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Joint Inspection and Inventory Control for Deteriorating Items with Time-Dependent Demand and Deteriorating rate

Yue Xie · Allen H. Tai · Wai-Ki Ching · Yong-Hong Kuo · Na Song

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Abstract In this paper, we consider inventory control problems for deteriorating items with maximum serviceable lifetime under mixed sales situation, both of the demand and the deterioration rates are depending on time. A model is presented to formulate the process of mixed sales that deteriorated items are sold to consumers together with serviceable items, where penalty cost for the sales of deteriorated products is included. From the literature search, this study is one of the first researches on the joint inspection and inventory control policies under the mixed sales situation with time-dependent demand and deterioration rate. In order to improve the inventory holder's profit, we design an additional ordering contract to solve the problem of insufficient supply due to the reduction of orders caused by the

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Yue Xie
College of Economics, Zhejiang University of Technology, Hangzhou, China.
E-mail: xieyue1988moon@163.com

Allen H. Tai
Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Hong Kong.
E-mail: allen.tai@polyu.edu.hk

Wai-Ki Ching
Advanced Modeling and Applied Computing Laboratory Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong.
E-mail: wching@hku.hk

Yong-Hong Kuo
Department of Industrial and Manufacturing Systems Engineering, The University of Hong Kong, Pokfulam Road, Hong Kong.
E-mail: yhkuo@hku.hk

Na Song(Corresponding Author)
School of Management and Economics, University of Electronic Science and Technology of China, Chengdu, China.
E-mail: songna@uestc.edu.cn

losses from the sales of deteriorated items. The optimal ordering time and ordering quantities are characterized for the additional ordering contract. We show that it would be more beneficial for the inventory holder to employ an additional order, especially when the supplier provides price discounts. Furthermore, two different inspection policies are considered in this study: (i) one inspection during the cycle; (ii) continuous monitoring in the cycle. The numerical results show that the net profit would increase if one inspection or continuous monitoring is conducted. These results provide useful insights to guide decision-making in inventory control problems of deteriorating products.

Keywords Deteriorating items · Mixed sales · Additional order · Inspection policy · Time-dependent demand and deterioration rate · Inventory system

1 Introduction

Studying inventory systems with deteriorating items is one of the main research directions in inventory control problems. Fresh food, fruits and chemicals deteriorate due to their physical nature. Most deteriorating items have their maximum serviceable lifetime, which means that they may become deteriorated before a predetermined date while must be no longer serviceable after a pre-determined date. Such deteriorated items must be scrapped. In this study, we consider an inventory system for deteriorating items with maximum lifetime. Moreover, the demand rate and the deterioration rate are considered as time dependent. The deterioration rate is assumed to be an increasing function of time. The demand rate is assumed to be an decreasing function of time, because the consumers would be less willing to buy the products which are much closer to their maximum lifetime.

In some situations, deteriorated items are not completely screened out from the inventory due to the insufficiencies of manpower and technology. Hence, these deteriorated items, together with good quality items, are sold to customers (i.e. mixed sales). Customers receiving deteriorated items may require a full refund or other compensations. Take Sam's club, a high-end membership store owned by Walmart, as a example. Most purchases made on SamsClub.com may be returned to any Sam's Club location in the U.S. in full. Furthermore, the inventory holder needs to provide after-sales service, such as an economic compensation or some discounts, in order to maintain a strong relationship with the customer. In other words, in order to prevent damage to business reputation, penalty cost is incurred by the inventory holder for the sales of deteriorated items. Hence, this work formulates the process of mixed sales that the deteriorated items are sold to customers together with serviceable items, where penalty cost for the sales of deteriorated items is included.

Since the demand rate would decrease over time and the deterioration rate would increase over time, the sales of the deteriorating items would be more difficult for the inventory holder as the time is closer to the end of the replenishment cycle. It is natural for the inventory holder to consider to make an additional order during the replenishment cycle, with the aim of keeping preferable freshness of the product items as well as narrowing the gap between the supply and the market demand. For example, at Metro, which is one of the leading international companies in the wholesale and food service sector, the computer system controls all the dynamics of commodity purchase, sales, and inventory, and maintain the inventory level to be in the most reasonable range. When the number of commodities is below the safety inventory level, the computer system can automatically generate an order to the supplier to ensure continuous supply of goods and low-cost operation.

Furthermore, in order to reduce the losses due to the sales of deteriorated items, the inventory manager may choose to perform inspections or continuously monitoring to screen out deteriorated items from the inventory. If inspection action is chosen, deteriorated items are screened out in one lot after an inspection is performed. There will be a portion of deteriorated items in the inventory before the inspection. After the inspection, the remaining items start to deteriorate again. Hence it is still possible that customers receive deteriorated items. In the other case, when continuous monitoring is chosen, once an item becomes deteriorated it will be screened out from the inventory. Therefore, all customers receive good quality items.

In the mixed sales scenario with penalty cost, how should the inventory holder rationally make inventory orders for the sales of deteriorated items with time-dependent demand and deteriorating rate? In order to reduce the chance of deteriorated items being bought by customers, it is natural to adopt inspection strategies. How to make the best inspection strategy? Is it profitable for the retailer to adopt relevant inspection strategies? Moreover, inventory holder is likely to reduce the inventory order to mitigate the loss caused by selling of deteriorating items. Reduction of orders may lead to shortfall of products supply. Is there any effective ordering mechanism to keep preferable freshness of the product items and make sure that the inventory level can meet the market demand? These problems are the main concerns of our study.

The key contributions of this study lie in the following aspects. First, we investigate an inventory system for deteriorating items with maximum lifetime. The penalty cost for the sales of deteriorated items incurred by the inventory holder is considered. The system is facing time-dependent demand and deterioration rate. Furthermore, we design an additional ordering contract to solve the problem of insufficient supply due to the reduction of orders caused by the losses from the sales of deteriorated items. We show that it is beneficial for the inventory holder to employ an additional order. In addition, during the replenishment cycle, two inspection policies can be considered: (i) one inspection during the cycle; (ii) continuous monitoring in the cycle. A dynamic programming approach is proposed for the case with one inspection, and closed-form solutions of optimal ordering quantity and replenishment cycle are obtained when the items are continuous monitored.

1.1 Literature Review

Study on inventory systems with deteriorating items has become one of the main research directions in inventory control problems since the work of Ghare and Schrader (1963) more than fifty years ago. They extended the classical economic order quantity (EOQ) model for an exponentially decaying inventory system. Literature reviews on this topic include Bakker et al. (2012), Ghiami and Williams (2015), Gilding (2014), Janssen et al. (2016) and Taleizadeh (2014). Chung and Wee (2011) considered a green supply chain inventory system for short life-cycle deteriorating products. Widyadana and Wee (2012) considered a production system of deteriorating items with rework process for defective items. Lee and Dye (2012) considered a deteriorating inventory with stock-dependent demand. The deterioration rate is under control as the inventory holder invests in preservation technology. Tiwari et al. (2018) considered a two-echelon supply chain for deteriorating items in which the retailer's warehouse capacity of display area is limited while the remaining items are stored in another warehouse with infinite capacity. Duan et al. (2018) investigated a joint pricing, production and inventory problems for deteriorating items under uncertain demand, in which the selling season is finite. Pervin et al. (2019) considered a multi-item deteriorating two-echelon inventory model with price- and stock-dependent demand. Khan et al. (2020) presented a two storage inventory model of non-instantaneous deteriorating items with partial backlogging shortages and advance payment. More studies on inventory systems with deteriorating items can be found in the literature, see for instance Tiwari et al. (2017), Mishra et al. (2018), Jaggi et al. (2019), Tiwari et al. (2020) and Roy et al. (2020).

In reality, most deteriorating items have their maximum lifetimes, such as fruits and vegetables, products may become deteriorated before that day while after that day the items must be no longer serviceable. Several papers study the inventory control problems for deteriorating items with maximum lifetime. For example, Wang et al. (2014) proposed an EOQ model for deteriorating items with maximum lifetime. When the credit period increases, both demand and default risk increase. Tiwari et al. (2018) considered a supplier-retailer-customer supply chain for deteriorating items under two-level partial trade credit. In their model, shortages are allowed and partially backlogged. Wu et al. (2018) proposed an EOQ model with a generalized deterioration rate which depends on time as well as the maximum lifetimes of products. Some other works on products with maximum lifetime include Chen and Teng (2015), Sarkar et al. (2015), Teng et al. (2016), Wu et al. (2016) and Wu and Chan (2014).

Most of the works mentioned above considered continuous monitoring, deteriorated items are screened out once they deteriorated during the period of their maximum lifetime. However, deteriorated items may not be completely screened out from the inventory due to the insufficiencies of manpower and technology. Hence, deteriorated items would be sold to customers together with good quality items, and the sales of deteriorated items would increase the cost incurred by the inventory holder. For example, in Tai et al. (2016), they studied the effect of inspection policies on optimal decisions for a deteriorating inventory system. It shows that to reduce the chance of selling deteriorated items to customers, inspections may be performed to the inventory to screen out deteriorated items. This study further considers the penalty cost for the sales of deteriorated items, and both of the demand and deterioration rate are deterministic and constant. This work was extended to the case of random maximum lifetime in Tai et al. (2019). Taleizadeh et al. (2019) also investigated the inventory problem for deteriorated items under the mixed sales situation. An inventory model with disparate inventory ordering policies consider a hybrid payment strategy was proposed in the study.

One assumption of classical EOQ model is that the demand rate is a constant. Some researchers considered extending the EOQ model by introducing different types of demand functions. Hung (2011) investigated an inventory model with time-dependent demand rate and deterioration rate. He showed that the optimal replenishment policy is actually independent of the demand. Ahmed et al. (2013) proposed deteriorating inventory models with ramp type demand rate. Khanra et al. (2011) considered an EOQ model for deteriorating items having time dependent quadratic demand. Maihami and Kamalabadi (2012) developed a joint pricing and inventory control for non-instantaneous deteriorating items where the demand depends on time and price. Sarkar and Chakrabarti (2013) investigated a deteriorating inventory with an exponential demand and permissible delay in payments. The deterioration follows the Weibull distribution. Tripathi and Pandey (2013) considered an EOQ model with constant deterioration rate and the Weibull demand rate. Recently, Pervin et al. (2018) proposed a deterministic inventory control model with stochastic deterioration. The demand rate and the inventory holding cost are both linear functions of time. Cárdenas-Barrón et al. (2018) investigated an EOQ model under both nonlinear stock dependent demand and nonlinear holding cost.

In addition, due to time-dependent demand and deterioration rate, the deteriorating product retailer often suffers from shortage, and then cause the retailer bearing losses. Additional orders can be provided with options when no item is serviceable (see Luo and Chen 2015, Wang and Chen 2013, Zhao et al. 2010, Wang and Chen 2016 etc.). But there are few researches on the inventory control policies under the mixed sales situation with time-dependent demand and deterioration rate when additional orders can be provided if no item is serviceable.

Time-dependent demand and deterioration rate are typical features of deteriorating items. However, there is a lack of research on the joint inspection and inventory control policies under the setting of time-dependent demand and deterioration rate in mixed sales scenario. Hence, this paper develops inventory models for deteriorating items to capture the following relevant and important facts: (1) deteriorated items are sold together with serviceable items, penalty cost is incurred by the inventory holder for the sales of deteriorated items; (2) the demand of a deteriorating item and the deterioration rate are time-varying. Further, an additional ordering contract is designed to solve the problem of insufficient supply and improve the inventory holder's net profit, two inspection policies are developed to reduce the losses from the sales of deteriorated items. The objective of this study is to determine and compare the optimal solutions and net profits resulting from these different policies to analyze their performance. Through the contribution of this work, it can be examined recently published research papers that compare other models and our present models which are described in Table 1.

The remainder of this paper is organized as follows. In Section 2, the notation and assumptions adopted in the development of proposed models are introduced. In Section 3, we first propose a basic model with time-dependent demand and deterioration rate. The penalty cost for the sales of deteriorated items is included. In Section 4, we then consider an additional ordering opportunity in the second model. In Section 5, we present a model in which an inspection is conducted during a replenishment cycle. We then consider an inventory system under continuous monitoring. Numerical examples and sensitivities analyses are given in Section 6. Finally, conclusions and future research issues are given in Section 7.

2 Notations and Assumptions

The following notations and assumptions are used throughout the entire paper.

Table 1: Surmised literature review of current research related to our present study

Author(s)	Mixed sales	Additional order	Inspection policy	Time dependent demand	Time dependent deterioration rate
Prasad et al. (2014)				√	√
Sarkar et al. (2015)					√
Wang et al. (2016)		√		√	
Tai et al. (2016)	√		√		
Wu et al. (2018)				√	√
Pervin et al. (2018)				√	√
Tai et al. (2019)	√		√		√
Taleizadeh et al. (2019)	√		√		
Our paper	√	√	√	√	√

2.1 Notations

The symbols are defined into three groups: parameters, decision variables and functions.

Parameters:

Q	the replenishment quantity (units)
D	demand before the product deteriorates (units per day)
T_1	the time when all items in the inventory are used up (days)
m	the maximum lifetime of a product item (days)
p	sales price of a product item ($ per unit)
c	purchasing cost ($ per unit)
h	inventory holding cost ($ per unit per day)
k	penalty cost for the sales of a deteriorated product ($ per unit)
b	purchasing cost for an additional order (per unit)
c_d	inspection cost ($ per unit)
d	cost of the initial installment for continuous monitoring ($ per unit)
g	management cost for continuous monitoring ($ per unit time)

Decision variables:

T	the length of a replenishment cycle (days)
τ	the time when an inspection is conducted, where $\tau < T$ (days)
t_1	the time when an additional order is placed, where $t_1 < T$ (days)

Functions:

$I(t)$	the inventory level of serviceable items at time t (units)
$J(t)$	the inventory level of deteriorated items at time t (units)
$\lambda(t)$	demand rate (units per day)
$\theta(t)$	deterioration rate (units per day)

Then we introduce some necessary assumptions in order to build up the mathematical model.

2.2 Assumptions

Our model is developed on the basis of the following assumptions:

1. We assume that the maximum lifetimes of all product items in the same batch are same, which is a known positive constant and denoted by m.
2. The demand rate is given as
$$\lambda(t) = D\left(1 - \frac{t}{m}\right), \tag{1}$$
which implies that the demand of the products decreases as the time increases. Here D is the initial demand when all items are serviceable. The demand goes to 0 when the products reach their maximum lifetime. In practice, the consumers would be less willing to buy the products which are much closer to their maximum lifetime. Similar functions of the demand rate have been adopted by Pervin et al. (2018).
3. The deterioration rate of an inventory system is a deterministic function of time t and the products' maximum lifetime m, which is given by
$$\theta(t) = \frac{1}{1+m-t}, \quad 0 \le t \le m. \tag{2}$$
This means that products may become deteriorated during the period of their maximum lifetime while after that day of their maximum lifetime the items must be no longer serviceable. This assumption was also adopted by Sarkar (2012), Chen and Teng (2015), Teng et al. (2016), Wang et al. (2014), Wu et al. (2016) and Wu and Chan (2014). As the replenishment cycle begins, the products begin to deteriorate. For example, fruits and vegetables, they begin to deteriorate once they are harvested.
4. If customers find that some of the items have been deteriorated just after they received the items, they could require a full refund or other compensations for the items which have been deteriorated, while not for the serviceable items. In other words, in order to prevent damage to business reputation, penalty cost is incurred for the sales of deteriorated items. Hence, the inventory holder receives revenue only from the sales of serviceable items.
5. We model a production system under the assumption that each inspection is perfect, which means that it will correctly screen out deteriorated items. This assumption was also adopted by Lee and Rosenblatt (1987), they gave some discussions of imperfect inspection.
6. The replenishment lead time and the ordering cost are negligible in this work and shortage is not allowed which means the replenishment cycle ends when one of the two cases occurs: (i) time t reaches m or (ii) all items in the inventory are used up, whichever comes first if there is no additional order.

3 The General Model

In this section, we begin with a general inventory model for mixed sales which means that deteriorated items are sold to consumers together with serviceable items. If a customer finds some of the items have been deteriorated, the customer receives a full refund for the deteriorated items, while not for the serviceable items. Further, the inventory holder needs to provide after-sales service, such as an economic compensation or some discounts, in order to maintain a strong relationship with the customer. We assume that the unit cost due to the sales of deteriorated products incurred by the inventory holder is denoted by k.

Let T_1 be the time when all items in the inventory are used up. Then we have $\int_0^{T_1} \lambda(x)dx = Q$, where Q is the initial replenishment quantity of the inventory holder and $\lambda(x)$ is the demand rate given in Eq. (1). The length of the replenishment cycle T should be

$$T = \min\{T_1, m\},$$

which is not greater than the maximum lifetime of the product item m. The total number of items sold during the time period $[0, T]$ is given by

$$\int_0^T \lambda(x)dx.$$

Since $\lambda(x) \geq 0$, we have

$$\int_0^T \lambda(x)dx \leq \int_0^m \lambda(x)dx = \frac{Dm}{2}. \tag{3}$$

If the initial replenishment quantity Q is greater than $\frac{Dm}{2}$, then we have $T_1 > m$, which means some unsold items will exceed their maximum lifetime. The inventory holder could enhance her profit by decreasing the initial replenishment quantity to avoid wastage. Hence we may take $Q \leq \frac{Dm}{2}$, which means $T_1 \leq m$. Then, we have

$$T = \min\{T_1, m\} = T_1,$$

which means that the replenishment cycle ends when all items in the inventory are used up. Hence we have

$$Q = \int_0^{T_1} \lambda(x)dx = DT_1 - \frac{DT_1^2}{2m}, \ \ T_1 \leq m. \tag{4}$$

Therefore, our problem of finding an optimal replenishment quantity Q can be redefined with the aim of finding an optimal replenishment cycle T_1 with $T_1 \leq m$,

Let $I(t)$ and $J(t)$ be the inventory levels of serviceable and deteriorated items at time t respectively, where $t \in [0, T_1]$. The chance that a customer receiving a serviceable or a deteriorated item is naturally corresponding to the proportion of serviceable and deteriorated items in the inventory. Hence the dynamic of $I(t)$ and $J(t)$ are governed by the following differential equations:

$$\begin{cases} I'(t) = -\dfrac{I(t)}{I(t)+J(t)}\lambda(t) - \theta(t)I(t), & 0 \leq t \leq T_1, \\ J'(t) = \theta(t)I(t) - \dfrac{J(t)}{I(t)+J(t)}\lambda(t), & 0 \leq t \leq T_1. \end{cases} \tag{5}$$

Solving the system of differential equations (5) with the boundary conditions $I(0) = Q$, $J(0) = 0$, we have

$$\begin{cases} I(t) = \dfrac{\left(Q - Dt + D\dfrac{t^2}{2m}\right)(1+m-t)}{1+m}, & 0 \le t \le T_1, \\ J(t) = \dfrac{\left(Q - Dt + D\dfrac{t^2}{2m}\right)t}{1+m}, & 0 \le t \le T_1. \end{cases} \tag{6}$$

The numbers of serviceable items and deteriorated items sold to customers in a replenishment cycle $[0, T_1]$ are denoted by Q_{1s} and Q_{1d} respectively, and we have

$$\begin{cases} Q_{1s} = \displaystyle\int_0^{T_1} \frac{I(t)}{I(t)+J(t)} \lambda(t)\, dt \\ \quad\;\; = DT_1 - \dfrac{1+2m}{2m(1+m)} D{T_1}^2 + \dfrac{1}{3m(1+m)} D{T_1}^3, \\ Q_{1d} = Q - Q_{1s}. \end{cases}$$

The revenue of the inventory holder is generated from the sales of serviceable items. Hence, the revenue per cycle is

$$TR_1 = pQ_{1s}, \tag{7}$$

where p is the sales price per unit item. The total cost of the inventory holder per cycle consists of three parts:

1. Purchasing cost cQ.
2. Penalty cost for the sales of deteriorated products kQ_{1d}.
3. Inventory holding cost per cycle $h \displaystyle\int_0^{T_1} \left(Q - \int_0^t \lambda(x)dx\right) dt$.

The total cost denoted as TC is given as follows:

$$TC_1 = cQ + kQ_{1d} + h \int_0^{T_1} \left(Q - \int_0^t \lambda(x)dx\right) dt. \tag{8}$$

Then, we denote the inventory holder's net profit as Π_1, and it has

$$\begin{aligned} \Pi_1 &= TR_1 - TC_1 \\ &= (p-c)DT_1 - \left[\frac{1+2m}{m(1+m)}(p+k) - \frac{(c+k)}{m} + h\right] \frac{DT_1^2}{2} \\ &\quad + \left[\frac{(p+k)}{m(1+m)} + \frac{h}{m}\right] \frac{DT_1^3}{3}. \end{aligned} \tag{9}$$

Our objective is to find the optimal replenishment quantity Q such that the inventory holder's profit is maximized. According to Eq. (4), Eq. (7) and Eq. (8), the problem of maximizing the inventory holder's profit can be redefined and given as follows:

$$\max_{T_1 \in (0,m]} \Pi_1. \tag{10}$$

Based on the analysis of the first derivative of Π_1, we have the following result in Proposition 1. The proof is presented in Appendix.

Proposition 1 *There are two cases under this general model.*

1) *If $p < c(1+m) + hm(1+m) + km$. The optimal replenishment cycle is given by*

$$T_1^* = \frac{(p-c)(1+m)}{p+k+h(1+m)}.$$

2) *If $p \geq c(1+m) + hm(1+m) + km$. The optimal replenishment cycle is given by*

$$T_1^* = m.$$

It satisfies $\Pi_1'(T_1^*) = 0$ in each case. Then we have

$$\Pi_1(T_1^*) = \max_{T_1 \in (0,m]} \Pi_1(T_1).$$

The optimal quantity is given by

$$Q^* = DT_1^* - \frac{1}{2m} DT_1^{*2} = D\frac{(p-c)(1+m)}{p+k+h(1+m)} - D\frac{(p-c)^2(1+m)^2}{2m[p+k+h(1+m)]^2}.$$

4 Additional Ordering Opportunity

In this section, the inventory holder is supposed to have one more ordering opportunity during the replenishment cycle $(0, T)$. The length of T is determined by the supplier based on his actual production conditions and production cycle, and it is assumed to be less than the double of the maximum lifetime. The inventory holder makes an initial order decision from one supplier at the beginning of the replenishment cycle, and the initial ordering quantity is denoted by Q_1. When the inventory level reaches zero at time t_1 ($t_1 < T$), the inventory holder is able to make an additional order at the instant. The additional ordering quantity is denoted by Q_2. The additional ordering opportunity may be provided by the supplier, which could promote the cooperation between the supplier and the inventory holder. The inventory holder could also choose to make an additional order from another supplier if an appropriate price is provided. The timelines of the model is shown in Figure 1.

We assume that the additional ordering cost per unit is b. The demand rate during the time period $[0, T]$ is given as

$$\lambda(t) = \begin{cases} D(1 - \frac{t}{m}), & t \in [0, t_1] \\ D(1 - \frac{t - t_1}{m}), & t \in [t_1, T] \end{cases} \tag{11}$$

The deterioration rate is given as

$$\theta(t) = \begin{cases} \frac{1}{1+m-t}, & t \in [0, t_1] \\ \frac{1}{1+m-(t-t_1)}, & t \in [t_1, T] \end{cases} \tag{12}$$

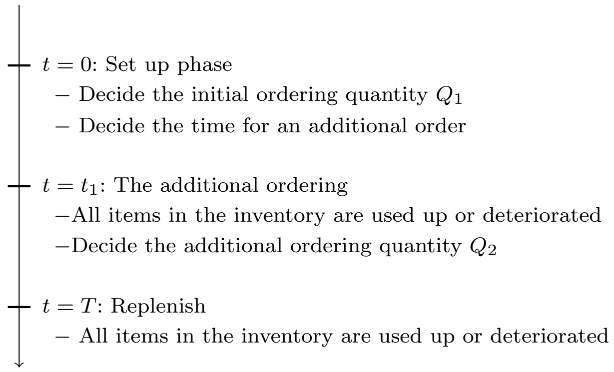


Fig. 1: Timeline of the replenishment cycle with one additional ordering opportunity.

Then, similar to Eq. (10), we denote Π_{21} and Π_{22} as the net profits during the first time period $t \in [0, t_1]$ and the second time period $t \in [t_1, T]$, which are then given by

$$\Pi_{21} = (p-c)Dt_1 - \left[\frac{1+2m}{m(1+m)}(p+k) - \frac{(c+k)}{m} + h\right]\frac{Dt_1^2}{2} + \left[\frac{(p+k)}{m(1+m)} + \frac{h}{m}\right]\frac{Dt_1^3}{3}$$

and

$$\begin{aligned}\Pi_{22} = (p-b)D(T-t_1) - \left[\frac{1+2m}{m(1+m)}(p+k) - \frac{(b+k)}{m} + h\right]\frac{D(T-t_1)^2}{2} \\ + \left[\frac{(p+k)}{m(1+m)} + \frac{h}{m}\right]\frac{D(T-t_1)^3}{3}.\end{aligned}$$

Then the total net profit is given by Π_2, and it has

$$\Pi_2 = \Pi_{21} + \Pi_{22}. \tag{13}$$

Our objective is to find the optimal time t_1 for an additional order such that the total net profit of the inventory holder denoted as Π_2 is maximized,

$$\max_{t_1 \in [0,T]} \Pi_2. \tag{14}$$

Denote Q_1^* as the optimal ordering quantity at the beginning of the replenishment cycle and Q_2^* as the optimal additional ordering quantity. Based on the analysis of the first derivative function of Π_2, we have the following results in Propositions 2, 3 and 4, which are corresponding to the situations of $c = b$, $c > b$ and $c < b$. The proofs are presented in Appendix.

Proposition 2 *Suppose that $c = b$.*

1) If $p < c(1+m) + hm(1+m) + km$, denote

$$m_1 = m + \frac{(p-c)(1+m)}{p+k+h(1+m)},$$

then the optimal time for an additional order is

$$t_1^* = \begin{cases} T/2, & T \in (0, m_1] \\ T-m \text{ or } m, & T \in (m_1, 2m) \end{cases}.$$

2) If $p \geq c(1+m) + hm(1+m) + km$, then the optimal time for an additional order is

$$t_1^* = T/2, \quad T \in (0, 2m).$$

The optimal ordering quantities are

$$\begin{cases} Q_1^* = Dt_1^* - \dfrac{1}{2m} Dt_1^{*2} \\ Q_2^* = D(T - t_1^*) - \dfrac{1}{2m} D(T - t_1^*)^2 \end{cases}.$$

Proposition 3 *Suppose that $c > b$. Denote*

$$\begin{aligned} m_{21} &= \frac{A-B}{2(p+k+h(1+m))}, \\ m_{22} &= m + \frac{(p-c)(1+m)}{p+k+h(1+m)}, \\ m_{23} &= \frac{A+B}{2(p+k+h(1+m))}, \\ m_{24} &= m + \frac{(p - \frac{(c+b)}{2})(1+m)}{p+k+h(1+m)}, \\ T_{2c} &= \frac{(T-m_{21})(T-m_{23})}{2(T-m_{24})}, \end{aligned}$$

where

$$A = [p+k+h(1+m)]m + (p-b)(1+m)$$

and

$$B = \sqrt{A^2 - 4(p+k+h(1+m))(c-b)(1+m)m}.$$

1) If $p < c(1+m) + hm(1+m) + km$, we have

$$0 < m_{21} < m < m_{22} < 2m.$$

Then the optimal time for an additional order is

$$t_1^* = \begin{cases} 0, & T \in (0, m_{21}] \\ T_{2c}, & T \in (m_{21}, m_{22}] \\ T-m, & T \in (m_{22}, 2m) \end{cases}$$

2) If $c(1+m)+hm(1+m)+km \leq p$, *we have*

$$0 < m_{21} < m < 2m < m_{22}.$$

Then the optimal time for an additional order is

$$t_1^* = \begin{cases} 0, & T \in (0, m_{21}] \\ T_{2c}, & T \in (m_{21}, 2m) \end{cases}$$

The optimal ordering quantities in each case are

$$\begin{cases} Q_1^* = Dt_1^* - \frac{1}{2m} Dt_1^{*2} \\ Q_2^* = D(T-t_1^*) - \frac{1}{2m} D(T-t_1^*)^2 \end{cases}.$$

Proposition 4 *Suppose that* $c < b$. *Denote*

$$\begin{aligned} m_{21} &= \frac{A-B}{2(p+k+h(1+m))}, \\ m_{23} &= \frac{A+B}{2(p+k+h(1+m))}, \\ m_{24} &= m + \frac{(p-\frac{(c+b)}{2})(1+m)}{p+k+h(1+m)}, \\ m_{25} &= m + \frac{(p-b)(1+m)}{p+k+h(1+m)}, \\ m_{31} &= \frac{E-F}{2(p+k+h(1+m))}, \\ T_{2c} &= \frac{(T-m_{21})(T-m_{23})}{2(T-m_{24})}, \end{aligned}$$

where

$$A = [p+k+h(1+m)]m + (p-b)(1+m),$$

$$B = \sqrt{A^2 - 4(p+k+h(1+m))(c-b)(1+m)m},$$

$$E = [p+k+h(1+m)]m + (p-c)(1+m)$$

and

$$F = \sqrt{E^2 - 4(p+k+h(1+m))(b-c)(1+m)m}.$$

1) If $p < b(1+m)+hm(1+m)+km$, *we have*

$$0 < m_{31} < m < m_{25} < 2m.$$

Then the optimal time for an additional order is

$$t_1^* = \begin{cases} T, & T \in (0, m_{31}] \\ T_{2c}, & T \in (m_{31}, m_{25}] \\ m, & T \in (m_{25}, 2m) \end{cases}$$

2) If $b(1+m)+hm(1+m)+km \le p$*, we have*

$$0 < m_{31} < m < 2m < m_{25}.$$

Then the optimal time for an additional order is

$$t_1^* = \begin{cases} T, & T \in (0, m_{31}] \\ T_{2c}, & T \in (m_{31}, 2m) \end{cases}$$

The optimal ordering quantities in each case are

$$\begin{cases} Q_1^* = Dt_1^* - \dfrac{1}{2m} Dt_1^{*2} \\ Q_2^* = D(T - t_1^*) - \dfrac{1}{2m} D(T - t_1^*)^2 \end{cases}.$$

Here the case of $c = b$ means that the purchasing price of the products remains unchanged. This suggests that the production process and the sales of the products is relatively stable. The supplier allows the inventory holder to make an additional order without any discount in price. If $c > b$, then the supplier encourages the inventory holder to make an additional order with a discount price. On one hand, this helps in extending business and it implies that the supplier may wish to strengthen business relations with the inventory holder. On the other hand, the supplier may estimate that the cost of production would be lower or the supply of the products would increase, hence he would provide a discount price to attract the inventory holder to make an additional order. The case of $c < b$ suggests that the supplier may estimate that the production cost would increase or the production of the items would be decreased which would result in the increase of the item's purchasing price. Hence the inventory holder would tend to increase the replenishment quantity at the beginning of the replenishment cycle.

5 Inspection Policy

In this section, we present the model which includes an inspection policy such that the net profit is maximized. Inspections can be carried out in the inventory at certain time during the replenishment cycle. Two special cases with one inspection and continuous monitoring are discussed in this section.

5.1 The Case with One Inspection

We have Q units of serviceable items arrive at the beginning of the replenishment cycle. The demand rate and the deterioration rate of the inventory system are $\lambda(t)$ and $\theta(t)$ during the whole replenishment cycle time $[0, T]$. The inventory level declines only due to the demand rate over time interval $[0, \tau)$. At time τ $(0 < \tau < T)$, one inspection is conducted and the inventory level reduces since the items which have deteriorated during the time interval $[0, \tau)$ are screened out, and the chance of delivering deteriorated items to customers will become lower after inspection. Then the inventory level reduces to zero owing to the demand during $(\tau, T]$. Our aim is to obtain the optimal inspection time such that the net profit

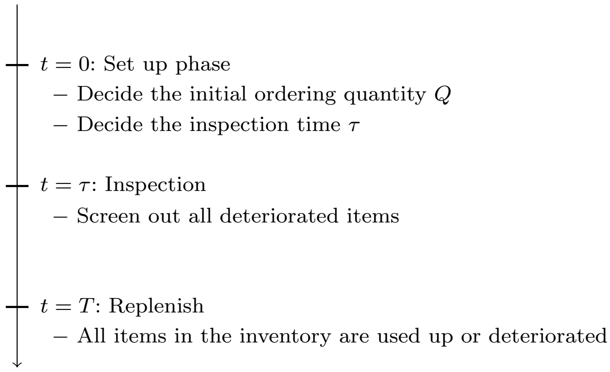


Fig. 2: Timeline of the replenishment cycle with one inspection.

of the inventory holder is maximized when the initial replenishment quantity Q is given. The timeline of the model with one inspection is shown in Figure 2.

Firstly, the inventory levels of serviceable and deteriorated items in the time period $[0, \tau]$ are governed by the following differential equations:

$$\begin{cases} I'(t) = -\dfrac{I(t)}{I(t)+J(t)}\lambda(t) - \theta(t)I(t), \quad 0 \le t \le \tau, \\ J'(t) = \theta(t)I(t) - \dfrac{J(t)}{I(t)+J(t)}\lambda(t), \quad 0 \le t \le \tau. \end{cases} \tag{15}$$

Solving the system of differential equations (15) with the boundary conditions $I(0) = Q$, $J(0) = 0$, we have

$$\begin{cases} I(t) = \dfrac{(Q - Dt + D\frac{t^2}{2m})(1+m-t)}{1+m}, \quad 0 \le t \le \tau, \\ J(t) = \dfrac{(Q - Dt + D\frac{t^2}{2m})t}{1+m}, \quad 0 \le t \le \tau. \end{cases} \tag{16}$$

At time τ, the quantity of the product items is given by

$$Q_\tau = I(\tau) + J(\tau) = \Big(Q - D\tau + D\frac{\tau^2}{2m}\Big),$$

and the number of the serviceable items is

$$Q_{I\tau} = I(\tau) = \frac{1+m-\tau}{1+m}\left(Q - D\tau + D\frac{\tau^2}{2m}\right).$$

Then if one inspection is conducted at time τ and the number of serviceable items is $Q_{I\tau}$ at time τ, we get the replenishment cycle for the inventory holder, which is shown in Lemma 1. Here, we define that the replenishment cycle ends when all items in the inventory are used up or deteriorated. The proof is presented in Appendix.

Lemma 1 *Let*

$$M_Q = D\frac{(m-\tau)^2}{2m}.$$

Then if one inspection is conducted at time τ and the number of serviceable items is $Q_{I\tau}$ at time τ, the optimal replenishment cycle T^* has

$$T^* = \begin{cases} m, & \text{if } Q_{I\tau} \geq M_Q; \\ m - \sqrt{(m-\tau)^2 - 2\frac{m}{D}Q_{I\tau}}, & \text{if } Q_{I\tau} < M_Q. \end{cases}$$

Lemma 1 indicates that the number of serviceable items $Q_{I\tau}$ at time τ should be less than M_Q in order to avoid wastage. Therefore, if the inventory holder chooses to conduct one inspection during a replenishment cycle, we have the following result in Lemma 2. The proof is presented in Appendix.

Lemma 2 *Denote*

$$Q_{\tau_1} = D\left(\frac{1+m}{2m}\right)\left(\frac{(m-\tau_1)^2}{1+m-\tau_1}\right) + D\tau_1 - D\frac{\tau_1^2}{2m},$$

where

$$\tau_1 = \frac{3(1+m) - \sqrt{(1+m)^2 + 8(1+m)}}{4},$$

and it has

$$Q_{\tau_1} > \frac{Dm}{2}.$$

The optimal initial replenishment quantity Q^* satisfies

$$Q^* \leq Q_{\tau_1}.$$

If the initial replenishment quantity Q is greater than Q_{τ_1}, then some unsold items will exceed their maximum lifetime whenever one inspection is conducted. Hence the inventory holder could enhance her profit by decreasing the initial replenishment quantity to avoid wastage.

Further, if the initial replenishment quantity Q is less than $Dm/2$, we introduce a result in Lemma 3.

Lemma 3 *If the initial replenishment quantity Q satisfies*

$$Q < \frac{Dm}{2},$$

set

$$\int_0^{T_1} \lambda(x)dx = Q, \;\; T_1 \leq m.$$

Solving the above equation yields

$$T_1 = \tau_m,$$

where $\tau_m = m - \sqrt{m^2 - 2\frac{m}{D}Q}$. The replenishment cycle with one inspection would be less than τ_m.

Next, one inspection would be conducted at time τ. The inventory levels of serviceable and deteriorated items, denoted by $I(t)$ and $J(t)$ in the time period $t \in [\tau, T]$, are given as follows:

$$\begin{cases} I(t+\tau) = I_1(t) \\ J(t+\tau) = J_1(t) \end{cases}, \tag{17}$$

where $I_1(t)$ and $J_1(t)$ are governed by the following differential equations:

$$\begin{cases} I_1'(t) = -\dfrac{I_1(t)}{I_1(t)+J_1(t)}\lambda_1(t) - \theta_1(t)I_1(t), \quad 0 \le t \le T-\tau \\ J_1'(t) = \theta_1(t)I_1(t) - \dfrac{J_1(t)}{I_1(t)+J_1(t)}\lambda_1(t), \quad 0 \le t \le T-\tau \end{cases}, \tag{18}$$

with

$$\begin{cases} \lambda_1(t) = D(1 - \dfrac{t}{m} - \dfrac{\tau}{m}) \\ \theta_1(t) = \dfrac{1}{1+m-\tau-t} \end{cases}. \tag{19}$$

Solving the system of differential equations (18) with the boundary conditions

$$\begin{cases} I_1(0) = Q_{I\tau} \\ J_1(0) = 0 \end{cases}, \tag{20}$$

we have

$$\begin{cases} I(t+\tau) = I_1(t) = \dfrac{(Q_{I\tau} - D\left(1-\frac{\tau}{m}\right)t + D\frac{t^2}{2m})(1+m-\tau-t)}{1+m-\tau}, \quad 0 \le t \le T-\tau, \\ J(t+\tau) = J_1(t) = \dfrac{(Q_{I\tau} - D(1-\frac{\tau}{m})t + D\frac{t^2}{2m})t}{1+m-\tau}, \quad 0 \le t \le T-\tau. \end{cases} \tag{21}$$

Based on Eqs. (16) and (21), the sales of serviceable items in the first time period $[0, \tau]$ and the second time period $[\tau, T]$ are given by Q_{31s} and Q_{32s}, respectively,

$$\begin{cases} Q_{31s} = \int_0^\tau (1 - \dfrac{t}{(1+m)})\lambda(t)dt \\ \quad = D\tau - \dfrac{1+2m}{2m(1+m)}D\tau^2 + \dfrac{1}{3m(1+m)}D\tau^3, \\ Q_{32s} = \int_0^{T-\tau} (1 - \dfrac{t}{1+m-\tau})\lambda_1(t)dt \\ \quad = Q_v - D\dfrac{1}{1+m-\tau}(1-\dfrac{\tau}{m})\dfrac{(T-\tau)^2}{2} + D\dfrac{1}{m(1+m-\tau)}\dfrac{(T-\tau)^3}{3}. \end{cases}$$

Here, according to Lemma 1, it has

$$Q_v = \begin{cases} D\dfrac{(m-\tau)^2}{2m}, \text{ if } Q_{I\tau} \ge D\dfrac{(m-\tau)^2}{2m}; \\ Q_{I\tau}, \qquad \text{ if } Q_{I\tau} < D\dfrac{(m-\tau)^2}{2m}. \end{cases}$$

The corresponding sales of deteriorated items during the two time periods are given by Q_{31d} and Q_{32d},

$$\begin{cases} Q_{31d} = \int_0^\tau (\frac{t}{1+m})\lambda(t)dt \\ \qquad = \frac{1}{2(1+m)}D\tau^2 + \frac{1}{3m(1+m)}D\tau^3, \\ Q_{32d} = \int_0^{T-\tau} (\frac{t}{1+m-\tau})\lambda_1(t)dt \\ \qquad = Q_v - Q_{32s}. \end{cases}$$

The revenue of the inventory holder is generated from the sales of serviceable items. Hence, the revenues during the time periods $[0, \tau]$ and $[\tau, T]$, which are denoted as TR_{31} and TR_{32}, are given as follows:

$$\begin{cases} TR_{31} = pQ_{31s}, \\ TR_{32} = pQ_{32s}. \end{cases}$$

The cost during the time period $[0, \tau]$, which is denoted as TC_{31}, consists of three parts:

1. Purchasing cost cQ.
2. Penalty cost for the sales of deteriorated products kQ_{31d}.
3. Inventory holding cost $h\int_0^\tau (Q - \int_0^t \lambda(x)dx)dt$.

The cost during the time period $[\tau, T]$, which is denoted as TC_{32}, consists of three parts:

1. Inspection cost $c_d Q_\tau$.
2. Penalty cost for the sales of deteriorated products kQ_{32d}.
3. Inventory holding cost $h\int_0^{T-\tau} (Q_{I\tau} - \int_0^t \lambda_1(x)dx)dt$.

Here, the holding costs during the two periods are

$$\begin{aligned} h\int_0^\tau (Q - \int_0^t \lambda(x)dx)dt &= h\int_0^\tau (Q - \int_0^t D(1 - \frac{x}{m})dx)dt \\ &= hQ\tau - hD\frac{\tau^2}{2} + hD\frac{\tau^3}{6m}, \end{aligned} \tag{22}$$

$$\begin{aligned} h\int_0^{T-\tau} (Q_{I\tau} - \int_0^t \lambda_1(x)dx)dt &= h\int_0^{T-\tau} (Q_{I\tau} - \int_0^t D(1 - \frac{\tau}{m} - \frac{x}{m})dx)dt \\ &= hQ_{I\tau}(T-\tau) - hD(1 - \frac{\tau}{m})\frac{(T-\tau)^2}{2} + hD\frac{(T-\tau)^3}{6m}. \end{aligned} \tag{23}$$

Then, Π_{31} and Π_{32}, which represent the net profits during the first time period $t \in [0, \tau]$ and the second time period $t \in [\tau, T]$, are then given, respectively, by

$$\begin{cases} \Pi_{31} = TR_{31} - TC_{31} \\ \qquad = pQ_{31s} - cQ - h\int_0^\tau (Q - \int_0^t \lambda(x)dx)dt - kQ_{31d}, \\ \Pi_{32} = TR_{32} - TC_{32} \\ \qquad = pQ_{32s} - h\int_0^{T-\tau} (Q_{I\tau} - \int_0^t \lambda_1(x)dx)dt - kQ_{32d} - c_d Q_\tau. \end{cases}$$

Then the total net profit is given by Π_3, and it has

$$\Pi_3 = \Pi_{31} + \Pi_{32}.$$

The objective is to obtain the optimal inspection time such that the total net profit of the inventory holder is maximized,

$$\max_{\tau\in(0,T)} \mathit{\Pi}_3,$$

where T can be determined from Lemma 1. Since the initial replenishment quantity Q is supposed to be less than or equal to Q_{τ_1} from Lemma 2, the problem with one inspection based on Lemma 3 can be redefined to find the optimal inspection time τ such that

$$\begin{cases} \max_{\tau\in(0,\tau_m)} \mathit{\Pi}_3, \text{ if } Q \in (0, \dfrac{Dm}{2}], \\ \max_{\tau\in(0,m)} \mathit{\Pi}_3, \text{ if } Q \in (\dfrac{Dm}{2}, Q_{\tau 1}]. \end{cases} \tag{24}$$

The problem would be solved by the following numerical method.

Step 1. Fix a number $n \in \mathbb{N}$.
Step 2. Set $\tau_i = \dfrac{iT'}{n+1}$ with $i = 1, \ldots, n$,
where $T' = m$ if $Q \in (\dfrac{Dm}{2}, Q_{\tau 1}]$ and $T' = \tau_m$ if $Q \in (0, \dfrac{Dm}{2}]$.
Step 3. Compute $M_Q = D\dfrac{(m-\tau_i)^2}{2m}$ and $Q_{I\tau} = \dfrac{1+m-\tau_i}{1+m}(Q - D\tau_i + D\dfrac{\tau_i^2}{2m})$.
If $Q_{I\tau} \geq M_Q$, $T = m$; If $Q_{I\tau} < M_Q$, $T = m - \sqrt{(m-\tau)^2 - 2\frac{m}{D}Q_{I\tau}}$.
Step 4. For each $i = 1, \ldots, n$, compute $\mathit{\Pi}_{31}(Q, \tau_i)$ and $\mathit{\Pi}_{32}(Q, \tau_i)$.
Step 5. Compute
$\mathit{\Pi}_3 = \max_{\tau_i} \Big[\mathit{\Pi}_{31}(Q, \tau_i) + \mathit{\Pi}_{32}(Q, \tau_i)\Big]$ and
$\tau_i^* = \operatorname{argmax}_{\tau_i} \Big[\mathit{\Pi}_{31}(Q, \tau_i) + \mathit{\Pi}_{32}(Q, \tau_i)\Big]$.
Step 6. Return $\mathit{\Pi}_3$ and τ_i^*.

5.2 The Case with Continuous Monitoring

In this section, we suppose that the items are continuous monitored during the replenishment cycle $(0, T)$ and deteriorated items are screened out instantaneously.

If the items in the inventory are continuously monitored, the inventory level of serviceable items with time-dependent demand $\lambda(t)$ and deterioration rate $\theta(t)$, which are given by Eqs. (1) and (2), is written as

$$I'(t) = -\lambda(t) - \theta(t)I(t), \quad 0 \leq t \leq T, \tag{25}$$

with boundary condition $I(0) = Q$, where Q is the initial replenishment quantity. Solving the differential equation of Eq. (25), one has

$$I(t) = \frac{1+m-t}{1+m}\Big(- \int_0^t \lambda(s) \left(\frac{1+m}{1+m-s}\right) ds + Q\Big), \quad 0 \leq t \leq T. \tag{26}$$

If $T > m$, the inventory holder could enhance her profit by decreasing the initial replenishment quantity. Hence, we assume that the serviceable items are sold out before the time m, that is $T \leq m$ and $I(T) = 0$. Then it has

$$Q = \frac{1+m}{m}(DT + D\ln(1+m-T) - D\ln(1+m)). \tag{27}$$

Based on Eq. (27), we have the following result in Lemma 4. The proof is presented in Appendix.

Lemma 4 *The optimal replenishment quantity under continuous monitoring satisfies*

$$Q^* \leq D(1+m)\left(1 - \frac{\ln(1+m)}{m}\right).$$

Otherwise, the inventory holder could enhance her profit by decreasing the initial replenishment quantity to avoid wastage.

The revenue of the inventory holder is generated from the sales of serviceable items. Hence, the revenue per cycle is

$$TR_4 = p\int_0^T \lambda(x)dx = PDT - PD\frac{T^2}{2m}. \quad (28)$$

The cost during the time period $[0, T]$, which is denoted as TC_4, consists of two parts:

1. Purchasing cost cQ.
2. Cost of continuous monitoring C_m.

Furthermore, the cost of continuous monitoring is assumed to be

$$C_m = dQ + gT,$$

where d is the unit cost of the initial installment for continuous monitoring and g is the management cost per unit time. We assume that the holding cost is included in the cost of continuous monitoring, since a professional Warehouse Management System is required if the continuous monitoring is conducted. Therefore, the total cost per cycle is

$$TC_4 = cQ + dQ + gT.$$

The average cost of one product is $\frac{TC_4}{Q}$. Since it can be proved that $\frac{TC_4}{Q} > c + d + g/D$ based on Eq. (27), which is shown in the proof of Proposition 5 in appendix, we suppose that the sales price p is high enough $(c + d + g/D < p)$ to ensure that the inventory holder's net profit is positive. The net profit is then given by Π_4, and we have

$$\begin{aligned}\Pi_4 &= TR_4 - TC_4\\ &= PDT - PD\frac{T^2}{2m}\\ &\quad - \frac{(c+d)(1+m)}{m}\left[DT + D\ln(1+m-T) - D\ln(1+m)\right] - gT. \quad (29)\end{aligned}$$

Our objective is to obtain the optimal initial replenishment quantity such that the net profit of the inventory holder is maximized:

$$\max_Q \Pi_4 \quad (30)$$

Based on the analysis of the first derivative of Π_4 with respect to the replenishment cycle T, we have the following result in Proposition 5. The proof is presented in Appendix.

Proposition 5 *Suppose that $c + d + \frac{g}{D} < p$ Denote*

$$y = \frac{G + \sqrt{G^2 - H}}{2pD},$$

where

$$G = pD + (c + d)(1 + m)D + mg,$$
$$H = 4pD(c + d)(1 + m)D.$$

The optimal replenishment cycle is

$$T^* = 1 + m - y,$$

and the optimal replenishment quantity is

$$Q^* = \frac{1 + m}{m}\left(DT^* + D\ln(1 + m - T^*) - D\ln(1 + m)\right).$$

6 Numerical Examples

In this section, we illustrate the effectiveness of the models proposed in Sections 3-5 through some numerical examples. We denote the general model as Model 1, the model with an additional order as Model 2, the case with one inspection as Model 3 and the case with continuous monitoring as Model 4. Assume that the parameters adopted in the numerical examples are summarized in Table 2. Here Q_i^*, τ_i^* and T_i^* represent the optimal replenishment quantity, inspection time and length of a replenishment cycle in Model i, respectively, $i = 1, 2, 3, 4$.

m	D	k	p	c
60 days	100 units/day	1 \$	20 \$/unit	4 \$/unit
h	c_d	d	g	
0.01 \$/unit/day	0.1 \$/unit	1 \$/unit	40\$ /unit time	

Table 2: Data of parameters

6.1 A Comparison of the Four Models

According to Proposition 1, the optimal replenishment cycle in Model 1 should be $T_1^* = 45.2$, the corresponding quantity is $Q^* = 2817$ and the optimal net profit is $\Pi_1^* = 27066$. These results could also be indicated in Figure 3, which are obtained using a numerical method.

In Model 2, we set the additional ordering cost per unit as $b = 4$, 1 and 7 for Case 1 ($c = b$), Case 2 ($c > b$) and Case 3 ($c < b$) respectively. For each replenishment cycle T, Figures 4, 5 and 6 show the optimal time for an additional order and the corresponding net profit for each case. Figure 4a shows that when

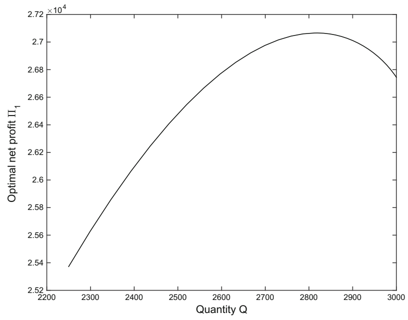
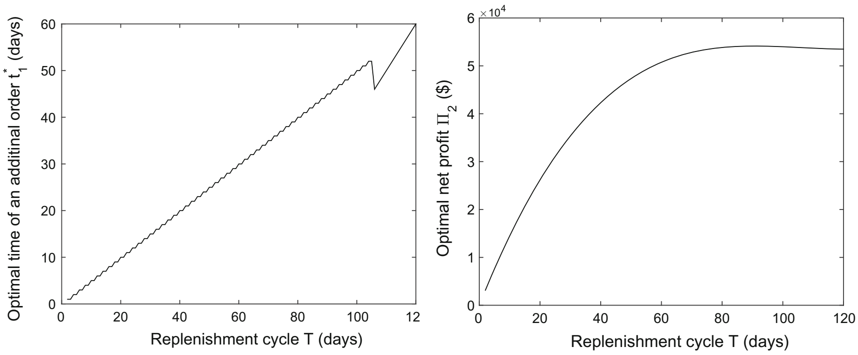


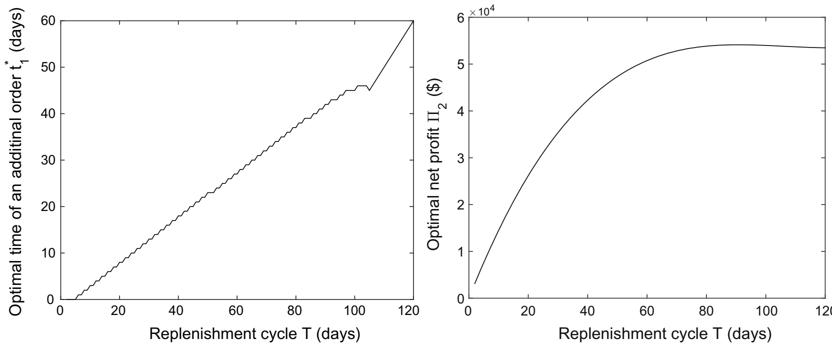
Fig. 3: Optimal net profit Π_1 for each quantity Q.

the purchasing price of the products remains unchanged ($c = b$), the strategy of the time for an additional order changes at the point of $T_{11} = 105$. Figure 5a shows that the strategy of the time for an additional order changes at the points of $T_{21} = 4$ and $T_{22} = 105$ in Case 2 ($c > b$). The inventory holder tends to wait for the discount price if $T \leq 4$, which indicates that if a discount price is provided, the supplier is suggested to set the time period for the additional order to be greater than m_{21} as shown in Proposition 3. Figure 6a shows that the strategy of the time for an additional order changes at the points $T_{31} = 5$ and $T_{32} = 96$ in Case 3 ($c < b$). The result implies that when the purchasing price of the products becomes higher, the inventory holder is suggested to abandon the chance of an additional order if the length of T is short and less than m_{31} as shown in Proposition 4. Furthermore, Figures 4b, 5b and 6b show that the optimal net profit of the inventory holder is concave to the replenishment cycle in each case, which indicates that the optimal replenishment cycle for the inventory holder could be found when an additional order is provided.



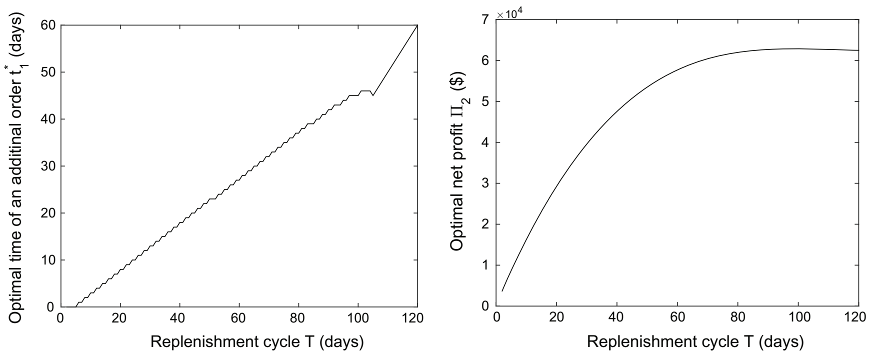
(a) Optimal time of an additional order t_1^* (b) Optimal net profit Π_2

Fig. 4: Optimal time of an additional order t_1^* and Optimal net profit Π_2 for each replenishment cycle T in Case 1.



(a) Optimal time of an additional order t_1^* (b) Optimal net profit Π_2

Fig. 5: Optimal time of an additional order t_1^* and Optimal net profit Π_2 for each replenishment cycle T in Case 2.



(a) Optimal time of an additional order t_1^* (b) Optimal net profit Π_2

Fig. 6: Optimal time of an additional order t_1^* and Optimal net profit Π_2 for each replenishment cycle T in Case 3.

Lemma 2 implies that the optimal replenishment quantity Q in Model 3 is supposed to be less than 3726 under the parameters in Table 2. The numerical results in Table 3 show that if Q is greater than 3700, the optimal net profit decreases as Q increases and the replenishment cycle tends to be the maximum of the lifetime. Hence, the optimal quantity Q would be less than $Q_{\tau 1}$ as shown in Lemma 2. Further, Figure 7 shows that the optimal net profit in Model 3 is a concave function of the quantity Q. Therefore, we can find the optimal quantity $Q^* = 3300$, the optimal inspection time $\tau^* = 16$, the length of replenishment cycle $T^* = 44$ and the optimal net profit is $\Pi_3 = 30418$.

Figure 8 shows that the optimal quantity in Model 4 is $Q^* = 4389$, the corresponding replenishment cycle is $T^* = 44.5$ and the optimal net profit is $\Pi_4 = 32260$. Comparing to the results in Figure 7, the optimal quantity Q would increase if the continuous monitoring is conducted and the net profit would increase if the inspection cost is not too high.

Q			$\Pi_3(Q,\tau^*)$
2100	$\tau^*=9$	$T=24$	25176
2300	$\tau^*=10$	$T=27$	26604
2500	$\tau^*=11$	$T=30$	27833
2700	$\tau^*=12$	$T=33$	28850
2900	$\tau^*=13$	$T=36$	29638
3100	$\tau^*=14$	$T=40$	30173
3300	$\tau^*=16$	$T=44$	$\boxed{30418}$
3500	$\tau^*=17$	$T=51$	30308
3700	$\tau^*=19$	$T=60$	29639
3900	$\tau^*=19$	$T=60$	28724
4100	$\tau^*=19$	$T=60$	27810
4300	$\tau^*=19$	$T=60$	26895

Table 3: Optimal value in Model 3.

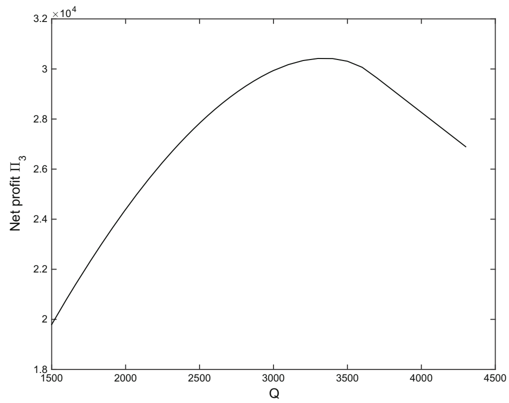


Fig. 7: Net profit Π_3 at optimal inspection time τ^* for each Q.

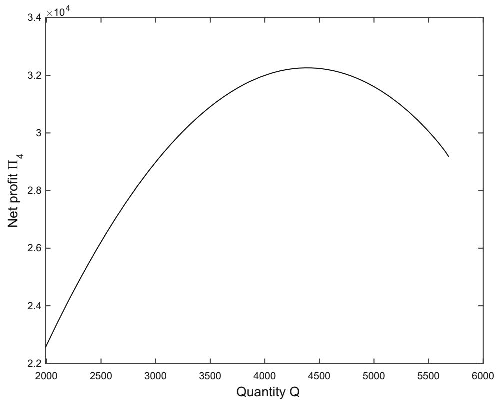


Fig. 8: Net profit Π_4 for each Q.

Since the optimal replenishment cycle in Model 1 is $T = 45$, we set $T = 45$ and make a comparison among the four models. Table 4 shows that the quantity Q would increase if one inspection or continuous monitoring is conducted and the net profit would increase if the inspection cost is not too high. In Model 2, the net profit of the inventory holder with price discounts for an additional order would be greater than those in other cases. Further, the net profit in each case of Model 2 would be higher than those in Models 1, 3, and 4. This implies that the inventory holder would prefer an additional order during the replenishment cycle, especially when a discount price is provided for the additional order. Otherwise, adopting one inspection or continuous monitoring would also be a good choice, since the penalty cost for the sales of deteriorated items would be saved and the net profit would increase if the cost for one inspection or continuous monitoring is not too high.

	Model 1	Model 2			Model 3	Model 4
		Case 1	Case 2	Case 3		
Q	2812	$--$	$--$	$--$	3350	4439
Q_1	$--$	1797	1667	1979	$--$	$--$
Q_2	$--$	1859	1979	1667	$--$	$--$
t_1^*	$--$	22	20	25	$--$	$--$
τ^*	$--$	$--$	$--$	$--$	16	$--$
Π	27065	45040	50765	39827	30417	32255

Table 4: Optimal value of the long-run average profit in each model.

6.2 Sensitivity Analysis

The sensitivity analysis is performed by changing each of the parameters by -30%, -15%, 15%, and 30%. We adopt the method that one parameter is changed at a time and the remaining parameters are kept constant.

The optimal profits for varying parameters in Models 1, 2, 3 and 4 are shown in Tables 5, 6, 7 and 8 respectively. Here, for Model 2, we choose Case 2 ($c > b$) and set $T = 45$ since the supplier usually provides price discounts in order to get more market shares. The results show that the net profit of Model 2, which is denoted as Π_2 in Table 6, is greater than other net profits in Tables 5, 7 and 8 under the same parameters.

From Figures 9, 10, 11 and 12, it is found that the optimal net profits Π_1, Π_2, Π_3 and Π_4 in the four models increase as D, m and p increase, but decrease as the other parameters increase. That is because the market demand rate D, the maximum lifetime m and the sale price p affect the incomes during the replenishment cycle while other parameters affect the inventory costs.

Further, the net profit Π_2 tends to rise much faster than the net profits in other models as D, m and p increase, and it tends to drop more slowly than the net profits in other models as k, c and h increase. This result indicates that the net profit under model 2 would be more stable with respect to the changes of parameters which would affect the inventory costs than the net profits under other three models. And the net profit under model 2 would be more sensitive to the changes of parameters which would affect the incomes than the net profits under other three models. Hence, the costs in Model 2 could be effective controlled, and the incomes under Model 2 would enhance substantially when the market works well. Therefore, the inventory holder is suggested to employ an additional order if the supplier provides price discounts.

In Model 3, the inventory holder would decide the optimal inspection time, and Figure 11 shows that the net profit is moderately sensitive to the change in the inspection cost c_d. This indicates that the optimal inspection time and the optimal quantity may remain unchanged when the inspection cost c_d varies within a certain range.

In Model 4, Figure 12 shows that the net profit decreases as d, g and c increase, and it is sensitive to the changes in parameters d and c, but moderately sensitive to the change in parameter g. Thus, reducing the costs which include the purchasing cost and the installation cost of equipment for continuous monitoring will result in a significant saving.

Π_1 Change / Parameter	-30% changed	-15% changed	15% changed	30% changed
D	18946	23006	31125	35185
m	19139	23114	30995	34902
k	27317	27191	26941	26817
p	15297	21141	33034	39029
c	30491	28767	25388	23739
h	27218	27141	26989	26913

Table 5: Sensitivity analysis of net profit in Model 1

Π_2 Change / Parameter	-30% changed	-15% changed	15% changed	30% changed
D	35535	43149	58379	65993
m	41560	46763	53907	56442
k	50952	50858	50670	50577
p	32673	41702	59827	68904
c	52808	51768	49764	48804
h	50878	50821	50707	50650

Table 6: Sensitivity analysis of net profit of Case 2 in Model 2

Π_3 Change / Parameter	-30% changed	-15% changed	15% changed	30% changed
D	21293	25855	34981	39544
m	21499	25973	34842	39235
k	30615	30515	30325	30232
p	17033	23670	37240	44077
c	34508	32454	28438	26496
h	30600	30507	30331	30244
c_d	30476	30447	30389	30361

Table 7: Sensitivity analysis of net profit in Model 3

Π_4 Change / Parameter	-30% changed	-15% changed	15% changed	30% changed
D	22051	27153	37363	42471
m	22663	27462	37056	41850
p	16014	23972	40737	49345
c	37733	34943	29676	27202
d	33586	32921	31604	30953
g	32795	32525	31992	31728

Table 8: Sensitivity analysis of net profit in Model 4

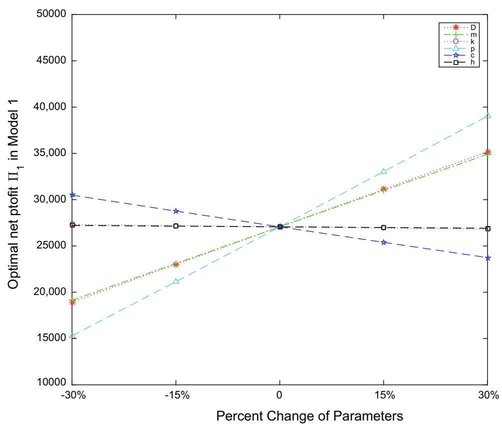


Fig. 9: Optimal net profit Π_1 for varying parameters in Model 1.

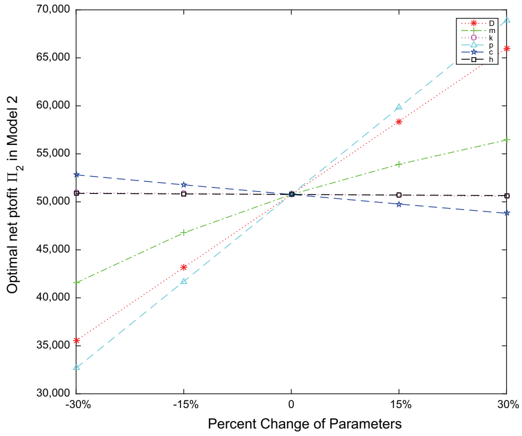


Fig. 10: Optimal net profit Π_2 for varying parameters in Model 2.

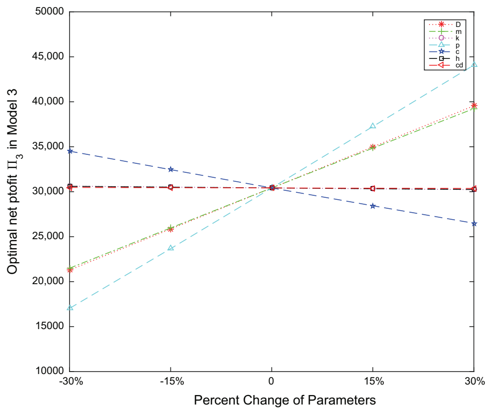


Fig. 11: Optimal net profit Π_3 for varying parameters in Model 3.

7 Conclusions

In this study, we consider inventory models for deteriorated items with maximum lifetime. The process of that deteriorated items are sold to consumers together with serviceable items during a replenishment cycle is modeled to derive the optimal order quantity and inspection strategies. The penalty cost for the sales of deteriorated products is considered. The inventory holder's profit is then improved by designing an additional order contract. We also derive the optimal ordering quantities and ordering time for the additional ordering contract to maximize the inventory holder's net profit. The optimal inspection time for one inspection policy and closed-form solutions of optimal ordering quantity and replenishment cycle are obtained when the items are continuous monitored. The results show that adopting one inspection or continuous monitoring would be a good choice, since the

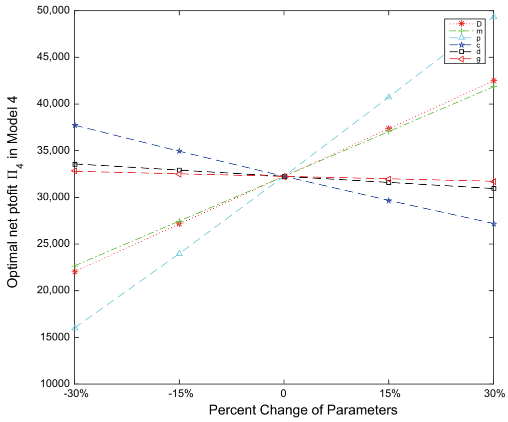


Fig. 12: Optimal net profit Π_4 for varying parameters in Model 4.

penalty cost for the sales of deteriorated items would be saved and the net profit would increase if the cost for one inspection or continuous monitoring is not too high. However, it would be more beneficial for the inventory holder to employ an additional order, especially when a discount is provided for an additional order. Further, if the supplier provides price discounts, the net profit of the inventory holder under model with additional order would be more stable with respect to the changes of parameters which would affect the inventory costs than the net profits under the models without additional order, and the incomes under model with additional order would be enhanced substantially when the market works well.

One limitation of this study is that we assume that the inspection process is performed perfectly in the sense that it will correctly screen out the deteriorated items. For future research, one may consider imperfect inspection with errors. Another research direction is to consider an extended inventory model for multiple items with different maximum lifetimes.

8 Appendix

8.1 Proof of Proposition 1

Based on Eq. (10), it has

$$\frac{\partial \Pi_1}{\partial T_1} = \frac{D[p+k+h(1+m)]}{m(1+m)} \left[T_1 - \frac{(p-c)(1+m)}{p+k+h(1+m)} \right] (T_1 - m).$$

Set $\dfrac{\partial \Pi_1}{\partial T_1} = 0$, it has $T_1 = \frac{(p-c)(1+m)}{p+k+h(1+m)}$ or $T_1 = m$.

Case 1 If $p < c(1+m) + hm(1+m) + km$, then $\frac{(p-c)(1+m)}{p+k+h(1+m)} < m$. Then it has

$$\frac{\partial \Pi_1}{\partial T_1} \geq 0 \text{ when } T_1 \in \left(0, \frac{(p-c)(1+m)}{p+k+h(1+m)}\right]$$
$$\frac{\partial \Pi_1}{\partial T_1} < 0 \text{ when } T_1 \in \left(\frac{(p-c)(1+m)}{p+k+h(1+m)}, m\right].$$

Therefore Π_1 reaches its maximum at $T_1^* = \dfrac{(p-c)(1+m)}{p+k+h(1+m)}$.

Case 2 If $p \geq c(1+m) + hm(1+m) + km$, then $\frac{(p-c)(1+m)}{p+k+h(1+m)} \geq m$. Then it has

$$\frac{\partial \Pi_1}{\partial T_1} \geq 0 \text{ when } T_1 \in (0, m].$$

Therefore Π_1 reaches its maximum at $T_1^* = m$.

The proof is completed.

8.2 Proof of Proposition 2

Suppose that $c = b$. Denote $m_1 = m + \frac{(p-c)(1+m)}{p+k+h(1+m)}$. Based on Eq. (14), it has

$$\frac{\partial \Pi_2}{\partial t_1} = \frac{-D[p+k+h(1+m)]}{m(1+m)}(T - 2t_1)(T - m_1)$$

and

$$\frac{\partial^2 \Pi_2}{\partial {t_1}^2} = \frac{2D[p+k+h(1+m)]}{m(1+m)}(T - m_1).$$

Case 1 If $p < c(1+m) + km + hm(1+m)$, then $\frac{(p-c)(1+m)}{p+k+h(1+m)} < m$ and $m_1 < 2m$.

a) If $0 < T < m_1$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Set $\dfrac{\partial \Pi_2}{\partial t_1} = 0$, it yields $t_1^* = \dfrac{T}{2}$.

b) If $T = m_1$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} = 0$ and $\dfrac{\partial \Pi_2}{\partial t_1} = 0$. Since it requires that $0 < t_1 \leq m$ and $0 < T - t_1 \leq m$, we have $T - m \leq t_1^* \leq m$.

c) If $m_1 < T < 2m$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} > 0$, and Π_2 is a convex function of t_1. Since $T - m < \dfrac{T}{2} < m$ and $m - \dfrac{T}{2} = \dfrac{T}{2} - (T - m)$, then $t_1^* = m$ or $T - m$.

Case 2 If $p \geq c(1+m) + km + hm(1+m)$, then $\frac{(p-c)(1+m)}{p+k+h(1+m)} \geq m$ and $m_1 \geq 2m$. Hence $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$ as $T \in (0, 2m)$, and Π_2 is a concave function of t_1. Set $\dfrac{\partial \Pi_2}{\partial t_1} = 0$, it yields $t_1^* = \dfrac{T}{2}$.

The proof is completed.

8.3 Proof of Proposition 3

Suppose that $c > b$. Denote

$$m_{21} = \frac{A - B}{2(p + k + h(1 + m))},$$
$$m_{22} = m + \frac{(p - c)(1 + m)}{p + k + h(1 + m)},$$
$$m_{23} = \frac{A + B}{2(p + k + h(1 + m))},$$
$$m_{24} = m + \frac{(p - \frac{(c+b)}{2})(1 + m)}{p + k + h(1 + m)},$$
$$m_{25} = m + \frac{(p - b)(1 + m)}{p + k + h(1 + m)},$$

where

$$A = (1 + 2m)(p + k) - (b + k)(1 + m) + hm(1 + m)$$

and

$$B = \sqrt{A^2 - 4(p + k + h(1 + m))(c - b)(1 + m)m}.$$

Based on Eq. (14), we have

$$\frac{\partial \Pi_2}{\partial t_1} = \frac{D[p + k + h(1 + m)]}{m(1 + m)} [2(T - m_{24})t_1 - (T - m_{21})(T - m_{23})]$$

and

$$\frac{\partial^2 \Pi_2}{\partial t_1^2} = \frac{2D[p + k + h(1 + m)]}{m(1 + m)}(T - m_{24}).$$

Set $\frac{\partial \Pi_2}{\partial t_1} = 0$ and we have

$$T_{2c} = \frac{(T - m_{21})(T - m_{23})}{2(T - m_{24})} = \frac{T}{2} + \frac{(c - b)(1 + m)(m - \frac{T}{2})}{2(T - m_{24})(p + k + h(1 + m))}.$$

Then it has

$$(T - T_{2c}) - m = \frac{(T - 2m)(T - m_{22})}{2(T - m_{24})}$$
$$T_{2c} - m = \frac{(T - 2m)(T - m_{25})}{2(T - m_{24})}$$
$$T - T_{2c} = \frac{T^2 - m_{22}T - m_{21}m_{23}}{2(T - m_{24})}.$$

Case 1 If $p < b(1 + m) + km + h(1 + m)$, it can be proved that

$$0 < m_{21} < m < m_{22} < m_{23} < m_{24} < m_{25} < 2m.$$

a) If $0 < T \leq m_{21}$, then $\frac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Since $T_{2c} \leq 0$, we have $t_1^* = 0$.

b) If $m_{21} < T \leq m$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Since $0 < T_{2c} < T$, we have $t_1^* = T_{2c}$.

c) If $m < T \leq m_{22}$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Since $T - m < T_{2c} < m$, we have $t_1^* = T_{2c}$.

d) If $m_{22} < T \leq m_{23}$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Since $0 < T_{2c} < T - m$, we have $t_1^* = T - m$.

e) If $m_{23} < T < m_{24}$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Since $T_{2c} < T - m$, we have $t_1^* = T - m$.

f) If $T = m_{24}$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} = 0$, and $\dfrac{\partial \Pi_2}{\partial t_1} < 0$. Since it requires that $T - m \leq t_1 \leq m$, we have $t_1^* = T - m$.

g) If $m_{24} < T \leq m_{25}$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} > 0$, and Π_2 is a convex function of t_1. Since $m < T_{2c}$, we have $t_1^* = T - m$.

h) If $m_{25} < T < 2m$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} > 0$, and Π_2 is a convex function of t_1. It shows that $\frac{T}{2} < T_{2c}$ and $T - m < T_{2c} < m$. Since $m - T_{2c} - T_{2c} - (T - m)) = T - 2T_{2c} < 0$, we have $t_1^* = T - m$.

Therefore, we have

$$t_1^* = \begin{cases} 0, & T \in (0, m_{21}] \\ T_{2c}, & T \in (m_{21}, m_{22}] \\ T - m, & T \in (m_{22}, 2m) \end{cases}$$

Case 2 If $b(1+m) + km + hm(1+m) \leq p < \frac{c+b}{2}(1+m) + km + hm(1+m)$, it can be proved that

$$0 < m_{21} < m < m_{22} < m_{23} < m_{24} < 2m < m_{25}.$$

a) If $0 < T \leq m_{21}$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Since $T_{2c} \leq 0$, we have $t_1^* = 0$.

b) If $m_{21} < T \leq m$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Since $0 < T_{2c} < T$, we have $t_1^* = T_{2c}$.

c) If $m < T \leq m_{22}$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Since $T - m < T_{2c} < m$, we have $t_1^* = T_{2c}$.

d) If $m_{22} < T \leq m_{23}$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Since $0 < T_{2c} < T - m$, we have $t_1^* = T - m$.

e) If $m_{23} < T < m_{24}$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Since $T_{2c} < T - m$, we have $t_1^* = T - m$.

f) If $T = m_{24}$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} = 0$, and $\dfrac{\partial \Pi_2}{\partial t_1} < 0$. Since it requires that $T - m \leq t_1 \leq m$, we have $t_1^* = T - m$.

g) If $m_{24} < T < 2m$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} > 0$, and Π_2 is a convex function of t_1. Since $T_{2c} > m$, we have $t_1^* = T - m$.

Then we have

$$t_1^* = \begin{cases} 0, & T \in (0, m_{21}] \\ T_{2c}, & T \in (m_{21}, m_{22}] \\ T - m, & T \in (m_{22}, 2m) \end{cases}$$

Case 3 If $\frac{c+b}{2}(1+m) + km + hm(1+m) \leq p < c(1+m) + km + hm(1+m)$, it can be proved that

$$0 < m_{21} < m < m_{22} < 2m < m_{24} < m_{23} < m_{25}.$$

a) If $0 < T \leq m_{21}$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Since $T_{2c} \leq 0$ and it requires that $0 \leq t_1 \leq T$, we have $t_1^* = 0$.

b) If $m_{21} < T \leq m$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Since $0 < T_{2c} < T$, we have $t_1^* = T_{2c}$.

c) If $m < T \leq m_{22}$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Since $T - m < T_{2c} < m$, we have $t_1^* = T_{2c}$.

d) If $m_{22} < T < 2m$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Since $0 < T_{2c} < T - m$, we have $t_1^* = T - m$.

Then we have

$$t_1^* = \begin{cases} 0, & T \in (0, m_{21}] \\ T_{2c}, & T \in (m_{21}, m_{22}] \\ T - m, & T \in (m_{22}, 2m) \end{cases}$$

Case 4 If $c(1+m) + km + hm(1+m) \leq p$, it can be proved that

$$0 < m_{21} < m < 2m < m_{22} < m_{24} < m_{23} < m_{25}.$$

a) If $0 < T \leq m_{21}$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Since $T_{2c} \leq 0$ and it requires that $0 \leq t_1 \leq T$, we have $t_1^* = 0$.

b) If $m_{21} < T \leq m$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Since $0 < T_{2c} < T/2$, we have $t_1^* = T_{2c}$.

c) If $m < T < 2m$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Since $0 < T - m < T_{2c} < m$, we have $t_1^* = T_{2c}$.

Then we have

$$t_1^* = \begin{cases} 0, & T \in (0, m_{21}] \\ T_{2c}, & T \in (m_{21}, 2m) \end{cases}$$

It shows that the results in Cases 1, 2 and 3 are the same, hence if $p < c(1+m) + km + hm(1+m)$,

$$t_1^* = \begin{cases} 0, & T \in (0, m_{21}] \\ T_{2c}, & T \in (m_{21}, m_{22}] \\ T - m, & T \in (m_{22}, 2m) \end{cases}.$$

The proof is completed.

8.4 Proof of Proposition 4

Suppose that $c < b$. Denote

$$
\begin{aligned}
m_{21} &= \frac{A-B}{2(p+k+h(1+m))},\\
m_{22} &= m + \frac{(p-c)(1+m)}{p+k+h(1+m)},\\
m_{23} &= \frac{A+B}{2(p+k+h(1+m))},\\
m_{24} &= m + \frac{(p-\frac{(c+b)}{2})(1+m)}{p+k+h(1+m)},\\
m_{25} &= m + \frac{(p-b)(1+m)}{p+k+h(1+m)},\\
m_{31} &= \frac{E-F}{2(p+k+h(1+m))},\\
m_{36} &= \frac{E+F}{2(p+k+h(1+m))},\\
T_{2c} &= \frac{(T-m_{21})(T-m_{23})}{2(T-m_{24})},
\end{aligned}
$$

where

$$
\begin{aligned}
A &= [p+k+h(1+m)]m + (p-b)(1+m),\\
B &= \sqrt{A^2 - 4(p+k+h(1+m))(c-b)(1+m)m},\\
E &= [p+k+h(1+m)]m + (p-c)(1+m)
\end{aligned}
$$

and

$$F = \sqrt{E^2 - 4(p+k+h(1+m))(b-c)(1+m)m}.$$

Based on Eq. (14), we have

$$\frac{\partial \Pi_2}{\partial t_1} = \frac{D[p+k+h(1+m)]}{m(1+m)}\left[2(T-m_{24})t_1 - (T-m_{21})(T-m_{23})\right]$$

and

$$\frac{\partial^2 \Pi_2}{\partial t_1^2} = \frac{2D[p+k+h(1+m)]}{m(1+m)}(T-m_{24}).$$

Set $\frac{\partial \Pi_2}{\partial t_1} = 0$ and we have

$$T_{2c} = \frac{(T-m_{21})(T-m_{23})}{2(T-m_{24})} = \frac{T}{2} + \frac{(c-b)(1+m)(m-\frac{T}{2})}{2(T-m_{24})(p+k+h(1+m))}.$$

Then it has

$$
\begin{aligned}
(T-T_{2c}) - m &= \frac{(T-2m)(T-m_{22})}{2(T-m_{24})}\\
T_{2c} - m &= \frac{(T-2m)(T-m_{25})}{2(T-m_{24})}\\
T - T_{2c} &= \frac{T^2 - m_{22}T - m_{21}m_{23}}{2(T-m_{24})}.
\end{aligned}
$$

Case 1 If $p < c(1+m) + km + h(1+m)$, it can be proved that

$$m_{21} < 0 < m_{31} < m < m_{25} < m_{24} < m_{23} < m_{22} < 2m.$$

a) If $0 < T \le m_{31}$, then $\frac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Since $T < T_{2c}$, we have $t_1^* = T$.
b) If $m_{31} < T \le m$, then $\frac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Since $0 < T_{2c} < T$, we have $t_1^* = T_{2c}$.
c) If $m < T \le m_{25}$, then $\frac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Since $T - m < T_{2c} < m$, we have $t_1^* = T_{2c}$.
d) If $m_{25} < T < m_{24}$, then $\frac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Since $m < T_{2c}$, we have $t_1^* = m$.
e) If $T = m_{24}$, then $\frac{\partial^2 \Pi_2}{\partial {t_1}^2} = 0$, and $\frac{\partial \Pi_2}{\partial t_1} > 0$. Since it requires that $T - m \le t_1 \le m$, we have $t_1^* = m$.
f) If $m_{24} < T \le m_{23}$, then $\frac{\partial^2 \Pi_2}{\partial {t_1}^2} > 0$, and Π_2 is a convex function of t_1. Since $T_{2c} < 0$ and it requires that $T - m \le t_1 \le m$, we have $t_1^* = m$.
g) If $m_{23} < T \le m_{22}$, then $\frac{\partial^2 \Pi_2}{\partial {t_1}^2} > 0$, and Π_2 is a convex function of t_1. Since $T_{2c} < T - m$, we have $t_1^* = m$.
h) If $m_{22} < T < 2m$, then $\frac{\partial^2 \Pi_2}{\partial {t_1}^2} > 0$, and Π_2 is a convex function of t_1. Since $T - m < T_{2c} < m$ and $T_{2c} < T/2$, we have $t_1^* = m$.

Therefore, we have

$$t_1^* = \begin{cases} T, & T \in (0, m_{31}] \\ T_{2c}, & T \in (m_{31}, m_{25}] \\ m, & T \in (m_{25}, 2m) \end{cases}$$

Case 2 If $c(1+m) + km + hm(1+m) \le p < \frac{c+b}{2}(1+m) + km + hm(1+m)$, it can be proved that

$$m_{21} < 0 < m_{31} < m < m_{25} < m_{24} < m_{23} < 2m < m_{22}.$$

a) If $0 < T \le m_{31}$, then $\frac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Since $T < T_{2c} < m$, we have $t_1^* = T$.
b) If $m_{31} < T \le m$, then $\frac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Since $0 < T_{2c} < T$, we have $t_1^* = T_{2c}$.
c) If $m < T \le m_{25}$, then $\frac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Since $T - m < T_{2c} < m$, we have $t_1^* = T_{2c}$.
d) If $m_{25} < T < m_{24}$, then $\frac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Since $m < T_{2c}$, we have $t_1^* = m$.
e) If $T = m_{24}$, then $\frac{\partial^2 \Pi_2}{\partial {t_1}^2} = 0$, and $\frac{\partial \Pi_2}{\partial t_1} > 0$. Since it requires that $T - m \le t_1 \le m$, we have $t_1^* = m$.

f) If $m_{24} < T \leq m_{23}$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} > 0$, and Π_2 is a convex function of t_1. Since $T_{2c} < 0 < T - m < m$, we have $t_1^* = m$.

g) If $m_{23} < T < 2m$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} > 0$, and Π_2 is a convex function of t_1. Since $0 < T_{2c} < T - m$ and it requires that $T - m \leq t_1 \leq m$, we have $t_1^* = m$.

Therefore, we have

$$t_1^* = \begin{cases} T, & T \in (0, m_{31}] \\ T_{2c}, & T \in (m_{31}, m_{25}] \\ m, & T \in (m_{25}, 2m) \end{cases}$$

Case 3 If $\frac{c+b}{2}(1+m) + km + hm(1+m) \leq p < b(1+m) + km + hm(1+m)$, it can be proved that

$$m_{21} < 0 < m_{31} < m < m_{25} < 2m < m_{23} < m_{24} < m_{22}.$$

a) If $0 < T \leq m_{31}$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Since $T < T_{2c}$, we have $t_1^* = T$.

b) If $m_{31} < T \leq m$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Since $0 < T_{2c} < T$, we have $t_1^* = T_{2c}$.

c) If $m < T \leq m_{25}$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Since $T - m < T_{2c} < m$, we have $t_1^* = T_{2c}$.

d) If $m_{25} < T < 2m$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Since $m < T_{2c}$ and it requires that $T - m \leq t_1 \leq m$, we have $t_1^* = m$.

Therefore, we have

$$t_1^* = \begin{cases} T, & T \in (0, m_{31}] \\ T_{2c}, & T \in (m_{31}, m_{25}] \\ m, & T \in (m_{25}, 2m) \end{cases}$$

Case 4 If $b(1+m) + km + hm(1+m) \leq p$, it can be proved that

$$m_{21} < 0 < m_{31} < m < 2m < m_{25} < m_{23} < m_{24} < m_{22}.$$

a) If $0 < T \leq m_{31}$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Since $T < T_{2c}$ and it requires that $0 \leq t_1 \leq T$, we have $t_1^* = T$.

b) If $m_{31} < T \leq m$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Since $0 < T_{2c} < T$ and it requires that $0 \leq t_1 \leq T$, we have $t_1^* = T_{2c}$.

c) If $m < T < 2m$, then $\dfrac{\partial^2 \Pi_2}{\partial {t_1}^2} < 0$, and Π_2 is a concave function of t_1. Since $T - m < T_{2c} < m$ and it requires that $T - m \leq t_1 \leq T$, we have $t_1^* = T_{2c}$.

Therefore, we have

$$t_1^* = \begin{cases} T, & T \in (0, m_{31}] \\ T_{2c}, & T \in (m_{31}, 2m) \end{cases}$$

It shows that the results in Cases 1, 2 and 3 are the same, hence if $p < b(1+m) + km + hm(1+m)$,

$$t_1^* = \begin{cases} T, & T \in (0, m_{31}] \\ T_{2c}, & T \in (m_{31}, m_{25}] \\ m, & T \in (m_{25}, 2m) \end{cases}$$

The proof is completed.

8.5 Proof of Lemma 1

Denote $T_s = T - \tau$.The total number of items sold during the time period $[\tau, T]$ is

$$Q_{\tau T} = \int_{\tau}^{T} \lambda(t)dt = \int_{0}^{T-\tau} \lambda_1(t)dt = D(1 - \frac{\tau}{m})T_s - D\frac{T_s^2}{2m}.$$

Since $0 \le T_s \le m-\tau$ and $\frac{\partial Q_{\tau T}}{T_s} > 0$, it implies that $Q_{\tau T}$ increases as T_s increases. Hence

$$\max_{T_s \in [0, m-\tau]} Q_{\tau T} = Q_{\tau T}(T_s = m - \tau) = D\frac{(m-\tau)^2}{2m} = M_Q$$

Then if $Q_{I\tau} \ge M_Q$, $T_s^* = m - \tau$ and $T^* = m$;
If $Q_{I\tau} < M_Q$, we have

$$Q_{I\tau} = D(1 - \frac{\tau}{m})T_s - D\frac{T_s^2}{2m}.$$

Then $T_s^* = m - \tau - \sqrt{(m-\tau)^2 - 2\frac{m}{D}Q_{I\tau}}$, and $T^* = m - \sqrt{(m-\tau)^2 - 2\frac{m}{D}Q_{I\tau}}$.

8.6 Proof of Lemma 2

Based on lemma 1, if $Q_{I\tau} > M_Q$, the inventory holder can increase the profit by decrease the quantity of Q. Therefore it has

$$Q_{I\tau} \le M_Q,$$

that is

$$\frac{1+m-\tau}{1+m}(Q - D\tau + D\frac{\tau^2}{2m}) \le D\frac{(m-\tau)^2}{2m}.$$

Then

$$Q \le V$$

where

$$V = D\frac{1+m}{2m}\frac{(m-\tau)^2}{1+m-\tau} + D\tau - D\frac{\tau^2}{2m}.$$

$$\frac{\partial V}{\partial \tau} = D\frac{m-\tau}{m(1+m-\tau)^2}(\tau - v_1)(\tau - v_2),$$

where $v_1 = \dfrac{3(1+m) - \sqrt{(1+m)^2 + 8(1+m)}}{4}$ and $v_1 = \dfrac{3(1+m) + \sqrt{(1+m)^2 + 8(1+m)}}{4}$.

$$\frac{\partial^2 V}{\partial \tau^2} = \frac{D}{m}\left[\frac{1+m}{(1+m-\tau)^3} - 1\right],$$

V is concave in $[0, 1+m-(1+m)^{1/3})$ and convex in $[1+m-(1+m)^{1/3}, m)$. Since

$$v_1 < 1+m-(1+m)^{1/3} < m < v_2,$$

V reaches its maximum at $\tau = v_1$. Hence $Q \leq V(\tau = v_1)$. The proof is completed.

8.7 Proof of Lemma 4

If the serviceable items are sold out before time m, it has

$$Q = \frac{1+m}{m}(Dt + D\ln(1+m-t) - D\ln(1+m)).$$

Denote $D_t = \frac{1+m}{m}(Dt + D\ln(1+m-t) - D\ln(1+m))$, then

$$\frac{\partial D_t}{\partial t} = D\frac{1+m}{m}(1 - \frac{1}{1+m-t}) > 0.$$

Hence

$$\max_{t\in[0,m]} D_t = D(1+m) - \frac{1+m}{m}\ln(1+m)),$$

and

$$Q \leq \max_{t\in[0,m]} D_t.$$

The proof is completed.

8.8 Proof of Proposition 5

Based on Eq. (27), we have

$$\frac{Q}{T} = D\frac{1+m}{m}(1 + \frac{1}{T}\ln\frac{1+m-T}{1+m}) \leq D\frac{1+m}{m}(1 + \ln\frac{m}{1+m}) < D.$$

Then the average cost of one product is

$$\frac{TC_4}{Q} = c + d + g\frac{T}{Q} > c + d + \frac{g}{D}.$$

Therefore, we suppose that the sales price is greater than $c + d + g/D$.

$$\begin{aligned}
\Pi_4 &= TR - TC \\
&= PDT - PD\frac{T^2}{2m} \\
&\quad - \frac{(c+d)(1+m)}{m}\left[DT + D\ln(1+m-T) - D\ln(1+m)\right] - gT.
\end{aligned}$$

We have

$$\begin{aligned}
\frac{\partial \Pi_4}{\partial T} &= \frac{1}{m}\left[pDm - pDT - (c+d)(1+m)D(1 - \frac{1}{1+m-T}) - gm\right] \\
&= \frac{1}{my}\left\{pDy^2 - [pD + (c+d)(1+m)D + gm]y + (c+d)(1+m)D\right\},
\end{aligned}$$

where $y = 1 + m - T$.

$$\frac{\partial^2 \Pi_4}{\partial T^2} = -\frac{D}{m}\left[p - (c+d)(1+m)\frac{1}{(1+m-T)^2}\right].$$

Denote

$$t_d = 1 + m - \sqrt{\frac{c+d}{p}(1+m)},$$

then $\frac{\partial^2 \Pi_4}{\partial T^2} < 0$ when $T \in (0, t_d)$, $\frac{\partial^2 \Pi_4}{\partial T^2} = 0$ when $T = t_d$, $\frac{\partial^2 \Pi_4}{\partial T^2} > 0$ when $T \in (t_d, m)$, which implies that Π_4 is concave in $(0, t_d)$ and convex in (t_d, m).

Denote

$$G = pD + (c+d)(1+m)D + mg,$$
$$H = 4pD(c+d)(1+m)D.$$

Set $\frac{\partial \Pi_4}{\partial T} = 0$, then $y_1 = \frac{G + \sqrt{G^2 - H}}{2pD}$ and $y_2 = \frac{G - \sqrt{G^2 - H}}{2pD}$. $T_{y1} = 1 + m - y_1$ and $T_{y2} = 1 + m - y_2$. It can be proved that $1 < y_1 < m + 1$ when $c + d + g/D < p$. It also has $y_2 < 1$.

a) If $c + d + g/D < p < (c+d)(1+m)$, then $0 < t_d < m$ and it can be proved that $0 < T_{y1} < t_d < m < T_{y2}$. Then the optimal replenishment cycle is $T^* = 1 + m - y_1$.
b) If $(c+d)(1+m) \leq p$, then $m \leq t_d$. It can be proved that $0 < T_{y1} < m < t_d$ and $m < T_{y2}$. Then the optimal replenishment cycle is $T^* = 1 + m - y_1$.

Therefore the optimal replenishment cycle is $T^* = 1 + m - y_1$ as $c + d + g/D < p$, and the optimal quantity is

$$Q^* = \frac{1+m}{m}(DT^* + D\ln(1 + m - T^*) - D\ln(1+m)).$$

The proof is completed.

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