Ann Oper Res (2016) 243:179–198
DOI 10.1007/s10479-015-1808-6

An Owen-type value for games with two-level communication structure

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Published online: 15 February 2015

Abstract We introduce an Owen-type value for games with two-level communication structure, which is a structure where the players are partitioned into a coalition structure such that there exists restricted communication between as well as within the a priori unions of the coalition structure. Both types of communication restrictions are modeled by an undirected communication graph. We provide an axiomatic characterization of the new value using an efficiency, two types of fairness (one for each level of the communication structure), and a new type of axiom, called fair distribution of the surplus within unions, which compares the effect of replacing a union in the coalition structure by one of its maximal connected components on the payoffs of these components. The relevance of the new value is illustrated by an example. We also show that for particular two-level communication structures the Owen value and the Aumann–Drèze value for games with coalition structure, the Myerson value for communication graph games, and the equal surplus division solution appear as special cases of this new value.

This research was supported by NWO (The Netherlands Organization for Scientific Research) Grant NL-RF 047.017.017. The research of Anna Khmelnitskaya was partially done during her stay at the University of Twente whose hospitality is appreciated and it was also partially supported by CRT Foundation and the University of Eastern Piedmont at Alessandria through the Lagrange Project 2010. We thank three anonymous referees for their useful comments.

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Keywords TU game · Coalition structure · Communication graph · Owen value · Myerson value

JEL Classification C71

1 Introduction

The study of TU games with coalition structure was initiated in the 1970's first by Aumann and Drèze (1974) and then Owen (1977). In these papers a coalition structure is given by a partition of the set of players. Another model of a game with limited cooperation presented by means of communication graphs was introduced in Myerson (1977). Various studies in both directions were done during the last three decades, but mostly either within one model or another. Vázquez-Brage et al. (1996) is the first study that combines both models by considering a TU game endowed with, independent of each other, both a coalition structure and a communication graph on the set of players. For this class of games they propose a solution by applying the Owen value for games with coalition structure to the Myerson restricted game of the game with communication graph.

Recently, Khmelnitskaya (2014) and Kongo (2007), independently from each other, have introduced another model of a TU game endowed with both a coalition structure and a communication graph, the so-called game with two-level communication structure. In contrast to Vázquez-Brage et al. (1996), in this model a two-level communication structure relates fundamentally to the given coalition structure and consists of a communication graph on the collection of the a priori unions in the coalition structure, as well as a communication graph within every union. It is assumed that communication is only possible either among the a priori unions or among single players within an a priori union. No communication and therefore no cooperation is allowed between single players from distinct elements of the coalition structure. Following Myerson (1977), who assumes that cooperation is possible only between connected players, as a solution Khmelnitskaya (2014) proposes to apply a two-step distribution procedure based on different combinations of known component efficient values, in particular, the Myerson value, the position value, the average tree solution, etc. These solutions are applied on both levels when first a priori unions collect their shares through upper level bargaining based only on cumulative interests of all members of every involved entire a priori union, and second, the players within each a priori union bargain how these shares are distributed over single players within each a priori union. The players within each union always have to distribute the total payoff that has been assigned to the union in the game between the unions, irrespectively of the communication links within the union. When some unions are not internally connected, this puts a severe restriction on the application of component efficient values. Khmelnitskaya (2014) provides a theoretical justification of solution concepts reflecting the two-stage distribution procedure when no cooperation between players belonging to different a priori unions is allowed and also reveals the conditions when the two-stage distribution procedure based on the application of component efficient values on both levels is feasible. In the solution proposed by Kongo (2007) the Myerson value is applied both on the level of the a priori unions with respect to the graph between the unions and for each union with respect to the graph within the union. Different from the framework of Khmelnitskaya (2014), aiming to avoid the just mentioned problem of the total distribution of the union share among its members, in Kongo (2007) it is assumed that within a stand-alone union a player can only cooperate with the players in his component of the internal communication graph, while a player in a union that is connected with other unions can cooperate with all players in his own union and in all the unions to which his union is connected, i.e., in this case a player can cooperate with other players within his own union, whether or not he is internally connected to these players. So, following this assumption the cooperation possibilities between two players belonging to the same a priori union depend on the connectedness of their union with other unions. The efficiency axiom in Kongo (2007), formulated as 'two-level component efficiency', reflects this particular assumption.

In this paper we follow a consistent approach where the role of the graph within a union does not depend on the graph between unions. We abide by the framework of Khmelnitskaya (2014), but we weaken the assumption concerning communication on the union level between a priori unions: similar as in Owen (1977), we allow for one a priori union among connected unions to be represented by a proper subcoalition. When unions are negotiating for their shares in the total payoff, following Myerson (1977) we assume that only unions are able to cooperate that are connected by the communication graph between the unions. Also, when a union is represented by a proper subcoalition of the union, this subcoalition can only cooperate with unions to which the union is connected by the graph between the unions. Similarly, a proper subset of players within one union can only cooperate when the players are connected by the communication graph within the union. However, as it was already mentioned above, the total payoff which has been assigned to (the representative of) a union in the game between the unions has to be distributed amongst the players of that union, irrespectively of the existing communication links within the union. So, deviating from Myerson (1977), we consider the union as an institution that distributes its total payoff amongst its members and so allows their members to cooperate as a whole beyond the bilateral communication links within the union. As a motivating example we discuss a governmental research budget that will be allocated amongst different research groups.

In case all unions are internally connected the payoff allocation will be determined by traditional efficiency and fairness axioms, in the context of the underlying model referred to as quotient component efficiency, quotient fairness and union fairness. For unions that are not internally connected, we introduce a new axiom to describe the effect that still the total payoff assigned to the union has to be distributed amongst its members. Consider a particular component within a union that is not internally connected. Suppose we know the total payoff of that component when the union is replaced by just the players in this component (and thus this reduced union is internally connected). Doing this for every connected component of this union, and compare this with the payoff allocation when the full union is present, we can compare how the (positive or negative) excess is shared among the different components. According to this new axiom, called fair distribution of the surplus within unions, this excess is shared proportional to the size of the component.

We show that the above mentioned four axioms characterize a new solution for the class of games with two-level communication structure. This new solution is an Owen-type value in the sense that it modifies the Owen value for games with two-level communication structure. As in Owen (1977), for each union we construct an internal game. To do so, first a game is obtained by applying Owen's procedure, but taking into account the communication graph between the unions. Then we take the Myerson restriction of this game taking into account the graph within the union, except that for the union as a whole we take the worth assigned to the union by Owen's procedure. The individual payoffs for the players within the union are then obtained by applying the Shapley value to this restricted internal game. As a corollary of this procedure we obtain the graph restricted analog of Owen's quotient game property that for each union the total payoff to the players of the union is equal to the Shapley payoff to this union in the Myerson restricted game (with respect to the communication graph between unions) of Owen's quotient game between the unions.

We also show that the Owen value and the Aumann–Drèze value for games with coalition structure, the Myerson value for communication graph games and the equal surplus division solution appear as special cases of this new value for particular two-level communication structures.

The paper is organized as follows. Basic definitions and notation are introduced in Sect. 2. Section 3 is devoted to the axioms that we require from a solution for games with two-level communication structure. In Sect. 4 we define an Owen-type value for such games and show that it is the unique solution satisfying these axioms. We also give the illustrating example. In Sect. 5 we consider several special cases and show that the new solution generalizes some well-known solutions for games in coalition structure and communication graph games.

2 Preliminaries

2.1 TU games and values

A situation in which a finite set of players can obtain certain payoffs by cooperating can be described by a *cooperative game with transferable utility*, or simply a TU game, being a pair $\langle N, v\rangle$, where $N \subset \mathbb{N}$ is a finite set of n players and $v\colon 2^N \to \mathbb{R}$ is a characteristic function on N such that $v(\emptyset) = 0$. For any coalition $S \subseteq N$, $v(S)$ is the worth of coalition S, i.e., the members of coalition S can obtain a total payoff of $v(S)$ by agreeing to cooperate.

We denote the set of all characteristic functions on player set N by $\mathcal{G}^N$. For simplicity of notation and if no ambiguity appears, we write v instead of $\langle N, v\rangle$. The *subgame* of v with respect to a player set $T \subseteq N$, $T \neq \emptyset$, is the game $v|_T$ defined as $v|_T(S) = v(S)$ for all $S \subseteq T$. We denote the cardinality of a given set A by $|A|$, along with lower case letters like $n = |N|$, $m = |M|$, $n_k = |N_k|$, $s = |S|$, $c = |C|$, $c' = |C'|$, and so on. For $K \subset \mathbb{N}$, we denote $\mathbb{R}^K$ as the k-dimensional vector space which elements $x \in \mathbb{R}^K$ have components x_i, $i \in K$. For every $x \in \mathbb{R}^N$ and $S \subseteq N$, we use the standard notation $x(S) = \sum_{i\in S} x_i$ and $x_S = (x_i)_S \in \mathbb{R}^S$.

For game $v \in \mathcal{G}^N$, a vector $x \in \mathbb{R}^N$ may be considered as a payoff vector assigning a payoff x_i to each player $i \in N$. A single-valued solution, called a *value*, is a mapping ξ that assigns for every $N \subset \mathbb{N}$ and every $v \in \mathcal{G}^N$ a payoff vector $\xi(v) \in \mathbb{R}^N$. A value ξ is *efficient* if $\sum_{i\in N} \xi_i(v) = v(N)$ for every $v \in \mathcal{G}^N$ and $N \subset \mathbb{N}$. The best-known efficient value is the Shapley value (Shapley 1953), given by

$$Sh_i(v) = \sum_{\{S\subseteq N | i\in S\}} \frac{(n-s)!(s-1)!}{n!}(v(S) - v(S\setminus\{i\})), \quad \text{for all } \ i \in N.$$

2.2 Games with coalition structure

A *coalition structure* on $N \subset \mathbb{N}$ is given by a partition $\mathcal{P} = \{N_1, \ldots, N_m\}$ of N. Elements of a partition will be called a priori *unions*. Let $\mathcal{C}^N$ denote the set of all coalition structures on N. A pair $\langle v, \mathcal{P}\rangle \in \mathcal{G}^N \times \mathcal{C}^N$ constitutes a *game with coalition structure*. A game with coalition structure represents situations in which a priori unions are formed. For partition $\mathcal{P} = \{N_1, \ldots, N_m\}$, we denote $M = \{1, \ldots, m\}$, and for every $i \in N$, we denote by $k(i)$ the index of the a priori union containing player i, so, $k(i)$ is defined by the relation $i \in N_{k(i)}$. For any payoff vector $x \in \mathbb{R}^N$, let $x^{\mathcal{P}} = \big(x(N_k)\big)_{k\in M} \in \mathbb{R}^M$ be the corresponding vector of total payoffs to the a priori unions.

A value for games with coalition structure is a mapping ξ that assigns for every N and every $\langle v, \mathcal{P}\rangle \in \mathcal{G}^N \times \mathcal{C}^N$ a payoff vector $\xi(v, \mathcal{P}) \in \mathbb{R}^N$. One of the best-known values for games with coalition structure is the *Owen value* (Owen 1977) that can be seen as a two-step procedure in which the Shapley value applies twice. Namely, the Owen value assigns to player $i \in N$ his Shapley value in the game $\bar{v}_{k(i)}$, i.e.,

$$Ow_i(v, \mathcal{P}) = Sh_i(\bar{v}_{k(i)}), \quad \text{for all } i \in N,$$

while for every a priori union $k \in M$, the game $\bar{v}_k \in \mathcal{G}^{N_k}$ on player set N_k is given by

$$\bar{v}_k(S) = Sh_k(\hat{v}_S), \quad S \subseteq N_k, \tag{2.1}$$

where for every $S \subseteq N_k$, the game $\hat{v}_S \in \mathcal{G}^M$ on the player set M of a priori unions is defined by

$$\hat{v}_S(Q) = \begin{cases} v(\cup_{h\in Q} N_h), & k \notin Q, \\ v(\cup_{h\in Q\setminus\{k\}} N_h \cup S), & k \in Q, \end{cases} \quad \text{for all } Q \subseteq M. \tag{2.2}$$

It is well-known that the Owen value is efficient and satisfies the quotient game property which means that for every a priori union the total payoff to the players within that union is determined by applying the Shapley value to the so-called *quotient game* being the game $v_{\mathcal{P}} \in \mathcal{G}^M$ in which the unions act as individual players,

$$v_{\mathcal{P}}(Q) = v(\cup_{k\in Q} N_k), \quad \text{for all } \ Q \subseteq M.$$

Notice that for every $k \in M$, the game $\hat{v}_{N_k}$ is equal to the quotient game $v_{\mathcal{P}}$.

Another well-known solution for games with coalition structure is the *Aumann–Drèze* value (Aumann and Drèze 1974) which assigns to every game $\langle v, \mathcal{P}\rangle \in \mathcal{G}^N \times \mathcal{C}^N$ the payoff vector

$$AD_i(v, \mathcal{P}) = Sh_i(v|_{N_{k(i)}}), \quad \text{for all } i \in N.$$

The Aumann–Drèze value assigns to a player i the Shapley payoff of player i in the subgame on the coalition N_k containing i. Notice that $\sum_{i\in N_k} AD_i(v, \mathcal{P}) = v(N_k)$, and thus, $\sum_{i\in N} AD_i(v, \mathcal{P}) = \sum_{k\in M} v(N_k)$. Therefore, the Aumann–Drèze value is not efficient. In fact, according to the Aumann–Drèze value it is assumed that every a priori union is a stand-alone coalition.

2.3 Communication graph games

For $N \subset \mathbb{N}$, a communication structure on N is specified by a *communication graph* $\langle N, \Gamma\rangle$ with $\Gamma \subseteq \Gamma^N = \{\{i, j\} \mid i, j \in N, \ i \neq j\}$, i.e., Γ is a collection of (unordered) pairs of nodes (players), where a pair $\{i, j\}$ represents a *link* between players $i, j \in N$, and $\langle N, \Gamma^N\rangle$ is the *complete* graph on N. Again, for simplicity of notation and if no ambiguity appears, we write graph Γ instead of $\langle N, \Gamma\rangle$. Let $\mathcal{L}^N$ denote the set of all communication graphs on N. A pair $\langle v, \Gamma\rangle \in \mathcal{G}^N \times \mathcal{L}^N$ constitutes a *game with (communication) graph structure*, or simply, a *graph game* on N. For given N, the *subgraph* of a graph $\Gamma \in \mathcal{L}^N$ with respect to set $S \subseteq N$, $S \neq \emptyset$, is the graph $\Gamma|_S \in \mathcal{L}^S$ defined by $\Gamma|_S = \{\{i, j\} \in \Gamma \mid i, j \in S\}$. For ease of notation given graph Γ and link $\{i, j\} \in \Gamma$, the subgraph $\Gamma\backslash\{\{i, j\}\}$ is denoted by $\Gamma|_{-ij}$.

For a graph Γ on N, a sequence of different nodes $(i_1, \ldots, i_k)$, $k \geq 2$, is a *path* from i_1 to i_k if for all $h = 1, \ldots, k-1$, $\{i_h, i_{h+1}\} \in \Gamma$. A graph Γ on a player set N is *connected* if for any two nodes in N there exists a path in Γ from one node to the other. For given graph Γ on N, we say that the player set $S \subseteq N$ is connected if the subgraph $\Gamma|_S$ is connected. For graph Γ on player set N and $S \subseteq N$, a subset $T \subseteq S$ is a maximal connected subset, or *component* of S, if (i) $\Gamma|_T$ is connected, and (ii) for every $i \in S\backslash T$ the subgraph $\Gamma|_{T\cup\{i\}}$ is not connected. For Γ on N and $S \subseteq N$, we denote by S/Γ the set of all components of S and by $(S/\Gamma)_i$ the component of S containing $i \in S$. Notice that S/Γ is a partition of S.

A value for communication graph games, a *graph game value*, is a mapping ξ that for every $N \subset \mathbb{N}$ and every $\langle v, \Gamma\rangle \in \mathcal{G}^N \times \mathcal{L}^N$ assigns a payoff vector $\xi(v, \Gamma) \in \mathbb{R}^N$. A graph game value ξ is *component efficient* if for any $\langle v, \Gamma\rangle \in \mathcal{G}^N \times \mathcal{L}^N$, $\sum_{i\in C} \xi_i(v, \Gamma) = v(C)$ for every $C \in N/\Gamma$. A well-known component efficient graph game value is the *Myerson value*. Following Myerson (1977), we assume that in a communication graph game $\langle v, \Gamma\rangle$ only connected coalitions are able to cooperate and to realise their worths. A non-connected coalition S can only realise the sum of the worths of its components in S/Γ. This yields the *restricted game* $v^\Gamma \in \mathcal{G}^N$ defined by

$$v^\Gamma(S) = \sum_{T\in S/\Gamma} v(T), \quad \text{for all } S \subseteq N.$$

Then the Myerson value for communication graph games is the graph game value μ that assigns to every communication graph game $\langle v, \Gamma\rangle$ the Shapley value of its restricted game v^Γ, i.e.,

$$\mu(v, \Gamma) = Sh_i(v^\Gamma).$$

Myerson (1977) shows that this value is the unique graph game value that is component efficient and satisfies the so-called Myerson fairness axiom. A graph game value ξ is *fair* if for every graph game $\langle v, \Gamma\rangle$ on any player set N, for every $\{h, k\} \in \Gamma$, $\xi_h(v, \Gamma) - \xi_h(v, \Gamma|_{-hk}) = \xi_k(v, \Gamma) - \xi_k(v, \Gamma|_{-hk})$.

3 Games with two-level communication structure

We now consider situations in which the players are partitioned into a coalition structure $\mathcal{P}$ and are linked to each other by communication graphs. First, there is a communication graph Γ_M between the a priori unions M in the partition $\mathcal{P}$. Second, for each a priori union N_k, $k \in M$, there is a communication graph Γ_k between the players in N_k. Given $\mathcal{P} \in \mathcal{C}^N$, a *two-level communication structure* on N is given by the tuple $\Gamma_\mathcal{P} = \langle \Gamma_M, \{\Gamma_k\}_{k\in M}\rangle$.

For $N \subset \mathbb{N}$ and $\mathcal{P} \in \mathcal{C}^N$, let $\mathcal{L}^N_\mathcal{P}$ be the set of all two-level communication structures on N with fixed $\mathcal{P}$, and let $\mathcal{L}^N_\mathcal{C} = \bigcup_{\mathcal{P}\in\mathcal{C}^N} \mathcal{L}^N_\mathcal{P}$ be the set of all two-level graph structures on N. A tuple $\langle v, \Gamma_\mathcal{P}\rangle \in \mathcal{G}^N \times \mathcal{L}^N_\mathcal{C}$ constitutes a *game with two-level communication structure* or simply *two-level graph game* on N. A value for games with two-level communication structure, a *two-level graph game value*, is a mapping ξ that assigns for every $N \subset \mathbb{N}$ and every two-level graph game $\langle v, \Gamma_\mathcal{P}\rangle \in \mathcal{G}^N \times \mathcal{L}^N_\mathcal{C}$ a payoff vector $\xi(v, \Gamma_\mathcal{P}) \in \mathbb{R}^N$.

We now state several axioms that can be satisfied by solutions for games with two-level communication structure. The first three axioms are generalizations of axioms used to characterize the Myerson value on the class of communication graph games. First, *quotient component efficiency* states an efficiency requirement for components with respect to the graph on the level of the a priori unions (in the sequel to be called shortly the *upper level*). It states that on the upper level the total payoff of the players in the a priori unions of a component $K \in M/\Gamma_M$ is equal to the worth of the unions in the component in the quotient game $v_\mathcal{P}$ on M.

Axiom 3.1 [Quotient component efficiency (QCE)] *For any player set* $N \subset \mathbb{N}$, *for every* $\langle v, \Gamma_\mathcal{P}\rangle \in \mathcal{G}^N \times \mathcal{L}^N_\mathcal{C}$, *it holds*

$$\sum_{k\in K}\sum_{i\in N_k}\xi_i(v,\Gamma_{\mathcal{P}})=v_{\mathcal{P}}(K),\quad \textit{for every } K\in M/\Gamma_M.$$

We compare this axiom with the efficiency axioms stated in Khmelnitskaya (2014) and Kongo (2007) for games with two-level communication structure. In Khmelnitskaya (2014) there are two efficiency axioms. The first one, called component efficiency in the quotient (CEQ), is similar to QCE for components with respect to the graph on the unions. The second axiom requires component efficiency with respect to the graph within the union for every union which is not internally connected. In Kongo (2007) there is one efficiency axiom, called two-level component efficiency (TCE), but this axiom states both requirements with respect to the graph on the unions for components containing at least two unions, as well as for every stand-alone union requirements for components of players with respect to its internal graph. To compare the requirements of the different axioms first notice that for every non-singleton component of the graph on the unions our QCE requires the same as the CEQ axiom used in Khmelnitskaya (2014) and the TCE axiom used in Kongo (2007), i.e., for a union connected to other unions it requires that the union is able to distribute not only its own worth, but also the surplus (positive or negative) obtained from its cooperation with other unions. Second, when a component of the graph on the unions is a singleton union, our QCE axiom requires the same as the CEQ axiom and the TCE axiom if the union is internally connected, namely efficiency (the total payoff to the players in the union is equal to its worth). However, for a component of the graph on the unions being a singleton union which is not internally connected, our QCE still requires efficiency, whereas the TCE axiom requires Myerson's component efficiency with respect to the internal graph and the CEQ axiom requires that the total payoff of the players in the union is equal to the sum of the worths of its components with respect to the internal graph. So, in contrast to Khmelnitskaya (2014) and Kongo (2007) for stand-alone unions we require that the payoff which a union obtains in the game between the unions is fully distributed amongst its members, irrespective whether the union itself is connected (of course the distribution depends on the internal graph). This deviates from the standard approach of component efficiency as introduced by Myerson (1977), but recently efficient values for one-level graph games have been motivated and studied in e.g. Casajus (2007), Hamiache (2012), Béal et al. (2012) and van den Brink et al. (2012). In fact, the difference between axioms QCE and CEQ stating efficiency requirements at the upper level between a priori unions is determined by different assumptions concerning the redistribution of total shares of a priori unions among their members. Following Myerson in Khmelnitskaya (2014) it is assumed that only connected players are able to cooperate, and therefore, only connected components within a priori unions are able to realize their worth. Different to that in this paper we assume that a priori unions are not only able, but also are responsible to distribute their payoffs obtained in the game between the unions among their members.

The next axiom applies the well-known Myerson fairness axiom between unions, i.e., it applies fairness on the upper level with respect to the quotient game. If a link $\{k,h\}\in\Gamma_M$ is removed from the graph Γ_M on the upper level, then the change in the total payoff to a priori union N_k is equal to the change in the total payoff to a priori union N_h. For $\Gamma_{\mathcal{P}}=\langle\Gamma_M,\{\Gamma_l\}_{l\in M}\rangle$ and link $\{k,h\}\in\Gamma_M$, we denote by $\Gamma_{\mathcal{P}}|_{-kh}$ the tuple $\langle\Gamma_M|_{-kh},\{\Gamma_l\}_{l\in M}\rangle$.

Axiom 3.2 [Quotient fairness (QF)] *For any player set $N\subset\mathbb{N}$, for every $\langle v,\Gamma_{\mathcal{P}}\rangle\in\mathcal{G}^N\times\mathcal{L}^N_C$ and every $\{k,h\}\in\Gamma_M$, it holds*

$$\sum_{i\in N_k}(\xi_i(v,\Gamma_{\mathcal{P}})-\xi_i(v,\Gamma_{\mathcal{P}}|_{-kh}))=\sum_{i\in N_h}(\xi_i(v,\Gamma_{\mathcal{P}})-\xi_i(v,\Gamma_{\mathcal{P}}|_{-kh})).$$

Quotient fairness is similar to 'fairness in the quotient' used by Vázquez-Brage et al. (1996), but within the different framework of only one communication graph between all players. The quotient fairness axiom is weaker than the 'between block fairness' in Kongo (2007) which not only requires quotient fairness, but also involves the additional requirement that when in Γ_M a link between two unions is deleted, within each of the two unions the change in payoff is the same for all players within that union.

In the next section it will be shown that the axioms above uniquely determine the total payoff to every a priori union N_k in the coalition structure $\mathcal{P}$, similar as in Myerson (1977) for a one-level communication graph. In fact, it follows that the total payoff to coalition N_k is equal to the Myerson payoff to union $k \in M$ of the quotient game $v_{\mathcal{P}}$ with respect to the upper level communication graph Γ_M between the unions.

The next two axioms will determine for every $k \in M$ the distribution of the total payoff assigned to coalition N_k amongst the players in N_k. The first one applies the Myerson fairness axiom within the unions, i.e., if a link $\{i, j\} \in \Gamma_k$ is removed from the communication graph Γ_k within the union N_k, then the change of payoff to player i is equal to the change of payoff to player j. For $\Gamma_{\mathcal{P}} = \langle \Gamma_M, \{\Gamma_h\}_{h \in M} \rangle$ and link $\{i, j\} \in \Gamma_k$, $k \in M$, we denote by $\Gamma_{\mathcal{P}}|^k_{-ij}$ the tuple $\langle \Gamma_M, \{\widehat{\Gamma}_h\}_{h \in M} \rangle$, where $\widehat{\Gamma}_h = \Gamma_h$ for $h \neq k$, and $\widehat{\Gamma}_k = \Gamma_k|_{-ij}$.

Axiom 3.3 [Union fairness (UF)] *For any player set $N \subset \mathbb{N}$, for every $\langle v, \Gamma_{\mathcal{P}} \rangle \in \mathcal{G}^N \times \mathcal{L}^N_C$, $k \in M$, and $\{i, j\} \in \Gamma_k$, it holds*

$$\xi_i(v, \Gamma_{\mathcal{P}}) - \xi_i(v, \Gamma_{\mathcal{P}}|^k_{-ij}) = \xi_j(v, \Gamma_{\mathcal{P}}) - \xi_j(v, \Gamma_{\mathcal{P}}|^k_{-ij}).$$

The union fairness axiom is the same as the 'within block fairness' axiom in Kongo (2007). Quotient fairness requires Myerson fairness on the upper level, while union fairness requires Myerson fairness on the lower level. In the '$(m + 1)$-tuple of deletion link axioms' used in Khmelnitskaya (2014), Myerson fairness can also be applied both on the upper level and the lower level. In this case the requirement of $(m + 1)$-tuple of deletion link axioms in Khmelnitskaya (2014) is similar to the total requirement of both quotient fairness and union fairness axioms.

As it was already mentioned before, the total payoff assigned to the players in N_k in the quotient game on the upper level has to be fully distributed over the players in N_k in the game within the union, also when the communication graph Γ_k partitions the union N_k into several components. So, within an a priori union N_k we have efficiency in the sense that the total payoff assigned to N_k is distributed, and thus, within N_k the component efficiency axiom does not hold. The last axiom determines the distribution of the total payoff to N_k among the several components of N_k in the communication graph Γ_k. For some $k \in M$ and component $C \in N_k/\Gamma_k$, let v^k_C denote the subgame $v|_{(N \setminus N_k) \cup C}$ of v with respect to the coalition $(N \setminus N_k) \cup C$. Further, let $\mathcal{P}^k_C$ denote the partition on $(N \setminus N_k) \cup C$ consisting of union C and all unions N_h in $\mathcal{P}$, $h \neq k$, and let $\Gamma_{\mathcal{P}^k_C} = \langle \Gamma_M, \{\widetilde{\Gamma}_h\}_{h \in M} \rangle$ with $\widetilde{\Gamma}_k = \Gamma_k|_C$ and $\widetilde{\Gamma}_h = \Gamma_h$ for all $h \in M \setminus \{k\}$ denote the two-level communication structure that is obtained from $\langle \Gamma_M, \{\Gamma_h\}_{h \in M} \rangle$ by replacing the communication graph Γ_k by its restriction on $C \subset N_k$.[1] This axiom applies the fair distribution of the surplus axiom for communication graph games, introduced recently in van den Brink et al. (2012), to graph games within the unions. As shown in Béal et al. (2013), there is a unique efficient and fair extension of the Myerson value on the class of connected graph games to the class of all graph games. Since

[1] Note that in this axiom we consider games with two-level communication structure where the player set N is replaced by $(N \setminus N_k) \cup C$. To be precise we therefore need to write such a game as a triple $\langle (N \setminus N_k) \cup C, v^k_C, \Gamma_{\mathcal{P}^k_C} \rangle$. Since the player set is clear from the context, we ignore the player set in the notation of a game.

the value proposed in van den Brink et al. (2012) is such an extension, it follows that this unique extension must satisfy the axiom of fair distribution of the surplus.

Axiom 3.4 [Fair distribution of the surplus within unions (FDSU)] *For any player set $N \subset \mathbb{N}$, for every $\langle v, \Gamma_{\mathcal{P}} \rangle \in \mathcal{G}^N \times \mathcal{L}_C^N$, $k \in M$, and any two components $C, C' \in N_k/\Gamma_k$, it holds*

$$\frac{1}{c} \sum_{i \in C} \left(\xi_i(v, \Gamma_{\mathcal{P}}) - \xi_i \left(v_C^k, \Gamma_{\mathcal{P}_C^k} \right) \right) = \frac{1}{c'} \sum_{i \in C'} \left(\xi_i(v, \Gamma_{\mathcal{P}}) - \xi_i \left(v_{C'}^k, \Gamma_{\mathcal{P}_{C'}^k} \right) \right).$$

Notice that this axiom only states a requirement for the distribution of the total payoff within a union N_k when N_k consists of multiple components with respect to the internal communication graph Γ_k, otherwise the requirement reduces to an identity. In case there are multiple components, fair distribution of the surplus within unions means that the excess (positive or negative), realized by the players of N_k when they all cooperate together in the game between the unions (instead of the cooperation within N_k being restricted to players within one component of N_k/Γ_k), is distributed to the components in proportion to the number of players in the components. In other words, considering only the players in component C or C' in Γ_k instead of all players in N_k, the change in the average payoff of the players in one of these components, say C, resulting from considering only the players in C, is the same as for the other component C' when considering only the players in C'.

Similar as 'balanced per capita contributions' introduced by Gómez-Rúa and Vidal-Puga (2011) for cooperative games with levels structure of cooperation [cf. Winter (1989)], the axiom of fair distribution of the surplus within unions equalizes changes in per capita payoffs in certain coalitions. Besides the fact that the model considered here is different, a major difference between these two axioms is that fair distribution of the surplus within unions equalizes the per capita change in payoff to a coalition (or component) if only this component from the components in its union stays in the game, while 'balanced per capita contributions' considers the effect on the per capita payoff in one coalition when the other coalition leaves. Although this boils down to the same if a union has two components, in case there are more than two components the two axioms consider different sets of players leaving the game.

It is not difficult to check that considering a component $C \in N_k/\Gamma_k$, the expression

$$\frac{1}{c} \sum_{i \in C} \left(\xi_i(v, \Gamma_{\mathcal{P}}) - \xi_i \left(v_C^k, \Gamma_{\mathcal{P}_C^k} \right) \right) = \frac{1}{n_k} \sum_{i \in N_k} \left(\xi_i(v, \Gamma_{\mathcal{P}}) - \xi_i \left(v_{(N_k/\Gamma_k)_i}^k, \Gamma_{\mathcal{P}_{(N_k/\Gamma_k)_i}^k} \right) \right)$$

provides an alternative representation of the fair distribution of the surplus within unions axiom.

4 An Owen-type value for two-level graph games

In this section we first show that there exists a two-level graph game value that satisfies the four axioms. After that we show that this solution is characterized by the four axioms, i.e., it is the unique two-level graph game value satisfying these axioms.

Analogously to the Owen value for games with coalition structure, we introduce an Owen-type value for the class of games with two-level communication structure. First, for every $k \in M$ and $S \subseteq N_k$ recall the game $\hat{v}_S \in \mathcal{G}^M$ on the player set M of a priori unions defined by (2.2), where the worth of a coalition Q of a priori unions of M equals to the worth of the union of all unions in Q, except that union N_k is replaced by $S \subseteq N_k$. We now take into account the communication graph Γ_M between the a priori unions. Instead of the game

$\bar{v}_k \in \mathcal{G}^{N_k}$ on player set N_k given by (2.1) we define a game $\widetilde{v}_k \in \mathcal{G}^{N_k}$ by taking the Myerson value of $\hat{v}$ with respect to Γ_M instead of the Shapley value of $\hat{v}$. So,

$$\widetilde{v}_k(S) = \mu_k(\hat{v}_S, \Gamma_M) = Sh_k(\hat{v}_S^{\Gamma_M}), \quad \text{for all } S \subseteq N_k.$$

Notice that $\widetilde{v}_k(N_k) = Sh_k(\hat{v}_{N_k}^{\Gamma_M}) = Sh_k(v_{\mathcal{P}}^{\Gamma_M})$, i.e., the worth of N_k in the game $\widetilde{v}_k$ is equal to the Myerson value of $k \in M$ (representing union N_k) in the quotient game with respect to the communication graph Γ_M. Next, recall again from Sect. 2.2 that without communication graphs the Owen value of a player $i \in N_k$ is the Shapley payoff to player i in the game $\bar{v}_k \in \mathcal{G}^{N_k}$. Taking into account the communication graph Γ_k within N_k, we take for player $i \in N_k$ its Shapley payoff in a modification of the Myerson restricted game $\widetilde{v}_k^{\Gamma_k}$ of the game $\widetilde{v}_k \in \mathcal{G}^{N_k}$. The modification concerns the worth of the coalition N_k itself for which we take its own worth $\widetilde{v}_k(N_k)$ instead of the sum of the worths of components $\sum_{C \in N_k/\Gamma_k} \widetilde{v}_k(C)$. This is because the players in N_k have to distribute the total payoff assigned to a priori union N_k in the restricted quotient game. The value constructed in this way is denoted by ψ, so,

$$\psi_i(v, \Gamma_{\mathcal{P}}) = Sh_i(\tilde{\tilde{v}}_{k(i)}), \quad \text{for all } i \in N, \; N \in \mathbb{N},$$

where for all $k \in M$, $\tilde{\tilde{v}}_k \in \mathcal{G}^{N_k}$ is defined by

$$\tilde{\tilde{v}}_k(S) = \begin{cases} \widetilde{v}_k^{\Gamma_k}(S), & S \subsetneq N_k, \\ \widetilde{v}_k(N_k) = Sh_k(v_{\mathcal{P}}^{\Gamma_M}), & S = N_k. \end{cases}$$

Analogously to the Owen value, the value ψ can be seen as a two-step procedure in which first every coalition $S \subseteq N_k$, $k \in M$, gets its Shapley value in the Myerson restriction of the quotient game $\hat{v}_S$ with respect to communication graph Γ_M, and second, every player i in a priori union N_k gets its Shapley payoff in the within a priori union game $\tilde{\tilde{v}}_k \in \mathcal{G}^{N_k}$. We now have the following proposition.

Proposition 4.1 *The two-level graph game value ψ satisfies QCE, QF, UF and FDSU.*

Proof *QCE*. First,

$$\sum_{i \in N_k} \psi_i(v, \Gamma_{\mathcal{P}}) = \sum_{i \in N_k} Sh_i(\tilde{\tilde{v}}_{k(i)}) = Sh_k(v_{\mathcal{P}}^{\Gamma_M}) = \mu_k(v_{\mathcal{P}}, \Gamma_M), \tag{4.1}$$

where the first equality follows by definition of ψ, the second equality follows from efficiency of the Shapley value, and the third equality follows from the definition of the Myerson value μ. Thus, for every $K \in M/\Gamma_M$ we have

$$\sum_{k \in K} \sum_{i \in N_k} \psi_i(v, \Gamma_{\mathcal{P}}) = \sum_{k \in K} \mu_k(v_{\mathcal{P}}, \Gamma_M) = v_{\mathcal{P}}(K),$$

where the first equality follows from (4.1) and the second equality follows from component efficiency of μ.

QF. We have

$$\begin{aligned} &\sum_{i \in N_k} \psi_i(v, \Gamma_{\mathcal{P}}) - \sum_{i \in N_k} \psi_i(v, \Gamma_{\mathcal{P}}|_{-kh}) \\ &= \mu_k(v_{\mathcal{P}}, \Gamma_M) - \mu_k(v_{\mathcal{P}}, \Gamma_M|_{-kh}) = \mu_h(v_{\mathcal{P}}, \Gamma_M) - \mu_h(v_{\mathcal{P}}, \Gamma_M|_{-kh}) \\ &= \sum_{i \in N_h} \psi_i(v, \Gamma_{\mathcal{P}}) - \sum_{i \in N_h} \psi_i(v, \Gamma_{\mathcal{P}}|_{-kh}), \end{aligned}$$

where the first and third equality follow from (4.1), and the second equality follows by fairness of μ.

UF. By definition

$$\tilde{\tilde{v}}_{k(i)} = \tilde{v}_{k(i)}^{\Gamma_{k(i)}} + w, \tag{4.2}$$

where $w \in \mathcal{G}^{N_{k(i)}}$ is given by

$$w(S) = \begin{cases} 0, & S \subsetneq N_{k(i)}, \\ Sh_k(v_{\mathcal{P}}^{\Gamma_M}) - \tilde{v}_{k(i)}^{\Gamma_{k(i)}}(N_{k(i)}), & S = N_{k(i)}, \end{cases}$$

i.e., game $\tilde{\tilde{v}}_{k(i)}$ is obtained by adding $(Sh_k(v_{\mathcal{P}}^{\Gamma_M}) - \tilde{v}_{k(i)}^{\Gamma_{k(i)}}(N_{k(i)}))$ times the unanimity game[2] of $N_{k(i)}$ to the game $\tilde{v}_{k(i)}^{\Gamma_{k(i)}}$. From this it follows that

$$\begin{aligned} \psi_i(v, \Gamma_{\mathcal{P}}) &= Sh_i(\tilde{\tilde{v}}_{k(i)}) = Sh_i(\tilde{v}_{k(i)}^{\Gamma_{k(i)}}) + \frac{Sh_k(v_{\mathcal{P}}^{\Gamma_M}) - \tilde{v}_{k(i)}^{\Gamma_{k(i)}}(N_{k(i)})}{n_{k(i)}} \\ &= \mu_i(\tilde{v}_{k(i)}, \Gamma_{k(i)}) + \frac{\mu_{k(i)}(v_{\mathcal{P}}, \Gamma_M) - \sum_{C \in N_{k(i)}/\Gamma_{k(i)}} \tilde{v}_{k(i)}(C)}{n_{k(i)}}, \end{aligned} \tag{4.3}$$

where the first equality follows by definition of the value ψ, the second equality follows from additivity of the Shapley value and (4.2), and the third equality follows by definition of μ and $\tilde{v}_{k(i)}^{\Gamma_{k(i)}}$. Hence,

$$\begin{aligned} \psi_i(v, \Gamma_{\mathcal{P}}) - \psi_i(v, \Gamma_{\mathcal{P}}|_{-ij}^{k(i)}) &= \mu_i(\tilde{v}_{k(i)}, \Gamma_{k(i)}) - \mu_i(\tilde{v}_{k(i)}, \Gamma_{k(i)}|_{-ij}) \\ &\quad + \frac{\mu_{k(i)}(v_{\mathcal{P}}, \Gamma_M) - \sum_{C \in N_{k(i)}/\Gamma_{k(i)}} \tilde{v}_{k(i)}(C)}{n_{k(i)}} \\ &\quad - \frac{\mu_{k(i)}(v_{\mathcal{P}}, \Gamma_M) - \sum_{C \in N_{k(i)}/\Gamma_{k(i)}|_{-ij}} \tilde{v}_{k(i)}(C)}{n_{k(i)}} \\ &= \mu_j(\tilde{v}_{k(i)}, \Gamma_{k(i)}) - \mu_j(\tilde{v}_{k(i)}, \Gamma_{k(i)}|_{-ij}) \\ &\quad + \frac{\mu_{k(i)}(v_{\mathcal{P}}, \Gamma_M) - \sum_{C \in N_{k(i)}/\Gamma_{k(i)}} \tilde{v}_{k(i)}(C)}{n_{k(i)}} \\ &\quad - \frac{\mu_{k(i)}(v_{\mathcal{P}}, \Gamma_M) - \sum_{C \in N_{k(i)}/\Gamma_{k(i)}|_{-ij}} \tilde{v}_{k(i)}(C)}{n_{k(i)}} \\ &= \psi_j(v, \Gamma_{\mathcal{P}}) - \psi_j(v, \Gamma_{\mathcal{P}}|_{-ij}^{k(i)}), \end{aligned}$$

where the first and third equality follow from (4.3), and the second equality follows by fairness of μ.

FDSU. By (4.3) we obtain that for every $C \in N_k/\Gamma_k$,

$$\sum_{i \in C} \psi_i(v, \Gamma_{\mathcal{P}}) = \sum_{i \in C} \mu_i(\tilde{v}_k, \Gamma_k) + \frac{c}{n_k}\left(\mu_k(v_{\mathcal{P}}, \Gamma_M) - \sum_{H \in N_k/\Gamma_k} \tilde{v}_k(H)\right).$$

[2] It is well known (Shapley 1953) that the collection of *unanimity games* $\{u_T\}_{\substack{T \subseteq N \\ T \neq \emptyset}}$, defined as $u_T(S) = 1$, if $T \subseteq S$, and $u_T(S) = 0$ otherwise, create a basis in $\mathcal{G}^N$.

Further,

$$\sum_{i\in C}\psi_i(v^k_C,\Gamma_{\mathcal{P}^k_C}) = \widetilde{\widetilde{(v^k_C)}_k}(C) = \widetilde{(v^k_C)}_k(C) = \sum_{i\in C}\mu_i(\tilde{v}_k,\Gamma_k),$$

where the first and second equality follow from the definition of ψ, efficiency of the Shapley value and C being the only component in $\Gamma_k|_C$, and the third equality follows from component efficiency of μ and because for all $S \subseteq C$ it holds that $\widetilde{(v^k_C)}_k(S) = \tilde{v}_k(S)$. Thus,

$$\sum_{i\in C}\left(\psi_i(v,\Gamma_{\mathcal{P}}) - \psi_i(v^k_C,\Gamma_{\mathcal{P}^k_C})\right) = \frac{c}{n_k}\left(\mu_k(v_{\mathcal{P}},\Gamma_M) - \sum_{H\in N_k/\Gamma_k}\tilde{v}_k(H)\right).$$

Hence, it follows that for any $C, C' \in N_k/\Gamma_k$,

$$\frac{1}{c}\sum_{i\in C}\left(\psi_i(v,\Gamma_{\mathcal{P}}) - \psi_i(v^k_C,\Gamma_{\mathcal{P}^k_C})\right) = \frac{1}{c'}\sum_{i\in C'}\left(\psi_i(v,\Gamma_{\mathcal{P}}) - \psi_i(v^k_{C'},\Gamma_{\mathcal{P}^k_{C'}})\right),$$

showing that ψ satisfies FDSU. □

Remark Note that (4.3) gives an alternative definition of the value ψ in graph game $\langle v,\Gamma_{\mathcal{P}}\rangle$ assigning to every player $i \in N$ his Myerson value in the corresponding Owen-type game $\tilde{v}_{k(i)}$ within $N_{k(i)}$ restricted by graph $\Gamma_{k(i)}$ and distributing the difference between the Myerson value of $N_{k(i)}$ in the quotient game on M and the sum of the worths of all its components in the Owen-type game $\tilde{v}_{k(i)}$ equally over the players within $N_{k(i)}$. In this sense ψ can be seen as combining elements of the Myerson value and equal division solution. This idea is similar to Kamijo (2009) who introduced a solution for games with coalition structure that allocates to every player its Shapley value in the game restricted to its own union and distributes the excess of the Shapley value of its union in the (quotient) game between the unions over the worth of this union equally among the players in this union.

The next theorem characterizes the value ψ as the unique solution satisfying the four axioms.

Theorem 4.2 *There is a unique two-level graph game value ξ satisfying QCE, QF, UF and FDSU.*

Proof By Proposition 4.1 we only need to show uniqueness. Let $\mathcal{P} = \{N_1,\ldots,N_m\} \in \mathcal{C}^N$ and $(v,\Gamma_{\mathcal{P}}) \in \mathcal{G}^N \times \mathcal{L}^N_{\mathcal{C}}$ with $\Gamma_{\mathcal{P}} = \langle\Gamma_M,\{\Gamma_h\}_{h\in M}\rangle$. For a solution ξ we denote by $\xi^k(v,\Gamma_{\mathcal{P}}) = \sum_{i\in N_k}\xi_i(v,\Gamma_{\mathcal{P}})$ the total payoff to the players in the union N_k, $k = 1,\ldots,m$. Suppose that solution ξ satisfies the four axioms. We determine the individual payoffs in three steps.

Step 1 We determine the 'union payoffs' in the game $(v,\Gamma_{\mathcal{P}}) \in \mathcal{G}^N \times \mathcal{L}^N_{\mathcal{C}}$ with $\Gamma_{\mathcal{P}} = \langle\Gamma_M,\{\Gamma_h\}_{h\in M}\rangle$ by induction on the number of links in Γ_M in a similar way as uniqueness of the Myerson value for one-level graph games is shown in Myerson (1977). When $|\Gamma_M| = 0$, then for all $k \in M$ the set of neighboring unions $\{h \in M \mid \{h,k\} \in \Gamma\} = \emptyset$, and thus, $\xi^k(v,\Gamma_{\mathcal{P}}) = v_{\mathcal{P}}(\{k\}) = v(N_k)$ by QCE.

Proceeding by induction, assume that the values $\xi^k(v,\Gamma'_{\mathcal{P}})$ have been determined whenever $\Gamma'_{\mathcal{P}} = \langle\Gamma',\{\Gamma_h\}_{h\in M}\rangle$ for every Γ' with $|\Gamma'| < |\Gamma_M|$. Let $Q \in M/\Gamma_M$ be a component in $\langle M,\Gamma_M\rangle$. If $Q \subseteq M$ is a singleton set $\{k\}$, then it follows from QCE that $\xi^k(v,\Gamma_{\mathcal{P}}) = v(N_k)$. If $q = |Q| \geq 2$, then there exists a spanning tree $\widetilde{\Gamma} \subseteq \Gamma_M|_Q$ on Q, i.e., $\langle Q,\widetilde{\Gamma}\rangle$ is connected and $\langle Q, \widetilde{\Gamma}\backslash\{k,h\}\rangle$ is not connected for all $\{k,h\} \in \widetilde{\Gamma}$. So, the number of links in $\widetilde{\Gamma}$ is $q-1$. By QF, for all $\{k,h\} \in \widetilde{\Gamma}$ it holds that

$$\xi^k(v, \Gamma_{\mathcal{P}}) - \xi^k(v, \Gamma_{\mathcal{P}}|_{-kh}) = \xi^h(v, \Gamma_{\mathcal{P}}) - \xi^h(v, \Gamma_{\mathcal{P}}|_{-kh}). \tag{4.4}$$

Moreover, by QCE it holds that

$$\sum_{k \in Q} \xi^k(v, \Gamma_{\mathcal{P}}) = v_{\mathcal{P}}(Q). \tag{4.5}$$

Since $|\Gamma_M|_{-hk}| = |\Gamma_M| - 1$, it follows by the induction hypothesis that all the values $\xi^k(v, \Gamma_{\mathcal{P}}|_{-kh})$, $\{k,h\} \in \widetilde{\Gamma}$, have been determined, and thus, (4.4) and (4.5) yield q linear equations in the q unknown payoffs $\xi^k(v, \Gamma_{\mathcal{P}})$, $k \in Q$. Since these equations are linearly independent, for every $Q \in M/\Gamma$ all payoffs $\xi^k(v, \Gamma_{\mathcal{P}})$, $k \in Q$, are uniquely determined.[3]

Step 2 Second, similarly as in Step 1, we determine for every $k \in M$, for every subset $C \subset N_k$ the 'union payoffs' in the game $(v_C, \Gamma_{\mathcal{P}^k_C})$, where v^k_C denotes the subgame $v|_{(N\backslash N_k)\cup C}$ of v with respect to the coalition $(N\backslash N_k) \cup C$, and $\Gamma_{\mathcal{P}^k_C}$ denotes the two-level communication structure $\langle \Gamma_M, \{\Gamma_h\}_{h \in M}\rangle$, where Γ_M is the communication graph on the partition $(\mathcal{P}\backslash\{N_k\}) \cup \{C\}$ (where the 'position' of N_k is taken over by C) and with the communication graph Γ_k replaced by its restriction on $C \subset N_k$. Note that now, for $k \in M$, the union payoff $\xi^k(v^k_C, \Gamma_{\mathcal{P}^k_C})$ is the total payoff to the players in $C \subset N_k$ in the game $(v^k_C, \Gamma_{\mathcal{P}^k_C})$.

Step 3 Third, we determine the individual payoffs in every coalition N_k, $k \in M$. Take some $k \in M$. If $|\Gamma_k| = 0$, then $\{i\} \in N_k/\Gamma_k$ for all $i \in N_k$. FDSU then implies that

$$\xi_i(v, \Gamma_{\mathcal{P}}) - \xi^k(v^k_{\{i\}}, \Gamma_{\mathcal{P}^k_{\{i\}}}) = \frac{\xi^k(v, \Gamma_{\mathcal{P}}) - \sum_{j \in N_k} \xi^k\left(v^k_{\{j\}}, \Gamma_{\mathcal{P}^k_{\{j\}}}\right)}{n_k}, \quad \text{for all } i \in N_k. \tag{4.6}$$

From Steps 1 and 2 above we know $\xi^k(v, \Gamma_{\mathcal{P}})$ and $\xi^k(v^k_{\{j\}}, \Gamma_{\mathcal{P}^k_{\{j\}}})$ for all $j \in N_k$. So, equation (4.6) determines $\xi_i(v, \Gamma_{\mathcal{P}})$ for all $i \in N_k$.

Now we proceed by induction similar as in Step 1, but first we show that for each component $C \in N_k/\Gamma_k$ the total payoff to the players in C is uniquely determined. The payoff $\xi^k(v, \Gamma_{\mathcal{P}})$ to the a priori union N_k has been determined already in Step 1, so,

$$\sum_{i \in N_k} \xi_i(v, \Gamma_{\mathcal{P}}) = \xi^k(v, \Gamma_{\mathcal{P}}). \tag{4.7}$$

If N_k is the unique component in N_k/Γ_k, then FDSU does not state any requirement. When N_k/Γ_k consists of multiple components, then for every component $C \in N_k/\Gamma_k$, FDSU states that

$$\frac{\sum_{i \in C} \xi_i(v, \Gamma_{\mathcal{P}}) - \xi^k(v^k_C, \Gamma_{\mathcal{P}^k_C})}{c} = \frac{\xi^k(v, \Gamma_{\mathcal{P}}) - \sum_{K \in N_k/\Gamma_k} \xi^k(v^k_K, \Gamma_{\mathcal{P}^k_K})}{n_k}. \tag{4.8}$$

Notice that every payoff ξ^k in this equation has been determined in either Step 1 or 2. We now prove the induction step similar as in Step 1 and as in Myerson (1977). Let $\Gamma'_{\mathcal{P}}$ denote the two-level graph structure $\langle \Gamma_M, \{\Gamma'_h\}_{h \in M}\rangle$ with $\Gamma'_h = \Gamma_h$ if $h \neq k$ and $\Gamma'_k = \Gamma'$ for some graph Γ' on N_k. Above we already showed that the payoffs in N_k are determined if $|\Gamma_k| = 0$. Now, assume that the values $\xi_i(v, \Gamma'_{\mathcal{P}})$ have been determined for every Γ' with $|\Gamma'| < |\Gamma_k|$. Let $C \in N_k/\Gamma_k$ be a component in (N_k, Γ_k). If $C \subseteq N_k$ is a singleton set $\{i\}$, then the payoff

[3] Note that in the proof of the induction step every possible spanning tree $\widetilde{\Gamma}$ yields the same solution for the values $\xi^k(v, \Gamma_{\mathcal{P}})$, $k \in Q$, because otherwise a solution does not exist, which contradicts Proposition 4.1.

$\xi_i(v, \Gamma_{\mathcal{P}})$ of the single player $i \in C$ follows from (4.8). If $c = |C| \geq 2$, then there exists a spanning tree $\widetilde{\Gamma} \subseteq \Gamma_k|_C$ on C. So, the number of links in $\widetilde{\Gamma}$ is $c-1$. By UF, for all $\{i, j\} \in \widetilde{\Gamma}$ it holds that

$$\xi_i(v, \Gamma_{\mathcal{P}}) - \xi_i\left(v, \Gamma_{\mathcal{P}}|^k_{-ij}\right) = \xi_j(v, \Gamma_{\mathcal{P}}) - \xi_j\left(v, \Gamma_{\mathcal{P}}|^k_{-ij}\right). \tag{4.9}$$

Since $|\Gamma_k|_{-ij}| = |\Gamma_k| - 1$, by the induction hypothesis it follows that all payoffs $\xi_i(v, \Gamma_{\mathcal{P}}|^k_{-ij})$, $\{i, j\} \in \widetilde{\Gamma}$, have been determined. If $C \neq N_k$, then the equations (4.8) and (4.9) yield c linearly independent equations with c unknown payoffs $\xi_i(v, \Gamma_{\mathcal{P}})$, $i \in C$. If $C = N_k$, then the Eqs. (4.7) and (4.9) yield c linearly independent equations in the c unknown payoffs $\xi_i(v, \Gamma_{\mathcal{P}})$, $i \in C$. Hence, for every $C \in N_k/\Gamma_k$, all payoffs $\xi_i(v, \Gamma_{\mathcal{P}})$, $i \in C$, are uniquely determined. □

Note that in the proof of Theorem 4.2 we use QCE and QF to determine the sum of the payoffs in every union, similar as done in Myerson (1977). In fact, we consider Γ_M as a one-level graph on M. But we apply FDSU to obtain uniqueness on the individual level. We cannot apply a similar proof as at the upper level using component efficiency to determine the individual payoffs inside each union because the total payoff to the players in each union should be equal to the total payoff to the union as determined in Step 1, which in general could differ from the sum of the payoffs that the components of the communication graph within the union obtain in the internal game. In fact, the latter observation puts a severe limitation on the application of the component efficient values on both levels in cases when communication graphs within the unions are not connected (cf. Khmelnitskaya (2014) for the detailed discussion on this the problem).

It remains to show that the four axioms are logically independent.

1. (*Equal division within the a priori unions*) Let the two-level graph game value $\xi^{(1)}$ assign in every $\langle v, \Gamma_{\mathcal{P}}\rangle \in \mathcal{G}^N \times \mathcal{L}^N_{\mathcal{C}}$ to every player $i \in N_k$, $k \in M$, payoff

   $$\xi^{(1)}_i(v, \Gamma_{\mathcal{P}}) = \frac{\widetilde{v}_k(N_k)}{n_k}.$$

   This value divides for each a priori union $k \in M$ the worth $\widetilde{v}_k(N_k)$ of coalition N_k in the restricted quotient game equally amongst the players in N_k. It satisfies quotient component efficiency, quotient fairness and union fairness, but does not satisfy fair distribution of the surplus within unions.
2. (*Equal division within the components of the a priori unions*) Let the two-level graph game value $\xi^{(2)}$ assign in every $\langle v, \Gamma_{\mathcal{P}}\rangle \in \mathcal{G}^N \times \mathcal{L}^N_{\mathcal{C}}$ to every player $i \in C$, $C \in N_k/\Gamma_k$, $k \in M$, payoff

   $$\xi^{(2)}_i(v, \Gamma_{\mathcal{P}}) = \frac{\tilde{\widetilde{v}}_k(C)}{c} + \frac{\widetilde{v}_k(N_k) - \sum_{H \in N_k/\Gamma_k} \tilde{\widetilde{v}}_k(H)}{n_k}.$$

   Each player $i \in C \in N_k/\Gamma_k$ gets an equal share in the worth $\tilde{\widetilde{v}}_k(C)$ of his component and an equal share in the surplus of N_k over the sum of the worths of the components in N_k/Γ_k. This value satisfies quotient component efficiency, quotient fairness and fair distribution of the surplus within unions, but it does not satisfy union fairness.
3. (*Equal division within the components of the upper-level structure*) Let the two-level graph game value $\xi^{(3)}$ be defined for every $\langle v, \Gamma_{\mathcal{P}}\rangle \in \mathcal{G}^N \times \mathcal{L}^N_{\mathcal{C}}$ by

   $$\xi^{(3)}_i(v, \Gamma_{\mathcal{P}}) = Sh_i(w_{k(i)}), \quad \text{for all } i \in N,$$

where for a priori union $k \in M$, belonging to a component $K \in M/\Gamma_M$ of the upper-level structure, the game $w_k \in \mathcal{G}^{N_k}$ is defined by

$$w_k(S) = \begin{cases} \tilde{v}_k^{\Gamma_k}(S), & S \subsetneq N_k, \\ \dfrac{1}{|K|} v(K), & S = N_k. \end{cases}$$

In this case every a priori union N_k gets an equal share in the worth of the component to which it belongs in the upper level structure. This value satisfies quotient component efficiency, union fairness and fair distribution of the surplus within unions, but it does not satisfy quotient fairness.

4. (*Efficient total payoff distribution*) Let the two-level graph game value $\xi^{(4)}$be defined for every $\langle v, \Gamma_{\mathcal{P}} \rangle \in \mathcal{G}^N \times \mathcal{L}_C^N$ by

$$\xi_i^{(4)}(v, \Gamma_{\mathcal{P}}) = Sh_i(w^*_{k(i)}), \quad \text{for all } i \in N,$$

where for a priori union $k \in M$ $w_k^* \in \mathcal{G}^{N_k}$ is defined by

$$w_k^*(S) = \begin{cases} \tilde{v}_k^{\Gamma_k}(S), & S \subsetneq N_k, \\ Sh_k(\bar{w}), & S = N_k, \end{cases}$$

with game $\bar{w}$ on M defined by $\bar{w}(Q) = v_{\mathcal{P}}^{\Gamma_M}(Q)$ for every $Q \subsetneq M$ and $\bar{w}(M) = v_{\mathcal{P}}(M) = v(N)$. In this case the total payoff is equal to the worth $v(N)$ of the grand coalition N of all players, i.e., $\xi^{(4)}$ is efficient. This value $\xi^{(4)}$ satisfies quotient fairness, union fairness and fair distribution of the surplus within unions, but it does not satisfy quotient component efficiency.

We now give an illustrating example.

Example 4.3 A governmental research fund is available for interdisciplinary research within a restricted list of different disciplines. To be succesfull, a proposal has to be submitted by researchers from at least two different disciplines, or by a consortium of different institutions containing researchers of at least two different disciplines. Suppose there are six different researchers located in three different universities, indexed U_1, U_2 and U_3, and in three different disciplines, indexed D_1, D_2 and D_3. University U_1 has three researchers, indexed 1,2,3. The researchers 1 and 2 are in the same faculty and in discipline D_1, researcher 3 is in another faculty and in discipline D_2. Researcher 4 is in discipline D_3 and in university U_2. Researchers 5 and 6 are in University U_3, in two different disciplines and in two different faculties. Two researchers can only cooperate with each other if they are in the same faculty, so, only the researchers 1 and 2 can cooperate. Two universities can only cooperate if there is a long-standing relation between them, there is such a relation between the universities U_1 and U_2, but such a relationship does not exist between the other pairs. These cooperation possibilities can be modeled as a two-level graph structure with coalition structure $\mathcal{P} = \{N_1, N_2, N_3\}$ given by $N_1 = \{1, 2, 3\}$, $N_2 = \{4\}$, and $N_3 = \{5, 6\}$. The graph on the union set $M = \{1, 2, 3\}$ is given by $\Gamma_M = \{1, 2\}$, the graph on N_1 is given by $\Gamma_1 = \{1, 2\}$, and the graphs on N_2 and N_3 are empty. The worth of a coalition is the grant that a coalition can obtain and is given by $v(S) = 2$ if $|S| = 2$ and contains researchers of at least two different disciplines, $v(S) = 6$ if $|S| = 3$ and contains researchers of at least two different disciplines, and $v(S) = 12$ if $|S| > 3$ and contains researchers of at least two different disciplines. It follows that $v(\{i\}) = 0$ for every i and $v(N_3) = 2$. Since N_3 is not connected to any of the other two unions, the worth of any coalition containing players from inside N_3 and outside N_3 is not relevant. It remains to consider $v(S)$ for $S \subseteq N_1 \cup N_2$. Since 1 and 2 are in discipline D_1, 3 in D_2 and 4 in D_3, it follows that for $S \subseteq N_1 \cup N_2$ we have that $v(S) = 2$ if $|S| = 2$, except that $v(\{1, 2\} = 0$, $v(S) = 6$ if $|S| = 3$ and $v(S) = 12$ if $S = N_1 \cup N_2$.

We now discuss various solution concepts. The solution based on the application of the Myerson value both on the upper level between a priori unions and on the lower level within each union discussed in Khmelnitskaya (2014) is not applicable in this case because the second necessary condition for application of component efficient values on both levels is violated (cf. Khmelnitskaya 2014, page 42, condition (ii)).

Next we consider the solution of Kongo (2007). His TCE axiom requires for the stand-alone coalition N_3 internal component efficiency, so, it follows immediately that the payoffs of the players 5 and 6 are given by $f_5^K = f_6^K = 0$. Next, for the component $\{1, 2\}$ on the level of unions the TCE axiom requires component efficiency on the level of the unions, so, the total payoff to the players in $N_1 \cup N_2$ is equal to $v(N_1 \cup N_2) = 12$. To distribute this payoff the players first get a payoff according to the Myerson payoff on the subgame within the unions. This yields payoff zero for all players, since player 4 is the only player in N_2 and can't generate any payoff on its own, and the internal Myerson restricted game of the subgame on N_1 is the null game, because any coalition is either not internally connected, or does not contain researchers of two different disciplines. So, this leaves a surplus of 12. By the between block fairness axiom (BBF) of Kongo (2007) it follows that the single player 4 in N_2 receives payoff of 6, whereas the remaining payoff of 6 is equally distributed between the three players in N_1. This gives payoff vector $f^K = (2, 2, 2, 6, 0, 0)^\top$. Notice that internal component efficiency on N_1 would require that every player in N_1 would get payoff of zero. However, the university board has still the governance structure that allows to distribute its total payoff obtained as a union among its researchers. On the other hand, when university U_3 acts as a union it can obtain a payoff of 2 by joining the efforts of its two researchers. However, for this stand-alone university Kongo's TCE axiom requires internal component efficiency, which illustrates the discrepancy between stand-alone unions and connected unions.

Finally we consider the solution proposed in this paper. Now QCE requires that total payoff to union N_3 is equal to 2 which is distributed equally among players 5 and 6 by the FDSU axiom. This gives payoffs $f_5 = f_6 = 1$. Although the players can not cooperate, the university acting as union can still obtain a grant of 2 and has the governance structure to allocate this grant to its researchers. The QCE axiom also requires that the total payoff to coalition $N_1 \cup N_2$ is equal to 12. When the link between union N_1 and union N_2 is deleted and, so, we have three stand-alone unions, then QCE states that the payoff to union N_1 is equal to $v(N_1) = 6$ and the payoff to union N_2 is $v(N_2) = 0$. By QF we have that the surplus of 6 generated by the link between N_1 and N_2 is equally distributed amongst the two unions, so, the total payoff to union N_1 is equal to 9 and the payoff to N_2 is 3. So, the single player 4 in N_2 gets payoff 3. It remains to distribute the payoff of 9 among the players in N_1. Any single player of N_1 acting on behalf of the union can obtain a payoff of 1 when cooperating with union $N_2 = \{4\}$ (their total payoff of 2 is divided equally). Further any coalition of two players of N_1 acting on behalf of the union can obtain a payoff of 3 when cooperating with union N_2 (their total payoff of 6 is divided equally between N_2 and the coalition of two players). Since only the players 1 and 2 are internally connected, the restricted internal game $\langle N_1, \tilde{\tilde{v}}_1 \rangle$, as defined in the beginning of this section, is given by $\tilde{\tilde{v}}_1(\{i\}) = 1$, $i \in N_1$, $\tilde{\tilde{v}}_1(\{1, 3\}) = \tilde{\tilde{v}}_1(\{2, 3\}) = 2$, $\tilde{\tilde{v}}_1(\{1, 2\}) = 3$, and $\tilde{\tilde{v}}_1(N_1) = 9$. Application of the Shapley value yields payoffs of $\frac{19}{6}$ to players 1 and 2 and payoff of $\frac{16}{6}$ to player 3. So, the payoff vector is $f = \left(\frac{19}{6},\ \frac{19}{6},\ \frac{16}{6},\ 3,\ 1,\ 1\right)^\top$. There are three differences compared with the payoff vector of Kongo's solution: (1) the players 5 and 6 get a positive payoff, (2) the payoff of the coalition $N_1 \cup N_2$ is distributed according to their number of members, and (3) the distribution of the total payoff to union N_1 reflects the internal cooperation possibilities.

We conclude this section with a remark on the efficiency axioms. In fact, within a union we require efficiency in the sense that every union is able to distribute the total payoff assigned to the unions, irrespective whether or not the union is internally connected. On the other hand, on the level of the unions we require component efficiency. An alternative is to require also efficiency on the level of the unions. For instance, let in the example above the total fund available be of 20 and the government always be able to distribute this total amount, whether or not unions are able to cooperate, so, $v(N) = 20$. A solution for this case is easily obtained by replacing the QCE axiom by the efficiency axiom together with an additional axiom of fair distribution of the suplus between unions. In this case we have efficiency on both levels. On the other hand, in general it is impossible to require component efficiency on both levels, as has been mentioned already before and has been illustrated in the example.

5 Comparison with other values

In this final section we consider several special cases of two-level structure $\Gamma_{\mathcal{P}}$ and its corresponding Owen-type value ψ and show that, for example, the Owen value, Aumann–Drèze value (for games in coalition structure), Myerson value (for communication graph games), and equal surplus division solutions can be obtained as special cases of this value. We distinguish two types of values, one depending on special communication graphs and the other depending on special partitions.

5.1 Special communication graphs

Two special cases of a communication graph are the complete and the empty graph. In this paper these two special cases can occur both on the upper level between the unions as on the lower level within the unions. We first discuss three special cases with an empty graph on the upper level and next three special cases with a complete graph on the upper level.

1. *Empty upper level structure, complete graph within the unions: the Aumann–Drèze valued.* Consider the case $\Gamma_{\mathcal{P}}$ with Γ_M the empty graph and every Γ_k, $k \in M$, the complete graph. In this case every a priori union N_k stands alone and the Myerson value applied to the quotient game with empty communication graph assigns to every a priori union N_k, $k \in M$, its own payoff $v(N_k)$. In the game $\widetilde{v}_k$ on N_k every coalition $S \subset N_k$ gets its own worth $v(S)$, thus, $\widetilde{v}_k(S) = v(S)$ for every $S \subseteq N_k$, $k \in M$. Within the union there is no restriction on the cooperation between the players, and thus, $\tilde{\widetilde{v}}_k(S) = v(S)$ for every $S \subseteq N_k$, $k \in M$. It follows that
$$\psi_i(v, \Gamma_{\mathcal{P}}) = Sh_i(v|_{N_{k(i)}}) = AD_i(v, \mathcal{P}), \quad \text{for all } i \in N,$$
i.e., every player i gets its Shapley value within the subgame of v on the a priori union N_k containing i, and therefore, in this case the value ψ is equal to the Aumann–Drèze value (Aumann and Drèze 1974).
2. *Empty two-level structure: equal surplus division.* Consider the case $\Gamma_{\mathcal{P}}$ with both Γ_M and every Γ_k, $k \in M$, empty. As in the previous case, every a priori coalition N_k, $k \in M$, stands alone and gets its own worth $v(N_k)$. Next, within a priori union N_k every player i is a stand alone component and $\tilde{\widetilde{v}}_k(\{i\}) = v(\{i\})$ for every $i \in N_k$. Then it follows from

fair distribution of the surplus within unions that for every $k \in M$ and $i \in N_k$,

$$\psi_i(v, \Gamma_{\mathcal{P}}) = v(\{i\}) + \frac{v(N_k) - \sum_{i \in N_k} v(\{i\})}{n_k}.$$

So, in this case the value ψ assigns within each a priori union N_k the equal surplus division solution on the subgame $v|_{N_k}$, first considered in Driessen and Funaki (1991) under the name of the center of the imputation set (CIS-value). In case v is zero-normalized, and thus, $v(\{i\}) = 0$ for every $i \in N$, the value ψ yields the equal division solution within each a priori union N_k.

3. *Empty upper level structure, connected graphs within the unions: the Myerson value.* Consider the case $\Gamma_{\mathcal{P}}$ with Γ_M the empty graph and every Γ_k, $k \in M$, connected, i.e., for every $k \in M$ union N_k itself is the only element in N_k/Γ_k. Again every a priori union N_k, $k \in M$, stands alone and gets its own worth $v(N_k)$ and in the game $\widetilde{v}_k$ every coalition $S \subset N_k$ gets its own worth $v(S)$, thus, $\widetilde{v}_k(S) = v(S)$ for every $S \subseteq N_k$, $k \in M$. Since Γ_k is connected, it follows that $\widetilde{v}_k^{\Gamma_k}(N_k) = \widetilde{v}_k(N_k) = v(N_k)$ and, therefore, $\tilde{\tilde{v}}_k = v^{\Gamma_k}$. So, ψ yields to every player i in every a priori union N_k the payoff of the Myerson value in the subgame on N_k with respect to the communication graph Γ_k within N_k. Even more, let $\Gamma = \cup_{k \in M} \Gamma_k$ be the communication graph between all players obtained by taking the union of all graphs within the unions. Then, by definition every N_k is a component of Γ, i.e., $N/\Gamma = \{N_1, \ldots, N_m\}$. By component efficiency of the Myerson value it follows immediately that for the case of an empty upper level structure and connected graphs within the unions the value ψ is equal to the Myerson value μ for the game v on N with respect to the (one-level) induced communication stucture $\Gamma = \cup_{k \in M} \Gamma_k$ on N.
4. *Complete two-level structure: the Owen valued.* Consider the case $\Gamma_{\mathcal{P}}$ with both Γ_M and every Γ_k, $k \in M$, complete graphs. In this case there is no restriction on the cooperation between a priori unions and within the a priori unions. Hence, for every $Q \subseteq M$, Q is the only component of the subgraph $\Gamma_M|_Q$ and also for every k and every $C \subseteq N_k$, C is the only component of the subgraph $\Gamma_k|_C$. Therefore, ψ reduces to the Owen value on $\mathcal{P}$: $\psi(v, \Gamma_{\mathcal{P}}) = Ow(v, \mathcal{P})$. Notice that in this case quotient component efficiency reduces to efficiency and fair distribution of the surplus within unions becomes redundant.
5. *Complete upper level structure, empty graphs within the unions: equal union surplus division.* Consider the case $\Gamma_{\mathcal{P}}$ with Γ_M the complete graph and Γ_k the empty graph for every $k \in M$. Again there is no restriction on the cooperation between the unions and, therefore,

$$\widetilde{v}_k(S) = \bar{v}_k(S), \quad \text{for all } k \in M \text{ and all } S \subseteq N_k.$$

On the other hand, within an a priori union N_k every player $i \in N_k$ is a stand alone component. With $\tilde{\tilde{v}}_k(\{i\}) = \bar{v}_k(\{i\})$ for all $i \in N_k$ and $\tilde{\tilde{v}}_k(N_k) = \bar{v}_k(N_k) = Sh_k(v_{\mathcal{P}})$, the Shapley value of a priori union k in the quotient game, from fair distribution of the surplus within unions it follows that for every $k \in M$ and $i \in N_k$,

$$\psi_i(v, \Gamma_{\mathcal{P}}) = \bar{v}_k(\{i\}) + \frac{Sh_k(v_{\mathcal{P}}) - \sum_{i \in N_k} \bar{v}_k(\{i\})}{n_k}.$$

So, within a priori union N_k every player i gets its stand alone value in the game $\bar{v}_k$ plus an equal share in the surplus of N_k in the quotient game.
6. *Complete upper level structure, connected graphs within the unions: the efficient Myerson-type value of* Casajus (2007). Consider the case $\Gamma_{\mathcal{P}}$ with Γ_M the complete graph and every Γ_k, $k \in M$, connected. Again, $\widetilde{v}_k(S) = \bar{v}_k(S)$ for all $k \in M$ and $S \subseteq N_k$. Because

of connectedness of every Γ_k, the value ψ is obtained by applying within every a priori union N_k the Myerson value μ to $\widetilde{v}_k = \bar{v}_k$ with respect to Γ_k, so, for every $k \in M$ and $i \in N_k$,

$$\psi_i(v, \Gamma_{\mathcal{P}}) = \mu_i(\bar{v}_k, \Gamma_k).$$

Furthermore, every Γ_k is connected and by definition (2.1) of $\bar{v}$, $\bar{v}_k^{\Gamma_k}(N_k) = \bar{v}_k(N_k) = Sh_k(v_{\mathcal{P}})$. Then from the efficiency of the Shapley value it follows that for every k, $\sum_{i \in N_k} \psi_i(v, \Gamma_{\mathcal{P}}) = Sh_k(v_{\mathcal{P}})$ and $\sum_{i \in N} \psi_i(v, \Gamma_{\mathcal{P}}) = \sum_{k \in M} Sh_k(v_{\mathcal{P}}) = v(N)$. So, ψ distributes the total worth $v(N)$ and, thus, meets efficiency.
In fact, in this case the two-level graph game value ψ yields the same payoffs as the so-called CO-value ϕ, introduced in Theorem 4.2 of Casajus (2007) as an efficient alternative for the Myerson value for games with a one-level communication graph. For a graph game $\langle v, \Gamma \rangle$ with communication graph Γ on N, Casajus (2007) considers the collection N/Γ of components of Γ, as a coalition structure $\mathcal{P}$ induced by the communication graph Γ. Let N_k be such a component. Then, within N_k the Shapley value is applied to the Myerson restricted game of $\bar{v}_k$. This gives the same payoffs as $\psi(v, \Gamma_{\mathcal{P}})$ for the two-level structure when Γ_M is taken to be the complete graph on M and for each $k \in M$ graphs Γ_k are connected. In this case the Casajus's graph $\Gamma = \cup_{k \in M} \Gamma_k$.

5.2 Special coalition structures

Finally we discuss the two special cases with respect to the coalition structure.

1. *Partition in singletons: the Myerson value*. When $\mathcal{P} = \{\{1\}, \ldots, \{n\}\}$ every a priori union consists of a single player i and there is no game within the unions. Hence, the value ψ reduces to the Myerson value with respect to the upper level graph structure Γ_M. Thus,

$$\psi_i(v, \Gamma_{\mathcal{P}}) = \mu_i(v, \Gamma_M) = Sh_i(v^{\Gamma_M}), \quad \text{for all } i \in N.$$

2. *Single a priori union: the efficient Myerson valued*. Consider the case $\mathcal{P} = \{N\}$, i.e., the grand coalition N itself is the singleton a priori union within the coalition structure $\mathcal{P}$. In this case $m = 1$ and denoting $\tilde{\tilde{v}} = \tilde{\tilde{v}}_1$ and $\Gamma = \Gamma_1$ for the single a priori union $N = N_1$ in $\mathcal{P}$ we have

$$\tilde{\tilde{v}}(S) = \begin{cases} v^{\Gamma}(S), & S \subsetneq N, \\ v(N), & S = N. \end{cases}$$

By definition, the value ψ assigns the Shapley value of the game $\tilde{\tilde{v}}$ on N, which equals to the Myerson value of $\langle v, \Gamma \rangle$ plus an equal split of the excess of the worth of the grand coalition over the total worth of all components in graph Γ, i.e.,

$$\psi_i(v, \Gamma_{\mathcal{P}}) = \mu_i(v, \Gamma) + \frac{v(N) - \sum_{C \in N/\Gamma} v(C)}{n}, \quad \text{for all } i \in N.$$

It appears that this is the unique value that satisfies union fairness (within the grand coalition N) and efficiency. Consider this case as just a one-level communication graph game $\langle v, \Gamma \rangle$ on N and recall that the Myerson value is the unique value that satisfies component efficiency and fairness. In fact, in case of $\mathcal{P} = \{N\}$ the value ψ yields the same payoffs as the efficient Myerson-type value of the game $\langle v, \Gamma \rangle$ for games with one-level communication graphs, recently studied in van den Brink et al. (2012).

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