A game-theoretic approach for downgrading the 1-median in the plane with Manhattan metric*

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Abstract

This paper deals with downgrading the 1-median, i.e., changing values of parameters within certain bounds such that the optimal objective value of the location problem with respect to the new values is maximized. We suggest a game-theoretic view at this problem which leads to a characterization of an optimal solution. This approach is demonstrated by means of the Downgrading 1-median problem in the plane with Manhattan metric and develop an $\mathcal{O}(n \log^2 n)$ time algorithm for this problem.

1 Introduction

This paper deals with a downgrading approach for the 1-median problem in the plane with Manhattan metric. Downgrading means that values of the parameters have to be changed such that the optimal objective value of the underlying location problem is maximized.

We suggest a game-theoretic interpretation of the problem which is applicable for many other downgrading problems. For the special case of downgrading the 1-median problem in the plane with Manhattan metric this approach leads to an $\mathcal{O}(n \log^2 n)$ time algorithm.

Location problems deal with finding an optimal location for a facility. The most well studied version is to minimize the total travel cost, i.e., the sum of weighted distances between clients and the facility. However, in practice the real demand of the clients may be unknown, i.e., for each client we are given a minimal and maximal demand and in addition information

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about an upper bound on the total demand of all clients is available. Then the following natural question occurs: What is the worst cast, i.e., maximize the total travel cost. As soon as the real demands are known, the facility will be located in an optimal way with respect to the fixed demands. Hence, the task is to find demands such that the total travel cost is largest possible even if the facility is located in an optimal way. Mathematically spoken, we have to maximize the optimal objective value of the location problem. The above mentioned application can be modelled in the following way: Each client has an initial demand (the lower bound) and the task is to distribute the remaining total demand among the clients such that their upper bound constraints are satisfied and the optimal objective value of the 1-median problem with respect to the new demands is maximized.

In this paper, the downgrading approach is applied to the 1-median problem in the plane with Manhattan metric, i.e., let $P_i = (x_i, y_i)$ $(i = 1, 2)$ be two points then the distance between these two points is equal to $d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$. Larson and Li [15] considered several applications where the Manhattan metric is used to measure the distance, e.g., urban transportation, plant and facility layout, locating power lines or the design of printed circuit boards.

The downgrading problem can be seen as a game where one player corresponds to the location planner while the second play is an adversary who seeks to maximize the cost of the location planner. Hence, the problem is a special max-min problem.

In contrast to downgrading problems, upgrading problem deal with minimizing the optimal objective value, i.e., the upgrading problem is a min-min problem. These two approaches of up- and downgrading have already been applied to several classical optimization problems like 1-center and 1-median problems in networks (Gassner [7], [8]), path problems (Fulkerson and Harding [6] and Hambrusch and Tu [11]), network flows (Phillips [17]), spanning trees and Steiner trees (Frederickson and Solis-Oba [5], Drangmeister et al. [4] and Krumke et al. [14]) and general 0/1-combinatorial optimization problems (Burkard, Klinz and Zhang [2] and Burkard, Lin and Zhang [3]).

The above algorithms for the above mentioned downgrading problems are mainly based on the special structure of the underlying optimization problem or use duality properties. In this paper we use a game-theoretic approach in order to characterize an optimal solution as Nash equilibrium of a two-person game. We then get properties of an optimal solution that are used to solve the downgrading problem.

In Subsection 1.1 the problem of downgrading the 1-median is defined in a formal way and in Subsection 1.2 some notation is introduced. Section

2 deals with a reformulation of the downgrading 1-median problem in the plane. In Section 3 we consider the special case where distances are measured by the Manhattan metric. For this case an $\mathcal{O}(n \log^2 n)$ time algorithm is suggested. Finally, in the Conclusion some remarks on related problems as well as on the corresponding upgrading 1-median problem are given.

1.1 Problem definition

The 1-median problem in the plane is given by a set $\mathcal{P} = \{P_1, \ldots, P_n\}$ of points with $P_i \in \mathbb{R}^2$ (for $i = 1, \ldots, n$) and a weight vector $w = (w_1, \ldots, w_n)$ with $w_i \in \mathbb{R}_+$. The task is to find a point $P \in \mathbb{R}^2$ such that

$$f(P) = \sum_{i=1}^{n} w_i d(P_i, P)$$

is minimized where $d(P_i, P)$ denotes the distance (e. g., Euclidean metric, Tchebychev metric or Manhattan metric) between P_i and P. If the distance is induced by an ℓ_p-norm then there exists an optimal solution P_0 that lies in the convex hull $\mathcal{C}(\mathcal{P})$ of $\mathcal{P}$ (see Juel and Love [12]). Hence, the 1-median problem is to solve

$$\min_{P \in \mathcal{C}(\mathcal{P})} f(P) = \sum_{i=1}^{n} w_i d(P_i, P).$$

The task of downgrading the 1-median is to increase the weights of the points such that the optimal 1-median objective value with respect to the modified weights is maximized. The downgrading version considered in this paper allows to change the weights within certain bounds. A weight modification $\delta = (\delta_i)_{P_i \in \mathcal{P}}$ is called feasible if a budget constraint is met and the modifications are within certain bounds: Let $c_i \in \mathbb{R}_+$ for $i = 1, \ldots, n$ denote the cost of changing the weight w_i by one unit and let $u_i \in \mathbb{R}_+$ for $i = 1, \ldots, n$ be an upper bound for the modification of w_i. Moreover, we are given a total budget B. Then δ is feasible if $\delta \in \Delta$ with

$$\Delta = \left\{ \delta \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^{n} c_i \delta_i \leq B \text{ and } 0 \leq \delta_i \leq u_i \text{ for all } i = 1, \ldots, n \right\}.$$

Let $\delta \in \Delta$ be a feasible weight modification and let $P \in \mathbb{R}^2$ be a point in the plane. Then

$$f_\delta(P) = \sum_{i=1}^{n} (w_i + \delta_i) d(P_i, P)$$

denotes the objective value of P with respect to weight vector $w + \delta$, i.e., after weight modification δ.

The downgrading 1-median problem in the plane is to find a feasible weight modification $\delta \in \Delta$ such that the optimal 1-median objective value for weight vector $w + \delta$

$$\min_{P \in \mathcal{C}(\mathcal{P})} f_\delta(P) = \min_{P \in \mathcal{C}(\mathcal{P})} \sum_{i=1}^{n} (w_i + \delta_i) d(P_i, P)$$

is maximized. Hence, the downgrading problem is to solve

$$\max_{\delta \in \Delta} \min_{P \in \mathcal{C}(\mathcal{P})} f_\delta(P).$$

1.2 Notation

Throughout this paper we will use the following notation:

Let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be a set of points with $P_i = (x_i, y_i)$ for $i = 1, \ldots, n$ then

$$V = \{(x_i, y_j) \mid i = 1, \ldots, n \text{ and } j = 1, \ldots, n\}$$

is called set of vertices. Observe that $\mathcal{P} \subseteq V$ and $|V| = \mathcal{O}(n^2)$.

Two vertices $Q = (x, y) \in V$ and $Q' = (x', y') \in V$ with $x < x'$ are called horizontally adjacent if $y = y'$ and there is no vertex $Q'' = (x'', y'') \in V$ with $x < x'' < x'$. Vertical adjacency is defined in an analogous way. Clearly, every vertex is horizontally (vertically) adjacent to one or two vertices. The set of points on the line segment between two adjacent vertices Q and Q' is denoted by $[Q, Q']$. The interior of $[Q, Q']$ is denoted by (Q, Q').

Every point $P = (x, y) \in \mathbb{R}^2$ defines four halfplanes:

$$\begin{aligned}
A^+(P) &= \{(\tilde{x}, \tilde{y}) \in \mathbb{R}^2 \mid \tilde{x} > x\} \\
A^-(P) &= \{(\tilde{x}, \tilde{y}) \in \mathbb{R}^2 \mid \tilde{x} < x\} \\
B^+(P) &= \{(\tilde{x}, \tilde{y}) \in \mathbb{R}^2 \mid \tilde{y} > y\} \\
B^-(P) &= \{(\tilde{x}, \tilde{y}) \in \mathbb{R}^2 \mid \tilde{y} < y\}
\end{aligned}$$

Let $X \subseteq \mathbb{R}^2$ then $w(X) = \sum_{P_i \in X \cap \mathcal{P}} w_i$ denotes the sum of weights of points in X. If $\delta \in \Delta$ then $(w + \delta)(X) = \sum_{P_i \in X \cap \mathcal{P}} (w_i + \delta_i)$. Finally, let $X, Y \subseteq \mathbb{R}^2$ then we define $X - Y = X \setminus Y$ and $w(X + \alpha Y) = w(X) + \alpha w(Y)$ for $\alpha \in \mathbb{R}$.

2 From Max-Min to Min-Max

In this section, we will reformulate the downgrading problem from a max-min problem to a min-max problem. Recall that the downgrading problem is to solve

$$\max_{\delta\in\Delta} \min_{P\in\mathcal{C}(\mathcal{P})} f_\delta(P).$$

Let $X \subset \mathbb{R}^2$ be a closed and convex set. Then a tuple (δ^*, P^*) with $P^* \in X$ and $\delta^* \in \Delta$ is called saddle-point in X if

$$f_\delta(P^*) \leq f_{\delta^*}(P^*) \leq f_{\delta^*}(P)$$

holds for all $\delta \in \Delta$ and $P \in X$. It is well-known that

$$\max_{\delta\in\Delta} \min_{P\in X} f_\delta(P) = \min_{P\in X} \max_{\delta\in\Delta} f_\delta(P) = f_{\delta^*}(P^*)$$

holds if and only if (δ^*, P^*) is a saddle-point in X (e. g., see the book of Rockafellar, Part VII [18]).

The existence of a saddle-point can be shown by using the famous min-max Theorem of Kakutani [13]: Observe that $\Delta \subset \mathbb{R}^n$ and $X \subset \mathbb{R}^2$ are closed and convex. Moreover, if $P \in X$ is fixed then $f_\delta(P) = \sum_{i=1}^n (w_i+\delta_i)d(P_i, P)$ is an affine-linear and therefore concave function in δ. Finally, we fix $\delta \in \Delta$. Since the distance $d(P_i, P)$ is a convex function in P also $(w_i + \delta_i)d(P_i, P)$ is convex for every fixed δ and herewith $f_\delta(P)$ is a convex function in P if $\delta \in \Delta$ is fixed. These assumptions guarantee the existence of a saddle-point in X.

Hence, we know that there exists a saddle-point (δ^*, P^*) in X with $\delta^* \in \Delta$ and $P^* \in X \subseteq \mathcal{C}(\mathcal{P})$ such that

$$\max_{\delta\in\Delta} \min_{P\in X} f_\delta(P) = \min_{P\in X} \max_{\delta\in\Delta} f_\delta(P) = f_{\delta^*}(P^*).$$

This implies for $X = \mathcal{C}(\mathcal{P})$ that there exists an optimal solution $\delta^* \in \Delta$ of the downgrading problem and a point $P^* \in \mathcal{C}(\mathcal{P})$ such that (δ^*, P^*) is a saddle-point. Therefore, we get the following theorem:

Theorem 2.1.

1. *If (δ^*, P^*) is a saddle-point in $\mathcal{C}(\mathcal{P})$ then δ^* is an optimal weight modification of Downgrading the 1-median in the plane.*

2. *There exists a saddle-point in every convex subset $X \subseteq \mathcal{C}(\mathcal{P})$.*

3. *(δ^*, P^*) is a saddle-point in X if and only if δ^* maximizes the 1-median objective value of P^* and P^* is 1-median with respect to $w + \delta^*$.*

The idea is to use the characterization of a saddle-point in order to find an optimal solution of Down1Med.

The above reformulation has the following game-theoretic interpretation: We are given a zero-sum game with two players. The set of strategies for player 1 is equal to Δ while the set of strategies for player 2 corresponds to $\mathcal{C}(\mathcal{P})$ (= set of potential locations). Let (δ^*, P^*) with $\delta^* \in \Delta$ and $P^* \in \mathcal{C}(\mathcal{P})$ be the strategies of the two players. If (δ^*, P^*) is a saddle-point then $f_{\delta^*}(P^*) = \max_{\delta \in \Delta} \min_{P \in \mathcal{C}(\mathcal{P})} f_\delta(P)$ holds. This means that $f_{\delta^*}(P^*)$ is the guaranteed minimum profit for player 1. With similar arguments we get, that $f_{\delta^*}(P^*)$ is the guaranteed maximum loss of player 2. Since the minimum profit and maximum loss are equal, there is no reason for one player to change his strategy. In terms of game theory, (δ^*, P^*) is a Nash equilibrium and δ^* is an optimal weight modification of Down1Med.

Observe that if (δ^*, P^*) is a saddle-point then P^* is in general not a vertex. This reflects the game-theoretic fact that in general there is no Nash equilibrium consisting of pure strategies.

Finally, let us consider the following subproblem: Fix a point $P \in \mathbb{R}^2$ and determine the maximum objective value for P that can be achieved by a feasible weight modification, i.e.,

$$\text{Max}(P): \qquad h(P) := \max_{\delta \in \Delta} \; f_\delta(P).$$

Observe that $\text{Max}(P)$ is a continuous knapsack problem that can be solved in linear time (Balas and Zemel [1]). Moreover, if (δ^*, P^*) is a saddle-point in X then

$$\max_{\delta \in \Delta} \min_{P \in X} f_\delta(P) = \min_{P \in X} \max_{\delta \in \Delta} f_\delta(P) = \min_{P \in X} h(P) = h(P^*).$$

Since $\text{Max}(P)$ is solvable efficiently, our algorithm is mainly based on solving $\text{Max}(P)$ for several points P. As soon as $h(P) < h(P')$ holds for two points P and P' we know that there is no saddle-point of the form (δ', P').

3 Downgrading the 1-median in the plane with Manhattan metric

In this section we consider the 1-median problem in the plane with Manhattan metric, i.e., let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ then

$$d(P, Q) = |x_1 - x_2| + |y_1 - y_2|.$$

The problem of downgrading the 1-median in the plane if the distances are measured by the Manhattan metric is called Down1Med for short.

The 1-median problem in the plane with Manhattan metric is called 1Med. The objective function of 1Med can be written as

$$f(P) = \sum_{i=1}^{n} w_i |x_i - x| + \sum_{i=1}^{n} w_i |y_i - y|$$

where $P = (x, y)$ and hence each sum can be minimized independently. Each of these two subproblems can be interpreted as 1-median problem on a path. Due to the well-known optimality criterion of Goldman [9] for the 1-median problem on trees, we get the following optimality criterion for 1Med:

Theorem 3.1. *Let $(\mathcal{P}, \tilde{w})$ be an instance of 1Med. Then $P_0 = (x, y)$ is an optimal solution of 1Med, i.e., a 1-median if and only if the following conditions hold:*

$$\tilde{w}(A^-(P_0)) \leq \frac{1}{2}\tilde{w}(\mathbb{R}^2) \tag{1}$$

$$\tilde{w}(A^+(P_0)) \leq \frac{1}{2}\tilde{w}(\mathbb{R}^2) \tag{2}$$

$$\tilde{w}(B^+(P_0)) \leq \frac{1}{2}\tilde{w}(\mathbb{R}^2) \tag{3}$$

$$\tilde{w}(B^-(P_0)) \leq \frac{1}{2}\tilde{w}(\mathbb{R}^2) \tag{4}$$

The previous theorem means that P_0 is 1-median if the total weight to the right (left) of P_0 is at most half of the total weight and the total weight above (below) P_0 is at most half of the total weight. Moreover, the theorem immediately implies that there exists an optimal solution among the vertices, i.e., $P_0 \in V$.

Now we are interested in objective values of two points P' and P''. Let $P' = (x', y')$ and $P'' = (x'', y')$ be two vertices with $x' \leq x''$ then

$$d(P, P') = \begin{cases} d(P, P'') + |x' - x''| & \text{if } P \in \mathbb{R}^2 - A^-(P'') \\ d(P, P'') - |x' - x''| & \text{if } P \in \mathbb{R}^2 - A^+(P') \subseteq A^-(P'') \\ d(P, P'') + d((x, y'), P') - d((x, y'), P'') & \text{if } P \in A^+(P') \cap A^-(P'') \end{cases}$$

holds for every $P = (x, y) \in \mathbb{R}^2$ and hence

$$d(P, P') \geq \begin{cases} d(P, P'') + |x' - x''| & \text{if } P \in \mathbb{R}^2 - A^-(P'') \\ d(P, P'') - |x' - x''| & \text{if } P \in A^-(P''). \end{cases}$$

Observe that we have equality if P' and P'' are horizontally adjacent. This observation immediately implies

$$f_\delta(P') \geq f_\delta(P'') + |x' - x''|(w + \delta)(\mathbb{R}^2 - 2A^-(P'')). \tag{5}$$

An analogous result holds for two vertices on a vertical line.

3.1 A first algorithm

Assume that we already knew which point $P \in \mathcal{C}(\mathcal{P})$ is a 1-median after an optimal weight modification $\delta^* \in \Delta$. Then the optimal strategy would be to maximize the objective value of P provided that P is 1-median after the modification, i.e.,

$$\text{Max\_Median}(P): \qquad g(P) := \qquad \max_{\delta \in \Delta} f_\delta(P)$$
$$\text{s.t.} \quad (1) \text{ to } (4) \text{ hold for } \tilde{w} = w + \delta.$$

Observe that Max_Median(P) maximizes the objective value of P and makes sure that P is a 1-median at the end while Max(P) only maximizes the objective value without guarantee that P is a 1-median. If δ^* is an optimal solution of Max(P) then P is in general not 1-median with respect to $w + \delta^*$. This observation holds for all vertices $P \in V$.

Unfortunately, we do not know the optimal point P a priori. However, since we know that there exists a 1-median among the set of vertices, we can solve Max_Median(P) for all vertices P and hence, Down1Med is equivalent to

$$\max_{Q \in V} \; g(Q).$$

Observe that $g(Q)$ can be determined in linear time because there are only five constraints in addition to the upper bound constraints [16]. Hence, Down1Med can be solved in $\mathcal{O}(n^3)$ time if $g(Q)$ is determined from scratch for every vertex. The drawback of this algorithm is the huge number of points for which Max_Median(P) has to be solved.

In the remaining of this paper we will, however, describe a more efficient algorithm that runs in $\mathcal{O}(n \log^2 n)$ time. The idea is to prune several vertices until the set of remaining potentially optimal points is small. Finally, Max_Median(P) is solved only for the small set of remaining vertices.

3.2 An efficient algorithm

The task of this subsection is to find a procedure that prunes a large number of vertices that are known to be not optimal. The procedure is based on the idea that there exists an optimal solution of Down1Med that implies a saddle-point (δ^*, P^*) in $\mathcal{C}(\mathcal{P})$. Observe that we are only allowed to apply Theorem 2.1 if we consider the whole (continuous) set $\mathcal{C}(\mathcal{P})$ because Theorem 2.1 does not hold for the discrete set V.

The main idea of the presented efficient algorithm is to successively reduce the search space for P^*. Let y' be the median among the y-coordinates of the vertices and let L' be the horizontal line through $(x', y') \in V$. Then the algorithm decides whether the upper or lower halfplane defined by L' can be pruned, i.e., there exists a saddle-point (δ^*, P^*) such that P^* lies in the unpruned region. Afterwards let x'' be the median among the x-coordinates of the remaining vertices and let L'' be the vertical line through $(x'', y'') \in V$. Again it can be decided whether the right or left halfplane defined by L'' can be pruned. Hence, after this step the number of x- and y-coordinates of unpruned vertices is halved. This procedure is repeated until either an optimal solution is already found or a singe cell, i.e., a rectangle that does not contain any vertex, remains. Since it is known that there exists a saddle-point (δ^*, P^*) that lies in the remaining cell, it follows that one of the corners must be a 1-median after an optimal weight modification. Therefore, $\max_{P \text{ is corner}} g(P)$ is an optimal solution of Down1Med. As described above, the main step is to decide which halfplane defined by a horizontal (vertical) line L has to be pruned. In order to make this decision, we solve a series of subproblems Max(P) for points P that lie on L.

Assume that we are given a line L and a point $P \in L$. If δ^* is an optimal solution of Max(P) such that (1) to (4) hold in point P for $w + \delta^*$ then δ^* is an optimal solution of Down1Med because (δ^*, P) is a saddle-point. The next Lemma states that there is still some information on the location of an optimal point P^* if only a part of the weight conditions is satisfied:

Lemma 3.2. *Let L be a horizontal line and let $P^* \in L$ with an optimal solution δ^* of Max(P^*) such that* (1) *and* (2) *hold in P^* for weight vector $w + \delta^*$. If condition* (3) *(*(4)*) is violated then $h(P) > h(P^*)$ holds for all points $P = (x, y)$ with $y < y^*$ ($y > y^*$).*

Observe that if $h(P) > h(P^*)$ holds for all $y < y^*$ then there is no saddle-point (δ, P) such that P lies below P^* and therefore we can prune a halfplane. Let us first prove the lemma:

Proof. Consider a point $P = (x, y)$ with $y < y^*$ and $x \leq x^*$. Since

$(w+\delta^*)(B^+(P^*)) > \frac{1}{2}(w+\delta^*)(\mathbb{R}^2)$ (because (3) is violated) and $B^-(P^*) \cap B^+(P^*) = \emptyset$ we get

$$(w+\delta^*)(B^-(P^*)) < \frac{1}{2}(w+\delta^*)(\mathbb{R}^2).$$

Moreover, conditions (1) and (2) imply

$$\begin{aligned}(w+\delta^*)(A^+(P^*)) &\leq \frac{1}{2}(w+\delta^*)(\mathbb{R}^2) \text{ and}\\ (w+\delta^*)(A^-(P^*)) &\leq \frac{1}{2}(w+\delta^*)(\mathbb{R}^2).\end{aligned}$$

Together with (5), we get

$$\begin{aligned}h(P) &\geq f_{\delta^*}(P)\\ &\geq f_{\delta^*}(P^*)+\\ &\qquad |x-x^*|\underbrace{(w+\delta^*)(\mathbb{R}^2-2A^-(P^*))}_{\geq 0}+|y-y^*|\underbrace{(w+\delta^*)(\mathbb{R}^2-2B^-(P^*))}_{>0}\\ &> f_{\delta^*}(P^*) = h(P^*).\end{aligned}$$

If $x > x^*$ then we have to replace $A^-(P^*)$ by $A^+(P^*)$. All further inequalities also hold true for this case. □

Observe that if we are given a point $P^* \in L$ together with an associated optimal solution δ^* of $\mathrm{Max}(P^*)$ that satisfies (1) and (2) then either

- (3) and (4) are also satisfied and hence we have found an optimal solution of Down1Med, or
- (3) is not satisfied and hence we can prune all points below P^*, or
- (4) is not satisfied and we can prune all points above P^*.

The previous lemma describes how to decide which halfplane can be pruned. However, in order to apply this lemma, we have to find a point $P^* \in L$ such that its associated optimal solution of $\mathrm{Max}(P^*)$ satisfies (1) and (2). Since we ignore the weight conditions in y-direction the obtained subproblem is 1-dimensional. Consider the following subproblem (restricted to L):

$$\min_{P \in L \cap \mathcal{C}(\mathcal{P})} \max_{\delta \in \Delta} f_\delta(P).$$

Since $L \cap \mathcal{C}(\mathcal{P})$ is a closed and convex set, Theorem 2.1 can be applied for $X = L \cap \mathcal{C}(\mathcal{P})$. The existence of a saddle-point (δ^*, P^*) in L is guaranteed and if (δ^*, P^*) is a saddle-point in L then δ^* is an optimal solution of $\mathrm{Max}(P^*)$ and P^* is 1-median restricted to L with respect to $(w + \delta^*)$. This latter condition, however, is equivalent to the fact that (1) and (2) hold for $w + \delta^*$ in point P^*. Hence, we first search for a saddle-point in L and then apply Lemma 3.2 to this point.

The task is therefore to find a saddle-point in L. A necessary condition for this saddle-point is $h(P^*) = \min_{P \in L \cap \mathcal{C}(\mathcal{P})} h(P)$.

Assume that we fix a point P^* on L but the determined optimal solution δ^* of $\mathrm{Max}(P^*)$ violates condition (1). Then the following Lemma states that there exists a saddle-point in L to the left of P^*.

Lemma 3.3. *Let L be a horizontal line. Moreover, let $P^* \in L$ and δ^* an optimal solution of Max(P^*) such that (1) ((2)) is violated. Then $h(P) > h(P^*)$ holds for all points $P = (x, y^*) \in L$ with $x > x^*$ $(x < x^*)$.*

The proof of Lemma 3.3 is similar to that of Lemma 3.2 and therefore omitted.

This observation immediately leads to binary search in the set of vertices in L. Consider a vertex P on L and solve $\mathrm{Max}(P)$. Either (1) and (2) are satisfied (in this case Lemma 3.2 can be applied to this point) or we can prune a half-ray of L (cf. Lemma 3.3) and continue with the unpruned vertices on L. Unfortunately, in general there does not exist a saddle-point (δ^*, P^*) in L such that P^* is a vertex. Hence, after at most $\mathcal{O}(\log n)$ applications of Lemma 3.3 either we find a vertex $P^* \in L$ whose optimal solution δ^* of $\mathrm{Max}(P^*)$ satisfies (1) and (2) or we end up with two adjacent vertices $[P', P'']$ such that $h(P) > h(P')$ for all points P on L to the left of P' and $h(P) > h(P'')$ for all points P on L to the right of P''. In this case there exists a saddle-point (δ^*, P^*) in L with $P^* \in [P', P'']$. Such an edge is called critical edge. In the next lemma we characterize this latter situation.

Lemma 3.4. *Let $[P', P'']$ be a critical edge on a horizontal line L. Then there exists a point $P^* \in [P', P'']$ with associated optimal solution δ^* of Max(P^*) such that*

$$(w + \delta^*)(A^+(P')) = (w + \delta^*)(A^-(P'')).$$

This lemma means that there exists a saddle-point (δ^*, P^*) in L such that the total weight with respect to $w + \delta^*$ to the right of P' is equal to the total weight to the left of P''.

Proof. Since all points $P \in L$ with $P \notin [P', P'']$ satisfy $h(P) > h(P')$ or $h(P) > h(P'')$ we know that there exists a saddle-point (δ^*, P^*) in L with $P^* \in [P', P'']$.

1. Case: $P^* = (x^*, y^*) \in (P', P'')$:
   Observe that there is no vertex $P = (x, y) \in V$ with $x = x^*$. Hence, Together with the property that (1) and (2) are satisfied for $w + \delta^*$ in P^* we immediately get the desired result.

2. Case: $P^* \in \{P', P''\}$:
   Recall that $[P', P'']$ is a critical edge because there exists an optimal solution δ' of $\mathrm{Max}(P')$ with
   $$(w + \delta')(A^+(P')) > \frac{1}{2}(w + \delta')(\mathbb{R}^2).$$
   On the other hand, there exists an optimal solution δ'' of $\mathrm{Max}(P'')$ with
   $$(w + \delta'')(A^-(P'')) > \frac{1}{2}(w + \delta'')(\mathbb{R}^2).$$
   Assume without loss of generality that $P^* = P'$. Then there exists an optimal solution $\tilde{\delta}$ of $\mathrm{Max}(P')$ with
   $$(w + \tilde{\delta})(A^+(P')) \leq \frac{1}{2}(w + \tilde{\delta})(\mathbb{R}^2). \tag{6}$$
   It is possible to find a convex combination $\xi = \alpha\delta' + (1 - \alpha)\tilde{\delta}$ with $0 \leq \alpha < 1$ such that ξ is again an optimal solution of $\mathrm{Max}(P')$ and (6) is satisfied with equality. □

Lemma 3.4 is used to find a saddle point (δ^*, P^*) in L with $P^* \in [P', P'']$. In fact we are not interested in the critical point P^* itself but only in its corresponding optimal solution δ^* of $\mathrm{Max}(P^*)$ because we would like to apply Lemma 3.2 in order to prune the upper or lower halfplane. In order to find an optimal weight modification δ^* of a saddle-point restricted to L, we define the following problem:

$$\begin{aligned} \mathrm{Max}^*(P): \qquad & \max_{\lambda \in \Delta} \; f_\lambda(P) \\ & \text{s.t.} \quad (w + \lambda)(A^+(P)) = (w + \lambda)(A^-(P)) \end{aligned}$$

The following lemma proves that $\mathrm{Max}^*(P)$ determines the desired solution δ^*.

Lemma 3.5. *Let $[P', P'']$ be a critical edge on the horizontal line L and let $\bar{P} \in (P', P'')$ be an arbitrary point in the interior of the edge where $\bar{\lambda}$ is an optimal solution of Max$^*(\bar{P})$. Then there exists a saddle-point $(\bar{\lambda}, P^*)$ on L with $P^* \in [P', P'']$.*

Proof. Since $\bar{P}$ lies in the interior of the edge and

$$(w + \bar{\lambda})(A^+(\bar{P})) = (w + \bar{\lambda})(A^-(\bar{P})) \tag{7}$$

holds by construction, we have $(w+\bar{\lambda})(A^+(P')) = (w+\bar{\lambda})(A^-(P''))$. Therefore, we get $f_{\bar{\lambda}}(\bar{P}) = f_{\bar{\lambda}}(P)$ and (1) and (2) are satisfied for $\bar{\lambda}$ in P for all $P \in [P', P'']$.

Now let (δ^*, P^*) be a saddle-point in L such that $P^* \in [P', P'']$ and $(w + \delta^*)(A^+(P')) = (w + \delta^*)(A^-(P''))$. Then

$$h(P^*) = f_{\delta^*}(P^*) = f_{\delta^*}(P)$$

holds for all $P \in [P', P'']$, especially for $P = \bar{P}$.

Since δ^* is a feasible solution of $\mathrm{Max}(\bar{P})$ we have $f_{\delta^*}(\bar{P}) \leq f_{\bar{\lambda}}(\bar{P})$. Moreover, $P^* \in [P', P'']$ implies $f_{\bar{\lambda}}(\bar{P}) = f_{\bar{\lambda}}(P^*)$. And finally, $\bar{\lambda}$ is a feasible solution of $\mathrm{Max}(P^*)$ and hence $f_{\bar{\lambda}}(P^*) \leq h(P^*)$. Putting all together, we get

$$h(P^*) = f_{\delta^*}(P^*) = f_{\delta^*}(\bar{P}) \leq f_{\bar{\lambda}}(\bar{P}) = f_{\bar{\lambda}}(P^*) \leq h(P^*)$$

which implies $f_{\bar{\lambda}}(P^*) = h(P^*)$, i.e., $\bar{\lambda}$ is an optimal solution of $\mathrm{Max}(P^*)$.

Since $\bar{\lambda}$ is an optimal solution of $\mathrm{Max}(P^*)$ and (1) and (2) are satisfied for $\bar{\lambda}$ in P^*, $(\bar{\lambda}, P^*)$ is a saddle-point in L. □

Given a horizontal line L, the procedure of pruning a halfplane can be summarized as follows: Let $V(L) = V$.

- Step 1 [Investigation of a vertex on L]: Let $P = (x, y) \in L \cap V(L)$ such that x is the median among the x-coordinates of points in $V(L)$. Let δ be an optimal solution of $\mathrm{Max}(P)$. If δ satisfies (1) and (2) then set $\delta^* = \delta$, $P^* = P$ and go to Step 4. If all vertices in $V(L)$ are already investigated then go to Step 3, otherwise go to Step 2.

- Step 2 [Prune half-ray]: If δ violates (1) ((2)) then delete all vertices in $V(L)$ that lie to the right (left) of P and go back to Step 1.

- Step 3 [Critical edge]: Let P' and P'' be the two remaining vertices in $V(L)$ and both vertices have already been investigated in Step 1. Choose an arbitrary point $\bar{P} \in (P', P'')$ and let $\bar{\lambda}$ be an optimal solution of $\mathrm{Max}^*(\bar{P})$. Set $\delta^* = \bar{\lambda}$, $P^* = P'$ (or $P^* = P''$) and go to Step 4.

- Step 4 [Prune halfplane]: If δ^* satisfied (3) and (4) in point P^* then (δ^*, P^*) is a saddle-point of Down1Med and hence an optimal weight modification δ^* is found. Otherwise, if (3) ((4)) is violated then prune all vertices below (above) L.

In an analogous way the above procedure can be applied to a vertical line in order to prune the left or right halfplane. If the procedure is iteratively applied to the horizontal line defined by the median of the y-coordinates of unpruned vertices and then to the vertical line defined by the median of the x-coordinates of unpruned vertices, then either during the pruning process an optimal solution if found (cf. Step 4) or we end up with a single cell, i.e., a rectangle that does not contain any vertex. We know that there exists a saddle-point (δ^*, P^*) such that P^* lies in the cell or on its boundary. Since P^* is known to be 1-median with respect to $w + \delta^*$ there also exists a vertex P' on the border of the cell that is also 1-median with respect to $w + \delta^*$ and hence $f_{\delta^*}(P') = f_{\delta^*}(P^*)$. This means, that at least one of the corners (that are vertices) of the remaining cell is a 1-median with respect to an optimal weight modification. Hence, we can try out all four corners.

Now we come back to the first algorithm described in Subsection 3.1. Let P_i for $i = 1, \ldots, 4$ be the corners of the remaining cell. For each $i = 1, \ldots, 4$ do the following: Maximize the objective value of P_i by a feasible weight modification provided that P_i is 1-median, i.e., let $\delta^{(i)}$ be an optimal solution of Max_Median(P_i). Then $g(P_i) = f_{\delta^{(i)}}(P_i)$ holds. Finally, let $\max_{i=1,\ldots,4} g(P_i) = f_{\delta^{(j)}}(P_j)$ then $\delta^* = \delta^{(j)}$ is an optimal solution of Down1Med. The previous discussion leads to the following main theorem:

Theorem 3.6. *Downgrading the* 1*-median in the plane with Manhattan metric can be done in* $\mathcal{O}(n \log^2 n)$ *time.*

Proof. The correctness of the algorithm described above follows from the previous discussion. We choose the median among the x- and y-coordinates of the set of unpruned vertices as vertical and horizontal test line, respectively. Hence, we have $\mathcal{O}(\log n)$ iterations.

In every iteration we use binary search (on the set of vertices on the line) to find a saddle-point on the line. Hence, we have to solve $\mathcal{O}(\log n)$ problems of type Max(P), i.e., continuous knapsack problems ($\mathcal{O}(n)$). Moreover, Max$^*(P)$ is solved at most once. Since Max$^*(P)$ has a constant number of constraints in addition to the upper bound constraints, it can also be solved in linear time [16]. Therefore, each iteration takes $\mathcal{O}(n \log n)$ time.

Finally, Max_Median(P) has to be solved for four vertices. Max_Median(P) can be solved in linear time because it has a constant number of constraints. Therefore, the total running time is $\mathcal{O}(n \log^2 n)$. $\square$

4 Conclusion

This paper considers the problem of maximizing the optimal 1-median objective value in the plane with Manhattan metric by changing the vertex weights within certain bounds. The suggested algorithm can directly be applied to the corresponding problem with Tchebychev metric because the Manhattan metric can be transformed into the Tchebychev metric by rotating the coordinate system by 45 degrees (e. g., see [10]).

The corresponding upgrading model, i.e., $\min_{\delta \in \Delta} \min_{P \in \mathbb{R}^2} f_\delta(P)$ can be solved by maximally improving the objective value of every vertex. This algorithm takes $\mathcal{O}(n^3)$. However, we assume that the running time can possibly be improved by using information of optimal solutions of adjacent vertices.

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