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Round-robin tournaments with homogeneous rounds

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Abstract We study single and double round-robin tournaments for n teams, where in each round a fixed number (g) of teams is present and each team present plays a fixed number (m) of matches in this round. In a single, respectively double, round-robin tournament each pair of teams play one, respectively two, matches. In the latter case the two matches should be played in different rounds. We give necessary combinatorial conditions on the triples (n, g, m) for which such round-robin tournaments can exist, and discuss three general construction methods that concern the cases $m = 1$, $m = 2$ and $m = g - 1$. For $n \leq 20$ these cases cover 149 of all 173 non-trivial cases that satisfy the necessary conditions. In 147 of these 149 cases a tournament can be constructed. For the remaining 24 cases the tournament does not exist in 2 cases, and is constructed in all other cases. Finally we consider the spreading of rounds for teams, and give some examples where well-spreading is either possible or impossible.

Keywords Round-robin tournament · Sports scheduling · Timetabling · Scheduling

1 Introduction

The most common form for sports competitions is the round-robin tournament. In such tournaments the number of matches between each pair of teams is the same: in a *single* round-robin tournament this number of matches is 1, while in a *double* round-robin tournament it is 2. If the competition contains n teams, there are $\frac{1}{2}n(n-1)$ matches in a single round-robin, and $n(n-1)$ matches in a double round-robin. Often the location of the match is important. In this case a match is *home* for one team, while it is *away* for the other team.

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If the matches are divided in *rounds*, the rounds are ordered in time, and a team plays at most in one match per round, we have the notion of break. We say that a team has a *break* between two of its consecutive matches, if it plays in both matches home, or in both matches away. It is well-known (Kirkman 1847; de Werra 1980, 1988) that if n is even, and there are $n-1$ rounds with each team playing a match, then there are at least $n-2$ breaks. For n odd, the usual round-robin tournament contains n rounds with $n-1$ teams present, playing one match. In this case we can find a round-robin tournament without breaks, and even more remarkable this schedule is essentially unique (see Frončck and Meszka 2003).

In some competitions the round-robin tournament has a different structure for the rounds: in a round only a (small) subset of all teams play one or more matches. The reason for this format can be two-fold: the match is played in a sports hall or court, which is rented for a (part of the) day by the league (see also van Weert and Schreuder 1998; Knust 2008) or the teams prefer to play twice or more on the same day; this preference is encountered regularly for youth teams. While often in round-robin tournaments breaks are an important issue, this is obviously not the case in this situation.

Our main motivation was provided by the Dutch Inline Skater Hockey League, where in each round 3 or 4 teams play once against each other, leading to 3 or 6 matches in a round. Note that scheduling a competition in this form gives the competition organizer the possibility to cope with problems due to unavailabilities of teams, an aspect important for the Dutch Inline Skater Hockey League, where teams can indicate what dates they are not available. Hence the construction of the round-robin tournaments in this case is in parallel to the construction of basic match schedules (see Schreuder 1992) in the usual round-robin tournaments: we make a schedule for dummy teams which are matched to the real teams in a second phase leading to the fixture list.[1]

This is the setting we will study: the round-robin tournament for n teams is divided in rounds which always contain a subset of teams of the same size (g), and all teams play an equal number (m) of matches in this round. The schedule in this form we will call an (n, g, m)-tournament. If necessary we will attach 'single' or 'double'; for example a double $(5, 4, 2)$-tournament.

In Sect. 2 we will derive some necessary conditions for the existence of an (n, g, m)-tournament. In Sects. 3 and 4 we will give our results for the single round-robin tournaments, while Sect. 5 will discuss the double round-robin tournaments. In Sect. 6 we will give some case studies on the well-spreading of (n, g, m)-tournaments.

In this work we mainly investigate the existence of (n, g, m)-tournaments by using different 'standard' techniques. The techniques used apply to any number (n) of teams. We tried to be exhaustive for 'small' n, namely $n \leq 20$: it is seldom that one encounters competitions with more than 20 teams. Our standard techniques settle the existence of most (n, g, m)-tournaments for $n \leq 20$. Most remaining cases were solved by hand, though we made an ILP formulation as well, and used CPLEX 11.2 to establish the existence of some double round-robin tournaments. The solutions constructed are usually far from unique; we noted that the solutions constructed by hand are usually much more regular than those found by CPLEX; examples of this can be found in Sect. 4.

[1] The dates represent the times that a round can be played. For most competitions of the Dutch Inline Skater Hockey League there are more dates than rounds, leading to even more possibilities for the competition organizer.

2 Conditions for the existence of (n, g, m)-tournaments

We consider round-robin tournaments for n teams, where in each round a subset of g teams play m matches. There are two cases that we call trivial, and which we skip in all our further considerations:

1. The case $g = 2$ and $m = 1$. In this case each match represents a round.
2. The case $g = n$ and $m = n - 1$. In this case a round is a complete single round-robin tournament.

For the remaining cases we can state integrality conditions for the existence of an (n, g, m)-tournament. This is formulated in Theorem 1. Before turning to this theorem, we consider an example. We look at Table 8 which gives a double $(7, 6, 2)$-tournament: this tournament for 7 teams has 7 rounds, where in each round there are 6 teams playing 2 matches. In total there are 42 matches, as it should be in a double round-robin tournament for 7 teams. In Table 8 there is a line between match 3 and match 4 of the round. Above and below this line there is a 'regular' round-robin tournament, in which one team per round has a bye. Hence this combination of two single $(7, 6, 1)$-tournaments gives one double $(7, 6, 2)$-tournament.

Theorem 1 *Let S be an (n, g, m)-tournament for a round-robin tournament. Let $d = 1$ if S is a single round-robin tournament, and $d = 2$ if S is a double round-robin tournament. Then the parameters n, g, m and d have the following relations.*

(a) *g is even or m is even.*
(b) *$dn(n - 1)$ is a multiple of gm.*
(c) *$d(n - 1)$ is a multiple of m.*

Proof

1. The number of matches in a round is $\frac{1}{2}gm$. Hence either g or m must be even.
2. The number of rounds in tournament S is $\frac{\frac{1}{2}dn(n-1)}{\frac{1}{2}gm}$, i.e. the total number of matches divided by the number of matches per round. Since this fraction is integer, we have that $dn(n - 1)$ is multiple of gm.
3. The number of rounds for a team is $\frac{d(n-1)}{m}$. Hence $d(n - 1)$ needs to be a multiple of m.

$\square$

Theorem 1 restricts the number of possible combinations considerably. Table 1 gives all possible combinations (g, m) for $n \leq 20$ in the single round-robin case, 80 cases in total. For example, for $n = 7$ there exist at most 3 single $(7, g, m)$-tournaments, namely for $(g, m) = (3, 2)$, $(g, m) = (6, 1)$, and $(g, m) = (7, 2)$. A similar table for double round-robin tournaments can be found in Sect. 5.

At this point it is appropriate to discuss a graph theoretical setting for the single round-robin tournaments. The existence of a single (n, g, m)-tournament is equivalent to a decomposition of the complete graph K_n in m-regular subgraphs on g vertices, such that all these subgraphs are edge-disjoint. There are some special cases of interest:

1. If $g = n$ and $m = 1$ this decomposition is called a 1-*factorization*. Necessarily n is even. This case corresponds to the regular round-robin tournament for n teams.
2. If $m = 2$ we have a decomposition of K_n in cycles. In our case the vertices in the components add up to g. If we require that all cycles have the same length g the decomposition is called a *cycle decomposition*.

Table 1 Parameters (g, m) for which a single (n, g, m)-tournament can exist

n	(g, m)								
4	(4, 1)								
5	(4, 1)	(5, 2)							
6	(6, 1)								
7	(3, 2)	(6, 1)	(7, 2)						
8	(4, 1)	(8, 1)							
9	(3, 2)	(4, 1)	(4, 2)	(6, 1)	(6, 2)	(6, 4)	(8, 1)	(9, 2)	(9, 4)
10	(6, 1)	(6, 3)	(10, 1)	(10, 3)					
11	(5, 2)	(10, 1)	(11, 2)						
12	(4, 1)	(6, 1)	(12, 1)						
13	(3, 2)	(4, 1)	(4, 3)	(6, 1)	(6, 2)	(12, 1)	(13, 2)	(13, 4)	(13, 6)
14	(14, 1)								
15	(3, 2)	(5, 2)	(6, 1)	(7, 2)	(10, 1)	(10, 7)	(14, 1)	(15, 2)	
16	(4, 1)	(4, 3)	(6, 1)	(6, 5)	(8, 1)	(8, 3)	(8, 5)	(10, 1)	(10, 3)
	(12, 1)	(12, 5)	(16, 1)	(16, 3)	(16, 5)				
17	(4, 1)	(4, 2)	(8, 1)	(8, 2)	(16, 1)	(17, 2)	(17, 4)	(17, 8)	
18	(6, 1)	(18, 1)							
19	(3, 2)	(6, 1)	(6, 3)	(9, 2)	(18, 1)	(19, 2)	(19, 6)		
20	(4, 1)	(10, 1)	(20, 1)						

3. If $m = g - 1$ we have that the teams present in a round all play against each other. Consequently the $(n, g, g-1)$-tournament is a collection of subsets of $\{1, 2, \ldots, n\}$, all of size g, such that each pair of teams is member of exactly 1 subset. Such collection of subsets is an $(n, g, 1)$-(*block*) *design*, see for example (Colbourn and Dinitz 2007).

In the next section we will describe the results based on these 'standard constructions', and some variations. Section 4 is devoted to the cases for which no standard construction exists.

3 Standard constructions for single round-robin tournaments

In the previous section we described three special cases, which we will call 'round-robin' construction, 'cycle' construction, and 'block-design' construction. We will describe the results of each of these combinatorial constructions in a separate subsection. Before turning to these constructions, we formulate a general construction, the merging of rounds. Note that this theorem is valid for both single and double round-robin tournaments.

Theorem 2 *Suppose an (n, g, m)-tournament exists with R rounds, where $g = n$, and m' is a multiple of m, as well as a divisor of mR. Then an (n, g, m')-tournament exists as well.*

Proof Starting with an (n, n, m)-tournament, we can unite groups of m'/m rounds to obtain the (n, n, m')-tournament. □

3.1 Round-robin constructions

The constructions for $(n, n, 1)$-tournaments (n even) and $(n, n-1, 1)$-tournaments (n odd) are well-known (Kirkman 1847; de Werra 1988). We will call these tournaments the *regular* round-robin tournaments (single or double) for the given number of teams. Based on a regular tournament we can construct other tournaments as well. The next theorem is valid for single and double round-robin tournaments.

Theorem 3 *Let M be the number of matches in the round-robin tournament. An $(n, g, 1)$-tournament exists in the following cases*:

(a) *For n even, g even, and g a divisor of n.*
(b) *For n odd, g even, and g is divisor of $n - 1$.*
(c) *For n even, g even, $\frac{1}{2}g$ a divisor of M, and $g \leq \frac{1}{2}n + 1$.*
(d) *For n odd, g even, $\frac{1}{2}g$ a divisor of M, and $g \leq \frac{1}{2}(n + 1)$.*

Proof

(a) Starting with a regular $(n, n, 1)$-tournament, we can split the $\frac{1}{2}n$ matches of a round in groups of $\frac{1}{2}g$ matches each. All the matches in a round contain different teams, hence we end up with an $(n, g, 1)$-tournament.
(b) The proof is similar to case (a), starting with a regular $(n, n - 1, 1)$-tournament.
(c) The construction resembles case (a), but we have to do a little more work. Consider a regular $(n, n, 1)$-tournament, and order the matches in this schedule according to the rounds. What we need to do is to refine this ordering such that picking the next $\frac{1}{2}g$ matches each time, results in an $(n, g, 1)$-tournament. (Note that the number of matches M is required to be a multiple of $\frac{1}{2}g$.) To construct the next round in the $(n, g, 1)$-tournament, we pick the next $\frac{1}{2}g$ consecutive matches: these fall within round (say) r and round $r + 1$. What could happen is that the matches in round $r + 1$ contain the same teams as those selected from round r. We show that the order of the matches can be adjusted to avoid this. Suppose k matches are in round r: these k matches contain $2k$ teams. Hence there are at most $2k$ matches in round $r + 1$ with those teams. Consequently there remain at least $\frac{1}{2}n - 2k$ matches in round $r + 1$ that can be used in the $(n, g, 1)$-tournament. The number of matches in round $r + 1$ is $\frac{1}{2}g - k$. Hence we need $\frac{1}{2}g - k \leq \frac{1}{2}n - 2k$, or $\frac{1}{2}g + k \leq \frac{1}{2}n$. Since $k < \frac{1}{2}g$, and $g \leq \frac{1}{2}n + 1$, this condition is satisfied.
(d) The analysis is similar to case (d), starting with a regular $(n, n - 1, 1)$-tournament.

□

As an example of case (c) let us describe how one obtains the first 5 rounds of the $(10, 6, 1)$-tournament from the first 3 rounds of a regular $(10, 10, 1)$-tournament; the remaining rounds are treated in the same way. For the regular tournament, we take the tournament constructed by the canonical factorization (de Werra 1980). The 15 matches in the first 3 round (separated by semi-colons) are:

1–10, 2–9, 3–8, 4–7, 5–6; 1–3, 2–10, 4–9, 5–8, 6–7;
1–5, 2–4, 3–10, 6–9, 7–8.

To construct the first round in the $(10, 6, 1)$-tournament, we simply take the matches 1–10, 2–9 and 3–8. The second round consists of the 4–7 and 5–6, and one match from the second round of the $(10, 10, 1)$-tournament, not containing the teams 4, 5, 6, 7. Since these teams are involved in at most 4 matches, there is always at least one match that remains: in this case we could choose 1–3 or 2–10. We choose 1–3, so the second round of the $(10, 6, 1)$-tournament consists of 4–7, 5–6 and 1–3. For the third round in the $(10, 6, 1)$-tournament, we can choose the 2–10, 4–9, 5–8. The fourth round we consist of 6–7, and two matches from round three of the (10, 10, 1)-tournament, not containing the teams 6 and 7. Hence we can take the matches 6–7, 1–5 and 2–4 as fourth round, and for the fifth round consists of the remaining matches 3–10, 6–9 and 7–8.

Theorem 3 settles the existence of 34 parameter combinations in Table 1, namely all cases with $m = 1$ except the cases $(9, 6, 1)$, $(15, 10, 1)$, $(16, 10, 1)$, and $(16, 12, 1)$. Applying Theorem 2 yields the cases $(10, 10, 3)$, $(16, 16, 3)$, and $(16, 16, 5)$.

3.2 Cycle constructions

The decomposition of graphs by cycles of fixed length has been studied extensively. This research lead to the end result for complete graphs (Alspach and Gavlas 2001; Šajna 2002), stating that the obvious necessary conditions are also sufficient, see Theorem 4.

Theorem 4 *Suppose n is odd and $3 \le g \le n$, such that g divides $\frac{1}{2}n(n-1)$. Then the complete graph K_n can be decomposed in edge-disjoint cycles of length g.*

Comparing this sufficient condition with the necessary conditions in Theorem 1, we see that for the existence of $(n, g, 2)$-tournaments the necessary conditions are sufficient. Hence all 22 $(n, g, 2)$-tournaments in Table 1 exist. Using Theorem 2 we get additionally the existence of all 6 (n, n, m)-tournaments for n odd and $m > 2$ even. Note that the results in Šajna (2002), Alspach and Gavlas (2001) use cycles of fixed length. This can be used to obtain the following construction.

Theorem 5 *Suppose n is odd, g is even and $\frac{1}{2}g$ divides n. Then a single $(n, g, 1)$-tournament exists.*

Proof According to Theorem 4 the complete graph K_n can be decomposed in cycles of length n. Choose in each of these cycles groups of $\frac{1}{2}g$ non-adjacent edges. Letting these edges correspond to the matches of a round between the corresponding nodes, we obtain the required $(n, g, 1)$-tournament. □

This construction yields several of the $(n, g, 1)$-tournaments constructed before. In addition it yields the existence of a single $(9, 6, 1)$-tournament and a single $(15, 10, 1)$-tournament.

3.3 Block design constructions

As explained in Sect. 2 we can use block designs for the construction of $(n, g, g-1)$-tournaments. We formulate this equivalence in the following theorem.

Theorem 6 *A single $(n, g, g-1)$-tournament exists if and only if an $(n, g, 1)$-design exists.*

Consequently, we can use the extensive tables (see for instance §II.1 in Colbourn and Dinitz 2007) on block designs to settle the existence of $(n, g, g-1)$-tournaments. For $n \le 20$ it yields the following results.

- A $(13, 4, 3)$-tournament and a $(16, 4, 3)$-tournament exist.
- A $(16, 6, 5)$-tournament does not exist.

Note that we have the first negative result. This implies that the necessary conditions in Theorem 1 are in general not sufficient. Another remark is on the $(16,4,3)$-tournament. Such a tournament was used in Post and Woeginger (2006) to obtain a regular round-robin tournament for 16 teams with at least 40 breaks. This block design is a so-called *resolvable* block design (see also Definition 2): there exist 5 groups of 4 rounds each, where these 4 rounds contain exactly all 16 teams. Splitting this resolution into 2 parts again, we obtain a $(16,8,3)$-tournament.

Concluding we settled the existence of 4 additional tournaments, leaving only 9 cases from Table 1 unaccounted for; these cases we will study in the next section.

4 Ad hoc constructions of single round-robin tournaments

In this section we study the 9 cases in Table 1 that remained unsolved. We devote a separate subsection to most of these cases.

4.1 The $(9,6,4)$-tournament

This tournament does not exist, which we can prove as follows. The 36 matches have to be divided over 3 rounds, and each teams appears in 2 rounds only. If we look at round 3, there are three teams not present, say the teams 1, 2, and 3. These all have to appear in the rounds 1 and 2, while all other teams (4 to 9) appear in exactly one of the rounds 1 and 2, say the teams 4, 5, 6 in round 1, and the teams 7, 8, 9 in round 2. Consequently the teams 1, 2, and 3 all have to play the teams 4, 5, and 6 in round one, and the teams 7, 8, and 9 in round 2. The last matches for the teams 1, 2, 3 are the 3 mutual matches, to be played in 2 rounds. This is impossible, because the $(3,3,1)$-tournament does not exist. Hence the claim follows.

4.2 The $(10,6,3)$-tournament

The tournament has 5 rounds. Group the 10 teams in pairs, and require that in round r the pairs $r-1, r$ and $r+1$ (cyclically) are present. With this basis a tournament can be constructed. The tournament in Table 2 has the rounds permuted, to improve the spreading.

Table 2 A single $(10,6,3)$-tournament

Round 1	Round 2	Round 3	Round 4	Round 5
1–5	1–7	3–6	1–2	5–9
2–6	2–8	4–8	4–10	6–10
3–4	9–10	5–7	3–9	7–8
1–6	1–8	3–8	1–4	5–10
3–5	7–9	4–7	2–9	8–9
2–4	2–10	5–6	3–10	6–7
1–3	1–9	3–7	4–9	5–8
2–5	2–7	6–8	1–10	7–10
4–6	8–10	4–5	2–3	6–9

Table 3 A single (15, 10, 7)-tournament

Round 1	Round 2	Round 3
group 1–group 2	group 1–group 3	group 2–group 3
1–2, 3–4, 5–1, 2–3, 4–5	1–3, 5–2, 4–1, 3–5, 2–4	6–8, 10–7, 9–6, 8–10,7–9
6–7, 8–9, 10–6,	11–12, 13–14, 15–11,	11–13, 15–12, 14–11,
7–8, 9–10	12–13, 14–15	13–15, 12–14

Table 4 A single (16, 8, 5)-tournament

Round 1	Round 2	Round 3
group 1–group 2	group 3–group 4	group 1–group 3
1–2, 3–4, 5–6, 7–8	9–10, 11–12, 13–14, 15–16	1–3, 2–4, 9–11, 10–12
Round 4	Round 5	Round 6
group 2–group 4	group 1–group 4	group 2–group 3
5–7, 6–8, 13–15, 14–16	1–4, 2–3, 13–16, 14–15	5–8, 6–7, 9–12, 10–11

4.3 The (15, 10, 7)-tournament

This tournament has 3 rounds, and each team plays in 2 of these rounds. We put the teams 1–5 in group 1, 6–10 in group 2 and 11–15 in group 3. We can require that in round 1 the groups 1 and 2 play, in round 2 the groups 1 and 3, and in round 3 the groups 2 and 3. In a round all matches between teams of different groups have to be played. Note that the teams within a group play according to a single (5, 5, 2)-tournament, which exists. Hence we can construct a schedule, which is given in Table 3.

4.4 The (16, 8, 5)-tournament

This tournament has 6 rounds, and each team appears in 3 rounds. This suggests to divide the teams in 4 groups of size 4 (1–4, 5–8, 9–12, and 13–16), and let 2 groups play in a round. Since there are 6 pairs of groups, this exactly fits. In a round the teams of one group have to play against all other teams in the other one, and one team of the same group. This leads to the tournament in Table 4.

4.5 The (16, 10, 1)-tournament and the (16, 12, 1)-tournament

These were constructed ad hoc from a (16, 16, 1)-tournament. Hence they both exist, but the (extensive) schedules are not presented here.

4.6 The (16, 10, 3)-tournament

There are 8 rounds, and each team plays in 5 rounds. We require in round r the teams $10r - 9$ to $10r$, where the team number is taken modulo 16 from $\{1, 2, \dots, 16\}$. So in the first round appear the teams 1 to 10, in round 2 the teams 11 to 16 and 1 to 4, and round 3 the teams 5 to 14, etcetera. With some care the schedule in Table 5 can be constructed.

Table 5 A single (16, 10, 3)-tournament

R1	R2	R3	R4	R5	R6	R7	R8
1–7	11–1	5–11	15–5	9–15	3–10	13–4	7–14
2–8	12–2	6–12	16–6	10–16	4–9	14–3	8–13
1–9	11–3	5–13	15–7	9–2	3–12	13–6	7–16
2–10	12–4	6–14	16–8	10–1	4–11	14–5	8–15
3–9	13–3	7–13	1–8	11–2	5–12	15–6	9–16
4–10	14–4	8–14	2–7	12–1	6–11	16–5	10–15
1–5	11–15	5–9	15–3	9–13	3–8	13–2	7–12
2–6	12–16	6–10	16–4	10–14	4–7	14–1	8–11
3–7	13–1	7–11	1–6	11–16	5–10	15–4	9–14
4–8	14–2	8–12	2–5	12–15	6–9	16–3	10–13
3–5	13–15	7–9	1–3	11–13	5–8	15–2	9–12
4–6	14–16	8–10	2–4	12–14	6–7	16–1	10–11
5–7	15–1	9–11	3–6	13–16	7–10	1–4	11–14
6–8	16–2	10–12	4–5	14–15	8–9	2–3	12–13
9–10	3–4	13–14	7–8	1–2	11–12	5–6	15–16

Table 6 A single (16, 12, 5)-tournament

Round 1	Round 2	Round 3	Round 4
(1, 2)–(5, 6)	(1, 2)–(7, 8)	(1, 2)–(11, 12)	(5, 6)–(11, 12)
(3, 4)–(7, 8)	(3, 4)–(5, 6)	(3, 4)–(9, 10)	(7, 8)–(9, 10)
(5, 6)–(9, 10)	(5, 6)–(13, 14)	(9, 10)–(13, 14)	(9, 10)–(15, 16)
(7, 8)–(11, 12)	(7, 8)–(15, 16)	(11, 12)–(15, 16)	(11, 12)–(13, 14)
(9, 10)–(1, 2)	(13, 14)–(1, 2)	(13, 14)–(3, 4)	(13, 14)–(7, 8)
(11, 12)–(3, 4)	(15, 16)–(3, 4)	(15, 16)–(1, 2)	(15, 16)–(5, 6)
1–2, 3–4	1–3, 2–4	1–4, 2–3	5–8, 6–7
5–6, 7–8	5–7, 6–8	9–11, 10–12	9–12, 10–11
9–10, 11–12	13–14, 15–16	13–15, 14–16	13–16, 14–15

4.7 The (16, 12, 5)-tournament

This tournament contains 4 rounds, and each team appears in 3 rounds. This suggests to divide the teams in 4 groups of 4, and per round to leave out one of these groups. Since each pair of groups appears twice, we let each team play against 2 teams from a different group, and one team from the own group. This leads to the schedule in Table 6. Here (a, b)-(c, d) denotes the matches a-c, b-d, a-d, and b-c.

4.8 The (19, 6, 3)-tournament

The (19, 6, 3)-tournament does not exist. This was established by formulating the construction problem as an IP problem, and let CPLEX run to establish infeasibility.

5 Existence of double round-robin tournaments

The construction of double round-robin tournaments follows along the lines of the single round-robin tournaments, but in addition there are two general constructions from the single round-robin case. These we formulate in the following theorem.

Theorem 7 *Suppose a single* (n, g, m)*-tournament exists. Then*

(a) *A double* (n, g, m)*-tournament exists.*
(b) *A double* $(n, g, 2m)$*-tournament exists.*

Proof

(a) To construct the double round-robin tournament, we can repeat the single tournament.
(b) To construct the double round-robin tournament, we can repeat the matches in one round. □

Case (b) is a construction that is usually not preferred as the two matches between two teams are played in the same round. More generally, we define a *proper* round-robin tournament as a tournament with only different matches in a round. We will study exclusively proper round-robin tournaments from now on. There exist parameters (n, g, m) for which the improper tournament exists, although a proper (n, g, m)-tournament does not exist. This is for example the case for the double $(15, 5, 4)$-tournament: since the single $(15, 5, 2)$-tournament exists, we can construct a double $(15, 5, 4)$-tournament. However, a proper double $(15, 5, 4)$-tournament corresponds to a $(15, 5, 2)$-design, which does not exist; see also Sect. 5.3.

The construction in case (a) is often preferred, or even required. However sometimes it is better to use two different single round-robin tournaments, instead of one, see Sect. 6.

In Table 7 we list all combinations (n, g, m) that satisfy the necessary conditions in Theorem 1 for the double round-robin case. Because of Theorem 7(a) we omit the triples for which the single tournament exists; hence we omit all triples from Table 1, except the cases $(9, 6, 4)$, $(16, 6, 5)$, and $(19, 6, 3)$. This leads to 93 cases to investigate for $n \leq 20$.

We will discuss the basic construction methods in this case, and establish the existence of double (n, g, m)-tournaments for $n \leq 20$.

5.1 Round-robin constructions

We can use Theorem 3 to construct tournaments from the regular double round-robin tournament. This leads to 9 $(n, g, 1)$-tournaments: only the cases $(12, 8, 1)$, $(13, 8, 1)$, $(15, 12, 1)$, $(18, 12, 1)$, and $(19, 12, 1)$ remain open. Using Theorem 2 leads to the existence of 12 (n, n, m)-tournaments with n even and $m > 1$.

5.2 Cycle constructions

For cycle constructions for the double round-robin tournament we need the directed version of Theorem 4, which can be found in Alspach et al. (2003).

Theorem 8 *Suppose* D_n *is the complete digraph on* n *vertices and* $M = n(n-1)$ *arcs, and* $3 \leq g \leq n$*, such that* g *divides* M*. Then* D_n *can be decomposed into edge-disjoint directed cycles of length* g*.*

Table 7 Parameters for which only a double (n, g, m)-tournament can exist

n	(g, m)								
4	(3, 2)	(4, 2)							
5	(4, 2)								
6	(3, 2)	(4, 1)	(5, 2)	(6, 2)					
7	(4, 1)	(4, 3)	(6, 2)	(7, 4)					
8	(4, 2)	(7, 2)	(8, 2)						
9	(6, 4)	(8, 2)							
10	(3, 2)	(4, 1)	(4, 3)	(5, 2)	(6, 2)	(9, 2)	(10, 2)	(10, 6)	
11	(4, 1)	(5, 4)	(10, 2)	(11, 4)					
12	(3, 2)	(4, 2)	(6, 2)	(8, 1)	(11, 2)	(12, 2)			
13	(4, 2)	(6, 4)	(8, 1)	(8, 3)	(12, 2)	(13, 8)			
14	(4, 1)	(7, 2)	(13, 2)	(14, 2)					
15	(4, 1)	(5, 4)	(6, 2)	(7, 4)	(10, 2)	(12, 1)	(12, 7)	(14, 2)	(15, 4)
16	(3, 2)	(4, 2)	(5, 2)	(6, 2)	(6, 5)	(8, 2)	(8, 6)	(10, 2)	(10, 6)
	(12, 2)	(12, 10)	(15, 2)	(16, 2)	(16, 6)	(16, 10)			
17	(8, 4)	(16, 2)							
18	(3, 2)	(4, 1)	(6, 2)	(9, 2)	(12, 1)	(17, 2)	(18, 2)		
19	(4, 1)	(4, 3)	(6, 2)	(6, 3)	(9, 4)	(12, 1)	(12, 3)	(18, 2)	(19, 4)
	(19, 12)								
20	(4, 2)	(5, 2)	(8, 1)	(10, 2)	(19, 2)	(20, 2)			

As in the single tournament case, this theorem leads to the conclusion that all double $(n, g, 2)$-tournaments, satisfying the necessary conditions in Theorem 1 exist. Hence in Table 7 we get the existence of the 43 remaining cases with $m = 2$. Using Theorem 2 we get the existence of the 6 (n, n, m)-tournaments, with n odd and $m > 2$ even.

By using Theorem 8 we can prove the 'even' version of Theorem 5:

Theorem 9 *Suppose n is even, g is even and $\frac{1}{2}g$ divides n. Then a double $(n, g, 1)$-tournament exists.*

From this theorem it follows that the $(12, 8, 1)$-tournament and the $(18, 12, 1)$-tournament exist.

5.3 Block design constructions

A theorem similar to Theorem 6 also exists for double round-robin tournaments.

Theorem 10 *A proper double $(n, g, g-1)$-tournament exists if and only if a $(n, g, 2)$-design exists.*

Hence we can settle these cases by using the tables for block designs.

- The $(7, 4, 3)$-tournament, the $(10, 4, 3)$-tournament, the $(11, 5, 4)$-tournament, the $(16, 6, 5)$-tournament, and the $(19, 4, 3)$-tournament exist.
- A (proper!) $(15, 5, 4)$-tournament does not exist. As noted before there exists an improper $(15, 5, 4)$-tournament.

5.4 The remaining cases

We have 15 cases left, 8 cases with m even and 7 cases with m odd. The cases with m even are: $(9,6,4)$, $(13,6,4)$, $(15,7,4)$, $(16,8,6)$, $(16,10,6)$, $(16,12,10)$, $(17,8,4)$, and $(19,9,4)$. Note that in all these cases an improper tournament exists. The proper cases we solved with CPLEX: they all exist. The 7 remaining cases with m odd remain are: (13, 8, 1), (13, 8, 3), (15, 12, 1), (15, 12, 7), (19, 6, 3), (19, 12, 1), and (19, 12, 3). These were all constructed manually, but are not presented here.

There is an interesting series among these remaining cases, already starting with the single $(7,3,2)$-tournament and the double $(11,5,4)$-tournament, and continuing with two cases in the list above: the $(15,7,4)$-tournament and the $(19,9,4)$-tournament. These tournaments are related to $(4n-1, 2n-1, n-1)$-designs, where in our case $n = 2, 3, 4, 5$. The blocks of the block design are used to determine the teams in the round.[2] For the $(7,3,2)$-tournament and the double $(11,5,4)$-tournament this determines the tournament as well (all teams in a round play against each other). For $n = 4, 5$ each pair of teams appears in $n-1$ rounds. To compose the tournament we have to choose which teams play against each other in which round. This can be done for $n = 4, 5$.

The $(4n-1, 2n-1, n-1)$-design is a symmetric block-design, also called a *Hadamard design* of order n. For small n the Hadamard designs are constructed by using different techniques, but for bigger values of n the existence is unknown. For one of the latest results on Hadamard designs we refer to Doković (2008).

6 Spreading of rounds for teams

Till now we discussed parameters (n, g, m) for which (n, g, m)-tournaments might exist. If such tournament exists, there are usually (but not always) many solutions. In this section we discuss an additional property that is important in practice, namely the spreading of rounds for a team. This is especially important in cases that a team appears in a few rounds only, few compared to the total number of rounds. In such cases we would prefer that the rounds in which a team appears are 'spread' over all rounds. Since we require this for all individual teams, it is not obvious that we can achieve that. Before continuing it is wise to give a definition of what we mean by 'well-spread'.

Definition 1 We call an (n, g, m)-tournament *well-spread* if after any number of rounds, the number of rounds played by the teams differ at most 1.

In the regular $(n, n, 1)$-tournaments (n even) the number of rounds played by the teams is the same after any number of rounds. For all other (n, g, m)-tournaments this is not the case. For this reason we allowed a difference of 1 in Definition 1. The regular round-robin tournaments are well-spread. The tournaments constructed by using Theorem 3(a) and (c) are well-spread as well. This is not the case for the tournaments constructed by Theorem 3(b) and (d).

There is another class of tournaments for which the rounds can be ordered such that the tournament is well-spread: the resolvable tournaments. This term is borrowed from the theory of block designs.

[2] Note that this is an extra restriction on the construction of the tournament.

Table 8 A double $(7, 6, 2)$-tournament

R 1	R 2	R 3	R 4	R 5	R 6	R 7
1–2	2–3	3–4	4–5	5–6	6–7	7–1
1–4	2–5	3–6	4–7	5–1	6–2	7–3
2–4	3–5	4–6	5–7	6–1	7–2	1–3
3–5	4–6	5–7	6–1	7–2	1–3	2–4
3–6	4–7	5–1	6–2	7–3	1–4	2–5
5–6	6–7	7–1	1–2	2–3	3–4	4–5

Definition 2 We call a (n, g, m)-tournament *resolvable* if it is possible to construct an (n, n, m)-tournament from it, by only merging rounds of the (n, g, m)-tournament to a new round in the (n, n, m)-tournament.

For a resolvable (n, g, m)-tournament g is a divisor of n. The regular $(n, n, 1)$-tournaments (n even) are (trivially) resolvable, as well as the tournament constructed from it by Theorem 3(a). Also in other cases resolvable tournaments can exist. We mentioned already the $(16, 4, 3)$-tournament in Sect. 3.3. The cases with $m = 2$ can be constructed from resolvable g-cycle systems (see Sect. V1.12 in Colbourn and Dinitz 2007). Such 3-cycle system exists in K_9, leading to a resolvable single $(9, 3, 2)$-tournament. (By accident this tournament also corresponds to a resolvable $(9, 3, 1)$-design.) For K_{15} there exist 3-cycle and 5-cycle systems, leading to well-spread $(15, 3, 2)$ and $(15, 5, 2)$-tournaments.

There are parameters (n, g, m) for which no well-spread tournament exists. A beautiful example of this situation is the single $(7, 3, 2)$-tournament. It corresponds to a $(7, 3, 1)$-design, which is unique (up to permutations of the teams). This tournament is given in Table 8 if one only selects the first 3 matches in each round.

Note that any 2 rounds have exactly 1 team in common; this reflects the fact that this design corresponds to a projective plane (Fano plane). Hence after 2 rounds, there is one team that played 2 rounds (team 2), there are 4 teams that played 1 round (the teams 1, 3, 4, 5) and 2 teams that didn't play at all (the teams 6 and 7). So the tournament is not well-spread. However for the double $(7, 3, 2)$-tournament we can find a well-spread tournament, by weaving two permuted single $(7, 3, 2)$-tournaments. Such tournament is given in Table 8 as a double $(7, 6, 2)$-tournament: by separating the first 3 and second 3 matches in a round to consecutive rounds, we obtain the well-spread double $(7, 3, 2)$-tournament.

Another example is the double $(6, 3, 2)$-tournament, which corresponds to a $(6, 3, 2)$-design. This design is unique and also has the property that any two subsets have one team in common. Hence the $(6, 3, 2)$-tournament can not be well-spread.

7 Conclusions

We studied an interesting extension of the regular single and double round-robin tournaments for n teams, which we called the (n, g, m)-tournaments. We derived necessary conditions on the parameters (n, g, m) for the existence of a (n, g, m)-tournament. For $n \leq 20$ there are 173 combinations of (n, g, m) satisfying the necessary conditions; only in 4 cases the (proper) (n, g, m)-tournament does not exist.

The (n, g, m)-tournaments appear in practice, especially for the case that 2 matches are played in a round for a subset of teams ($m = 2$). From the theory of graph decomposition we established that such tournaments exist, if the number of teams in a round is a divisor of the total number of matches in the tournament. The case that more matches are played in a round can be interesting as well. One could think of a competition where many short matches are played in a round, for example in a bridge competition.

Our focus was on the existence of the tournaments. We only touched on one additional property, namely the spreading of rounds for a team. Another requirement could be the spreading of the matches of a team within a round (see Knust 2008).

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