Comparing the notions of optimality in CP-nets, strategic games and soft constraints

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Abstract

The notion of optimality naturally arises in many areas of applied mathematics and computer science concerned with decision making. Here we consider this notion in the context of three formalisms used for different purposes in reasoning about multi-agent systems: strategic games, CP-nets, and soft constraints. To relate the notions of optimality in these formalisms we introduce a natural qualitative modification of the notion of a strategic game. We show then that the optimal outcomes of a CP-net are exactly the Nash equilibria of such games. This allows us to use the techniques of game theory to search for optimal outcomes of CP-nets and vice-versa, to use techniques developed for CP-nets to search for Nash equilibria of the considered games. Then, we relate the notion of optimality used in the area of soft constraints to that used in a generalization of strategic games, called graphical games. In particular we prove that for a natural class of soft constraints that includes weighted constraints every optimal solution is both a Nash equilibrium and Pareto efficient joint strategy. For a natural mapping in the other direction we show that Pareto efficient joint strategies coincide with the optimal solutions of soft constraints.

1 Introduction

The concept of optimality is prevalent in many areas of applied mathematics and computer science. It is of relevance whenever we need to choose among several alternatives that are not equally preferable. For example, in constraint optimization, each solution of a constraint satisfaction problem has a quality level associated with it and the aim is to choose an optimal solution, that is, a solution with an optimal quality level. In turn, in strategic games, two concepts of optimality have been commonly used: Nash equilibrium and Pareto efficient outcome.

Some formalisms proposed in AI employ 'their own' concept of an optimal outcome. The aim of this paper is to clarify the status of such notions of optimality used in CP-nets and soft constraints. To this end we use tools and techniques from game theory, more specifically theory of strategic games.

This allows us to gain new insights into the relationship between these formalisms which hopefully will lead to further cross-fertilization among these three different approaches to modelling optimality.

1.1 Background

Game theory, notably the theory of *strategic games*, forms one of the main tools in the area of multi-agent systems since they formalize in a simple and powerful way the idea that the agents interact with each other while pursuing their own interests. Each agent has a set of strategies and a payoff function on the set of joint strategies. The agents choose their strategies simultaneously with the aim of maximizing one's payoff.

The most commonly used concept of optimality is that of a Nash equilibrium. Intuitively, it is an outcome that is optimal for each player under the assumption that only he may reconsider his action. Another concept of optimality is that of Pareto efficient joint strategies, which are those in which no player can improve his payoff without decreasing the payoff of some other player. Sometimes it is useful to consider constrained Nash equilibria, that is, Nash equilibria that satisfy some additional requirements, see e.g. [12]. For example, Pareto efficient Nash equilibria are Nash equilibria which are also Pareto efficient among the Nash equilibria.

In turn, *CP-nets* (Conditional Preference nets) are an elegant formalism for representing conditional and qualitative preferences, see [6, 5]. They model such preferences under a *ceteris paribus* (that is, 'all else being equal') assumption. A CP-net exploits the idea of conditional independence to provide a compact representation of preference problems. Preference elicitation in such a framework appears to be natural and intuitive.

Research on CP-nets has been focused on their modeling capabilities and on algorithms for solving various natural problems related to their use. Also, computational complexity of these problems was extensively studied. One of the fundamental problems is that of finding an optimal outcome, i.e., one that cannot be improved in the presence of the adopted preference statements. This is in general a complex problem since it was found that finding optimal outcomes and testing for their existence is in general NP-hard, see [6, 5]. In contrast, for so-called acyclic CP-nets this is an easy problem which can be solved by a linear time algorithm, see [6, 5].

Finally, *soft constraints*, see e.g. [4], are a quantitative formalism which allow us to express constraints and preferences. While constraints state which combinations of variable values are acceptable, soft constraints allow for several levels of acceptance. An example are fuzzy constraints, see [8] and [21], where acceptance levels are between 0 and 1, and where the quality of a solution is the minimal level over all the constraints. An optimal solution is the one with the highest quality. The research in this area has dealt mainly with the algorithms for finding optimal solutions and with the relationship between modelling formalisms, see [19].

1.2 Main results

We consider the notions of optimality in two preference modelling frameworks, that is, CP-nets and soft constraints, and in strategic games. Although apparently there is no connection among these different ways of modelling preferences, we show that in fact there is a strong relationship. This is surprising and interesting on its own. Moreover, it might be exploited for a cross-fertilization among these three frameworks.

In particular, we start by considering the relationship between CP-nets and strategic games, and we show how game-theoretic techniques can be fruitfully used to study CP-nets. Our approach is based on the observation that the ceteris-paribus principle, typical of CP-nets, implies that an optimal outcome is worsened if a worsening change (to some variable) is made. This is exactly the idea behind Nash equilibria and the desired results easily follow once this observation is made formal by introducing an appropriate modification of strategic games. In this modification each player has at his disposal a preference relation on his set of strategies, parametrized by a joint strategy of his opponents. We call such games *strategic games with parametrized preferences*.

The cornerstone of our approach are two results closely relating CP-nets to such games. They show that the optimal outcomes of a CP-net are exactly the Nash equilibria of an appropriately defined strategic game with parametrized preferences. This allows us to transfer techniques of game theory to CP-nets, and vice-versa.

In strategic games techniques have been studied which iteratively reduce the game by eliminating some players' strategies, thus obtaining a smaller game while maintaining its Nash equilibria. In [11], for example, interesting results concerning the order in which such reductions are applied are described. We introduce two counterparts of such game-theoretic techniques that allow us to reduce a CP-net while maintaining its optimal outcomes. We also introduce a method of simplifying a CP-net by eliminating so-called redundant variables from the variables parent sets. Both techniques simplify the search for optimal outcomes in a CP-net.

In the other direction, we can use the techniques developed to reason about optimal outcomes of a CP-net to search for Nash equilibria of strategic games with parametrized preferences. We illustrate this point by introducing the notion of a hierarchical game with parametrized preferences and by explaining that such games have a unique Nash equilibrium that can be found in linear time.

In the final part of the paper we consider the relationship between strategic games and soft constraints, such as fuzzy, weighted and hard constraints. The appropriate notion of a strategic game is here that of a *graphical game*, see [13]. This is due to the fact that (soft) constraints usually involve only a small subset of the problem variables. This is in analogy with the fact that in a graphical game a player's payoff function depends only on a (usually small) number of other players.

We consider a natural mapping that associates with each soft constraint satisfaction problem (in short, a soft CSP or an SCSP) a graphical game. This mapping creates a direct corresponce between constraints and players' neighbourhoods. We show that, when using such a mapping, in general no relation exists between the notions of optimal solutions in soft CSPs and Nash equilibria in the corresponding games. On the other hand, for the class of strictly monotonic SCSPs (which includes in particular weighted constraints), every optimal solution corresponds to both a Nash equilibrium and Pareto efficient joint strategy. We also show that this mapping, when applied to a consistent CSP (that is, a satisfiable hard constraint satisfaction problem), defines a bijection between the solutions of the CSP and the set of joint strategies that are both Nash equilibria and Pareto efficient.

The latter holds in general, and not just for a subclass, if we consider a mapping from graphical games to soft CSPs which is independent of the constraint structure. This mapping, however, is less appealing from the computational complexity point of view since it requires that one considers all possible complete assignments, the number of which may be exponential in the size of the SCSP.

None of these two mappings are surjective, thus they cannot be used to pass from a generic graphical game to an SCSP. We also consider a mapping which goes in this direction. This mapping creates a soft constraint for each player, by looking at his neighbourhood. We show that this mapping defines a bijection between Pareto efficient joint strategies and optimal solutions of the SCSP.

The study of the relations among preference models coming from different fields such as AI and game theory has only recently gained attention. In [10] a mapping from the graphical games to hard CSPs has been defined, and it has been shown that the Nash equilibria of these games coincide with the solutions of the CSPs. We can use this mapping, together with our mapping from the graphical games to SCSPs, to identify the Pareto efficient Nash equilibria of the given game. In fact, these equilibria correspond to the optimal solutions of the SCSP obtained by joining the soft and hard constraints generated by the two mappings. The mapping of [10] leads to interesting results on the complexity of deciding whether a game has a pure Nash equilibrium or other kinds of desirable joint strategies.

In [14] a mapping from distributed constraint optimization problems (DCOPs) to graphical games is introduced, where the optimization criterion is to maximize the sum of utilities. By using this mapping, it is shown that the optimal solutions of the given DCOP are Nash equilibria of the generated game. This result is in line with our finding regarding strictly monotonic SCSPs, which include the class of problems considered in [14].

1.3 Organization of the paper

The paper is organized as follows. In Section 2 we introduce CP-nets, soft constraints, and strategic games. Next, in Section 3 we introduce a modification of the classical notion of strategic games considered in this paper. In Section 4 we show how to pass from CP-nets to so defined strategic games, while in Section 5 we deal with the opposite direction.

Then in Section 6 we show how to apply techniques developed in game theory to reason about CP-nets, while in Section 7 we study the other direction. Next, in Section 8 and 9 we study the relationship between soft CSPs and strategic games by relating optimal solutions of soft CSPs to Nash equilibria and Pareto efficient joint strategies. Finally, in Section 10 we summarize the main contributions of the paper.

Preliminary results of this research were reported in [2] and [3].

2 Preliminaries

In this section we recall the main notions regarding CP-nets, soft constraints, and strategic games.

2.1 CP-nets

CP-nets [6, 5] (for Conditional Preference nets) are a graphical model for compactly representing conditional and qualitative preference relations. They exploit conditional preferential independence by decomposing an agent's preferences via the ***ceteris paribus*** assumption. Informally, CP-nets are sets of ***ceteris paribus (cp)*** preference statements. For instance, the statement *"I prefer red wine to white wine if meat is served."* asserts that, given two meals that differ *only* in the kind of wine served *and* both containing meat, the meal with a red wine is preferable to the meal with a white wine. On the other hand, this statement does not order two meals with a different main course. Many users' preferences appear to be of this type.

CP-nets bear some similarity to Bayesian networks. Both utilize directed graphs where each node stands for a domain variable, and assume a set of *features* (variables) $F = \{X_1, \ldots, X_n\}$ with the corresponding finite domains $\mathcal{D}(X_1), \ldots, \mathcal{D}(X_n)$. For each feature X_i, a user specifies a (possibly empty) set of ***parent*** features $Pa(X_i)$ that can affect her preferences over the values of X_i. This defines a directed graph, called ***dependency graph***, in which each node X_i has $Pa(X_i)$ as its immediate predecessors. A CP-net is said to be ***acyclic*** if its dependency graph does not contain cycles.

Given this structural information, the user explicitly specifies her preference over the values of X_i for *each complete assignment* on $Pa(X_i)$. In this paper this preference is assumed to take the form of a linear order over $\mathcal{D}(X_i)$ [6, 5].[1] Each such specification is called below a ***preference statement*** for the variable X_i. These conditional preferences over the values of X_i are captured by a *conditional preference table* which is annotated with the node X_i in the CP-net. An ***outcome*** is an assignment of values to the variables with each value taken from the corresponding domain.

[1] In this we follow [5], where ties among values are initially allowed, (that is linear preorders are admitted) but in presentation only total orders are used. If ties are admitted, the notion of an optimal outcome of a CP-net has to be appropriately modified.

As an example, consider a CP-net whose features are A, B, C and D, with binary domains containing f and $\overline{f}$ if F is the name of the feature, and with the following preference statements:

$d : a \succ \overline{a}$, $\overline{d} : a \succ \overline{a}$,
$a : b \succ \overline{b}$, $\overline{a} : \overline{b} \succ b$,
$b : c \succ \overline{c}$, $\overline{b} : \overline{c} \succ c$,
$c : d \succ \overline{d}$, $\overline{c} : \overline{d} \succ d$.

Here the preference statement $d : a \succ \overline{a}$ states that $A = a$ is preferred to $A = \overline{a}$, given that $D = d$. From the structure of these preference statements we see that $Pa(A) = \{D\}, Pa(B) = \{A\}, Pa(C) = \{B\}, Pa(D) = \{C\}$ so the dependency graph is cyclic.

An ***acyclic*** CP-net is one in which the dependency graph is acyclic. As an example, consider a CP-net whose features and domains are as above and with the following preference statements:

$a \succ \overline{a}$,
$b \succ \overline{b}$,
$(a \wedge b) \vee (\overline{a} \wedge \overline{b}) : c \succ \overline{c}$, $(a \wedge \overline{b}) \vee (\overline{a} \wedge b) : \overline{c} \succ c$,
$c : d \succ \overline{d}$, $\overline{c} : \overline{d} \succ d$.

Here, the preference statement $a \succ \overline{a}$ represents the unconditional preference for $A = a$ over $A = \overline{a}$. Also each preference statement for the variable C is a actually an abbreviated version of two preference statements. In this example we have $Pa(A) = \emptyset, Pa(B) = \emptyset, Pa(C) = \{A, B\}, Pa(D) = \{C\}$.

A ***worsening flip*** is a transition between two outcomes that consists of a change in the value of a single variable to one which is less preferred in the unique preference statement for that variable. By analogy we define an ***improving flip***. For example, in the acyclic CP-net described in the previous paragraph, passing from $abcd$ to $ab\overline{c}d$ is a worsening flip since c is better than $\overline{c}$ given a and b. We say that an outcome α is ***better*** than the outcome β (or, equivalently, β is ***worse*** than α), written as $\alpha \succ \beta$, iff there is a chain of worsening flips from α to β. This definition induces a strict preorder over the outcomes. In the acyclic CP-net described in the previous paragraph, the outcome $\overline{a}b\overline{c}\overline{d}$ is worse than $abcd$.

An ***optimal*** outcome is one for which no better outcome exists. So an outcome is optimal iff no improving flip from it exists. In general, a CP-net does not need to have an optimal outcome. As an example consider two features A and B with the respective domains $\{a, \overline{a}\}$ and $\{b, \overline{b}\}$ and the following preference statements:

$a : b \succ \overline{b}$, $\overline{a} : \overline{b} \succ b$,
$b : \overline{a} \succ a$, $\overline{b} : a \succ \overline{a}$.

It is easy to see that then

$ab \succ a\overline{b} \succ \overline{a}\overline{b} \succ \overline{a}b \succ ab$.

Finding optimal outcomes and testing for optimality is known to be NP-hard [6, 5]. However, in acyclic CP-nets there is a unique optimal outcome and it can be found in linear time [6, 5]. We simply sweep through the CP-net, following the arrows in the dependency graph, assigning at each step the most preferred value in the preference relation. For instance, in the CP-net above, we would choose $A = a$ and $B = b$, then $C = c$ and then $D = d$. The optimal outcome is therefore $abcd$.

Determining whether one outcome is better than another according to this order (a so-called *dominance query*) is also NP-hard even for acyclic CP-nets, see [9]. Whilst tractable special cases exist, there are also acyclic CP-nets in which there are exponentially long chains of worsening flips between two outcomes [9].

Hard constraints are enough to find optimal outcomes of a CP-net and to test whether a CP-net has an optimal outcome. In fact, given a CP-net one can define a set of hard constraints (called ***optimality constraints***) such that their solutions are the optimal outcomes of the CP-net, see [7, 20].

Indeed, take a CP-net N and consider a linear order $\succ$ over the elements of the domain of a variable X used in a preference statement for X. Let φ be the disjunction of the corresponding assignments used in the preference statements that use $\succ$. Then for each of such linear order $\succ$ the corresponding optimality constraint is $\varphi \rightarrow X = a_j$, where a_j is the undominated element of $\succ$. The optimality constraints $opt(N)$ corresponding to N consist of the entire set of such optimality constraints, each for one such linear order $\succ$.

For example, the preference statements $a \succ \overline{a}$ and $(a \wedge \overline{b}) \vee (\overline{a} \wedge b) : \overline{c} \succ c$ from the above CP-net map to the hard constraints $A = a$ and $(A = a \wedge B = \overline{b}) \vee (A = \overline{a} \wedge B = b) \rightarrow C = \overline{c}$, respectively.

It has been shown that an outcome is optimal in the strict preorder over the outcomes induced by a CP-net N iff it is a satisfying assignment for $opt(N)$.

A CP-net is called ***eligible*** iff it has an optimal outcome. Even if the strict preorder induced by a CP-net has cycles, the CP-net may still be useful if it is eligible. All acyclic CP-nets are trivially eligible as they have a unique optimal outcome. We can thus test eligibility of any (even cyclic) CP-net by testing the consistency of the optimality constraints $opt(N)$. That is, a CP-net N is eligible iff $opt(N)$ is consistent.

2.2 Soft constraints

Soft constraints, see e.g. [4], allow us to express constraints and preferences. While constraints state which combinations of variable values are acceptable, soft constraints (also called *preferences*) allow for several levels of acceptance. A technical way to describe soft constraints is via the use of an algebraic structure called a c-semiring.

A ***c-semiring*** is a tuple $\langle A, +, \times, \mathbf{0}, \mathbf{1} \rangle$, where:

- A is a set, called the ***carrier*** of the semiring, and $\mathbf{0}, \mathbf{1} \in A$;
- $+$ is commutative, associative, idempotent, $\mathbf{0}$ is its unit element, and $\mathbf{1}$ is its absorbing element;
- $\times$ is associative, commutative, distributes over $+$, $\mathbf{1}$ is its unit element and $\mathbf{0}$ is its absorbing element.

Elements $\mathbf{0}$ and $\mathbf{1}$ represent, respectively, the highest and lowest preference. While the operator $\times$ is used to combine preferences, the operator $+$ induces a partial order on the carrier A defined by

$$a \leq b \text{ iff } a + b = b.$$

Given a c-semiring $S = \langle A, +, \times, \mathbf{0}, \mathbf{1} \rangle$, and a set of variables V, each variable x with a domain $D(x)$, a ***soft constraint*** is a pair $\langle def, con \rangle$, where $con \subseteq V$ and $def : \times_{y \in con} D(y) \to A$. So a constraint specifies a set of variables (the ones in con), and assigns to each tuple of values from $\times_{y \in con} D(y)$, the Cartesian product of the variable domains, an element of the semiring carrier A.

A ***soft constraint satisfaction problem*** (in short, a ***soft CSP*** or ***SCSP***) is a tuple $\langle C, V, D, S \rangle$ where V is a set of variables, with the corresponding set of domains D, C is a set of soft constraints over V and S is a c-semiring. Given an SCSP a ***solution*** is an instantiation of all the variables. The ***preference*** of a solution s is the combination by means of the $\times$ operator of all the preference levels given by the constraints to the corresponding subtuples of the solution, or more formally,

$$\times_{c \in C} def_c(s \downarrow_{con_c}),$$

where $\times$ is the multiplicative operator of the semiring and $def_c(s \downarrow_{con_c})$ is the preference associated by the constraint c to the projection of the solution s on the variables in con_c.

A solution is called ***optimal*** if there is no other solution with a strictly higher preference.

Three widely used instances of SCSPs are:

- **Classical CSPs** (in short **CSPs**), based on the c-semiring $\langle\{0,1\},\vee,\wedge,0,1\rangle$. They model the customary CSPs in which tuples are either allowed or not. So CSPs can be seen as a special case of SCSPs.

- **Fuzzy CSPs**, based on the ***fuzzy c-semiring*** $\langle[0,1],max,min,0,1\rangle$. In such problems, preferences are the values in $[0,1]$, combined by taking the minimum and the goal is to maximize the minimum preference.

- **Weighted CSPs**, based on the ***weighted c-semiring*** $\langle\Re_+,min,+,\infty,0\rangle$. Preferences are costs ranging over non-negative reals, which are aggregated using the sum. The goal is to minimize the total cost.

A simple example of a fuzzy CSP is the following one:

- three variables: x, y, and z, each with the domain $\{a,b\}$;

- two constraints: C_{xy} (over x and y) and C_{yz} (over y and z) defined by:
  $C_{xy} := \{(aa,0.4),(ab,0.1),(ba,0.3),(bb,0.5)\}$,
  $C_{yz} := \{(aa,0.4),(ab,0.3),(ba,0.1),(bb,0.5)\}$.

The unique optimal solution of this problem is bbb (an abbreviation for $x=y=z=b$). Its preference is 0.5.

2.3 Strategic games

Let us recall now the notion of a strategic game, see, e.g., [17]. A strategic game for a set $N=\{1,\dots,n\}$ of n players ($n>1$) is a tuple

$$(S_1,\dots,S_n,p_1,\dots,p_n),$$

where for each $i\in[1..n]$

- S_i is the non-empty set of ***strategies*** available to player i,

- p_i is the ***payoff function*** for the player i, so $p_i : S_1\times\dots\times S_n\rightarrow\mathcal{R}$, where $\mathcal{R}$ is the set of real numbers.

Given a sequence of non-empty sets $S_1, \ldots, S_n$ and $s \in S_1 \times \ldots \times S_n$ we denote the ith element of s by s_i, abbreviate $N \setminus \{i\}$ to $-i$, and use the following standard notation of game theory, where $i \in [1..n]$ and $I := i_1, \ldots, i_k$ is a subsequence of $1, \ldots, n$:

- $s_I := (s_{i_1}, \ldots, s_{i_k})$,
- $(s'_i, s_{-i}) := (s_1, \ldots, s_{i-1}, s'_i, s_{i+1}, \ldots, s_n)$, where we assume that $s'_i \in S_i$,
- $S_I := S_{i_1} \times \ldots \times S_{i_k}$.

A joint strategy s is called

- a ***pure Nash equilibrium*** (from now on, simply ***Nash equilibrium***) if
$$p_i(s) \geq p_i(s'_i, s_{-i}) \tag{1}$$
for all $i \in [1..n]$ and all $s'_i \in S_i$,
- ***Pareto efficient*** if for no joint strategy s', $p_i(s') \geq p_i(s)$ for all $i \in [1..n]$ and $p_i(s') > p_i(s)$ for some $i \in [1..n]$.

Pareto efficiency can be alternatively defined by considering the following strict ***Pareto order*** $<_P$ on the n-tuples of reals:

$$(a_1, \ldots, a_n) <_P (b_1, \ldots, b_n) \text{ iff } \forall i \in [1..n]\ a_i \leq b_i \text{ and } \exists i \in [1..n]\ a_i < b_i.$$

Then a joint strategy s is Pareto efficient iff the n-tuple $(p_1(s), \ldots, p_n(s))$ is a maximal element in the $<_P$ order on such n-tuples of reals.

To clarify these notions consider the classical Prisoner's Dilemma game represented by the following bimatrix representing the payoffs to both players:

	C_2	N_2
C_1	$3, 3$	$0, 4$
N_1	$4, 0$	$1, 1$

Each player i represents a prisoner, who has two strategies, C_i (cooperate) and N_i (not cooperate). Table entries represent payoffs for the players (where the first component is the payoff of player 1 and the second one that of player 2).

The two prisoners gain when both cooperate (with a profit of 3 each). However, if only one of them cooperates, the other one will gain more (with a profit of 4). If none of them cooperates, both gain very little (a profit of 1 each), but more than the "cheated" prisoner whose cooperation is not returned (that is, 0).

Here the unique Nash equilibrium is (N_1, N_2), while the other three joint strategies (C_1, C_2), (C_1, N_2) and (N_1, C_2) are Pareto efficient.

A natural modification of the concept of strategic games, called graphical games, was proposed in [13]. These games stress the locality in taking decision. In a graphical game the payoff of each player depends only on the strategies of its neighbours in a given in advance graph structure over the set of players.

Formally, a ***graphical game*** for n players with the corresponding strategy sets $S_1, \ldots, S_n$ is defined by assuming a neighbour function *neigh* that given a player i yields its set of neighbours $neigh(i)$. The payoff for player i is then a function p_i from $\times_{j \in neigh(i) \cup \{i\}} S_j$ to $\mathcal{R}$.

By using the canonic extensions of these payoff functions to the Cartesian product of all strategy sets one can then extend the previously introduced concepts, notably that of a Nash equilibrium, to the graphical games. Further, when all pairs of players are neighbours, a graphical game reduces to a strategic game.

3 Strategic games with parametrized preferences

In game theory it is customary to study strategic games defined as above, in quantitative terms. A notable exception is [18] in which instead of payoff functions the linear quasi-orders on the sets of joint strategies are used.

For our purposes we need a different approach. To define it we first introduce the concept of a ***preference*** on a set A which in this paper denotes a strict linear order on A. We then assume that each player has to his disposal a preference relation $\succ(s_{-i})$ on his set of strategies *parametrized* by a joint strategy s_{-i} of his opponents. So in our approach

- for each $i \in [1..n]$ player i has a finite, non-empty, set S_i of strategies available to him,
- for each $i \in [1..n]$ and $s_{-i} \in S_{-i}$ player i has a preference relation $\succ(s_{-i})$ on his set of strategies S_i.

In what follows such a ***strategic game with parametrized preferences*** (in short a ***game with parametrized preferences***, or just a ***game***) for n players is represented by a tuple

$$(S_1, \ldots, S_n, \succ(s_{-1}), \ldots, \succ(s_{-n})),$$

where each s_{-i} ranges over S_{-i}.

It is straightforward to transfer to the case of games with parametrized preferences the basic notions concerning strategic games. In particular the following notions will be of importance for us (for the original definitions see, e.g., [18]), where G is a game with parametrized preferences specified as above:

- A strategy s_i is a ***best response*** for player i to a joint strategy s_{-i} of his opponents if $s_i \succ(s_{-i})\ s'_i$, for all $s'_i \neq s_i$.
- A strategy s_i is ***never a best response*** for player i if it is not a best response to any joint strategy s_{-i} of his opponents.
- A strategy s'_i is ***strictly dominated*** by a strategy s_i if $s_i \succ(s_{-i})\ s'_i$, for all $s_{-i} \in S_{-i}$.

So according to this terminology a joint strategy s is a ***Nash equilibrium*** of G iff each s_i is a best response to s_{-i}. Note, however, that in our setup the underlying preferences are strict, so the above notions of a best response and Nash equilibrium correspond in the customary setting of strategic games to the notions of a unique best response and a strict Nash equilibrium. In particular, note that s is a Nash equilibrium of G iff for all $i \in [1..n]$ and all $s'_i \neq s_i$

$$s_i \succ(s_{-i})\ s'_i,$$

because to each joint strategy s_{-i} a unique best response exists.

To clarify these definitions let us return to the above example of the strategic game that models the Prisoner's Dilemma. To view this game as a game with parametrized preferences we abstract from the numerical values and simply stipulate that

$$\begin{array}{ll} \succ(C_2) :=\ N_1 \succ C_1, & \succ(N_2) :=\ N_1 \succ C_1, \\ \succ(C_1) :=\ N_2 \succ C_2, & \succ(N_1) :=\ N_2 \succ C_2. \end{array}$$

These orders reflect the fact that for each strategy of the opponent each player considers his 'not cooperate' strategy better than his 'cooperate' strategy.

It is easy to check that:

- for each player i the strategy C_i is strictly dominated by N_i (since $N_i \succ(C_{3-i})C_i$ and $N_i \succ(N_{3-i})C_i$),
- for each player i the strategy N_i is a best response to the strategy N_{3-i} of his opponent,

- (as a result) (N_1, N_2) is a unique Nash equilibrium of this game with parametrized preferences.

The framework of the games with parametrized preferences allows us to discuss only some aspects of the customary strategic games. In particular it does not allow us to introduce the notion of a mixed strategy, since the outcomes of playing different strategies by a player, given the joint strategy chosen by the opponents, cannot be aggregated. Also the notion of a Pareto efficient outcome does not have a counterpart in this framework because in general two joint strategies cannot be compared. For example, in the above modelling of the Prisoner's Dilemma game we cannot compare the joint strategies (N_1, N_2) and (C_1, C_2).

In the field of strategic games two techniques of reducing a game have been considered — by means of iterated elimination of strategies strictly dominated by a mixed strategy or of iterated elimination of never best responses to a mixed strategy (see, e.g., [18].) These techniques can be easily transferred to the games with parametrized preferences provided we limit ourselves to strict dominance by a pure strategy and never best responses to a pure strategy.

First, given such a game

$$G := (S_1, \ldots, S_n, \succ(s_{-1}), \ldots, \succ(s_{-n})),$$

where each s_{-i} ranges over S_{-i}, and sets of strategies $S'_1, \ldots, S'_n$ such that $S'_i \subseteq S_i$ for $i \in [1..n]$, we say that

$$G' := (S'_1, \ldots, S'_n, \succ(s_{-1}), \ldots, \succ(s_{-n})),$$

where each s_{-i} now ranges over S'_{-i}, is a ***subgame*** of G, and identify in the context of G' each preference relation $\succ(s_{-i})$ with its restriction to S'_i.

We then introduce the following two notions of reduction between a game

$$G := (S_1, \ldots, S_n, \succ(s_{-1}), \ldots, \succ(s_{-n})),$$

where each s_{-i} ranges over S_{-i} and its subgame

$$G' := (S'_1, \ldots, S'_n, \succ(s_{-1}), \ldots, \succ(s_{-n})),$$

where each s_{-i} ranges over S'_{-i}:

- $G \rightarrow_{NBR} G'$
  when $G \neq G'$ and for all $i \in [1..n]$ each $s_i \in S_i \setminus S'_i$ is never a best response for player i in G,

- $G \rightarrow_S G'$

  when $G \neq G'$ and for all $i \in [1..n]$ each $s'_i \in S_i \setminus S'_i$ is strictly dominated in G by some $s_i \in S_i$.

In the literature it is customary to consider more specific reduction relations in which, respectively, *all* never best responses or *all* strictly dominated strategies are eliminated. The advantage of using the above versions is that we can prove the relevant property of both reductions by just one simple lemma, since by definition a strictly dominated strategy is never a best response and consequently $G \rightarrow_S G'$ implies $G \rightarrow_{NBR} G'$.

Lemma 1 *Suppose that $G \rightarrow_{NBR} G'$. Then s is a Nash equilibrium of G iff it is a Nash equilibrium of G'.*

Proof. $(\Rightarrow)$ By definition each s_i is a best response to s_{-i} to G. So no s_i is eliminated in the reduction of G to G'.

$(\Leftarrow)$ Suppose s is not a Nash equilibrium of G. So some s_i is not a best response to s_{-i} in G. Let s'_i be a best response to s_{-i} in G. (s'_i exists since $\succ(s_{-i})$ is a linear order.)

So s'_i is not eliminated in the reduction of G to G' and s'_i is a best response to s_{-i} in G'. But this contradicts the fact that s is a Nash equilibrium of G'. $\Box$

Theorem 1 *Suppose that $G \rightarrow^*_{NBR} G'$, i.e., G' is obtained by an iterated elimination of never best responses from the game G.*

(i) Then s is a Nash equilibrium of G iff it is a Nash equilibrium of G'.

(ii) If each player in G' has just one strategy, then the resulting joint strategy is a unique Nash equilibrium of G.

Proof.
(i) By the repeated application of Lemma 1.

(ii) It suffices to note that $(s_1, \ldots s_n)$ is a unique Nash equilibrium of the game in which each player i has just one strategy, s_i. $\Box$

The above theorem allows us to reduce a game without affecting its (possibly empty) set of Nash equilibria or even, occasionally, to find its unique Nash equilibrium. In the latter case one says that the original game was ***solved*** by an iterated elimination of never best responses (or of strictly dominated strategies).

As an example let us return to the Prisoner's Dilemma game with parametrized preferences defined above. In this game each strategy C_i is strictly dominated by N_i, so the game can be solved by either reducing it in two steps (by removing in each step one C_i strategy) or in one step (by removing both C_i strategies) to a game in which each player i has exactly one strategy, N_i.

Finally, let us mention that [11] and [22] proved that all iterated eliminations of strictly dominated strategies yield the same final outcome. An analogous result for the iterated elimination of never best responses was established in [1]. Both results carry over to our framework of games with parametrized preferences by a direct modification of the proofs.

4 From CP-nets to strategic games

Consider now a CP-net with the set of variables $\{X_1, \ldots, X_n\}$ with the corresponding finite domains $\mathcal{D}(X_1), \ldots, \mathcal{D}(X_n)$. We write each preference statement for the variable X_i as $X_I = a_I : \succ_i$, where for the subsequence $I = i_1, \ldots, i_k$ of $1, \ldots, n$:

- $Pa(X_i) = \{X_{i_1}, \ldots, X_{i_k}\}$,
- $X_I = a_I$ is an abbreviation for $X_{i_1} = a_{i_1} \wedge \ldots \wedge X_{i_k} = a_{i_k}$,
- $\succ_i$ is a preference over $\mathcal{D}(X_i)$.

We also abbreviate $\mathcal{D}(X_{i_1}) \times \ldots \times \mathcal{D}(X_{i_k})$ to $\mathcal{D}(X_I)$.

By definition, the preference statements for a variable X_i are exactly all statements of the form $X_I = a_I : \succ(a_I)$, where a_I ranges over $\mathcal{D}(X_I)$ and $\succ(a_I)$ is a preference on $\mathcal{D}(X_i)$ that depends on a_I.

We now associate with each CP-net N a game $\mathcal{G}(N)$ with parametrized preferences as follows:

- each variable X_i corresponds to a player i,
- the strategies of player i are the elements of the domain $\mathcal{D}(X_i)$ of X_i.

To define the parametrized preferences, consider a player i. Suppose $Pa(X_i) = \{X_{i_1}, \ldots, X_{i_k}\}$ and let $I := i_1, \ldots, i_k$. So I is a subsequence of $1, \ldots, i-1, i+1, \ldots, n$ and consequently each joint strategy a_{-i} of the opponents of player i uniquely determines a sequence a_I. Given now an arbitrary a_{-i} we associate with it the preference relation $\succ(a_I)$ on $\mathcal{D}(X_i)$ where $X_I = a_I : \succ(a_I)$ is the unique preference statement for X_i determined by a_I.

In words, the preference of a player i over his strategies, assuming the joint strategy a_{-i} of its opponents, coincides with the preference given by the CP-net over the domain of X_i, assuming the assignment to its parents a_I (which coincides with the projection of a_{-i} over I). This completes the definition of $\mathcal{G}(N)$.

As an example consider the first CP-net of Section 2. The corresponding game has four players A, B, C, D, each with two strategies indicated with f, $\bar{f}$ for player F. The preference of each player on his strategies will depend only on the strategies chosen by the players which correspond to his parents in the CP-net. Consider for example player B. His preference over his strategies b and $\bar{b}$, given the joint strategy of his opponents $s_{-B} = dac$, is $b \succ \bar{b}$. Notice that, for example, the same order holds for the opponents joint strategy $s_{-B} = \bar{d}a\bar{c}$, since the strategy chosen by the only player corresponding to his parent, A, has not changed.

We have then the following result.

Theorem 2 *An outcome of a CP-net N is optimal iff it is a Nash equilibrium of the game $\mathcal{G}(N)$.*

Proof. ($\Rightarrow$) Take an optimal outcome o of N. Consider a player i in the game $\mathcal{G}(N)$ and the corresponding variable X_i of N. Suppose $Pa(X_i) = \{X_{i_1}, \ldots, X_{i_k}\}$. Let $I := i_1, \ldots, i_k$, and let $X_I = o_I : \succ(o_I)$ be the corresponding preference statement for X_i. By definition there is no improving flip from o to another outcome, so o_i is the maximal element in the order $\succ(o_I)$.

By the construction of the game $\mathcal{G}(N)$, each outcome in N is a joint strategy in $\mathcal{G}(N)$. Also, two outcomes are one flip away iff the corresponding joint strategies differ only in a strategy of one player. Given the joint strategy o considered above, we thus have that, if we modify the strategy of player i, while leaving the strategies of the other players unchanged, this change is worsening in $\succ(o_{-i})$, since $\succ(o_{-i})$ coincides with $\succ(o_I)$. So by definition o is a Nash equilibrium of $\mathcal{G}(N)$.

($\Leftarrow$) Take a Nash equilibrium s of the game $\mathcal{G}(N)$. Consider a variable X_i of N. Suppose $Pa(X_i) = \{X_{i_1}, \ldots, X_{i_k}\}$. Let $I := i_1, \ldots, i_k$, and let $X_I = s_I : \succ(s_I)$ be the corresponding preference statement for X_i.

By definition for every strategy $s'_i \neq s_i$ of player i, we have $s_i \succ(s_{-i})\ s'_i$, so $s_i \succ(s_I)\ s'_i$ since $\succ(s_{-i})$ coincides with $\succ(s_I)$. So by definition s is an optimal outcome for N. $\square$

5 From strategic games to CP-nets

We now associate with each game G with parametrized preferences a CP-net $\mathcal{N}(G)$ as follows:

- each variable X_i corresponds to a player i,
- the domain $\mathcal{D}(X_i)$ of the variable X_i consists of the set of strategies of player i,
- we stipulate that $Pa(X_i) = \{X_1, X_{i-1}, \ldots, X_{i+1}, \ldots, X_n\}$, where n is the number of players in G.

Next, for each joint strategy s_{-i} of the opponents of player i we take the preference statement $X_{-i} = s_{-i} : \succ(s_{-i})$, where $\succ(s_{-i})$ is the preference relation on the set of strategies of player i associated with s_{-i}.

This completes the definition of $\mathcal{N}(G)$. As an example of this construction let us return to the Prisoner's Dilemma game with parametrized preferences from Section 2.3. In the corresponding CP-net we have then two variables X_1 and X_2 corresponding to players 1 and 2, with the respective domains $\{C_1, N_1\}$ and $\{C_2, N_2\}$. To explain how each parametrized preference translates to a preference statement take for example $\succ(C_2) := \ N_1 \succ C_1$. It translates to $X_2 = C_2 : \ N_1 \succ C_1$.
We have now the following counterpart of Theorem 2.

Proposition 1 *A joint strategy is a Nash equilibrium of the game G iff it is an optimal outcome of the CP-net $\mathcal{N}(G)$.*

Proof. It suffices to notice that $\mathcal{G}(\mathcal{N}(G)) = G$ and use Theorem 2. $\square$

The disadvantage of the above construction of the CP-net $\mathcal{N}(G)$ from a game G is that it always produces a CP-net in which all sets of parent features are of size $n-1$ where n is the number of features of the CP-net. This can be rectified by reducing each set of parent features to a minimal one as follows.

Given a CP-net N, consider a variable X_i with the parents $Pa(X_i)$, and take a variable $Y \in Pa(X_i)$. Suppose that for all assignments a to $Pa(X) - \{Y\}$ and any two values $y_1, y_2 \in \mathcal{D}(Y)$, the orders $\succ(a, y_1)$ and $\succ(a, y_2)$ on $\mathcal{D}(X_i)$ coincide.

We say then that Y is *redundant* in the set of parents of X_i. It is easy to see that by removing all redundant variables from the set of parents of X_i and by modifying the corresponding preference statements for X_i accordingly, the strict preorder $\succ$ over the outcomes of the CP-nets is not changed.

Given a CP-net, if for all its variable X_i the set $Pa(X_i)$ does not contain any redundant variable, we say that the CP-net is ***reduced***.

By iterating the above construction every CP-net can be transformed to a reduced CP-net. As an example consider a CP-net with three features, X, Y and Z, with the respective domains $\{a_1, a_2\}, \{b_1, b_2\}$ and $\{c_1, c_2\}$. Suppose now that $Pa(X) = Pa(Y) = \emptyset, Pa(Z) = \{X, Y\}$ and that

$\succ(a_1, b_1) = \succ(a_2, b_1),\ \ \succ(a_1, b_2) = \succ(a_2, b_2),$

$\succ(a_1, b_1) = \succ(a_1, b_2),\ \ \succ(a_2, b_1) = \succ(a_2, b_2).$

Then both X and Y are redundant in the set of parents of Z, so we can reduce the CP-net by reducing $Pa(Z)$ to $\emptyset$. Z becomes an independent variable in the reduced CP-net with the order over its domain which coincides with $\succ(a_1, b_1)$ (which is the same as the other three orders on the domain of Z).

In what follows for a CP-net N we denote by $r(N)$ the corresponding reduced CP-net. The following result, depicted in Figure 1, summarizes the relevant properties of $r(N)$ and relates it to the constructions of $\mathcal{G}(N)$ and $\mathcal{N}(G)$.

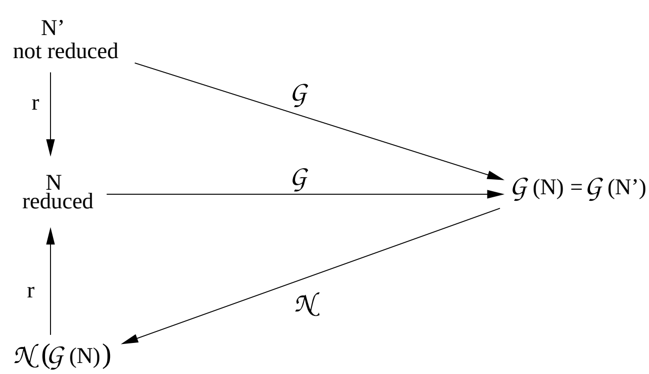

Figure 1: Relation between a CP-net N, its reduced form and corresponding games

Proposition 2

(i) *Each CP-net N and its reduced form $r(N)$ have the same order $\succ$ over the outcomes.*

(ii) *For each CP-net N and its reduced form $r(N)$ we have $\mathcal{G}(N) = \mathcal{G}(r(N))$.*

(iii) *Each reduced CP-net N is a reduced CP-net corresponding to the game $\mathcal{G}(N)$. Formally: $N = r(\mathcal{N}(\mathcal{G}(N)))$.*

Proof.

Statements (i) and (ii) easily follow from the definition of function r and from from the construction of the game corresponding to a CP net. We will thus write explicitly only the proof of statement (iii).

(iii) Given a reduced CP-net N, consider the CP-net $\mathcal{N}(\mathcal{G}(N))$. For each variable X_i, $Pa(X_i)$ in N is a subset of $Pa(X_i)$ in $\mathcal{N}(\mathcal{G}(N))$, which is the set of all variables except X_i. However, by the construction of the game corresponding to a CP-net and of the CP-net corresponding to a game, in each conditional preference table, if the assignments to the common parents are the same, the preference orders over X_i are the same.

Let us now reduce $\mathcal{N}(\mathcal{G}(N))$ to obtain $N' = r(\mathcal{N}(\mathcal{G}(N)))$. Then $Pa(X_i)$ in N' coincides with $Pa(X_i)$ in N. Indeed, suppose there is a parent of X_i in N which is not in N'. Since N is reduced, such a parent is not redundant in N. Thus the reduction r, when applied to $\mathcal{N}(\mathcal{G}(N))$, does not remove this parent since the orders in the conditional preference tables of N and $\mathcal{N}(\mathcal{G}(N))$ are the same.

Further, suppose there is a parent of X_i in N' which is not in N. Since N' is reduced, such a parent is not redundant in N'. Thus it is also not redundant in $\mathcal{N}(\mathcal{G}(N))$. By the construction of $\mathcal{N}(\mathcal{G}(N))$, this parent is not redundant in N either. $\square$

Part (i) states that the reduction procedure r preserves the order over the outcomes. Part (ii) states that the construction of a game corresponding to a CP-net does not depend on the redundancy of the given CP-net. Finally, part (iii) states that the reduced CP-net N can be obtained 'back' from the game $\mathcal{G}(N)$.

6 Game-theoretic techniques in CP-nets

Thanks to the established connections between CP-nets and games with parametrized preferences, we can now transfer to CP-nets the techniques of iterated elimination of strictly dominated strategies or of never best responses considered in Section 2.3. To introduce them in the context of

CP-nets consider a CP-net N with the set of variables $\{X_1, \ldots, X_n\}$ with the corresponding finite domains $\mathcal{D}(X_1), \ldots, \mathcal{D}(X_n)$.

- We say that an element d_i from the domain $\mathcal{D}(X_i)$ of the variable X_i is a ***best response*** to a preference statement

$$X_I = a_I : \succ_i$$

  for X_i if $d_i \succ_i d'_i$ for all $d'_i \in \mathcal{D}(X_i)$ such that $d_i \neq d'_i$.

- We say that an element d_i from the domain of the variable X_i is a ***never a best response*** if it is not a best response to any preference statement for X_i.

- Given two elements d_i, d'_i from the domain $\mathcal{D}(X_i)$ of the variable X_i we say that d'_i is ***strictly dominated*** by d_i if for all preference statements $X_I = a_I : \succ_i$ for X_i we have

$$d_i \succ_i d'_i.$$

By a ***subnet*** of a CP-net N we mean a CP-net obtained from N by removing some elements from some variable domains followed by the removal of all preference statements that refer to a removed element.

Then we introduce the following relation between a CP-net N and its subnet N':

$$N \rightarrow_{NBR} N'$$

when $N \neq N'$ and for each variable X_i each removed element from the domain of X_i is never a best response in N, and also introduce an analogous relation $N \rightarrow_S N'$ for the case of strictly dominated elements. Since each strictly dominated element is never a best response, $N \rightarrow_S N'$ implies $N \rightarrow_{NBR} N'$.

The following counterpart of Theorem 1 then holds.

Theorem 3 *Suppose that $N \rightarrow^*_{NBR} N'$, i.e., the CP-net N' is obtained by an iterated elimination of never best responses from the CP-net N.*

(i) Then s is an optimal outcome of N iff it is an optimal outcome of N'.

(ii) If each variable in N' has a singleton domain, then the resulting outcome is a unique optimal outcome of N.

□

To illustrate the use of this theorem reconsider the first CP-net from Section 2, i.e., the one with the preference statements
$d : a \succ \overline{a}$, $\overline{d} : a \succ \overline{a}$,
$a : b \succ \overline{b}$, $\overline{a} : \overline{b} \succ b$,
$b : c \succ \overline{c}$, $\overline{b} : \overline{c} \succ c$,
$c : d \succ \overline{d}$, $\overline{c} : \overline{d} \succ d$.
Denote it by N.

We can reason about it using the iterated elimination of strictly dominated strategies (which coincides here with the iterated elimination of never best responses, since each domain has exactly two elements).

We have the following chain of reductions:

$$N \rightarrow_S N_1 \rightarrow_S N_2 \rightarrow_S N_3 \rightarrow_S N_4,$$

where

- N_1 results from N by removing $\overline{a}$ (from the domain of A) and the preference statements $d : a \succ \overline{a}$, $\overline{d} : a \succ \overline{a}$, $\overline{a} : \overline{b} \succ b$,
- N_2 results from N_1 by removing $\overline{b}$ and the preference statements $a : b \succ \overline{b}$, $\overline{b} : \overline{c} \succ c$,
- N_3 results from N_2 by removing $\overline{c}$ and the preference statements $b : c \succ \overline{c}$ $\overline{c} : \overline{d} \succ d$,
- N_4 results from N_3 by removing $\overline{d}$ from the domain of D and the preference statement $c : d \succ \overline{d}$.

Indeed, in each step the removed element is strictly dominated in the considered CP-net. So using the iterated elimination of strictly dominated elements we reduced the original CP-net to one in which each variable has a singleton domain and consequently found a unique optimal outcome of the original CP-net N.

Finally, the following result shows that the introduced reduction relation on CP-nets is complete for acyclic CP-nets.

Theorem 4 *For each acyclic CP-net N a subnet N' with the singleton domains exists such that $N \rightarrow^*_{NBR} N'$. The outcome associated with N' is a unique optimal outcome of N and hence N' is unique.*

Proof. First note that if N is an acyclic CP-net with some non-singleton domain, then $N \rightarrow_{NBR} N'$ for some subnet N' of N. Indeed, suppose N is such a CP-net. By acyclicity a variable X exists with a non-singleton domain with no parent variable that has a non-singleton domain. So there exists in N exactly one preference statement for X, say $X_I = a_I : \succ_i$, where X_I is the sequence of parent variables of X. Reduce the domain of X to the maximal element in $\succ_i$. Then for the resulting subnet N' we have $N \rightarrow_{NBR} N'$. Since N' is also acyclic and has one variable less with a non-singleton domain, by iterating this procedure we obtain a subnet N' with the singleton domains such that $N \rightarrow^*_{NBR} N'$.

The claim that the outcome associated with N' is a unique optimal outcome of N is a consequence of Theorem 3(ii). □

The singleton domains obtained via the use of the $\rightarrow_{NBR}$ reduction correspond to the unique optimal outcome of an acyclic CP-net, as defined in [6, 5].

7 CP-net techniques in strategic games

The established relationship between CP-nets and strategic games with parametrized preferences also allows us to exploit the techniques developed for the CP-nets when studying such games.

One natural idea is to consider a counterpart of the notion of an acyclic CP-net. We call a game with parametrized preferences ***hierarchical*** if the CP-net $r(\mathcal{N}(G))$ is acyclic.

We can introduce this notion directly, without using the CP-nets, by considering a partition of players $1, \ldots, n$ in the game

$$(S_1, \ldots, S_n, \succ(s_{-1}), \ldots, \succ(s_{-n})),$$

where each s_{-i} ranges over S_{-i}, into levels $1, \ldots, k$ such that for each player i at level j and each $s_{-i} \in S_{-i}$ the preference $\succ(s_{-i})$ depends only on the entries in s_{-i} associated with the players from levels $< j$.

So a game is hierarchical if the players can be partitioned into levels $1, 2, \ldots, k$, such that each player at level j can express his preferences without taking into account the players at his level or higher levels (lower levels are more important).

We have then the following counterpart of Theorem 4.

Theorem 5 *For each hierarchical game G a subgame G' with the singleton strategy sets exists such that $G \rightarrow^*_{NBR} G'$. The resulting joint strategy associated with G' is a unique Nash equilibrium of G and hence G' is unique.*

Proof. By an analogous argument as the one used in the proof of Theorem 4. □

Given a hierarchical game G, by definition the CP-net $r(\mathcal{N}(G))$ is acyclic. Thus we know that it has a unique optimal outcome which can be found in linear time. This means that the unique Nash equilibrium of G can be found in linear time by the usual CP-net techniques applied to $r(\mathcal{N}(G))$.

Hierarchical games naturally represent multi-agent scenarios in which agents (that is, players of the game) can be partitioned into levels such that each agent can determine his preferences without consulting agents at his level or lower levels. Informally, agents at one level are 'more important' than agents at lower levels in the sense that they can take their decisions without consulting them.

A more general class of games is obtained by analogy to graphical games. We define a ***graphical game with parametrized preferences*** as follows. Given a neighbour function *neigh* we assume that for each player i and a joint strategy s_i of his opponents, the preference $\succ(s_{-i})$ depends only on the entries in s_{-i} associated with the players from $neigh(i)$. Equivalently, we may just use the preference relations $\succ_s^i$ for each player i and each joint strategy s of the neighbours of i. Hierarchical games are then graphical games with parametrized preferences with acyclic neighbour graphs.

Given a CP-net N and the corresponding game $\mathcal{G}(N)$, the dependency graph of N uniquely determines the neighbour function *neigh* between the players in $\mathcal{G}(N)$. This allows us to associate with each CP-net N a graphical game with parametrized preferences. Conversely, each graphical game G with parametrized preferences uniquely determines a CP-net. It is obtained by proceeding as in Section 5 but by stipulating that the parent relation corresponds to the neighbour function *neigh*, that is, by putting

$$Pa(X_i) := \{X_j \mid j \in neigh(i)\}.$$

The counterparts of Theorems 1 and 2 then hold for CP-nets and graphical games with parametrized preferences.

Note that we arrived at the concept of a hierarchical game through the analogy with the acyclic CP-nets. To see a natural example of such games consider the problem of spreading a technology in a social network, inspired by the problems studied in [16] for the case of infinite number of players. We assume that the players (users) are connected in a network, which is a directed graph, and that there are k technologies (for example mobile telephone companies) $t_1, \ldots, t_k$. Assume further that each user, given two technologies, prefers to use the one that is used by more of his neighbours in the network (for instance to cut down on the telephone costs).

We model this situation as a graphical game with parametrized preferences. We assume that each player i has k strategies, $t_1, \ldots, t_k$, and for each joint strategy s of the neighbours of i we define the preference relation $\succ_s^i$ by putting

$$t_k \succ_s^i t_l \text{ iff } |s(t_k)| > |s(t_l)| \text{ or } (|s(t_k)| = |s(t_l)| \text{ and } k < l), \tag{2}$$

where $s(X)$ is the set of components of s that are equal to the strategy X. So we assume that in the case of a tie player i prefers a technology with the lower index.

We can now analyze the process of selecting a technology by exploiting the relation between hierarchical games and CP-nets. Namely, suppose that the above defined graphical game G with parametrized preferences is hierarchical. Then by virtue of Theorem 5 $G \rightarrow_{NBR}^{*} G'$, where in G' each player has a single strategy, t_1. The resulting joint strategy is then a unique Nash equilibrium of G. Additionally, by the corresponding order independence result mentioned at the end of Section 3, G' is a unique outcome of iterating the $\rightarrow_{NBR}$ reduction.

This corresponds to an informal statement that when the neighbour function describes an acyclic graph, eventually technology t_1 is adopted by everybody. Because of the nature of the preference relations used above, this result actually holds for a larger class of graphical games with parametrized preferences.

They correspond to the following class of directed graphs. We call a directed graph ***well-structured*** if levels can be assigned to its nodes in such a way that each node has at least as many incoming edges from the nodes with strictly lower levels than from the other nodes. Of course, each directed acyclic graph is well-structured but other examples exist, see, e.g., Figure 2.

We have then the following result.

Theorem 6 *Consider a graphical game G with parametrized preferences in which each player has k strategies, $t_1, \ldots, t_k$, the preference relations $\succ_s^i$ are defined by (2), and the neighbour function describes a well-structured graph. Then $G \rightarrow_{NBR}^{*} G'$, where in G' each player has a single strategy, t_1, and the resulting joint strategy is a unique Nash equilibrium of G.*

Proof. We prove by induction on the level m that

$$G \rightarrow_{NBR}^{*} G', \tag{3}$$

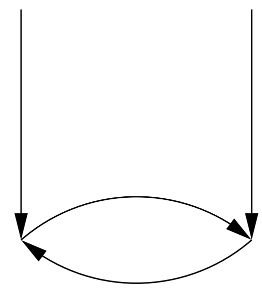

Figure 2: A well-structured graph that is not acyclic

where in G' each player of level $\leq m$ has a single strategy, t_1. This yields then the desired conclusion about Nash equilibrium by Theorem 1.

The claim holds for the lowest level, say 0, as then each player of level 0 has no neighbours and hence his strategies $t_2, \ldots, t_k$ can be eliminated as never best responses.

Suppose (3) holds for some level m. So we have $G \rightarrow^*_{NBR} G'$, where in G' each player of level $\leq m$ has a single strategy, t_1. Consider the players of level $m+1$ in the game G'. Each of them has at least as many neigbours with the single strategy t_1 than with other sets of strategies. So each joint strategy of his neighbours has at least as many t_1s as other strategies. Hence $G' \rightarrow^*_{NBR} G''$, where in G'' each player of level $\leq m+1$ has a single strategy, t_1. Consequently $G \rightarrow^*_{NBR} G''$, which establishes the induction step. $\Box$

The above example shows that graphical games with parametrized preferences can be used to provide a natural qualitative analysis of some problems studied in social networks. Expressing the process of selecting a technology using games with parametrized preferences, Nash equilibria and elimination of never best responses is more natural than using CP-nets. On the other hand we arrived at the relevant result about adoption of a single technology by searching for an analogue of Theorem 4 about acyclic CP-nets.

8 From SCSPs to graphical games

In this and the next section we relate the notion of optimality in soft constraints and graphical games. To obtain an appropriate match we assume that in graphical games payoffs are elements of a linearly ordered set A instead of the set of real numbers. (This precludes the use of mixed strategies but they are not needed here.) We denote then such games by

$$(S_1, \ldots, S_n, neigh, p_1, \ldots, p_n, A),$$

where *neigh* is the given neighbour function.

In this section we define two mappings from SCSPs to a specific kind of graphical games. In what follows we focus on SCSPs based on c-semirings with the carrier linearly ordered by $\leq$ (e.g. fuzzy or weighted) and compare the concepts of optimal solutions in SCSPs with Nash equilibria and Pareto efficient joint strategies in the graphical games. In both mappings we identify the players with the variables. Since soft constraints link variables, the resulting game players are naturally connected, which explains why we use graphical games.

8.1 Local mapping

Given a SCSP $P := \langle C, V, D, S \rangle$ we define the corresponding graphical game for $n = |V|$ players as follows:

- the players: one for each variable;
- the strategies of player i: all values in the domain of the corresponding variable x_i;
- the neighbourhood function: $j \in neigh(i)$ iff the variables x_i and x_j appear together in some constraint from C;
- the payoff function of player i:

  Let $C_i \subseteq C$ be the set of constraints involving x_i and let X be the set of variables that appear together with x_i in some constraint in C_i (i.e., $X = \{x_j \mid j \in neigh(i)\}$). Then given an assignment s to all variables in $X \cup \{x_i\}$ the payoff of player i w.r.t. s is defined by:

$$p_i(s) := \Pi_{c \in C_i} \mathrm{def}_c(s \downarrow_{\mathrm{con}_c}).$$

We denote the resulting graphical game by $L(P)$ to emphasize the fact that the payoffs are obtained using *local* information about each variable, by looking only at the constraints in which it is involved.

One could think of a different mapping where players correspond to constraints. However, such a mapping can be obtained by applying the local mapping L to the hidden variable encoding [15] of the SCSP in input.

8.1.1 General case

In general, the concepts of optimal solutions of a SCSP P and the Nash equilibria of the derived game $L(P)$ are unrelated. Indeed, consider the fuzzy CSP defined at the end of Section 2.2. The corresponding game has:

- three players, x, y, and z;
- each player has two strategies, a and b;
- the neighbourhood function is defined by:

$$neigh(x) := \{y\},\ neigh(y) := \{x, z\},\ neigh(z) := \{y\};$$

- the payoffs of the players are defined as follows:
  - for player x:
    $p_x(aa*) := 0.4$, $p_x(ab*) := 0.1$, $p_x(ba*) := 0.3$, $p_x(bb*) := 0.5$;
  - for player y:
    $p_y(aaa) := 0.4$, $p_y(aab) := 0.3$, $p_y(abb) := 0.1$, $p_y(bbb) := 0.5$,
    $p_y(bba) := 0.5$, $p_y(baa) := 0.3$, $p_y(bab) := 0.3$, $p_y(aba) := 0.1$;
  - for player z:
    $p_z(*aa) := 0.4$, $p_z(*ab) := 0.3$, $p_z(*ba) := 0.1$, $p_z(*bb) := 0.5$;

where $*$ stands for either a or b and where to facilitate the analysis we use the canonical extensions of the payoff functions p_x and p_z to the functions on $\{a, b\}^3$.

This game has two Nash equilibria: aaa and bbb. However, only bbb is an optimal solution of the fuzzy SCSP.

One could thus think that in general the set of Nash equilibria is a superset of the set of optimal solutions of the corresponding SCSP. However, this is not the case. Indeed, consider a fuzzy CSP with as before three variables, x, y and z, each with the domain $\{a, b\}$, but now with the constraints:

$C_{xy} := \{(aa, 0.9), (ab, 0.6), (ba, 0.6), (bb, 0.9)\}$,
$C_{yz} := \{(aa, 0.1), (ab, 0.2), (ba, 0.1), (bb, 0.2)\}$.

Then aab, abb, bab and bbb are all optimal solutions but only aab and bbb are Nash equilibria of the corresponding graphical game.

8.1.2 SCSPs with strictly monotonic combination

Next, we consider the case when the multiplicative operator $\times$ is strictly monotonic. Recall that given a c-semiring $\langle A, +, \times, \mathbf{0}, \mathbf{1}\rangle$, the operator $\times$ is ***strictly monotonic*** if for any $a, b, c \in A$ such that $a < b$ we have $c \times a < c \times b$. (The symmetric condition is taken care of by the commutativity of $\times$.)

Note for example that in weighted CSP $\times$ is strictly monotonic, as $a < b$ in the carrier means that $b < a$ as reals, so for any c we have $c + b < c + a$, i.e., $c \times a < c \times b$ in the carrier. In contrast, the fuzzy CSPs $\times$ are not strictly monotonic, as $a < b$ does not imply that $min(a, c) < min(b, c)$ for all c.

So consider now a c-semiring with a linearly ordered carrier and a strictly monotonic multiplicative operator. As in the previous case, given an SCSP P, it is possible that a Nash equilibrium of $L(P)$ is not an optimal solution of P. Consider for example a weighted SCSP P with

- two variables, x and y, each with the domain $D = \{a, b\}$;
- one constraint $C_{xy} := \{(aa, 3), (ab, 10), (ba, 10), (bb, 1)\}$.

The corresponding game $L(P)$ has:

- two players, x and y, who are neighbours of each other;
- each player has two strategies, a and b;
- the payoffs defined by:
  $p_x(aa) := p_y(aa) := 7$, $p_x(ab) := p_y(ab) := 0$,
  $p_x(ba) := p_y(ba) := 0$, $p_x(bb) := p_y(bb) := 9$.

Notice that, in a weighted CSP we have $a \leq b$ in the carrier iff $b \leq a$ as reals, so when passing from the SCSP to the corresponding game, we have complemented the costs w.r.t. 10, when making them payoffs. In general, given a weighted CSP, we can define the payoffs (which must be maximized) from the costs (which must be minimized) by complementing the costs w.r.t. the greatest cost used in any constraint of the problem.

Here $L(P)$ has two Nash equilibria, aa and bb, but only bb is an optimal solution. Thus, as in the fuzzy case, we have that there can be a Nash equilibrium of $L(P)$ that is not an optimal solution of P. However, in contrast to the fuzzy case, the set of Nash equilibria of $L(P)$ is now a superset of the set of optimal solutions of P. In fact, a stronger result holds.

Theorem 7 *Consider a SCSP P defined on a c-semiring $\langle A, +, \times, \mathbf{0}, \mathbf{1}\rangle$, where A is linearly ordered and $\times$ is strictly monotonic, and the corresponding graphical game $L(P)$. Then*

(i) Every optimal solution of P is a Nash equilibrium of $L(P)$.

(ii) Every optimal solution of P is a Pareto efficient joint strategy in $L(P)$.

Proof.
(i) We prove that if a joint strategy s is not a Nash equilibrium of game $L(P)$, then it is not an optimal solution of SCSP P.

Let a be the strategy of player x in s, and let $s_{neigh(x)}$ and s_Y be, respectively, the joint strategy of the neighbours of x, and of all other players, in s. That is, $V = \{x\} \cup neigh(x) \cup Y$ and we write s as $(a, s_{neigh(x)}, s_Y)$.

By assumption there is a strategy b for x such that the payoff $p_x(s')$ for the joint strategy $s' := (b, s_{neigh(x)}, s_Y)$ is higher than $p_x(s)$. (We use here the canonical extension of p_x to the Cartesian product of all the strategy sets).

So by the definition of the mapping L

$$\Pi_{c\in C_x}\mathrm{def}_c(s\downarrow_{\mathrm{con}_c}) < \Pi_{c\in C_x}\mathrm{def}_c(s'\downarrow_{\mathrm{con}_c}),$$

where C_x is the set of all the constraints involving x in SCSP P. But the preference of s and s' is the same on all the constraints not involving x and $\times$ is strictly monotonic, so we conclude that

$$\Pi_{c\in C}\mathrm{def}_c(s\downarrow_{\mathrm{con}_c}) < \Pi_{c\in C}\mathrm{def}_c(s'\downarrow_{\mathrm{con}_c}).$$

This means that s is not an optimal solution of P.

(ii) We prove that if a joint strategy s is not Pareto efficient in the game $L(P)$, then it is not an optimal solution of SCSP P.

Since s is not Pareto efficient, there is a joint strategy s' such that $p_i(s) \leq p_i(s')$ for all $i \in [1..n]$ and $p_i(s) < p_i(s')$ for some $i \in [1..n]$. Let us denote with $I = \{i \in [1..n]$ such that $p_i(s) < p_i(s')\}$. By the definition of the mapping L, we have:

$$\Pi_{c\in C_i}\mathrm{def}_c(s\downarrow_{\mathrm{con}_c}) < \Pi_{c\in C_i}\mathrm{def}_c(s'\downarrow_{\mathrm{con}_c}),$$

for all $i \in I$ and where C_i is the set of all the constraints involving the variable corresponding to player i in SCSP P. Since the preference of s and

s' is the same on all the constraints not involving any $i \in I$, and since $\times$ is strictly monotonic, we have:

$$\Pi_{c\in C}\mathrm{def}_c(s\downarrow_{\mathrm{con}_c}) < \Pi_{c\in C}\mathrm{def}_c(s'\downarrow_{\mathrm{con}_c}).$$

This means that s is not an optimal solution of P. $\square$

To see that there may be joint strategies that are both Nash equilibria and Pareto efficient but do not correspond to the optimal solutions, consider a weighted SCSP P with

- two variables, x and y, each with domain $D = \{a, b\}$;
- constraint $C_x := \{(a, 2), (b, 1)\}$;
- constraint $C_y := \{(a, 4), (b, 7)\}$;
- constraint $C_{xy} := \{(aa, 0), (ab, 10), (ba, 10), (bb, 0)\}$.

The corresponding game $L(P)$ has:

- two players, x and y, who are neighbours of each other;
- each player has two strategies: a and b;
- the payoffs defined by: $p_x(aa) := 8$, $p_y(aa) := 6$, $p_x(ab) := p_y(ab) := 0$, $p_x(ba) := p_y(ba) := 0$, $p_x(bb) := 9$, $p_y(bb) := 3$.

As above, when passing from an SCSP to the corresponding game, we have complemented the costs w.r.t. 10, when turning them to payoffs. $L(P)$ has two Nash equilibria: aa and bb. They are also both Pareto efficient. However, only aa is optimal in P.

8.1.3 Classical CSPs

Note that in the classical CSPs $\times$ is not strictly monotonic, as $a < b$ implies that $a = 0$ and $b = 1$ but $c \wedge a < c \wedge b$ does not hold then for $c = 0$. In fact, the above result does not hold for classical CSPs. Indeed, consider a CSP with:

- three variables: x, y, and z, each with the domain $\{a, b\}$;

- two constraints: C_{xy} (over x and y) and C_{yz} (over y and z) defined by:

  $C_{xy} := \{(aa, 1), (ab, 0), (ba, 0), (bb, 0)\}$,

  $C_{yz} := \{(aa, 0), (ab, 0), (ba, 1), (bb, 0)\}$.

This CSP has no solutions in the classical sense, i.e., each optimal solution, in particular *baa*, has preference 0. However, *baa* is not a Nash equilibrium of the resulting graphical game, since the payoff of player x increases when he switches to the strategy a.

On the other hand, if we restrict the domain of L to consistent CSPs, that is, CSPs with at least one solution with value 1, then it yields games in which the set of Nash equilibria that are also Pareto efficient joint strategies coincides with the set of solutions of the CSP.

Theorem 8 *Consider a consistent CSP P and the corresponding graphical game $L(P)$. Then an instantation of the variables of P is a solution of P iff it is a Nash equilibrium and Pareto efficient joint strategy in $L(P)$.*

Proof. Consider a solution s of P. In the resulting game $L(P)$ the payoff to each player is maximal, namely 1. So the joint strategy s is both a Nash equilibrium and Pareto efficient. Conversely, every Pareto efficient joint strategy in $L(P)$ corresponds to solution of P. □

There are other ways to relate CSPs and games so that the CSP solutions and the Nash equilibria coincide. This is what is done in [10], where a mapping from the strategic games to CSPs is defined. Notice that our mapping goes in the opposite direction and it is not the reverse of the one in [10]. In fact, the mapping in [10] is not reversible.

8.2 Global mapping

The mapping L is in some sense 'local', since it considers the neighbourhood of each variable. An alternative 'global' mapping considers all constraints. More precisely, given a SCSP $P = \langle C, V, D, S\rangle$, with a linearly ordered carrier A of S, we define the corresponding game on $n = |V|$ players, $GL(P) = (S_1, \ldots, S_n, p_1, \ldots, p_n, A)$ by using the following payoff function p_i for player i:

- given an assignment s to *all* variables in V

$$p_i(s) := \Pi_{c \in C} \mathrm{def}_c(s \downarrow_{\mathrm{con}_c}).$$

Notice that in the resulting game the payoff functions of all players are the same. Then the following result analogous to Theorem 8 holds.

Theorem 9 *Consider an SCSP P over a linearly ordered carrier, and the corresponding graphical game $GL(P)$. Then an instantiation of the variables of P is an optimal solution of P iff it is a Nash equilibrium and Pareto efficient in $GL(P)$.*

Proof. An optimal solution of P, say s, is a joint strategy for which all players have the same, highest, payoff. So no other joint strategy exists for which some player is better off and consequently s is both a Nash equilibrium and Pareto efficient. Conversely, every Pareto efficient joint strategy in $GL(P)$ has the highest payoff, so it corresponds to an optimal solution of P. $\square$

The global mapping GL has the advantage of providing a precise relationship between the optimal solutions and joint strategies that are both Nash equilibria and Pareto efficient. However, it has an obvious disadvantage from the computational point of view, since it requires to consider all the complete assignments of the SCSP.

9 From graphical games to SCSPs

Next, we define a mapping from graphical games to SCSPs. To define it we limit ourselves to SCSPs defined on c-semirings which are the Cartesian product of linearly ordered c-semirings (see Section 2.2).

9.1 The mapping

Given a graphical game $G = (S_1, \ldots, S_n, neigh, p_1, \ldots, p_n, A)$ we define the corresponding SCSP $L'(G) = \langle C, V, D, S \rangle$, as follows:

- each variable x_i corresponds to a player i;
- the domain $D(x_i)$ of the variable x_i consists of the set of strategies of player i, i.e., $D(x_i) := S_i$;
- the c-semiring is
  $\langle A_1 \times \cdots \times A_n, (+_1, \ldots, +_n), (\times_1, \ldots, \times_n), (\mathbf{0}_1, \ldots, \mathbf{0}_n), (\mathbf{1}_1, \ldots, \mathbf{1}_n) \rangle$,
  the Cartesian product of n *arbitrary* linearly ordered semirings;

- soft constraints: for each variable x_i, one constraint $\langle \text{def}, \text{con} \rangle$ such that:
  - con $= neigh(x_i) \cup \{x_i\}$;
  - def : $\Pi_{y \in \text{con}} D(y) \to A_1 \times \cdots \times A_n$ such that for any $s \in \Pi_{y \in \text{con}} D(y)$, $\text{def}(s) := (d_1, \ldots, d_n)$ with $d_j = \mathbf{1}_j$ for every $j \neq i$ and $d_i = f(p_i(s))$, where $f : A \to A_i$ is an order preserving mapping from payoffs to preferences (i.e., if $r > r'$ then $f(r) > f(r')$ in the c-semiring's ordering).

To illustrate it consider again the previously used Prisoner's Dilemma game:

	C_2	N_2
C_1	$3, 3$	$0, 4$
N_1	$4, 0$	$1, 1$

Recall that in this game the only Nash equilibrium is (N_1, N_2), while the other three joint strategies are Pareto efficient.

We shall now construct a corresponding SCSP based on the Cartesian product of two weighted semirings. This SCSP according to the mapping L' has:[2]

- two variables: x_1 and x_2, each with the domain $\{c, n\}$;
- two constraints, both on x_1 and x_2:
  - constraint c_1 with $\text{def}(cc) := \langle 7, 0 \rangle$, $\text{def}(cn) := \langle 10, 0 \rangle$, $\text{def}(nc) := \langle 6, 0 \rangle$, $\text{def}(nn) := \langle 9, 0 \rangle$;
  - constraint c_2 with $\text{def}(cc) := \langle 0, 7 \rangle$, $\text{def}(cn) := \langle 0, 6 \rangle$, $\text{def}(nc) := \langle 0, 10 \rangle$, $\text{def}(nn) := \langle 0, 9 \rangle$;

The optimal solutions of this SCSPs are: cc, with preference $\langle 7, 7 \rangle$, nc, with preference $\langle 10, 6 \rangle$, cn, with preference $\langle 6, 10 \rangle$. The remaining solution, nn, has a lower preference in the Pareto ordering. Indeed, its preference $\langle 9, 9 \rangle$ is dominated by $\langle 7, 7 \rangle$, the preference of cc (since preferences are here costs and have to be minimized). Thus the optimal solutions coincide here with the Pareto efficient joint strategies of the given game. This is true in general.

[2] Recall that in the weighted semiring $\mathbf{1}$ equals 0.

Theorem 10 *Consider a graphical game G and a corresponding SCSP $L'(G)$. Then the optimal solutions of $L'(G)$ coincide with the Pareto efficient joint strategies of G.*

Proof. In the definition of the mapping L' we stipulated that the mapping f maintains the ordering from the payoffs to preferences. As a result each joint strategy s corresponds to the n-tuple of preferences $(f(p_1(s)), \ldots, f(p_n(s)))$ and the Pareto orderings on the n-tuples $(p_1(s), \ldots, p_n(s))$ and $(f(p_1(s)), \ldots, f(p_n(s)))$ coincide. Consequently a sequence s is an optimal solution of the SCSP $L'(G)$ iff $(f(p_1(s)), \ldots, f(p_n(s)))$ is a maximal element of the corresponding Pareto ordering. $\square$

We notice that L' is injective and, thus, can be reversed on its image. When such a reverse mapping is applied to these specific SCSPs, payoffs correspond to projecting of the players' valuations to a subcomponent.

9.2 Pareto efficient Nash equilibria

As mentioned earlier, in [10] a mapping is defined from the graphical games to CSPs such that Nash equilibria coincide with the solutions of CSP. Instead, our mapping is from the graphical games to SCSPs, and is such that Pareto efficient joint strategies and the optimal solutions coincide.

Since CSPs can be seen as a special instance of SCSPs, where only **1**, **0**, the top and bottom elements of the semiring, are used, it is possible to add to any SCSP a set of hard constraints. Therefore we can merge the results of the two mappings into a single SCSP, which contains the soft constraints generated by L' and also the hard constraints generated by the mapping in [10], Below we denote these hard constraints by $H(G)$. We recall that each constraint in $H(G)$ corresponds to a player, has the variables corresponding to the player and it neighbours and allows only tuples corresponding to the strategies in which the player has no so-called regrets. If we do this, then the optimal solutions of the new SCSP with preference higher than **0** are the Pareto efficient Nash equilibria of the given game, that is, those Nash equilibria which dominate or are incomparable with all other Nash equilibria according to the Pareto ordering. Formally, we have the following result.

Theorem 11 *Consider a graphical game G and the SCSP $L'(G) \cup H(G)$. If the optimal solutions of $L'(G) \cup H(G)$ have global preference greater than* ***0****, they correspond to the Pareto efficient Nash equilibria of G.*

Proof. Given any solution s, let p be its preference in $L'(G)$ and p' in $L'(G) \cup H(G)$. By the construction of the constraints $H(G)$ we have that p' equals p if s is a Nash equilibrium and p' equals $\mathbf{0}$ otherwise. The remainder of the argument is as in the proof of Theorem 10. □

For example, in the Prisoner's Dilemma game, the mapping in [10] would generate just one constraint on x_1 and x_2 with nn as the only allowed tuple. In our setting, when using as the linearly ordered c-semirings the weighted semirings, this would become a soft constraint with

$$\text{def}(cc) := \text{def}(cn) := \text{def}(nc) = \langle \infty, \infty \rangle, \ \text{def}(nn) := \langle 0, 0 \rangle.$$

With this new constraint, all solutions have the preference $\langle \infty, \infty \rangle$, except for nn which has the preference $\langle 9, 9 \rangle$ and thus is optimal. This solution corresponds to the joint strategy (N_1, N_2) with the payoff $(1, 1)$ (and thus preference $(9, 9)$). This is the only Nash equilibrium and thus the only Pareto efficient Nash equilibrium.

This method allows us to identify among Nash equilibria the 'optimal' ones. One may also be interested in knowing whether there exist Nash equilibria which are also Pareto efficient joint strategies. For example, in the Prisoners' Dilemma example, there are no such Nash equilibria. To find any such joint strategies we can use the two mappings separately, to obtain, given a game G, both an SCSP $L'(G)$ and a CSP $H(G)$ (using the mapping in [10]). Then we should take the intersection of the set of optimal solutions of $L'(G)$ and the set of solutions of $H(G)$.

10 Conclusions

In this paper we related three formalisms that are commonly used to reason about optimal outcomes: strategic games, CP-nets and soft constraints. To this end we modified the concept of strategic games to games with parametrized preferences and showed that the optimal outcomes in CP-nets are exactly Nash equilibria of such games. This allowed us to exploit game-theoretic techniques in search for the optimal outcomes of CP-nets. In the other direction, we showed how the notion of an acyclic CP-net naturally leads to the concept of a hierarchical game. Such games have a unique Nash equilibrium.

We also considered the relation between graphical games and various classes of soft constraints. While for soft constraints there is only one notion of optimality, for graphical games there are at least two. In this paper we have considered Nash equilibria and Pareto efficient joint strategies. We showed that for a natural (local) mapping from soft CSPs to graphical games in general no relation exists between the notions of optimal solutions of soft CSPs and Nash equilibria. On the other hand, when in the SCSPs the preferences are combined using a strictly monotonic operator, the optimal solutions of the SCSP are included both in the Nash equilibria of the game and in the set of Pareto efficient joint strategies. In general the inclusions cannot be reversed. We have also exhibited a (global) mapping from the graphical games to a class of SCSPs such that the Pareto efficient joint strategies of the game coincide with the optimal solutions of the SCSP.

For the reverse direction we showed that for a natural mapping from the graphical games to a class of SCSPs the optimal solutions coincide with Pareto efficient joint strategies. Moreover, if we add suitable hard constraints to the soft constraints, optimal solutions coincide with Pareto efficient Nash equilibria.

The results of this paper clarify the relationship between various notions of optimality used in strategic games, CP-nets and soft constraints. These results can be used in a number of ways. One obvious way is to try to exploit computational results existing for one of these areas in another. This has been pursued already in [10] for games versus hard constraints. Using our results this can also be done for strategic games versus CP nets or soft constraints. For example, finding a Pareto efficient joint strategy involves mapping a game into a soft CSP and then solving it. Similar approach can also be applied to Pareto efficient Nash equilibria, which can be found by solving a suitable soft CSP.

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