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CORRECTION

Correction to: Directed Lovász Local Lemma and Shearer's Lemma

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1 Corrected proof of the lopsidepended case in [3]

The proof of Theorem 1a in our article [3] has a mistake in the way Equation (16) is used. We give below the full corrected proof together with a new version of Definition 3 of that article, suitable for the new proof. The full version of the corrected article can be found in [2].

New version of Definition 3 in [3] A labeled rooted forest $\mathcal{F}$ is called *feasible* if:

1. the labels of its roots are *pairwise distinct*,
2. the labels of any two siblings (i.e. vertices with a common parent) are distinct and
3. an internal vertex labeled by E_j has at most $|\Gamma_j|+1$ children, with labels whose indices are in $\Gamma_j \cup \{j\}$.

Corrected proof of Theorem 1a in [3]. We may assume, without loss of generality, that $\Pr[E_j] < \chi_j \prod_{i\in\Gamma_j}(1-\chi_i)$ for all $j \in \{1,\ldots,m\}$, i.e. that the hypothesis is given in terms

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of a strict inequality. Indeed, otherwise consider an event B, such that B and $E_1, \ldots, E_m$, are mutually independent, where $\Pr[B] = 1 - \delta$, for arbitrary small $\delta > 0$. We can now perturb the events a little, by considering e.g. $E_j \cap B$, $j = 1, \ldots, m$. As a consequence, we can also assume without loss of generality that for some other small enough $\epsilon > 0$, we have that $Pr[E_j] \leq (1-\epsilon)\chi_j \prod_{i \in \Gamma_j}(1-\chi_i)$. Note that to obtain this new event B, it might be necessary in some case to enlarge the probability space by adding one more random variable that is independent from $X_1, \ldots, X_l$.

It suffices to prove that $\hat{P}_n$ is inverse exponential in n. Specifically, we show that $\hat{P}_n \leq (1-\epsilon)^n$. We use the following notation: $\mathbf{n} = (n_1, \ldots, n_m)$, where $n_1, \ldots, n_m \geq 0$ are such that $\sum_{i=1}^m n_i = n$ and $\mathbf{n} - (1)_j := (n_1, \ldots, n_j - 1, \ldots, n_m)$.

Let $Q_{\mathbf{n},j}$ be the probability that ValAlg is successful when started on a *tree* whose root is labeled with E_j and has $\sum_{i=1}^m n_i = n$ nodes labeled with $E_1, \ldots, E_m$. Observe that to obtain a bound for $\hat{P}_n$ we need to add over all possible forests with n nodes in total. Thus, it holds that:

$$\hat{P}_n \leq \sum_{\mathbf{n}} \sum_{\mathbf{n}^1 + \ldots + \mathbf{n}^m = \mathbf{n}} \left(Q_{\mathbf{n}^1,1} \cdots Q_{\mathbf{n}^m,m} \right).$$

Our aim is to show that $Q_{\mathbf{n},j}$ is exponentially small to n, for any given sequence of $\mathbf{n}$ and any $j \in \{1, \ldots, m\}$. Thus, by ignoring polynomial in n factors, the same will hold for $\hat{P}_n$ (recall that the number of variables and the number of events are considered constants, asymptotics are in terms of the number of steps n only).

Let $\Gamma_j^+ := \Gamma_j \cup \{j\}$, and assume that, for each $j \in \{1, \ldots, m\}$, $|\Gamma_j^+| = k_j$. Observe now that $Q_{\mathbf{n},j}$ is bounded from above by a function, denoted again by $Q_{\mathbf{n},j}$ (to avoid overloading the notation), which follows the recurrence:

$$Q_{\mathbf{n},j} = (1-\epsilon)\Pr[E_j] \cdot \sum_{\mathbf{n}^1 + \cdots + \mathbf{n}^{k_j} = \mathbf{n} - (1)_j} \left(Q_{\mathbf{n}^1, j_1} \cdots Q_{\mathbf{n}^{k_j}, j_{k_j}} \right), \tag{1}$$

with initial conditions $Q_{\mathbf{n},j} = 0$ when $n_j = 0$ and there exists an $i \neq j$ such that $n_i \geq 1$; and with $Q_{\mathbf{0},j} = 1$, where 0 is a sequence of m zeroes.

To solve the above recurrence, we introduce, for $j = 1, \ldots, m$, the *multivariate generating functions*:

$$Q_j(\mathbf{t}) = \sum_{\mathbf{n}: n_j \geq 1} Q_{\mathbf{n},j} \mathbf{t}^{\mathbf{n}}, \tag{2}$$

where $\mathbf{t} = (t_1, \ldots, t_m)$, $\mathbf{t}^{\mathbf{n}} := t_1^{n_1} \cdots t_m^{n_m}$.

By multiplying both sides of (1) by $\mathbf{t}^n$ and adding all over suitable $\mathbf{n}$, we get the system of equations $\mathbf{Q}$:

$$Q_j(\mathbf{t}) = t_j f_j(\mathbf{Q}), \tag{3}$$

where, for $\mathbf{x} = (x_1, \ldots, x_m)$ and $j = 1 \ldots, m$:

$$f_j(\mathbf{x}) = (1-\epsilon) \cdot \chi_j \cdot \left(\prod_{i \in \Gamma_j} (1-\chi_i) \right) \cdot \left(\prod_{i \in \Gamma_j^+} (x_i + 1) \right). \tag{4}$$

To solve the system, we will directly use the result of Bender and Richmond in [1] (Theorem 2). Let g be any m-ary projection function on some of the m coordinates. In the sequel we take $g := pr_s^m$, the (m)-ary projection on the s-th coordinate. Let also $\mathcal{B}$ be the set of trees

$B = (V(B), E(B))$ whose vertex set is $\{0, 1, \ldots, m\}$ and with edges directed towards 0. By [1], we get:

$$[\mathbf{t^n}]g(\mathbf{Q}(\mathbf{t})) = \frac{1}{\prod_{j=1}^{m} n_j} \sum_{B\in\mathcal{B}} [\mathbf{x^{n-1}}] \frac{\partial(g, f_1^{n_1}, \ldots, f_m^{n_m})}{\partial B}, \tag{5}$$

where the term for a tree $B \in \mathcal{B}$ is defined as:

$$[\mathbf{x^{n-1}}] \prod_{r\in V(B)} \left\{ \left(\prod_{(i,r)\in E(B)} \frac{\partial}{\partial x_i} \right) f_r^{n_r}(\mathbf{x}) \right\}, \tag{6}$$

where $r \in \{0, \ldots, m\}$ and $f_0^{n_0} := g$.

We consider a tree $B \in \mathcal{B}$ such that (6) is not equal to 0. Thus, $(i, 0) \neq E(B)$, for all $i \neq s$. On the other hand, $(s, 0) \in E(B)$, lest vertex 0 is isolated, and each vertex has out-degree *exactly* one, lest a cycle is formed or connectivity is broken. From vertex 0, we get $\frac{\partial pr_s^m(\mathbf{x})}{\partial x_s} = 1$. Since our aim is to prove that $\hat{P}_n$ is exponentially small in n, we are are interested only in factors of (6) that are exponential in n, and we can thus ignore the derivatives (except the one for vertex 0), as they introduce only polynomial (in n) factors to the product. Thus, we have that (6) is equal to the coefficient of $\mathbf{x^{n-1}}$ in:

$$\prod_{j=1}^{m} \left\{ (1-\epsilon)^{n_j} \cdot \chi_j^{n_j} \cdot \left(\prod_{i\in\Gamma_j} (1-\chi_i)^{n_j} \right) \cdot \left(\prod_{i\in\Gamma_j^+} (x_i+1)^{n_j} \right) \right\}. \tag{7}$$

We now group the factors of (7) according to the i's. We have already argued each vertex i has out-degree 1. Thus, the exponent of the term $x_i + 1$ is $n_i + \sum_{j:i\in\Gamma_j} n_j$ and the product of (7) is equal to:

$$\prod_{i=1}^{m} \left\{ (1-\epsilon)^{n_i} \cdot \chi_i^{n_i} \cdot (1-\chi_i)^{\sum_{j:i\in\Gamma_j} n_j} \cdot (x_i+1)^{n_i+\sum_{j:i\in\Gamma_j} n_j} \right\}. \tag{8}$$

Using the binomial theorem and by ignoring polynomial factors, we get that the coefficient of $\mathbf{x^{n-1}}$ in (8) is:

$$\prod_{i=1}^{m} \left\{ (1-\epsilon)^{n_i} \cdot \chi_i^{n_i} \cdot (1-\chi_i)^{\sum_{j:i\in\Gamma_j} n_j} \cdot \binom{n_i + \sum_{j:i\in\Gamma_j} n_j}{n_i} \right\}. \tag{9}$$

By expanding $(\chi_i + 1 - \chi_i)^{n_i+\sum_{j:i\in\Gamma_j} n_j}$, we get that (9) is at most:

$$\prod_{i=1}^{m} (1-\epsilon)^{n_i} = (1-\epsilon)^{\sum_{i=1}^{n} n_i} = (1-\epsilon)^n. \tag{10}$$

Thus, $\hat{P}_n$ is inverse exponential in n. □

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