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Partially ordered cooperative games: extended core and Shapley value

J. Puerto · F.R. Fernández · Y. Hinojosa

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Abstract In this paper we analyze cooperative games whose characteristic function takes values in a partially ordered linear space. Thus, the classical solution concepts in cooperative game theory have to be revisited and redefined: the core concept, Shapley–Bondareva theorem and the Shapley value are extended for this class of games. The classes of standard, vector-valued and stochastic cooperative games among others are particular cases of this general theory.

Keywords Cooperative games · Core · Shapley value · Partial order

1 Introduction

Game theorists have tried over years to keep connected the development of Game Theory to actual applications. This orientation is very important from an economic point of view and also from the practitioners point of view. Nevertheless, nowadays game theory is not only an economic tool but also an interesting mathematical discipline. In this regard, games themselves are mathematical objects worth to be investigated. One may argue that in mathematical-economics the stress is put in the applicability however there are many aspects within the Theory of Games that remain open and beg for further analysis.

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J. Puerto (✉) · F.R. Fernández
Departamento de Estadística e Investigación Operativa, Facultad de Matemáticas, Universidad de Sevilla, 41012 Sevilla, Spain
e-mail: puerto@us.es

F.R. Fernández
e-mail: fernande@us.es

Y. Hinojosa
Departamento de Economía Aplicada I, Facultad CCEE, Universidad de Sevilla, Sevilla, Spain
e-mail: yhinojos@us.es

One of these situations can be identified in the allocation of divisible entities among the agents operating in an optimization problem. One of the branches of the theory of games, namely TU-cooperative game theory, has covered partially this field. Indeed, all this theory is founded in that either worth or cost will be allocated to the players. This means that any object to be allocated is given a value through a "utility" and only this virtual utility can be decomposed and finally allocated. One can think of situations, as for instance the following story, that fits bad to the paradigm above. Three guys are in the middle of the desert and have in common one orange, one grapefruit and one watermelon. Each one of them must go in a different direction and needs the fruits as future refreshment. It is clearly not optimal to cut the fruits in pieces since conservation would be impossible. Moreover, here the monetary value is of no interest. This is of course, also, the case of heritages of goods having familiar or subjective interest.

It is clear to us that considering the problem of how to allocate mathematical objects as a whole requires a titanic effort (as considered as a single project). Therefore, one can try to proceed by stages. In a first stage we restrict ourselves to the problem of how to allocate elements of a partially ordered linear space.

The standard analysis of cooperative TU-games assumes that the payoff of any coalition is valued by a real number. Here we replace this assumption allowing the payoffs to be elements of any partially ordered linear space. Different extensions of cooperative games have being the games with a continuous of player by Aumann and Shapley (1974) or the fuzzy coalition theory by Aubin (1987). In recent years another productive line of research has been to impose different structures on the set of coalitions for the players in the games. This analysis gives rise to the so called cooperative games on combinatorial structures (see Bilbao 2000). Our analysis is completely different instead of imposing conditions on the argument of the characteristic function we extend the nature of the payoffs. Particular instances of this model have been already considered in (Fernández et al. 2002a; Granot 1977; Nishizaki and Sakawa 2001; Suijs 2000; Suijs et al. 1998, 1999; Timmer 2001).

Our goal is to extend the solution concepts of the classical cooperative game theory to this new class of games. Specifically, we study the two most widely used solution concepts within the theory of cooperative games: (1) the core set (set solution), and; (2) the Shapley value. The partially ordered cooperative game theory includes as a particular instance the standard TU-games, as well as some other classes of games such as vector-valued and stochastic games.

The results in this paper are summarized in the following: (1) introduction of a new class of cooperative games whose characteristic function ranges on any linear space, (2) definition of different core concepts according to the domination relationship defined on the space, (3) characterization of non-emptiness of the core; and, (4) extension of the Shapley value to this class of games and its characterization by potentials and axiomatically.

The paper is organized as follows. In Sect. 2 we present the basic concepts and definitions concerning partially ordered cooperative games. Section 3 contains the results concerning the core. We present the notion of core set and its characterization in terms of coalitional dominance. We also prove an analogous to Bondareva–Shapley Theorem that holds for partially ordered cooperative games using a general form of duality by Jahn (1983). Section 4 introduces the extended Shapley value. It also characterizes this value using extended potentials and a set of axioms. The paper ends with the references cited in the text.

2 Basic concepts

Let $\aleph$ be a linear space over the real field. A partially ordered cooperative game (N, v) is a set $N = \{1, 2, \ldots, n\}$ of players and a map $v : 2^N \cup \{\emptyset\} \to \aleph$ on the set 2^N of all subsets of N such that $v(\emptyset) = \ominus_\aleph$ ($\ominus_\aleph$ is the null vector in the space $\aleph$). The elements of the set N are called players and the function v is the characteristic function of the game. The function $v(S)$ is the worth of the coalition S. We denote by $PO^n(\aleph)$ the family of all the partially ordered cooperative games defined on the space $\aleph$.

We assume that there exists a partial order $\succeq$ (reflexivity and transitivity) defined on the set $\aleph$ (see Roubens and Vincke 1985). We represent by $\succ$ the corresponding strict partial order and by $\sim$ the indifference relationship. Associated with $\succeq$ there is another binary reflexive relation $\succsim$ defined by:

$$X \succsim Y \quad \text{iff} \quad not(Y \succeq X), \qquad \forall X \neq Y,$$

that is important to be considered in our analysis. We require to this partial order a natural densedness condition:

$$\text{for any } X \neq Y \in \aleph, \quad X \succsim Y \quad \Rightarrow \quad \exists Z \in \aleph, \quad X \succsim Z \succ Y. \tag{1}$$

It is worth noting that the well-known scalar, vector-valued (Fernández et al. 2002a) and stochastic cooperative games (Fernández et al. 2002b; Suijs et al. 1999) are particular cases of this formulation, just considering $\aleph = \mathbb{R}$ with the $\geq$ order over the reals, $\mathbb{R}^n$ with the component-wise order and $L^1(\mathbb{R})$ with the stochastic dominance order, respectively. The cases above are examples that can be found in the literature of game theory although one can think of many other interesting structures where this approach can be applied.

If players agree on cooperation then an interesting question which arises is how the worth $v(N)$ should be allocated among the various players. The natural extension of the idea of allocation (or preimputation) used in scalar games to the partially ordered cooperative games consists of using an allocation

$$X = (X_1, \ldots, X_n)$$

where $X_i \in \aleph$, $i = 1, \ldots, n$, stands for the payoff of the i-th player. We assume that the worth of $v(N)$ must be allocated to the players in N. This is the well-known efficiency principle. Therefore, in the rest of the paper we will consider only allocations that satisfy

$$\sum_{i=1}^{n} X_i \sim v(N).$$

The set of allocations of a partially ordered cooperative game (N, v) is denoted by $I^*(N, v)$. Formally,

$$I^*(N, v) = \left\{ (X_1, \ldots, X_n) \in \aleph^n : \sum_{i=1}^{n} X_i \sim v(N) \right\}.$$

Among all the allocations of the game $(N, v) \in PO^n(\aleph)$ we are interested in those which cannot be dominated by the worth given to the coalitions. Thus, some kind of ordering concept is necessary to perform these comparisons.

3 Core solutions

The minimum requirement imposed on allocations so that players do not refuse them is the following: each individual player i gets a payoff X_i being not worse than the worth $v(i)$ given by the characteristic function of the game.

The set of all the allocations that fulfill this property, $I(N,v)$, is called imputation set of the game.

$$I(N,v) = \{X \in I^*(N,v) : X_i \succsim v(i)\ \forall i\}.$$

Keeping tracks of the development followed in the standard theory the next step is to impose the collective rationality to those imputations proposed as good allocations. This idea was first suggested by (Gillies 1959) and later formalized (for scalar games) under the name of core of the game.

Definition 3.1 The core of the partially ordered cooperative game $(N,v) \in PO^n(\aleph)$ is defined as the set of allocations such that $X_S := \sum_{i\in S} X_i$ is as least as preferred as $v(S)$, for every coalition S and it is denoted by

$$core(N,v;\succeq) = \{X \in I^*(N,v)/X_S \succeq v(S)\ \forall S \subset N\}.$$

Notice that $\subset$ stands for strict inclusion, while $\subseteq$ will be used in the paper for the regular inclusion.

Example 3.1 Let us consider the partially ordered lineal space of continuous functions $X : [0,2] \longrightarrow \mathbb{R}$ such that $X(t) = I_{[0,1]}(t)f_1(t) + I_{[1,2]}(t)f_2(t)$ where f_i $i = 1,2$ are affine functions,

$$I_A(t) = \begin{cases} 1 & \text{if } t \in A, \\ 0 & \text{otherwise;} \end{cases}$$

and the partial order given for any X, Y in this space by $X \succeq Y$ if $X(t) \geq Y(t)$ for any $t \in [0,2]$.

We consider the three-person game (N,v) whose characteristic function is given by:

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
$f_1(t)$	2	$3+t$	$1+3t$	$6+t$	$3+3t$	$4+5t$	$10+2t$
$f_2(t)$	$3/2+t/2$	$7/2+t/2$	$7/2+t/2$	7	$5+t$	$17/2+t/2$	12

It is clear that the allocation given by

	X_1	X_2	X_3
$f_1(t)$	3	$5-t$	$2+3t$
$f_2(t)$	$7/2-t/2$	$7/2+t/2$	5

belongs to $core(N,v,\succeq)$.

In order to characterize the core we need to introduce the coalitional dominance induced by the relationship $\succsim$. Let $X, Y \in I^*(N,v)$ and $S \subseteq N$ a coalition. Y dominates X through S according to $\succsim$ and we will denote $Y \overset{S}{dom}_{\succsim} X$ if $Y_S \succ X_S$ and $v(S) \succsim Y_S$. This concept leads us to consider the notion of non-dominated imputation by allocations.

Definition 3.2 An imputation $X \in I(N, v)$ of the game (N, v) is non-dominated by allocations if for any coalition $S \subseteq N$ it does not exist an allocation $Y \in I^*(N, v)$ such that $Y \overset{S}{dom_{\succsim}} X$. This set is given by:

$$NDIA(N, v, \succsim) = \{X \in I(N, v) / \nexists\, S \subseteq N, Y \in I^*(N, v), Y \not\succ X : Y \overset{S}{dom_{\succsim}} X\}$$

This set exhibits a close relation with the concept of core given in Definition 3.1.

Theorem 3.1 *The following relationship* $NDIA(N, v; \succsim) = core(N, v; \succeq)$ *holds.*

Proof Let us assume that $X \notin NDIA(N, v; \succsim)$ then it must exist a coalition $S \subset N$ and $Y \in I^*(N, v)$ such that:

$$v(S) \succsim Y_S \succ X_S.$$

This implies that $X_S \succeq v(S)$ does not hold and hence $X \notin core(N, v; \succeq)$.

Conversely, let $X \notin core(N, v; \succeq)$. Then, it exists $S \subseteq N$ such that $X_S \succeq v(S)$ does not hold. Therefore, $v(S) \succsim X_S$. Now, we can apply the densedness property (1) and it must exists $Y \in \aleph$ satisfying:

$$v(S) \succsim Y \succ X_S,$$

and hence $X \notin NDIA(N, v; \succeq)$. □

Our next result is a first sufficient condition for non-emptiness of the core. Let u be a function $u : \aleph \to \mathbb{R}$ satisfying for any $X_1, X_2 \in \aleph$, $X_1 \succeq X_2 \Rightarrow u(X_1) \geq u(X_2)$. ($u$ agrees with the partial order $\succeq$.) We define the set

$$C(N, v_u) = \{X \in I^*(N, v) : u(v(S)) \leq u(X_S),\ \forall S \subseteq N\}.$$

It is worth noting that this set may be used as solution concept if players agree on allocating the worth in the game through a utility function u.

Lemma 3.1 *For any u that agrees with the partial order* $\succeq$, *the relationship*

$$core(N, v, \succeq) \subseteq C(N, v_u),$$

holds.

Proof Let us assume that $X \in core(N, v, \succeq)$ but $X \notin C(N, v_u)$. Then, it must exist a coalition S such that $u(X_S) < u(v(S))$. However, this is not possible because u agrees with the partial order $\succeq$. □

We can give alternative conditions on the non-emptiness of the core. Let us assume that the partial order $\succeq$ is defined by the family of functions U. That means that $X \succeq Y \Leftrightarrow u(X) \geq u(Y),\ \forall u \in U$. Then, we can establish the following theorem.

Theorem 3.2 *It holds that*:

$$core(N, v, \succeq) = \bigcap_{u \in U} C(N, v_u).$$

Proof The inclusion $core(N, v, \succeq) \subseteq \bigcap_{u \in U} C(N, v_u)$ is clear by the definition of $C(N, v_u)$ and Lemma 3.1.

Then, let us assume that $X \in \bigcap_{u \in U} C(N, v_u)$ and $X \notin core(N, v, \succeq)$. Thus, it must exist $S \subseteq N$ such that $not(X_S \succeq v(S))$. This is equivalent to that it exists $\bar{u} \in U$ such that $\bar{u}(v(S)) > \bar{u}(X_S)$. However, this means that $X \notin \bigcap_{u \in U} C(N, v_u)$. □

Let us consider a utility function u on $\aleph$. Associated with u we define the following scalar cooperative game (N, v_u) where the characteristic function is given by $v_u(S) = u(v(S))$ for any $S \subseteq N$. Theorem 3.2 is particularly important when the cone that characterizes the partial order $\succeq$ is finitely generated. ($X \succeq Y \Leftrightarrow u_j(X) \geq u_j(Y),\ j = 1, \ldots, k$.) Then we get the following lemma. (Recall that the core in a scalar cooperative game is not empty if and only if the game is balanced. See Owen (1995).)

Lemma 3.2 *$core(N, v, \succeq) = \emptyset$ if the scalar game (N, v_{u_j}) is not balanced for some $j = 1, \ldots, k$.*

Notice that the very important case of a partial order defined by individual utilities, described above, fits into this category. It is worth noting that for the core to be empty it suffices that a game (N, v_{u_i}) has empty core, although some other games (N, v_{u_j}), $j \neq i$ may have nonempty core.

Example 3.2 (*Vector-valued games,* Fernández et al. 2002a) Let us consider the space $(\mathbb{R}^k, \geq)$, where for any $x, y \in \mathbb{R}^k$, $x \geq y$ means $x_i \geq y_i$ for $i = 1, \ldots, k$. The game (N, v) whose characteristic function v is defined: $v : 2^N \to \mathbb{R}^k$, $v(\emptyset) = 0$ is called vector-valued game.

In this case the partial order is generated by the utility functions $u_i(x) = x_i$, $i = 1, \ldots, k$. Therefore, by Theorem 3.2 $core(N, v, \geq) = \bigcap_{i=1}^{k} C(N, v_{u_i})$.

In this particular case, any element $X \in core(N, v, \geq)$ is a $k \times n$ matrix whose i-th row X^i is a core allocation of the scalar game (N, v_{u_i}).

3.1 A necessary and sufficient condition for non-emptiness of the core

In the following we address a characterization of $core(N, v, \succeq)$ similar to the one known as Bondareva–Shapley Theorem. First of all, we would like to recall the idea behind that theorem. The theorem states a primal minimization problem whose feasible set defines the imputations in the core while the objective function is just the value obtained by the grand coalition with a given imputation. To this problem (primal feasibility problem) it is associated a dual maximization problem. The feasible set of this problem is taken as a definition of balancedness. Proving strong duality between these two problems allows to characterize the non-emptiness of the core as soon as the dual problem is bounded from above. This is essentially what is done by Bondareva–Shapley theorem. This analysis is not only privative of the family of games defined over the real line. In general, every time that we are able to define the core of a game as the feasible set of a primal minimization problem and strong duality is proven with respect to a dual maximization problem we can do a similar argument and a necessary and sufficient condition for the core is generated. This is the argument in our approach.

In order to be able to prove such characterization we assume that $\succeq$ is induced by a convex cone $D_\aleph$, that is $X \succeq Y \Leftrightarrow X - Y \in D_\aleph$. We also consider the space Z being the $2^n - 2$-fold Cartesian product of the linear space $\aleph$, i.e. $Z = \aleph^{2^n - 2}$. The space Z is partially ordered by the convex cone $D_Z = (D_{\aleph})^{2^n-2}$. (The reader may notice that this is not a restriction because all the interesting cases fall into the considered case.) Let $\aleph^*$ and Z^* be the topological dual of the spaces $\aleph$ and Z. For any $Y \in \aleph$ and $\hat{Y} \in \aleph^*$ we denote by $\langle \hat{Y}, Y \rangle$ the pairing between the elements of the primal and the dual spaces, e.g. the action of the continuous linear functional $\hat{Y}$ on Y. Therefore, $\langle \hat{Y}, Y \rangle = \hat{Y}(Y)$. (The analogous definition holds for the pairing between Z and Z^*). The ordering cone of the topological dual space $\aleph^*$ is given by:

$$D_{\aleph^*} := \{\hat{Y} \in \aleph^* : \langle \hat{Y}, Y \rangle \geq 0, \ \forall Y \in D_{\aleph}\}$$

and the quasi-interior of $D_{\aleph^*}$ is given by:

$$D^{\#}_{\aleph^*} := \{\hat{Y} \in \aleph^* : \langle \hat{Y}, Y \rangle > 0, \ \forall Y \in D_{\aleph} \setminus \{\ominus_{\aleph}\}\},$$

where $\ominus_{\aleph}$ denotes the zero of the space $\aleph$.

Let us define two linear mappings from $\aleph^n$ into $\aleph$ and Z, respectively, as follows:

$$\begin{array}{llll} C: & \aleph^n & \longrightarrow & \aleph \\ & X = (X_1, \ldots, X_n) & \longrightarrow & C(X) = \sum_{i=1}^{n} X_i \end{array}$$

$$\begin{array}{llll} A: & \aleph^n & \longrightarrow & Z \\ & X = (X_1, \ldots, X_n) & \longrightarrow & A(X) = \left(\sum_{i \in S} X_i\right)_{S \subset N}. \end{array}$$

We denote by * applied to an operator its adjoint operator, namely C^*, A^* and T^* denote the adjoint mappings of C, A and T, respectively. Finally, let $L(Z, \aleph)$ be the linear space over $\mathbb{R}$ of continuous linear mappings from Z into $\aleph$.

Definition 3.3 The partially ordered cooperative game $(N, v, \succeq)$ is balanced if for any $\hat{Y} \in D^{\#}_{\aleph^*}$ there exists $T \in L(Z, \aleph)$ such that:

1. $(C - TA)^*(\hat{Y}) \in (D_{\aleph^*})^n$,
2. $T^*(\hat{Y}) \in D_{Z^*}$,
3. $v(N) \succeq T((v(S))_{S \subset N})$ and does not exist T' such that $T'((v(S))_{S \subset N}) \succeq T((v(S))_{S \subset N})$

We note in passing that the above balancedness condition is similar to the one in the scalar case and reduces to the usual one when we consider scalar cooperative games. It states that it must exist a maximal (non-dominated by the partial order $\succeq$) element T being inferior that $v(N)$. It is based on a feasibility condition on a dual problem that will appear in the proof of the characterization theorem. The reader may notice that there exist in the literature some other extensions of the concept of balancedness to class of games without side payment as for instance in (Billera 1970; Kannai 1992; Keiding and Thorlund-Petersen 1987; Shapley 1973) and the more recent by (Predtetchinski and Herings 2004) where a necessary and sufficient condition on the non-emptiness of the core of a cooperative game without side payment is given.

Let us consider the following set

$$FA = \{X \in \aleph^n : A(X) - (v(S)_{S \subset N}) \in D_Z, X \in (D_{\aleph})^n\}.$$

First of all, we assume that $(v(S))_{S\subset N} \neq \ominus_Z$. Moreover, we impose that our continuous linear map A verifies the Slater type stability condition (see Jahn 1983, p. 346), i.e. there exists $(X_1, \dots, X_n) \in \aleph^n$ such that $A(X) - (v(S))_{S\subset N} \in \overset{\circ}{D}_Z$, the topological interior of D_Z. These are stability conditions to ensure a certain type of duality defined on the game payoff function. We assume further that

$$\{C(X) : X \in FA\} + D_\aleph \quad \text{is convex with non-empty algebraic interior.} \tag{2}$$

This hypothesis ensures that any minimal element of the set $\{C(X) : X \in FA\}$ is properly minimal (see Jahn 1984). It is worth noting that our problem always fulfills condition (2) in finite dimension spaces. Thus, in the scalar case it always holds without explicitly imposed.

Theorem 3.3 *The game $(N, v, \succeq)$ is balanced if and only if core$(N, v, \succeq) \neq \emptyset$.*

Proof Let us consider the following problem:

$$\begin{aligned}(P):\quad & \text{“min” } C(X)\\ & \text{s.t.:}\quad A(X) - (v(S))_{S\subset N} \in D_Z,\\ & \qquad\quad X \in (D_\aleph)^n\end{aligned}$$

where “min” must be understood in the sense of minimal points in the order $\succeq$. Under our hypothesis on stability of the map A we can apply the dual by (Jahn 1983) which for problem (P) turns out to be:

$$\begin{aligned}(D):\quad & \text{“max” } T((v(S))_{S\subset N})\\ & \text{s.t.:}\quad (C - TA)^*(\hat{Y}) \in (D_{\aleph^*})^n,\\ & \qquad\quad T^*(\hat{Y}) \in D_{Z^*},\\ & \qquad\quad T \in L(Z, \aleph),\ \hat{Y} \in D^{\#}_{\aleph^*}.\end{aligned}$$

These two dual problems satisfy that any maximal solution of (D) corresponds to a properly minimal element of (P) and conversely any properly minimal element of (P) is a maximal element of (D) by (2).

Let us assume that $(N, v, \succeq)$ is balanced with T. This element corresponds to a maximal solution of (D). Then, by duality there exists a feasible solution $\hat{X}$ to (P) such that it is properly minimal and $C(\hat{X}) = T((v(S))_{S\subset N})$. Therefore, we have $v(N) - C(\hat{X}) \succeq \ominus_\aleph$; and there exists a core allocation in *core*$(N, v, \succeq)$.

Conversely, assume w.l.o.g. that $v(N)$ is a minimal element of (P). Since we have imposed the condition (2) on (P) any minimal element is properly minimal. Then, by duality there exists a maximal element $\hat{T}$ of (D) such that $v(N) = T((v(S))_{S\subset N})$. This implies that the game is balanced. □

The application of this characterization to the well-known componentwise partial order of $\mathbb{R}^m$ is natural. The reader should notice that for $m = 1$ it reduces to the standard case of $\mathbb{R}$ with the natural order, and therefore the definition of balancedness will give us the classic one for real valued cooperative games.

Example 3.3 (Balancedness of vector-valued games with the componentwise order of $\mathbb{R}^m$) The elements in the case of $\aleph = \mathbb{R}^m$ with the componentwise order given by the cone $D_\aleph = \mathbb{R}^m_+$ are the following. The dual space $\aleph^* = \mathbb{R}^m$, and the dual cone $D_{\aleph^*} = \mathbb{R}^m_+$. This implies that $Z=\mathbb{R}^{m(2^n-2)}$. The map $C:\mathbb{R}^{(m\cdot n)\times 1}\to\mathbb{R}^{m\times 1}$ is given by $C=(c_i^j)_{\substack{i=1\dots m\\ j=1\dots m\times n}}$ where

$$c_i^j=\begin{cases}1 & \text{if } j=(k-1)m+i,\ \text{for } k=1,2,\dots n,\\ 0 & \text{otherwise.}\end{cases}$$

The map $A:\mathbb{R}^{(m\cdot n)\times 1}\longrightarrow\mathbb{R}^{(m(2^n-2))\times 1}$ is given by $A=(A_S)^t_{S\subset N}$ (^{t} denotes the transpose), where for any $S\subset N$, $A_S=(a^j_{i,S})_{\substack{i=1\dots m\\ j=1\dots m\cdot n}}$, and

$$a^j_{i,S}=\begin{cases}1 & \text{if } j=(k-1)m+i,\ \text{for } k=1,2,\dots n,\ \text{and } k\in S\\ 0 & \text{otherwise.}\end{cases}$$

Notice that since we consider the componentwise order in $\mathbb{R}^m$, in the adapted problem (P) (see page 150), there are m constraints per each coalition $S\subset N\setminus\emptyset$. Therefore, the dimension of A is $((2^n-2)\cdot m)\times(m\cdot n)$.

Finally, T is an element of $L(Z,\mathbb{R}^m)$, the space of continuous linear maps from Z to $\mathbb{R}^m$. This means that T is a matrix of dimensions $m\times[(2^n-2)\cdot m]$. To simplify, we write $T=(T_1,\dots,T_m)^t$ where $T_i=(T_i^S)_{S\subset N}\in\mathbb{R}^{1\times[(2^n-2)m]}$, and $T_i^S=(t_i^{S,j})_{j=1\dots m}\in\mathbb{R}^{1\times m}$.

In order to compute $G:=C-TA$, we must compute the matrix $E:=T\times A=(T_1A,T_2A,\dots,T_mA)^t$, being

$$T_iA=\sum_{S\subset N}T_i^SA^S\in\mathbb{R}^{1\times(mn)}.$$

Notice that the row vector $T_iA:=(e_i^j)_{j=1\dots m}$ is given by:

$$e_i^j=\sum_{S\subset N}\sum_{r=1}^{m}t_i^{S,r}a^j_{r,S},\qquad \text{for } j=1\dots mn.$$

According to the definition of $a^j_{r,S}$ we obtain that

$$a^j_{r,S}=1\quad\text{iff}\quad j=(k-1)m+r,\quad \text{for some } k=1,2,\dots,n,\ \text{and } k\in S,$$

and thus

$$e_i^j=\sum_{\substack{S\subset N\\ k\in S}}t_i^{S,(k-1)m+i}.$$

Hence, we obtain $G:=(C-TA):=(g_i^j)=(c_i^j-e_i^j)_{\substack{i=1\dots m\\ j=1\dots m\times n}}$.

Here the adjoint G^* of G is the transpose matrix, e.g. $G^*=G^t$. Now we are in conditions to write down the balancedness conditions. For the sake of simplicity we first write conditions 2, and 3, and then condition 1.

(2) For any $\hat{Y}\in\mathbb{R}^m_+\setminus\{0\}$, $T^t\hat{Y}\ge 0$. This is equivalent to $t_i^{S,j}\ge 0$ for any i, j and $S\subset N$.
(3) For any $i=1,\dots,m$ it must hold

$$v_i(N)\ge\sum_{S\subset N}\sum_{k=1}^{n}t_i^{S,(k-1)m+i}v_i(S).$$

(1) For any $\hat{Y} \in \mathbb{R}^m_+ \setminus \{0\}$, $G^t\hat{Y} \geq 0$. Satisfying this condition for any vector $\hat{Y}$ is equivalent to satisfy it for the elements of the canonical basis of $\mathbb{R}^m$. Applying to the elements in this basis, the condition is equivalent to

$$1 - \sum_{\substack{S \subset N \\ k \in S}} t_i^{S,(k-1)m+i} \geq 0, \quad \text{for any } i = 1\ldots m,\ k = 1\ldots mn.$$

Notice that the payoffs can be considered zero-normalized without loss of generality. In addition, since conditions (2), and (3) hold then it suffices to check for equality in the expression above. Thus, condition (1) is

$$1 - \sum_{\substack{S \subset N \\ k \in S}} t_i^{S,(k-1)m+i} = 0, \quad \text{for any } i = 1\ldots m,\ k = 1\ldots mn.$$

Example 3.4 (Balancedness for scalar games) Finally, when $m = 1$ the conditions (1), (2), and (3) reduce to:

$$\begin{aligned} &(1) \quad \sum_{\substack{S \subset N \\ k \in S}} t^S = 1, \quad \text{for } k = 1\ldots n; \\ &(2) \quad t^S \geq 0, \quad \text{for } S \subset N, \\ &(3) \quad v(N) \geq \sum_{S \subset N} t^S v(S), \end{aligned}$$

the balancedness condition for standard cooperative games.

More sophisticated partial orders give rise to different balancedness conditions although in any case all of them are derived under the same theory developed above.

Remark 3.1 Checking balancedness for this class of games can be very difficult. Thus, other properties of these games may be more useful to ensure non-emptiness of the core. We prove that the natural extension of convexity is sufficient.

Analogous to the concept for standard games, convexity of the game $(N, v, \succeq)$ can be defined in terms of monotonicity, in the considered order, of the first differences. We define the marginal contribution, d_i, of player i in the game v, as:

$$\begin{aligned} &d_i : 2^N \longrightarrow \aleph, \\ &\qquad S \longrightarrow d_i(S) = \begin{cases} v(S \cup \{i\}) - v(S) & \text{if } i \notin S, \\ v(S) - v(S \setminus \{i\}) & \text{otherwise.} \end{cases} \end{aligned} \tag{3}$$

The game $(N, v, \succeq)$ is said to be $\succeq$-convex, if for each $i \in N$,

$$d_i(T) \succeq d_i(S)$$

holds for any $S \subset T$.

Using now the standard argument one can proof that for any permutation π of N, the marginal worth vector $d^\pi \in \aleph^n$ defined by

$$d_i^\pi = v(P_i^\pi \cup \{i\}) - v(P_i^\pi), \quad \text{for all } i \in N, \tag{4}$$

where $P_i^\pi = \{j \in N : \pi(j) < \pi(i)\}$; belongs to $core(N, v, \succeq)$. Thus, we can state the following result.

Theorem 3.4 *If the game* $(N, v, \succeq)$ *is* $\succeq$*-convex, the* $core(N, v, \succeq) \neq \emptyset$.

4 The extended Shapley value

The analysis of the core of partially ordered cooperative games has shown that it may be empty (as it also happens in the standard case). This fact leads us to consider another kind of solution concept. In this occasion we look for point solutions rather that for solution sets. The use of a partial order $\succeq$ induced by $D_\aleph$ modifies the standard analysis of point solutions for this class of games. In this case, there are two different relations that play an important role. On the one hand, there is the '$=$' relation between elements of $\aleph$. On the second hand, there is the $\sim$ relation, $x \sim y$ iff $x \succeq y$ and $y \succeq x$. In the scalar case the natural order of the reals implies that given two reals x and y, they are equivalent in the order, i.e. $x \sim y$ if and only if $x = y$. In the general case this might not be the case. Nevertheless, from the preference induced by $D_\aleph$, if $x \sim y$ there is no reason to prefer x over y nor the other way around. In this sense, the order modifies our perception of the space identifying many points as equivalent. As for applications of the extended Shapley value, one can think of coalitions sharing stock options which payoffs are clearly random variables. Depending on the partial order considered on the space $L^1(\mathbb{R})$, each player will have a unique portfolio or a set of indifferent portfolios given by his extended value.

First of all, we note that the binary operator $\sim$ induces an equivalence relation on $\aleph$. Let us consider the set $[\sim] := \{x \in \aleph : x \sim \ominus_\aleph\}$. This set is a linear subspace of $\aleph$, therefore the quotient space $\aleph/[\sim]$ is also a linear space where the cone $D_{\aleph/[\sim]}$ induces a strict partial order, i.e. for any $x, y \in \aleph/[\sim]$, the following relation holds: $x \sim y$ iff $x = y$.

Now, looking for point solutions means that we are interested in functions φ defined on the set of all the partially ordered cooperative games on $\aleph$ such that for any $v \in PO^n(\aleph)$ we have $\varphi(v) \in \aleph^n$. The reader may notice that in fact, in the original space, these point solutions are sets of the form $\varphi(v) + [\sim]$.

Among all of those functions we consider the most widely accepted: the Shapley value. According to the usual definition given by Shapley (1953), we define the extended Shapley value of $v \in PO^n(\aleph)$ as the unique function $\varphi = (\varphi_1, \ldots, \varphi_n)$ such that:

$$\varphi_i[v] \sim \sum_{\substack{T \subseteq N \\ i \in N}} \frac{(t-1)!(n-t)!}{n!}[v(T) - v(T - \{i\})], \quad i = 1, \ldots, n. \tag{5}$$

Formula (5) coincides with the Shapley value for games defined on $\mathbb{R}$ when the linear space considered is the real line.

Example 4.1 Let us consider a vector valued game on $\aleph = \mathbb{R}^3$ with the partial order induced by the cone $D_\aleph = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 \geq 0, x_3 \geq 0\}$.

Let $(v, \{1, 2, 3\})$ be a three player partially ordered cooperative game with characteristic function given by:

$$\begin{array}{c|cccccccc} & \emptyset & \{1\} & \{2\} & \{3\} & \{1,2\} & \{1,3\} & \{2,3\} & N \\ \hline v & \begin{pmatrix}0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\end{pmatrix} & \begin{pmatrix}1\\-1\\0\end{pmatrix} & \begin{pmatrix}-2\\2\\0\end{pmatrix} & \begin{pmatrix}5\\5\\1/2\end{pmatrix} & \begin{pmatrix}6\\4\\1/3\end{pmatrix} & \begin{pmatrix}3\\-3\\1/4\end{pmatrix} & \begin{pmatrix}10\\0\\1\end{pmatrix} \end{array}$$

In this situation, we have that $[\sim] = \{x = (x_1, x_2, 0) \in \mathbb{R}^3 : x_1 + x_2 = 0\}$, and therefore, $\aleph/[\sim] = \mathbb{R}^2$ and $D_{\aleph/[\sim]} = (\mathbb{R}^+)^2$. The game $\bar{v}$ induced by v on $\aleph/[\sim]$ is given by:

$$\begin{array}{c|cccccccc} & \emptyset & \{1\} & \{2\} & \{3\} & \{1,2\} & \{1,3\} & \{2,3\} & N \\ \hline v & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}10\\1/2\end{pmatrix} & \begin{pmatrix}10\\1/3\end{pmatrix} & \begin{pmatrix}0\\1/4\end{pmatrix} & \begin{pmatrix}10\\1\end{pmatrix} \end{array}$$

Hence, the Shapley value of $\bar{v}$ is:

$$\varphi_1[\bar{v}] = \begin{pmatrix}20/3\\7/18\end{pmatrix}, \qquad \varphi_2[\bar{v}] = \begin{pmatrix}5/3\\25/72\end{pmatrix}, \qquad \varphi_3[\bar{v}] = \begin{pmatrix}5/3\\19/72\end{pmatrix}.$$

Finally, the extended Shapley value of v is:

$$\varphi_1[v] = \{x \in \mathbb{R}^3 : x_1 + x_2 = 20/3,\ x_3 = 7/18\},$$
$$\varphi_2[v] = \{x \in \mathbb{R}^3 : x_1 + x_2 = 5/3,\ x_3 = 25/72\},$$
$$\varphi_3[v] = \{x \in \mathbb{R}^3 : x_1 + x_2 = 5/3,\ x_3 = 19/72\}.$$

Example 3.1 (Continued) In this example, $[\sim] = \emptyset$. Therefore, $\aleph = \aleph/[\sim]$ and thus, the induced game coincides with the original one. The extended Shapley value is an element in $\aleph^3$ given by:

$$\varphi_1(v) = \begin{pmatrix}7/2 - t\\5/2\end{pmatrix}, \qquad \varphi_2(v) = \begin{pmatrix}9/2 + t/2\\21/4 - t/4\end{pmatrix}, \qquad \varphi_3(v) = \begin{pmatrix}2 + 5t/2\\17/4 + t/4\end{pmatrix}.$$

This is clearly an imputation of the game but it does not belong to $core(N, v, \succeq)$.

Remark 4.1 If the game $(N, v, \succeq)$ is $\succeq$-convex then formula (5) states the relationship between the extended Shapley value and the core of these games. Since the extended Shapley value is a convex combination of all marginal worth vectors (see (4) for a definition), it belongs to $core(N, v, \succeq)$. (See Theorem 3.4.)

In the following for the sake of readability and without loss of generality, we will denote the quotient space $\aleph/[\sim]$ as $\aleph$, and $D_{\aleph/[\sim]}$ as $D_\aleph$; thus assuming that the partial order is strict.

Originally Shapley in his paper (Shapley 1953) introduced his value axiomatically. The original proof makes use of the existence of a finite basis of the linear space of the n-player cooperative games: the unanimity games. In the class of games considered in this paper this kind of analysis is clearly not possible. In infinite dimension linear spaces the argument based on a basis can not be applied simply because there may not exist such a basis (neither finite nor infinite). Nevertheless, we justify the extended Shapley value using a different approach based on an extension of the concept of potentials (see Hart and Mas-Colell 1989). This approach leads us to two different characterizations that extend the ideas of preservation of differences and consistency as introduced in (Hart and Mas-Colell 1989). Finally, we also analyze how to extend Shapley's axioms to the case of partially ordered cooperative games. We show that for those spaces $\aleph$ where there exist Schauder bases it is still possible to define a new class of unanimity games that permit a characterization of the extended Shapley value with almost the same original Shapley's axioms.

Let $PO(\aleph)$ be the class of all the games (N, v) defined on $\aleph$. A function $P : PO(\aleph) \to \aleph$ with $P(\emptyset, v) = \ominus_\aleph$ is called an extended potential if it satisfies the equation:

$$\sum_{i \in N} P(N, v) - P(N \setminus \{i\}, v) \sim v(N), \tag{6}$$

for all games $(N, v) \in PO(\aleph)$. Usually, the difference $P(N, v) - P(N \setminus \{i\}, v)$ is called the marginal contribution of player i in the game and is abbreviated as:

$$D^i P(N, v) \sim P(N, v) - P(N \setminus \{i\}, v).$$

Once more, extended potentials reduce to usual potential in the sense of Hart and Mas-Colell (1989) when the linear space $\aleph$ equals $\mathbb{R}$.

First of all, it is straightforward to check that the expression (6) can be equivalently rewritten as:

$$P(N, v) \sim \frac{1}{n}\left[v(N) + \sum_{i \in N} P(N \setminus \{i\}, v)\right]. \tag{7}$$

The first theorem in this section proves the existence and uniqueness of the extended potential function. Moreover, its marginal contribution vector is the extended Shapley value.

Theorem 4.1 *There exists a unique potential function P whose expression for any game $(N, v) \in PO(\aleph)$ is*:

$$P(N, v) \sim \sum_{S \subseteq N} \frac{(s-1)!(n-s)!}{n!} v(S). \tag{8}$$

The resulting marginal contribution vector $(DP^i)_{i \in N}$ is the extended Shapley value of the game (N, v).

Proof The proof for existence and uniqueness of the extended potential function is similar to the one for the potential, although we include it for completeness. As usual, $|N| = n$ and $|S| = s$.

We can assume without loss of generality that the order $\succeq$ is strict. Otherwise we consider the quotient space. Starting with $P(\emptyset, v) = \ominus_\aleph$ and applying the formula (7) we can determine uniquely $P(N, v)$ for any v. Indeed, this is done applying recursively the formula to (S, v) for any $S \subset N$.

Now, it remains to prove that the potential has the expression given in (8). Notice that it suffices to prove that formula (8) satisfies the expression (7) that uniquely defines the potential. Replacing (8) into (7) we get:

$$\sum_{S \subseteq N} \frac{(s-1)!(n-s)!}{n!} v(S) = \frac{1}{n}\left\{v(N) + \sum_{i \in N}\left[\sum_{S \subset N\setminus\{i\}} \frac{(s-1)!(n-s)!}{(n-1)!} v(S)\right]\right\},$$

and this can be rewritten as,

$$\begin{aligned}
&\frac{v(N)(n-1)!}{n!} + \sum_{S \subset N} \frac{(s-1)!(n-s)!}{n!} v(S) \\
&\quad = \frac{v(N)}{n} + \frac{1}{n}\left[\sum_{S:1 \in S} \frac{(s-1)!(n-s)!}{(n-1)!} v(S) + \cdots + \sum_{S:n \in S} \frac{(s-1)!(n-s)!}{(n-1)!} v(S)\right].
\end{aligned}$$

Both sides of the equation above are equal. Thus, the formula (8) defines the potential function.

Finally, using this formula the marginal contribution vector $(D^i P)_{i\in N}$ is easily seen to satisfy $D^i P = \varphi_i$ for all $i \in N$. □

It is important to remark that there is a subtle difference in this derivation of Shapley value with respect to the original one by Hart and Mas-Colell (1989). In their original approach they reduce their construction to unanimity games to get an expression of the potential function in terms of the Harsanyi's dividend of all the coalitions. (See the proof of Theorem A in (Hart and Mas-Colell 1989).) That approach makes use of the existence of a basis of the linear space of the games (for a fixed number of players). Obviously this approach is avoided with the argument followed in the proof of Theorem 4.1 where no existence of any basis of $\aleph$ is required. In this regard, this proof seems to give more generality to the potential analysis of games: it can be applied not only to games with a continuum of players as already noted in (Hart and Mas-Colell 1989) but also to games where the payoffs are different from the reals.

We have proven the existence of a unique extended potential function whose marginal contribution vector coincides with the extended Shapley value in the class of partially ordered cooperative games. This result allows us to obtain the extended Shapley value using those characterizations based on potential. This is the case for the preservation of differences principle (see Hart and Mas-Colell 1989 for a discussion on this principle).

Let $x^i(S)$ be the payoff of player i in the game (S, v) for $i \in S$. Consider a set of differences $\{d^{ij}\} \subset \aleph$ where

$$d^{ij} = x^i(N \setminus \{j\}) - x^j(N \setminus \{i\}), \quad \text{for all } i, j.$$

The following theorem characterizes the extended Shapley value. Its proof follows the line of Theorem 3.4 in (Hart and Mas-Colell 1989) and is therefore omitted.

Theorem 4.2 *The extended Shapley value is the unique payoff vector* $(x^i)_{i\in N} \subset \aleph$ *that satisfies*:

1. $\sum_{i\in N} x^i(N) = v(N)$.
2. $x^i(N) - x^j(N) = d^{ij}$ *for all* i, j.

In the same way the extended Shapley value can also be characterized as the unique point solution φ that satisfies the properties of consistency with respect to the reduced game and is standard for two-person games. The interested reader is referred to the paper by Hart and Mas-Colell (1989) for the definitions of these properties. For the sake of brevity the proof and the details of these results are omitted.

4.1 Axiomatic approach to the extended Shapley value

In the sequel we address an axiomatic characterization of the extended Shapley value that resembles very much the original derivation given by Shapley (1953). In doing that we consider a set of three axioms *dummy player, symmetry, and linear-continuity*. Before to proceed we would like to point out some important remarks. First of all, the reader may have noticed that comparing with the standard Shapley value we have added a certain form of continuity. It will be clear later that it is crucial because we may have to deal with spaces of infinite dimension. Second, we have to mention the scope of this approach. This characterization uses the existence of a basis in the linear space where the game takes its values. Therefore, its applicability confines to those spaces having basis. In particular, it applies to all the spaces of finite dimension, any Hilbert space, separable spaces, spaces having Schauder basis (as for instance the spaces of continuous functions of any order), etcetera.

It is clear that this approach is more restrictive than the one based on potentials. On the other hand, it applies to the important class of vector valued games (Fernández et al. 2002a) (i.e. those games whose values are taken on $\mathbb{R}^m$). In addition, it is also interesting because it shows us what unanimity games mean in the class of partially ordered cooperative games.

Let $v \in PO^n(\aleph)$ be an n-person partially ordered cooperative game defined on $\aleph$. The game (N, v) can be identified with a vector $X \in \aleph^{2^n-1}$, $X = (X_{\{1\}}, X_{\{2\}}, \ldots, X_S, \ldots, X_N)$ such that $v(S) = X_S \in \aleph$. It is clear that $PO^n(\aleph)$ has structure of linear space over the reals. Assume that $\aleph$ admits a basis $\mathcal{B} = \{p^i\}_{i\in I}$. Therefore for any $x \in \aleph$ there exists a unique sequence λ^i, $i \in I$ (possibly infinite) such that $x = \sum_{i\in I} \lambda^i p^i$.

For any $p^i \in \mathcal{B}$ and $S \subset N$, let the unanimity game $w_S^{p^i}$ be defined by

$$w_S^{p^i}(T) = \begin{cases} \ominus_\aleph & \text{if } S \not\subset T, \\ p^i & \text{if } S \subseteq T. \end{cases}$$

Remark 4.2 Notice that unanimity games must be a basis of $PO^n(\aleph)$. Thus, their values must be the elements of the basis $\mathcal{B}$. This is an important difference with respect to standard case where the games take value on the reals. The usual unanimity games take values 0, 1 since a basis for $\mathbb{R}$ is 1. It is also interesting to point out that the standard cooperative games take value on the linear space $\mathbb{R}$ over the field $\mathbb{R}$. (The linear space coincides with the field over which it is defined. This is a very particular structure.)

Axioms. Let φ be a value and (N, u), (N, v) two partially ordered cooperative games.

A1. *Dummy player.*

$$\sum_{i\in S} \varphi_i[v] \sim v(S),$$

for any S such that $v(S) = v(S \cap T)$ for all $T \subseteq N$. (S is a carrier for v.)

A2. *Symmetry.* For any permutation π, and $i \in N$,

$$\varphi_{\pi(i)}[\pi v] \sim \varphi_i[v],$$

where the game πv means the game u such that, for any $S = \{i_1, \ldots, i_S\}$, $u(\{\pi(i_1), \ldots, \pi(i_S)\}) = v(S)$.

A3. *Linearity.*

$$\varphi_i[u + v] \sim \varphi_i[u] + \varphi_i[v].$$

A4. *Continuity.* φ_i must be continuous, i.e. for any sequence $(x^n)_{n\geq 1} \subset \aleph$ it holds

$$\varphi_i\left(\lim_{n\to\infty} x^n\right) = \lim_{n\to\infty} \varphi_i(x^n).$$

The reader may notice that axioms A3 and A4 imply the following new axiom:

A3′ *Linear-Continuity.* Let $(u^n)_{n\in\mathbb{N}} \subset PO^n(\aleph)$ be a denumerable sequence of partially ordered cooperative games then

$$\varphi_i\left(\sum_{n\in\mathbb{N}} u^n\right) = \sum_{n\in\mathbb{N}} \varphi_i(u^n).$$

Remark 4.3 We assume that the games are normalized with respect to the cone cone($\mathcal{B}$), induced by the positive linear combinations of the elements of the basis of $\aleph$. This means that $v(\{i\}) \notin -\text{cone}(\mathcal{B})$ for any $i \in N$. Notice that it does not mean loss of generality since we can translate all the payoffs by a fix vector maintaining the ordering relationships among them.

Theorem 4.3 *The extended Shapley value is the unique value defined on all the partially ordered cooperative games satisfying Axioms* A1, A2, *and* A3′.

Proof It is clear that if S is a carrier for $w_S^{p^j}$ then any superset T such that $S \subseteq T$ is also a carrier. Thus, by axiom A1 we get that $\sum_{i\in T} \varphi_i[w_S^{p^j}] = p^j$ for any $S \subseteq T$. This fact together with Remark 4.3 implies that $\varphi_i[w_S^{p^j}] = \ominus_\aleph$ for $i \notin S$. Now, by using A2 and the linearity implied by A3′ we can deduce that for any $c > 0$

$$\varphi_i[cw_S^{p^j}] = \begin{cases} \frac{cp^j}{s} & \text{if } i \in S,\\ \ominus_\aleph & \text{otherwise.}\end{cases}$$

Let $p_S^i = (\ominus_\aleph, \dots, \overset{S}{p^i}, \dots, \ominus_\aleph)^t \in \aleph^{2^n-1}$, it is clear that since $\mathcal{B}$ is a basis of $\aleph$ then $\bar{\mathcal{B}} = (p_S^i)_{i\in I, S\subseteq N}$ is a basis of $\aleph^{2^n-1}$.

For any game $(N, v) \in PO^n(\aleph)$, let $(d_S^{p^i})_{i\in I, S\subseteq N}$ be the unique set of scalar in the representation of v in the basis $\bar{\mathcal{B}}$. This is,

$$v = \sum_{p^i\in\mathcal{B}} \sum_{S\subseteq N} d_S^{p^i} p_S^i.$$

Let $U \subset N$ be any coalition, then

$$v(U) = \sum_{p^i} \sum_{T\subseteq U} d_T^{p^i} p^i = \sum_{p^i} \sum_{T\subseteq U} \Bigg(\sum_{\substack{S\subseteq U\\ S\supseteq T}} (-1)^{s-t} \Bigg) d_T^{p^i} p^i,$$

$$= \sum_{p^i} \sum_{S\subseteq U} \Bigg(\sum_{T\subseteq S} (-1)^{s-t} d_T^{p^i} \Bigg) p^i$$

$$\Bigg(\text{let } c_S^{p^i} = \sum_{T\subseteq S} (-1)^{s-t} d_T^{p^i} \Bigg) = \sum_{p^i} \sum_{S\subseteq U} c_S^{p^i} p^i = \sum_{p^i} \sum_{S\subseteq N} c_S^{p^i} w_S^{p^i}(U).$$

Therefore, using axiom A3′ we have

$$\varphi_j[v] = \sum_{p^i} \sum_{S\subseteq N} c_S^{p^i} \varphi_j[w_S^{p^i}] = \sum_{p^i} \sum_{\substack{S\subseteq N\\ j\in S}} c_S^{p^i} \frac{p^i}{s}$$

$$= \sum_{p^i} \sum_{\substack{S\subseteq N\\ j\in S}} \Bigg(\sum_{T\subseteq S} (-1)^{s-t} d_T^{p^i} \Bigg) \frac{p^i}{s} = \sum_{p^i} \sum_{T\subseteq N} \Bigg(\sum_{\substack{S\subseteq N\\ T\cup\{j\}\subseteq S}} (-1)^{s-t} \frac{d_T^{p^i}}{s} \Bigg) p^i$$

$$\left(\text{let } \gamma_j(T) = \sum_{\substack{S \subseteq N \\ T \cup \{j\} \subseteq S}} (-1)^{s-t} \frac{1}{s}\right) = \sum_{p^i} p^i \sum_{\substack{T \subseteq N \\ j \in T}} \gamma_j(T)[d_T^{p^i} - d_{T\setminus\{j\}}^{p^i}]$$

$$\left(\text{since } \gamma_j(T) = \frac{(t-1)!(n-t)!}{n!}\right) = \sum_{\substack{T \subseteq N \\ j \in T}} \frac{(t-1)!(n-t)!}{n!} \sum_{p^i} [d_T^{p^i} - d_{T\setminus\{j\}}^{p^i}] p^i$$

$$= \sum_{\substack{T \subseteq N \\ j \in T}} \frac{(t-1)!(n-t)!}{n!} (v(T) - v(T \setminus \{j\})).$$

Therefore, the extended Shapley value is uniquely characterized by axioms A1, A2, and A3′. □

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