 Q) = \lambda_1 \lambda_2 Q$; (6) $Q_1 \otimes Q_2 = Q_2 \otimes Q_1$;
(7) $(Q_1 \otimes Q_2) \otimes Q_3 = Q_1 \otimes (Q_2 \otimes Q_3)$; (8) $(Q_1 \otimes Q_2)^\lambda = Q_1^\lambda \otimes Q_2^\lambda$;
(9) $Q^{\lambda_1+\lambda_2} = Q^{\lambda_1} \otimes Q^{\lambda_2}$; (10) $(Q^{\lambda_1})^{\lambda_2} = Q^{\lambda_1 \lambda_2}$;

The proofs of Theorems 2.1 and 2.2 are easy to derive, which are omitted here.

2.2 A novel score function for *q*-ROFNs

Definition 2.5 Let $Q = \langle u, v \rangle$ be a q-ROFN, the score function S is defined as.

$$S(Q) = \frac{1}{2}\left(1 + u^q - v^q - \frac{1}{2}\sin\frac{(\pi_Q)^q \pi}{2} \right), \tag{5}$$

where $\pi_Q = (1 - u^q - v^q)^{\frac{1}{q}}$.

Theorem 2.3 *For any q-ROFN* $Q = \langle u, v \rangle$, *the score function S monotonically increases w.r.t.u and monotonically decreases w.r.t.v, respectively.*

Proof Calculate the partial derivatives of S w.r.t.u and v, respectively, i.e.,

$$\begin{aligned} \frac{\partial S}{\partial u} &= \frac{qu^{q-1}}{2}\left(1+\frac{\pi}{4}\cos\frac{(1-u^q-v^q)\pi}{2}\right) \geq 0; \\ \frac{\partial S}{\partial v} &= \frac{qv^{q-1}}{2}\left(\frac{\pi}{4}\cos\frac{(1-u^q-v^q)\pi}{2}-1\right) \leq 0. \end{aligned} \tag{6}$$

Hence the score function S monotonically increases w.r.t.u and monotonically decreases w.r.t.v.

Corollary 2.1 *For any q-ROFN $Q = \langle u, v\rangle$, the score function S satisfies.*

(1) $0 \leq S(Q) \leq 1$;
(2) $S(Q) = 1$ *iff* $Q = \langle 1, 0\rangle$;$S(Q) = 0$ *iff* $Q = \langle 0, 1\rangle$.

Corollary 2.2 *Let* $Q_1 = \langle u_1, v_1\rangle$ *and* $Q_2 = \langle u_2, v_2\rangle$ *be two q-ROFNs, if* $Q_1 \geq Q_2$ $(\Leftrightarrow u_1 \geq u_2$ and $v_1 \leq v_2)$, *then* $S(Q_1) \geq S(Q_2)$.

Apparently, Corollarys 2.1 and 2.2 can be directly derived from Theorem 2.3.

Theorem 2.4 *Let* $Q_1 = \langle u_1, v_1\rangle$ *and* $Q_2 = \langle u_2, v_2\rangle$ *be two q-ROFNs, if* $u_1 = v_1$,$u_2 = v_2$ *and* $\pi_{Q_1} \geq \pi_{Q_2}$, *then* $S(Q_1) \leq S(Q_2)$.

Definition 2.6 Let $Q_1 = \langle u_1, v_1\rangle$ and $Q_2 = \langle u_2, v_2\rangle$ be two q-ROFNs,

(1) If $S(Q_1) > S(Q_2)$, then $Q_1 \succ Q_2$;
(2) If $S(Q_1) < S(Q_2)$, then $Q_1 \prec Q_2$;
(3) If $S(Q_1) = \mathrm{S}(Q_2)$, then $Q_1 \approx Q_2$.

In order to illustrate the rationality and superiority of the proposed score function, we compare it with the existing score functions, as shown in Table 1.

As can be seen from Table 1, the proposed score function S has higher distinguishing ability for q-ROFN in contrast with the score functions $S_{\text{Liu - Wang}}$ and S_{Li} defined by Liu and Wang (2018) and Li et al. (2019), respectively. In other words, when $S_{\text{Liu - Wang}}$ and S_{Li} do not work at all, the proposed score function S can also present significant differences of alternatives. Furthermore, if we continue to use the accuracy function $H_{\text{Liu - Wang}}$ (Liu and Wang 2018) for the two cases in Table 1, we can obtain the ranking results consistent with the proposed score function S. This shows that our method is effective, and compared with the cumbersome recognition process introduced by Liu and Wang (2018) and Li et al. (2019), the proposed score function is more straightforward.

2.3 Improved *q*-ROF entropy axiomatic definition and formula

q-ROFE is a requisite tool to measure the fuzziness and uncertainty of q-ROFSs. In this subsection, we elaborate on the defects of the axiomatic definition of q-ROFE (Peng and Liu 2019) and revise them. Subsequently, we construct a q-ROFE formula based on the revised axiomatic definition.

Table 1 Comparison with existing score functions

Cases	Score values	Ranking results
$Q_1 = \langle 0.5, 0.5\rangle$ $Q_2 = \langle 0.3, 0.3\rangle$	$S_{\text{Liu - Wang}}(Q_1) = S_{\text{Liu - Wang}}(Q_2) = 0\,(q = 1)$	$Q_1 \approx Q_2\,(q = 1)$
	$S_{\text{Liu - Wang}}(Q_1) = S_{\text{Liu - Wang}}(Q_2) = 0\,(q = 2)$	$Q_1 \approx Q_2\,(q = 2)$
	$S_{\text{Liu - Wang}}(Q_1) = S_{\text{Liu - Wang}}(Q_2) = 0\,(q = 3)$	$Q_1 \approx Q_2\,(q = 3)$
	$S_{\text{Li}}(Q_1) = S_{\text{Li}}(Q_2) = 0.5000\,(q = 1)$	$Q_1 \approx Q_2\,(q = 1)$
	$S_{\text{Li}}(Q_1) = S_{\text{Li}}(Q_2) = 0.5000\,(q = 2)$	$Q_1 \approx Q_2\,(q = 2)$
	$S_{\text{Li}}(Q_1) = S_{\text{Li}}(Q_2) = 0.5000\,(q = 3)$	$Q_1 \approx Q_2\,(q = 3)$
	$S(Q_1) = 0.5000,\ S(Q_2) = 0.3531\,(q = 1)$	$Q_1 \succ Q_2\,(q = 1)$
	$S(Q_1) = 0.3232,\ S(Q_2) = 0.2599\,(q = 2)$	$Q_1 \succ Q_2\,(q = 2)$
	$S(Q_1) = 0.2690,\ S(Q_2) = 0.2509\,(q = 3)$	$Q_1 \succ Q_2\,(q = 3)$
$Q_1 = \langle 0.5, 0.1\rangle$ $Q_2 = \langle 0.6, 0.2\rangle$	$S_{\text{Liu - Wang}}(Q_1) = S_{\text{Liu - Wang}}(Q_2) = 0.4000\,(q = 1)$	$Q_1 \approx Q_2\,(q = 1)$
	$S_{\text{Liu - Wang}}(Q_1) = 0.2400,\ S_{\text{Liu - Wang}}(Q_2) = 0.3200\,(q = 2)$	$Q_1 \prec Q_2\,(q = 2)$
	$S_{\text{Liu - Wang}}(Q_1) = 0.1240,\ S_{\text{Liu - Wang}}(Q_2) = 0.2080\,(q = 3)$	$Q_1 \prec Q_2\,(q = 3)$
	$S_{\text{Li}}(Q_1) = 0.6429,\ S_{\text{Li}}(Q_2) = 0.6667\,(q = 1)$	$Q_1 \prec Q_2\,(q = 1)$
	$S_{\text{Li}}(Q_1) = 0.5729,\ S_{\text{Li}}(Q_2) = 0.5858\,(q = 2)$	$Q_1 \prec Q_2\,(q = 2)$
	$S_{\text{Li}}(Q_1) = 0.5455,\ S_{\text{Li}}(Q_2) = 0.5476\,(q = 3)$	$Q_1 \prec Q_2\,(q = 3)$
	$S(Q_1) = 0.5531,\ S(Q_2) = 0.6227\,(q = 1)$	$Q_1 \prec Q_2\,(q = 1)$
	$S(Q_1) = 0.3906,\ S(Q_2) = 0.4577\,(q = 2)$	$Q_1 \prec Q_2\,(q = 2)$
	$S(Q_1) = 0.3169,\ S(Q_2) = 0.3693\,(q = 3)$	$Q_1 \prec Q_2\,(q = 3)$

Definition 2.7 Peng and Liu (2019).

For any $A, B \in q\text{ - ROFSs(X)}$, where $X = \{x_i | i = 1, 2, \ldots, n\}$ is a universe of discourse and $q\text{ - ROFSs(X)}$ is the set of all the q-ROFSs on X, the mapping $E : q\text{ - ROFSs(X)} \to [0, 1]$ is a q-ROFE if E satisfies the conditions as follows:

(C1) $E(A) = 0$ iff A is a crisp set;
(C2) $E(A) = 1$ iff $u_A(x_i) = v_A(x_i)$ for any $x_i \in X$;
(C3) $E(A) = E(A^C)$, where $A^C = \{\langle x_i, v_A(x_i), u_A(x_i)\rangle | x_i \in X\}$;
(C4) $E(A) \le E(B)$ if A is less fuzzy than B, i.e.,
$u_A(x_i) \le u_B(x_i) \le v_B(x_i) \le v_A(x_i)$ or $v_A(x_i) \le v_B(x_i) \le u_B(x_i) \le u_A(x_i)$ for any $x_i \in X$.

We now point out that the above axiomatic definition has several shortcomings:

(1) The condition (C2) only indicates that when the information contained in MD and N-MD is equal, the entropy is the largest, but it doesn't emphasize the amount of information contained in MD and N-MD. For example, $A = \{\langle x, 0, 0\rangle | x \in X\}$ and $B = \{\langle x, 0.4, 0.4\rangle | x \in X\}$ are two q-ROFSs on X. It is evident that A contains more unknown information than B, hence $E(A)$ should be larger than $E(B)$ in intuitive sense.
(2) For the condition (C4), it is clear that

$$u_A(x_i) \le u_B(x_i) \le v_B(x_i) \le v_A(x_i) \text{ or } v_A(x_i) \le v_B(x_i) \le u_B(x_i) \le u_A(x_i)$$

$$\Leftrightarrow \left|(u_A(x_i))^q - (v_A(x_i))^q\right| \geq \left|(u_B(x_i))^q - (v_B(x_i))^q\right|$$
$$\Leftrightarrow d(A_i, A_i^C) \geq d(B_i, B_i^C)$$
$$\Leftrightarrow S(A_i, A_i^C) \leq S(B_i, B_i^C),$$

where $d(A_i, A_i^C) = \left|(u_A(x_i))^q - (v_A(x_i))^q\right|$ is the distance between $A_i = \langle u_A(x_i), v_A(x_i)\rangle$ and its complement $A_i^C = \langle v_A(x_i), u_A(x_i)\rangle$, and $S(A_i, A_i^C) = 1 - d(A_i, A_i^C)$ indicates the similarity measure between A_i and A_i^C; the similar goes for $d(B_i, B_i^C)$ and $S(B_i, B_i^C)$.

It can be concluded that the condition (C4) characterizes the relationship between the similarity measure S_{x_i} and the entropy E_i. To put it more precisely, the higher the similarity measure S_{x_i}, the more fuzzy the ith q-ROFN, thus generating a larger entropy value E_i. On the other hand, it is well known that the HD π_{x_i} is the direct reflection of the uncertainty of ith q-ROFN. Hence the constraint (C4) is incomplete without considering the effect of HD on q-ROFE. Besides, the cases with equal similarity measure cannot be distinguished.

In view of the above shortcomings, now we introduce a new axiomatic definition of q-ROFE which takes the similarity measure and the HD into account.

Definition 2.8 For any $A, B \in q$ - ROFSs(X), where $X = \{x_i | i = 1, 2, \ldots, n\}$ is a universe of discourse and q - ROFSs(X) is the set of all the q-ROFSs on X, the mapping $E : q$ - ROFSs(X) $\to [0, 1]$ is a q-ROFE if E satisfies the conditions as follows:

(C1) $E(A) = 0$ iff A is a crisp set;
(C2') $E(A) = 1$ iff $u_A(x_i) = v_A(x_i) = 0$, for any $x_i \in X$;
(C3) $E(A) = E(A^C)$, where $A^C = \{\langle x_i, v_A(x_i), u_A(x_i)\rangle | x_i \in X\}$;
(C4') $E(A) \geq \frac{1}{n}\sum_{i=1}^{n} \pi_A(x_i)$;
(C5') $E(A) \leq E(B)$ if one of the following conditions holds for any $x_i \in X$:

(1) If $(\pi_A(x_i))^q = (\pi_B(x_i))^q$, then $\left|(u_A(x_i))^q - (v_A(x_i))^q\right| \geq \left|(u_B(x_i))^q - (v_B(x_i))^q\right|$;
(2) If $\left|(u_A(x_i))^q - (v_A(x_i))^q\right| = \left|(u_B(x_i))^q - (v_B(x_i))^q\right|$, then $(\pi_A(x_i))^q \leq (\pi_B(x_i))^q$.

The improved axiomatic definition of q-ROFE has several advantages:

(1) The condition (C2') states that when the information of a q-ROFS is completely unknown, the q-ROFE reaches its maximum, which is 1. Further, it's clear that (C1) and (C2') present the sufficient and necessary conditions for obtaining the maximum and minimum.
(2) The contribution of unknown information to the entropy is absolute, in other words, for each $A_i = \langle u_A(x_i), v_A(x_i)\rangle$ separated from the q-ROFS $A = \{\langle x_i, u_A(x_i), v_A(x_i)\rangle | x_i \in X\}$, its entropy $E(A_i)$ must not be less than its HD $\pi_A(x_i)$. Now let $E(A) = \frac{1}{n}\sum_{i=1}^{n} E(A_i)$, then $E(A) \geq \frac{1}{n}\sum_{i=1}^{n} \pi_A(x_i)$ holds. Therefore, we reach the conclusion that (C4') provides the maximum lower bound of q-ROFE.

(3) The condition (C5') reflects that each individual entropy E_i is a function containing the similarity measure S_{x_i} and the HD π_{x_i} and increases monotonically w.r.t.S_{x_i} and π_{x_i}, respectively. It is very intuitive, the higher the similarity measure S_{x_i}, the more fuzzy the ith q-ROFN, thus leading to a larger entropy value E_i; similarly, the larger the HD π_{x_i}, the higher the unknown degree of the ith q-ROFN, so there must be a larger entropy value E_i corresponding to it.

(4) Quite evidently, compared with Definition 2.7, the proposed axiomatic definition of q-ROFE considers the HD part, which can absorb more fuzzy information and generate objective results.

Theorem 2.5 *For any* $A \in q$ - ROFSs(X),$E(A) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\cos\frac{\pi\left|(u_A(x_i))^q-(v_A(x_i))^q\right|}{2}+(\pi_A(x_i))^q}{2}\right)^{\frac{1}{q}}$ *is an entropy.*

Proof Since $0 \le \left|\left(u_A\left(x_i\right)\right)^q - \left(v_A\left(x_i\right)\right)^q\right| \le 1$ and $0 \le \left(\pi_A\left(x_i\right)\right)^q \le 1$ for any $x_i \in X$, then.

$$0 \le \frac{\cos\frac{\pi\left|(u_A(x_i))^q-(v_A(x_i))^q\right|}{2} + \left(\pi_A\left(x_i\right)\right)^q}{2} \le 1. \tag{7}$$

So we can get $0 \le E(A) \le 1$.

(C1) Given that $0 \le \left|\left(u_A\left(x_i\right)\right)^q - \left(v_A\left(x_i\right)\right)^q\right| \le 1$ and $0 \le \left(\pi_A\left(x_i\right)\right)^q \le 1$ for any $x_i \in X$,

$$E(A) = 0 \Leftrightarrow \cos\frac{\pi\left|\left(u_A\left(x_i\right)\right)^q - \left(v_A\left(x_i\right)\right)^q\right|}{2} + \left(\pi_A\left(x_i\right)\right)^q = 0$$

$$\Leftrightarrow \begin{cases} \cos\frac{\pi\left|\left(u_A\left(x_i\right)\right)^q - \left(v_A\left(x_i\right)\right)^q\right|}{2} = 0 \\ \left(\pi_A\left(x_i\right)\right)^q = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \left|\left(u_A\left(x_i\right)\right)^q - \left(v_A\left(x_i\right)\right)^q\right| = 1 \\ 1 - \left(u_A\left(x_i\right)\right)^q - \left(v_A\left(x_i\right)\right)^q = 0 \end{cases}$$

$$\Leftrightarrow u_A\left(x_i\right) = 1, v_A\left(x_i\right) = 0 \text{ or } u_A\left(x_i\right) = 0, v_A\left(x_i\right) = 1$$

$$\Leftrightarrow A \text{ is a crisp set}$$

(C2') Since $0 \le \left|\left(u_A\left(x_i\right)\right)^q - \left(v_A\left(x_i\right)\right)^q\right| \le 1$ and $0 \le \left(\pi_A\left(x_i\right)\right)^q \le 1$ for any $x_i \in X$, then

$$
\begin{aligned}
E(A)=1 &\Leftrightarrow \frac{\cos\frac{\pi\left|(u_A(x_i))^q-(v_A(x_i))^q\right|}{2}+\left(\pi_A\left(x_i\right)\right)^q}{2}=1\\
&\Leftrightarrow \left|\left(u_A\left(x_i\right)\right)^q-\left(v_A\left(x_i\right)\right)^q\right|=0,\left(\pi_A\left(x_i\right)\right)^q=1\\
&\Leftrightarrow u_A\left(x_i\right)=v_A\left(x_i\right)=0.
\end{aligned}
$$

(C3) Since $A^C=\left\{\left\langle x_i, v_A\left(x_i\right), u_A\left(x_i\right)\right\rangle \middle| x_i\in X\right\}$, then

$$
\begin{aligned}
E(A) &= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\cos\frac{\pi\left|(u_A(x_i))^q-(v_A(x_i))^q\right|}{2}+\left(\pi_A\left(x_i\right)\right)^q}{2}\right)^{\frac{1}{q}}\\
&= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\cos\frac{\pi\left|(v_A(x_i))^q-(u_A(x_i))^q\right|}{2}+\left(\pi_A\left(x_i\right)\right)^q}{2}\right)^{\frac{1}{q}}=E\left(A^C\right).
\end{aligned}
\tag{8}
$$

(C4') To prove $E(A)\geq \frac{1}{n}\sum\limits_{i=1}^{n}\pi_A\left(x_i\right)$, we just need to verify

$$
\left(\frac{\cos\frac{\pi\left|(u_A(x_i))^q-(v_A(x_i))^q\right|}{2}+\left(\pi_A\left(x_i\right)\right)^q}{2}\right)^{\frac{1}{q}}\geq \pi_A\left(x_i\right).
\tag{9}
$$

Besides,

$$
\begin{aligned}
&\left(\frac{\cos\frac{\pi\left|(u_A(x_i))^q-(v_A(x_i))^q\right|}{2}+\left(\pi_A\left(x_i\right)\right)^q}{2}\right)^{\frac{1}{q}}\geq \pi_A\left(x_i\right)\\
&\Leftrightarrow \frac{\cos\frac{\pi\left|(u_A(x_i))^q-(v_A(x_i))^q\right|}{2}+\left(\pi_A\left(x_i\right)\right)^q}{2}\geq\left(\pi_A\left(x_i\right)\right)^q
\end{aligned}
$$

$$
\begin{aligned}
&\Leftrightarrow \cos\frac{\pi\left|\left(u_A\left(x_i\right)\right)^q-\left(v_A\left(x_i\right)\right)^q\right|}{2}\geq\left(\pi_A\left(x_i\right)\right)^q\\
&\Leftrightarrow \cos\frac{\pi\left(\left(u_A\left(x_i\right)\right)^q-\left(v_A\left(x_i\right)\right)^q\right)}{2}+\left(u_A\left(x_i\right)\right)^q+\left(v_A\left(x_i\right)\right)^q-1\geq 0.
\end{aligned}
$$

Apparently, we only need to prove

$$
\cos\frac{\pi\left(\left(u_A\left(x_i\right)\right)^q-\left(v_A\left(x_i\right)\right)^q\right)}{2}+\left(u_A\left(x_i\right)\right)^q+\left(v_A\left(x_i\right)\right)^q-1\geq 0.
\tag{10}
$$

Now we consider the function $f(x,y)=\cos\frac{\pi(x-y)}{2}+x+y-1$, where $0\leq x,y\leq 1$ and $x+y\leq 1$. Let $\frac{\partial f}{\partial x}=-\frac{\pi}{2}\sin\frac{\pi(x-y)}{2}+1=0$ and $\frac{\partial f}{\partial y}=\frac{\pi}{2}\sin\frac{\pi(x-y)}{2}+1=0$, then it's clear that $f(x,y)$ has no critical points in the constraint region, i.e., the minimum points of $f(x,y)$ are only derived on the boundary $x=0$ or $y=0$ or $x+y=1$. Further, we can easily verify that $f(x,y)$ reaches its minimum at $(0,0)$ or $(0,1)$ or $(1,0)$, which is 0. Thus, we have proved $f(x,y)\geq 0$, i.e., Eq. (10) holds.

(C5') For any $x_i \in X$, the following two cases are straightforward:

(1) If $\left(\pi_A\left(x_i\right)\right)^q = \left(\pi_B\left(x_i\right)\right)^q$, and given $\left|\left(u_A\left(x_i\right)\right)^q - \left(v_A\left(x_i\right)\right)^q\right| \geq \left|\left(u_B\left(x_i\right)\right)^q - \left(v_B\left(x_i\right)\right)^q\right|$, then

$$\left(\frac{\cos\frac{\pi\left|\left(u_A\left(x_i\right)\right)^q-\left(v_A\left(x_i\right)\right)^q\right|}{2} + \left(\pi_A\left(x_i\right)\right)^q}{2}\right)^{\frac{1}{q}} \leq \left(\frac{\cos\frac{\pi\left|\left(u_B\left(x_i\right)\right)^q-\left(v_B\left(x_i\right)\right)^q\right|}{2} + \left(\pi_B\left(x_i\right)\right)^q}{2}\right)^{\frac{1}{q}}; \tag{11}$$

(2) If $\left|\left(u_A\left(x_i\right)\right)^q - \left(v_A\left(x_i\right)\right)^q\right| = \left|\left(u_B\left(x_i\right)\right)^q - \left(v_B\left(x_i\right)\right)^q\right|$, and given $\left(\pi_A\left(x_i\right)\right)^q \leq \left(\pi_B\left(x_i\right)\right)^q$, then

$$\left(\frac{\cos\frac{\pi\left|\left(u_A\left(x_i\right)\right)^q-\left(v_A\left(x_i\right)\right)^q\right|}{2} + \left(\pi_A\left(x_i\right)\right)^q}{2}\right)^{\frac{1}{q}} \leq \left(\frac{\cos\frac{\pi\left|\left(u_B\left(x_i\right)\right)^q-\left(v_B\left(x_i\right)\right)^q\right|}{2} + \left(\pi_B\left(x_i\right)\right)^q}{2}\right)^{\frac{1}{q}}. \tag{12}$$

Hence $E(A) \leq E(B)$ holds.

Thus, we have completed the proof of Theorem 2.5. □

3 Generalized *q*-ROF interactive Hamacher PA for processing MADM

In this section, we first introduce the APA and its weight form (WAPA) to remedy the deficiencies of the PA and its weight form (WPA). Then we apply the q-ROF interactive Hamacher operations to the WAPA and propose the q-ROFIHWAPA. Moreover, we present a MADM algorithm and its application example based on the q-ROFIHWAPA. Finally, according to the results of the application example, we propose a method to determine the parameter carried by the q-ROFIHWAPA.

3.1 APA and WAPA

Definition 3.1 Yager (2001).
Let $a_1, a_2, \ldots, a_n$ be n real numbers, the PA:$\mathbb{R}^n \to \mathbb{R}$ is defined as follows:

$$\mathrm{PA}(a_1, a_2, \ldots, a_n) = \sum\nolimits_{i=1}^n \frac{1 + T(a_i)}{\sum\nolimits_{r=1}^n \left(1 + T(a_r)\right)} a_i, \tag{13}$$

where $T(a_i) = \sum\limits_{j=1, j\neq i}^{n} \mathrm{Sup}(a_i, a_j)$, and $\mathrm{Sup}(a_i, a_j)$ denotes the support for a_i from a_j. In particular,$\mathrm{Sup}(a_i, a_j)$ satisfies the following properties:

(1) $\mathrm{Sup}(a_i, a_j) \in [0, 1]$;
(2) $\mathrm{Sup}(a_i, a_j) = \mathrm{Sup}(a_j, a_i)$;
(3) $\mathrm{Sup}(a_i, a_j) \geq \mathrm{Sup}(a_s, a_t)$, if $\left|a_i - a_j\right| < \left|a_s - a_t\right|$.

The most noteworthy feature of the PA is to endow each data with certain credibility by the support and strengthening among the input arguments, so as to highlight the role of the data close to the overall information and weaken the influence of the data deviating from the overall information. To be concrete, if a_i is close to the overall information, its total support $T(a_i)$ from other arguments is large, and thus a_i obtain high credibility; otherwise, such data should be evaluated a small weight. However, given the input arguments, the total support of each data from other data is correspondingly fixed, so that the nonlinear weight of each data is also determined. In view of this, we can match different adjustment coefficients for the total supports $T(a_i)\,(i = 1, 2, \ldots, n)$ to make them dynamic, and thus yielding various weights.

Definition 3.2 Yager (2001). Let $a_1, a_2, \ldots, a_n$ be n real numbers with the weights $\omega_1, \omega_2, \ldots, \omega_n$ such that $\omega_i \geq 0$ and $\sum\limits_{i=1}^{n} \omega_i = 1$, the WPA:$\mathbb{R}^n \to \mathbb{R}$ is defined as follows:

$$\mathrm{WPA}(a_1, a_2, \ldots, a_n) = \sum\nolimits_{i=1}^{n} \frac{\omega_i\left(1 + T^{\omega}(a_i)\right)}{\sum\nolimits_{r=1}^{n} \omega_r\left(1 + T^{\omega}(a_r)\right)} a_i, \tag{14}$$

where $T^{\omega}(a_i) = \sum\limits_{j=1, j\neq i}^{n} \omega_j \mathrm{Sup}(a_i, a_j)$, and $\mathrm{Sup}(a_i, a_j)$ is exactly the same as Definition 3.1.

Obviously, when $\omega_i = \frac{1}{n}\,(i = 1, 2, \ldots, n)$, the weighted nonlinear weights $\frac{\omega_i\left(1+T^{\omega}(a_i)\right)}{\sum\limits_{r=1}^{n} \omega_r\left(1+T^{\omega}(a_r)\right)}\,(i = 1, 2, \ldots, n)$ degenerate to $\frac{1+\frac{1}{n}T(a_i)}{\sum\limits_{r=1}^{n}\left(1+\frac{1}{n}T(a_r)\right)}\,(i = 1, 2, \ldots, n)$, which are not equal to $\frac{1+T(a_i)}{\sum\limits_{r=1}^{n}\left(1+T(a_r)\right)}\,(i = 1, 2, \ldots, n)$, i.e., the WPA does not satisfy reducibility.

Considering the above deficiencies, we now propose the APA and WAPA.

Definition 3.3 Let $a_1, a_2, \ldots, a_n$ be n real numbers, the APA:$\mathbb{R}^n \to \mathbb{R}$ is defined as follows:

$$\mathrm{APA}(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} \frac{1 + \left(T(a_i)\right)^{\lambda}}{\sum\nolimits_{r=1}^{n}\left(1 + \left(T(a_r)\right)^{\lambda}\right)} a_i, \tag{15}$$

where $\lambda \geq 1$, and $T(a_i)$ is exactly the same as Definition 3.1.

Compared with the classical PA, the added parameter λ in Definition 3.3 can be regarded as an adjustment coefficient for the total supports $T(a_i)\,(i = 1, 2, \ldots, n)$.

Definition 3.4 Let $a_1, a_2, \ldots, a_n$ be n real numbers with the weights $\omega_1, \omega_2, \ldots, \omega_n$ such that $\omega_i \geq 0$ and $\sum\limits_{i=1}^{n} \omega_i = 1$, the WAPA:$\mathbb{R}^n \to \mathbb{R}$ is defined as follows:

$$\mathrm{WAPA}(a_1, a_2, \ldots, a_n) = \sum\nolimits_{i=1}^{n} \frac{\omega_i^{\lambda} + \left(T^{\omega}(a_i)\right)^{\lambda}}{\sum\limits_{r=1}^{n}\left(\omega_r^{\lambda} + \left(T^{\omega}(a_r)\right)^{\lambda}\right)} a_i, \tag{16}$$

where $\lambda \geq 1$, and $T^{\omega}(a_i)$ is exactly the same as Definition 3.2.

Remark 3.1 The following special cases can be directly derived from Definition 3.4.

(1) If $\lambda = 1$, then the WAPA is called the revised WPA (RWPA);
(2) If $\omega_1 = \omega_2 = \cdots = \omega_n = \frac{1}{n}$, then the WAPA degenerates into the APA;
(3) If $\lambda = 1$ and $\omega_1 = \omega_2 = \cdots = \omega_n = \frac{1}{n}$, then the WAPA degenerates into the PA.

Theorem 3.1 *The WAPA satisfies the following properties*:

(1) (*Idempotency*) *If* $a_1 = a_2 = \cdots = a_n = a$, *then* $\mathrm{WAPA}(a_1, a_2, \ldots, a_n) = a$;
(2) (*Monotonicity*) *If* $a_i'(i = 1, 2, \ldots, n)$ *are another set of real numbers*, *which have exactly the same weights as* $a_i(i = 1, 2, \ldots, n)$, *and* $T^{\omega}(a_i) = T^{\omega}(a_i')$ *and* $a_i \le a_i'$ *for each* i, *then*.

$$\mathrm{WAPA}\left(a_1, a_2, \ldots, a_n\right) \le \mathrm{WAPA}\left(a_1', a_2', \ldots, a_n'\right),$$

(3) (*Boundedness*)$\min_i \{a_i\} \le \mathrm{WAPA}(a_1, a_2, \ldots, a_n) \le \max_i \{a_i\}$;
(4) (*Commutativity*)$\mathrm{WAPA}(a_1, a_2, \ldots, a_n)$ *is not changed if* $a_1, a_2, \ldots, a_n$ *and the weights* $\omega_1, \omega_2, \ldots, \omega_n$ *are permuted simultaneously.*

The proof of Theorem 3.1 *is easy to derive*, *which is omitted here*.

3.2 *q*-ROFIHWAPA

Definition 3.5 Let $Q_1, Q_2, \ldots, Q_n$ be n q-ROFNs with the weights $\omega_1, \omega_2, \ldots, \omega_n$ such that $\omega_i \ge 0$ and $\sum_{i=1}^{n} \omega_i = 1$, the q-ROFIHWAPA:$\mathcal{Q}^n \to \mathcal{Q}$ is defined as follows:

$$q\text{ - ROFIHWAPA}(Q_1, Q_2, \ldots, Q_n) = \mathop{\oplus}_{i=1}^{n} \frac{\omega_i^{\lambda} + \left(T^{\omega}(Q_i)\right)^{\lambda}}{\sum_{r=1}^{n}\left(\omega_r^{\lambda} + \left(T^{\omega}(Q_r)\right)^{\lambda}\right)} Q_i, \tag{17}$$

where $\lambda \ge 1$,$T^{\omega}(Q_i) = \sum_{j=1, j\ne i}^{n} \omega_j \mathrm{Sup}(Q_i, Q_j)$, and $\mathrm{Sup}(Q_i, Q_j)$ denotes the support for Q_i from Q_j. Especially, $\mathrm{Sup}(Q_i, Q_j)$ satisfies the following properties:

(1) $\mathrm{Sup}(Q_i, Q_j) \in [0, 1]$;
(2) $\mathrm{Sup}(Q_i, Q_j) = \mathrm{Sup}(Q_j, Q_i)$;
(3) $\mathrm{Sup}(Q_i, Q_j) \ge \mathrm{Sup}(Q_s, Q_t)$, if $d\left(Q_i, Q_j\right) < d\left(Q_s, Q_t\right)$.

Let $\Delta_i^{\omega} = \frac{\omega_i^{\lambda} + \left(T^{\omega}(Q_i)\right)^{\lambda}}{\sum_{r=1}^{n}\left(\omega_r^{\lambda} + \left(T^{\omega}(Q_r)\right)^{\lambda}\right)}$, Eq. (17) is simplified to

$$q\text{ - ROFIHWAPA}(Q_1, Q_2, \ldots, Q_n) = \mathop{\oplus}_{i=1}^{n} \Delta_i^{\omega} Q_i. \tag{18}$$

Lemma 3.1 *Let* $Q_i = \langle u_i, v_i \rangle$ $(i = 1, 2, \ldots, n)$ *be n q-ROFNs*, *and* $\gamma > 0$, *we get*.

$$\mathop{\oplus}_{i=1}^{n} Q_i = \left\langle \left(\frac{\prod_{i=1}^{n}\left(1 + (\gamma - 1)u_i^q\right) - \prod_{i=1}^{n}\left(1 - u_i^q\right)}{\prod_{i=1}^{n}\left(1 + (\gamma - 1)u_i^q\right) + (\gamma - 1)\prod_{i=1}^{n}\left(1 - u_i^q\right)} \right)^{\frac{1}{q}}, \left(\frac{\gamma\prod_{i=1}^{n}\left(1 - u_i^q\right) - \gamma\prod_{i=1}^{n}\left(1 - u_i^q - v_i^q\right)}{\prod_{i=1}^{n}\left(1 + (\gamma - 1)u_i^q\right) + (\gamma - 1)\prod_{i=1}^{n}\left(1 - u_i^q\right)} \right)^{\frac{1}{q}} \right\rangle. \tag{19}$$

Proof We prove Eq. (19) by the mathematical induction.

Setting $n = 2$, it is clear that

$$\overset{2}{\underset{i=1}{\oplus}} Q_i = Q_1 \oplus Q_2$$

$$= \left\langle \left(\frac{\prod_{i=1}^{2}\left(1+(\gamma-1)u_i^q\right) - \prod_{i=1}^{2}\left(1-u_i^q\right)}{\prod_{i=1}^{2}\left(1+(\gamma-1)u_i^q\right) + (\gamma-1)\prod_{i=1}^{2}\left(1-u_i^q\right)} \right)^{\frac{1}{q}}, \left(\frac{\gamma\prod_{i=1}^{2}\left(1-u_i^q\right) - \gamma\prod_{i=1}^{2}\left(1-u_i^q-v_i^q\right)}{\prod_{i=1}^{2}\left(1+(\gamma-1)u_i^q\right) + (\gamma-1)\prod_{i=1}^{2}\left(1-u_i^q\right)} \right)^{\frac{1}{q}} \right\rangle, \tag{20}$$

and thus Eq. (19) holds for $n = 2$.
Suppose that Eq. (19) holds for $n = k$, i.e.,

$$\overset{k}{\underset{i=1}{\oplus}} Q_i = \left\langle \left(\frac{\prod_{i=1}^{k}\left(1+(\gamma-1)u_i^q\right) - \prod_{i=1}^{k}\left(1-u_i^q\right)}{\prod_{i=1}^{k}\left(1+(\gamma-1)u_i^q\right) + (\gamma-1)\prod_{i=1}^{k}\left(1-u_i^q\right)} \right)^{\frac{1}{q}}, \left(\frac{\gamma\prod_{i=1}^{k}\left(1-u_i^q\right) - \gamma\prod_{i=1}^{k}\left(1-u_i^q-v_i^q\right)}{\prod_{i=1}^{k}\left(1+(\gamma-1)u_i^q\right) + (\gamma-1)\prod_{i=1}^{k}\left(1-u_i^q\right)} \right)^{\frac{1}{q}} \right\rangle. \tag{21}$$

Then we derive

$$\overset{k+1}{\underset{i=1}{\oplus}} Q_i = \left(\overset{k}{\underset{i=1}{\oplus}} Q_i\right) \oplus Q_{k+1}$$

$$= \left\langle \left(\frac{\prod_{i=1}^{k}\left(1+(\gamma-1)u_i^q\right) - \prod_{i=1}^{k}\left(1-u_i^q\right)}{\prod_{i=1}^{k}\left(1+(\gamma-1)u_i^q\right) + (\gamma-1)\prod_{i=1}^{k}\left(1-u_i^q\right)} \right)^{\frac{1}{q}}, \left(\frac{\gamma\prod_{i=1}^{k}\left(1-u_i^q\right) - \gamma\prod_{i=1}^{k}\left(1-u_i^q-v_i^q\right)}{\prod_{i=1}^{k}\left(1+(\gamma-1)u_i^q\right) + (\gamma-1)\prod_{i=1}^{k}\left(1-u_i^q\right)} \right)^{\frac{1}{q}} \right\rangle \oplus \left\langle u_{k+1}, v_{k+1} \right\rangle$$

$$= \left\langle \left(\frac{\prod_{i=1}^{k+1}\left(1+(\gamma-1)u_i^q\right) - \prod_{i=1}^{k+1}\left(1-u_i^q\right)}{\prod_{i=1}^{k+1}\left(1+(\gamma-1)u_i^q\right) + (\gamma-1)\prod_{i=1}^{k+1}\left(1-u_i^q\right)} \right)^{\frac{1}{q}}, \left(\frac{\gamma\prod_{i=1}^{k+1}\left(1-u_i^q\right) - \gamma\prod_{i=1}^{k+1}\left(1-u_i^q-v_i^q\right)}{\prod_{i=1}^{k+1}\left(1+(\gamma-1)u_i^q\right) + (\gamma-1)\prod_{i=1}^{k+1}\left(1-u_i^q\right)} \right)^{\frac{1}{q}} \right\rangle, \tag{22}$$

and it follows that Eq. (19) holds for $n = k + 1$.
Thus, it is concluded that Eq. (19) holds for all n. □

Theorem 3.2 *Let $Q_i = \langle u_i, v_i \rangle\, (i = 1, 2, \ldots, n)$ in Eq.* (18)*, the aggregated value of the q-ROFIHWAPA is shown in Eq.* (23)*, which is still a q-ROFN.*

$$q\text{ - ROFIHWAPA}\left(Q_1, Q_2, \ldots, Q_n\right)$$

$$= \left\langle \left(\frac{\prod_{i=1}^{n}\left(1+(\gamma-1)u_i^q\right)^{\Delta_i^{\omega}} - \prod_{i=1}^{n}\left(1-u_i^q\right)^{\Delta_i^{\omega}}}{\prod_{i=1}^{n}\left(1+(\gamma-1)u_i^q\right)^{\Delta_i^{\omega}} + (\gamma-1)\prod_{i=1}^{n}\left(1-u_i^q\right)^{\Delta_i^{\omega}}} \right)^{\frac{1}{q}}, \left(\frac{\gamma\prod_{i=1}^{n}\left(1-u_i^q\right)^{\Delta_i^{\omega}} - \gamma\prod_{i=1}^{n}\left(1-u_i^q-v_i^q\right)^{\Delta_i^{\omega}}}{\prod_{i=1}^{n}\left(1+(\gamma-1)u_i^q\right)^{\Delta_i^{\omega}} + (\gamma-1)\prod_{i=1}^{n}\left(1-u_i^q\right)^{\Delta_i^{\omega}}} \right)^{\frac{1}{q}} \right\rangle. \tag{23}$$

Proof According to scalar multiplication rule for q-ROFNs, we have.

$$\Delta_i^{\omega} Q_i = \left\langle \left(\frac{\left(1+(\gamma-1)u_i^q\right)^{\Delta_i^{\omega}} - \left(1-u_i^q\right)^{\Delta_i^{\omega}}}{\left(1+(\gamma-1)u_i^q\right)^{\Delta_i^{\omega}} + (\gamma-1)\left(1-u_i^q\right)^{\Delta_i^{\omega}}} \right)^{\frac{1}{q}}, \left(\frac{\gamma\left(1-u_i^q\right)^{\Delta_i^{\omega}} - \gamma\left(1-u_i^q-v_i^q\right)^{\Delta_i^{\omega}}}{\left(1+(\gamma-1)u_i^q\right)^{\Delta_i^{\omega}} + (\gamma-1)\left(1-u_i^q\right)^{\Delta_i^{\omega}}} \right)^{\frac{1}{q}} \right\rangle. \tag{24}$$

In addition, by Lemma 3.1, we obtain

$$\bigoplus_{i=1}^{n} \Delta_i^{\omega} Q_i = \left\langle \left(\frac{\prod_{i=1}^{n}\left(1+(\gamma-1)u_i^q\right)^{\Delta_i^{\omega}} - \prod_{i=1}^{n}\left(1-u_i^q\right)^{\Delta_i^{\omega}}}{\prod_{i=1}^{n}\left(1+(\gamma-1)u_i^q\right)^{\Delta_i^{\omega}} + (\gamma-1)\prod_{i=1}^{n}\left(1-u_i^q\right)^{\Delta_i^{\omega}}} \right)^{\frac{1}{q}}, \left(\frac{\gamma\prod_{i=1}^{n}\left(1-u_i^q\right)^{\Delta_i^{\omega}} - \gamma\prod_{i=1}^{n}\left(1-u_i^q-v_i^q\right)^{\Delta_i^{\omega}}}{\prod_{i=1}^{n}\left(1+(\gamma-1)u_i^q\right)^{\Delta_i^{\omega}} + (\gamma-1)\prod_{i=1}^{n}\left(1-u_i^q\right)^{\Delta_i^{\omega}}} \right)^{\frac{1}{q}} \right\rangle. \tag{25}$$

Because of the closure of $\mathcal{Q}$ under the addition and scalar multiplication, for $Q_i \in \mathcal{Q}\,(i=1,2,\ldots,n)$, we get

$$q\text{ - ROFIHWAPA}\left(Q_1, Q_2, \ldots, Q_n\right) = \bigoplus_{i=1}^{n} \Delta_i^{\omega} Q_i \in \mathcal{Q}$$

which means $q\text{ - ROFIHWAPA}\left(Q_1, Q_2, \ldots, Q_n\right)$ is still a q-ROFN.
Hence the proof of Theorem 3.2 is completed.

Theorem 3.3 *The q-ROFIHWAPA satisfies the following properties*:

(1) (*Idempotency*) *If* $Q_1 = Q_2 = \cdots = Q_n = Q$, *then* $q\text{ - ROFIHWAPA}\left(Q_1, Q_2, \ldots, Q_n\right) = Q$;
(2) (*Boundedness*) *If* $Q^- = \langle 0, 1\rangle$ *and* $Q^+ = \langle 1, 0\rangle$, *then* $Q^- \le q\text{ - ROFIHWAPA}\left(Q_1, Q_2, \ldots, Q_n\right) \le Q^+$;
(3) (*Commutativity*)$q\text{ - ROFIHWAPA}\left(Q_1, Q_2, \ldots, Q_n\right)$ *is not changed if* $Q_1, Q_2, \ldots, Q_n$ *and the weights* $\omega_1, \omega_2, \ldots, \omega_n$ *are permuted simultaneously.*

Proof

(1) If $Q_i = Q\,(i=1,2,\ldots,n)$, then.

$$\begin{aligned} q\text{ - ROFIHWAPA}\left(Q_1, Q_2, \ldots, Q_n\right) &= q\text{ - ROFIHWAPA}(Q, Q, \ldots, Q) \\ &= \bigoplus_{i=1}^{n} \frac{\omega_i^{\lambda} + \left(T^{\omega}(Q_i)\right)^{\lambda}}{\sum_{r=1}^{n}\left(\omega_r^{\lambda} + \left(T^{\omega}(Q_r)\right)^{\lambda}\right)} Q = \frac{\sum_{i=1}^{n}\left(\omega_i^{\lambda} + \left(T^{\omega}(Q_i)\right)^{\lambda}\right)}{\sum_{r=1}^{n}\left(\omega_r^{\lambda} + \left(T^{\omega}(Q_r)\right)^{\lambda}\right)} Q = Q. \end{aligned} \tag{26}$$

(2) This is straightforward because Q^- and Q^+ are the bottom and top of the q-ROFNs, respectively.
(3) If $\left(Q_1', Q_2', \ldots, Q_n'\right)$ is the permutation of $\left(Q_1, Q_2, \ldots, Q_n\right)$ and $\left(\omega_1', \omega_2', \ldots, \omega_n'\right)$ is the permutation of $\left(\omega_1, \omega_2, \ldots, \omega_n\right)$, then

$$q\text{ - ROFIHWAPA}\left(Q_1, Q_2, \ldots, Q_n\right) = \bigoplus_{i=1}^{n} \frac{\omega_i^{\lambda} + \left(T^{\omega}(Q_i)\right)^{\lambda}}{\sum_{r=1}^{n}\left(\omega_r^{\lambda} + \left(T^{\omega}(Q_r)\right)^{\lambda}\right)} Q_i$$
$$= \bigoplus_{i=1}^{n} \frac{\omega_i'^{\lambda} + \left(T^{\omega}(Q_i')\right)^{\lambda}}{\sum_{r=1}^{n}\left(\omega_r'^{\lambda} + \left(T^{\omega}(Q_r')\right)^{\lambda}\right)} Q_i' = q\text{ - ROFIHWAPA}\left(Q_1', Q_2', \ldots, Q_n'\right). \tag{27}$$

Thus, we have proved Theorem 3.3.

Example 3.1 Let $Q_1 = \langle 0.8, 0.7\rangle$, $Q_2 = \langle 0.5, 0.9\rangle$, $Q_3 = \langle 0.7, 0.7\rangle$ and $Q_4 = \langle 0.6, 0.5\rangle$ be four q-ROFNs, whose weights are 0.4, 0.3,0.2 and 0.1, respectively. Now we fuse Q_1,Q_2,Q_3 and Q_4 by the q-ROFIHWAPA, where we set $q = 3$, $\lambda = 2$ and $\gamma = 3$.

(1) Calculate the supports $\mathrm{Sup}\left(Q_i, Q_j\right) = 1 - d\left(Q_i, Q_j\right)$ $(i,j = 1,2,3,4,\ i \neq j)$, where $d\left(Q_i, Q_j\right)$ is the normalized Hamming distance between Q_i and Q_j, which is given in Eq. (2).

$$\mathrm{Sup}\left(Q_1, Q_2\right) = \mathrm{Sup}\left(Q_2, Q_1\right) = 0.6130,\ \mathrm{Sup}\left(Q_1, Q_3\right) = \mathrm{Sup}\left(Q_3, Q_1\right) = 0.8310,\ \mathrm{Sup}\left(Q_1, Q_4\right) = \mathrm{Sup}\left(Q_4, Q_1\right) = 0.4860,$$
$$\mathrm{Sup}\left(Q_2, Q_3\right) = \mathrm{Sup}\left(Q_3, Q_2\right) = 0.6140,\ \mathrm{Sup}\left(Q_2, Q_4\right) = \mathrm{Sup}\left(Q_4, Q_2\right) = 0.3960,\ \mathrm{Sup}\left(Q_3, Q_4\right) = \mathrm{Sup}\left(Q_4, Q_3\right) = 0.6550.$$

(2) Calculate the total weighted supports $T\left(Q_i\right)$ $(i = 1,2,3,4)$ by combining the weights 0.4, 0.3,0.2 and 0.1:

$$T\left(Q_1\right) = 0.3987,\ T\left(Q_2\right) = 0.4076,\ T\left(Q_3\right) = 0.5821,\ T\left(Q_4\right) = 0.4442.$$

(3) Calculate the weighted nonlinear weights Δ_i^{ω} $(i = 1,2,3,4)$:

$$\Delta_1^{\omega} = 0.2747,\ \Delta_2^{\omega} = 0.2206,\ \Delta_3^{\omega} = 0.3262,\ \Delta_4^{\omega} = 0.1785.$$

(4) By Eq. (23), we get

$$q\text{ - ROFIHWAPA}\left(Q_1, Q_2, Q_3, Q_4\right) = \langle 0.6851, 0.7542\rangle.$$

3.3 A MADM algorithm based on the q-ROFIHWAPA

For a q-ROF MADM problem, let $A_1, A_2, \ldots, A_m$ be m alternatives, and let $C_1, C_2, \ldots, C_n$ be n attributes, whose weights are $\omega_1, \omega_2, \ldots, \omega_n$, respectively, such that $\omega_j \geq 0$ and $\sum_{j=1}^{n} \omega_j = 1$. Assume that the evaluation value of the alternative A_i regarding the attribute C_j provided by the DM is a q-ROFN $\hat{Q}_{ij} = \left\langle \hat{u}_{ij}, \hat{v}_{ij}\right\rangle$, and then the q-ROF decision matrix $\widehat{EX} = \left(\hat{Q}_{ij}\right)_{m\times n}$ is established as i and j traverse.

Next, we present the detailed operation steps:

Step 1 Transform the q-ROF decision matrix $\widehat{EX} = \left(\hat{Q}_{ij}\right)_{m\times n}$ into the normalized decision matrix $EX = \left(Q_{ij}\right)_{m\times n}$, where

$$Q_{ij} = \left\langle u_{ij}, v_{ij} \right\rangle = \begin{cases} \left\langle \hat{u}_{ij}, \hat{v}_{ij} \right\rangle, \text{ for benefit - type attribute } C_j \\ \left\langle \hat{v}_{ij}, \hat{u}_{ij} \right\rangle, \text{ for cost - type attribute } C_j \end{cases}, i = 1, 2, \dots, m, j = 1, 2, \dots, n. \tag{28}$$

Step 2 Calculate the supports between the jth attribute and the tth attribute $\mathrm{Sup}_{jt} = \left(\mathrm{Sup}(Q_{ij}, Q_{it})\right)_{m\times 1}$,

where

$$\mathrm{Sup}(Q_{ij}, Q_{it}) = 1 - d(Q_{ij}, Q_{it}),\ i = 1, 2, \dots, m,\ j, t = 1, 2, \dots, n,\ j \neq t. \tag{29}$$

Herein, we assume that $d(Q_{ij}, Q_{it})$ is the normalized Hamming distance between Q_{ij} and Q_{it}, which is given in Eq. (2).

Step 3 Calculate the total weighted support matrix $T = \left(T^{\omega}\left(Q_{ij}\right)\right)_{m\times n}$ by combining the attributive weights $\omega_1, \omega_2, \dots, \omega_n$, where

$$T^{\omega}\left(Q_{ij}\right) = \sum_{t=1, t\neq j}^{n} \omega_t \mathrm{Sup}(Q_{ij}, Q_{it}),\ i = 1, 2, \dots, m,\ j, t = 1, 2, \dots, n,\ j \neq t. \tag{30}$$

Step 4 Fix the parameter λ and calculate the weighted nonlinear weight matrix $\Delta_{\lambda}^{\omega} = \left(\Delta_{\lambda}^{\omega}\left(Q_{ij}\right)\right)_{m\times n}$, where

$$\Delta_{\lambda}^{\omega}\left(Q_{ij}\right) = \frac{\omega_j^{\lambda} + \left(T^{\omega}\left(Q_{ij}\right)\right)^{\lambda}}{\sum_{r=1}^{n} \left(\omega_r^{\lambda} + \left(T^{\omega}\left(Q_{ir}\right)\right)^{\lambda}\right)}, i = 1, 2, \dots, m,\ j, r = 1, 2, \dots, n. \tag{31}$$

Step 5 Use the q-ROFIHWAPA (Eq. 23) to aggregate all the individual attribute values of the alternative A_i into the overall attribute value Q_i.
Step 6 By Eq. (5) and Definition 2.6, calculate the score values $S\left(Q_i\right)(i = 1, 2, \dots, m)$ of the overall attribute values $Q_i(i = 1, 2, \dots, m)$ and rank the alternatives to select the best alternative.
Step 7 End.

3.4 Application example

Example 3.2 Assume that a chain supermarket enterprise intends to choose one of the four locations A_1,A_2,A_3 and A_4 to open a branch store based on the following attributes: the population density (C_1), the consumption capacity (C_2) and the commercial potential (C_3), whose weights are 0.40, 0.35 and 0.25, respectively. The q-ROF decision matrix $\widehat{EX} = \left(\hat{Q}_{ij}\right)_{4\times 3}$ relied on the DM's preferences is established in Table 2, where $q > 1$.

Now we present the detailed operation steps to solve this practical example. Without loss of generality, we can set $q = 2$.

Step 1 Transform the q-ROF decision matrix $\widehat{EX} = \left(\hat{Q}_{ij}\right)_{4\times3}$ into the normalized decision matrix $EX = \left(Q_{ij}\right)_{4\times3}$. Because all the attributes are benefit-type attributes, we get $EX = \widehat{EX}$.

Step 2 Calculate the supports between the jth attribute and the tth attribute $\mathrm{Sup}_{jt} = \left(\mathrm{Sup}(Q_{ij}, Q_{it})\right)_{4\times1}$ $(j, t = 1, 2, 3, j \neq t)$ as shown below:

$$\mathrm{Sup}_{12} = \mathrm{Sup}_{21} = \begin{pmatrix} 0.8500 \\ 0.8900 \\ 0.8800 \\ 0.5500 \end{pmatrix}, \mathrm{Sup}_{13} = \mathrm{Sup}_{31} = \begin{pmatrix} 0.6000 \\ 0.9200 \\ 0.8000 \\ 0.8000 \end{pmatrix}, \mathrm{Sup}_{23} = \mathrm{Sup}_{32} = \begin{pmatrix} 0.4500 \\ 0.8600 \\ 0.8000 \\ 0.3500 \end{pmatrix}.$$

Step 3 Calculate the total weighted support matrix $T = \left(T^{\omega}\left(Q_{ij}\right)\right)_{4\times3}$ by combining the attributive weights 0.40, 0.35 and 0.25 as shown below:

$$T = \begin{pmatrix} 0.4475 & 0.4525 & 0.3975 \\ 0.5415 & 0.5710 & 0.6690 \\ 0.5080 & 0.5520 & 0.6000 \\ 0.3925 & 0.3075 & 0.4425 \end{pmatrix}.$$

Step 4 Let $\lambda = 1.5$, we can get the weighted nonlinear weight matrix $\Delta^{\omega}_{1.5} = \left(\Delta^{\omega}_{1.5}\left(Q_{ij}\right)\right)_{4\times3}$ as shown below:

$$\Delta^{\omega}_{1.5} = \begin{pmatrix} 0.3837 & 0.3553 & 0.2610 \\ 0.3320 & 0.3254 & 0.3426 \\ 0.3376 & 0.3387 & 0.3237 \\ 0.3850 & 0.2914 & 0.3236 \end{pmatrix}.$$

Step 5 Use the q-ROFIHWAPA (Eq. 23) to aggregate all the individual attribute values of the alternative A_i into the overall attribute value Q_i as shown below, where we let $\gamma = 3$.

$$Q_1 = \langle 0.6720, 0.4407\rangle, Q_2 = \langle 0.5348, 0.2167\rangle, Q_3 = \langle 0.4740, 0.2826\rangle, Q_4 = \langle 0.6840, 0.4500\rangle$$

Step 6 Based on Eq. (5), we can derive the score values $S\left(Q_i\right)(i = 1, 2, 3, 4)$ of the overall attribute values $Q_i(i = 1, 2, 3, 4)$ as shown below:

$$S\left(Q_1\right) = 0.4966,\ S\left(Q_2\right) = 0.4030,\ S\left(Q_3\right) = 0.3505,\ S\left(Q_4\right) = 0.5090.$$

Further, the ranking result of the alternatives A_1,A_2,A_3 and A_4 obtained by Definition 2.6 is $A_4 \succ A_1 \succ A_2 \succ A_3$. Thus, the best alternative is A_4.

Table 2 The q-ROF decision matrix $\widehat{EX}$ from the DM

	C_1	C_2	C_3
A_1	$\langle 0.7, 0.4\rangle$	$\langle 0.8, 0.4\rangle$	$\langle 0.3, 0.5\rangle$
A_2	$\langle 0.5, 0.3\rangle$	$\langle 0.6, 0.2\rangle$	$\langle 0.5, 0.1\rangle$
A_3	$\langle 0.4, 0.4\rangle$	$\langle 0.4, 0.2\rangle$	$\langle 0.6, 0.2\rangle$
A_4	$\langle 0.6, 0.4\rangle$	$\langle 0.9, 0.3\rangle$	$\langle 0.4, 0.6\rangle$

Step 7 End.

In the above operation, we assume $\lambda = 1.5$ in advance, which is, of course, very subjective. In order to reflect the influence of the parameter λ on the ranking results, we use MATLAB software to draw the variation trend of the score value of each alternative with respect to λ for this MADM problem, which is shown in Fig. 1.

It can be seen from Fig. 1 that the variation of λ leads to different ranking results of the alternatives, i.e.,

(1) When $\lambda \in [1, 1.8983)$, the ranking result of these four alternatives is $A_4 \succ A_1 \succ A_2 \succ A_3$;
(2) When $\lambda = 1.8983$, the ranking result of these four alternatives is $A_4 \approx A_1 \succ A_2 \succ A_3$;
(3) When $\lambda = (1.8983, 6.9737)$, the ranking result of these four alternatives is $A_1 \succ A_4 \succ A_2 \succ A_3$;
(4) When $\lambda = 6.9737$, the ranking result of these four alternatives is $A_1 \succ A_4 \approx A_2 \succ A_3$;
(5) When $\lambda = (6.9737, 7.7253)$, the ranking result of these four alternatives is $A_1 \succ A_2 \succ A_4 \succ A_3$;
(6) When $\lambda = 7.7253$, the ranking result of these four alternatives is $A_1 \succ A_2 \succ A_4 \approx A_3$;
(7) When $\lambda = (7.7253, 8]$, the ranking result of these four alternatives is $A_1 \succ A_2 \succ A_3 \succ A_4$.

Thus, we can conclude that if the parameter λ carried by the q-ROFIHWAPA is determined subjectively in advance, it may affect the accuracy of the decision making result and even result in a wrong decision making result. Given this, in what follows, we propose an entropy weight fitting method to determine the parameter λ.

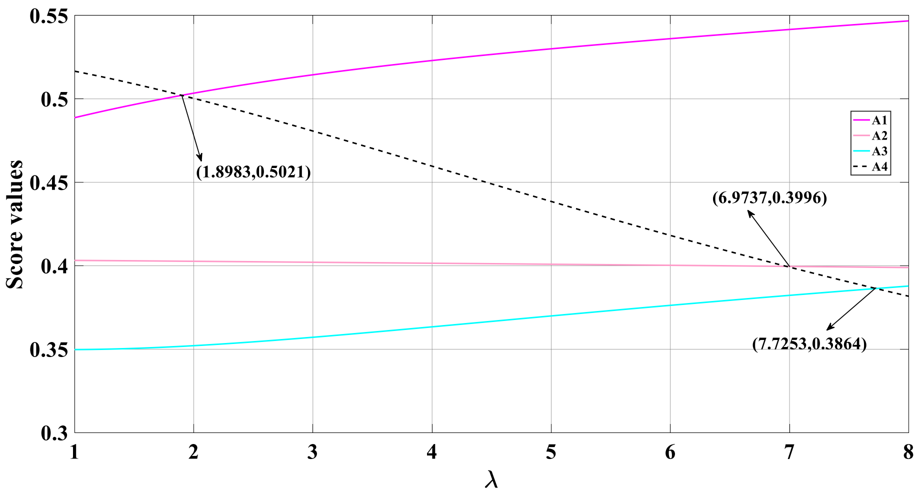

Fig. 1 Score values of the alternatives when $\lambda \in [1, 8]$

3.5 Entropy weight fitting method to determine the parameter λ carried by the q-ROFIHWAPA

Let $Q_i = \langle u_i, v_i \rangle$ $(i = 1, 2, \ldots, n)$ be n q-ROFNs with the weights $\omega_1, \omega_2, \ldots, \omega_n$ such that $\omega_i \geq 0$ and $\sum_{i=1}^{n} \omega_i = 1$.

Step 1 Calculate the entropy weights $\omega_1^E, \omega_2^E, \ldots, \omega_n^E$ for the q-ROFNs $Q_1, Q_2, \ldots, Q_n$, where $\omega_i^E = \frac{1-E_i}{\sum_{r=1}^{n}(1-E_r)}$, and $E_i = \left(\frac{\cos \frac{\pi \left| u_i^q - v_i^q \right|}{2} + \pi_i^q}{2} \right)^{\frac{1}{q}}$ denotes the entropy of Q_i.

Step 2 Taking the importance of different q-ROFNs into account, we use the subjective weights $\omega_1, \omega_2, \ldots, \omega_n$ to revise the entropy weights $\omega_1^E, \omega_2^E, \ldots, \omega_n^E$, so as to obtain the revised entropy weights $\omega_1^{RE}, \omega_2^{RE}, \ldots, \omega_n^{RE}$, where

$$\omega_i^{RE} = \frac{\omega_i \left(1 - E_i\right)}{\sum_{r=1}^{n} \omega_r \left(1 - E_r\right)}. \tag{32}$$

Step 3 Calculate the weighted nonlinear weights $\Delta_1^{\omega}, \Delta_2^{\omega}, \ldots \Delta_n^{\omega}$ for the q-ROFNs $Q_1, Q_2, \ldots, Q_n$, where

$$\Delta_i^{\omega} = \frac{\omega_i^{\lambda} + \left(T^{\omega}(Q_i)\right)^{\lambda}}{\sum_{r=1}^{n} \left(\omega_r^{\lambda} + \left(T^{\omega}(Q_r)\right)^{\lambda} \right)}, \ \lambda \geq 1. \tag{33}$$

Step 4 According to the consistency of the two sets of weights, the parameter λ can be fitted by the following model:

$$\begin{aligned} &\min \sum_{i=1}^{n} \left| \omega_i^{RE} - \Delta_i^{\omega} \right| \\ &s.t.\ \lambda \geq 1. \end{aligned} \tag{34}$$

Step 5 End.

We label the above process to determine the parameter λ carried by the q-ROFIHWAPA as entropy weight fitting method.

Now let's reconsider the "Location Selection" issue in Example 3.2 based on our proposed entropy weight fitting method, which is as follows:

Step 1' Calculate the entropy matrix $E = \left(E\left(Q_{ij}\right)\right)_{4\times3}$ of the normalized decision matrix $EX = \left(Q_{ij}\right)_{4\times3}$ as shown below:

$$E = \begin{pmatrix} 0.7806 & 0.6815 & 0.9024 \\ 0.9024 & 0.8592 & 0.9137 \\ 0.9165 & 0.9440 & 0.8592 \\ 0.8459 & 0.5127 & 0.8459 \end{pmatrix}.$$

Step 2' Calculate the revised entropy weight matrix $W^{RE} = \left(\omega^{RE}\left(Q_{ij}\right)\right)_{4\times 3}$ by combining the attributive weights 0.40, 0.35 and 0.25 as shown below:

$$W_{RE} = \begin{pmatrix} 0.3925 & 0.4984 & 0.1091 \\ 0.3553 & 0.4485 & 0.1962 \\ 0.3786 & 0.2222 & 0.3992 \\ 0.2277 & 0.6300 & 0.1423 \end{pmatrix}.$$

Step 3' Fit the parameter vector $\tilde{\lambda} = \left(\lambda_1, \lambda_2, \lambda_3, \lambda_4\right)^{\mathrm{T}}$ corresponding to the alternatives A_1, A_2, A_3 and A_4 on the basis of the attributive weights 0.40, 0.35 and 0.25, total weighted support matrix $T = \left(T^{\omega}\left(Q_{ij}\right)\right)_{4\times 3}$.
(It has been calculated in Step3 of Example 3.2) and revised entropy weight matrix $W^{RE} = \left(\omega^{RE}\left(Q_{ij}\right)\right)_{4\times 3}$, i.e.,

$$\tilde{\lambda} = (18.2727, 1.0000, 3.9697, 1.0000)^{\mathrm{T}}.$$

Step 4' Calculate the weighted nonlinear weight matrix $\Delta^{\omega}_{\tilde{\lambda}} = \left(\Delta^{\omega}_{\lambda_i}\left(Q_{ij}\right)\right)_{4\times 3}$ by the parameter vector $\tilde{\lambda} = \left(\lambda_1, \lambda_2, \lambda_3, \lambda_4\right)^{\mathrm{T}}$ as shown below:

$$\Delta^{\omega}_{\tilde{\lambda}} = \begin{pmatrix} 0.4552 & 0.4986 & 0.0463 \\ 0.3385 & 0.3311 & 0.3304 \\ 0.2773 & 0.3236 & 0.3991 \\ 0.3699 & 0.3069 & 0.3232 \end{pmatrix}.$$

Step 5' Use the q-ROFIHWAPA (Eq. 23) to aggregate all the individual attribute values of the alternative A_i into the overall attribute value Q_i as shown below, where we let $\gamma = 3$.

$$Q_1 = \langle 0.7436, 0.4126\rangle, Q_2 = \langle 0.5354, 0.2182\rangle, Q_3 = \langle 0.4898, 0.2691\rangle, Q_4 = \langle 0.6904, 0.4482\rangle.$$

Step 6' Based on Eq. (5), we can derive the score values $S\left(Q_i\right)(i = 1, 2, 3, 4)$ of the overall attribute values $Q_i (i = 1, 2, 3, 4)$ as shown below:

$$S\left(Q_1\right) = 0.5860, S\left(Q_2\right) = 0.4032, S\left(Q_3\right) = 0.3632, S\left(Q_4\right) = 0.5166.$$

Further, the ranking result of the alternatives A_1,A_2,A_3 and A_4 obtained by Definition 2.6 is $A_1 \succ A_4 \succ A_2 \succ A_3$. Thus, the best alternative is A_1.

Step 7' End.

4 Generalized *q*-ROF interactive Hamacher HMs

In this section, we introduce the WCHM and WGCHM on the basis of the HM and GHM, respectively, which can eliminate the redundancy of the DGWBM and DGWBGM. Then we extend the WCHM and WGCHM to q-ROF environment and propose the q-ROFIHWCHM and q-ROFIHWGCHM based on the interactive Hamacher operation rules.

4.1 WCHM and WGCHM

Definition 4.1 Zhang et al. (2017). Let $a_1, a_2, \ldots, a_n$ be n non-negative real numbers with the weights $\omega_1, \omega_2, \ldots, \omega_n$ such that $\omega_i \geq 0$ and $\sum_{i=1}^{n} \omega_i = 1$, the DGWBM:$(\mathbb{R}\backslash\mathbb{R}^-)^n \to \mathbb{R}\backslash\mathbb{R}^-$ is defined as follows:

$$\mathrm{DGWBM}\left(a_1, a_2, \ldots, a_n\right) = \left(\sum_{\tau_1,\tau_2,\ldots,\tau_n=1}^{n}\left(\prod_{j=1}^{n}\omega_{\tau_j}a_{\tau_j}^{r_j}\right)\right)^{\frac{1}{\sum_{j=1}^{n} r_j}}, \tag{35}$$

where $R = \left(r_1, r_2, \ldots, r_n\right)$ is the parameter vector, such that $r_j \geq 0\,(j = 1, 2, \ldots, n)$ and $\sum_{j=1}^{n} r_j \neq 0$.

Definition 4.2 Zhang et al. (2017). Let $a_1, a_2, \ldots, a_n$ be n non-negative real numbers with the weights $\omega_1, \omega_2, \ldots, \omega_n$ such that $\omega_i \geq 0$ and $\sum_{i=1}^{n} \omega_i = 1$, the DGWBGM:$(\mathbb{R}\backslash\mathbb{R}^-)^n \to \mathbb{R}\backslash\mathbb{R}^-$ is defined as follows:

$$\mathrm{DGWBGM}\left(a_1, a_2, \ldots, a_n\right) = \frac{1}{\sum_{j=1}^{n} r_j}\left(\prod_{\tau_1,\tau_2,\ldots,\tau_n=1}^{n}\left(\sum_{j=1}^{n} r_j a_{\tau_j}\right)^{\prod_{j=1}^{n}\omega_{\tau_j}}\right), \tag{36}$$

where $R = \left(r_1, r_2, \ldots, r_n\right)$ is the parameter vector, such that $r_j \geq 0\,(j = 1, 2, \ldots, n)$ and $\sum_{j=1}^{n} r_j \neq 0$.

Liu and Liu (2021) exhibited the development course of BMs as shown in Fig. 2 (cited from Liu and Liu (2021).

In fact, Eqs. (35 and 36) are equivalent to Eqs. (37 and 38), respectively, i.e.,

$$\mathrm{DGWBM}(a_1, a_2, \ldots, a_n) = \left(\frac{\sum_{\tau_1,\tau_2,\ldots,\tau_n=1}^{n}\left(\prod_{j=1}^{n}\omega_{\tau_j}a_{\tau_j}^{r_j}\right)}{\sum_{\tau_1,\tau_2,\ldots,\tau_n=1}^{n}\left(\prod_{j=1}^{n}\omega_{\tau_j}\right)}\right)^{\frac{1}{\sum_{j=1}^{n} r_j}} \tag{37}$$

and

$$\mathrm{DGWBGM}(a_1, a_2, \ldots, a_n) = \frac{1}{\sum_{j=1}^{n} r_j}\left(\prod_{\tau_1,\tau_2,\ldots,\tau_n=1}^{n}\left(\sum_{j=1}^{n} r_j a_{\tau_j}\right)^{\prod_{j=1}^{n}\omega_{\tau_j}}\right)^{\frac{1}{\sum_{\tau_1,\tau_2,\ldots,\tau_n=1}^{n}\left(\prod_{j=1}^{n}\omega_{\tau_j}\right)}}. \tag{38}$$

Definition 4.3 Yu and Wu (2012).

Let $a_1, a_2, \ldots, a_n$ be n non-negative real numbers, the HM:$(\mathbb{R}\backslash\mathbb{R}^-)^n \to \mathbb{R}\backslash\mathbb{R}^-$ is defined as follows:

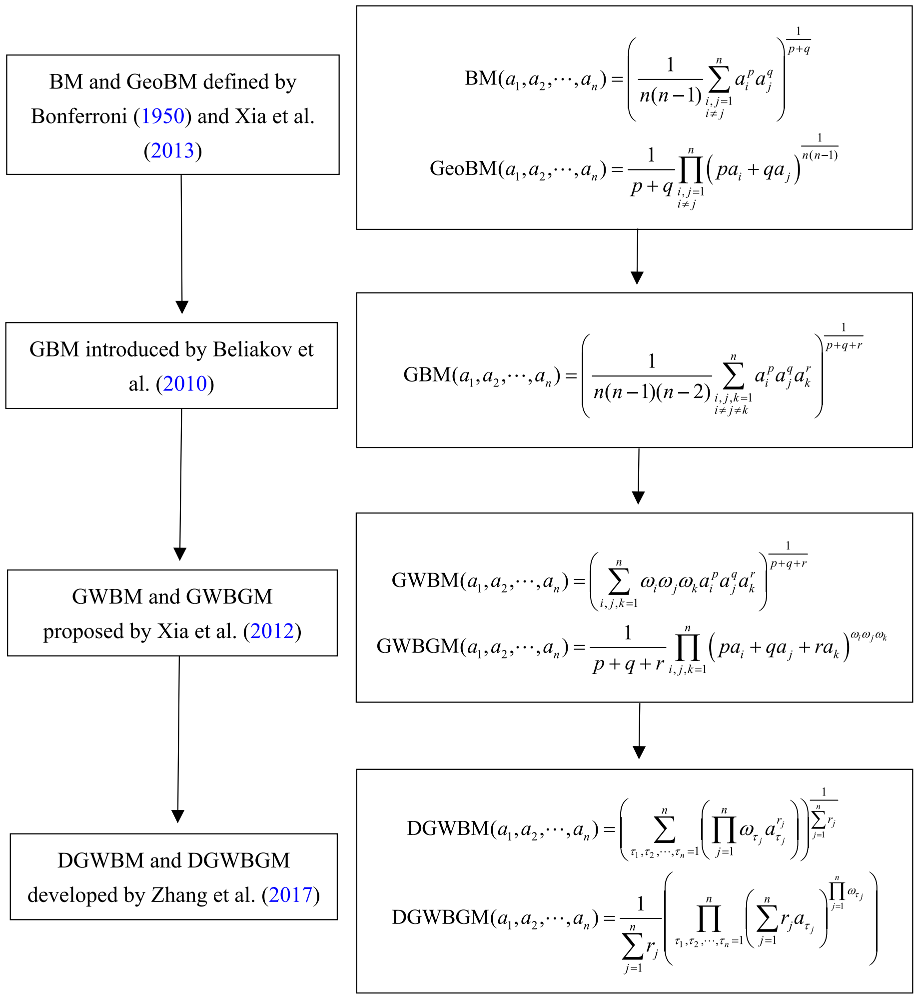

Fig. 2 The development course of BMs

$$\mathrm{HM}(a_1, a_2, \ldots, a_n) = \left(\frac{2}{n(n+1)}\sum_{i=1,j=i}^{n} a_i^p a_j^q\right)^{\frac{1}{p+q}}, \tag{39}$$

where $\hat{R} = (p, q)$ is the parameter vector, such that $p, q \geq 0$ and $p + q \neq 0$.

Definition 4.4 Yu (2013).

Let $a_1, a_2, \ldots, a_n$ be n non-negative real numbers, the GHM:$(\mathbb{R}\backslash\mathbb{R}^-)^n \to \mathbb{R}\backslash\mathbb{R}^-$ is defined as follows:

$$\mathrm{GHM}(a_1, a_2, \dots, a_n) = \frac{1}{p+q}\left(\prod_{i=1,j=i}^{n}\left(pa_i + qa_j\right)\right)^{\frac{2}{n(n+1)}}, \tag{40}$$

where $\hat{R} = (p, q)$ is the parameter vector, such that $p, q \geq 0$ and $p + q \neq 0$.

Stimulated by the development of BMs, we propose the WCHM and WGCHM on the basis of the HM and GHM, respectively, which eliminate the redundancy of the DGWBM and DGWBGM (Eqs. 37 and 38), i.e., the case of $\tau_1 > \tau_2 > \cdots > \tau_n$.

Definition 4.5 Let $a_1, a_2, \dots, a_n$ be n non-negative real numbers with the weights $\omega_1, \omega_2, \dots, \omega_n$ such that $\omega_i \geq 0$ and $\sum_{i=1}^{n} \omega_i = 1$, the WCHM:$(\mathbb{R}\backslash\mathbb{R}^-)^n \to \mathbb{R}\backslash\mathbb{R}^-$ is defined as follows:

$$\mathrm{WCHM}\left(a_1, a_2, \dots, a_n\right) = \left(\frac{\sum\limits_{\tau_1=1,\tau_2=\tau_1,\dots,\tau_n=\tau_{n-1}}^{n}\left(\prod\limits_{j=1}^{n}\omega_{\tau_j} a_{\tau_j}^{r_j}\right)}{\sum\limits_{\tau_1=1,\tau_2=\tau_1,\dots,\tau_n=\tau_{n-1}}^{n}\left(\prod\limits_{j=1}^{n}\omega_{\tau_j}\right)}\right)^{\frac{1}{\sum_{j=1}^{n} r_j}}, \tag{41}$$

where $R = \left(r_1, r_2, \dots, r_n\right)$ is the parameter vector, such that $r_j \geq 0\,(j = 1, 2, \dots, n)$ and $\sum_{j=1}^{n} r_j \neq 0$.

Definition 4.6 Let $a_1, a_2, \dots, a_n$ be n non-negative real numbers with the weights $\omega_1, \omega_2, \dots, \omega_n$ such that $\omega_i \geq 0$ and $\sum_{i=1}^{n} \omega_i = 1$, the WGCHM:$(\mathbb{R}\backslash\mathbb{R}^-)^n \to \mathbb{R}\backslash\mathbb{R}^-$ is defined as follows:

$$\mathrm{WGCHM}(a_1, a_2, \dots, a_n) = \frac{1}{\sum\limits_{j=1}^{n} r_j}\left(\prod_{\tau_1=1,\tau_2=\tau_1,\dots,\tau_n=\tau_{n-1}}^{n}\left(\sum_{j=1}^{n} r_j a_{\tau_j}\right)^{\prod_{j=1}^{n}\omega_{\tau_j}}\right)^{\frac{1}{\sum_{\tau_1=1,\tau_2=\tau_1,\dots,\tau_n=\tau_{n-1}}^{n}\left(\prod_{j=1}^{n}\omega_{\tau_j}\right)}}, \tag{42}$$

where $R = \left(r_1, r_2, \dots, r_n\right)$ is the parameter vector, such that $r_j \geq 0\,(j = 1, 2, \dots, n)$ and $\sum_{j=1}^{n} r_j \neq 0$.

Theorem 4.1 *The WCHM and WGCHM satisfy the following properties*:

(1) (*Idempotency*) *If* $a_1 = a_2 = \cdots = a_n = a$, *then*

$$\mathrm{WCHM}(a_1, a_2, \dots, a_n) = a \text{ (resp. } \mathrm{WGCHM}(a_1, a_2, \dots, a_n) = a);$$

(2) (*Monotonicity*) *If* $a_i'(i = 1, 2, \dots, n)$ *are another set of non-negative real numbers, which have exactly the same weights as* $a_i(i = 1, 2, \dots, n)$, *and* $a_i \leq a_i'$ *for each* i, *then.*

$$\mathrm{WCHM}(a_1, a_2, \ldots, a_n) \le \mathrm{WCHM}(a_1', a_2', \ldots, a_n') \text{ (resp. } \mathrm{WGCHM}(a_1, a_2, \ldots, a_n) \le \mathrm{WGCHM}(a_1', a_2', \ldots, a_n'))$$

(3) (*Boundedness*) $\min_i \{a_i\} \le \mathrm{WCHM}(a_1, a_2, \ldots, a_n) \le \max_i \{a_i\}$

$$(\text{resp.} \min_i \{a_i\} \le \mathrm{WGCHM}(a_1, a_2, \ldots, a_n) \le \max_i \{a_i\});$$

(4) (*Commutativity*)$\mathrm{WCHM}(a_1, a_2, \ldots, a_n)$(*resp.*$\mathrm{WGCHM}(a_1, a_2, \ldots, a_n)$) *is not changed if* $a_1, a_2, \cdots, a_n$ *and the weights* $\omega_1, \omega_2, \ldots, \omega_n$ *are permuted simultaneously.*

The proof of Theorem 4.1 is easy to derive, which is omitted here.

4.2 *q*-ROFIHWCHM and *q*-ROFIHWGCHM

The dual generalized PF weighted BM (DGPFWBM) introduced by Zhang et al. (2017) and 2-dimensional uncertain linguistic DGWBM (2DULDGWBM) introduced by Liu and Liu (2021), which are directly derived from the DGWBM, are not idempotent. In view of these facts, we propose such a WCHM for q-ROFNs.

Definition 4.7 Let $Q_1, Q_2, \ldots, Q_n$ be n q-ROFNs with the weights $\omega_1, \omega_2, \ldots, \omega_n$ such that $\omega_i \ge 0$ and $\sum_{i=1}^{n} \omega_i = 1$, the q-ROFIHWCHM:$\mathcal{Q}^n \to \mathcal{Q}$ is defined as follows:

$$q\text{ - ROFIHWCHM}(Q_1, Q_2, \ldots, Q_n) = \left(\frac{\bigoplus_{\tau_1=1, \tau_2=\tau_1, \ldots, \tau_n=\tau_{n-1}}^{n} \left(\prod_{j=1}^{n} \omega_{\tau_j} \right) \left(\bigotimes_{j=1}^{n} Q_{\tau_j}^{r_j} \right)}{\sum_{\tau_1=1, \tau_2=\tau_1, \ldots, \tau_n=\tau_{n-1}}^{n} \left(\prod_{j=1}^{n} \omega_{\tau_j} \right)} \right)^{\frac{1}{\sum_{j=1}^{n} r_j}}, \tag{43}$$

where $R = (r_1, r_2, \ldots, r_n)$ is the parameter vector, such that $r_j \ge 0\,(j = 1, 2, \ldots, n)$ and $\sum_{j=1}^{n} r_j \ne 0$.

Lemma 4.1 *Let* $Q_i = \langle u_i, v_i \rangle (i = 1, 2, \ldots, n)$ *be n q-ROFNs, and* $\gamma > 0$, *we get.*

$$\bigotimes_{i=1}^{n} Q_i = \left\langle \left(\frac{\gamma \prod_{i=1}^{n} (1 - v_i^q) - \gamma \prod_{i=1}^{n} (1 - u_i^q - v_i^q)}{\prod_{i=1}^{n} (1 + (\gamma - 1) v_i^q) + (\gamma - 1) \prod_{i=1}^{n} (1 - v_i^q)} \right)^{\frac{1}{q}}, \left(\frac{\prod_{i=1}^{n} (1 + (\gamma - 1) v_i^q) - \prod_{i=1}^{n} (1 - v_i^q)}{\prod_{i=1}^{n} (1 + (\gamma - 1) v_i^q) + (\gamma - 1) \prod_{i=1}^{n} (1 - v_i^q)} \right)^{\frac{1}{q}} \right\rangle. \tag{44}$$

The proof of Lemma 4.1 is similar to that of Lemma 3.1, which is omitted here.

Theorem 4.2 *Let* $Q_i = \langle u_i, v_i \rangle (i = 1, 2, \ldots, n)$ *in Eq.* (43), *the aggregated value of the q-ROFIHWCHM is shown in Eq.* (45), *which is still a q-ROFN.*

$$q\text{ - ROFIHWCHM}\big(Q_1,Q_2,\ldots,Q_n\big)=\left\langle\begin{array}{l}\left(\dfrac{\gamma(\Phi^\rho-\Psi^\rho+\gamma\Omega^\rho)^\xi-\gamma(\gamma\Omega^\rho)^\xi}{(\Phi^\rho+(\gamma-1)(\gamma+1)\Psi^\rho-(\gamma-1)\gamma\Omega^\rho)^\xi+(\gamma-1)(\Phi^\rho-\Psi^\rho+\gamma\Omega^\rho)^\xi}\right)^{\frac{1}{q}},\\ \left(\dfrac{(\Phi^\rho+(\gamma-1)(\gamma+1)\Psi^\rho-(\gamma-1)\gamma\Omega^\rho)^\xi-(\Phi^\rho-\Psi^\rho+\gamma\Omega^\rho)^\xi}{(\Phi^\rho+(\gamma-1)(\gamma+1)\Psi^\rho-(\gamma-1)\gamma\Omega^\rho)^\xi+(\gamma-1)(\Phi^\rho-\Psi^\rho+\gamma\Omega^\rho)^\xi}\right)^{\frac{1}{q}}\end{array}\right\rangle, \tag{45}$$

where

$$\xi=\frac{1}{\sum\limits_{j=1}^{n}r_j},$$

$$\rho=\frac{1}{\sum\limits_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod\limits_{j=1}^{n}\omega_{\tau_j}\right)},$$

$$\varpi=\prod_{j=1}^{n}\omega_{\tau_j},$$

$$\Phi=\prod_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod_{j=1}^{n}\left(1+(\gamma-1)v_{\tau_j}^q\right)^{r_j}+(\gamma-1)(\gamma+1)\prod_{j=1}^{n}\left(1-v_{\tau_j}^q\right)^{r_j}-(\gamma-1)\gamma\prod_{j=1}^{n}\left(1-u_{\tau_j}^q-v_{\tau_j}^q\right)^{r_j}\right)^{\varpi},$$

$$\Psi=\prod_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod_{j=1}^{n}\left(1+(\gamma-1)v_{\tau_j}^q\right)^{r_j}-\prod_{j=1}^{n}\left(1-v_{\tau_j}^q\right)^{r_j}+\gamma\prod_{j=1}^{n}\left(1-u_{\tau_j}^q-v_{\tau_j}^q\right)^{r_j}\right)^{\varpi},$$

$$\Omega=\prod_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\gamma\prod_{j=1}^{n}\left(1-u_{\tau_j}^q-v_{\tau_j}^q\right)^{r_j}\right)^{\varpi}.$$

Proof Let $\xi=\frac{1}{\sum_{j=1}^{n}r_j}$, $\varpi=\prod\limits_{j=1}^{n}\omega_{\tau_j}$ and $\rho=\dfrac{1}{\sum\limits_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod\limits_{j=1}^{n}\omega_{\tau_j}\right)}$, Eq. (43) is simplified to.

$$q\text{ - ROFIHWCHM}\big(Q_1,Q_2,\ldots,Q_n\big)=\left(\rho\left(\bigoplus_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\varpi\left(\bigotimes_{j=1}^{n}Q_{\tau_j}^{r_j}\right)\right)\right)^{\xi}. \tag{46}$$

Lemma 4.1 implies that

$$\bigotimes_{j=1}^{n}Q_{\tau_j}^{r_j}=\left\langle\left(\frac{\gamma\prod\limits_{j=1}^{n}\left(1-v_{\tau_j}^q\right)^{r_j}-\gamma\prod\limits_{j=1}^{n}\left(1-u_{\tau_j}^q-v_{\tau_j}^q\right)^{r_j}}{\prod\limits_{j=1}^{n}\left(1+(\gamma-1)v_{\tau_j}^q\right)^{r_j}+(\gamma-1)\prod\limits_{j=1}^{n}\left(1-v_{\tau_j}^q\right)^{r_j}}\right)^{\frac{1}{q}},\left(\frac{\prod\limits_{j=1}^{n}\left(1+(\gamma-1)v_{\tau_j}^q\right)^{r_j}-\prod\limits_{j=1}^{n}\left(1-v_{\tau_j}^q\right)^{r_j}}{\prod\limits_{j=1}^{n}\left(1+(\gamma-1)v_{\tau_j}^q\right)^{r_j}+(\gamma-1)\prod\limits_{j=1}^{n}\left(1-v_{\tau_j}^q\right)^{r_j}}\right)^{\frac{1}{q}}\right\rangle. \tag{47}$$

Then we have Eq. (48) by Lemma 3.1, i.e.,

$$\bigoplus_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\varpi\left(\bigotimes_{j=1}^{n}Q_{\tau_j}^{r_j}\right)=\left\langle\left(\frac{\Phi-\Psi}{\Phi+(\gamma-1)\Psi}\right)^{\frac{1}{q}},\left(\frac{\gamma\Psi-\gamma\Omega}{\Phi+(\gamma-1)\Psi}\right)^{\frac{1}{q}}\right\rangle, \tag{48}$$

where

$$\Phi=\prod_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod_{j=1}^{n}\left(1+(\gamma-1)v_{\tau_j}^{q}\right)^{r_j}+(\gamma-1)(\gamma+1)\prod_{j=1}^{n}\left(1-v_{\tau_j}^{q}\right)^{r_j}-(\gamma-1)\gamma\prod_{j=1}^{n}\left(1-u_{\tau_j}^{q}-v_{\tau_j}^{q}\right)^{r_j}\right)^{\varpi},$$

$$\Psi=\prod_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod_{j=1}^{n}\left(1+(\gamma-1)v_{\tau_j}^{q}\right)^{r_j}-\prod_{j=1}^{n}\left(1-v_{\tau_j}^{q}\right)^{r_j}+\gamma\prod_{j=1}^{n}\left(1-u_{\tau_j}^{q}-v_{\tau_j}^{q}\right)^{r_j}\right)^{\varpi},$$

$$\Omega=\prod_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\gamma\prod_{j=1}^{n}\left(1-u_{\tau_j}^{q}-v_{\tau_j}^{q}\right)^{r_j}\right)^{\varpi}.$$

Therefore, we can obtain

$$\begin{aligned}&\left(\rho\left(\underset{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}{\overset{n}{\oplus}}\varpi\left(\overset{n}{\underset{j=1}{\otimes}}Q_{\tau_j}^{r_j}\right)\right)\right)^{\xi}\\&=\left\langle\begin{array}{l}\left(\dfrac{\gamma(\Phi^{\rho}-\Psi^{\rho}+\gamma\Omega^{\rho})^{\xi}-\gamma(\gamma\Omega^{\rho})^{\xi}}{(\Phi^{\rho}+(\gamma-1)(\gamma+1)\Psi^{\rho}-(\gamma-1)\gamma\Omega^{\rho})^{\xi}+(\gamma-1)(\Phi^{\rho}-\Psi^{\rho}+\gamma\Omega^{\rho})^{\xi}}\right)^{\frac{1}{q}},\\\left(\dfrac{(\Phi^{\rho}+(\gamma-1)(\gamma+1)\Psi^{\rho}-(\gamma-1)\gamma\Omega^{\rho})^{\xi}-(\Phi^{\rho}-\Psi^{\rho}+\gamma\Omega^{\rho})^{\xi}}{(\Phi^{\rho}+(\gamma-1)(\gamma+1)\Psi^{\rho}-(\gamma-1)\gamma\Omega^{\rho})^{\xi}+(\gamma-1)(\Phi^{\rho}-\Psi^{\rho}+\gamma\Omega^{\rho})^{\xi}}\right)^{\frac{1}{q}}\end{array}\right\rangle.\end{aligned}\tag{49}$$

Due to the closure of $\mathcal{Q}$ under the addition, multiplication, scalar multiplication and power, for $Q_i\in\mathcal{Q}\,(i=1,2,\ldots,n)$, we get

$$q\text{ - ROFIHWCHM}\left(Q_1,Q_2,\ldots,Q_n\right)=\left(\rho\left(\underset{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}{\overset{n}{\oplus}}\varpi\left(\overset{n}{\underset{j=1}{\otimes}}Q_{\tau_j}^{r_j}\right)\right)\right)^{\xi}\in\mathcal{Q}$$

which implies q - ROFIHWCHM$\left(Q_1,Q_2,\ldots,Q_n\right)$ is still a q-ROFN.

This completes the proof of Theorem 4.2.

Theorem 4.3 *The q-ROFIHWCHM satisfies the following properties*:

(1) (*Idempotency*) *If* $Q_1=Q_2=\cdots=Q_n=Q$, *then* q - ROFIHWCHM$\left(Q_1,Q_2,\ldots,Q_n\right)=Q$;
(2) (*Boundedness*) *If* $Q^-=\langle 0,1\rangle$ *and* $Q^+=\langle 1,0\rangle$, *then* $Q^-\le q$ - ROFIHWCHM$\left(Q_1,Q_2,\ldots,Q_n\right)\le Q^+$;
(3) (*Commutativity*) q - ROFIHWCHM$\left(Q_1,Q_2,\ldots,Q_n\right)$ *is not changed if* $Q_1,Q_2,\ldots,Q_n$ *and the weights* $\omega_1,\omega_2,\ldots,\omega_n$ *are permuted simultaneously.*

Proof

(1) If $Q_i=Q\,(i=1,2,\ldots,n)$, then by the interactive Hamacher operation properties for q-ROFNs presented in Theorem 2.2, we have.

$$q\text{ - ROFIHWCHM}\left(Q_1,Q_2,\ldots,Q_n\right)=q\text{ - ROFIHWCHM}(Q,Q,\ldots,Q)$$

$$=\left(\frac{\bigoplus_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod_{j=1}^{n}\omega_{\tau_j}\right)\left(Q^{\sum_{j=1}^{n}r_j}\right)}{\sum_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod_{j=1}^{n}\omega_{\tau_j}\right)}\right)^{\frac{1}{\sum_{j=1}^{n}r_j}}$$

$$=\left(\frac{\sum_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod_{j=1}^{n}\omega_{\tau_j}\right)}{\sum_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod_{j=1}^{n}\omega_{\tau_j}\right)}\left(Q^{\sum_{j=1}^{n}r_j}\right)\right)^{\frac{1}{\sum_{j=1}^{n}r_j}}=Q. \tag{50}$$

(2) This is clear because Q^- and Q^+ are the bottom and top of the q-ROFNs, respectively.
(3) If $\left(Q_1',Q_2',\ldots,Q_n'\right)$ is the permutation of $\left(Q_1,Q_2,\ldots,Q_n\right)$ and $\left(\omega_1',\omega_2',\ldots,\omega_n'\right)$ is the permutation of $\left(\omega_1,\omega_2,\ldots,\omega_n\right)$, then

$$q\text{ - ROFIHWCHM}\left(Q_1,Q_2,\ldots,Q_n\right)=\left(\frac{\bigoplus_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod_{j=1}^{n}\omega_{\tau_j}\right)\left(\bigotimes_{j=1}^{n}Q_{\tau_j}^{r_j}\right)}{\sum_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod_{j=1}^{n}\omega_{\tau_j}\right)}\right)^{\frac{1}{\sum_{j=1}^{n}r_j}}$$

$$=\left(\frac{\bigoplus_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod_{j=1}^{n}\omega'_{\tau_j}\right)\left(\bigotimes_{j=1}^{n}Q'^{r_j}_{\tau_j}\right)}{\sum_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod_{j=1}^{n}\omega'_{\tau_j}\right)}\right)^{\frac{1}{\sum_{j=1}^{n}r_j}}=q\text{ - ROFIHWCHM}\left(Q_1',Q_2',\ldots,Q_n'\right). \tag{51}$$

Therefore, we have proved Theorem 4.3.

Now we explore the case where $R=\left(\lambda_1,\lambda_2,\ldots,\lambda_l,0,0,\ldots,0\right)$ for the q-ROFIHWCHM.

If $R=\left(\lambda_1,\lambda_2,\ldots,\lambda_l,0,0,\ldots,0\right)$, where $\lambda_j\geq 0\,(j=1,2,\ldots,l)$ and $\sum_{j=1}^{l}\lambda_j\neq 0$, then Eq. (43) degenerates into the following:

$$q\text{ - ROFIHWCHM}\left(Q_1,Q_2,\ldots,Q_n\right)=\left(\bigoplus_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_l=\tau_{l-1}}^{n}\mathcal{K}_{\tau_1\tau_2\cdots\tau_l}\left(\bigotimes_{j=1}^{l}Q_{\tau_j}^{\lambda_j}\right)\right)^{\frac{1}{\sum_{j=1}^{l}\lambda_j}}, \tag{52}$$

where

$$\mathcal{K}_{\tau_1\tau_2\cdots\tau_l} = \frac{\sum\limits_{\tau_{l+1}=\tau_l,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod\limits_{j=1}^{n}\omega_{\tau_j}\right)}{\sum\limits_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod\limits_{j=1}^{n}\omega_{\tau_j}\right)}.$$

Since

$$\sum_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_l=\tau_{l-1}}^{n}\mathcal{K}_{\tau_1\tau_2\cdots\tau_l} = \sum_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_l=\tau_{l-1}}^{n}\frac{\sum\limits_{\tau_{l+1}=\tau_l,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod\limits_{j=1}^{n}\omega_{\tau_j}\right)}{\sum\limits_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod\limits_{j=1}^{n}\omega_{\tau_j}\right)} = 1,$$

then Eq. (52) is called the q-ROF interactive Hamacher weighted ($l-$ parameter) HM (q-ROFIHW($l-$ P) HM).

More specifically, if $l = 2$, then Eq. (52) degenerates into the following:

$$q\text{ - ROFIHWCHM}\left(Q_1, Q_2, \ldots, Q_n\right) = \left(\bigoplus_{\tau_1=1,\tau_2=\tau_1}^{n}\mathcal{K}_{\tau_1\tau_2}\left(Q_{\tau_1}^{\lambda_1}\otimes Q_{\tau_2}^{\lambda_2}\right)\right)^{\frac{1}{\lambda_1+\lambda_2}}, \tag{53}$$

where

$$\mathcal{K}_{\tau_1\tau_2} = \frac{\sum\limits_{\tau_3=\tau_2,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod\limits_{j=1}^{n}\omega_{\tau_j}\right)}{\sum\limits_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod\limits_{j=1}^{n}\omega_{\tau_j}\right)},$$

which is the q-ROF interactive Hamacher weighted HM (q-ROFIHWHM);

if $l = 1$, then Eq. (52) degenerates into the following:

$$q\text{ - ROFIHWCHM}\left(Q_1, Q_2, \ldots, Q_n\right) = \left(\bigoplus_{\tau_1=1}^{n}\mathcal{K}_{\tau_1}Q_{\tau_1}^{\lambda_1}\right)^{\frac{1}{\lambda_1}}, \tag{54}$$

where

$$\mathcal{K}_{\tau_1} = \frac{\sum\limits_{\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod\limits_{j=1}^{n}\omega_{\tau_j}\right)}{\sum\limits_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod\limits_{j=1}^{n}\omega_{\tau_j}\right)},$$

which is the q-ROF interactive Hamacher generalized weighted average (q-ROFIHGWA).

Definition 4.8 Let $Q_1, Q_2, \ldots, Q_n$ be n q-ROFNs with the weights $\omega_1, \omega_2, \ldots, \omega_n$ such that $\omega_i \geq 0$ and $\sum_{i=1}^{n} \omega_i = 1$, the q-ROFIHWGCHM:$\mathcal{Q}^n \to \mathcal{Q}$ is defined as follows:

$$q\text{ - ROFIHWGCHM}\left(Q_1, Q_2, \ldots, Q_n\right) = \frac{1}{\sum_{j=1}^{n} r_j}\left(\bigotimes_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\bigoplus_{j=1}^{n} r_j Q_{\tau_j}\right)^{\prod_{j=1}^{n}\omega_{\tau_j}}\right)^{\frac{1}{\sum_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod_{j=1}^{n}\omega_{\tau_j}\right)}}, \tag{55}$$

where $R = \left(r_1, r_2, \ldots, r_n\right)$ is the parameter vector, such that $r_j \geq 0\,(j = 1, 2, \ldots, n)$ and $\sum_{j=1}^{n} r_j \neq 0$.

Theorem 4.4 *Let $Q_i = \langle u_i, v_i\rangle (i = 1, 2, \ldots, n)$ in Eq. (55), the aggregated value of the q-ROFIHWGCHM is shown in Eq. (56), which is still a q-ROFN.*

$$q\text{ - ROFIHWGCHM}\left(Q_1, Q_2, \ldots, Q_n\right) = \left\langle \begin{array}{l} \left(\frac{(\Gamma^\rho + (\gamma-1)(\gamma+1)\Lambda^\rho - (\gamma-1)\gamma\Omega^\rho)^\xi - (\Gamma^\rho - \Lambda^\rho + \gamma\Omega^\rho)^\xi}{(\Gamma^\rho + (\gamma-1)(\gamma+1)\Lambda^\rho - (\gamma-1)\gamma\Omega^\rho)^\xi + (\gamma-1)(\Gamma^\rho - \Lambda^\rho + \gamma\Omega^\rho)^\xi}\right)^{\frac{1}{q}}, \\ \left(\frac{\gamma(\Gamma^\rho - \Lambda^\rho + \gamma\Omega^\rho)^\xi - \gamma(\gamma\Omega^\rho)^\xi}{(\Gamma^\rho + (\gamma-1)(\gamma+1)\Lambda^\rho - (\gamma-1)\gamma\Omega^\rho)^\xi + (\gamma-1)(\Gamma^\rho - \Lambda^\rho + \gamma\Omega^\rho)^\xi}\right)^{\frac{1}{q}} \end{array}\right\rangle, \tag{56}$$

where

$$\xi = \frac{1}{\sum_{j=1}^{n} r_j},$$

$$\rho = \frac{1}{\sum_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod_{j=1}^{n}\omega_{\tau_j}\right)},$$

$$\varpi = \prod_{j=1}^{n}\omega_{\tau_j},$$

$$\Gamma = \prod_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod_{j=1}^{n}\left(1+(\gamma-1)u_{\tau_j}^q\right)^{r_j} + (\gamma-1)(\gamma+1)\prod_{j=1}^{n}\left(1-u_{\tau_j}^q\right)^{r_j} - (\gamma-1)\gamma\prod_{j=1}^{n}\left(1-u_{\tau_j}^q - v_{\tau_j}^q\right)^{r_j}\right)^{\varpi},$$

$$\Lambda = \prod_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod_{j=1}^{n}\left(1+(\gamma-1)u_{\tau_j}^q\right)^{r_j} - \prod_{j=1}^{n}\left(1-u_{\tau_j}^q\right)^{r_j} + \gamma\prod_{j=1}^{n}\left(1-u_{\tau_j}^q - v_{\tau_j}^q\right)^{r_j}\right)^{\varpi},$$

$$\Omega = \prod_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\gamma\prod_{j=1}^{n}\left(1-u_{\tau_j}^q - v_{\tau_j}^q\right)^{r_j}\right)^{\varpi}.$$

Proof Let $\xi = \frac{1}{\sum_{j=1}^{n} r_j}$,$\varpi = \prod_{j=1}^{n}\omega_{\tau_j}$ and $\rho = \frac{1}{\sum_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod_{j=1}^{n}\omega_{\tau_j}\right)}$, Eq. (55) is simplified to.

$$q\text{ - ROFIHWGCHM}\left(Q_1, Q_2, \ldots, Q_n\right) = \xi\left(\bigotimes_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\bigoplus_{j=1}^{n} r_j Q_{\tau_j}\right)^{\varpi}\right)^{\rho}. \tag{57}$$

Using Lemma 3.1, we get

$$\bigoplus_{j=1}^{n} r_j Q_{\tau_j} = \left\langle \left(\frac{\prod_{j=1}^{n}\left(1+(\gamma-1)u_{\tau_j}^q\right)^{r_j} - \prod_{j=1}^{n}\left(1-u_{\tau_j}^q\right)^{r_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)u_{\tau_j}^q\right)^{r_j} + (\gamma-1)\prod_{j=1}^{n}\left(1-u_{\tau_j}^q\right)^{r_j}}\right)^{\frac{1}{q}}, \left(\frac{\gamma\prod_{j=1}^{n}\left(1-u_{\tau_j}^q\right)^{r_j} - \gamma\prod_{j=1}^{n}\left(1-u_{\tau_j}^q - v_{\tau_j}^q\right)^{r_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)u_{\tau_j}^q\right)^{r_j} + (\gamma-1)\prod_{j=1}^{n}\left(1-u_{\tau_j}^q\right)^{r_j}}\right)^{\frac{1}{q}} \right\rangle. \tag{58}$$

Then we obtain Eq. (59) by Lemma 4.1, i.e.,

$$\bigotimes_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\bigoplus_{j=1}^{n} r_j Q_{\tau_j}\right)^{\varpi}=\left\langle\left(\frac{\gamma\Lambda-\gamma\Omega}{\Gamma+(\gamma-1)\Lambda}\right)^{\frac{1}{q}},\left(\frac{\Gamma-\Lambda}{\Gamma+(\gamma-1)\Lambda}\right)^{\frac{1}{q}}\right\rangle, \tag{59}$$

where

$$\Gamma=\prod_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod_{j=1}^{n}\left(1+(\gamma-1)u_{\tau_j}^{q}\right)^{r_j}+(\gamma-1)(\gamma+1)\prod_{j=1}^{n}\left(1-u_{\tau_j}^{q}\right)^{r_j}-(\gamma-1)\gamma\prod_{j=1}^{n}\left(1-u_{\tau_j}^{q}-v_{\tau_j}^{q}\right)^{r_j}\right)^{\varpi},$$

$$\Lambda=\prod_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod_{j=1}^{n}\left(1+(\gamma-1)u_{\tau_j}^{q}\right)^{r_j}-\prod_{j=1}^{n}\left(1-u_{\tau_j}^{q}\right)^{r_j}+\gamma\prod_{j=1}^{n}\left(1-u_{\tau_j}^{q}-v_{\tau_j}^{q}\right)^{r_j}\right)^{\varpi},$$

$$\Omega=\prod_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\gamma\prod_{j=1}^{n}\left(1-u_{\tau_j}^{q}-v_{\tau_j}^{q}\right)^{r_j}\right)^{\varpi}.$$

Thus, we have

$$\xi\left(\bigotimes_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\bigoplus_{j=1}^{n} r_j Q_{\tau_j}\right)^{\varpi}\right)^{\rho}=\left\langle\begin{array}{c}\left(\frac{(\Gamma^{\rho}+(\gamma-1)(\gamma+1)\Lambda^{\rho}-(\gamma-1)\gamma\Omega^{\rho})^{\xi}-(\Gamma^{\rho}-\Lambda^{\rho}+\gamma\Omega^{\rho})^{\xi}}{(\Gamma^{\rho}+(\gamma-1)(\gamma+1)\Lambda^{\rho}-(\gamma-1)\gamma\Omega^{\rho})^{\xi}+(\gamma-1)(\Gamma^{\rho}-\Lambda^{\rho}+\gamma\Omega^{\rho})^{\xi}}\right)^{\frac{1}{q}},\\ \left(\frac{\gamma(\Gamma^{\rho}-\Lambda^{\rho}+\gamma\Omega^{\rho})^{\xi}-\gamma(\gamma\Omega^{\rho})^{\xi}}{(\Gamma^{\rho}+(\gamma-1)(\gamma+1)\Lambda^{\rho}-(\gamma-1)\gamma\Omega^{\rho})^{\xi}+(\gamma-1)(\Gamma^{\rho}-\Lambda^{\rho}+\gamma\Omega^{\rho})^{\xi}}\right)^{\frac{1}{q}}\end{array}\right\rangle. \tag{60}$$

By the closure of $\mathcal{Q}$ under the addition, multiplication, scalar multiplication and power, it's clear that the aggregated value of the q-ROFIHWGCHM is still a q-ROFN.

This completes the proof of Theorem 4.4.

It is easy to verify that the q-ROFIHWGCHM remains idempotent, bounded and commutative whose proofs are omitted here.

Next, we consider the case where $R=\left(\lambda_1,\lambda_2,\ldots,\lambda_l,0,0,\ldots,0\right)$ for the q-ROFIHWGCHM.

If $R=\left(\lambda_1,\lambda_2,\ldots,\lambda_l,0,0,\ldots,0\right)$, where $\lambda_j\geq 0\,(j=1,2,\ldots,l)$ and $\sum_{j=1}^{l}\lambda_j\neq 0$,, then Eq. (55) degenerates into the following:

$$q\text{ - ROFIHWGCHM}\left(Q_1,Q_2,\ldots,Q_n\right)=\frac{1}{\sum_{j=1}^{l}\lambda_j}\left(\bigotimes_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_l=\tau_{l-1}}^{n}\left(\bigoplus_{j=1}^{l}\lambda_j Q_{\tau_j}\right)^{\mathcal{K}_{\tau_1\tau_2\cdots\tau_l}}\right), \tag{61}$$

where.

$\mathcal{K}_{\tau_1\tau_2\cdots\tau_l}=\dfrac{\sum_{\tau_{l+1}=\tau_l,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod_{j=1}^{n}\omega_{\tau_j}\right)}{\sum_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod_{j=1}^{n}\omega_{\tau_j}\right)}$, which is called the q-ROF interactive Hamacher weighted geometric ($l-$ parameter) HM (q-ROFIHWG ($l-$ P) HM).

In particular, if $l=2$, then Eq. (61) degenerates into the following:

$$q\text{ - ROFIHWGCHM}\left(Q_1, Q_2, \ldots, Q_n\right) = \frac{1}{\lambda_1 + \lambda_2}\left(\underset{\tau_1=1,\tau_2=\tau_1}{\overset{n}{\otimes}}\left(\lambda_1 Q_{\tau_1} \oplus \lambda_2 Q_{\tau_2}\right)^{\mathcal{K}_{\tau_1\tau_2}}\right), \tag{62}$$

where

$$\mathcal{K}_{\tau_1\tau_2} = \frac{\sum\limits_{\tau_3=\tau_2,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod\limits_{j=1}^{n}\omega_{\tau_j}\right)}{\sum\limits_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod\limits_{j=1}^{n}\omega_{\tau_j}\right)},$$

which is the q-ROF interactive Hamacher weighted GHM (q-ROFIHWGHM); if $l = 1$, then Eq. (61) degenerates into the following:

$$q\text{ - ROFIHWGCHM}\left(Q_1, Q_2, \ldots, Q_n\right) = \frac{1}{\lambda_1}\left(\underset{\tau_1=1}{\overset{n}{\otimes}}\left(\lambda_1 Q_{\tau_1}\right)^{\mathcal{K}_{\tau_1}}\right), \tag{63}$$

where

$$\mathcal{K}_{\tau_1} = \frac{\sum\limits_{\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod\limits_{j=1}^{n}\omega_{\tau_j}\right)}{\sum\limits_{\tau_1=1,\tau_2=\tau_1,\ldots,\tau_n=\tau_{n-1}}^{n}\left(\prod\limits_{j=1}^{n}\omega_{\tau_j}\right)},$$

which is the q-ROF interactive Hamacher generalized weighted geometric average (q-ROFIHGWGA).

Example 4.1 Let $Q_1 = \langle 0.9, 0.6\rangle$, $Q_2 = \langle 0.7, 0.8\rangle$ and $Q_3 = \langle 0.5, 0.7\rangle$ be three q-ROFNs, whose weights are 0.5, 0.3 and 0.2. Now we fuse Q_1, Q_2 and Q_3 by the q-ROFIHWCHM and q-ROFIHWGCHM, respectively, where we set $q = 3$, $\gamma = 3$ and $R = (1, 1, 1)$.

We first get

$$\xi = \frac{1}{1+1+1} = \frac{1}{3},$$

$$\rho = \frac{1}{\sum\limits_{\tau_1=1,\tau_2=\tau_1,\tau_3=\tau_2}^{3}\omega_{\tau_1}\omega_{\tau_2}\omega_{\tau_3}} = 2.4390,$$

$$\Omega = \prod_{\tau_1=1,\tau_2=\tau_1,\tau_3=\tau_2}^{3}\left(3 * \left(1 - u_{\tau_1}^3 - v_{\tau_1}^3\right) * \left(1 - u_{\tau_2}^3 - v_{\tau_2}^3\right) * \left(1 - u_{\tau_3}^3 - v_{\tau_3}^3\right)\right)^{\omega_{\tau_1}\omega_{\tau_2}\omega_{\tau_3}} = 0.0925.$$

(1) Since

$$\Phi = \prod_{\tau_1=1,\tau_2=\tau_1,\tau_3=\tau_2}^{3} \begin{pmatrix} \left(1+(3-1)v_{\tau_1}^3\right) * \left(1+(3-1)v_{\tau_2}^3\right) * \left(1+(3-1)v_{\tau_3}^3\right) + \\ (3-1)*(3+1)*\left(1-v_{\tau_1}^3\right) * \left(1-v_{\tau_2}^3\right) * \left(1-v_{\tau_3}^3\right) - \\ (3-1)*3*\left(1-u_{\tau_1}^3-v_{\tau_1}^3\right) * \left(1-u_{\tau_2}^3-v_{\tau_2}^3\right) * \left(1-u_{\tau_3}^3-v_{\tau_3}^3\right) \end{pmatrix}^{\omega_{\tau_1}\omega_{\tau_2}\omega_{\tau_3}} = 2.2182,$$

$$\Psi = \prod_{\tau_1=1,\tau_2=\tau_1,\tau_3=\tau_2}^{3} \begin{pmatrix} \left(1+(3-1)v_{\tau_1}^3\right) * \left(1+(3-1)v_{\tau_2}^3\right) * \left(1+(3-1)v_{\tau_3}^3\right) - \\ \left(1-v_{\tau_1}^3\right) * \left(1-v_{\tau_2}^3\right) * \left(1-v_{\tau_3}^3\right) + \\ 3*\left(1-u_{\tau_1}^3-v_{\tau_1}^3\right) * \left(1-u_{\tau_2}^3-v_{\tau_2}^3\right) * \left(1-u_{\tau_3}^3-v_{\tau_3}^3\right) \end{pmatrix}^{\omega_{\tau_1}\omega_{\tau_2}\omega_{\tau_3}} = 1.7246,$$

we derive q - ROFIHWCHM$\left(Q_1, Q_2, Q_3\right) = \langle 0.8455, 0.6666\rangle$ with the help of Eq. (45).

(2) Since

$$\Gamma = \prod_{\tau_1=1,\tau_2=\tau_1,\tau_3=\tau_2}^{3} \begin{pmatrix} \left(1+(3-1)u_{\tau_1}^3\right) * \left(1+(3-1)u_{\tau_2}^3\right) * \left(1+(3-1)u_{\tau_3}^3\right) + \\ (3-1)*(3+1)*\left(1-u_{\tau_1}^3\right) * \left(1-u_{\tau_2}^3\right) * \left(1-u_{\tau_3}^3\right) - \\ (3-1)*3*\left(1-u_{\tau_1}^3-v_{\tau_1}^3\right) * \left(1-u_{\tau_2}^3-v_{\tau_2}^3\right) * \left(1-u_{\tau_3}^3-v_{\tau_3}^3\right) \end{pmatrix}^{\omega_{\tau_1}\omega_{\tau_2}\omega_{\tau_3}} = 2.5241,$$

$$\Lambda = \prod_{\tau_1=1,\tau_2=\tau_1,\tau_3=\tau_2}^{3} \begin{pmatrix} \left(1+(3-1)u_{\tau_1}^3\right) * \left(1+(3-1)u_{\tau_2}^3\right) * \left(1+(3-1)u_{\tau_3}^3\right) - \\ \left(1-u_{\tau_1}^3\right) * \left(1-u_{\tau_2}^3\right) * \left(1-u_{\tau_3}^3\right) + \\ 3*\left(1-u_{\tau_1}^3-v_{\tau_1}^3\right) * \left(1-u_{\tau_2}^3-v_{\tau_2}^3\right) * \left(1-u_{\tau_3}^3-v_{\tau_3}^3\right) \end{pmatrix}^{\omega_{\tau_1}\omega_{\tau_2}\omega_{\tau_3}} = 2.3366,$$

we derive q - ROFIHWGCHM$\left(Q_1, Q_2, Q_3\right) = \langle 0.7709, 0.7643\rangle$ by Eq. (56).

5 A novel MADM algorithm based on the introduced means

In this section, we use the q-ROFIHWAPA and q-ROFIHWCHM (resp. q-ROFIHWGCHM) to devise a novel q-ROF MADM algorithm.

Let $A_1, A_2, \ldots, A_m$ be m alternatives; let $C_1, C_2, \ldots, C_n$ be n attributes, whose weights are $\omega_1, \omega_2, \ldots, \omega_n$, respectively, such that $\omega_j \geq 0$ and $\sum_{j=1}^{n} \omega_j = 1$; $\widehat{EX} = \left(\hat{Q}_{ij}\right)_{m\times n} = \left(\left\langle \hat{u}_{ij}, \hat{v}_{ij}\right\rangle\right)_{m\times n}$ is the q-ROF decision matrix. In view of DM's lack of cognition for alternatives, which leads to the extreme evaluation values, we can draw support from the weighted nonlinear weights carried by the q-ROFIHWAPA to weaken the influence of these unreasonable data. In addition, to capture the correlations among attributes, we use the q-ROFIHWCHM (resp. q-ROFIHWGCHM) to aggregate attribute values of each alternative. Next, we present the detailed operation steps of this algorithm.

Step 1 Transform the q-ROF decision matrix $\widehat{EX} = \left(\hat{Q}_{ij}\right)_{m\times n}$ into the normalized decision matrix $EX = \left(Q_{ij}\right)_{m\times n}$, where

$$Q_{ij} = \langle u_{ij}, v_{ij} \rangle = \begin{cases} \langle \hat{u}_{ij}, \hat{v}_{ij} \rangle, \text{for benefit - type attribute } C_j \\ \langle \hat{v}_{ij}, \hat{u}_{ij} \rangle, \text{for cost - type attribute } C_j \end{cases}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n. \tag{64}$$

Step 2 Calculate the entropy matrix $E = \left(E\left(Q_{ij}\right)\right)_{m\times n}$ of the normalized decision matrix $EX = \left(Q_{ij}\right)_{m\times n}$,

where

$$E\left(Q_{ij}\right) = \left(\frac{\cos\frac{\pi\left|(u_{ij})^q - (v_{ij})^q\right|}{2} + \left(\pi_{ij}\right)^q}{2}\right)^{\frac{1}{q}}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n. \tag{65}$$

Step 3 Combine the attributive weights $\omega_1, \omega_2, \ldots, \omega_n$ to calculate the revised entropy weight matrix $W^{RE} = \left(\omega^{RE}\left(Q_{ij}\right)\right)_{m\times n}$, where

$$\omega^{RE}\left(Q_{ij}\right) = \frac{\omega_j\left(1 - E\left(Q_{ij}\right)\right)}{\sum_{r=1}^{n} \omega_r\left(1 - E\left(Q_{ir}\right)\right)}, \ i = 1, 2, \ldots, m, \ r, j = 1, 2, \ldots, n. \tag{66}$$

Step 4 Calculate the supports between the jth attribute and the tth attribute $\mathrm{Sup}_{jt} = \left(\mathrm{Sup}(Q_{ij}, Q_{it})\right)_{m\times 1}$, where

$$\mathrm{Sup}(Q_{ij}, Q_{it}) = 1 - d(Q_{ij}, Q_{it}), \ i = 1, 2, \ldots, m, \ j, t = 1, 2, \ldots, n, \ j \neq t. \tag{67}$$

Herein, we assume that $d(Q_{ij}, Q_{it})$ is the normalized Hamming distance between Q_{ij} and Q_{it} as shown in Eq. (2).

Step 5 Draw on the attributive weights $\omega_1, \omega_2, \ldots, \omega_n$ to calculate the total weighted support matrix $T = \left(T^{\omega}\left(Q_{ij}\right)\right)_{m\times n}$, where

$$T^{\omega}\left(Q_{ij}\right) = \sum_{t=1, t\neq j}^{n} \omega_t \mathrm{Sup}(Q_{ij}, Q_{it}), \ i = 1, 2, \ldots, m, \ j, t = 1, 2, \ldots, n, \ j \neq t. \tag{68}$$

Step 6 Fit the parameter vector $\tilde{\lambda} = \left(\lambda_1, \lambda_2, \ldots, \lambda_m\right)^{\mathrm{T}}$ corresponding to the alternatives $A_1, A_2, \ldots, A_m$ on the basis of the attributive weights $\omega_1, \omega_2, \ldots, \omega_n$, total weighted support matrix $T = \left(T^{\omega}\left(Q_{ij}\right)\right)_{m\times n}$ and revised entropy weight matrix $W^{RE} = \left(\omega^{RE}\left(Q_{ij}\right)\right)_{m\times n}$, where

$$\lambda_i = \arg \min_{\lambda_i \in [1, +\infty)} \sum_{j=1}^{n} \left|\omega^{RE}\left(Q_{ij}\right) - \Delta^{\omega}_{\lambda_i}\left(Q_{ij}\right)\right| \tag{69}$$

and

$$\Delta_{\lambda_i}^{\omega}\left(Q_{ij}\right) = \frac{\omega_j^{\lambda_i} + \left(T^{\omega}\left(Q_{ij}\right)\right)^{\lambda_i}}{\sum_{r=1}^{n}\left(\omega_r^{\lambda_i} + \left(T^{\omega}\left(Q_{ir}\right)\right)^{\lambda_i}\right)}, \; i = 1,2,\ldots,m, \; r,j = 1,2,\ldots,n. \tag{70}$$

Step 7 Calculate the weighted nonlinear weight matrix $\Delta_{\tilde{\lambda}}^{\omega} = \left(\Delta_{\lambda_i}^{\omega}\left(Q_{ij}\right)\right)_{m\times n}$ by the parameter vector $\tilde{\lambda} = \left(\lambda_1, \lambda_2, \ldots, \lambda_m\right)^{\mathrm{T}}$.
Step 8 Use the q-ROFIHWCHM (Eq. (45)) (resp. q-ROFIHWGCHM (Eq. (56))) to aggregate all the individual attribute values of the alternative A_i into the overall attribute value Q_i(Note: In this step, each alternative corresponds to a set of weighted nonlinear weights).
Step 9 By Eq. (5) and Definition 2.6, calculate the score values $S\left(Q_i\right)(i = 1,2,\ldots,m)$ of the overall attribute values $Q_i(i = 1,2,\ldots,m)$ and rank the alternatives to select the best alternative.
Step 10 End.

6 A case study

In this section, we present an application example to illustrate the effectiveness and superiority of the introduced algorithm.

6.1 The application of the proposed algorithm

Example 6.1 Suppose that four enterprises A_1, A_2, A_3 and A_4 are evaluated based on the following attributes: the growth potential (C_1), the profitability (C_2), the operation capability (C_3) and the solvency (C_4), whose weights are 0.3, 0.4, 0.2 and 0.1, respectively. Assume that the evaluation value of the alternative A_i regarding the attribute C_j provided by the DM is a q-ROFN $\hat{Q}_{ij} = \left\langle \hat{u}_{ij}, \hat{v}_{ij} \right\rangle$, and then the q-ROF decision matrix $\widehat{EX} = \left(\hat{Q}_{ij}\right)_{4\times4}$ is established as i and j traverse, which is shown in Table 3, where $q > 2$.

Now we apply the developed algorithm to solve this practical example. Without loss of generality, we can set $q = 3$.

Step 1 Transform the q-ROF decision matrix $\widehat{EX} = \left(\hat{Q}_{ij}\right)_{4\times4}$ into the normalized decision matrix $EX = \left(Q_{ij}\right)_{4\times4}$. In fact, all the attributes are benefit-type attributes, i.e.,$EX = \widehat{EX}$.
Step 2 Calculate the entropy matrix $E = \left(E\left(Q_{ij}\right)\right)_{4\times4}$ of the normalized decision matrix $EX = \left(Q_{ij}\right)_{4\times4}$ as shown below:

$$E = \begin{pmatrix} 0.9522 & 0.9882 & 0.9083 & 0.8402 \\ 0.8398 & 0.8393 & 0.9104 & 0.7203 \\ 0.9843 & 0.9952 & 0.9515 & 0.9377 \\ 0.8354 & 0.9457 & 0.9843 & 0.7078 \end{pmatrix}.$$

Table 3 The q-ROF decision matrix $\widehat{EX}$ from the DM

	C_1	C_2	C_3	C_4
A_1	$\langle 0.6, 0.1\rangle$	$\langle 0.4, 0.1\rangle$	$\langle 0.7, 0.4\rangle$	$\langle 0.4, 0.8\rangle$
A_2	$\langle 0.8, 0.3\rangle$	$\langle 0.8, 0.2\rangle$	$\langle 0.7, 0.3\rangle$	$\langle 0.9, 0.6\rangle$
A_3	$\langle 0.4, 0.3\rangle$	$\langle 0.3, 0.1\rangle$	$\langle 0.2, 0.6\rangle$	$\langle 0.5, 0.6\rangle$
A_4	$\langle 0.8, 0.6\rangle$	$\langle 0.6, 0.4\rangle$	$\langle 0.4, 0.3\rangle$	$\langle 0.9, 0.4\rangle$

Step 3 Combine the attributive weights 0.3, 0.4, 0.2 and 0.1 to calculate the revised entropy weight matrix $W^{RE} = \left(\omega^{RE}\left(Q_{ij}\right)\right)_{4\times4}$ as shown below:

$$W^{RE} = \begin{pmatrix} 0.2688 & 0.0883 & 0.3437 & 0.2992 \\ 0.3038 & 0.4062 & 0.1132 & 0.1768 \\ 0.2086 & 0.0856 & 0.4295 & 0.2763 \\ 0.4774 & 0.2099 & 0.0303 & 0.2824 \end{pmatrix}.$$

Step 4 Calculate the supports between the jth attribute and the tth attribute $\mathrm{Sup}_{jt} = \left(\mathrm{Sup}(Q_{ij}, Q_{it})\right)_{4\times1} (j, t = 1, 2, 3, 4, j \neq t)$ as shown below:

$$\mathrm{Sup}_{12} = \mathrm{Sup}_{21} = \begin{pmatrix} 0.8480 \\ 0.9810 \\ 0.9370 \\ 0.5520 \end{pmatrix}, \mathrm{Sup}_{13} = \mathrm{Sup}_{31} = \begin{pmatrix} 0.8100 \\ 0.8310 \\ 0.8110 \\ 0.3630 \end{pmatrix}, \mathrm{Sup}_{14} = \mathrm{Sup}_{41} = \begin{pmatrix} 0.4890 \\ 0.5940 \\ 0.7500 \\ 0.7830 \end{pmatrix},$$

$$\mathrm{Sup}_{23} = \mathrm{Sup}_{32} = \begin{pmatrix} 0.6580 \\ 0.8310 \\ 0.7850 \\ 0.8110 \end{pmatrix}, \mathrm{Sup}_{24} = \mathrm{Sup}_{42} = \begin{pmatrix} 0.4890 \\ 0.5750 \\ 0.6870 \\ 0.4870 \end{pmatrix}, \mathrm{Sup}_{34} = \mathrm{Sup}_{43} = \begin{pmatrix} 0.5520 \\ 0.4250 \\ 0.8830 \\ 0.2980 \end{pmatrix}.$$

Step 5 Draw on the attributive weights 0.3, 0.4, 0.2 and 0.1 to calculate the total weighted support matrix $T = \left(T^{\omega}\left(Q_{ij}\right)\right)_{4\times4}$ as shown below:

$$T = \begin{pmatrix} 0.5501 & 0.4349 & 0.5614 & 0.4527 \\ 0.6180 & 0.5180 & 0.6242 & 0.4932 \\ 0.6120 & 0.5068 & 0.6456 & 0.6764 \\ 0.3717 & 0.3765 & 0.4631 & 0.4893 \end{pmatrix}.$$

Step 6 Fit the parameter vector $\tilde{\lambda} = \left(\lambda_1, \lambda_2, \lambda_3, \lambda_4\right)^{\mathrm{T}}$ corresponding to the alternatives A_1, A_2, A_3 and A_4 on the basis of the attributive weights 0.3, 0.4, 0.2 and 0.1, total weighted support matrix $T = \left(T^{\omega}\left(Q_{ij}\right)\right)_{4\times4}$ and revised entropy weight matrix $W^{RE} = \left(\omega^{RE}\left(Q_{ij}\right)\right)_{4\times4}$, i.e.,

$$\tilde{\lambda} = (4.3523, 1.1782, 6.0498, 3.0450)^{\mathrm{T}}.$$

Step 7 Calculate the weighted nonlinear weight matrix $\Delta^{\omega}_{\tilde{\lambda}} = \left(\Delta^{\omega}_{\lambda_i}\left(Q_{ij}\right)\right)_{4\times4}$ by the parameter vector $\tilde{\lambda} = \left(\lambda_1, \lambda_2, \lambda_3, \lambda_4\right)^{\mathrm{T}}$ as shown below:

$$\Delta_{\tilde{\lambda}}^{\omega} = \begin{pmatrix} 0.3333 & 0.1896 & 0.3437 & 0.1334 \\ 0.2855 & 0.2823 & 0.2554 & 0.1768 \\ 0.2192 & 0.0856 & 0.2991 & 0.3962 \\ 0.1845 & 0.2778 & 0.2553 & 0.2824 \end{pmatrix}.$$

Step 8 Use the q-ROFIHWCHM (Eq. (45)) (resp. q-ROFIHWGCHM (Eq. (56))) to aggregate all the individual attribute values of the alternative A_i into the overall attribute value Q_i(Assume that the DM clings to a global perspective, i.e., he/she believes that the correlations among all attributes should be considered; we let $\gamma = 3$ and $R = (1, 1, 1, 1)$):

(1) By the q-ROFIHWCHM, we derive $Q_1 = \langle 0.6260, 0.4112 \rangle$, $Q_2 = \langle 0.8340, 0.3623 \rangle$, $Q_3 = \langle 0.4263, 0.5581 \rangle$, $Q_4 = \langle 0.7672, 0.4497 \rangle$;
(2) By the q-ROFIHWGCHM, we derive $Q_1 = \langle 0.6113, 0.4410 \rangle$, $Q_2 = \langle 0.8127, 0.4258 \rangle$, $Q_3 = \langle 0.4271, 0.5582 \rangle$, $Q_4 = \langle 0.7163, 0.5544 \rangle$.

Step 9 By Eq. (5), calculate the score values $S(Q_i)(i = 1, 2, 3, 4)$ of the overall attribute values Q_i $(i = 1, 2, 3, 4)$:

(1) Based on the q-ROFIHWCHM, we get $S(Q_1) = 0.3679$, $S(Q_2) = 0.6282$, $S(Q_3) = 0.2211$, $S(Q_4) = 0.5157$;
(2) Based on the q-ROFIHWGCHM, we get $S(Q_1) = 0.3511$, $S(Q_2) = 0.5873$, $S(Q_3) = 0.2213$, $S(Q_4) = 0.4326$.

Further, the ranking result of the alternatives A_1, A_2, A_3 and A_4 obtained by Definition 2.6 is $A_2 \succ A_4 \succ A_1 \succ A_3$. Thus, the best alternative is A_2.

Step 10 End.

6.2 The impact of q, γ and R on the ranking results

6.2.1 The impact of q on the ranking results

In the previous subsection, $q = 3$ is set in advance according to the DM's evaluation information. We now study the ranking results derived from the different parameter q, which are shown in Tables 4, 5 (Let $\gamma = 3$ and $R = (1, 1, 1, 1)$).

From Table 4, based on the q-ROFIHWCHM, the ranking results are identical when $q = 3, 5$ and 7, i.e., $A_2 \succ A_4 \succ A_1 \succ A_3$, and the ranking result is $A_2 \succ A_4 \succ A_3 \succ A_1$ when $q = 10$. Although the ranking results are slightly different,A_2 is always the best alternative. The same goes for the Table 5, which is not described again. Liu et al. (2020), Liu and Wang (2019) pointed out that the fuzzy environment parameter q should be the smallest positive integer, such that $\hat{u}_{ij}^q + \hat{v}_{ij}^q \le 1$, where $\hat{Q}_{ij} = \langle \hat{u}_{ij}, \hat{v}_{ij} \rangle$ is the evaluation value of the alternative A_i regarding the attribute C_j provided by the DM. As a matter of fact, the larger the parameter q, the more serious the information distortion. Just taking the pair $\hat{Q}_{34} = \langle 0.5, 0.6 \rangle$ as an example, when $q = 3, 5, 7$ and 10, its HDs are 0.8702, 0.9772, 0.9948 and 0.9993, respectively. Obviously, the larger q, the higher its uncertainty, and it is almost

Table 4 Score values and ranking results derived from the different parameter q based on the q-ROFIHWCHM

q - values	Score values	Ranking results
$q = 3$	$S_1 = 0.3679$, $S_2 = 0.6282$, $S_3 = 0.2211$, $S_4 = 0.5157$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$q = 5$	$S_1 = 0.2768$, $S_2 = 0.4557$, $S_3 = 0.2292$, $S_4 = 0.3685$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$q = 7$	$S_1 = 0.2457$, $S_2 = 0.3802$, $S_3 = 0.2409$, $S_4 = 0.3281$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$q = 10$	$S_1 = 0.2154$, $S_2 = 0.3222$, $S_3 = 0.2478$, $S_4 = 0.2985$	$A_2 \succ A_4 \succ A_3 \succ A_1$

Table 5 Score values and ranking results derived from the different parameter q based on the q-ROFIHWGCHM

q - values	Score values	Ranking results
$q = 3$	$S_1 = 0.3511$, $S_2 = 0.5873$, $S_3 = 0.2213$, $S_4 = 0.4326$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$q = 5$	$S_1 = 0.2683$, $S_2 = 0.4555$, $S_3 = 0.2301$, $S_4 = 0.3690$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$q = 7$	$S_1 = 0.2414$, $S_2 = 0.3822$, $S_3 = 0.2410$, $S_4 = 0.3338$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$q = 10$	$S_1 = 0.2152$, $S_2 = 0.3252$, $S_3 = 0.2478$, $S_4 = 0.3043$	$A_2 \succ A_4 \succ A_3 \succ A_1$

$S_i(i = 1, 2, 3, 4)$ are the abbreviations of the score values $S\left(Q_i\right)(i = 1, 2, 3, 4)$, and we still use this notation in the following tables.

completely indeterminate when $q = 7$ and 10. Of course, this bad situation also occurs in other evaluation values. Thus,q selected according to Liu et al.'s viewpoint (Liu and Wang 2019; Liu et al. 2020) greatly reduces the overall information loss, which in turn leads to more accurate decision results.

6.2.2 The impact of γ on the ranking results.

In what follows, we analyze the ranking results derived from the different parameter γ, which are shown in Tables 6, 7 (Let $q = 3$ and $R = (1, 1, 1, 1)$).

From Tables 6, 7, the change of the parameter γ does not affect the ranking results educed by the q-ROFIHWCHM and q-ROFIHWGCHM, which are always $A_2 \succ A_4 \succ A_1 \succ A_3$, i.e., A_2 is the best alternative. In addition, the score values based on the q-ROFIHWCHM are relatively large when γ is large, and the opposite is true for the q-ROFIHWGCHM. The reason for this phenomenon is that these two means have their own emphasis, i.e., the arithmetic mean centers upon the whole, while the geometric mean centers upon the individual.

6.2.3 The impact of R on the ranking results

In fact, for the Example 6.1 we can embed different numbers of parameters to R so as to mirror different types of correlations. Let $R = \left(\lambda_1, 0, 0, 0\right)$ if these four attributes are considered to be independent of each other; let $R = \left(\lambda_1, \lambda_2, 0, 0\right)$ if these four attributes are deemed pairwise interrelated; let $R = \left(\lambda_1, \lambda_2, \lambda_3, 0\right)$ if the correlations among any three of these four attributes are to be reflected; let $R = \left(\lambda_1, \lambda_2, \lambda_3, \lambda_4\right)$ if the correlations among all

Table 6 Score values and ranking results derived from the different parameter γ based on the q-ROFIHW-CHM

γ - values	Score values	Ranking results
$\gamma = 1$	$S_1 = 0.3689$, $S_2 = 0.6284$, $S_3 = 0.2193$, $S_4 = 0.5161$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$\gamma = 2$	$S_1 = 0.3677$, $S_2 = 0.6280$, $S_3 = 0.2199$, $S_4 = 0.5153$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$\gamma = 3$	$S_1 = 0.3679$, $S_2 = 0.6282$, $S_3 = 0.2211$, $S_4 = 0.5157$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$\gamma = 5$	$S_1 = 0.3693$, $S_2 = 0.6292$, $S_3 = 0.2242$, $S_4 = 0.5174$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$\gamma = 10$	$S_1 = 0.3737$, $S_2 = 0.6318$, $S_3 = 0.2343$, $S_4 = 0.5213$	$A_2 \succ A_4 \succ A_1 \succ A_3$

Table 7 Score values and ranking results derived from the different parameter γ based on the q-ROFIHW-GCHM

γ - values	Score values	Ranking results
$\gamma = 1$	$S_1 = 0.3566$, $S_2 = 0.6065$, $S_3 = 0.2204$, $S_4 = 0.4891$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$\gamma = 2$	$S_1 = 0.3544$, $S_2 = 0.5936$, $S_3 = 0.2211$, $S_4 = 0.4560$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$\gamma = 3$	$S_1 = 0.3511$, $S_2 = 0.5873$, $S_3 = 0.2213$, $S_4 = 0.4326$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$\gamma = 5$	$S_1 = 0.3442$, $S_2 = 0.5809$, $S_3 = 0.2210$, $S_4 = 0.4005$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$\gamma = 10$	$S_1 = 0.3294$, $S_2 = 0.5751$, $S_3 = 0.2191$, $S_4 = 0.3554$	$A_2 \succ A_4 \succ A_1 \succ A_3$

attributes are to be mirrored, where λ_1,λ_2,λ_3 and λ_4 are all positive numbers. Next, we analyze the influence of R on the ranking results educed by the q-ROFIHWCHM and q-ROFIHWGCHM for all types of correlations mentioned above, which are shown in Tables 8, 9 (Let $q = 3$ and $\gamma = 3$).

As shown in Table 8, the ranking results educed by the q-ROFIHWCHM are all $A_2 \succ A_4 \succ A_1 \succ A_3$ regardless of parameter changes for every pre-assumed correlation structure, i.e.,A_2 is the best solution. As for Table 9, except for the case where these four attributes are considered to be independent of each other, the ranking results educed by the q-ROFIHWGCHM are affected by the parameter changes. Specifically, in the other three cases, when only the correlations are characterized without the intensity, the ranking results educed by the q-ROFIHWGCHM are completely consistent with those obtained by the q-ROFIHWCHM, i.e.,$A_2 \succ A_4 \succ A_1 \succ A_3$; but once the correlation strength is enhanced, the ranking results change to $A_2 \succ A_1 \succ A_4 \succ A_3$ or $A_2 \succ A_1 \succ A_3 \succ A_4$.On the other hand, with regard to each correlation structure, the score values based on the q-ROFIHWCHM relatively large when R is embedded with large parameters, and the opposite is true for the q-ROFIHWGCHM. The reason for the different ranking results and the opposite trend of the score values educed by the q-ROFIHWCHM and q-ROFIHWGCHM is that their expressions are dissimilar, in other words, one is on the basis of the arithmetic mean and the other is on the basis of the geometric mean, which reflect different emphasis. From Tables 8, 9, it is clear that A_2 is always the best alternative no matter how the ranking results change.

Table 8 Score values and ranking results derived from the different parameter vector R based on the q-ROFIHWCHM

R - values	Score values	Ranking results
$R=(1,0,0,0)$	$S_1=0.3639$, $S_2=0.5689$, $S_3=0.2272$, $S_4=0.4186$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$R=(3,0,0,0)$	$S_1=0.3605$, $S_2=0.5723$, $S_3=0.2262$, $S_4=0.4180$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$R=(5,0,0,0)$	$S_1=0.3614$, $S_2=0.5749$, $S_3=0.2361$, $S_4=0.4587$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$R=(1,1,0,0)$	$S_1=0.3654$, $S_2=0.5678$, $S_3=0.2181$, $S_4=0.4214$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$R=(3,3,0,0)$	$S_1=0.3692$, $S_2=0.5786$, $S_3=0.2329$, $S_4=0.4635$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$R=(5,5,0,0)$	$S_1=0.3782$, $S_2=0.5806$, $S_3=0.2764$, $S_4=0.4963$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$R=(1,1,1,0)$	$S_1=0.3683$, $S_2=0.5826$, $S_3=0.2169$, $S_4=0.4570$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$R=(3,3,3,0)$	$S_1=0.3883$, $S_2=0.5988$, $S_3=0.2557$, $S_4=0.5005$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$R=(5,5,5,0)$	$S_1=0.4029$, $S_2=0.6021$, $S_3=0.3236$, $S_4=0.5207$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$R=(1,1,1,1)$	$S_1=0.3679$, $S_2=0.6282$, $S_3=0.2211$, $S_4=0.5157$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$R=(3,3,3,3)$	$S_1=0.4167$, $S_2=0.6468$, $S_3=0.2792$, $S_4=0.5468$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$R=(5,5,5,5)$	$S_1=0.4381$, $S_2=0.6544$, $S_3=0.3490$, $S_4=0.5599$	$A_2 \succ A_4 \succ A_1 \succ A_3$

Table 9 Score values and ranking results derived from the different parameter vector R based on the q-ROFIHWGCHM

R - values	Score values	Ranking results
$R=(1,0,0,0)$	$S_1=0.3688$, $S_2=0.5734$, $S_3=0.2316$, $S_4=0.4746$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$R=(3,0,0,0)$	$S_1=0.3731$, $S_2=0.5712$, $S_3=0.2362$, $S_4=0.4588$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$R=(5,0,0,0)$	$S_1=0.3728$, $S_2=0.5656$, $S_3=0.2356$, $S_4=0.3824$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$R=(1,1,0,0)$	$S_1=0.3731$, $S_2=0.5720$, $S_3=0.2246$, $S_4=0.4509$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$R=(3,3,0,0)$	$S_1=0.3722$, $S_2=0.5591$, $S_3=0.2231$, $S_4=0.3483$	$A_2 \succ A_1 \succ A_4 \succ A_3$
$R=(5,5,0,0)$	$S_1=0.3593$, $S_2=0.5263$, $S_3=0.2122$, $S_4=0.2561$	$A_2 \succ A_1 \succ A_4 \succ A_3$
$R=(1,1,1,0)$	$S_1=0.3725$, $S_2=0.5767$, $S_3=0.2220$, $S_4=0.4374$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$R=(3,3,3,0)$	$S_1=0.3466$, $S_2=0.5341$, $S_3=0.2104$, $S_4=0.2792$	$A_2 \succ A_1 \succ A_4 \succ A_3$
$R=(5,5,5,0)$	$S_1=0.2922$, $S_2=0.4835$, $S_3=0.1939$, $S_4=0.2065$	$A_2 \succ A_1 \succ A_4 \succ A_3$
$R=(1,1,1,1)$	$S_1=0.3511$, $S_2=0.5873$, $S_3=0.2213$, $S_4=0.4326$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$R=(3,3,3,3)$	$S_1=0.2833$, $S_2=0.5022$, $S_3=0.2029$, $S_4=0.2430$	$A_2 \succ A_1 \succ A_4 \succ A_3$
$R=(5,5,5,5)$	$S_1=0.2350$, $S_2=0.4560$, $S_3=0.1848$, $S_4=0.1827$	$A_2 \succ A_1 \succ A_3 \succ A_4$

6.3 Comparison with the existing MADM methods

In this subsection, we illustrate the rationality and superiority of the developed algorithm by comparing it with some extant q-ROF MADM methods.

(1) Compare with the MADM methods using the q-ROF Hamacher means

Let $Q_i = \langle u_i, v_i \rangle \, (i = 1, 2, \dots, n)$ be n q-ROFNs with the weights $\omega_1, \omega_2, \dots, \omega_n$ such that $\omega_i \geq 0$ and $\sum_{i=1}^{n} \omega_i = 1$, the aggregated values of several Hamacher means are reviewed as follows:

(1) Weighted q-ROF Hamacher average (Wq-ROFHA) (Darko and Liang 2020):

$$\mathrm{W}q\text{ - ROFHA}\left(Q_1, Q_2, \dots, Q_n\right) = \mathop{\oplus}_{i=1}^{n} \omega_i Q_i$$
$$= \left\langle \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+(\gamma-1)u_i^q\right)^{\omega_i} - \prod_{i=1}^{n}\left(1-u_i^q\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+(\gamma-1)u_i^q\right)^{\omega_i} + (\gamma-1)\prod_{i=1}^{n}\left(1-u_i^q\right)^{\omega_i}}}, \frac{\sqrt[q]{\gamma}\prod_{i=1}^{n} v_i^{\omega_i}}{\sqrt[q]{\prod_{i=1}^{n}\left(1+(\gamma-1)\left(1-v_i^q\right)\right)^{\omega_i} + (\gamma-1)\prod_{i=1}^{n} v_i^{q\omega_i}}} \right\rangle, \tag{71}$$

where $\gamma > 0$;

(2) q-ROF weighted Hamacher BM (q-ROFWHBM) (Liu and Wang 2019):t

$$q\text{ - ROFWHBM}\left(Q_1, Q_2, \dots, Q_n\right) = \left(\frac{1}{n(n-1)} \mathop{\oplus}_{\substack{i,j=1 \\ i \neq j}}^{n} \left(n\omega_i Q_i\right)^s \otimes \left(n\omega_j Q_j\right)^t \right)^{\frac{1}{s+t}}$$
$$= \left\langle \left(\frac{\gamma\left(\frac{u'-u''}{u'+(\gamma-1)u''}\right)^{\frac{1}{s+t}}}{(\gamma-1)\left(\frac{u'-u''}{u'+(\gamma-1)u''}\right)^{\frac{1}{s+t}} + \left(\gamma-(\gamma-1)\left(\frac{u'-u''}{u'+(\gamma-1)u''}\right)\right)^{\frac{1}{s+t}}} \right)^{\frac{1}{q}}, \left(\frac{\left(1+(\gamma-1)\left(\frac{\gamma v'}{(\gamma-1)v'+v''}\right)\right)^{\frac{1}{s+t}} - \left(1-\left(\frac{\gamma v'}{(\gamma-1)v'+v''}\right)\right)^{\frac{1}{s+t}}}{\left(1+(\gamma-1)\left(\frac{\gamma v'}{(\gamma-1)v'+v''}\right)\right)^{\frac{1}{s+t}} + (\gamma-1)\left(1-\left(\frac{\gamma v'}{(\gamma-1)v'+v''}\right)\right)^{\frac{1}{s+t}}} \right)^{\frac{1}{q}} \right\rangle, \tag{72}$$

where

$$u' = \left(1 + (\gamma - 1)\frac{a'_{ij} - a''_{ij}}{a'_{ij} + (\gamma - 1)a''_{ij}}\right)^{\frac{1}{n(n-1)}},$$

$$u'' = \left(1 - \frac{a'_{ij} - a''_{ij}}{a'_{ij} + (\gamma - 1)a''_{ij}}\right)^{\frac{1}{n(n-1)}},$$

$$v' = \left(\frac{\gamma b'_{ij}}{(\gamma - 1)b'_{ij} + b''_{ij}}\right)^{\frac{1}{n(n-1)}},$$

$$v'' = \left(\gamma - (\gamma - 1)\frac{\gamma b'_{ij}}{(\gamma - 1)b'_{ij} + b''_{ij}}\right)^{\frac{1}{n(n-1)}},$$

$$a'_{ij} = \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1 + (\gamma - 1)\left(\frac{x_i x_j}{\gamma + (1-\gamma)\left(x_i + x_j - x_i x_j\right)}\right)\right),$$

$$a''_{ij} = \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1 - \left(\frac{x_i x_j}{\gamma + (1-\gamma)\left(x_i + x_j - x_i x_j\right)}\right)\right),$$

$$b'_{ij} = \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(\frac{y_i + y_j + (\gamma - 2)y_i y_j}{1 - (1-\gamma)y_i y_j}\right),$$

$$b''_{ij} = \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(\gamma - (\gamma - 1)\left(\frac{y_i + y_j + (\gamma - 2)y_i y_j}{1 - (1-\gamma)y_i y_j}\right)\right),$$

$$x_i = \frac{\gamma\left(\frac{\left(1+(\gamma-1)u_i^q\right)^{n\omega_i} - \left(1-u_i^q\right)^{n\omega_i}}{\left(1+(\gamma-1)u_i^q\right)^{n\omega_i} + (\gamma-1)\left(1-u_i^q\right)^{n\omega_i}}\right)^s}{(\gamma - 1)\left(\frac{\left(1+(\gamma-1)u_i^q\right)^{n\omega_i} - \left(1-u_i^q\right)^{n\omega_i}}{\left(1+(\gamma-1)u_i^q\right)^{n\omega_i} + (\gamma-1)\left(1-u_i^q\right)^{n\omega_i}}\right)^s + \left(\gamma - (\gamma - 1)\frac{\left(1+(\gamma-1)u_i^q\right)^{n\omega_i} - \left(1-u_i^q\right)^{n\omega_i}}{\left(1+(\gamma-1)u_i^q\right)^{n\omega_i} + (\gamma-1)\left(1-u_i^q\right)^{n\omega_i}}\right)^s},$$

$$x_j = \frac{\gamma\left(\frac{\left(1+(\gamma-1)u_j^q\right)^{n\omega_j} - \left(1-u_j^q\right)^{n\omega_j}}{\left(1+(\gamma-1)u_j^q\right)^{n\omega_j} + (\gamma-1)\left(1-u_j^q\right)^{n\omega_j}}\right)^t}{(\gamma - 1)\left(\frac{\left(1+(\gamma-1)u_j^q\right)^{n\omega_j} - \left(1-u_j^q\right)^{n\omega_j}}{\left(1+(\gamma-1)u_j^q\right)^{n\omega_j} + (\gamma-1)\left(1-u_j^q\right)^{n\omega_j}}\right)^t + \left(\gamma - (\gamma - 1)\frac{\left(1+(\gamma-1)u_j^q\right)^{n\omega_j} - \left(1-u_j^q\right)^{n\omega_j}}{\left(1+(\gamma-1)u_j^q\right)^{n\omega_j} + (\gamma-1)\left(1-u_j^q\right)^{n\omega_j}}\right)^t},$$

$$y_i = \frac{\left(1 + (\gamma - 1)\frac{\gamma v_i^{n\omega_i}}{(\gamma-1)v_i^{qn\omega_i} + \left(1+(\gamma-1)\left(1-v_i^q\right)\right)^{n\omega_i}}\right)^s - \left(1 - \frac{\gamma v_i^{n\omega_i}}{(\gamma-1)v_i^{qn\omega_i} + \left(1+(\gamma-1)\left(1-v_i^q\right)\right)^{n\omega_i}}\right)^s}{\left(1 + (\gamma - 1)\frac{\gamma v_i^{n\omega_i}}{(\gamma-1)v_i^{qn\omega_i} + \left(1+(\gamma-1)\left(1-v_i^q\right)\right)^{n\omega_i}}\right)^s + (\gamma - 1)\left(1 - \frac{\gamma v_i^{n\omega_i}}{(\gamma-1)v_i^{qn\omega_i} + \left(1+(\gamma-1)\left(1-v_i^q\right)\right)^{n\omega_i}}\right)^s},$$

$$y_j = \frac{\left(1 + (\gamma - 1)\frac{\gamma v_j^{n\omega_j}}{(\gamma-1)v_j^{qn\omega_j} + \left(1+(\gamma-1)\left(1-v_j^q\right)\right)^{n\omega_j}}\right)^t - \left(1 - \frac{\gamma v_j^{n\omega_j}}{(\gamma-1)v_j^{qn\omega_j} + \left(1+(\gamma-1)\left(1-v_j^q\right)\right)^{n\omega_j}}\right)^t}{\left(1 + (\gamma - 1)\frac{\gamma v_j^{n\omega_j}}{(\gamma-1)v_j^{qn\omega_j} + \left(1+(\gamma-1)\left(1-v_j^q\right)\right)^{n\omega_j}}\right)^t + (\gamma - 1)\left(1 - \frac{\gamma v_j^{n\omega_j}}{(\gamma-1)v_j^{qn\omega_j} + \left(1+(\gamma-1)\left(1-v_j^q\right)\right)^{n\omega_j}}\right)^t},$$

such that $\gamma > 0$, $s \geq 0$, $t \geq 0$ and $s + t \neq 0$;

(3) Weighted q-ROF Hamacher MSM (Wq-ROFHMSM) (Darko and Liang 2020):

$$\mathrm{W}q\text{ - ROFHMSM}\left(Q_1, Q_2, \ldots, Q_n\right) = \left(\frac{\bigoplus_{1 \le \tau_1 < \cdots < \tau_k \le n} \left(\bigotimes_{j=1}^{k} \omega_{\tau_j} Q_{\tau_j} \right)}{C_n^k} \right)^{\frac{1}{k}}$$

where

$$= \left\langle \sqrt[q]{\frac{\gamma\left(\prod_{1\le\tau_1<\cdots<\tau_k\le n}(x+(\gamma^2-1)y)^{\frac{1}{C_n^k}} - \prod_{1\le\tau_1<\cdots<\tau_k\le n}(x-y)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}}{\left(\prod_{1\le\tau_1<\cdots<\tau_k\le n}(x+(\gamma^2-1)y)^{\frac{1}{C_n^k}} + (\gamma^2-1)\prod_{1\le\tau_1<\cdots<\tau_k\le n}(x-y)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}} + (\gamma-1)\left(\prod_{1\le\tau_1<\cdots<\tau_k\le n}(x+(\gamma^2-1)y)^{\frac{1}{C_n^k}} - \prod_{1\le\tau_1<\cdots<\tau_k\le n}(x-y)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}}}, \sqrt[q]{\frac{\left(\prod_{1\le\tau_1<\cdots<\tau_k\le n}(m+(\gamma^2-1)z)^{\frac{1}{C_n^k}} + (\gamma^2-1)\prod_{1\le\tau_1<\cdots<\tau_k\le n}(m-z)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}} - \left(\prod_{1\le\tau_1<\cdots<\tau_k\le n}(m+(\gamma^2-1)z)^{\frac{1}{C_n^k}} - \prod_{1\le\tau_1<\cdots<\tau_k\le n}(m-z)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}}{\left(\prod_{1\le\tau_1<\cdots<\tau_k\le n}(m+(\gamma^2-1)z)^{\frac{1}{C_n^k}} + (\gamma^2-1)\prod_{1\le\tau_1<\cdots<\tau_k\le n}(m-z)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}} + (\gamma-1)\left(\prod_{1\le\tau_1<\cdots<\tau_k\le n}(m+(\gamma^2-1)z)^{\frac{1}{C_n^k}} - \prod_{1\le\tau_1<\cdots<\tau_k\le n}(m-z)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}}} \right\rangle,$$

$$x = \prod_{j=1}^{k} \left(\frac{\left(1+(\gamma-1)u_{\tau_j}^q\right)^{\omega_{\tau_j}} + \left(\gamma^2-1\right)\left(1-u_{\tau_j}^q\right)^{\omega_{\tau_j}}}{\left(1+(\gamma-1)u_{\tau_j}^q\right)^{\omega_{\tau_j}} + (\gamma-1)\left(1-u_{\tau_j}^q\right)^{\omega_{\tau_j}}} \right),$$

$$y = \prod_{j=1}^{k} \left(\frac{\left(1+(\gamma-1)u_{\tau_j}^q\right)^{\omega_{\tau_j}} - \left(1-u_{\tau_j}^q\right)^{\omega_{\tau_j}}}{\left(1+(\gamma-1)u_{\tau_j}^q\right)^{\omega_{\tau_j}} + (\gamma-1)\left(1-u_{\tau_j}^q\right)^{\omega_{\tau_j}}} \right),$$

$$m = \prod_{j=1}^{k} \left(\frac{\left(1+(\gamma-1)\left(1-v_{\tau_j}^q\right)\right)^{\omega_{\tau_j}} + \left(\gamma^2-1\right)v_{\tau_j}^{q\omega_{\tau_j}}}{\left(1+(\gamma-1)\left(1-v_{\tau_j}^q\right)\right)^{\omega_{\tau_j}} + (\gamma-1)v_{\tau_j}^{q\omega_{\tau_j}}} \right),$$

$$z = \prod_{j=1}^{k} \left(\frac{\left(1+(\gamma-1)\left(1-v_{\tau_j}^q\right)\right)^{\omega_{\tau_j}} - v_{\tau_j}^{q\omega_{\tau_j}}}{\left(1+(\gamma-1)\left(1-v_{\tau_j}^q\right)\right)^{\omega_{\tau_j}} + (\gamma-1)v_{\tau_j}^{q\omega_{\tau_j}}} \right),$$

such that $\gamma > 0$.

We now use the Wq-ROFHA (Darko and Liang 2020), q-ROFWHBM (Liu and Wang 2019) and Wq-ROFHMSM (Darko and Liang 2020) to solve Example 6.1. Then the overall attribute value of each alternative aggregated by these means are presented in Table 10, and the score values and ranking results are shown in Table 11 (Let $q = 3$ and $\gamma = 3$, where γ is the (interactive) Hamacher operation parameter).

From Table 11, when the four attributes are considered to be independent of each other or pairwise interrelated, the ranking result educed by the Wq-ROFHA (Darko and Liang 2020) or q-ROFWHBM (Liu and Wang 2019) is exactly the same as those educed by the q-ROFIHWCHM and q-ROFIHWGCHM, i.e.,$A_2 \succ A_4 \succ A_1 \succ A_3$. For the case where the correlations among any three of the four attributes are to be reflected, the ranking result educed by the Wq-ROFHMSM (Darko and Liang 2020) is $A_2 \succ A_1 \succ A_3 \succ A_4$, which is slightly different from the ranking $A_2 \succ A_4 \succ A_1 \succ A_3$ educed by the q-ROFIHWCHM and q-ROFIHWGCHM. Even so,A_2 is still the best choice. Therefore, our introduced algorithm is indeed effective.

In the following, we point out the irrationality of the Wq-ROFHA (Darko and Liang 2020), q-ROFWHBM (Liu and Wang 2019) and Wq-ROFHMSM (Darko and Liang 2020) in terms of information fusion.

Case 1 We adjust $\hat{Q}_{21}$ from $\langle 0.8, 0.3\rangle$ to $\langle 0.8, 0\rangle$, and the other evaluation values $\hat{Q}_{22}$,$\hat{Q}_{23}$ and $\hat{Q}_{24}$ are exactly the same as those in Table 3. With regard to the alternative A_2, the overall attribute values Q_2', Q_2'' and Q_2''', which are aggregated by the Wq-ROFHA (Darko and Liang 2020) when $q = 3$ and $\gamma = 3$,5 and 7, are $\langle 0.7969, 0\rangle$, $\langle 0.7960, 0\rangle$ and $\langle 0.7955, 0\rangle$, respectively. So the N-MDs of $\hat{Q}_{22}$,$\hat{Q}_{23}$ and $\hat{Q}_{24}$ and the Hamacher operation parameter γ are invalid in this case.

Case 2 We adjust $\hat{Q}_{21}$ from $\langle 0.8, 0.3\rangle$ to $\langle 0.8, 0\rangle$ and $\hat{Q}_{22}$ from $\langle 0.8, 0.2\rangle$ to $\langle 0.8, 0\rangle$, and besides, the other evaluation values $\hat{Q}_{23}$ and $\hat{Q}_{24}$ are exactly the same as those in Table 3. As far as the alternative A_2 is concerned, the overall attribute values $\breve{Q}_2'$,$\breve{Q}_2''$ and $\breve{Q}_2'''$, which are aggregated by the q-ROFWHBM (Liu and Wang 2019) when $q = 3$,$s = t = 1$ and $\gamma = 3$,5 and 7, are $\langle 0.7900, 0\rangle$,$\langle 0.7945, 0\rangle$ and $\langle 0.7984, 0\rangle$, respectively. This implies that the N-MDs of $\hat{Q}_{23}$ and $\hat{Q}_{24}$ and the Hamacher operation parameter γ do not work at all.

Case 3 We adjust $\hat{Q}_{21}$ from $\langle 0.8, 0.3\rangle$ to $\langle 0.8, 0\rangle$,$\hat{Q}_{22}$ from $\langle 0.8, 0.2\rangle$ to $\langle 0.8, 0\rangle$, and $\hat{Q}_{23}$ from $\langle 0.7, 0.3\rangle$ to $\langle 0.7, 0\rangle$, and the evaluation value $\hat{Q}_{24}$ is exactly the same as that in Table 3. Regarding the alternative A_2, the overall attribute values $\overset{\frown}{Q}_2'$,$\overset{\frown}{Q}_2''$ and $\overset{\frown}{Q}_2'''$, which are aggregated by the Wq-ROFHMSM (Darko and Liang 2020) when $q = 3$,$k = 3$ and $\gamma = 3$,5 and 7, are $\langle 0.4880, 0\rangle$,$\langle 0.4601, 0\rangle$ and $\langle 0.4402, 0\rangle$, respectively. Therefore, the N-MD of $\hat{Q}_{24}$ and the Hamacher operation parameter γ are fruitless in this case.

We now recalculate *Case 1* using the proposed method. For the alternative A_2, the overall attribute values $\bar{Q}_2'$,$\bar{Q}_2''$ and $\bar{Q}_2'''$, which are aggregated by the q-ROFIHWCHM when $q = 3$, $R = (1, 0, 0, 0)$ and $\gamma = 3$,5 and 7, are $\langle 0.7947, 0.2096\rangle$,$\langle 0.7945, 0.2097\rangle$ and $\langle 0.7944, 0.2097\rangle$, respectively; the overall attribute values $\not{Q}_2'$,$\not{Q}_2''$ and $\not{Q}_2'''$, which are aggregated by the q-ROFIHWGCHM when $q = 3$, $R = (1, 0, 0, 0)$ and $\gamma = 3$,5 and 7, are $\langle 0.7971, 0.1776\rangle$,$\langle 0.7972, 0.1751\rangle$ and $\langle 0.7972, 0.1732\rangle$, respectively. It is clear that the N-MD of $\hat{Q}_{21}$ no longer dominates the overall attribute value of the alternative A_2.Thus, compared with the Wq-ROFHA (Darko and Liang 2020), it is more reasonable to use the q-ROFIHWCHM or q-ROFIHWGCHM to fuse information. The same is true when compared with the q-ROFWHBM (Liu and Wang 2019) and Wq-ROFHMSM (Darko and Liang 2020), which are not illustrated here.

In addition, we note that the q-ROFWHBM (Liu and Wang 2019) and Wq-ROFHMSM (Darko and Liang 2020) are not idempotent. This means that when the evaluation values

Table 10 Overall attribute values derived from different means

Means	Q_1	Q_2	Q_3	Q_4
Wq-ROFHA (Darko and Liang 2020)	$\langle 0.5472, 0.1651 \rangle$	$\langle 0.7969, 0.2746 \rangle$	$\langle 0.3522, 0.2414 \rangle$	$\langle 0.6957, 0.4290 \rangle$
q-ROFIHWCHM ($R = (1, 0, 0, 0)$)	$\langle 0.5950, 0.2373 \rangle$	$\langle 0.7947, 0.2291 \rangle$	$\langle 0.3723, 0.4848 \rangle$	$\langle 0.7037, 0.5492 \rangle$
q-ROFIHWGCHM ($R = (1, 0, 0, 0)$)	$\langle 0.6006, 0.2198 \rangle$	$\langle 0.7972, 0.2841 \rangle$	$\langle 0.3768, 0.4732 \rangle$	$\langle 07414, 0.4940 \rangle$
q-ROFWHBM (Liu and Wang 2019) ($s = t = 1$)	$\langle 0.5438, 0.6863 \rangle$	$\langle 0.7900, 0.7231 \rangle$	$\langle 0.3472, 0.7566 \rangle$	$\langle 0.6925, 0.7864 \rangle$
q-ROFIHWCHM ($R = (1, 1, 0, 0)$)	$\langle 0.6036, 0.2960 \rangle$	$\langle 0.7947, 0.3172 \rangle$	$\langle 0.3836, 0.5255 \rangle$	$\langle 0.7008, 0.5169 \rangle$
q-ROFIHWGCHM ($R = (1, 1, 0, 0)$)	$\langle 0.6107, 0.2652 \rangle$	$\langle 0.7965, 0.2935 \rangle$	$\langle 0.3937, 0.5135 \rangle$	$\langle 0.7201, 0.4753 \rangle$
Wq-ROFHMSM (Darko and Liang 2020) ($k = 3$)	$\langle 0.3222, 0.8091 \rangle$	$\langle 0.4880, 0.8393 \rangle$	$\langle 0.2110, 0.8526 \rangle$	$\langle 0.4133, 0.8834 \rangle$
q-ROFIHWCHM ($R = (1, 1, 1, 0)$)	$\langle 0.6154, 0.3498 \rangle$	$\langle 0.8052, 0.3376 \rangle$	$\langle 0.4029, 0.5469 \rangle$	$\langle 0.7259, 0.4816 \rangle$
q-ROFIHWGCHM ($R = (1, 1, 1, 0)$)	$\langle 0.6181, 0.3310 \rangle$	$\langle 0.8011, 0.3316 \rangle$	$\langle 0.4120, 0.5401 \rangle$	$\langle 0.7121, 0.4998 \rangle$

$Q_i(i = 1, 2, 3, 4)$ are the overall attribute values of the alternatives $A_i(i = 1, 2, 3, 4)$

Table 11 Score values and ranking results derived from different means

Means	Score values	Ranking results
Wq-ROFHA (Darko and Liang 2020)	$S'_1 = 0.5797$, $S'_2 = 0.7427$, $S'_3 = 0.5148$, $S'_4 = 0.6289$	$A_2 \succ A_4 \succ A_1 \succ A_3$
q-ROFIHWCHM ($R = (1, 0, 0, 0)$)	$S_1 = 0.3639$, $S_2 = 0.5689$, $S_3 = 0.2272$, $S_4 = 0.4186$	$A_2 \succ A_4 \succ A_1 \succ A_3$
q-ROFIHWGCHM ($R = (1, 0, 0, 0)$)	$S_1 = 0.3688$, $S_2 = 0.5734$, $S_3 = 0.2316$, $S_4 = 0.4746$	$A_2 \succ A_4 \succ A_1 \succ A_3$
q-ROFWHBM (Liu and Wang 2019) ($s = t = 1$)	$S''_1 = -0.1624$, $S''_2 = 0.1149$, $S''_3 = -0.3913$, $S''_4 = -0.1542$	$A_2 \succ A_4 \succ A_1 \succ A_3$
q-ROFIHWCHM ($R = (1, 1, 0, 0)$)	$S_1 = 0.3654$, $S_2 = 0.5678$, $S_3 = 0.2181$, $S_4 = 0.4214$	$A_2 \succ A_4 \succ A_1 \succ A_3$
q-ROFIHWGCHM ($R = (1, 1, 0, 0)$)	$S_1 = 0.3731$, $S_2 = 0.5720$, $S_3 = 0.2246$, $S_4 = 0.4509$	$A_2 \succ A_4 \succ A_1 \succ A_3$
Wq-ROFHMSM (Darko and Liang 2020) ($k = 3$)	$S'_1 = 0.2519$, $S'_2 = 0.2625$, $S'_3 = 0.1948$, $S'_4 = 0.1906$	$A_2 \succ A_1 \succ A_3 \succ A_4$
q-ROFIHWCHM ($R = (1, 1, 1, 0)$)	$S_1 = 0.3683$, $S_2 = 0.5826$, $S_3 = 0.2169$, $S_4 = 0.4570$	$A_2 \succ A_4 \succ A_1 \succ A_3$
q-ROFIHWGCHM ($R = (1, 1, 1, 0)$)	$S_1 = 0.3725$, $S_2 = 0.5767$, $S_3 = 0.2220$, $S_4 = 0.4374$	$A_2 \succ A_4 \succ A_1 \succ A_3$

S' and S'' are different score functions, which are from (Darko and Liang 2020) and (Liu and Wang 2019), respectively are all equal for a certain alternative, the q-ROFWHBM (Liu and Wang 2019) and Wq-ROFHMSM (Darko and Liang 2020) can yield discordant overall attribute values, respectively, which seems counter- intuitive.

(2) Compare with the MADM method using the q-ROF power weighed MSM (q-ROFPWMSM) (Liu et al. 2020).

Let $Q_i = \langle u_i, v_i \rangle$ $(i = 1, 2, \ldots, n)$ be n q-ROFNs with the weights $\omega_1, \omega_2, \ldots, \omega_n$ such that $\omega_i \geq 0$ and $\sum_{i=1}^{n} \omega_i = 1$, the aggregated value of the q-ROFPWMSM is reviewed as

$$q\text{ - ROFPWMSM}\left(Q_1, Q_2, \ldots, Q_n\right) = \left(\frac{\bigoplus_{1 \leq \tau_1 < \cdots < \tau_k \leq n} \left(\bigotimes_{j=1}^{k} n\varpi_{\tau_j} Q_{\tau_j} \right)}{C_n^k} \right)^{\frac{1}{k}}$$

$$= \left\langle \begin{array}{l} \left(\left(1 - \left(\prod_{1 \leq \tau_1 < \cdots < \tau_k \leq n} \left(1 - \prod_{j=1}^{k} \left(1 - \left(1 - u_{\tau_j}^{q} \right)^{n\varpi_{\tau_j}} \right) \right) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{q}} \right)^{\frac{1}{k}}, \\ \left(1 - \left(1 - \prod_{1 \leq \tau_1 < \cdots < \tau_k \leq n} \left(1 - \prod_{j=1}^{k} \left(1 - \left(v_{\tau_j}^{n\varpi_{\tau_j}} \right)^{q} \right) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right)^{\frac{1}{q}} \end{array} \right\rangle, \tag{74}$$

where $\varpi_i = \frac{\omega_i(1+T(Q_i))}{\sum_{r=1}^{n} \omega_r(1+T(Q_r))}$,$T(Q_i) = \sum_{j=1, j \neq i}^{n} \mathrm{Sup}(Q_i, Q_j)$,$\mathrm{Sup}(Q_i, Q_j) = 1 - d(Q_i, Q_j)$, and $d(Q_i, Q_j)$ is the normalized Hamming distance between Q_i and Q_j as given in Eq. (2).

If we use the q-ROFPWMSM (Liu et al. 2020) to tackle Example 6.1 (Let $q = 3$ and $k = 3$), then the overall attribute values of the alternatives are as follows:$Q_1 = \langle 0.5167, 0.5869 \rangle$, $Q_2 = \langle 0.7692, 0.5231 \rangle$, $Q_3 = \langle 0.3372, 0.6053 \rangle$ and $Q_4 = \langle 0.6666, 0.5278 \rangle$; the score values of the alternatives are as follows:$S_1'' = -0.0642$,

$S_2'' = 0.3119$, $S_3'' = -0.1834$ and $S_4'' = 0.1492$, where S'' is the score function in (Liu et al. 2020) and (Liu and Wang 2019). Hence the ranking result educed by the q-ROFPWMSM (Liu et al. 2020) when $q = 3$ and $k = 3$ is $A_2 \succ A_4 \succ A_1 \succ A_3$. This ranking result is completely consistent with those educed by the q-ROFIHWCHM and q-ROFIHWGCHM when $q = 3$,$\gamma = 3$ and $R = (1, 1, 1, 0)$, which have been shown in Table 11. In fact, the MADM method using the q-ROFPWMSM (Liu et al. 2020) when $q = 3$ and $k = 3$ and our proposed algorithm using the q-ROFIHWCHM or q-ROFIHWGCHM when $q = 3$,$\gamma = 3$ and $R = (1, 1, 1, 0)$ have in common that they not only weaken the impacts of the extreme evaluation values, but also reflect the correlations among any three of the four attributes (Note: For the MADM method using the q-ROFPWMSM (Liu et al. 2020), each evaluation value is assigned with a degree of importance (weighted nonlinear weight) by the q-ROFPWA (Liu et al. 2020); however, with regard to our developed method, before aggregating all the individual attribute values of the alternatives into the overall attribute values with the q-ROFIHWCHM or q-ROFIHWGCHM, each data has been endowed with a degree of importance (weighted nonlinear weight) by the q-ROFIHWAPA). So the effectiveness of our proposed algorithm is authenticated by the consistent ranking again.

Next, we illustrate the degrees of importance distributed by the q-ROFIHWAPA to evaluation values are more reasonable than those assigned by the q-ROFPWA (Liu et al. 2020). We adjust $\hat{Q}_{21}$ from $\langle 0.8, 0.3\rangle$ to $\langle 0.8, 0.7\rangle$,$\hat{Q}_{22}$ from $\langle 0.8, 0.2\rangle$ to $\langle 0.8, 0.6\rangle$ and $\hat{Q}_{23}$ from $\langle 0.7, 0.3\rangle$ to $\langle 0.01, 0.01\rangle$, and the evaluation value $\hat{Q}_{24}$ is exactly the same as that in Table 3, which is $\langle 0.9, 0.6\rangle$. As a matter of fact, the new evaluation values on the alternative A_2 are very balanced except for $\hat{Q}_{23}$. When $q = 3$, the degrees of importance distributed by the q-ROFPWA (Liu et al. 2020) to evaluation values $\hat{Q}_{21}$,$\hat{Q}_{22}$,$\hat{Q}_{23}$ and $\hat{Q}_{24}$ are 0.3272, 0.4561, 0.1146 and 0.1021, respectively; the degrees of importance distributed by the q-ROFIHWAPA to these four evaluation values are 0.3002, 0.3182, 0.0896 and 0.2921, respectively. Apparently, compared with the q-ROFPWA (Liu et al. 2020), the q-ROFIHWAPA is more capable of exploring the importance of original information, i.e., it assigns a less degree of credibility to the extreme data $\hat{Q}_{23}$ and assigns the relatively equilibrious degrees of credibility to the balanced evaluation values $\hat{Q}_{21}$,$\hat{Q}_{22}$ and $\hat{Q}_{24}$, which is consistent with our intuition.

On the other hand, we point out that the q-ROFPWMSM (Liu et al. 2020) does not satisfy the idempotency, i.e., using it to aggregate the identical evaluation information will lead to counter-intuition.

Now we make a summary about the characteristics of the above-mentioned methods from the following perspectives:

(P1) Whether it can reflect the attributive correlations;
(P2) Whether it can reflect the correlations between pairwise attributes;
(P3) Whether it can reflect the correlations among multiple attributes;
(P4) Whether it weakens the impacts of the extreme evaluation values;
(P5) Whether it weakens the impacts of the extreme evaluation values more reasonably;
(P6) Whether it considers the interactions between MD and N-MD;
(P7) Whether it has the characteristic of generality (It can generate different methods by different operation parameters);
(P8) Whether it satisfies the idempotency.

For convenience, we present their characteristics in Table 12.

From Table 12, we easily find that the proposed method has too many advantages compared with the MADM methods constructed on the existing means. Thus, it is more suitable to tackle the q-ROF MADM.

Table 12 The characteristics of different MADM methods

Methods	(P1)	(P2)	(P3)	(P4)	(P5)	(P6)	(P7)	(P8)
Wq-ROFHA (Darko and Liang 2020)	✗	✗	✗	✗	✗	✗	✓	✓
q-ROFWHBM (Liu and Wang 2019)	✓	✓	✗	✗	✗	✗	✓	✗
Wq-ROFHMSM (Darko and Liang 2020)	✓	✓	✓	✗	✗	✗	✓	✗
q-ROFPWMSM (Liu et al. 2020)	✓	✓	✓	✓	✗	✗	✗	✗
The proposed method	✓	✓	✓	✓	✓	✓	✓	✓

7 Conclusions

As an extension of IFS and PFS, q-ROFS has a strong ability to characterize the vagueness and uncertainty. Considering this, we take it as the background to implement MADM analysis. The specific work focuses on the following aspects:

(1) We introduce the q-ROF interactive Hamacher operations, improved score function and new q-ROFE formula, which serve as the theoretical basis of the full text.
(2) We propose the APA and its weight form (WAPA) to remedy the deficiencies of the PA and its weight form (WPA). Then the q-ROFIHWAPA is obtained with the help of the q-ROF interactive Hamacher operations, and the basic properties are analyzed. Further, we present a MADM algorithm and its application example based on the q-ROFIHWAPA. Finally, according to the results of the application example, we develop the entropy weight fitting method to determine the parameter carried by the q-ROFIHWAPA. By this means the weighted nonlinear weights derived from the q-ROFIHWAPA are more objective.
(3) Inspired by the development of BMs, we define the WCHM and WGCHM on the basis of the HM and GHM, respectively, which can eliminate the redundancy of the DGWBM and DGWBGM, i.e., the case of $\tau_1 > \tau_2 > \cdots > \tau_n$. Subsequently, we develop the q-ROFIHWCHM and q-ROFIHWGCHM by combining them with the q-ROF interactive Hamacher operations, and the common properties and special cases are also investigated.
(4) We establish a MADM model relied on the q-ROFIHWAPA and q-ROFIHWCHM (resp. q-ROFIHWGCHM). More precisely, before aggregating all the individual attribute values of the alternatives into the overall attribute values with the q-ROFIHWCHM or q-ROFIHWGCHM, the weight of each data has been replaced with the weighted nonlinear weight carried by the q-ROFIHWAPA. Then a practical example is presented to illustrate that the introduced algorithm (i) can reflect the correlations among multiple attributes; (ii) weakens the impacts of the extreme evaluation values more reasonably; (iii) considers the interactions between the MD and N-MD of different q-ROFNs; (iv) has the characteristic of generality (It can generate different methods by different operations).

In the following, we point out several points for future research:

(1) Propose more advanced operations for q-ROFNs.
As a matter of fact, the q-ROFIHWAPA, q-ROFIHWCHM and q-ROFIHWGCHM introduced in this paper have the following disadvantages: (i) when there is at least one $\langle 1, 0\rangle$ in a set of q-ROFNs, the fusion results derived from the q-ROFIHWAPA and q-ROFIHWCHM are both $\langle 1, 0\rangle$ regardless of the other values; (ii) when there is at least one $\langle 0, 1\rangle$ in a set of q-ROFNs, the fusion result derived from the q-ROFIHWGCHM is always $\langle 0, 1\rangle$ regardless of the other values. Therefore, it is necessary to explore more advanced q-ROF operation rules to eliminate these deficiencies.
(2) Develop the generalized WCHMs.
Dutta and Guha (Dutta and Guha 2015) proposed the partitioned BM (PBM) on the basis of such an assumption that all attributes are separated into some partitions, the attributes in the same partition are interrelated to each other, and the attributes in different partitions are independent. Similarly, we can introduce the weighted parti-

tioned coordinated HM (WPCHM) and weighted partitioned geometric coordinated HM (WPGCHM). However, it has to be mentioned that their expressions will be quite complicated. Further, we can also study the prioritization between partitions.

(3) Use a certain kind of fuzzy information to express the weights instead of the real number.

For MADM or multi-attribute group decision making (MAGDM), it has become a convention that the weights of attributes or DMs are quantified in real numbers. To better characterize ambiguities and uncertainties, in fact, the q-ROFNs, linguistic values and other forms can be used to express the views on the attributes or DMs. Also, the corresponding decision algorithm will be presented.

Acknowledgements This work was supported by the National Natural Science Foundation of China (Grant No. 11771111).

Author contributions JL: Conceptualization, Methodology, Investigation, Writing-original draft.MC: Conceptualization, Methodology, Supervision, Writing-review & editing.SP: Figures 1, Data analysis.All authors reviewed the manuscript.

Declarations

Conflict of interest The authors declare no competing interests.

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