# Approximations for Generalized Unsplittable Flow on Paths with Application to Power Systems Optimization 

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#### Abstract

The Unsplittable Flow on a Path (UFP) problem has garnered considerable attention as a challenging combinatorial optimization problem with notable practical implications. Steered by its pivotal applications in power engineering, the present work formulates a novel generalization of UFP, wherein demands and capacities in the input instance are monotone step functions over the set of edges. As an initial step towards tackling this generalization, we draw on and extend ideas from prior research to devise a quasi-polynomial time approximation scheme (QPTAS) under the premise that the demands and capacities lie in a quasi-polynomial range. Second, retaining the same assumption, an efficient logarithmic approximation is introduced for the single-source variant of the problem. Finally, we round up the contributions by designing a (kind of) black-box reduction that, under some mild conditions, allows to translate LP-based approximation algorithms for the studied problem into their counterparts for the Alternating Current Optimal Power Flow (AC OPF) problem - a fundamental workflow in operation and control of power systems.


Keywords Unsplittable Flow Problem • QPTAS • LP Rounding • Logarithmic Approximation • Power Systems Engineering • AC Optimal Power Flow.

[^0][^1]
## Nomenclature

| Notation | Description |
| :---: | :--- |
| $\mathcal{G}$ Line graph <br> $\mathcal{V}$ Set of vertices (indexed by $i$ or $j$ ) <br> $\mathcal{E}$ Set of $m$ edges (indexed by $e$ or $(i, j)$ ) <br> $\mathcal{I}$ Set of $n$ users (indexed by $k$ ) <br> $\mathcal{Q}$ Grouping (of cardinality Q) of users based on utility-to-demand ratio <br> $\mathcal{I}^{q}$ Set of users in group $q \in \mathcal{Q}$ <br> $\mathcal{L}^{q}$ Set of users with "large" demands in group $q \in \mathcal{Q}$ <br> $\mathcal{S}^{q}$ Set of users with "small" demands in group $q \in \mathcal{Q}$ <br> $d$ Number of dimensions related to the generalized UFP <br> $u_{k}$ User $k$ 's utility value <br> $x_{k}$ Decision variable for user $k$ <br> $f_{k}^{r}(\cdot)$ User $k$ 's demand function over $\mathcal{E}$ in dimension $r \in\{1,2, \ldots, d\}$ <br> $e_{k}^{r}, \hat{e}_{k}^{r}$ User $k$ 's demand function's binding edges in dimension $r \in\{1,2, \ldots, d\}$ <br> $c^{r}(\cdot)$ Capacity function over $\mathcal{E}$ in dimension $r \in\{1,2, \ldots, d\}$ <br> $T_{1}, \ldots, T_{d}$ $d$ positive integers <br> $T$ Maximum among $T_{1}, \ldots, T_{d}$ <br> $C_{1}, \ldots, C_{d}$ $d$ integers each greater than 1 <br> $P_{r}$ Number of edge partitions in dimension $r \in\{1,2, \ldots, d\}$ <br> $\epsilon$ Constant in $(0,1)$ |  |

Summary of key notations related to $A C$ OPF

| $\mathcal{T}$ | Graph of a radial distribution network |
| :---: | :--- |
| $\mathcal{V}^{+}$ | Set of vertices excluding the root 0 |
| $\mathcal{V}_{i}^{+}$ | Set of vertices in $\mathcal{V}^{+}$excluding the node $i$ |
| $\mathcal{N}$ | Set of all users (electrical loads) (indexed by $k$ ) |
| $\mathcal{U}_{j}$ | Set of users at node $j$ |
| $\mathcal{N}_{j}$ | Set of users residing on the subpath rooted at node $j$ |
| $\mathcal{F}$ | Set of users with elastic power demands |
| $\mathcal{P}_{j}$ | The (unique) path from node $j$ to the root 0 |
| $s_{k}$ | User $k$ 's complex power demand |
| $z_{i, j}$ | Impedance of power line (i,j) |
| $V_{j}$ | Voltage at node $j$ |
| $v_{j}$ | Voltage magnitude square at node $j$ |
| $I_{i, j}$ | Current traversing through line $(i, j)$ |
| $l_{i, j}$ | Squared magnitude of current flowing through line $(i, j)$ |
| $S_{i, j}$ | Complex power flowing from node $i$ to node $j$ |

## 1 Introduction

The UFP, in its most generic form, takes as input a capacitated line graph along with a collection of flow requests, each parameterized by a demand, a profit (utility)
and a pair of source-sink vertices. Constrained by edge capacities, the pursued objective is to compute a maximum profit subset of requests routable simultaneously. Despite its apparent simplicity, UFP specializes to a number of classical NP-hard combinatorial problems, including the Knapsack problem (when the graph comprises a solitary edge) and the Maximum Edge-disjoint Path problem (when all demands and capacities are set to unity). On the practical side, this problem underlies a spectrum of real-world applications in communication networks (Bar-Noy et al., 2001), space missions (Hall and Magazine, 1994), the Web (Albers et al., 1999) and data centers management (Bansal et al., 2006), to name a few.

Recently, several studies have revisited UFP generalizing it from different perspectives. In (Momke and Wiese, 2015), UFP is extended to the Storage Allocation problem, where the requests are additionally characterized by a vertical position (i.e., height) and a coupling constraint is imposed enforcing a non-overlapping drawing of them. Another line of work in (Adamaszek et al., 2016), adapted the problem to the setting of a submodular objective function stimulated by theoretical and practical appeal thereof. Expanding the application scope further, this paper introduces a novel generalization of UFP. Previously, Cook et al. (Cook et al., 1998) developed an interesting framework linking electrical power transmission with the unsplittable flow on general graphs. Solidifying this nexus, we establish a formal bond between UFP and the AC OPF, an essential problem in power systems engineering introduced by Carpentier in 1962 (Carpentier, 1962) (see Section 5 for particulars). Formally, the proposed generalization is defined in what follows.

Generalization of UFP: In the d-dimensional Unsplittable Stairstep Flow on a $P$ ath ( $d$-USFP) problem, defined here for a fixed positive integer $d \in \mathbb{Z}_{+}$, given is a line network $\mathcal{G}=(\mathcal{V}, \mathcal{E})$ rooted at node 0 and a set $\mathcal{I}$ of $n$ users. Assuming an ascending ordering of the edges by distance from the root (i.e., $e_{1}<e_{2}<$ $\ldots<e_{m}$, where $\left.e_{i}=(i-1, i)\right)$, each user demand is captured by a $d$-dimensional vector $f_{k}=\left(f_{k}^{1}, \ldots, f_{k}^{d}\right)$, where for $\forall r, f_{k}^{r}: \mathcal{E} \rightarrow \mathbb{R}_{+}$are either monotone nonincreasing or monotone non-decreasing step functions over $\mathcal{E}$ (e.g., $f_{k}^{r}$ is monotone non-decreasing if $f_{k}^{r}(e) \leq f_{k}^{r}\left(e^{\prime}\right)$ whenever $e \leq e^{\prime}$ ). As with UFP, if $f_{k}$ is satisfied (routed), $u_{k} \geq 0$ is the perceived utility for customer $k$ and each edge $e \in \mathcal{E}$ is associated with a capacity, which in the current context is a $d$-dimensional vector $c=\left(c^{1}, \ldots, c^{d}\right)$, where $c^{r}: \mathcal{E} \rightarrow \mathbb{R}_{+}$is a monotone non-decreasing function on $\mathcal{E}$. With this input, $d$-USFP then takes the form

$$
\begin{align*}
(d-\operatorname{USFP}[\mathcal{I}, c]) & \max _{x} \sum_{k \in \mathcal{I}} u_{k} x_{k}, \\
\text { s.t. } & \sum_{k \in \mathcal{I}} f_{k}^{r}(e) x_{k} \leq c^{r}(e), \quad \forall e \in \mathcal{E}, r \in\{1, \ldots, d\}  \tag{1}\\
& x_{k} \in\{0,1\}, \quad \forall k \in \mathcal{I} . \tag{2}
\end{align*}
$$

In the above formulation, we may assume without loss of generality (by reversing the order on $\mathcal{E}$ if necessary) that $f_{k}^{r}(\cdot)$ is monotone non-decreasing for $\forall k \in$ $\mathcal{I}, r \in\{1, \ldots, d\}$. While $d$-USFP can be defined for any such $f_{k}^{r}(\cdot)$, this paper confines the scope to functions of separable form. More precisely, given positive integers $T_{1}, \ldots, T_{d}$, monotone (non-decreasing) functions $b^{r, t}: \mathcal{E} \rightarrow \mathbb{R}_{+}$, for $t=$
$1, \ldots, T_{r}, r \in[d]$ as well as non-negative numbers $a_{k}^{r, t} \in \mathbb{R}_{+}$and edges $e_{k}^{r}, \hat{e}_{k}^{r} \in \mathcal{E}$, for $t=1, \ldots, T_{r}$, it is assumed that

$$
f_{k}^{r}(e)=\sum_{t=1}^{T_{r}} a_{k}^{r, t} \tilde{b}_{k}^{r, t}(e), \text { where } \tilde{b}_{k}^{r, t}(e)= \begin{cases}0 & \text { if } e<e_{k}^{r},  \tag{3}\\ b^{r, t}(e) & \text { if } e_{k}^{r} \leq e<\hat{e}_{k}^{r} \\ b^{r, t}\left(\hat{e}_{k}^{r}\right) & \text { otherwise }\end{cases}
$$

This choice of functions ${ }^{1}$ stems from the relevant structural properties of OPF constraints, as elaborated in Section 5.2. Yet, even with this condition in place, $d$-USFP remains substantially more complicated than UFP as it entails the packing of monotone step functions of special type (rather than intervals) within a given capacity function. Hence, known techniques for UFP, if amenable, have to be extrapolated in a non-trivial manner to deal with $d$-USFP. In the proceeding paragraphs, we briefly review these techniques.

Related Work: As noted previously, UFP is NP-hard since it specializes to the Knapsack problem. In fact, even under the setting of uniform profits and capacities, it has proven to be strongly NP-hard (Chrobak et al., 2012). In light of this hurdle, most of the prior studies attempted simplified variants of UFP, with the two predominantly common ones being the uniform capacity UFP (UCUFP) and the UFP with the no-bottleneck assumption (UFP-NBA).

For UCUFP, which was also studied under the name of Resource Allocation problem, the first constant factor approximation was presented in (Phillips et al., 2000), attaining a 6 -approximation via LP rounding techniques. This factor was then refined by (Calinescu et al., 2002) to $(2+\epsilon)$. The approach therein decouples the instance into small and large requests, subsequently tackling the former in a fashion analogous to (Phillips et al., 2000), while the latter through dynamic programming.

Ensuing from a more general case, UFP-NBA restricts the maximum demand to be at most the minimum capacity of any edge. The crux of this condition is rooted in the integrality gap of the natural LP relaxation of UFP, which was shown to be $\Omega(n)$ in (Chakrabarti et al., 2007), whereas that of UFP-NBA is $\mathrm{O}(1)$. For UFP-NBA, the first constant factor approximation was derived in (Chakrabarti et al., 2007). Improving upon this, Chekuri et al. (Chekuri et al., 2007) obtained a $(2+\epsilon)$-approximation. These both studies broadly follow the aforementioned framework of decomposing the requests into small and large.

Turning to UFP, in 2006 Bansal et al. (Bansal et al., 2006) developed a deterministic QPTAS under the assumption that the capacities and demands are bounded by $2^{\text {polylog(n) }}$, thereby ruling out UFP's APX-hardness and hinting to the likely existence of a PTAS. The first polynomial-time approximation algorithm for UFP, yielding $\mathrm{O}(\log n)$ guarantee, was introduced in (Bansal et al., 2014). The algorithm is combinatorial, thus allowing to bypass the $\Omega(n)$ integrality gap of the natural LP relaxation. Later on, this result was extended in (Anagnostopoulos et al., 2014) to a $(2+\epsilon)$-approximation. Beating the barrier of 2 , Grandoni et al. (Grandoni et al., 2018) provided a dynamic programming-based polynomial time algorithm with an approximation factor of $\left(\frac{5}{3}+\epsilon\right)$, which in (Grandoni

[^2]et al., 2022b) was subsequently improved to $1+\frac{1}{1+e}+\epsilon<1.269$ (in expectation) via a novel randomized sketching technique. Closing the search for a PTAS, very recently Grandoni et al. (Grandoni et al., 2022a) devised a polynomial time $(1+\epsilon)$-approximation algorithm which tackles UFP by rephrasing the problem as a solitary game and is the best possible result unless $\mathrm{P}=\mathrm{NP}$.

Contributions and Paper Outline: As such, this study advances extant research in the following two aspects:
> We introduce a practically-driven generalization of UFP and initiate the search for its efficient approximations. As a first step in this direction, we extend the ideas in (Bansal et al., 2006) to construct a QPTAS for separable $d$ USFP, under the assumption that the demands and capacities lie in a quasipolynomial range. Second, relying on the same assumption, we devise an LPbased $\mathrm{O}(d \log n)$-approximation for the single-source setting of the problem (i.e., when all the requests share the same origin). The algorithm hinges on a simple reduction allowing to transform the problem to an easier instance with only $\mathrm{O}(d \log n)$ constraints.
$>$ A (kind of) black-box reduction is derived that, under some practical assumptions, translates an LP-based approximation for separable $d$-USFP into its analog for OPF on line distribution networks. This result complements the strand of research in (Karapetyan et al., 2018; Chau et al., 2018; Khonji et al., 2019; Elbassioni et al., 2019; Karapetyan et al., 2021) concerned with developing efficient approximations tailored for combinatorial optimization of AC electric power systems.

The remainder of this article is organized as follows. Section 2 covers the adopted notation along with a basic result on partitioning of the studied step functions. Section 3 presents the QPTAS for separable $d$-USFP. In Section 4 we provide the logarithmic approximation for single-source separable $d$-USFP. Section 5 contains an overview of AC OPF problem, followed by its mathematical formulation and the proposed reduction procedure producing LP-based approximations for OPF on line networks. Lastly, Section 6 concludes the paper with a discussion on applications and connotations of present contributions as well as prospective directions for further developments.

## 2 Notational Convention and Preliminaries

In what follows, unless otherwise explicitly mentioned, constants or variables are denoted in normal font (e.g., $C, d$ ), while sets in calligraphic capital letters (e.g., $\mathcal{E})$. We let $\mathbf{0}$ and $\mathbf{1}$ symbolize the vectors of all zeros and ones, respectively, and as a shorthand, we shall write $[n]$ to encode the range $\{1, \ldots, n\}$ for an integer $n$. Unless stated differently, we designate the operators ${ }^{-}$, _ to capture the maximum and minimum values of a variable/parameter/function, respectively. Given a complex number $\nu \in \mathbb{C}$, we let $|\nu|$ be its magnitude, $\arg (\nu)$ be the phase angle that it makes with the real axis, $\nu^{*}$ be its complex conjugate and write $\nu^{\mathrm{R}} \triangleq \operatorname{Re}(\nu)$, $\nu^{I} \triangleq \operatorname{Im}(\nu)$ for its real and imaginary components, respectively. With a slight abuse of notation, we shall also use the superscript * to mark the optimal solutions.

In line with (Bansal et al., 2006), we suppose the range of demands and capacities is quasi-polynomial. Mathematically,

$$
\max \left\{\frac{\max _{e \in \mathcal{E}, k \in \mathcal{I}, r \in[d]} f_{k}^{r}(e)}{\min _{e \in \mathcal{E}, k \in \mathcal{I}, r \in[d]: f_{k}^{r}(e)>0} f_{k}^{r}(e)}, \frac{\max _{e \in \mathcal{E}, r \in[d]} c^{r}(e)}{\min _{e \in \mathcal{E}, r \in[d]: c^{r}(e)>0} c^{r}(e)}\right\}=2^{\operatorname{polylog}(n)}
$$

This assumption is leveraged both, in the QPTAS and the logarithmic approximation, however, one can possibly discard it with techniques from (Batra et al., 2015).

The proposed approximations employ the following simple, yet crucial, lemma which, in a sense, states that the line can be partitioned into logarithmic (in $n$ ) number of regions such that, for each user $k$, the function $f_{k}(\cdot)$ is roughly constant in each region.

Lemma 1 For any $C_{r}>1, r \in[d], \mathcal{E}$ can be partitioned along each coordinate $r \in[d]$ into $P_{r}<T_{r} \log _{C_{r}}\left(\frac{b^{r}}{b^{r}}\right)$ intervals $\mathcal{E}^{r}=\bigcup_{p=1}^{P_{r}} \mathcal{E}_{p}^{r}$, where $\mathcal{E}_{p}^{r}:=$ $\left\{e_{\underline{i}(p, r)}, e_{\underline{i}(p, r)+1}, \ldots, e_{\bar{i}(p, r)}\right\}$, and

$$
\cdots<e_{\bar{i}(p-1, r)}<e_{\underline{i}(p, r)}<e_{\underline{i}(p, r)+1}<\cdots<e_{\bar{i}(p, r)}<e_{\underline{i}(p+1, r)}<\cdots,
$$

with the following property:

$$
\begin{equation*}
\bar{f}_{k}^{p, r} \leq C_{r} \cdot \underline{f}_{k}^{p, r}, \quad \forall k \in \mathcal{I}, \quad \forall p \in\left[P_{r}\right], \forall r \in[d], \tag{4}
\end{equation*}
$$

where $\underline{b}^{r}:=\min _{e \in \mathcal{E}, t \in\left[T_{r}\right]: b^{r, t}(e)>0} b^{r, t}(e), \bar{b}^{r}:=\max _{e \in \mathcal{E}, t \in\left[T_{r}\right]} b^{r, t}(e)=b^{r, t}$ $\left(e_{n}\right), \underline{f}_{k}^{p, r}:=\min _{e \in \mathcal{E}_{p}^{r}: f_{k}^{r}(e)>0} f_{k}^{r}(e)$ and $\bar{f}_{k}^{p, r}:=\max _{e \in \mathcal{E}_{p}^{r}} f_{k}^{r}(e)=f_{k}^{r}\left(e_{\bar{i}(p, r)}\right)$.

Proof: Fix $r \in[d]$. For $t \in\left[T_{r}\right]$, let $j^{t, 1} \in \mathcal{V}$ be the smallest index such that $b^{r, t}\left(\left(j^{t, 1}, j^{t, 1}+1\right)\right)>0$, and for $\ell^{\prime}=2,3, \ldots$, let $j^{t, \ell^{\prime}} \in \mathcal{V}$, be the smallest index such that

$$
\begin{equation*}
b^{r, t}\left(\left(j^{t, \ell^{\prime}}, j^{t, \ell^{\prime}}+1\right)\right)>C_{r} \cdot b^{r, t}\left(\left(j^{t, \ell^{\prime}-1}, j^{t, \ell^{\prime}-1}+1\right)\right) . \tag{5}
\end{equation*}
$$

Let $\bar{\ell}$ be the largest index for which (5) is possible (if no such index exists, then the lemma follows with $P_{r}=1$ ), and set $\ell_{t}:=\bar{\ell}+1$ and $j^{t, \ell_{t}}:=m$. The inequality in (5) implies that $b^{r, t}\left(e_{n}\right)>C_{r}^{\ell_{t}-1} \cdot b^{r, t}\left(\left(j^{t, 1}, j^{t, 1}+1\right)\right)$ which implies in turn that

$$
\begin{equation*}
\ell_{t} \leq \log _{C_{r}} \frac{b^{r, t}\left(e_{n}\right)}{b^{r, t}\left(\left(j^{t, 1}, j^{t, 1}+1\right)\right)} \leq \log _{C_{r}}\left(\frac{\bar{b}^{r}}{\underline{b}^{r}}\right) \tag{6}
\end{equation*}
$$

Moreover, (5) implies

$$
\begin{equation*}
\frac{b^{r, t}((j-1, j))}{b^{r, t}\left(\left(j^{\prime}-1, j^{\prime}\right)\right)} \leq C_{r}, \quad \forall j, j^{\prime} \in\left\{j^{t, \ell^{\prime}}+1, \ldots, j^{t, \ell^{\prime}+1}\right\}, \forall \ell^{\prime}=2, \ldots, \ell_{t}-1 \tag{7}
\end{equation*}
$$

The set $\bigcup_{t \in\left[T_{r}\right]}\left\{j^{t, \ell^{\prime}}: \ell^{\prime} \in\left[\ell_{t}\right]\right\} \subseteq \mathcal{V}$ defines a partition of $\mathcal{E}$ into $P_{r} \leq$ $\sum_{t=1}^{T_{r}}\left(\ell_{t}-1\right)$ intervals $\mathcal{E}_{1}^{r}, \ldots, \mathcal{E}_{P_{r}}^{r}$. By (6),

$$
\begin{equation*}
P_{r}<T_{r} \log _{C_{r}}\left(\frac{\bar{b}^{r}}{\underline{b}^{r}}\right) . \tag{8}
\end{equation*}
$$

Consider any interval $\mathcal{E}_{p}^{r}:=\left\{e_{\underline{i}(1, p)}, e_{\underline{i}(1, p)+1}, \ldots, e_{\bar{i}(1, p)}\right\}$ in the partition. Then by (3) and (7), for any $e, e^{\prime} \in \mathcal{E}_{p}^{r}$, we have

$$
\tilde{b}^{r, t}(e) \leq C_{r} \cdot \tilde{b}^{r, t}\left(e^{\prime}\right), \quad \text { whenever } e^{\prime} \geq e_{k}^{r}
$$

and thus, it follows form (3) that, whenever $f_{k}^{r}\left(e^{\prime}\right)>0$ (and hence $e^{\prime} \geq e_{k}^{r}$ ), we have

$$
f_{k}^{r}(e)=\sum_{t=1}^{T_{r}} a_{k}^{r, t} \tilde{b}_{k}^{r, t}(e) \leq \sum_{t=1}^{T_{r}} a_{k}^{r, t} C_{r} \tilde{b}_{k}^{r, t}\left(e^{\prime}\right) \leq C_{r} f_{k}^{r}\left(e^{\prime}\right),
$$

as required by (4).

## 3 A QPTAS for separable $d$-USFP

This section presents an LP-based approach that arrives at a QPTAS for separable $d$-USFP with the main result stated in Theorem 1. The high-level idea behind the provided scheme is to segment the users' demand functions in each partition of edges guaranteed by Lemma 1 into "large" and "small", then effectively combine their solutions by exploiting monotonicity and separability of these functions. As the number of "large" demands in the optimal solution turns to be provably bounded, we guess the corresponding decision variables through exhaustive search. On the other hand, the situation with "small" demands is more complicated since their presence in the optimal solution can be significant. However, as shown in Lemma 2, for such demands, a given fractional solution $\tilde{x}$ for separable $d$-USFP can be rounded to an integral one that fits within $\tilde{x}$ 's resource requirements without a notable sacrifice in the objective value.

For exposition clarity, the analysis is arranged into two subsections, which are then further dissected into more concise paragraphs. We proceed by exploring the properties of near-optimal solutions.

### 3.1 Structure of Near-optimal Solutions

Discretizing the instance: Let $u_{\text {max }}:=\max _{k \in \mathcal{I}} u_{k}$ and $\epsilon \in(0,1)$ be a given constant. Define $\hat{\mathcal{I}}:=\left\{k \in \mathcal{I}: u_{k} \geq \frac{\epsilon u_{\max }}{n}\right\}$. Note that $u_{\max } \leq$ Opt for a feasible instance, where OPT is the value of an optimal solution for $d$-USFP $[\mathcal{I}, c]$. It follows that $\sum_{k \in \mathcal{I} \backslash \hat{\mathcal{I}}} u_{k} \leq \epsilon u_{\max } \leq \epsilon$ OРт and hence, $\sum_{k \in \hat{\mathcal{I}}} u_{k} \geq(1-\epsilon)$ OРт.

For $k \in \hat{\mathcal{I}}$ and $r \in[d]$, let $\underline{f}_{k}^{r}:=\min _{e: f_{k}^{r}(e)>0} f_{k}^{r}(e), \bar{f}_{k}^{r}:=\max _{e} f_{k}^{r}(e)=$ $f_{k}^{r}\left(e_{n}\right), \underline{f}^{r}:=\min _{k} \underline{f}_{k}^{r}$ and $\bar{f}^{r}:=\max _{k} \bar{f}_{k}^{r}$. We consider discrete levels of function values: for $l=-\infty, 0,1,2, \ldots,\left\lceil\log _{(1+\epsilon)} \frac{n \bar{f}^{r}}{\underline{f}^{r}}\right\rceil$ let $F_{l}^{r}:=(1+\epsilon)^{l} \underline{f}^{r}$, and $F^{r}:=$ $\left\{F_{l}^{r}: l=-\infty, 0,1,2, \ldots,\left\lceil\log _{(1+\epsilon)} \frac{n \bar{f}^{r}}{\underline{f}^{r}}\right\rceil\right\}$ with $\bar{F}:=\max \left\{\left|F^{r}\right|: r \in[d]\right\}$.

Partitioning the instance: For each $r \in[d]$, we assume the partition of $\mathcal{E}$ guaranteed by Lemma 1, and let $\underline{a}^{r}:=\min _{k \in \hat{\mathcal{I}}, t \in\left[T_{r}\right]: a_{k}^{r, t}>0} a_{k}^{r, t}$ and $\bar{a}^{r}:=\max _{k \in \hat{\mathcal{I}}, t \in\left[T_{r}\right]}$ $a_{k}^{r, t}$. Note that if $a_{k}^{r, t}>0$ and $k \in \hat{\mathcal{I}}$, then $\frac{\epsilon u_{\text {max }}}{n \bar{a}^{r}} \leq \frac{u_{k}}{a_{k}^{r, t}} \leq \frac{u_{\text {max }}}{\underline{a}^{r}}$. We partition the users in $\hat{\mathcal{I}}$ into $Q:=\prod_{r=1}^{d} \prod_{t=1}^{T_{r}} Q_{r, t}$ groups, where $Q_{r, t}:=\left\lceil\log \frac{n \bar{a}^{r}}{\epsilon \underline{a}^{r}}\right\rceil+1$ :

$$
\begin{equation*}
\mathcal{I}^{q}=\left\{k \in \hat{\mathcal{I}}: 2^{q_{r, t}-1} L \leq \frac{u_{k}}{a_{k}^{r, t}}<2^{q_{r, t}} L \text { for all } t \in\left[T_{r}\right], r \in[d]\right\}, \tag{9}
\end{equation*}
$$

for $q=\left(q_{r, t}: \quad t \in\left[T_{r}\right], r \in[d]\right) \in \mathcal{Q}:=\prod_{r=1}^{d} \prod_{t=1}^{T_{r}}\left\{1, \ldots, Q_{t, r}-1, \infty\right\}$, where ${ }^{2}$ $L:=\frac{\epsilon u_{\text {max }}}{n \bar{a}^{r}}$. Let $\bar{Q}:=\max _{t, r} Q_{r, t}$. Then $Q \leq \bar{Q}^{\sum_{r=1}^{d} T_{r}}$.

Structure of the optimal solution: Consider an optimal solution $x^{*}$ to separable $d$-USFP $[\mathcal{I}, c]$. For $q \in \mathcal{Q}$, let $\mathcal{T}^{*}=\left\{k \in \hat{\mathcal{I}}: x_{k}^{*}=1\right\}$. Then $\left(f^{*}\right)^{q, r}(e):=$ $\sum_{k \in \mathcal{T} * \cap \mathcal{I}^{q}} f_{k}^{r}(e)$, for $r \in[d]$, defines a monotone non-decreasing function on $\mathcal{E}$. We call such a function a "profile" defined by the optimal solution in group $\mathcal{I}^{q}$. For $p \in\left[P_{r}\right]$, let $\left(h^{*}\right)^{q, p, r}=\max _{e \in \mathcal{E}_{p}^{r}}\left(f^{*}\right)^{q, r}(e)$ be the peak demand defined by the optimal solution (from group $q$ ) within the interval $\mathcal{E}_{p}^{r}$.

For $q \in \mathcal{Q}$, let $\left(\mathcal{L}^{*}\right)^{q}:=\left\{k \in \mathcal{I}^{q} \cap \mathcal{T}^{*}: \underline{f}_{k}^{p, r}>\epsilon^{2}\left(h^{*}\right)^{q, p, r}\right.$ for some $p \in\left[P_{r}\right], r \in$ [d]\} be the set of "large" demands within group $\mathcal{I}^{q}$ in the optimal solution, and let $\mathcal{S}^{q}:=\mathcal{I}^{q} \cap \mathcal{T}^{*} \backslash\left(\mathcal{L}^{*}\right)^{q}$ be the set of "small" demands within the same group. Note that, by definition of $\left(h^{*}\right)^{q, p, r}$ and the monotonicity of $f_{k}^{r}(\cdot)$, there cannot be more than $\frac{1}{\epsilon^{2}}$ demands $k$ in $\mathcal{I}^{q} \cap \mathcal{T}^{*}$ such that $\underline{f}_{k}^{p, r}>\epsilon^{2}\left(h^{*}\right)^{q, p, r}$, and hence $\left|\left(\mathcal{L}^{*}\right)^{q}\right| \leq \frac{\sum_{r=1}^{d} P_{r}}{\epsilon^{2}}$. The situation with small demands is more complicated as their number in the optimal solution can be high. However, with a small loss in the objective value, the profile defined by such small demands can be restricted into one that admits a small description. This motivates the following definition (generalizing that of in (Bansal et al., 2006)).
Definition 1 (( $h, \epsilon$ )-restricted profile) Let $\epsilon>0$ be such that $1 / \epsilon \in \mathbb{Z}_{+}$. For $r \in[d]$ and $p \in\left[P_{r}\right]$, let $h=\left(h^{p, r}\right)_{p \in\left[P_{r}\right], r \in[d]}$ be a given vector of numbers such that $h^{p, r} \in F^{r}$ and $h^{p, r} \geq h^{p-1, r}$, for all $p=2, \ldots, P_{r}$ and $r \in[d]$. An $(h, \epsilon)$ restricted profile $g=\left(g^{r}\right)_{r \in[d]}$ is vector of monotone functions $g^{r}: \mathcal{E} \rightarrow \mathbb{R}_{+}$such that $g^{r}(e) \in\left\{l \epsilon h^{p, r}: l \in\{0,1 \ldots, 1 / \epsilon\}, p \in\left[P_{r}\right]\right\}$ (see Figure 3 in Section A for pictorial interpretation of an ( $h, \epsilon)$-restricted profile).
Accordingly, the total number of $(h, \epsilon)$-restricted profiles is at most $m^{\sum_{r=1}^{d} P_{r} / \epsilon}$. For $q \in \mathcal{Q}$ and for $p \in\left[P_{r}\right]$, define

$$
\begin{equation*}
H^{q, p, r}:=\sum_{t=1}^{T_{r}} \frac{b^{r, t}\left(e_{\bar{i}(p, r)}\right)}{2^{q_{r, t}} L} \tag{10}
\end{equation*}
$$

Note that

$$
\begin{align*}
\forall p \in\left[P_{r}\right]: H^{q, p, r}>0 & \Leftrightarrow \exists t \in\left[T_{r}\right]: q_{r, t} \neq \infty \\
& \Leftrightarrow \forall k \in \mathcal{I}^{q} \exists t \in\left[T_{r}\right]: a_{k}^{r, t}>0 \\
& \Leftrightarrow \forall k \in \mathcal{I}^{q}: f_{k}^{r}\left(e_{n}\right)>0 . \tag{11}
\end{align*}
$$

[^3]Let $\mathcal{H}^{q}:=\left\{r \in[d]: H^{q, P_{r}, r}>0\right\}$, and $\alpha:=\frac{\sum_{r=1}^{d} P_{r}}{\sum_{r \in \mathcal{H}^{q}} P_{r}}$. Assume $\mathcal{H}^{q} \neq \emptyset$ since otherwise, $f_{k}^{r}\left(e_{n}\right)=0$ for all $k \in \mathcal{I}^{q}$ and hence all the users in $\mathcal{I}^{q}$ can be taken in the solution without affecting the constraints.

In proving Theorem 1, we shall resort to the below Lemma, which builds on top of the findings in (Bansal et al., 2006) and is proved in Section A.

Lemma 2 Fix $q \in \mathcal{Q}$ and $\epsilon \in(0,1)$. Let $\mathcal{S}^{q} \subseteq \mathcal{I}^{q}$ be a set of demands within group $q$ such that $f_{k}^{p, r} \leq B^{p, q, r}$ for all $k \in \mathcal{S}^{q}, p \in\left[P_{r}\right], r \in[d]$, and some numbers $B^{p, q, r} \in \mathbb{R}_{+}^{-k}$. Let $h^{q}=\left(h^{q, p, r}\right)_{p \in\left[P_{r}\right], r \in[d]}$ be a given vector of numbers such that $h^{q, p, r} \in F^{r}$ and $h^{q, p, r} \geq h^{q, p-1, r}$, for all $p=2, \ldots, P_{r}$ and $r \in[d]$, and $\left(\tilde{x}_{k}\right)_{k \in \mathcal{S}^{q}} \in[0,1]^{\mathcal{S}^{q}}$ be such that

$$
\begin{equation*}
\sum_{k \in \mathcal{S}^{q}} \bar{f}_{k}^{p, r} \tilde{x}_{k} \leq(1+\epsilon) h^{q, p, r}, \quad \forall p \in\left[P_{r}\right], \forall r \in[d] . \tag{12}
\end{equation*}
$$

Then we can find in polynomial time an integral vector $\left(\hat{x}_{k}\right)_{k \in \mathcal{S}^{q}} \in\{0,1\}^{\mathcal{S}^{q}}$ and an ( $h, \epsilon$ )-restricted profile $g^{q}$, such that
(i) $\sum_{k \in \mathcal{S}^{q}} f_{k}^{r}(e) \hat{x}_{k} \leq g^{q, r}(e) \leq \sum_{k \in \mathcal{S}^{q}} f_{k}^{r}(e) \tilde{x}_{k}$ for all $e \in \mathcal{E}, r \in[d]$, and
(ii) $\sum_{k \in \mathcal{S}^{q}} u_{k} \hat{x}_{k} \geq \sum_{k \in \mathcal{S}^{q}} u_{k} \tilde{x}_{k}-\sum_{r \in \mathcal{H}^{q}}\left(\sum_{p=1}^{P_{r}}\left(\frac{C_{r}}{H^{q, p, r}}\left(\epsilon h^{q, p, r}+B^{q, p, r}\right)\right)+\frac{\alpha P_{r} B^{q, P_{r}, r}}{\epsilon H^{q, P_{r}, r}}\right)$.

In other terms, Lemma 2 establishes that, when all demands are small, a given fractional solution $\tilde{x}$ for separable $d$-USFP can be rounded to an integral solution $\hat{x}$ that fits within a capacity profile with a small description, losing only a small part of the utility of $\tilde{x}$.

### 3.2 Approximation Scheme

The featured QPTAS, formally stated in Alg. 1, proceeds as follows. As Opt $\geq$ $u_{\text {max }}$, by restricting the set of demands to $\hat{\mathcal{I}}$ (defined in Section 3.1) we lose only a value of at most $\epsilon$ Opt from the optimal solution. Next, the algorithm discretizes the instance and partitions the users in $\hat{\mathcal{I}}$ into $Q$ groups $\left(\mathcal{I}^{q}\right)_{q \in \mathcal{Q}}$, as described in Section 3.1. Additionally, Alg. 1 partitions $\mathcal{E}$ into intervals $\mathcal{E}^{r}$ satisfying assumption (4), as per Lemma 1 (with $C_{r}=2$ ).

Then for each group $q \in \mathcal{Q}$, Alg. 1 guesses the set of large demands $\mathcal{L}^{q} \subseteq \mathcal{I}^{q}$ in the optimal solution, and the peaks $h^{q, p, r}$, within $1+\epsilon$, of the small demands in the optimal solution within the interval $\mathcal{E}_{p}^{r}$. Let $\mathcal{L}=\left(\mathcal{L}^{q}\right)_{q \in \mathcal{Q}}$ and $h^{q}=\left(h^{q, p, r}\right)_{p \in\left[P_{r}\right], r \in[d]}$ where $h^{q, p, r} \in F^{r}$. Define the set of small demands within group $q \in \mathcal{Q}$ as

$$
\begin{equation*}
\mathcal{S}^{q}:=\left\{k \in \mathcal{I}^{q}: \underline{f}_{k}^{p, r} \leq B^{q, p, r} \text { for all } p \in\left[P_{r}\right], r \in[d]\right\}, \tag{13}
\end{equation*}
$$

where $B^{q, p, r}:=\epsilon^{2}\left[h^{q, p, r}+\sum_{k \in \mathcal{L}^{q}} \bar{f}_{k}^{p, r}\right]$.
Let $T:=\max _{r} T_{r}$ and $M:=\max \left\{\max _{r} \frac{\bar{a}^{r}}{\underline{a}^{r}}, \max _{r} \frac{\bar{b}^{r}}{\underline{b}^{r}}\right\}$.
Theorem 1 For any fixed $\varepsilon \in(0,1)$, Alg. 1 attains a ( $1-\varepsilon$ )-approximation for separable d-USFP in time $\left(\frac{n m \log (d n T M)}{\varepsilon}\right)^{d T \cdot O\left(\log \frac{d n T M}{\varepsilon}\right)^{d T}}$.

```
Algorithm \(1 d\)-USFP-QPTAS
Require: An approximation parameter \(\epsilon \in(0,1)\); separable \(d\)-USFP input \(\left(f_{k}^{r}\right)_{k \in \mathcal{I}}, r \in[d]\)
    satsifying (3); capacities \(\left(c^{r}\right)_{r \in[d]}\)
Ensure: An integral solution \(\hat{x}\) to \(d\)-USFP such that \(\sum_{k \in \mathcal{I}} u_{k} \hat{x}_{k} \geq(1-O(\epsilon))\) OPT
    for each selection \(\left(\mathcal{L}=\left(\mathcal{L}^{q}\right)_{q \in \mathcal{Q}}, h=\left(h^{q}=\left(h^{q, p, r}\right)_{p \in\left[P_{r}\right], r \in[d]}\right)_{q \in \mathcal{Q}}\right)\) such that \(\mathcal{L}^{q} \subseteq \mathcal{I}\),
    \(\left|\mathcal{L}^{q}\right| \leq \frac{\sum_{r=1}^{d} P_{r}}{\epsilon^{2}}\) and \(h^{q, p, r} \in F^{r}\) do
        if \(\sum_{k \in \mathcal{L}} f_{k}^{r}(e)+\sum_{p \in\left[P_{r}\right], q \in \mathcal{Q}} h^{q, p, r} \leq c^{r}(e) \forall e \in \mathcal{E}, r \in[d]\) then
            \(\hat{x}_{k}^{\prime} \leftarrow 1 \forall k \in \mathcal{L}\)
            for \(q \in \mathcal{Q}\) do
                Let \(\mathcal{S}^{q}\) be given by (13)
                for every \((h, \epsilon)\)-restricted profile \(g^{q}\) do
                    \(\left(\hat{x}_{k}^{\prime}\right)_{k \in \mathcal{S}^{q}} \leftarrow\) Integral vector returned by applying Lemma 2 with vector \(h^{q}\),
    and \(\left(\tilde{x}_{k}\right)_{k \in \mathcal{S}^{q}}=\left(x_{k}^{\prime}\right)_{k \in \mathcal{S}^{q}}\)
            if \(\sum_{k \in \mathcal{I}} u_{k} \hat{x}_{k}^{\prime}>\sum_{k \in \mathcal{I}} u_{k} \hat{x}_{k}\) then
                \(\hat{x} \leftarrow \hat{x}^{\prime}\)
    return \(\hat{x}\)
```

Proof: Let $\epsilon:=\frac{\varepsilon}{2 \beta+1}$, where $\beta=\max _{r \in \mathcal{H}^{q}} 2\left(2 C_{r}+\alpha P_{r}\right)=O\left(d^{3}(T \log M)^{2}\right)$. The number of possible choices for each $\mathcal{L}^{q}$ in step 1 of Alg. 1 is at most $n^{\sum_{r=1}^{d} P_{r} / \epsilon^{2}}$. Thus, using $Q \leq \bar{Q}^{\sum_{r=1}^{d} T_{r}}$, and $\bar{Q}=O\left(\log \frac{n M}{\epsilon}\right)$, the number of possible choices for $\mathcal{L}$ is at most

$$
\begin{equation*}
n^{\sum_{r=1}^{d} P_{r} Q / \epsilon^{2}} \leq n^{\sum_{r=1}^{d} P_{r} \bar{Q}^{\sum_{r=1}^{d} T_{r}} / \epsilon^{2}}=n^{d T \log M \cdot O\left(\log \frac{n M}{\epsilon}\right)^{d T / \epsilon^{2}}} . \tag{14}
\end{equation*}
$$

The number of choices for each $h^{q}=\left(h^{q, p, r}\right)_{p \in\left[P_{r}\right], r \in[d]}$ is

$$
\bar{F}^{\sum_{r=1}^{d} P_{r}}=O\left(\left(\frac{\log (n T M)}{\epsilon}\right)^{d T \log M}\right)
$$

and the number of choices for $Q$ in step 4 is

$$
\begin{equation*}
\bar{Q}^{\sum_{r=1}^{d} T_{r}} \leq O\left(\log \frac{n M}{\epsilon}\right)^{d T} \tag{15}
\end{equation*}
$$

giving at most

$$
\begin{equation*}
\left(O\left(\frac{\log (n T M)}{\epsilon}\right)^{d T \log M}\right)^{Q}=\left(O\left(\frac{\log (n T M)}{\epsilon}\right)^{d T \log M}\right)^{O\left(\log \frac{n M}{\epsilon}\right)^{d T}} \tag{16}
\end{equation*}
$$

choices for $h=\left(h^{q}\right)_{q \in \mathcal{Q}}$ in step 1. The number of choices for the $\epsilon$-restricted profiles in step 6 is bounded from above by $m^{\sum_{r=1}^{d} P_{r} / \epsilon}=m^{O\left(\frac{d T \log M}{x \epsilon}\right)}$. The bound on the running time of Alg. 1 follows from this and (14),(15),(16).

We now argue that the solution $\hat{x}$ outputted by Alg. 1 is $(1-O(\epsilon))$-approximation for separable $d$-USFP. Let $x^{*}$ be an optimal solution for $d$-USFP of objective value OPT $\triangleq \sum_{k \in \mathcal{I}} u_{k} x_{k}^{*}$. By the definition of $\hat{\mathcal{I}}$, we have

$$
\begin{equation*}
\sum_{k \in \mathcal{I} \backslash \hat{\mathcal{I}}} u_{k} \leq \epsilon \text { ОРт } \tag{17}
\end{equation*}
$$

Define $\mathcal{T}^{*} \triangleq\left\{k \in \hat{\mathcal{I}} \mid x_{k}^{*}=1\right\}$ and $\left(h^{*}\right)^{q, p, r}=\sum_{k \in \mathcal{T}^{*} \cap \mathcal{I}^{q}} \bar{f}_{k}^{p, r}$, for $p \in\left[P_{r}\right]$, $r \in[d]$ and $q \in \mathcal{Q}$. Let $\left(\mathcal{L}^{*}\right)^{q}:=\left\{k \in \mathcal{I}^{q} \cap \mathcal{T}^{*}: \underline{f}_{k}^{p, r}>\epsilon^{2}\left(h^{*}\right)^{q, p, r}\right.$ for some $p \in$ $\left[P_{r}\right]$, and some $\left.r \in[d]\right\}$ be the set of "large" demands within group $\mathcal{I}^{q}$ in the optimal solution, and let $\left(\mathcal{S}^{*}\right)^{q}:=\mathcal{I}^{q} \cap \mathcal{T}^{*} \backslash\left(\mathcal{L}^{*}\right)^{q}$ be the set of "small" demands within the same group. Note by this definition that $\left|\left(\mathcal{L}^{*}\right)^{q}\right| \leq \frac{\sum_{r=1}^{d} P_{r}}{\epsilon^{2}}$, and thus $\mathcal{L}^{*}=\left(\left(\mathcal{L}^{*}\right)^{q}\right)_{q \in \mathcal{Q}}$ and $h=\left(h^{q}\right)_{q \in \mathcal{Q}}$ will be one of the guesses considered by the algorithm in step 1. Let us focus on this particular iteration of the loop in step 1. Let $h^{q, p, r}=(1+\epsilon)^{\underline{\ell}} f^{r}$, where $\underline{\ell}$ is the smallest integer (including $-\infty$ ) such that $h^{q, p, r}+\sum_{k \in\left(\mathcal{L}^{*}\right)^{q}} \bar{f}_{k}^{p, r} \geq\left(h^{*}\right)^{q, p, r}$. Note that $h^{q, p, r} \in F^{r}$, and

$$
\begin{equation*}
\frac{1}{1+\epsilon} h^{q, p, r}+\sum_{k \in\left(\mathcal{L}^{*}\right)^{q}} \bar{f}_{k}^{p, r} \leq\left(h^{*}\right)^{q, p, r} \leq h^{q, p, r}+\sum_{k \in\left(\mathcal{L}^{*}\right)^{q}} \bar{f}_{k}^{p, r} \tag{18}
\end{equation*}
$$

Note that for any $k \in\left(\mathcal{S}^{*}\right)^{q}, q \in \mathcal{Q}, p \in\left[P_{r}\right]$, and $r \in[d]$, we have by (18),

$$
\underline{f}_{k}^{p, r} \leq \epsilon^{2}\left(h^{*}\right)^{q, p, r} \leq \epsilon^{2}\left(h^{q, p, r}+\sum_{k \in\left(\mathcal{L}^{*}\right)^{q}} \bar{f}_{k}^{p, r}\right)
$$

and hence $\left(\mathcal{S}^{*}\right)^{q} \subseteq \mathcal{S}^{q}$. Note also that

$$
\begin{align*}
B^{q, p, r} & =\epsilon^{2}\left[h^{q, p, r}+\sum_{k \in\left(\mathcal{L}^{*}\right)^{q}} \bar{f}_{k}^{p, r}\right] \\
& \leq \epsilon^{2}\left[h^{q, p, r}+(1+\epsilon) \sum_{k \in\left(\mathcal{L}^{*}\right)^{q}} \bar{f}_{k}^{p, r}\right] \leq \epsilon^{2}(1+\epsilon)\left(h^{*}\right)^{q, p, r} \tag{19}
\end{align*}
$$

For each $q \in \mathcal{Q}$, there is an $(h, \epsilon)$-restricted profile $g^{q}$ and an integral solution $\left(\hat{x}_{k}^{\prime}\right)_{k \in \mathcal{S}^{q}}$ that satisfy Lemma 2 (applied with $\hat{x} \leftarrow \hat{x}^{\prime}$ and $\tilde{x} \leftarrow x^{*}$ ). Since all the possible ( $h, \epsilon$ )-restricted profiles are probed, the profile $g^{q}$ will be identified in one of the iterations of the loop in step 6 of Alg. 1. Let us consider this iteration. By condition (ii) of Lemma 2 and (19),

$$
\begin{align*}
\sum_{k \in \mathcal{S}^{q}} u_{k} \hat{x}_{k}^{\prime} & \geq \sum_{k \in \mathcal{S}^{q}} u_{k} x_{k}^{*}-\sum_{r \in \mathcal{H}^{q}}\left(\sum_{p=1}^{P_{r}}\left(\frac{C_{r}\left(\epsilon h^{q, p, r}+B^{q, p, r}\right)}{H^{q, p, r}}\right)+\frac{\alpha P_{r} B^{q, P_{r}, r}}{\epsilon H^{q, P_{r}, r}}\right) \\
& =\sum_{k \in \mathcal{S}^{q}} u_{k} x_{k}^{*}-\sum_{r \in \mathcal{H}^{q}}\left(\sum_{p=1}^{P_{r}}\left(\frac{C_{r} \epsilon(1+\epsilon)^{2}\left(h^{*}\right)^{q, p, r}}{H^{q, p, r}}\right)+\frac{\alpha P_{r} \epsilon^{2}(1+\epsilon)\left(h^{*}\right)^{q, P_{r}, r}}{\epsilon H^{q, P_{r}, r}}\right) \\
& =\sum_{k \in \mathcal{S}^{q}} u_{k} x_{k}^{*}-\epsilon(1+\epsilon) \sum_{r \in \mathcal{H}^{q}}\left(\sum_{p=1}^{P_{r}}\left(\frac{C_{r}(1+\epsilon)\left(h^{*}\right)^{q, p, r}}{H^{q, p, r}}\right)+\frac{\alpha P_{r}\left(h^{*}\right)^{q, P_{r}, r}}{H^{q, P_{r}, r}}\right) . \tag{20}
\end{align*}
$$

On the other hand, for $k \in \mathcal{S}^{q}$ and $r \in[d]$ such that $f_{k}^{r}\left(e_{n}\right)>0$ (and hence $H^{q, p, r}>0$ for all $p \in\left[P_{r}\right]$ by (11)), we have $u_{k} \geq 2^{q_{r, t}-1} L a_{k}^{r, t}$ and thus

$$
\begin{equation*}
u_{k} \frac{b^{r, t}\left(e_{\bar{i}(p, r)}\right)}{2^{q_{r, t}-1} L} \geq a_{k}^{r, t} b^{r, t}\left(e_{\bar{i}(p, r)}\right) \geq a_{k}^{r, t} \tilde{b}^{r, t}\left(e_{\bar{i}(p, r)}\right) \tag{21}
\end{equation*}
$$

Summing up (21) over $t \in\left[T_{r}\right]$, we get $u_{k} \geq \frac{\bar{f}_{h}^{p, r}}{2 H^{q, p, r}}$. Recall that $\left(h^{*}\right)^{q, p, r}=$ $\sum_{k \in \mathcal{T}^{*} \cap \mathcal{I}^{q}} \bar{f}_{k}^{p, r}$, then summing this inequality over $k \in \mathcal{T}^{*} \cap \mathcal{I}^{q}$ yields

$$
\begin{equation*}
\mathrm{OPT}^{q}:=\sum_{k \in \mathcal{T}^{*} \cap \mathcal{I}^{q}} u_{k} \geq \sum_{k \in \mathcal{T}^{*} \cap \mathcal{I}^{q}} \frac{\bar{f}_{k}^{p, r}}{2 H^{q, p, r}}=\frac{\left(h^{*}\right)^{q, p, r}}{2 H^{q, p, r}} \tag{22}
\end{equation*}
$$

Summing (22), over $r \in \mathcal{H}^{q}$ and $p \in\left[P_{r}\right]$ gives

$$
\begin{align*}
\mathrm{OPT}^{q} & \geq \sum_{r \in \mathcal{H}^{q}} \sum_{p=1}^{P_{r}} \frac{\left(h^{*}\right)^{q, p, r}}{2 H^{q, p, r}} \\
& =\frac{1}{\beta} \cdot \sum_{r \in \mathcal{H}^{q}} \sum_{p=1}^{P_{r}} \frac{\beta\left(h^{*}\right)^{q, p, r}}{2 H^{q, p, r}} \\
& \geq \frac{1}{\beta} \cdot \sum_{r \in \mathcal{H}^{q}} \sum_{p=1}^{P_{r}} \frac{2\left(2 C_{r}+\alpha P_{r}\right)\left(h^{*}\right)^{q, p, r}}{2 H^{q, p, r}} \\
& \geq \frac{1}{\beta} \cdot \sum_{r \in \mathcal{H}^{q}} \sum_{p=1}^{P_{r}}\left(\frac{2 C_{r}\left(h^{*}\right)^{q, p, r}}{H^{q, p, r}}+\frac{\alpha P_{r}\left(h^{*}\right)^{q, p, r}}{H^{q, p, r}}\right) \\
& \geq \frac{1}{\beta} \cdot \sum_{r \in \mathcal{H}^{q}}\left(\sum_{p=1}^{P_{r}} \frac{(1+\epsilon) C_{r}\left(h^{*}\right)^{q, p, r}}{H^{q, p, r}}+\sum_{p=1}^{P_{r}} \frac{\alpha P_{r}\left(h^{*}\right)^{q, p, r}}{H^{q, p, r}}\right) \\
& \geq \frac{1}{\beta} \cdot \sum_{r \in \mathcal{H}^{q}}\left(\sum_{p=1}^{P_{r}}\left(\frac{C_{r}}{H^{q, p, r}}(1+\epsilon)\left(h^{*}\right)^{q, p, r}\right)+\frac{\alpha P_{r}\left(h^{*}\right)^{q, P_{r}, r}}{H^{q, P_{r}, r}}\right) \tag{23}
\end{align*}
$$

where $\beta=\max _{r \in \mathcal{H}^{q}} 2\left(2 C_{r}+\alpha P_{r}\right)$ as defined previously. Thus, it follows from (20) and (23) that

$$
\begin{equation*}
\sum_{k \in \mathcal{S}^{q}} u_{k} \hat{x}_{k}^{\prime} \geq \sum_{k \in \mathcal{S}^{q}} u_{k} x_{k}^{*}-\epsilon(1+\epsilon) \beta \mathrm{OPT}^{q} \geq \sum_{k \in\left(\mathcal{S}^{*}\right)^{q}} u_{k} x_{k}^{*}-\epsilon(1+\epsilon) \beta \mathrm{OPT}^{q} . \tag{24}
\end{equation*}
$$

Summing (24) over all $q \in \mathcal{Q}$ and using (17) and (24) gives

$$
\begin{aligned}
\sum_{k \in \mathcal{I}} u_{k} \hat{x}_{k}^{\prime} & =\sum_{q \in \mathcal{Q}}\left(\sum_{k \in\left(\mathcal{L}^{*}\right)^{q}} u_{k} \hat{x}_{k}^{\prime}+\sum_{k \in \mathcal{S}^{q}} u_{k} \hat{x}_{k}^{\prime}\right) \\
& \geq \sum_{q \in \mathcal{Q}}\left(\sum_{k \in\left(\mathcal{L}^{*}\right)^{q}} u_{k} x_{k}^{*}+\sum_{k \in\left(\mathcal{S}^{*}\right)^{q}} u_{k} x_{k}^{*}-\epsilon(1+\epsilon) \beta \mathrm{OPT}^{q}\right) \\
& =\sum_{k \in \mathcal{T}^{*}} u_{k} x_{k}^{*}-\epsilon(1+\epsilon) \beta \sum_{k \in \mathcal{T}^{*}} u_{k} \\
& =\sum_{k \in \hat{\mathcal{I}}} u_{k} x_{k}^{*}-\epsilon(1+\epsilon) \beta \sum_{k \in \mathcal{T}^{*}} u_{k} \\
& \geq \sum_{k \in \mathcal{I}} u_{k} x_{k}^{*}-\epsilon(2 \beta+1) \mathrm{OPT}=(1-\varepsilon) \mathrm{OPT} .
\end{aligned}
$$

It follows that the solution $\hat{x}$ returned by Alg. 1 satisfies

$$
\sum_{k \in \mathcal{I}} u_{k} \hat{x}_{k} \geq \sum_{k \in \mathcal{I}} u_{k} \hat{x}_{k}^{\prime} \geq(1-\varepsilon) \mathrm{OPT}
$$

thus concluding the proof.
Note that the running time is quasi-polynomial if $M=2^{\text {polylog }(m, n)}$ and $d=$ $O(1), T=O(1)$.

## 4 A Logarithmic approximation for single-source separable $d$-USFP

Notwithstanding its theoretical appeal, the QPTAS devised in Sec. 3 is computationally prohibitive even for modest problem sizes, hence is of limited practicality. This section presents an efficient logarithmic approximation for single-source separable $d-\operatorname{USFP}[\mathcal{I}, c]$ with a running time complexity dominated by solving an LP. Before stating the result formally, we rewrite the problem in a suitable matrix notation and briefly outline the underlying technique. Notice that $d$-USFP[ $\mathcal{I}, c]$ can be cast as a general packing integer program (PIP) of the form

$$
\begin{align*}
\left(\mathcal{P}\left[\left(A^{r}\right)_{r \in[d]}, u,\left(c^{r}\right)_{r \in[d]}\right]\right) & \max _{x} u^{T} x \\
\text { s.t. } & A^{r} x \leq c^{r}, \quad \forall r \in[d]  \tag{25}\\
& x \in\{0,1\}^{n}, \tag{26}
\end{align*}
$$

where $u \in \mathbb{R}_{+}^{n}$ is the utility vector, $c^{r} \in \mathbb{R}_{+}^{m}$ denotes the edge capacities in dimension $r \in[d]$ and $A^{r} \in \mathbb{R}_{+}^{m \times n}$ resembles the edge-demand incidence relation for the corresponding dimension $r \in[d]$, with the rows signifying the edges and the columns the demands (i.e., $A_{i k}^{r}=f_{k}^{r}\left(e_{i}\right)$ for $\forall i \in[m], k \in \mathcal{I}$ ).

Exploiting the special structure of $\mathcal{P}$ induced by the monotonicity and separability of demands, we develop a simple grouping and scaling method allowing to reduce the problem to an easier instance with only logarithmically many constraints. Recall that an analogously named technique was derived in (Kolliopoulos and Stein, 2001) for the single-source unsplittable flow problem. Deviating from the setting in (Kolliopoulos and Stein, 2001) of partitioning the instance in the demand space, the proposed approach, instead, decomposes the edges into disjoint segments, each defining a subproblem of $\mathcal{P}$ where each capacity and demand varies within a preset range. These subproblems, after certain alterations, are then reconsolidated, effectively formulating the compacted problem with $\mathrm{O}(d \log n)$ number of constraints. It's noteworthy that this reduction subroutine holds irrespective of the rather restrictive NBA condition, which is stipulated in (Kolliopoulos and Stein, 2001). Thereafter, invoking the standard randomized rounding algorithm on the natural LP relaxation of the reduced problem ensures the claimed approximation factor. Formally, the preceding analysis culminates in Theorem 3.

In proving Theorem 3, we capitalize on several established results on randomized rounding and its derandomization (codified in the theorem to follow) as a unified black box technique and thereby omit the intricate particulars.

Theorem 2 ((Srinivasan, 1999; Raghavan and Tompson, 1987) ) Let $\mathcal{B}$ be a PIP of the form $\max \left\{u^{T} x: A x \leq c, x \in\{0,1\}^{n}\right\}$, where $A \in[0,1]^{m \times n}$,
$u \in[0,1]^{n}$ and $c \in[1, \infty)^{m}$ with $\max _{j} u_{j}=1$. Then, there exists an algorithm outputting in deterministic polynomial time a feasible solution to $P$ of value

$$
\Omega\left(\max \left\{\frac{\mathrm{OPT}_{L}}{m^{1 / \nu}},\left(\frac{\mathrm{OPT}_{L}}{m^{1 / \nu}}\right)^{\frac{\nu}{\nu-1}}\right\}\right)
$$

where $\mathrm{Opt}_{L}$ is the optimum of the linear relaxation of $\mathcal{B}$ and $\nu=\min _{j} c_{j}$.
Theorem 3 There is an $O(d \log n)$-approximation for single-source separable $d$ $U S F P$, provided the edge capacities and demands are bounded by $2^{\mathrm{polylog}(n)}$.

Proof: Let $\Lambda=\left(\left(A^{r}\right)_{r \in[d]}, u,\left(c^{r}\right)_{r \in[d]}\right)$ be an input instance of $\mathcal{P}$ with Opt denoting the value of its optimal solution $x^{*}$. From $\Lambda$, construct an augmented instance $\Lambda^{\prime}=\left(\left(\left[A^{r} c^{r}\right]\right)_{r \in[d]},(u, 0),\left(c^{r}\right)_{r \in[d]}\right)$, which essentially models the outcome of incorporating a dummy request with a utility of 0 and a demand equal to edge capacities. This auxiliary step, meant to streamline the proof, incurs no loss of generality as neither $x^{*}$ nor its structure is affected in the aftermath. Thus, to elude cumbersome notation, $\left(A^{r}\right)_{r \in[d]}$ and $u$ are hereafter assumed implicitly of the augmented form as in $\Lambda^{\prime}$.

At a loss of only a constant factor in Opt, we shall now transform $\mathcal{P}$ to a problem with $\mathrm{O}(d \log n)$ constraints. Let $\Pi$ denote the LP relaxation of $\mathcal{P}$, obtained by allowing $x$ to lie in $[0,1]^{n}$. Fix a constant $C>1$, along with the corresponding partitions $\left(\mathcal{E}^{r}\right)_{r \in[d]}$ guaranteed by Lemma 1 , and denote by $A^{r, p}$ the submatrix of $A^{r}$ restricted to the rows in $\left\{i \in[m] \mid e_{i} \in \mathcal{E}_{p}^{r}\right\}$. Observe that each interval $\mathcal{E}_{p}^{r}$ in $\left(\mathcal{E}^{r}\right)_{r \in[d]}$ naturally defines a subproblem $\Pi\left[A^{r, p}, u, c^{r, p}\right]$, where $\frac{\max _{i} A_{i j}^{r, p}}{\min _{i: A_{i j}^{r, p}>0} A_{i j}^{r, p}} \leq C$ for $\forall j \in[n+1]$ and, by introduced ancillary demands, $\frac{\max _{i} c_{i}^{r, p}}{\min _{i: c_{i}^{r, p}}^{r, p} c_{i}^{r, p}} \leq C$. Given $\Pi\left[A^{r, p}, u, c^{r, p}\right]$, compose a simplified instance $\Pi\left[\bar{A}^{r, p}, u, \underline{c}^{r, p}\right]$, with $\underline{c}^{r, p}:=\min _{i} c_{i}^{r, p} \cdot \mathbf{1}$ and $\bar{A}^{r, p}$ standing for the matrix whose $i, j$-th entry equals $\max _{i} A_{i j}^{r, p}$ if $A_{i j} \neq 0$ and 0 otherwise. In a sense, this amounts to setting each demand to its maximum, therein flattening out the step functions into lines, and uniforming the edge capacities across the interval. Consider an optimal solution $y^{*}$ of $\Pi\left[A^{r, p}, u, c^{r, p}\right]$ and set $\tilde{y}:=\frac{y^{*}}{C^{2}}$. As a corollary, $\tilde{y}$ becomes a feasible solution for $\Pi\left[\bar{A}^{r, p}, u, \underline{c}^{r, p}\right]$. On the other hand, any feasible solution to $\Pi\left[\bar{A}^{r, p}, u, \underline{c}^{r, p}\right]$ translates into that of $\Pi\left[A^{r, p}, u, c^{r, p}\right]$ of the same value. Taken together and generalized over all the partitions, these observations imply that

$$
\begin{equation*}
{\widetilde{\mathrm{OPT}_{\Pi}}}_{\Pi} \geq \frac{\mathrm{OPT}_{\Pi}}{C^{2}} \geq \frac{\mathrm{OPT}^{2}}{C^{2}} \tag{27}
\end{equation*}
$$

where $\widetilde{\mathrm{OPT}}_{\Pi}$ and $\mathrm{OPT}_{\Pi}$ are the optimal objective values of $\Pi\left[\left(\bar{A}^{r}\right)_{r \in[d]}, u,\left(\underline{c}^{r}\right)_{r \in[d]}\right]$ and $\Pi\left[\left(A^{r}\right)_{r \in[d]}, u,\left(c^{r}\right)_{r \in[d]}\right]$, respectively. Furthermore, a finer inspection of the former problem can render the majority of its constraints redundant. Indeed, by construction, each subproblem $\Pi\left[\bar{A}^{r, p}, u, \underline{c}^{r, p}\right]$ of $\Pi\left[\left(\bar{A}^{r}\right)_{r \in[d]}, u,\left(\underline{c}^{r}\right)_{r \in[d]}\right]$ boils down to a single Knapsack inequality ${ }^{3}$ since both, demands and capacities, are levelled therein, and all the requests share the same origin. Compounding these

[^4]$\tilde{m}=O(d \log n)$ inequalities into $\tilde{A} \in \mathbb{R}_{+}^{\tilde{m} \times n+1}$ and $\tilde{c} \in \mathbb{R}_{+}^{\tilde{m} \times 1}$, formulate a new $\operatorname{PIP} \mathcal{P}[\tilde{A}, u, \tilde{c}]$ minding that $\widetilde{\mathrm{OPT}}_{\Pi}$ is the optimum value of its linear relaxation.

Henceforth, it remains to invoke Theorem 2 on $\mathcal{P}[\tilde{A}, u, \tilde{c}]$ after some proper scaling. In particular, without loss of generality, assume for $\forall i, j, \tilde{A}_{i, j} \leq \tilde{c}_{i}$ since otherwise we might as well set the corresponding $j$-th decision variable to 0 . This being so, scale down each row $i$ of $\tilde{A}$ and $\tilde{c}$ by $\max _{j} \tilde{A}_{i, j}$, consequently letting $\tilde{A} \in[0,1]^{\tilde{m} \times n+1}$ and $\tilde{c}=\mathbf{1}$ (due to the dummy requests). Next, scaling $u$ such that $\max _{j} u_{j}=1$, conforms $\mathcal{P}[\tilde{A}, u, \tilde{c}]$ to the form in Theorem 2. Accordingly, we obtain a feasible integral solution to $\mathcal{P}[\tilde{A}, u, \tilde{c}]$, and hence to $\mathcal{P}\left[\left(A^{r}\right)_{r \in[d]}, u,\left(c^{r}\right)_{r \in[d]}\right]$, of value $\frac{\widetilde{\mathrm{OPT}}_{\Pi}}{O(d \log n)}$, which together with (27) yields the theorem.

Remark: For the sake of variety, the result in this section was provided in an existential form, rather than in an algorithmic variant as in Section 3. However, the algorithm is straightforward and follows immediately from the proof. Also, it should be noted that, at an additional $\operatorname{loss}$ of $\mathrm{O}(\log n)$ factor, one can possibly extend this result to separable $d$-USFP through the approach in (Bansal et al., 2014) of decomposing the given instance into one in which all the demands intersect.

## 5 From Unsplittable Flows to Electrical Flows: Application to Power Systems

In this section, we develop a reduction procedure that can be applied to LP-based approximations for separable $d$-USFP to produce approximations for AC OPF on line distribution networks. To this end, Section 5.1 first outlines the pertinent background on OPF and formulates the problem mathematically, then Section 5.2 expounds the proposed reduction.

### 5.1 AC OPF and its Exact Relaxation for Radial Networks

The AC OPF problem, introduced by Carpentier in 1962 (Carpentier, 1962), lies at the heart of techniques routinely deployed in power systems for performance optimization and control (see e.g., (Frank et al., 2012) for a comprehensive survey on OPF). As such, the input of OPF comprises an electrical network, such as the one depicted in Fig. 1, represented by an undirected graph where nodes stand for electric buses, whereas the edges model power lines. Among the buses, some correspond to AC generators while others to demand nodes (loads). The objective is to determine an operating point, optimal with respect to a given objective (e.g., minimizing generation cost), that satisfies user demands while meeting operational (engineering) constraints (e.g. line thermal limit) and physical properties (imposed by Ohm's and Kirchoff's laws) of the electrical network.

From computational perspective, OPF is notoriously toilsome due mainly to the existence of non-convex constraints involving complex-valued entities of power system parameters such as current, voltage and power. Recently, there has been a major progress on tackling OPF through convex relaxations (Bose et al., 2015; Huang et al., 2017; Gan et al., 2015; Low, 2014a,b). These papers focus chiefly on radial (i.e., tree) networks, since they are fairly common in real-world, and derive


Fig. 1 An example of a radial electrical network.
sufficient conditions under which the convex relaxation is exact (i.e., equivalent to the original non-convex problem); for example, relaxing the rank-1 constraint in the semidefinite programming (SDP) formulation (Bose et al., 2015), or relaxing the equality constraints in the second order cone programming (SOCP) formulation (Huang et al., 2017; Gan et al., 2015; Low, 2014a,b). While these results yield polynomial time algorithms for OPF, their scope is limited to the case with continuously adjustable power injection constraints; control variables responsible for modulating power loads are fractional and defined in terms of buses). In a more general setting, however, it is often necessary to account for discrete (or a mix of discrete and continuous) variables (Chapman et al., 2013; Mhanna et al., 2016; Karapetyan et al., 2021; Khonji et al., 2020). Specifically, certain loads and devices, e.g., TV, vacuum cleaner or washing machine, operate only under a particular supply of electricity; are either switched on with a fixed power consumption rate or turned off. This combinatoric structure renders a substantially more complicated instance of OPF. Concretely, as demonstrated in (Khonji et al., 2018), OPF with discrete demands in a delta network is hard to approximate within any polynomial guarantees unless $\mathrm{P}=\mathrm{NP}$. Prior studies on OPF with discrete control variables, e.g., (Briglia et al., 2017; Lin and Lin, 2008; Hijazi et al., 2017), mainly resort to heuristic techniques, which, per se, are devoid of any optimality guarantees or theoretical guidance.

With the above background in view, we next provide a model of an electrical network and define OPF formally. Recall from the convention in Sec. 2 that given a complex number $\nu \in \mathbb{C}$ we let $|\nu|$ be its magnitude, $\arg (\nu)$ be the phase angle that it makes with the real axis, $\nu^{*}$ be its complex conjugate and write $\nu^{\mathrm{R}} \triangleq$ $\operatorname{Re}(\nu), \nu^{\mathrm{I}} \triangleq \operatorname{Im}(\nu)$ for its real and imaginary components, respectively. Consider a radial distribution network represented by a line graph $\mathcal{T}=(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}=$ $\{0,1, \ldots, m\}$ denotes the electric buses, whereas $\mathcal{E}$ symbolizes the distribution lines. Each line $e \in \mathcal{E}$ is characterized by a complex impedance $z_{e} \in \mathcal{C}$, with a nonnegative real part resembling the resistance of the line (to the flow of current) and imaginary part quantifying the reactance (inductance if positive and capacitance if negative). In the setup under study, a substation generator is attached to the root of $\mathcal{T}$, node 0 . By convention, it is assumed that power flows from the root to the nodes. Let $\mathcal{V}^{+} \triangleq \mathcal{V} \backslash\{0\}$ and $\mathcal{V}_{i}^{+} \triangleq \mathcal{V}_{i} \backslash\{i\}$. When referring to an edge, we


Fig. 2 Conservation of power flow at node $j$.
shall use the (ordered) pair of subscripts $(i, j)$ and $e$ interchangeably, where it is assumed that $i$ is the parent of $j$ in $\mathcal{T}$.

At each node $j \in \mathcal{V}^{+}$, attached is a set $\mathcal{U}_{j}$ of users (electrical loads). Let $\mathcal{N} \triangleq$ $\cup_{j \in \mathcal{V}}+\mathcal{U}_{j}$ be the set of all users $(|\mathcal{N}|=\tilde{n})$, while $\mathcal{N}_{j}$ be those residing in the subpath rooted at node $j \in \mathcal{V}^{+}$. Among these users, some have inelastic (discrete) power demands, denoted by $\mathcal{I} \subseteq \mathcal{N}$. A discrete demand is either completely satisfied or dropped. An example is an appliance that is either switched on with a fixed power consumption rate or switched off. The rest of users, denoted by $\mathcal{F} \triangleq \mathcal{N} \backslash \mathcal{I}$, have elastic demands which can be partially satisfied. The demand of user $k$ is represented by a complex-valued number $s_{k} \in \mathbb{C}$; the real part $s_{k}^{\mathrm{R}}$ denotes the so-called active power while the imaginary part $s_{k}^{\mathrm{I}}$ captures the reactive power; the apparent power is defined as the magnitude $\left|s_{k}\right|=\sqrt{\left(s_{k}^{\mathrm{R}}\right)^{2}+\left(s_{k}^{\mathrm{I}}\right)^{2}}$ of $s_{k}$. Additionally, each user $k \in \mathcal{I}$ is associated with a number $u_{k} \in \mathbb{R}_{+}$indicating the utility of user $k$ if her demand $s_{k}$ is fully satisfied.

Denote the unique path from node $j$ to the root 0 by $\mathcal{P}_{j}$. For each user $k \in \mathcal{U}_{j}$, define $\mathcal{P}_{k} \triangleq \mathcal{P}_{j}$. With a slight abuse of notation, we interchangeably refer as $\mathcal{P}_{j}$ to the set of edges as well as the set nodes on the path from $j$ to the root.

A steady-state power flow in a distribution network is generally described by a system of equations. For radial networks (which include paths), these can be framed through the Branch Flow (a.k.a. DistFlow) Model (BFM) (Baran and $\mathrm{Wu}, 1989$ ). Under BFM, OPF in $\mathcal{T}$ is embodied by the following mixed-integer programming problem.

$$
\begin{align*}
& \text { Input : } v_{0} ;\left(\underline{v}_{j}, \bar{v}_{j}\right)_{j \in \mathcal{V}^{+}} ;\left(\bar{S}_{e}, \bar{\ell}_{e}, z_{e}\right)_{e \in \mathcal{E}} ;\left(s_{k}\right)_{k \in \mathcal{N}} \\
& \text { Output : } s_{0} ;\left(v_{j}\right)_{j \in \mathcal{V}^{+}} ;\left(S_{e}, \ell_{e}\right)_{e \in \mathcal{E}} ;\left(x_{k}\right)_{k \in \mathcal{N}} \\
&(\mathrm{OPF}) \max _{s_{0}, x, v, \ell, S} f_{\mathrm{OPF}}\left(s_{0}, x\right), \\
& \text { s.t. } \ell_{i, j}=\frac{\left|S_{i, j}\right|^{2}}{v_{i}}, \forall(i, j) \in \mathcal{E}  \tag{28}\\
& S_{i, j}=\sum_{k \in \mathcal{U}_{j}} s_{k} x_{k}+\sum_{t:(j, t) \in \mathcal{E}} S_{j, t}+z_{i, j} \ell_{i, j}, \quad \forall(i, j) \in \mathcal{E} \tag{29}
\end{align*}
$$

$$
\begin{align*}
& S_{0,1}=-s_{0}  \tag{30}\\
& v_{j}=v_{i}+\left|z_{i, j}\right|^{2} \ell_{i, j}-2 \operatorname{Re}\left(z_{i, j}^{*} S_{i, j}\right), \quad \forall(i, j) \in \mathcal{E}  \tag{31}\\
& \underline{v}_{j} \leq v_{j} \leq \bar{v}_{j}, \quad \forall j \in \mathcal{V}^{+}  \tag{32}\\
& \left|S_{e}\right| \leq \bar{S}_{e},\left|-S_{e}+z_{e} \ell_{e}\right| \leq \bar{S}_{e}, \quad \forall e \in \mathcal{E}  \tag{33}\\
& \ell_{e} \leq \bar{\ell}_{e} \quad \forall e \in \mathcal{E}  \tag{34}\\
& x_{k} \in\{0,1\}, \quad \forall k \in \mathcal{I}, \quad x_{k} \in[0,1], \quad \forall k \in \mathcal{F}  \tag{35}\\
& v_{j} \in \mathbb{R}_{+}, \forall j \in \mathcal{V}^{+} \quad \ell_{e} \in \mathbb{R}_{+}, \quad S_{e} \in \mathbb{C}, \quad \forall e \in \mathcal{E} \tag{36}
\end{align*}
$$

The variables: In the above formulation, the complex variable $S_{i, j}$ represents the power output at node $i$ along the edge $(i, j), z_{i, j}^{*}$ denotes the complex conjugate of $z_{i, j}$, and $v_{j} \triangleq\left|V_{j}\right|^{2}$ and $\ell_{e} \triangleq\left|I_{e}\right|^{2}$ define the voltage and current magnitude squares at node $j$ and link $e$, respectively. Note that in BFM phase angles for the voltages and currents, $\arg \left(V_{j}\right)$ and $\arg \left(I_{e}\right)$, are eliminated from the formulation. However, as proved in (Farivar and Low, 2013), this relaxation is exact for radial networks. That is, one can (in polynomial time) uniquely recover the phase angles once a solution to the relaxation is obtained. Finally, each user demand $k \in \mathcal{N}$ is assigned a control variable $x_{k}$; if $k \in \mathcal{I}$, then $x_{k} \in\{0,1\}$, otherwise, $x_{k} \in[0,1]$ for $k \in \mathcal{F}$. Define vectors $S \triangleq\left(S_{e}\right)_{e \in \mathcal{E}}, \ell \triangleq\left(\ell_{e}\right)_{e \in \mathcal{E}}, x \triangleq\left(x_{k}\right)_{k \in \mathcal{N}}, v=\left(v_{i}\right)_{i \in \mathcal{V}^{+}}$.

The objective: OPF seeks to assign values to the control vector $x$, complex power vector $S$ as well as current and voltage magnitude vectors $\ell$ and $v$, such that the following concave non-negative objective function ${ }^{4}$

$$
f_{\mathrm{OPF}}\left(s_{0}, x\right)=f_{0}\left(s_{0}^{\mathrm{R}}\right)+f_{1}\left(\left(s_{k}^{\mathrm{R}} x_{k}\right)_{k \in \mathcal{F}}\right)+\sum_{k \in \mathcal{I}} u_{k} x_{k},
$$

is maximized, without violating the physical and operating constraints described below.

The constraints: Let $\underline{v}_{j}, \bar{v}_{j} \in \mathbb{R}^{+}$be respectively the minimum and maximum allowable voltage magnitude squares at node $j$, and $\bar{S}_{e}, \bar{l}_{e} \in \mathbb{R}^{+}$be the maximum allowable apparent power and current magnitude on edge $e \in \mathcal{E}$, respectively. As customary, it is assumed that the generator voltage $v_{0} \in \mathbb{R}^{+}$is given. In the above formulation, Eqn. (28) is immediate from the definition of the magnitude of the complex power $S_{i, j}=V_{i} I_{i, j}^{*}$. Eqn. (29) (in complex variables) captures the power flow conservation rule at node $j$ (see Figure 2). The rule equates the power output at node $i$ along the edge $(i, j)$ minus the power lost on that line $\left(z_{i, j} \ell_{i, j}=z_{i, j}\left|I_{i, j}\right|^{2}\right)$ to the total power consumed by the loads at node $j$ (namely, $\sum_{k \in \mathcal{U}_{j}} s_{k} x_{k}$ ) plus the total power output on the lines outgoing from $j$ (which is $\sum_{t:(j, t) \in \mathcal{E}} S_{j, t}$ ). Eqn. (30) is the special case of Eqn. (29) applied to node 0

[^5](assuming an artificial edge $(0,0)$ ), where the demand $s_{0}$ is negated to indicate power generation (rather than consumption). Eqn. (31) is a consequence of Ohm's law: $V_{i}-V_{j}=z_{i, j} I_{i, j}$, and the definition of power $S_{i, j}=V_{i} I_{i, j}^{*}$. The inequalities in (32) and (34) limit the voltage and current magnitudes at each node and on each line, respectively, to the allowable range. While those in (33) cap the apparent power on each link in both directions by the capacity of the link: $\left|S_{i, j}\right| \leq \bar{S}_{i, j}$ and $\left|S_{j, i}\right| \leq \bar{S}_{i, j}$, where $S_{j, i}=V_{j} I_{j, i}^{*}=-V_{j} I_{i, j}^{*}=-\left(V_{i}-z_{i, j} I_{i, j}\right) I_{i, j}^{*}=-S_{i, j}+$ $z_{i, j}\left|I_{i, j}\right|^{2}$.

### 5.1.1 Assumptions

In tackling OPF, we shall rely on the following practical assumptions.
A0: $f_{0}(\cdot)$ is non-decreasing in $s_{0}^{\mathrm{R}}$. Recall that by definition $\left.f_{0}\left(s_{0}^{\mathrm{R}}\right)=Y-c\left(-s_{0}^{\mathrm{R}}\right)\right)$, where $c\left(-s_{0}^{\mathrm{R}}\right)=c\left(S_{0,1}^{\mathrm{R}}\right)$ captures the active power generation cost. As is customary in power systems literature (Huang et al., 2017; Farivar and Low, 2013; Gan et al., 2015; Zhang and Tse, 2013), we treat the generation cost $c(\cdot)$ as a non-decreasing convex function of $S_{0,1}^{\mathrm{R}}$. Consequently, one can set $Y$ to be a sufficiently large number such that $f_{0}(\cdot)$ is non-negative and non-decreasing in $s_{0}^{\mathrm{R}}$ ).
A1: $z_{e} \geq 0$ for all $e \in \mathcal{E}$, which naturally holds in distribution networks.
A2: $\underline{v}_{j} \leq v_{0} \leq \bar{v}_{j}$ for all $j \in \mathcal{V}^{+}$. Typically in a distribution network, $v_{0}=1$ (per unit), $\underline{v}_{j}=(0.95)^{2}$ and $\bar{v}_{j}=(1.05)^{2}$; in other words, a $5 \%$ deviation from the nominal voltage is allowed.
A3: $\operatorname{Re}\left(z_{e}^{*} s_{k}\right) \geq 0$ for all $k \in \mathcal{N}, e \in \mathcal{E}$. Equivalently, the angle difference between $z_{e}$ and $s_{k}$ is at most $\frac{\pi}{2}$.
A4: $\left|\arg \left(s_{k}\right)-\arg \left(s_{k^{\prime}}\right)\right| \leq \frac{\pi}{2}$ for any $k, k^{\prime} \in \mathcal{N}$. In practical settings, the socalled load power factor usually varies between 0.8 to 1 (Korovesis et al., 2004) and thus the maximum phase angle difference between any pair of demands is restricted to be in the range of $\left[0,36^{\circ}\right]$. We also assume $s_{k}^{\mathrm{R}} \geq 0$ for all $k \in \mathcal{N}$, which always holds in power systems (assuming no power generation at non-root nodes in $\mathcal{V}^{+}$).
A5: The range of impedances and demands is quasi-polynomial, that is,

$$
\max \left\{\frac{\max _{e \in \mathcal{E}} z_{e}^{\mathrm{R}}}{\min _{e: z_{e}^{\mathrm{R}}>0} z_{e}^{\mathrm{R}}}, \frac{\max _{e \in \mathcal{E}} z_{e}^{\mathrm{I}}}{\min _{e: z_{e}^{\mathrm{I}}>0} z_{e}^{I}}, \frac{\max _{k \in \mathcal{N}} s_{k}^{\mathrm{R}}}{\min _{k: s_{k}^{\mathrm{R}}>0} s_{k}^{\mathrm{R}}}, \frac{\max _{k \in \mathcal{N}} s_{k}^{\mathrm{I}}}{\min _{k: s_{k}^{\mathrm{I}}>0} s_{k}^{\mathrm{I}}}\right\}=2^{\operatorname{polylog}(m, \tilde{n})} .
$$

Assumptions A3 and A4 are motivated, from a theoretical point of view, by the inapproximability results in (Khonji et al., 2018) (if either one is invalid, the problem cannot be approximated within any polynomial factor unless $\mathrm{P}=\mathrm{NP}$ ). Assumption A3 holds in reasonable practical settings (Huang et al., 2017). As clarified in the next subsection, by performing an axis rotation, A4 implies $s_{k} \geq 0$. Clearly, under this and A1, the reverse power constraint in (33) is implied by the forward power one ( $\left|S_{e}\right| \leq \bar{S}_{e}$ ). Similarly, under A1, A2 and A3, the voltage upper bounds in (32) can be dropped, as elaborated in subsection 5.1.2. Lastly, A5 is required merely for the analysis of the featured approximations and may possibly be bypassed with techniques from (Batra et al., 2015).

### 5.1.2 Rotational Invariance of OPF

In the below lemma, it is argued that complex quantities in the OPF formulation (namely, $z_{e}, s_{k}$ ) can be rotated by a fixed angle without affecting the problem's structure. This property allows to replace A0 and A4 by the ones listed below.
$\mathrm{A} 0^{\prime}: f_{0}\left(s_{0}^{\mathrm{R}} \cos \phi+s_{0}^{\mathrm{I}} \sin \phi\right)$ is non-decreasing in $s_{0}^{\mathrm{R}}, s_{0}^{\mathrm{I}}$.
$\mathrm{A}^{\prime}: s_{k} \geq 0$ for all $k \in \mathcal{N}$.
Note that A1 and A4' already imply A3.
Lemma 3 Assume A4 and suppose that $s_{k}$, for all $k \in \mathcal{N}$, and $z_{e}$, for all $e \in$ $\mathcal{E}$, are rotated by an angle $\phi \triangleq \min \left\{\max _{k \in \mathcal{N}}-\arg \left(s_{k}\right), 0\right\} \in\left[0, \frac{\pi}{2}\right]$. Denote the resulting OPF problem by $\mathrm{OPF}^{\phi}$ :
$\left(\mathrm{OPF}^{\phi}\right) \max _{s_{0}, x, v, \ell, S} f_{\mathrm{OPF}}\left(s_{0} e^{-\mathbf{i} \phi}, x\right)$,

$$
\text { s.t. }(28)-(36), \text { with } z_{e} \text { replaced by } z_{e} e^{\mathbf{i} \phi} \text {, and } s_{k} \text { replaced by } s_{k} e^{\mathbf{i} \phi}
$$

Then $\mathrm{OPF}^{\phi}$ is equivalent to OPF and satisfies assumptions $\mathrm{A}^{\prime}, \mathrm{A} 1, \mathrm{~A} 2, \mathrm{~A} 3$ and A4 ${ }^{\prime}$.

Proof: One can easily show that a feasible solution $F=\left(s_{0}, x, v, \ell, S\right)$ to $\left(\mathrm{OPF}^{\phi}\right)$ can be converted to a feasible solution $\overline{\bar{F}}=\left(\overline{\bar{s}}_{0}, x, v, \ell, \overline{\bar{S}}\right)$ to OPF, such that $\overline{\bar{S}}_{i, j} \triangleq S_{i, j} e^{-\mathbf{i} \phi}, \overline{\bar{s}}_{0} \triangleq s_{0} e^{-\mathbf{i} \phi}$ are rotated by $\phi$, and vise versa. Moreover, the two objective functions are equal. It is immediate to see that assumptions A0' A1, A2, A 3 , and $A 4^{\prime}$ hold for $\mathrm{OPF}^{\phi}$.

Hereafter, we implicitly consider the rotated problem which, with a slight abuse of notation, is simply denoted by OPF.

### 5.1.3 Exact Second Order Cone Relaxation

As observed from the preceding formulation, OPF's feasible set is non-convex due to the quadratic equality constraint (28). Replacing this by $\ell_{i, j} \geq \frac{\left|S_{i, j}\right|^{2}}{v_{i}}$, one obtains an SOCP relaxation of OPF ${ }^{5}$, defined below and denoted by COPF.

$$
\begin{align*}
& (\mathrm{COPF}) \max _{s_{0}, x, v, \ell, S} f_{\mathrm{OPF}}\left(s_{0}, x\right) \\
& \text { s.t. }(29)-(36) \\
& \quad \ell_{i, j} \geq \frac{\left|S_{i, j}\right|^{2}}{v_{i}}, \forall(i, j) \in \mathcal{E} . \tag{37}
\end{align*}
$$

Let RCOPF be the relaxation of COPF where the integrality constraints in (35) are replaced by $x_{k} \in[0,1]$ for all $k \in \mathcal{N}$. For a given $\hat{x} \in[0,1]^{\tilde{n}}$, define by $\operatorname{COPF}[\hat{x}]$ the restriction of COPF where $x=\hat{x}$.

[^6]Recently, studies in (Low, 2014b; Huang et al., 2017; Gan et al., 2015) presented sufficient conditions for COPF to have an optimal solution in which Cons. (37) holds with equality. For current purposes, we avail of the following lemma which is a slightly simplified version of that in (Huang et al., 2017) and is proved in Section B.

Lemma 4 Under assumptions A0, A1, A2, and A3, for any given $x^{\prime} \in[0,1]^{\tilde{n}}$, there exists an optimal solution $F^{\prime}=\left(s_{0}^{\prime}, x^{\prime}, v^{\prime}, \ell^{\prime}, S^{\prime}\right)$ of $\mathrm{COPF}\left[x^{\prime}\right]$ that satisfies $\ell_{i, j}=\frac{\left|S_{i, j}^{\prime}\right|^{2}}{v_{i}^{\prime}}$ for all $(i, j) \in \mathcal{E}$. Such a solution can be found in polynomial time.

### 5.2 Reduction Scheme

Having defined OPF formally, we next present the developed technique that obtains approximations for OPF on path distribution networks from LP-based approximations intended for separable $d$-USFP.

Lemma 5 Let $F^{\prime}=\left(s_{0}^{\prime}, x^{\prime}, v^{\prime}, \ell^{\prime}, S^{\prime}\right)$ be a feasible solution for RCOPF. Let $\bar{x} \in$ $[0,1]^{\tilde{n}}$ be such that

$$
\begin{align*}
\sum_{k \in \mathcal{I}} u_{k} \bar{x}_{k} & \geq \sum_{k \in \mathcal{I}} u_{k} x_{k}^{\prime}-\varepsilon f_{\mathrm{OPF}}\left(s_{0}^{\prime}, x^{\prime}\right), \text { for some } \varepsilon \in[0,1] \\
\sum_{k \in \mathcal{N}} \operatorname{Re}\left(\sum_{(h, t) \in \mathcal{P}_{k} \cap \mathcal{P}_{j}} z_{h, t}^{*} s_{k}\right) \bar{x}_{k} & \leq \sum_{k \in \mathcal{N}} \operatorname{Re}\left(\sum_{(h, t) \in \mathcal{P}_{k} \cap \mathcal{P}_{j}} z_{h, t}^{*} s_{k}\right) x_{k}^{\prime} \quad \forall(i, j) \in \mathcal{E},  \tag{38}\\
\sum_{k \in \mathcal{N}_{j}} s_{k}^{\mathrm{R}} \bar{x}_{k} & \leq \sum_{k \in \mathcal{N}_{j}} s_{k}^{\mathrm{R}} x_{k}^{\prime} \quad \forall(i, j) \in \mathcal{E},  \tag{39}\\
\sum_{k \in \mathcal{N}_{j}} s_{k}^{\mathrm{I}} \bar{x}_{k} & \leq \sum_{k \in \mathcal{N}_{j}} s_{k}^{\mathrm{I}} x_{k}^{\prime} \quad \forall(i, j) \in \mathcal{E},  \tag{41}\\
\bar{x}_{k} & =x_{k}^{\prime} \quad \forall k \in \mathcal{F}, \tag{42}
\end{align*}
$$

where $f_{\mathrm{OPF}}(\cdot)$ is the objective function of OPF. Then, under assumptions $\mathrm{A}^{\prime}{ }^{\prime}$, A1, A2, A3 and $\mathrm{A}^{\prime}, \operatorname{RcOPF}[\bar{x}]$ has a feasible solution $\tilde{F}=(\tilde{s} 0, \tilde{x}, \tilde{v}, \tilde{\ell}, \tilde{S})$ such that $f_{\mathrm{OPF}}\left(\tilde{s}_{0}, \tilde{x}\right) \geq(1-\varepsilon) f_{\mathrm{OPF}}\left(s_{0}^{\prime}, x^{\prime}\right)$, where $\operatorname{RCOPF}[\bar{x}]$ denotes the restriction of RCOPF with $x$ set to $\bar{x}$.

Observe that, in Lemma 5 (which is proved in Section C), the inequalities (39), (40) and (41) taken together form a single-source separable $d$-USFP with $d=3$. Indeed, for $k \in \mathcal{I}$ and $e=(i, j) \in \mathcal{E}$, define
$f_{k}^{1}(e)=\operatorname{Re}\left(\sum_{e^{\prime} \in \mathcal{P}_{k} \cap \mathcal{P}_{j}} z_{e^{\prime}}^{*} s_{k}\right), \quad f_{k}^{2}(e)=\left\{\begin{array}{ll}s_{k}^{\mathrm{R}} & \text { if } k \in \mathcal{N}_{j} \\ 0 & \text { otherwise },\end{array} \quad f_{k}^{3}(e)= \begin{cases}s_{k}^{\mathrm{I}} & \text { if } k \in \mathcal{N}_{j} \\ 0 & \text { otherwise } .\end{cases}\right.$
Note that $f_{k}^{1}$ is monotone non-decreasing on $\mathcal{E}$ when ordered by distance from the root, while $f_{k}^{2}$ and $f_{k}^{3}$ are monotone non-decreasing considering the reverse order on $\mathcal{E}$. Moreover, these functions are of the form (3) (i.e., separability condition in
$d$-USFP). For $r=2$ (similarly, for $r=3$ ), set $T_{2}=1, a_{k}^{2,1}:=s_{k}^{\mathrm{R}}, e_{k}^{2}=\hat{e}_{k}^{2}:=e_{j(k)}$, $b^{2,1}(e):=1$. As for $r=1$, note that

$$
\begin{equation*}
f_{k}^{1}(e)=\left(\sum_{e^{\prime} \in \mathcal{P}_{k} \cap \mathcal{P}_{j}} z_{e^{\prime}}^{\mathrm{R}}\right) s_{k}^{\mathrm{R}}+\left(\sum_{e^{\prime} \in \mathcal{P}_{k} \cap \mathcal{P}_{j}} z_{e^{\prime}}^{\mathrm{I}}\right) s_{k}^{\mathrm{I}} \text { for } \forall e=(i, j) \in \mathcal{E} . \tag{43}
\end{equation*}
$$

Thus, setting $T_{1}=2, a_{k}^{1,1}:=\operatorname{Re}\left(s_{k}\right), a_{k}^{1,2}:=\operatorname{Im}\left(s_{k}\right), e_{k}^{1}:=e_{1}, \hat{e}_{k}^{1}:=e_{j(k)}$, $\tilde{b}^{1,1}((i, j)):=\sum_{e^{\prime} \in \mathcal{P}_{j}} \operatorname{Re}\left(z_{e^{\prime}}\right)$ and $\tilde{b}^{1,2}((i, j)):=\sum_{e^{\prime} \in \mathcal{P}_{j}} \operatorname{Im}\left(z_{e^{\prime}}\right)$ writes $f_{k}^{1}(e)$ in the form (3).

The above arguments coupled with Lemma 5, imply the following theorem.

Theorem 4 Under assumptions $\mathrm{A}^{\prime}$, $\mathrm{A} 1, \mathrm{~A} 2, \mathrm{~A} 3, \mathrm{~A}^{\prime}$, and A 5 , there is a quasipolynomial time algorithm that for any $\epsilon \in(0,1)$ produces a $(1-\varepsilon)$-approximation for OPF on line networks with single substation generator.

Proof: Let $\widehat{\text { OPT }}$ be the optimal objective value of OPF. Consider the approximation scheme detailed in Alg. 2, which is the analog of Alg. 1 for OPF.

```
Algorithm 2 QPTAS-OPF
Require: An approximation parameter \(\epsilon \in(0,1)\); OPF input \(v_{0} ;\left(\underline{v}_{j}, \bar{v}_{j}\right)_{j \in \mathcal{V}+} ;\left(\bar{S}_{e}, \bar{\ell}_{e}, z_{e}\right)_{e \in \mathcal{E}}\)
Ensure: A solution \(\hat{F}\) to OPF such that \(f_{\mathrm{OPF}}(\hat{F}) \geq(1-O(\epsilon)) \widehat{\mathrm{OPT}}\)
    for each selection \(\left(\mathcal{L}=\left(\mathcal{L}^{q}\right)_{q \in \mathcal{Q}}, h=\left(h^{q}=\left(h^{q, p, r}\right)_{p \in\left[P_{r}\right], r \in[d]}\right)_{q \in \mathcal{Q}}\right)\) such that \(\mathcal{L}^{q} \subseteq \mathcal{I}\),
    \(\left|\mathcal{L}^{q}\right| \leq \frac{\sum_{r=1}^{d} P_{r}}{\epsilon^{2}}\) and \(h^{q, p, r} \in F^{r}\) do
        if \(\operatorname{RCOPF}[\mathcal{L}, h]\) is feasible then
            \(F^{\prime} \leftarrow\) Solution of \(\operatorname{RcOPF}[\mathcal{L}, h]\)
            for \(q \in \mathcal{Q}\) do
                    Let \(\mathcal{S}^{q}\) be given by (13)
                    for every \((h, \epsilon)\)-restricted profile \(g^{q}\) do
                    \(\left(\hat{x}_{k}\right)_{k \in \mathcal{S}^{q}} \leftarrow\) Integral vector returned by applying Lemma 2 with vector \(h^{q}\),
    and \(\left(\tilde{x}_{k}\right)_{k \in \mathcal{S}^{q}}=\left(x_{k}^{\prime}\right)_{k \in \mathcal{S}^{q}}\)
            \(\bar{x}_{k} \leftarrow \begin{cases}\hat{x}_{k} & \text { if } k \in \bigcup_{q \in \mathcal{Q}} \mathcal{S}^{q}, \\ x_{k}^{\prime} & \text { if } k \in \mathcal{N} \backslash\left(\bigcup_{q \in \mathcal{Q}} \mathcal{S}^{q}\right)\end{cases}\)
            \(\tilde{F} \leftarrow\) Solution of \(\operatorname{COPF}[\bar{x}]\)
            if \(f_{\text {OPF }}(\tilde{F})>f_{\text {OPF }}\left(\hat{F}^{\prime}\right)\) then
                \(\hat{F}^{\prime} \leftarrow \tilde{F}\)
    Apply Lemma 4 to convert \(\hat{F}^{\prime}\) to a feasible solution \(\hat{F}\) for OPF
    return \(\hat{F}\)
```

Similar to Alg. 1, the algorithm guesses the set of large demands $\mathcal{L}^{q} \subseteq \mathcal{I}^{q}$ in the optimal solution for each group $q \in \mathcal{Q}$, and the peaks $h^{q, p, r}$, within $1+\epsilon$, of the small demands in the optimal solution within the interval $\mathcal{E}_{p}^{r}$. Let $\mathcal{L}=\left(\mathcal{L}^{q}\right)_{q \in \mathcal{Q}}$ and $h^{q}=\left(h^{q, p, r}\right)_{p \in\left[P_{r}\right], r \in[d]}$ where $h^{q, p, r} \in F^{r}$. Define a restrictive version of $\operatorname{RCOPF}$, denoted by $\operatorname{RCOPF}[\mathcal{L}, h]$, which enforces that $x_{k}=1$ for all $k \in \mathcal{L}^{q}$ and $q \in \mathcal{Q}$ and that the peak total contribution of the small demands in group $q$ within the interval $\mathcal{E}_{p}^{r}$ is at most $h^{q, p, r}: \sum_{k \in \mathcal{S}^{q}} \bar{f}_{k}^{p, r} x_{k} \leq(1+\epsilon) h^{q, p, r}$.

$$
\begin{align*}
(\operatorname{RCOPF}[\mathcal{L}, h]) & \max _{s_{0}, x, v, \ell, S} f_{\mathrm{OPF}}\left(s_{0}, x\right), \\
\text { s.t. } & (29)-(34),(36),(37)  \tag{44}\\
& \sum_{k \in \mathcal{S}^{q}} \bar{f}_{k}^{p, r} x_{k} \leq h^{q, p, r}, \forall p \in\left[P_{r}\right], \forall r \in[d], \forall q \in \mathcal{Q}  \tag{45}\\
& x_{k}=0, \quad \forall k \in \mathcal{I} \backslash \bigcup_{q \in \mathcal{Q}}\left(\mathcal{L}^{q} \cup \mathcal{S}^{q}\right)  \tag{46}\\
& x_{k}=1, \quad \forall k \in \mathcal{L}^{q}, \forall q \in \mathcal{Q}  \tag{47}\\
& x_{k} \in[0,1], \quad \forall k \in \mathcal{F} \cup\left(\bigcup_{q \in \mathcal{Q}} \mathcal{S}^{q}\right) . \tag{48}
\end{align*}
$$

Here, the set of small demands within group $q \in \mathcal{Q}$ is

$$
\begin{equation*}
\mathcal{S}^{q}=\left\{k \in \mathcal{I}^{q}: \underline{f}_{k}^{p, r} \leq B^{q, p, r} \text { for all } p \in\left[P_{r}\right], r \in[d]\right\} \tag{49}
\end{equation*}
$$

where $B^{q, p, r}=\epsilon^{2}\left[h^{q, p, r}+\sum_{k \in \mathcal{L}^{q}} \bar{f}_{k}^{p, r}\right]$. Given a feasible solution $F^{\prime}=\left(s_{0}^{\prime}, x^{\prime}\right.$, $\left.v^{\prime}, \ell^{\prime}, S^{\prime}\right)$ to $\operatorname{RcOPF}[\mathcal{L}, h]$, Alg. 2 applies Lemma 2 with $\tilde{x}=x^{\prime}$. By the lemma, one can find (in polynomial time) an integral solution $\hat{x}$ satisfying conditions (i) and (ii). Next, the algorithm recalculates $s_{0}, S, \ell, v$ utilizing the program $\operatorname{COPF}[\bar{x}]$ given in Section 5.1.3, and then applies Lemma 4 to obtain a feasible solution to OPF.

Define

$$
\begin{align*}
\tilde{M}:= & \max \left\{\frac{\bar{z}}{\underline{z}}, \max _{r} \frac{\bar{f}^{r}}{\underline{f}^{r}}\right\} \\
& =\max \left\{\frac{\bar{z}}{\underline{z}}, \max _{k, k^{\prime} \in \mathcal{I}} \frac{s_{k}^{\mathrm{R}}}{s_{k^{\prime}}^{\mathrm{R}}}, \max _{k, k^{\prime} \in \mathcal{I}} \frac{s_{k}^{\mathrm{I}}}{s_{k^{\prime}}^{\mathrm{I}}}, \max _{k, k^{\prime} \in \mathcal{I},( }^{(i, j),\left(i^{\prime}, j^{\prime}\right) \in \mathcal{E}} \frac{\operatorname{Re}\left(\sum_{e^{\prime} \in \mathcal{P}_{k} \cap \mathcal{P}_{j}} z_{e^{\prime}}^{*} s_{k}\right)}{\operatorname{Re}\left(\sum_{e^{\prime} \in \mathcal{P}_{k^{\prime}} \cap \mathcal{P}_{j^{\prime}}} z_{e^{\prime}}^{*} s_{k^{\prime}}\right)}\right\} \tag{50}
\end{align*}
$$

where $\underline{z}:=\min \left\{\min _{e: \operatorname{Re}\left(z_{e}\right)>0} \operatorname{Re}\left(z_{e}\right), \min _{e: \operatorname{Im}\left(z_{e}\right)>0} \operatorname{Im}\left(z_{e}\right)\right\}$ and $\bar{z}:=\max _{e \in \mathcal{E}}$ $\max \left\{\operatorname{Re}\left(z_{e}\right), \operatorname{Im}\left(z_{e}\right)\right\}$.

In what follows, we prove that, for any fixed $\varepsilon \in(0,1)$, Alg. 2 arrives at a $(1-\varepsilon)$-approximation in time $\left(\frac{\tilde{n} \log (\tilde{n} m \tilde{M})}{\varepsilon}\right)^{O\left(\log ^{9}\left(\frac{\tilde{n} m \tilde{M}}{\varepsilon}\right) / \varepsilon^{2}\right)}$.

Let $\epsilon:=\frac{\varepsilon}{3(2 \beta+1)}$, where $\beta=\max _{r \in \mathcal{H}^{q}} 2\left(2 C_{r}+\alpha P_{r}\right)=O\left(\log ^{2}(m \tilde{M})\right)$. The number of possible choices for each $\mathcal{L}^{q}$ in step 1 of $\operatorname{Alg} .2$ is at most $\tilde{n}^{\sum_{r=1}^{d} P_{r} / \epsilon^{2}}$, where $\tilde{n}=|\mathcal{N}|$. Thus, with $d=3, P_{1}=O(\log (m \tilde{M})), P_{2}=P_{3}=1, T_{1}=2$, $T_{2}=T_{3}=1, Q \leq \bar{Q}^{\sum_{r=1}^{d} T_{r}}$, and $\bar{Q}=O\left(\log \frac{\tilde{n} \tilde{M}}{\epsilon}\right)$, hence the number of possible choices for $\mathcal{L}$ is at most

$$
\begin{equation*}
\tilde{n}^{\sum_{r=1}^{d} P_{r} Q / \epsilon^{2}} \leq \tilde{n}^{\sum_{r=1}^{d} P_{r} \bar{Q}^{\sum_{r=1}^{d} T_{r}} / \epsilon^{2}}=\tilde{n}^{O\left(\log (m \tilde{M}) \log ^{4}\left(\frac{\tilde{n} \tilde{M}}{\epsilon}\right) / \epsilon^{2}\right)} \tag{51}
\end{equation*}
$$

The number of choices for each $h^{q}=\left(h^{q, p, r}\right)_{p \in\left[P_{r}\right], r \in[d]}$ is

$$
\bar{F}^{\sum_{r=1}^{d} P_{r}}=O\left(\left(\frac{\log (\tilde{n} \tilde{M})}{\epsilon}\right)^{\log (m \tilde{M})}\right)
$$

and that of for $Q$ in step 4 is

$$
\begin{equation*}
\bar{Q}^{\sum_{r=1}^{d} T_{r}} \leq \log ^{4}\left(\frac{\tilde{n} \tilde{M}}{\epsilon}\right) \tag{52}
\end{equation*}
$$

giving at most

$$
\begin{align*}
O\left(\left(\left(\frac{\log (\tilde{n} \tilde{M})}{\epsilon}\right)^{\log (m \tilde{M})}\right)^{Q}\right) & =O\left(\left(\left(\frac{\log (\tilde{n} \tilde{M})}{\epsilon}\right)^{\log (m \tilde{M})}\right)^{\bar{Q}^{\sum_{r=1}^{d} T_{r}}}\right) \\
& =O\left(\left(\frac{\log (\tilde{n} \tilde{M})}{\epsilon}\right)^{\log (m \tilde{M}) \log ^{4}\left(\frac{\tilde{n} \tilde{M}}{\epsilon}\right)}\right) \tag{53}
\end{align*}
$$

choices for $h=\left(h^{q}\right)_{q \in \mathcal{Q}}$ in step 1 . The number of choices for the $\epsilon$-restricted profiles in step 6 is bounded from above by $m^{\sum_{r=1}^{d} P_{r} / \epsilon}=m^{O(\log (m \tilde{M}) / \epsilon)}$. Thus, the bound on the running time follows from this and (51),(53),(52).

We now argue that the solution $\hat{F}$ outputted by Alg. 2 is $(1-O(\epsilon))$-approximation for OPF. Let $F^{*}=\left(s_{0}^{*}, x^{*}, v^{*}, \ell^{*}, S^{*}\right)$ be an optimal solution for OPF of objective value $\widehat{\mathrm{OPT}}=f_{\mathrm{OPF}}\left(F^{*}\right)$. By the definition of $\hat{\mathcal{I}}$, we have

$$
\begin{equation*}
\sum_{k \in \mathcal{I} \backslash \hat{\mathcal{I}}} u_{k} \leq \epsilon \widehat{\mathrm{OPT}} \leq \epsilon f_{\mathrm{OPF}}\left(F^{*}\right) \tag{54}
\end{equation*}
$$

Define $\mathcal{T}^{*} \triangleq\left\{k \in \hat{\mathcal{I}} \mid x_{k}^{*}=1\right\}$ and $\left(h^{*}\right)^{q, p, r}=\sum_{k \in \mathcal{T}^{*} \cap \mathcal{I}^{q}} \bar{f}_{k}^{p, r}$, for $p \in\left[P_{r}\right]$, $r \in[d]$ and $q \in \mathcal{Q}$. Let $\left(\mathcal{L}^{*}\right)^{q}:=\left\{k \in \mathcal{I}^{q} \cap \mathcal{T}^{*}: \underline{f}_{k}^{p, r}>\epsilon^{2}\left(h^{*}\right)^{q, p, r}\right.$ for some $p \in$ [ $\left.P_{r}\right]$, and some $\left.r \in[d]\right\}$ be the set of large demands within group $\mathcal{I}^{q}$ in the optimal solution, and let $\left(\mathcal{S}^{*}\right)^{q}:=\mathcal{I}^{q} \cap \mathcal{T}^{*} \backslash\left(\mathcal{L}^{*}\right)^{q}$ be the set of "small" demands within the same group. Note by this definition that $\left|\left(\mathcal{L}^{*}\right)^{q}\right| \leq \frac{\sum_{r=1}^{d} P_{r}}{\epsilon^{2}}$, and thus $\mathcal{L}^{*}=$ $\left(\left(\mathcal{L}^{*}\right)^{q}\right)_{q \in \mathcal{Q}}$ and $h=\left(h^{q}\right)_{q \in \mathcal{Q}}$ will be one of the guesses considered by the algorithm in step 1. Let us focus on this particular iteration of the loop in step 1. Let $h^{q, p, r}=(1+\epsilon)^{\ell^{\prime}} \underline{f}^{r}$, where $\ell^{\prime}$ is the smallest integer (including $-\infty$ ) such that $h^{q, p, r}+\sum_{k \in \mathcal{L}^{q}} \bar{f}_{k}^{\overline{p, r}} \geq\left(h^{*}\right)^{q, p, r}$. Note that $h^{q, p, r} \in F^{r}$, and

$$
\begin{equation*}
\frac{1}{1+\epsilon} h^{q, p, r}+\sum_{k \in\left(\mathcal{L}^{*}\right)^{q}} \bar{f}_{k}^{p, r} \leq\left(h^{*}\right)^{q, p, r} \leq h^{q, p, r}+\sum_{k \in\left(\mathcal{L}^{*}\right)^{q}} \bar{f}_{k}^{p, r} . \tag{55}
\end{equation*}
$$

Moreover, for any $k \in\left(\mathcal{S}^{*}\right)^{q}, q \in \mathcal{Q}, p \in\left[P_{r}\right]$, and $r \in[d]$, we have by (55),

$$
\underline{f}_{k}^{p, r} \leq \epsilon^{2}\left(h^{*}\right)^{q, p, r} \leq \epsilon^{2}\left(h^{q, p, r}+\sum_{k \in\left(\mathcal{L}^{*}\right)^{q}} \bar{f}_{k}^{p, r}\right)
$$

and hence $\left(\mathcal{S}^{*}\right)^{q} \subseteq \mathcal{S}^{q}$. Note also that

$$
\begin{align*}
B^{q, p, r} & =\epsilon^{2}\left[h^{q, p, r}+\sum_{k \in\left(\mathcal{L}^{*}\right)^{q}} \bar{f}_{k}^{p, r}\right] \\
& \leq \epsilon^{2}\left[h^{q, p, r}+(1+\epsilon) \sum_{k \in\left(\mathcal{L}^{*}\right)^{q}} \bar{f}_{k}^{p, r}\right] \leq \epsilon^{2}(1+\epsilon)\left(h^{*}\right)^{q, p, r} . \tag{56}
\end{align*}
$$

Furthermore, $x^{*}$ is feasible for the constraint (45) as

$$
\sum_{k \in \mathcal{S}^{q}} \bar{f}_{k}^{p, r} x_{k}^{*}=\sum_{k \in\left(\mathcal{S}^{*}\right)^{q}} \bar{f}_{k}^{p, r} x_{k}^{*}=\sum_{k \in\left(\mathcal{S}^{*}\right)^{q}} \bar{f}_{k}^{p, r}=\left(h^{*}\right)^{q, p, r}-\sum_{k \in\left(\mathcal{L}^{*}\right)^{q}} \bar{f}_{k}^{p, r} \leq h^{q, p, r} .
$$

It follows that $F^{*}$ is feasible for $\operatorname{R} 1[\mathcal{L}, h]$, implying by (54) that the solution $F^{\prime}$ obtained in step 3 of the algorithm satisfies

$$
\begin{equation*}
f_{\mathrm{OPF}}\left(F^{\prime}\right) \geq(1-\epsilon) f_{\mathrm{OPF}}\left(F^{*}\right) \tag{57}
\end{equation*}
$$

For each $q \in \mathcal{Q}$, there is an $(h, \epsilon)$-restricted profile $g^{q}$ and an integral solution $\left(\hat{x}_{k}\right)_{k \in \mathcal{S}^{q}}$ that satisfy Lemma 2 . Since all the possible $(h, \epsilon)$-restricted profiles are probed, the profile $g^{q}$ will be found in one of the iterations in the loop in line 6. Let us consider this iteration. By condition (i) of the lemma, $\sum_{k \in \mathcal{S}^{q}} f_{k}^{r}(e) \hat{x}_{k} \leq$ $\sum_{k \in \mathcal{S}^{q}} f_{k}^{r}(e) x_{k}^{\prime}$ for all $e \in \mathcal{E}$ and $r \in[d]$, which implies that conditions (39)-(42) of Lemma 5 hold for the vector $\bar{x}$, defined in line 8 of Alg. 2.

At this point, following exactly the same lines as in the proof of Theorem 1, it can be shown that

$$
\sum_{k \in \mathcal{I}} u_{k} \bar{x}_{k} \geq \sum_{k \in \hat{\mathcal{I}}} u_{k} x_{k}^{\prime}-3 \epsilon(2 \beta+1) f_{\text {OPF }}\left(F^{\prime}\right) .
$$

Thus condition (38) in Lemma 5 is satisfied with $\varepsilon=3 \epsilon(2 \beta+1)$ implying that $\tilde{F}$ is a feasible solution for COPF, and hence for OPF by Lemma 4, with $f_{\text {OPF }}(\tilde{F}) \geq$ $(1-\varepsilon) f_{\mathrm{OPF}}\left(F^{\prime}\right) \geq(1-\varepsilon) f_{\mathrm{OPF}}\left(F^{*}\right)$.
Remark: Following arguments analogous to those in the above proof, it is conceivable to generalize the logarithmic approximation devised in Section 4 to OPF on line networks with single substation generator, provided assumptions $\mathrm{AO}^{\prime}, \mathrm{A} 1$, A2, A3, A4 ${ }^{\prime}$ and A5 hold. To this end, however, an additional constant factor would be lost in the approximation ratio for bounding the capacities (i.e., the right hand sides of inequalities (39), (40) and (41)).

## 6 Concluding Remarks

This study defined a novel generalization of UFP, dubbed as $d$-USFP, and bridged it with AC OPF, which is a fundamental problem in power systems engineering. In a preliminary step towards tackling this extended problem, we devised a QPTAS and an efficient logarithmic approximation for its single-source variant. Leveraging the connection between separable $d$-USFP and AC OPF, a (kind of) black-box reduction is developed that, under some mild conditions, allows one to convert an approximation for the former problem to that of for AC OPF on line distribution networks with discrete demands. It's noteworthy that this reduction applies only to algorithms that depend on LP-rounding techniques, hence the focus of the present study on LP-based approximations. Whereas for future work, it would be interesting to generalize and extend the known alternative techniques (e.g., the surveyed combinatorial and dynamic programming based ones) to $d$-USFP, consequently improving upon the current results. As from power systems perspective, one future avenue to explore, would be extension of the established framework to a more practical setting with multiple generation sources and tree networks.

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## Appendix

## A Proof of Lemma 2

Proof: For $r \in[d]$, consider the graph of the fractional profile $\sum_{k \in \mathcal{S}^{q}} f_{k}^{r}(e) \tilde{x}_{k}$ illustrated in Figure 3. For $p \in\left[P_{r}\right]$, slice the region between the horizontal axis and horizontal line at height $h^{q, p, r}$ with $\frac{1}{\epsilon}+1$ horizontal lines, with inter-distance $\epsilon h^{q, p, r}$. The intersections of the optimal profile with these lines define a monotone function $g^{q, r}$, as pictured in Figure 3, with $g^{q, r}(e) \in\left\{l \epsilon h^{p, r}: l \in\{0,1 \ldots, 1 / \epsilon\}, p \in\left[P_{r}\right]\right\}$, for all $e \in \mathcal{E}$. We adopt a greedy procedure, explained in Algorithm 3 below, to remove a set of demands from $\mathcal{S}^{q}$ in each interval $\mathcal{E}_{p}^{r}$ such that the remaining set of demands fractionally fits below $g^{q, r}$ (see lines 2-9). The algorithm proceeds by removing the "left-most" set of demands that minimally ensures that the remaining ones in $\mathcal{S}^{q}$ can be packed under capacity $g^{q, r}$. This defines an intermediate fractional vector $\overline{\bar{x}}$ for separable $d$-USFP-R $\left[\mathcal{S}^{q}, g^{q}\right]$, where $g^{q}=\left(g^{q, r}\right)_{r \in[d]}$, which can be converted to a basic feasible solution (BFS) with the same or better objective value. Lastly, the fractional components of $\overline{\bar{x}}$ are rounded down yielding an integral solution $\hat{x}$.

We first show that condition (i) holds when $\hat{x}$ is replaced by $\bar{x}$. For $r \in[d]$, let $\mathcal{J}^{r}\left(e_{i}\right)$ be the set of demands $k \in \mathcal{S}^{q}$ for which $\bar{x}_{k}$ was set to 0 in step 7 when considering edge $e_{i} \in \mathcal{E}$.

```
Algorithm 3 MODIFY
Require: \(q \in \mathcal{Q}\); a restricted profile \(\operatorname{RP}_{\epsilon}(h ; w ; g)\); a set of users \(\mathcal{S}^{q} \subseteq \mathcal{I}^{q}\); a fractional vector
``` \(\left(\tilde{x}_{k}\right)_{k \in \mathcal{S}^{q}} \in[0,1]^{\mathcal{S}^{q}}\)
Ensure: A integral vector \(\left(\hat{x}_{k}\right)_{k \in \mathcal{S}^{q}} \in\{0,1\} \mathcal{S}^{q}\) satisfying conditions (i) and (ii) of lemma 2
        \(\bar{x} \leftarrow \tilde{x} ; t \leftarrow 0\)
    for \(r=1\) to \(d\) do
        for \(p=1, \ldots, P_{r}\) do
            \(i \leftarrow \underline{i}(p, r)\)
            while \(t<\epsilon h^{q, p, r}\) do
                if \(\exists k \in \mathcal{S}^{q}\) such that \(\tilde{x}_{k} f_{k}^{r}\left(e_{i}\right)>0\) then
                    \(\bar{x}_{k}=0\)
                    \(t \leftarrow t+\tilde{x}_{k} f_{k}^{r}\left(e_{i}\right)\)
                    else \(i \leftarrow i+1\)
    Convert \(\bar{x}\) to a BFS \(\overline{\bar{x}}\) for \(d\)-USFP-R \(\left[\mathcal{S}, g^{q}\right]\) with \(\sum_{k \in \mathcal{S}} u_{k} \overline{\bar{x}}_{k} \geq \sum_{k \in \mathcal{S}} u_{k} \bar{x}_{k}\)
    \(\left(\hat{x}_{k}\right)_{k \in \mathcal{S}^{q}} \leftarrow\left(\left\lfloor\overline{\bar{x}}_{k}\right\rfloor\right)_{k \in \mathcal{S}^{q}}\)
    return \(\hat{x}\)


Fig. 3 A profile and its \((h, \epsilon)\)-restriction.

Consider an edge \(e \in \mathcal{E}_{p}^{r}\) such that \(\sum_{k \in \mathcal{S}^{q}} f_{k}^{r}(e) \bar{x}_{k}>0\). Note that \(0 \leq \sum_{k \in \mathcal{S}^{q}} f_{k}^{r}(e) \tilde{x}_{k}-\) \(g^{q, r}(e) \leq \epsilon h^{q, p, r}\) by (12) and the definition of \(g^{q, r}\). By the monotonicity of \(f_{k}^{r}(\cdot)\) and the condition of the while-loop in step 5 we have
\[
\begin{aligned}
\sum_{k \in \mathcal{S}^{q}} f_{k}^{r}(e) \bar{x}_{k} & =\sum_{k \in \mathcal{S}^{q}} f_{k}^{r}(e) \tilde{x}_{k}-\sum_{i: e_{i} \leq e} \sum_{k \in \mathcal{J}^{r}\left(e_{i}\right)} f_{k}^{r}(e) \tilde{x}_{k} \\
& \leq \sum_{k \in \mathcal{S}^{q}} f_{k}^{r}(e) \tilde{x}_{k}-\sum_{i: e_{i} \leq e} \sum_{k \in \mathcal{J}^{r}\left(e_{i}\right)} f_{k}^{r}\left(e_{i}\right) \tilde{x}_{k} \\
& \leq \sum_{k \in \mathcal{S}^{q}} f_{k}^{r}(e) \tilde{x}_{k}-\epsilon h^{q, p, r} \\
& \leq g^{q, r}(e) .
\end{aligned}
\]

Since \(\bar{x}\) is feasible for \(d\)-USFP-R \(\left[\mathcal{S}^{q}, g^{q}\right]\), one can obtain a BFS \(\overline{\bar{x}}\) for the same linear program with \(\sum_{k \in \mathcal{S}^{q}} u_{k} \overline{\bar{x}}_{k} \geq \sum_{k \in \mathcal{S}} u_{k} \bar{x}_{k}\) as in step 10 of procedure Modify. Then, round down the fractional components in \(\overline{\bar{x}}\) to obtain an integral solution \(\hat{x}\). Note that, for all \(e \in \mathcal{E}\),
\[
\sum_{k \in \mathcal{S}^{q}} f_{k}^{r}(e) \hat{x}_{k} \leq \sum_{k \in \mathcal{S}^{q}} f_{k}^{r}(e) \overline{\bar{x}}_{k} \leq g^{q, r}(e) \leq \sum_{k \in \mathcal{S}^{q}} f_{k}^{r}(e) \tilde{x}_{k},
\]
and hence (i) holds.
Note that the total fractional utility of demands removed by Algorithm 3 in steps 2-9 is
\[
\begin{aligned}
\sum_{r \in[d], e \in \mathcal{E}} \sum_{k \in \mathcal{J}^{r}(e)} u_{k} \tilde{x}_{k} & =\sum_{r \in[d]::} \sum_{H^{P_{r}, r}>0}^{P_{r}} \sum_{p=1} \sum_{e \in \mathcal{E}_{p}^{r}} u_{k \in \mathcal{J}^{r}(e)} \tilde{x}_{k} \\
& \leq \sum_{r \in \mathcal{H}^{q}} \sum_{p=1}^{P_{r}} \sum_{e \in \mathcal{E}_{p}^{r}} \sum_{k \in \mathcal{J}^{r}(e)} \frac{1}{H^{q, p, r}} \sum_{t=1}^{T_{r}} a_{k}^{r, t} b^{r, t}\left(e_{\bar{i}(p, r)}\right) \tilde{x}_{k} \\
& =\sum_{r \in \mathcal{H}^{q}} \sum_{p=1}^{P_{r}} \frac{1}{H^{q, p, r}} \sum_{e \in \mathcal{E}_{p}^{r}} \sum_{k \in \mathcal{J}^{r}(e)} f_{k}^{r}\left(e_{\bar{i}(p, r)}\right) \tilde{x}_{k} \\
& \leq \sum_{r \in \mathcal{H}^{q}} \sum_{p=1}^{P_{r}} \frac{C_{r}}{H^{q, p, r}}\left(\sum_{e \in \mathcal{E}_{p}^{r}} \sum_{k \in \mathcal{J}^{r}(e)} f_{k}^{r}(e) \tilde{x}_{k}\right) \\
& \leq \sum_{r \in \mathcal{H}^{q}} \sum_{p=1}^{P_{r}} \frac{C_{r}}{H^{q, p, r}}\left(\epsilon h^{q, p, r}+B^{q, p, r}\right),
\end{aligned}
\]
where we use the fact that \(k \in \mathcal{I}^{q}\) in the first inequality, property (4) in the second inequality, and \(\underline{f}_{k}^{p, r} \leq B^{q, p, r}\) and the condition of the while-loop in step 5 in the last inequality. (Note that we sum above over \(r \in[d]\) such that in \(H^{q, P_{r}, r}>0\) since \(k \in \mathcal{J}^{r}(e)\) implies that \(f_{k}^{r}(e)>0\), which in turn implies by (11) that \(H^{q, P_{r}, r}>0\).)

It follows that
\[
\begin{align*}
\sum_{k \in \mathcal{S}^{q}} u_{k} \bar{x}_{k} & \geq \sum_{k \in \mathcal{S}^{q}} u_{k} \tilde{x}_{k}-\sum_{r, e} \sum_{k \in \mathcal{J}^{r}(e)} u_{k} \tilde{x}_{k} \\
& \geq \sum_{k \in \mathcal{S}^{q}} u_{k} \tilde{x}_{k}-\sum_{r \in \mathcal{H}^{q}} \sum_{p=1}^{P_{r}} \frac{C_{r}}{H^{q, p, r}}\left(\epsilon h^{q, p, r}+B^{q, p, r}\right) . \tag{58}
\end{align*}
\]

By the monotonicity of the functions \(f_{k}^{r}(\cdot), d\)-USFP-R \(\left[\mathcal{S}, g^{q}\right]\) has only \(\frac{1}{\epsilon} \sum_{r=1}^{d} P_{r}\) nonredundant packing inequalities of the form (1). It follows that the BFS \(\overline{\bar{x}}\) computed in step 10 has at most \(\frac{1}{\epsilon} \sum_{r=1}^{d} P_{r}\) fractional components \(\overline{\bar{x}} \in(0,1)\). Thus,
\[
\begin{align*}
\sum_{k \in \mathcal{S}^{q}} u_{k} \hat{x}_{k} & =\sum_{k \in \mathcal{S}^{q}} u_{k} \overline{\bar{x}}_{k}-\sum_{k \in \mathcal{S}^{q}: \overline{\bar{x}}_{k} \in(0,1)} u_{k} \overline{\bar{x}}_{k} \\
& \geq \sum_{k \in \mathcal{S}^{q}} u_{k} \bar{x}_{k}-\frac{1}{\epsilon} \sum_{r=1}^{d} P_{r} \cdot \max _{k} u_{k} \overline{\bar{x}}_{k} \\
& \geq \sum_{k \in \mathcal{S}^{q}} u_{k} \bar{x}_{k}-\frac{1}{\epsilon} \frac{\sum_{r=1}^{d} P_{r}}{\sum_{r \in \mathcal{H}^{q}} P_{r}} \sum_{r \in \mathcal{H}^{q}} \frac{P_{r} B^{q, P_{r}, r}}{H^{q}, P_{r}, r} \tag{59}
\end{align*}
\]
where we use in the last inequality that \(\overline{\bar{x}}_{k} \leq 1\) and
\[
\begin{aligned}
u_{k} & \leq \frac{\sum_{r \in \mathcal{H}^{q}} P_{r} \sum_{t=1}^{P_{r}} a_{k}^{r, t} b^{r, t}\left(e_{n}\right) / H^{q, P_{r}, r}}{\sum_{r \in \mathcal{H}^{q}} P_{r}}=\frac{\sum_{r \in \mathcal{H}^{q}} P_{r} f_{k}^{r}\left(e_{n}\right) / H^{q, P_{r}, r}}{\sum_{r \in \mathcal{H}^{q}} P_{r}} \\
& \leq \frac{\sum_{r \in \mathcal{H}^{q}} P_{r} B^{q, P_{r}, r} / H^{q, P_{r}, r}}{\sum_{r \in \mathcal{H}^{q}} P_{r}},
\end{aligned}
\]
for \(k \in \mathcal{S}^{q}\). Condition (ii) follows from (58) and (59).
```

Algorithm 4 FORWARD-BACKWARD-SWEEP
Require: A feasible solution $F^{\prime}=\left(s_{0}^{\prime}, x^{\prime}, v^{\prime}, \ell^{\prime}, S^{\prime}\right)$ to $\operatorname{COPF}^{\prime}\left[x^{\prime}\right]$ such that $\ell_{h, t}^{\prime}>\frac{\left|S_{h, t}^{\prime}\right|^{2}}{v_{h}^{\prime}}$ for
some $(h, t) \in \mathcal{E}$
Ensure: A feasible solution $\tilde{F}=(\tilde{s} 0, \tilde{x}, \tilde{v}, \tilde{\ell}, \tilde{S})$ to $\operatorname{COPF}^{\prime}\left[x^{\prime}\right]$ such that $\tilde{x}=x^{\prime}$ and $\sum_{e \in \mathcal{E}} \tilde{\ell}_{e}<$
$\sum_{e \in \mathcal{E}} \ell_{e}^{\prime}$
$\tilde{x} \leftarrow x^{\prime} ; \tilde{v}_{0} \leftarrow v_{0}$
Number nodes $\mathcal{V}=\{0,1, \ldots, m\}$ in a breadth-first search order
for $j=m, m-1, \ldots, 1$ do /* Forward sweep */
Let $i$ be s.t. $(i, j) \in \mathcal{E}$
$\tilde{\ell}_{i, j} \leftarrow \frac{\left|S_{i, j}^{\prime}\right|^{2}}{v_{i}^{\prime}}$
$\tilde{S}_{i, j} \leftarrow \sum_{k \in \mathcal{N}_{j}} s_{k} \tilde{x}_{k}+\sum_{t:(j, t) \in \mathcal{E}} \tilde{S}_{j, t}+z_{i, j} \tilde{\ell}_{i, j}$
$\tilde{s}_{0} \leftarrow-\tilde{S}_{0,1}$
for $j=1,2, \ldots, m$ do /* Backward sweep */
Let $i$ be s.t. $(i, j) \in \mathcal{E}$
$\tilde{v}_{j} \leftarrow \tilde{v}_{i}+\left|z_{i, j}\right|^{2} \tilde{\ell}_{i, j}-2 \operatorname{Re}\left(z_{i, j}^{*} \tilde{S}_{i, j}\right)$
return $\tilde{F}$

```

\section*{B Proof of Lemma 4}

Proof: The analysis follows the same lines as in (Gan et al., 2015; Low, 2014b; Huang et al., 2017) and is sketched here for completeness. Let \(F^{\prime \prime}=\left(s_{0}^{\prime \prime}, x^{\prime}, v^{\prime \prime}, \ell^{\prime \prime}, S^{\prime \prime}\right)\) be an optimal solution of \(\operatorname{COPF}\left[x^{\prime}\right]\), which can be found (to within any desired accuracy) in polynomial time, by solving a convex program. Consider the following problem.
\[
\begin{align*}
\left(\mathrm{COPF}^{\prime}\left[x^{\prime}\right]\right) & \min _{s_{0}, x, v, \ell, S} \sum_{e \in \mathcal{E}} \ell_{e} \\
\text { s.t. } & (29)-(34),(36),(37) \\
& x=x^{\prime}  \tag{60}\\
& f_{\mathrm{OPF}}\left(s_{0}, x\right) \geq f_{\mathrm{OPF}}\left(s_{0}^{\prime \prime}, x^{\prime}\right) \tag{61}
\end{align*}
\]

Clearly, \(\mathrm{COPF}^{\prime}\left[x^{\prime}\right]\) is feasible as \(F^{\prime \prime}\) satisfies all its constraints. Hence, it has an optimal solution \(F^{\prime}=\left(s_{0}^{\prime}, x^{\prime}, v^{\prime}, \ell^{\prime}, S^{\prime}\right)\), which we claim satisfies the statement of the lemma. Suppose, for the sake of contradiction, that there exists an edge \((h, t)\) such that \(\ell_{h, t}^{\prime}>\frac{\left|S_{h, t}^{\prime}\right|^{2}}{v_{h}^{\prime}}\). In the sequel, we construct a feasible solution \(\tilde{F}=\left(\tilde{s}_{0}, x^{\prime}, \tilde{v}, \tilde{\ell}, \tilde{S}\right)\) for \(\operatorname{COPF}^{\prime}\left[x^{\prime}\right]\) such that \(\sum_{e \in \mathcal{E}} \tilde{\ell}_{e}<\sum_{e \in \mathcal{E}} \ell_{e}^{\prime}\), leading to a contradiction.

Apply the forward-backward sweep algorithm, illustrated in Alg. 4, on the solution \(F^{\prime}\) to obtain a feasible solution \(\tilde{F}\).

We show the feasibility of the solution \(\tilde{F}\). By Steps 6,7 and 10 of Alg. 4, all equality constraints of \(\mathrm{COPF}^{\prime}\left[x^{\prime}\right]\) are satisfied. By Step 5 and the feasibility of \(F^{\prime}\), we also have
\[
\begin{equation*}
\tilde{\ell}_{e} \leq \ell_{e}^{\prime} \leq \bar{\ell}_{e} \text { for all } e \in \mathcal{E} \tag{62}
\end{equation*}
\]

Next, by rewriting \(\tilde{S}_{i, j}\), recursively substituting from the leaves, we get
\[
\begin{equation*}
\tilde{S}_{i, j}=\sum_{k \in \mathcal{N}_{j}} s_{k} \tilde{x}_{k}+\sum_{e \in \mathcal{E}_{j} \cup\{(i, j)\}} z_{e} \tilde{\ell}_{e} \tag{63}
\end{equation*}
\]

Write \(\Delta \ell_{e} \triangleq \tilde{\ell}_{e}-\ell_{e}^{\prime} \leq 0, \Delta S_{e} \triangleq \tilde{S}_{e}-S_{e}^{\prime}\), and \(\Delta\left|S_{e}\right|^{2} \triangleq\left|\tilde{S}_{e}\right|^{2}-\left|S_{e}^{\prime}\right|^{2}\), for \(e \in \mathcal{E}\). Let \(\widehat{S}_{j} \triangleq \sum_{k \in \mathcal{N}_{j}} s_{k} x_{k}^{\prime}, \tilde{L}_{i, j} \triangleq \sum_{e \in \mathcal{E}_{j} \cup\{(i, j)\}} z_{e} \tilde{\ell}_{e}\), and \(L_{i, j}^{\prime} \triangleq \sum_{e \in \mathcal{E}_{j} \cup\{(i, j)\}} z_{e} \ell_{e}^{\prime}\). Note by (63) that \(\tilde{S}_{i, j}=\widehat{S}_{j}+\tilde{L}_{i, j}\) and, similarly, \(S_{i, j}=\widehat{S}_{j}+L_{i, j}^{\prime}\). It follows that, for all \((i, j) \in \mathcal{E}\),
\[
\begin{equation*}
\Delta S_{i, j}=\tilde{L}_{i, j}-L_{i, j}^{\prime}=\sum_{e \in \mathcal{E}_{j} \cup\{(i, j)\}} z_{e} \Delta \ell_{e} \leq 0, \tag{64}
\end{equation*}
\]
where the inequality follows by assumption A 1 . In particular, for \((i, j)=(0,1)\), we obtain
\[
\begin{equation*}
\tilde{s}_{0}^{\mathrm{R}}=-\tilde{S}_{0,1}^{\mathrm{R}} \geq-S_{0,1}^{\prime \mathrm{R}}=s_{0}^{\prime \mathrm{R}} \tag{65}
\end{equation*}
\]
implying by A 0 that \(f_{0}\left(\tilde{s}_{0}^{\mathrm{R}}\right) \geq f_{0}\left(s_{0}^{\prime \mathrm{R}}\right)\) and hence (61) is satisfied.
Furthermore,
\[
\begin{align*}
\Delta\left|S_{i, j}\right|^{2}= & \left|\tilde{S}_{i, j}\right|^{2}-\left|S_{i, j}^{\prime}\right|^{2}  \tag{66}\\
= & \left(\tilde{S}_{i, j}^{\mathrm{R}}\right)^{2}-\left(S_{i, j}^{\prime \mathrm{R}}\right)^{2}+\left(\tilde{S}_{i, j}^{\mathrm{I}}\right)^{2}-\left(S_{i, j}^{\prime \mathrm{I}}\right)^{2}  \tag{67}\\
= & \Delta S_{i, j}^{\mathrm{R}}\left(\tilde{S}_{i, j}^{\mathrm{R}}+S_{i, j}^{\prime \mathrm{R}}\right)+\Delta S_{i, j}^{\mathrm{I}}\left(\tilde{S}_{i, j}^{\mathrm{I}}+S_{i, j}^{\prime \mathrm{I}}\right)  \tag{68}\\
= & \sum_{e \in \mathcal{E}_{j} \cup\{(i, j)\}} z_{e}^{\mathrm{R}} \Delta \ell_{e}\left(2 \widehat{S}_{j}^{\mathrm{R}}+\tilde{L}_{i, j}^{\mathrm{R}}+L_{i, j}^{\prime \mathrm{R}}\right) \\
& +\sum_{e \in \mathcal{E}_{j} \cup\{(i, j)\}} z_{e}^{\mathrm{I}} \Delta \ell_{e}\left(2 \widehat{S}_{j}^{\mathrm{I}}+\tilde{L}_{i, j}^{\mathrm{I}}+L_{i, j}^{\prime \mathrm{I}}\right)  \tag{69}\\
= & \sum_{e \in \mathcal{E}_{j} \cup\{(i, j)\}} 2 \Delta \ell_{e} \operatorname{Re}\left(z_{e}^{*} \widehat{S}_{j}\right)+\sum_{e \in \mathcal{E}_{j} \cup\{(i, j)\}} \Delta \ell_{e} \operatorname{Re}\left(z_{e}^{*} \tilde{L}_{i, j}\right) \\
& +\sum_{e \in \mathcal{E}_{j} \cup\{(i, j)\}} \Delta \ell_{e} \operatorname{Re}\left(z_{e}^{*} L_{i, j}^{\prime}\right) \leq 0, \tag{70}
\end{align*}
\]
where Eqn. (70) follows by A1, A3 (or A4') and \(\Delta \ell_{e} \leq 0\). Therefore, by the feasibility of \(S_{e}\),
\[
\begin{equation*}
\left|\tilde{S}_{e}\right| \leq\left|S_{e}^{\prime}\right| \leq \bar{S}_{e} \quad \text { for all } e \in \mathcal{E} \tag{71}
\end{equation*}
\]

Note that, by \(\underset{\tilde{S}}{ } 1\), the inequalities in (71) also imply that the reverse power constraint in (33) is satisfied for \(\tilde{S}\).

Rewrite Cons. (31) by recursively substituting \(\tilde{v}_{j}\), for \(j\) moving away from the root, and then substituting for \(\tilde{S}_{h, t}\) using (63):
\[
\begin{align*}
\tilde{v}_{j}= & v_{0}-2 \sum_{(h, t) \in \mathcal{P}_{j}} \operatorname{Re}\left(z_{h, t}^{*} \tilde{S}_{h, t}\right)+\sum_{(h, t) \in \mathcal{P}_{j}}\left|z_{h, t}\right|^{2} \tilde{\ell}_{h, t} \\
= & v_{0}-2 \sum_{(h, t) \in \mathcal{P}_{j}} \operatorname{Re}\left(z_{h, t}^{*}\left(\sum_{k \in \mathcal{N}_{t}} s_{k} \tilde{x}_{k}+\sum_{e \in \mathcal{E}_{t} \cup\{(h, t)\}} z_{e} \tilde{\ell}_{e}\right)\right)+\sum_{(h, t) \in \mathcal{P}_{j}}\left|z_{h, t}\right|^{2} \tilde{\ell}_{h, t}, \\
= & v_{0}-2 \sum_{k \in \mathcal{N}} \operatorname{Re}\left(\sum_{(h, t) \in \mathcal{P}_{k} \cap \mathcal{P}_{j}} z_{h, t}^{*} s_{k}\right) \tilde{x}_{k}-2 \sum_{(h, t) \in \mathcal{P}_{j}} \operatorname{Re}\left(z_{h, t}^{*} \sum_{e \in \mathcal{E}_{t}} z_{e} \tilde{\ell}_{e}\right) \\
& -2 \sum_{(h, t) \in \mathcal{P}_{j}}\left|z_{h, t}\right|^{2} \tilde{\ell}_{h, t}+\sum_{(h, t) \in \mathcal{P}_{j}}\left|z_{h, t}\right|^{2} \tilde{\ell}_{h, t} \tag{72}
\end{align*}
\]
where the last statement follows from exchanging the summation operators, and \(z_{e}^{*} z_{e}=\left|z_{e}\right|^{2}\). Thus,
\[
\begin{align*}
\tilde{v}_{j}= & v_{0}-2 \sum_{k \in \mathcal{N}} \operatorname{Re}\left(\sum_{(h, t) \in \mathcal{P}_{k} \cap \mathcal{P}_{j}} z_{h, t}^{*} s_{k}\right) \tilde{x}_{k} \\
& -\left(2 \sum_{(h, t) \in \mathcal{P}_{j}} \operatorname{Re}\left(z_{h, t}^{*} \sum_{e \in \mathcal{E}_{t}} z_{e} \tilde{\ell}_{e}\right)+\sum_{(h, t) \in \mathcal{P}_{j}}\left|z_{h, t}\right|^{2} \tilde{\ell}_{h, t}\right) \\
\leq & v_{0}<\bar{v}_{j} \tag{73}
\end{align*}
\]
where the first inequality follows by A 1 and A 3 , and the last inequality follows by A2. Since \(\tilde{\ell}_{e} \leq \ell_{e}^{\prime}\) and \(\tilde{x}=x^{\prime}\), we get by A1 and the feasibility of \(F^{\prime}\),
\[
\begin{align*}
\tilde{v}_{j} \geq & v_{0}-2 \sum_{k \in \mathcal{N}} \operatorname{Re}\left(\sum_{(h, t) \in \mathcal{P}_{k} \cap \mathcal{P}_{j}} z_{h, t}^{*} s_{k}\right) x_{k}^{\prime} \\
& -\left(2 \sum_{(h, t) \in \mathcal{P}_{j}} \operatorname{Re}\left(z_{h, t} \sum_{e \in \mathcal{E}_{t}} z_{e} \ell_{e}\right)+\sum_{(h, t) \in \mathcal{P}_{j}}\left|z_{h, t}\right|^{2} \ell_{h, t}\right) \\
= & v_{j}^{\prime} \geq \underline{v}_{j} . \tag{74}
\end{align*}
\]

By Ineqs. (71) and (74), \(\tilde{\ell}_{i, j}=\frac{\left|S_{i, j}^{\prime}\right|^{2}}{v_{i}^{\prime}} \geq \frac{\left|\tilde{S}_{i, j}\right|^{2}}{\tilde{v}_{i}}\), hence, \(\tilde{F}\) is feasible.
Finally, by the first inequality in (62) and the fact that \(\ell_{h, t}^{\prime}>\frac{\left|S_{h, t}\right|^{2}}{v_{h}}=\tilde{\ell}_{h, t}\), we have \(\sum_{e \in \mathcal{E}} \tilde{\ell}_{e}<\sum_{e \in \mathcal{E}} \ell_{e}^{\prime}\), contradicting the optimality of \(F^{\prime}\) for \(\operatorname{COPF}^{\prime}\left[x^{\prime}\right]\).

\section*{C Proof of Lemma 5}

Proof: The argument is similar to that in Lemma 4. We apply a slightly modified version of Alg. 4 on the solution \(F^{\prime}\) to obtain a feasible solution \(F\). Replace steps 1 and 5 in Alg. 4, respectively, by:
\[
\begin{equation*}
1: \tilde{x} \leftarrow \bar{x} ; \tilde{v}_{0} \leftarrow v_{0}, \text { and } 5: \tilde{\ell}_{i, j} \leftarrow \ell_{i, j}^{\prime} \tag{75}
\end{equation*}
\]

By Steps 6, 7 and 10 of the (modified) algorithm, all equality constraints of ( \(\mathrm{RCOPF}[\bar{x}]\) ) are satisfied. By (modified) Step 5 and the feasibility of \(F^{\prime}\), we also have
\[
\begin{equation*}
\tilde{\ell}_{e}=\ell_{e}^{\prime} \leq \bar{\ell}_{e} \text { for all } e \in \mathcal{E} \tag{76}
\end{equation*}
\]

Write \(\Delta S_{e} \triangleq \tilde{S}_{e}-S_{e}^{\prime}\), and \(\Delta\left|S_{e}\right|^{2} \triangleq\left|\tilde{S}_{e}\right|^{2}-\left|S_{e}^{\prime}\right|^{2}\), for \(e \in \mathcal{E}\). Let \(S_{j}^{\prime} \triangleq \sum_{k \in \mathcal{N}_{j}} s_{k} x_{k}^{\prime}\), \(\tilde{S}_{j} \triangleq\) \(\sum_{k \in \mathcal{N}_{j}} s_{k} \tilde{x}_{k}\), and \(\tilde{L}_{i, j} \triangleq \sum_{e \in \mathcal{E}} \cup\{(i, j)\} z_{e} \tilde{\ell}_{e}\). Note by (63) that \(\tilde{S}_{i, j}=\tilde{S}_{j}+\tilde{L}_{i, j}\) and, \(S_{i, j}^{\prime}=\) \(S_{j}^{\prime}+\tilde{L}_{i, j}\). It follows that, for all \((i, j) \in \mathcal{E}\),
\[
\begin{equation*}
\Delta S_{i, j}=\tilde{S}_{j}-S_{j}^{\prime}=\sum_{k \in \mathcal{N}_{j}} s_{k} \bar{x}_{k}-\sum_{k \in \mathcal{N}_{j}} s_{k} x_{k}^{\prime} \leq 0 \tag{77}
\end{equation*}
\]
where the inequality follows from (40) and (41). In particular, for \((i, j)=(0,1)\), we obtain
\[
\begin{equation*}
\tilde{s}_{0}^{\mathrm{R}}=-\tilde{S}_{0,1}^{\mathrm{R}} \geq-S_{0,1}^{\prime \mathrm{R}}=s_{0}^{\prime \mathrm{R}} \tag{78}
\end{equation*}
\]
implying by \(\mathrm{A}^{\prime}\) that \(\left.f_{0}\left(\tilde{s}_{0}^{\mathrm{R}} \cos \phi+\tilde{s}_{0}^{\mathrm{I}} \sin \phi\right) \geq f_{0}\left(s_{0}^{\prime \mathrm{R}} \cos \phi+s_{0}^{\prime \mathrm{I}} \sin \phi\right)\right)\) and hence \(f_{\mathrm{OPF}}\left(\tilde{s}_{0}, \tilde{x}\right) \geq\) \((1-\varepsilon) f_{\text {OPF }}\left(s_{0}^{\prime}, x^{\prime}\right)\) follows from (38) and (42).

Furthermore,
\[
\begin{aligned}
\Delta\left|S_{i, j}\right|^{2} & =\left|\tilde{S}_{i, j}\right|^{2}-\left|S_{i, j}^{\prime}\right|^{2} \\
& =\left(\tilde{S}_{i, j}^{\mathrm{R}}\right)^{2}-\left(S_{i, j}^{\mathrm{R}}\right)^{2}+\left(\tilde{S}_{i, j}^{\mathrm{I}}\right)^{2}-\left(S_{i, j}^{\prime \mathrm{I}}\right)^{2} \\
& =\Delta S_{i, j}^{\mathrm{R}}\left(\tilde{S}_{i, j}^{\mathrm{R}}+S_{i, j}^{\prime \mathrm{R}}\right)+\Delta S_{i, j}^{\mathrm{I}}\left(\tilde{S}_{i, j}^{\mathrm{I}}+S_{i, j}^{\prime \mathrm{I}}\right) \\
& =\Delta S_{i, j}^{\mathrm{R}}\left(\tilde{S}_{j}^{\mathrm{R}}+S_{j}^{\prime \mathrm{R}}+2 \tilde{L}_{i, j}^{\mathrm{R}}\right)+\Delta S_{i, j}^{\mathrm{I}}\left(\tilde{S}_{j}^{\mathrm{I}}+S_{j}^{\prime \mathrm{I}}+2 \tilde{L}_{i, j}^{\mathrm{I}}\right) \leq 0,
\end{aligned}
\]
where the last inequality follows by \(\mathrm{A} 1, \mathrm{~A} 4^{\prime}\) and (77). Therefore,
\[
\begin{equation*}
\left|\tilde{S}_{i, j}\right| \leq\left|S_{i, j}^{\prime}\right| \leq \bar{S}_{i, j} \tag{79}
\end{equation*}
\]

Next, we show \(\underline{v}_{j} \leq \tilde{v}_{j} \leq \bar{v}_{j}\). As in (72), rewrite Cons. (31) by recursively substituting \(v_{j}^{\prime}\), for \(j\) moving away from the root, and then substituting for \(\tilde{S}_{h, t}\) using (63):
\[
\begin{align*}
v_{j}^{\prime}= & v_{0}-2 \sum_{k \in \mathcal{N}} \operatorname{Re}\left(\sum_{(h, t) \in \mathcal{P}_{k} \cap \mathcal{P}_{j}} z_{h, t}^{*} s_{k}\right) x_{k}^{\prime} \\
& -\left(2 \sum_{(h, t) \in \mathcal{P}_{j}} \operatorname{Re}\left(z_{h, t}^{*} \sum_{e \in \mathcal{E}_{t}} z_{e} \ell_{e}^{\prime}\right)+\sum_{(h, t) \in \mathcal{P}_{j}}\left|z_{h, t}\right|^{2} \ell_{h, t}^{\prime}\right) \tag{80}
\end{align*}
\]

A similar equation can be derived for \(\tilde{v}_{j}\), where \(x^{\prime}\) and \(\ell^{\prime}\) in (80) are replaced by \(\tilde{x}\) and \(\tilde{\ell}\), respectively. By assumptions A2 and A3, we have
\[
\begin{aligned}
& \tilde{v}_{j}= v_{0} \\
&-2 \sum_{k \in \mathcal{N}} \operatorname{Re}\left(\sum_{(h, t) \in \mathcal{P}_{k} \cap \mathcal{P}_{j}} z_{h, t}^{*} s_{k}\right) \tilde{x}_{k} \\
&-\left(2 \sum_{(h, t) \in \mathcal{P}_{j}} \operatorname{Re}\left(z_{h, t}^{*} \sum_{e \in \mathcal{E}_{t}} z_{e} \tilde{\ell}_{e}\right)+\sum_{(h, t) \in \mathcal{P}_{j}}\left|z_{h, t}\right|^{2} \tilde{\ell}_{h, t}\right) \\
& \leq v_{0}<\bar{v}_{j} .
\end{aligned}
\]

Moreover, since \(\tilde{\ell}_{e}=\ell_{e}^{\prime}\) and \(\tilde{x}=\bar{x}\) satisfies (39), we get by A1 and the feasibility of \(F^{\prime}\),
\[
\begin{align*}
\tilde{v}_{j} \geq & v_{0}-2 \sum_{k \in \mathcal{N}} \operatorname{Re}\left(\sum_{(h, t) \in \mathcal{P}_{k} \cap \mathcal{P}_{j}} z_{h, t}^{*} s_{k}\right) x_{k}^{\prime} \\
& -\left(2 \sum_{(h, t) \in \mathcal{P}_{j}} \operatorname{Re}\left(z_{h, t} \sum_{e \in \mathcal{E}_{t}} z_{e} \ell_{e}^{\prime}\right)+\sum_{(h, t) \in \mathcal{P}_{j}}\left|z_{h, t}\right|^{2} \ell_{h, t}^{\prime}\right) \\
= & v_{j}^{\prime} \geq \underline{v}_{j} . \tag{81}
\end{align*}
\]

Finally, by inequalities (79) and (81), \(\tilde{\ell}_{i, j}=\ell_{i, j}^{\prime}=\frac{\left|S_{i, j}^{\prime}\right|^{2}}{v_{i}^{\prime}} \geq \frac{\left|\tilde{S}_{i, j}\right|^{2}}{\tilde{v}_{i}}\), hence \(\tilde{\ell}_{i, j}\) satisfies Cons. (37).```


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[^2]:    1 Note that, for $\forall r \in\{1, \ldots, d\}$, the capacity function $c^{r}(\cdot)$ adheres to this form trivially with $T_{r}=a^{r, t}=1, b^{r, t}(\cdot)=c^{r}(\cdot)$ for $\forall t$ and $e^{r}=e_{1}, \hat{e}^{r}=e_{m}$.

[^3]:    ${ }^{2}$ For clarity, it is assumed in (9) that the strict inequality is replaced by an inequality when $a_{k}^{r, t}=0$.

[^4]:    3 This inequality is captured by the first constraint appearing in the subproblem, and thus can be extracted in $O(1)$ time.

[^5]:    ${ }^{4}$ Traditionally, the objective is to minimize the generation $\operatorname{cost} c\left(S_{0,1}^{\mathrm{R}}\right)$, which is typically a non-decreasing convex function of the active generation power $S_{0,1}^{\mathrm{R}}$. In the discrete demand case under study, we combine the minimization of the generation cost with the utility maximization of the satisfied demands by using the function $f_{\text {OPF }}\left(s_{0}, x\right)$, where $f_{0}\left(s_{0}^{\mathrm{R}}\right) \triangleq Y-c\left(S_{0,1}^{\mathrm{R}}\right)=Y-$ $c\left(-s_{0}^{\mathrm{R}}\right)$ ), for a sufficiently large number $Y$, is a nonnegative concave function, non-decreasing in $\left.s_{0}^{\mathrm{R}}\right)$.

[^6]:    ${ }^{5}$ Note that Cons. (37) can be rewritten as

    $$
    \left\|\left(\begin{array}{c}
    2 S_{i, j}^{\mathrm{R}} \\
    2 S_{i, j}^{I} \\
    \ell_{i, j}-v_{i}
    \end{array}\right)\right\|_{2} \leq \ell_{i, j}+v_{i}
    $$

