A Zariski Topology for Bicomodules and Corings*

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Abstract

In this paper we introduce and investigate *top (bi)comodules* of corings, that can be considered as dual to top (bi)modules of rings. The *fully coprime spectra* of such (bi)comodules attains a *Zariski topology*, defined in a way dual to that of defining the Zariski topology on the prime spectra of (commutative) rings. We restrict our attention in this paper to duo (bi)comodules (satisfying suitable conditions) and study the interplay between the coalgebraic properties of such (bi)comodules and the introduced Zariski topology. In particular, we apply our results to introduce a Zariski topology on the fully coprime spectrum of a given non-zero coring considered canonically as duo object in its category of bicomodules.

1 Introduction

Several papers considered the so called *top modules,* i.e. modules (over commutative rings) whose spectrum of *prime submodules* attains a Zariski topology, e.g. [Lu1999], [MMS1997], [Zha1999]. Dually, we introduce and investigate *top (bi)comodules* for corings and study their properties (restricting our attention in this first paper to duo (bi)comodules satisfying suitable conditions). In particular, we extend results of [NT2001] on the topology defined on the spectrum of *(fully) coprime subcoalgebras* of a given coalgebra over a base field to the general situation of a topology on the *fully coprime spectrum* of a given non-zero bicomodule over a given pair of non-zero corings.

Throughout, R is a commutative ring with $1_R \neq 0_R$ and A, B are R-algebras. With *locally projective* modules, we mean those in the sense of [Z-H1976] (see also [Abu2006]). We denote by $\mathcal{C} = (\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ a non-zero A-coring with ${}_A\mathcal{C}$ flat and by $\mathcal{D} = (\mathcal{D}, \Delta_{\mathcal{D}}, \varepsilon_{\mathcal{D}})$ a non-zero B-coring with $\mathcal{D}_B$ flat, so that the categories ${}^{\mathcal{D}}\mathbb{M}^{\mathcal{C}}$ of $(\mathcal{D}, \mathcal{C})$-bicomodules, $\mathbb{M}^{\mathcal{C}}$ of right $\mathcal{C}$-comodules and ${}^{\mathcal{D}}\mathbb{M}$ of left $\mathcal{D}$-comodules are Grothendieck.

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After this brief introduction, we include in the second section some preliminaries and extend some of our results in [Abu2006] on fully coprimeness in categories of comodules to fully coprimeness in categories of bicomodules.

In the third and main section, we introduce a Zariski topology for bicomodules. Let M be a given non-zero $(\mathcal{D},\mathcal{C})$-bicomodule and consider the *fully coprime spectrum*

$$\mathrm{CPSpec}(M) := \{K \mid K \subseteq M \text{ is a } \textit{fully } M\textit{-coprime } (\mathcal{D},\mathcal{C})\text{-subbicomodule}\}.$$

For every $(\mathcal{D},\mathcal{C})$-subbicomodule $L \subseteq M$, set $\mathcal{V}_L := \{K \in \mathrm{CPSpec}(M) \mid K \subseteq L\}$ and $\mathcal{X}_L := \{K \in \mathrm{CPSpec}(M) \mid K \subsetneqq L\}$. As in the case of the spectra of prime submodules of modules over (commutative) rings (e.g. [Lu1999], [MMS1997], [Zha1999]), the class of *varieties* $\xi(M) := \{\mathcal{V}_L \mid L \subseteq M \text{ is a } (\mathcal{D},\mathcal{C})\text{-subbicomodule}\}$ satisfies all axioms of closed sets in a topological space with the exception that $\xi(M)$ is *not necessarily* closed under finite unions. We say M is a *top bicomodule*, iff $\xi(M)$ is closed under finite unions, equivalently iff $\tau_M := \{\mathcal{X}_L \mid L \subseteq M \text{ is a } (\mathcal{D},\mathcal{C})\text{-subbicomodule}\}$ is a topology (in this case we call $\mathbf{Z}_M := (\mathrm{CPSpec}(M), \tau_M)$ a *Zariski topology* of M). We then restrict our attention to the case in which M is a *duo bicomodule* (i.e. every subbicomodule of M is fully invariant) satisfying suitable conditions. For such a bicomodule M we study the interplay between the coalgebraic properties of M and the topological properties of $\mathbf{Z}_M$.

In the fourth section we give some applications and examples. Our main application will be to non-zero corings which turn out to be *duo* bicomodules in the canonical way. We also give some concrete examples that establish some of the results in section three.

It is worth mentioning that several properties of the Zariski topology for bicomodules and corings are, as one may expect, dual to those of the classical Zariski topology on the prime spectrum of commutative rings (e.g. [AM1969], [Bou1998]).

This paper is a continuation of [Abu2006]. The ideas in both papers can be transformed to investigate the notion of coprimeness in the sense of Annin [Ann2002] in categories of (bi)comodules and define a Zariski topology on the spectrum of *coprime sub*(*bi*)*comodules* of a given (bi)comodule. Moreover, different notions of primeness and coprimeness in these papers can be investigated in categories of (bi)modules over rings, which can be seen as bicomodules over the ground rings that should be considered with the trivial coring structures (a different approach has been taken in the recent work [Wij2006], where several primeness and coprimeness conditions are studied in categories of modules and then applied to categories of comodules of *locally projective* coalgebras over commutative rings). More generally, such (co)primeness notions can be developed in more general Grothendieck categories. These and other applications will be considered in forthcoming papers.

2 Preliminaries

All rings and their modules in this paper are assumed to be unital. For a ring T, we denote with $Z(T)$ the *center* of T and with T^{op} the opposite ring of T. For basic definitions and results on corings and comodules, the reader is referred to [BW2003]. A reference for the topological terminology and other results we use could be any standard book in general topology (notice that in our case, a *compact space* is not necessarily Hausdorff; such spaces are called *quasi-compact* by some authors, e.g. [Bou1966, I.9.1.]).

2.1. (e.g. [BW2003, 17.8.]) For any A-coring $\mathcal{C}$, the dual module ${}^*\mathcal{C} := \mathrm{Hom}_{A-}(\mathcal{C}, A)$ (resp. $\mathcal{C}^* := \mathrm{Hom}_{-A}(\mathcal{C}, A)$) is an A^{op}-rings with unity $\varepsilon_{\mathcal{C}}$ and multiplication

$$(f *^l g)(c) := \sum f(c_1 g(c_2)) \text{ (resp. } (f *^r g)(c) := \sum g(f(c_1)c_2)).$$

2.2. Let M be a non-zero $(\mathcal{D}, \mathcal{C})$-bicomodule. Then M is a $({}^*\mathcal{C}, \mathcal{D}^*)$-bimodule with actions

$$f \rightharpoonup m := \sum m_{<0>} f(m_{<1>}) \text{ and } m \leftharpoonup g := \sum g(m_{<-1>}) m_{<0>},\ f \in {}^*\mathcal{C},\ g \in \mathcal{D}^*,\ m \in M.$$

Moreover, the set ${}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}} := {}^{\mathcal{D}}\mathrm{End}^{\mathcal{C}}(M)^{op}$ of $(\mathcal{D}, \mathcal{C})$-bicolinear endomorphisms of M is a ring with multiplication the *opposite* composition of maps, so that M is canonically a $({}^*\mathcal{C} \otimes_R \mathcal{D}^{*op}, {}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}})$-bimodule. A $(\mathcal{D}, \mathcal{C})$-subbicomodule $L \subseteq M$ is called *fully invariant,* iff it is a right ${}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$-submodule as well. We call $M \in {}^{\mathcal{D}}\mathbb{M}^{\mathcal{C}}$ *duo* (*quasi-duo*), iff every (simple) $(\mathcal{D}, \mathcal{C})$-subbicomodule of M is fully invariant. If ${}_A\mathcal{C}$ and $\mathcal{D}_B$ are locally projective, then ${}^{\mathcal{D}}\mathbb{M}^{\mathcal{C}} \simeq {}^{\mathcal{D}}\mathrm{Rat}^{\mathcal{C}}({}_{(\mathcal{D}^*)^{op}}\mathbb{M}_{({}^*\mathcal{C})^{op}}) = {}^{\mathcal{D}}\mathrm{Rat}^{\mathcal{C}}({}_{{}^*\mathcal{C}}\mathbb{M}_{\mathcal{D}^*})$ (the category of $(\mathcal{D}, \mathcal{C})$-*birational* $({}^*\mathcal{C}, \mathcal{D}^*)$-*bimodules*, e.g. [Abu2003, Theorem 2.17.]).

Notation. Let M be a $(\mathcal{D}, \mathcal{C})$-bicomodule. With $\mathcal{L}(M)$ (resp. $\mathcal{L}_{f.i.}(M)$) we denote the lattice of (fully invariant) $(\mathcal{D}, \mathcal{C})$-subbicomodules of M and with $\mathcal{I}_r({}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}})$ (resp. $\mathcal{I}({}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}})$) the lattice of right (two-sided) ideals of ${}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$. With $\mathcal{I}_r^{f.g.}({}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}) \subseteq \mathcal{I}_r({}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}})$ (resp. $\mathcal{L}^{f.g.}(M) \subseteq \mathcal{L}(M)$) we denote the subclass of finitely generated right ideals of ${}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$ (the subclass of $(\mathcal{D}, \mathcal{C})$-subbicomodules of M which are finitely generated as (B, A)-bimodules). For $\varnothing \neq K \subseteq M$ and $\varnothing \neq I \subseteq {}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$ we set

$$\mathrm{An}(K) := \{f \in {}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}} \mid f(K) = 0\} \text{ and } \mathrm{Ke}(I) := \{m \in M \mid f(m) = 0 \text{ for all } f \in I\}.$$

In what follows we introduce some notions for an object in ${}^{\mathcal{D}}\mathbb{M}^{\mathcal{C}}$:

Definition 2.3. We say that a non-zero $(\mathcal{D}, \mathcal{C})$-bicomodule M is

self-injective, iff for every $(\mathcal{D}, \mathcal{C})$-subbicomodule $K \subseteq M$, every $f \in {}^{\mathcal{D}}\mathrm{Hom}^{\mathcal{C}}(K, M)$ extends to some $(\mathcal{D}, \mathcal{C})$-bicolinear endomorphism $\widetilde{f} \in {}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$;

self-cogenerator, iff M cogenerates M/K in ${}^{\mathcal{D}}\mathbb{M}^{\mathcal{C}}$ $\forall$ $(\mathcal{D}, \mathcal{C})$-subbicomodule $K \subseteq M$;

intrinsically injective, iff $\mathrm{AnKe}(I) = I$ for every finitely generated right ideal $I \lhd_r {}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$.

simple, iff M has no non-trivial $(\mathcal{D}, \mathcal{C})$-subbicomodules;

subdirectly irreducible[1], iff M contains a unique simple $(\mathcal{D}, \mathcal{C})$-subbicomodule that is contained in every non-zero $(\mathcal{D}, \mathcal{C})$-subbicomodule of M (equivalently, iff $\bigcap_{0 \neq K \in \mathcal{L}(M)} \neq 0$).

semisimple, iff $M = \mathrm{Corad}(M)$, where $\mathrm{Corad}(M) := \sum\{K \subseteq M \mid K$ is a simple $(\mathcal{D}, \mathcal{C})$-subbicomodule$\}$ $(:= 0$, if M has no simple $(\mathcal{D}, \mathcal{C})$-subbicomodules).

Notation. Let M be a non-zero $(\mathcal{D}, \mathcal{C})$-bicomodule. We denote with $\mathcal{S}(M)$ ($\mathcal{S}_{f.i.}(M)$) the class of simple $(\mathcal{D}, \mathcal{C})$-subbicomodules of M (non-zero fully invariant $(\mathcal{D}, \mathcal{C})$-subbicomodules of M with no non-trivial fully invariant $(\mathcal{D}, \mathcal{C})$-subbicomodules). Moreover, we denote with $\mathrm{Max}_r({}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}})$ ($\mathrm{Max}({}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}})$) the class of maximal right (two-sided) ideals of ${}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$. The Jacobson radical (prime radical) of ${}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$ is denoted by $\mathrm{Jac}({}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}})$ ($\mathrm{Prad}({}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}})$).

[1] *Subdirectly irreducible* comodules were called *irreducible* in [Abu2006]. However, we observed that such a terminology may cause confusion, so we choose to change it in this paper to be consistent with the terminology used for modules (e.g. [Wis1991, 9.11., 14.8.]).

2.4. Let M be a non-zero $(\mathcal{D},\mathcal{C})$-bicomodule. We M has *Property* $\mathbf{S}$ $(\mathbf{S}_{f.i.})$, iff $\mathcal{S}(L)\neq\varnothing$ $(\mathcal{S}_{f.i.}(L)\neq\varnothing)$ for every (fully invariant) non-zero $(\mathcal{D},\mathcal{C})$-subbicomodule $0\neq L\subseteq M$. Notice that if M has $\mathbf{S}$, then M is subdirectly irreducible if and only if $L_1\cap L_2\neq 0$ for any two non-zero $(\mathcal{D},\mathcal{C})$-subbicomodules $0\neq L_1,L_2\subseteq M$.

Lemma 2.5. *Let M be a non-zero $(\mathcal{D},\mathcal{C})$-bicomodule. If $B\otimes_R A^{op}$ is left perfect and ${}_A\mathcal{C}$, $\mathcal{D}_B$ are locally projective, then*

1. *every finite subset of M is contained in a $(\mathcal{D},\mathcal{C})$-subbicomodule $L\subseteq M$ that is finitely generated as a (B,A)-bimodule.*

2. *every non-zero $(\mathcal{D},\mathcal{C})$-subbicomodule $0\neq L\subseteq M$ has a simple $(\mathcal{D},\mathcal{C})$-subbicomodule, so that M has Property $\mathbf{S}$. If moreover, M is quasi-duo, then M has Property $\mathbf{S}_{f.i.}$.*

3. $\mathrm{Corad}(M)\subseteq^e M$ *(an essential $(\mathcal{D},\mathcal{C})$-subbicomodule).*

Proof. 1. It's enough to show the assertion for a single element $m\in M$. Let $\varrho_M^{\mathcal{C}}(m)=\sum\limits_{i=1}^{n}m_i\otimes_A c_i$ and $\varrho_M^{\mathcal{D}}(m_i)=\sum\limits_{j=1}^{k_i}d_{i,j}\otimes_B m_{ij}$ for each $i=1,...,n$. Since ${}_A\mathcal{C}$, $\mathcal{D}_B$ are locally projective, the $({}^*\mathcal{C},\mathcal{D}^*)$-subbimodule $L:={}^*\mathcal{C}\rightharpoonup m\leftharpoonup\mathcal{D}^*\subseteq M$ is by [Abu2003, Theorem 2.17.] a $(\mathcal{D},\mathcal{C})$-subbicomodule. Moreover, $\{m_{i,j}\mid 1\leq i\leq n,\ 1\leq j\leq k_i\}$ generates ${}_BL_A$, since

$$f\rightharpoonup m\leftharpoonup g=[\sum_{i=1}^{n}m_i f(c_i)]\leftharpoonup g=\sum_{i=1}^{n}\sum_{j=1}^{k_i}g(d_{i,j})m_{i,j}f(c_i)\ \forall\ f\in\ {}^*\mathcal{C}\text{ and }g\in\mathcal{D}^*.$$

2. Suppose $0\neq L\subseteq M$ is a $(\mathcal{D},\mathcal{C})$-subbicomodule with no simple $(\mathcal{D},\mathcal{C})$-subbicomodules. By "1", L contains a non-zero $(\mathcal{D},\mathcal{C})$-subbicomodule $0\neq L_1\subsetneqq M$ that is finitely generated as a (B,A)-bimodule. Since L contains no simple $(\mathcal{D},\mathcal{C})$-subbicomodules, for every $n\in\mathbb{N}$ we can pick (by induction) a non-zero $(\mathcal{D},\mathcal{C})$-subbicomodule $0\neq L_{n+1}\subsetneqq L_n$ that is finitely generated as a $B\otimes_R A^{op}$-module. In this way we obtain an infinite chain $L_1\supsetneqq L_2\supsetneqq...\supsetneqq L_n\supsetneqq L_{n+1}\supsetneqq....$ of finitely generated $B\otimes_R A^{op}$-submodules of L (a contradiction to the assumption that $B\otimes_R A^{op}$ is left perfect, see [Fai1976, Theorem 22.29]). Consequently, L should contain at least one simple $(\mathcal{D},\mathcal{C})$-subbicomodule. Hence M has property $\mathbf{S}$. The last statement is obvious.

3. For every non-zero $(\mathcal{D},\mathcal{C})$-subbicomodule $0\neq L\subseteq M$, we have by "1" $L\cap\mathrm{Corad}(M)=\mathrm{Corad}(L)\neq 0$, hence $\mathrm{Corad}(M)\subseteq^e M$.■

Given a non-zero $(\mathcal{D},\mathcal{C})$-bicomodule M, we have the following annihilator conditions. The proofs are similar to the corresponding results in [Wis1991, 28.1.], hence omitted:

2.6. Let M be a non-zero $(\mathcal{D},\mathcal{C})$-bicomodule and consider the order-reversing mappings

$$\mathrm{An}(-):\mathcal{L}(M)\to\mathcal{I}_r({}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}})\text{ and }\mathrm{Ke}(-):\mathcal{I}_r({}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}})\to\mathcal{L}(M).\qquad(1)$$

1. $\mathrm{An}(-)$ and $\mathrm{Ke}(-)$ restrict to order-reversing mappings

$$\mathrm{An}(-) : \mathcal{L}_{f.i.}(M) \to \mathcal{I}({}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}) \text{ and } \mathrm{Ke}(-) : \mathcal{I}({}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}) \to \mathcal{L}_{f.i.}(M). \tag{2}$$

2. For a $(\mathcal{D},\mathcal{C})$-subbicomodule $K \subseteq M$: $\mathrm{Ke}(\mathrm{An}(K)) = K$ if and only if M/K is M-cogenerated. So, if M is self-cogenerator, then the map $\mathrm{An}(-)$ in (1) and its restriction in (2) are injective.

3. If M is self-injective, then

   (a) $\mathrm{An}(\bigcap\limits_{i=1}^{n} K_i) = \sum\limits_{i=1}^{n} \mathrm{An}(K_i)$ for any $(\mathcal{D},\mathcal{C})$-subbicomodules $K_1, ..., K_n \subseteq M$ (i.e. $\mathrm{An}(-)$ in (1) and its restriction in (2) are lattice anti-morphisms).

   (b) M is intrinsically injective.

Remarks 2.7. Let M be a non-zero $(\mathcal{D},\mathcal{C})$-bicomodule. If M is self-cogenerator and ${}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$ is *right-duo* (i.e. every right ideal is a two-sided ideal), then M is duo. On the otherhand, if M is intrinsically injective and M is duo, then ${}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$ is right-duo. If M is self-injective and duo, then every fully invariant $(\mathcal{D},\mathcal{C})$-subbicomodule of M is also duo.

Fully coprime (fully cosemiprime) bicomodules

2.8. Let M be a non-zero $(\mathcal{D},\mathcal{C})$-bicomodule. For any R-submodules $X, Y \subseteq M$ we set

$$(X \overset{(\mathcal{D},\mathcal{C})}{:_M} Y) := \bigcap\{f^{-1}(Y) \mid f \in \mathrm{An}_{{}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}}(X)\} = \bigcap_{f \in \mathrm{An}(X)} \{\mathrm{Ker}(\pi_Y \circ f : M \to M/Y)\}.$$

If $Y \subseteq M$ is a $(\mathcal{D},\mathcal{C})$-subbicomodule (and $f(X) \subseteq X$ for all $f \in {}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$), then $(X \overset{(\mathcal{D},\mathcal{C})}{:_M} Y) \subseteq M$ is a (fully invariant) $(\mathcal{D},\mathcal{C})$-subbicomodule. If $X, Y \subseteq M$ are $(\mathcal{D},\mathcal{C})$-subbicomodules, then we call $(X \overset{(\mathcal{D},\mathcal{C})}{:_M} Y) \subseteq M$ the *internal coproduct* of X and Y in M.

Lemma 2.9. *Let $X, Y \subseteq M$ be any R-submodules. Then*

$$(X :_M^{\mathcal{C}} Y) \subseteq \mathrm{Ke}(\mathrm{An}(X) \circ^{op} \mathrm{An}(Y)), \tag{3}$$

with equality in case M is self-cogenerator and $Y \subseteq M$ is a $(\mathcal{D},\mathcal{C})$-subbicomodule.

Definition 2.10. Let M be a non-zero $(\mathcal{D},\mathcal{C})$-bicomodule. We call a non-zero fully invariant $(\mathcal{D},\mathcal{C})$-subbicomodule $0 \neq K \subseteq M$:

fully M-coprime, iff for any fully invariant $(\mathcal{D},\mathcal{C})$-subbicomodules $X, Y \subseteq M$ with $K \subseteq (X \overset{(\mathcal{D},\mathcal{C})}{:_M} Y)$, we have $K \subseteq X$ or $K \subseteq Y$;

fully M-cosemiprime, iff for any fully invariant $(\mathcal{D},\mathcal{C})$-subbicomodule $X \subseteq M$ with $K \subseteq (X \overset{(\mathcal{D},\mathcal{C})}{:_M} X)$, we have $K \subseteq X$;

In particular, we call M *fully coprime* (*fully cosemiprime*), iff M is fully M-coprime (fully M-cosemiprime).

The fully coprime coradical

The *prime spectra* and the associated *prime radicals* for rings play an important role in the study of structure of rings. Dually, we define the *fully coprime spectra* and the *fully coprime coradicals* for bicomodules.

Definition 2.11. Let M be a non-zero $(\mathcal{D},\mathcal{C})$-bicomodule. We define the *fully coprime spectrum* of M as

$$\mathrm{CPSpec}(M) := \{0 \neq K \mid K \subseteq M \text{ is a fully } M\text{-coprime } (\mathcal{D},\mathcal{C})\text{-subbicomodule}\}$$

and the *fully coprime coradical* of M as $\mathrm{CPcorad}(M) := \sum\limits_{K\in\mathrm{CPSpec}(M)} K$ ($:= 0$, in case $\mathrm{CPSpec}(M) = \varnothing$). Moreover, we set

$$\mathrm{CSP}(M) := \{K \mid K \subseteq M \text{ is a fully } M\text{-cosemiprime } (\mathcal{D},\mathcal{C})\text{-subbicomodule}\}.$$

Remark 2.12. We should mention here that the definition of fully coprime (bi)comodules we present is motivated by the modified version of the definition of coprime modules (in the sense of Bican et. al. [BJKN80]) as presented in [RRW2005]. (Fully) coprime coalgebras over base fields were introduced first in [NT2001] and considered in [JMR] using the *wedge product* of subcoalgebras.

2.13. Let M be a non-zero $(\mathcal{D},\mathcal{C})$-bicomodule and $L \subseteq M$ a fully invariant non-zero $(\mathcal{D},\mathcal{C})$-subbicomodule. Then L is called E-*prime* (E-*semiprime*), iff $\mathrm{An}(K) \lhd {}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$ is prime (semiprime). With $\mathrm{EP}(M)$ ($\mathrm{ESP}(M)$) we denote the class of E-prime (E-semiprime) $(\mathcal{D},\mathcal{C})$-subbicomodules of M.

The results of [Abu2006] on comodules can be reformulated (with slight modifications of the proofs) for bicomodules. We state only two of them that are needed in the sequel.

Proposition 2.14. *Let M be a non-zero $(\mathcal{D},\mathcal{C})$-bicomodule. If M is self-cogenerator, then* $\mathrm{EP}(M) \subseteq \mathrm{CPSpec}(M)$ *and* $\mathrm{ESP}(M) \subseteq \mathrm{CSP}(M)$, *with equality if M is intrinsically injective. If moreover ${}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$ is right Noetherian, then*

$$\mathrm{Prad}({}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}) = \mathrm{An}(\mathrm{CPcorad}(M)) \textit{ and } \mathrm{CPcorad}(M) = \mathrm{Ke}(\mathrm{Prad}({}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}));$$

in particular, M is fully cosemiprime if and only if $M = \mathrm{CPcorad}(M)$.

Proposition 2.15. *Let M be a non-zero $(\mathcal{D},\mathcal{C})$-bicomodule and $0 \neq L \subseteq M$ a fully invariant $(\mathcal{D},\mathcal{C})$-subbicomodule. If M is self-injective, then*

$$\mathrm{CPSpec}(L) = \mathcal{M}_{f.i.}(L) \cap \mathrm{CPSpec}(M) \textit{ and } \mathrm{CSP}(L) = \mathcal{M}_{f.i.}(L) \cap \mathrm{CSP}(M); \qquad (4)$$

hence $\mathrm{CPcorad}(L) := L \cap \mathrm{CPcorad}(M)$.

Remark 2.16. Let M be a non-zero $(\mathcal{D},\mathcal{C})$-bicomodule. Then every $L \in \mathcal{S}_{f.i.}(M)$ is trivially a fully coprime $(\mathcal{D},\mathcal{C})$-bicomodule. If M is self-injective, then $\mathcal{S}_{f.i.}(M) \subseteq \mathrm{CPSpec}(M)$ by Proposition 2.15; hence if M has Property $\mathbf{S}_{f.i.}$, then every *fully invariant* non-zero $(\mathcal{D},\mathcal{C})$-subbicomodule $L \subseteq M$ contains a fully M-coprime $(\mathcal{D},\mathcal{C})$-subbicomodule $K \subseteq L$ (in particular, $\varnothing \neq \mathrm{CPSpec}(L) \subseteq \mathrm{CPSpec}(M) \neq \varnothing$).

3 Top Bicomodules

In what follows we introduce *top (bi)comodules*, which can be considered (in some sense) as dual to *top (bi)modules,* [Lu1999], [MMS1997], [Zha1999]. We define a Zariski topology on the fully coprime spectrum of such (bi)comodules in a way dual to that of defining the classical Zariski topology on the prime spectrum of (commutative) rings.

As before, $\mathcal{C}$ is a non-zero A-coring and $\mathcal{D}$ is a non-zero B-coring with ${}_A\mathcal{C}$, $\mathcal{D}_B$ flat. Moreover, M is a non-zero $(\mathcal{D},\mathcal{C})$-bicomodule.

Notation. For every $(\mathcal{D},\mathcal{C})$-subbicomodule $L \subseteq M$ set

$$\mathcal{V}_L := \{K \in \mathrm{CPSpec}(M) \mid K \subseteq L\},\ \mathcal{X}_L := \{K \in \mathrm{CPSpec}(M) \mid K \nsubseteq L\}.$$

Moreover, we set

$$\begin{array}{lllllll} \xi(M) & := & \{\mathcal{V}_L \mid L \in \mathcal{L}(M)\}; & & \xi_{f.i.}(M) & := & \{\mathcal{V}_L \mid L \in \mathcal{L}_{f.i.}(M)\}; \\ \tau_M & := & \{\mathcal{X}_L \mid L \in \mathcal{L}(M)\}; & & \tau_M^{f.i.} & := & \{\mathcal{X}_L \mid L \in \mathcal{L}_{f.i.}(M)\}. \\ \mathbf{Z}_M & := & (\mathrm{CPSpec}(M), \tau_M); & & \mathbf{Z}_M^{f.i.} & := & (\mathrm{CPSpec}(M), \tau_M^{f.i.}). \end{array}$$

Lemma 3.1. *1. $\mathcal{X}_M = \emptyset$ and $\mathcal{X}_{\{0\}} = \mathrm{CPSpec}(M)$.*

2. If $\{L_\lambda\}_\Lambda \subseteq \mathcal{L}(M)$, then $\mathcal{X}_{\sum\limits_{\Lambda} L_\lambda} \subseteq \bigcap\limits_{\Lambda} \mathcal{X}_{L_\lambda} \subseteq \bigcup\limits_{\Lambda} \mathcal{X}_{L_\lambda} = \mathcal{X}_{\bigcap\limits_{\Lambda} L_\lambda}$.

3. For any $L_1, L_2 \in \mathcal{L}_{f.i.}(M)$, we have $\mathcal{X}_{L_1+L_2} = \mathcal{X}_{L_1} \cap \mathcal{X}_{L_2} = \mathcal{X}_{(L_1 \overset{(\mathcal{D},\mathcal{C})}{:_M} L_2)}$.

Proof. Notice that "1" and "2" and the inclusion $\mathcal{X}_{L_1+L_2} \subseteq \mathcal{X}_{L_1} \cap \mathcal{X}_{L_2}$ in (3) are obvious. If $K \in \mathcal{X}_{L_1} \cap \mathcal{X}_{L_2}$, and $K \notin \mathcal{X}_{(L_1 \overset{(\mathcal{D},\mathcal{C})}{:_M} L_2)}$, then $K \subseteq L_1$ or $K \in L_2$ since K is fully M-coprime, hence $K \notin \mathcal{X}_{L_1}$ or $K \notin \mathcal{X}_{L_2}$ (a contradiction, hence $\mathcal{X}_{L_1} \cap \mathcal{X}_{L_2} \subseteq \mathcal{X}_{(L_1 \overset{(\mathcal{D},\mathcal{C})}{:_M} L_2)}$). Since $L_2 \subseteq M$ is a fully invariant, we have $L_1 + L_2 \subseteq (L_1 \overset{(\mathcal{D},\mathcal{C})}{:_M} L_2)$, hence $\mathcal{X}_{(L_1 \overset{(\mathcal{D},\mathcal{C})}{:_M} L_2)} \subseteq \mathcal{X}_{L_1+L_2}$ and we are done.■

Remark 3.2. Let $L_1, L_2 \subseteq M$ be *arbitrary* $(\mathcal{D},\mathcal{C})$-subbicomodules. If $L_1, L_2 \subseteq M$ are not fully invariant, then it is not evident that there exists a $(\mathcal{D},\mathcal{C})$-subbicomodule $L \subseteq M$ such that $\mathcal{X}_{L_1} \cap \mathcal{X}_{L_2} = \mathcal{X}_L$. So, for an arbitrary $(\mathcal{D},\mathcal{C})$-bicomodule M, the set $\xi(M)$ is not necessarily closed under finite unions.

The remark above motivates the following

Definition 3.3. We call M a *top bicomodule*, iff $\xi(M)$ is closed under finite unions.

As a direct consequence of Lemma 3.1 we get

Theorem 3.4. $\mathbf{Z}_M^{f.i.} := (\mathrm{CPSpec}(M), \tau_M^{f.i.})$ *is a topological space. In particular, if M is duo, then M is a top $(\mathcal{D},\mathcal{C})$-bicomodule (i.e.* $\mathbf{Z}_M := (\mathrm{CPSpec}(M), \tau_M)$ *is a topological space).*

To the end of this section, M is *duo*, *self-injective* and has Property **S**, so that $\varnothing \neq \mathcal{S}(L) = \mathcal{S}_{f.i.}(L) \subseteq \mathrm{CPSpec}(M)$ for every non-zero $(\mathcal{D},\mathcal{C})$-subbicomodule $0 \neq L \subseteq M$ (by Remark 2.16), and hence a top $(\mathcal{D},\mathcal{C})$-bicomodule.

Remarks 3.5. Consider the Zariski topology $\mathbf{Z}_M := (\mathrm{CPSpec}(M), \tau_M)$.

1. $\mathbf{Z}_M$ is a T_0 (Kolmogorov) space.

2. $\mathcal{B} := \{\mathcal{X}_L \mid L \in \mathcal{L}^{f.g.}(M)\}$ is a basis of open sets for the Zariski topology $\mathbf{Z}_M$: any $K \in \mathrm{CPSpec}(M)$ is contained in some $\mathcal{X}_L$ for some $L \in \mathcal{L}^{f.g.}(M)$ (e.g. $L = 0$); and if $L_1, L_2 \in \mathcal{L}^{f.g.}(M)$ and $K \in \mathcal{X}_{L_1} \cap \mathcal{X}_{L_2}$, then setting $L := L_1 + L_2 \in \mathcal{L}^{f.g.}(M)$, we have $K \in \mathcal{X}_L \subseteq \mathcal{X}_{L_1} \cap \mathcal{X}_{L_2}$.

3. Let $L \subseteq M$ be a $(\mathcal{D}, \mathcal{C})$-subbicomodule.

   (a) L is simple if and only if L is fully M-coprime and $\mathcal{V}_L = \{L\}$.

   (b) Assume $L \in \mathrm{CPSpec}(M)$. Then $\overline{\{L\}} = \mathcal{V}_L$; in particular, L is simple if and only if $\{L\}$ is closed in $\mathbf{Z}_M$.

   (c) $\mathcal{X}_L = \mathrm{CPSpec}(M)$ if and only if $L = 0$.

   (d) If $\mathcal{X}_L = \emptyset$, then $\mathrm{Corad}(M) \subseteq L$.

4. Let $0 \neq L \overset{\theta}{\hookrightarrow} M$ be a non-zero $(\mathcal{D}, \mathcal{C})$-bicomodule and consider the embedding $\mathrm{CPSpec}(L) \overset{\widetilde{\theta}}{\hookrightarrow} \mathrm{CPSpec}(M)$ (compare Proposition 2.15). Since $\theta^{-1}(\mathcal{V}_N) = \mathcal{V}_{N \cap L}$ for every $N \in \mathcal{L}(M)$, the induced map $\boldsymbol{\theta} : \mathbf{Z}_L \to \mathbf{Z}_M$, $K \mapsto \theta(K)$ is continuous.

5. Let $M \overset{\theta}{\simeq} N$ be an isomorphism of non-zero $(\mathcal{D}, \mathcal{C})$-bicomodules. Then we have bijections $\mathrm{CPSpec}(M) \longleftrightarrow \mathrm{CPSpec}(N)$ and $\mathrm{CSP}(M) \longleftrightarrow \mathrm{CSP}(N)$; in particular, $\theta(\mathrm{CPcorad}(M)) = \mathrm{CPcorad}(N)$. Moreover, $\mathbf{Z}_M \approx \mathbf{Z}_N$ are *homeomorphic spaces.*

Theorem 3.6. *The following are equivalent:*

1. $\mathrm{CPSpec}(M) = \mathcal{S}(M)$;

2. $\mathbf{Z}_M$ *is discrete;*

3. $\mathbf{Z}_M$ *is a* T_2 *(Hausdorff space)* .

4. $\mathbf{Z}_M$ *is a* T_1 *(Fréchet space).*

Proof. (1) $\Rightarrow$ (2). For every $K \in \mathrm{CPSpec}(M) = \mathcal{S}(M)$, we have $\{K\} = \mathcal{X}_{\mathcal{Y}_K}$ whence open, where $\mathcal{Y}_K := \sum\{L \in \mathrm{CPSpec}(M) \mid K \nsubseteq L\}$.

(2) $\Rightarrow$ (3) & (3) $\Rightarrow$ (4) : Every discrete topological space is T_2 and every T_2 space is T_1.

(4) $\Rightarrow$ (1) Let $\mathbf{Z}_M$ be T_1 and suppose $K \in \mathrm{CPSpec}(M) \backslash \mathcal{S}(M)$, so that $\{K\} = \mathcal{V}_L$ for some $L \in \mathcal{L}(M)$. Since K is not simple, there exists by assumptions and Remark 2.16 $K_1 \in \mathcal{S}(K) \subseteq \mathrm{CPSpec}(M)$ with $K_1 \subsetneqq K$, i.e. $\{K_1, K\} \subsetneqq \mathcal{V}_L = \{K\}$, a contradiction. Consequently, $\mathrm{CPSpec}(M) = \mathcal{S}(M)$.■

Proposition 3.7. *Let M be self-cogenerator and* ${}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$ *be Noetherian with every prime ideal maximal (e.g. a biregular ring*[2]*).*

[2]a ring in which every two-sided ideal is generated by a central idempotent (see [Wis1991, 3.18(6,7)]).

1. *$\mathcal{S}(M) = \mathrm{CPSpec}(M)$ (hence M is subdirectly irreducible $\Leftrightarrow |\mathrm{CPSpec}(M)| = 1$).*

2. *If $L \subseteq M$ is a $(\mathcal{D},\mathcal{C})$-subbicomodule, then $\mathcal{X}_L = \varnothing$ if and only if $\mathrm{Corad}(M) \subseteq L$.*

Proof. 1. Notice that $\mathcal{S}(M) \subseteq \mathrm{CPSpec}(M)$ by Remark 2.16. If $K \in \mathrm{CPSpec}(M)$, then $\mathrm{An}(K) \lhd {}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$ is prime by Proposition 2.14, whence maximal by assumption and it follows then that $K = \mathrm{Ke}(\mathrm{An}(K))$ is simple (if $0 \neq K_1 \subsetneqq K$, for some $K_1 \in \mathcal{L}(M)$, then $\mathrm{An}(K) \subsetneqq \mathrm{An}(K_1) \subsetneqq {}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$ since $\mathrm{Ke}(-)$ is injective, a contradiction).

2. If $L \subseteq M$ is a $(\mathcal{D},\mathcal{C})$-subbicomodule, then it follows from "1" that $\mathcal{X}_L = \varnothing$ if and only if $\mathrm{Corad}(M) = \mathrm{CPcorad}(M) \subseteq L$.■

Remark 3.8. Proposition 3.7 corrects [NT2001, Lemma 2.6.], which is absurd since it assumes C^* PID, while C is not (fully) coprime (but C^* domain implies C is (fully) coprime!!).

Theorem 3.9. *If $|\mathcal{S}(M)|$ is countable (finite), then $\mathbf{Z}_M$ is Lindelof (compact). The converse holds, if $\mathcal{S}(M) = \mathrm{CPSpec}(M)$.*

Proof. Assume $\mathcal{S}(M) = \{S_{\lambda_k}\}_{k\geq 1}$ is countable (finite). Let $\{\mathcal{X}_{L_\alpha}\}_{\alpha\in I}$ be an open cover of $\mathrm{CPSpec}(M)$ (i.e. $\mathrm{CPSpec}(M) \subseteq \bigcup\limits_{\alpha\in I} \mathcal{X}_{L_\alpha}$). Since $\mathcal{S}(M) \subseteq \mathrm{CPSpec}(M)$ we can pick for each $k \geq 1$, some $\alpha_k \in I$ such that $S_{\lambda_k} \nsubseteq L_{\alpha_k}$. If $\bigcap\limits_{k\geq 1} L_{\alpha_k} \neq 0$, then it contains by Property **S** a simple $(\mathcal{D},\mathcal{C})$-subbicomodule $0 \neq S \subseteq \bigcap\limits_{k\geq 1} L_{\alpha_k}$, (a contradiction, since $S = S_{\lambda_k} \nsubseteq L_{\alpha_k}$ for some $k \geq 1$). Hence $\bigcap\limits_{k\geq 1} L_{\alpha_k} = 0$ and we conclude that $\mathrm{CPSpec}(M) = \mathcal{X}_{\bigcap\limits_{k\geq 1} L_{\alpha_k}} = \bigcup\limits_{k\geq 1} \mathcal{X}_{L_{\alpha_k}}$ (i.e. $\{\mathcal{X}_{L_{\alpha_k}} \mid k \geq 1\} \subseteq \{\mathcal{X}_{L_\alpha}\}_{\alpha\in I}$ is a countable (finite) subcover). Notice that if $\mathcal{S}(M) = \mathrm{CPSpec}(M)$, then $\mathbf{Z}_M$ is discrete by Theorem 3.6 and so $\mathbf{Z}_M$ is Lindelof (compact) if and only if $\mathrm{CPSpec}(M)$ is countable (finite).■

Definition 3.10. A collection $\mathcal{G}$ of subsets of a topological space $\mathbf{X}$ is *locally finite*, iff every point of $\mathbf{X}$ has a neighbourhood that intersects only finitely many elements of $\mathcal{G}$.

Proposition 3.11. *Let $\mathcal{K} = \{K_\lambda\}_\Lambda \subseteq \mathcal{S}(M)$ be a non-empty family of simple $(\mathcal{D},\mathcal{C})$-subbicomodules. If $|\mathcal{S}(L)| < \infty$ for every $L \in \mathrm{CPSpec}(M)$, then $\mathcal{K}$ is locally finite.*

Proof. Let $L \in \mathrm{CPSpec}(M)$ and set $F := \sum\{K \in \mathcal{K} \mid K \nsubseteq L\}$. Since $|\mathcal{S}(L)| < \infty$, there exists a finite number of simple $(\mathcal{D},\mathcal{C})$-subbicomodules $\{S_{\lambda_1}, .., S_{\lambda_n}\} = \mathcal{K} \cap \mathcal{V}_L$. If $L \subseteq F$, then $0 \neq L \subseteq \sum\limits_{i=1}^{n} S_{\lambda_i} \subseteq (S_{\lambda_1} :_M^{(\mathcal{D},\mathcal{C})} \sum\limits_{i=2}^{n} S_{\lambda_i})$ and it follows by induction that $0 \neq L \subsetneqq S_{\lambda_i}$ for some $1 \leq i \leq n$ (a contradiction, since S_{λ_i} is simple), whence $L \in \mathcal{X}_F$. It is clear then that $\mathcal{K} \cap \mathcal{X}_F = \{K_{\lambda_1}, .., K_{\lambda_n}\}$ and we are done.■

Definition 3.12. ([Bou1966], [Bou1998]) A topological space $\mathbf{X}$ is said to be *irreducible* (*connected*), iff $\mathbf{X}$ is not the (disjoint) union of two proper closed subsets; equivalently, iff the intersection of any two non-empty open subsets is non-empty (the only subsets of $\mathbf{X}$ that are open and closed are $\varnothing$ and $\mathbf{X}$). A maximal irreducible subspace of $\mathbf{X}$ is called an *irreducible component*.

Proposition 3.13. CPSpec(M) *is irreducible if and only if* CPcorad(M) *is fully M-coprime.*

Proof. Let CPSpec(M) be irreducible. By Remark 2.16, CPcorad$(M) \neq 0$. Suppose that CPcorad(M) is not fully M-coprime, so that there exist $(\mathcal{D}, \mathcal{C})$-subbicomodules $X, Y \subseteq M$ with CPcorad$(M) \subseteq (X :_M^{(\mathcal{D},\mathcal{C})} Y)$ but CPcorad$(M) \nsubseteq X$ and CPcorad$(M) \nsubseteq Y$. It follows then that CPSpec$(M) = \mathcal{V}_{(X :_M^{(\mathcal{D},\mathcal{C})} Y)} = \mathcal{V}_X \cup \mathcal{V}_Y$ a union of proper closed subsets, a contradiction. Consequently, CPcorad(M) is fully M-coprime.

On the otherhand, assume CPcorad$(M) \in$ CPSpec(M) and suppose that CPSpec$(M) = \mathcal{V}_{L_1} \cup \mathcal{V}_{L_2} = \mathcal{V}_{(L_1 :_M^{(\mathcal{D},\mathcal{C})} L_2)}$ for some $(\mathcal{D}, \mathcal{C})$-subbicomodules $L_1, L_2 \subseteq M$. It follows then that CPcorad$(M) \subseteq L_1$, so that $\mathcal{V}_{L_1} =$ CPSpec(M); or CPcorad$(M) \subseteq L_2$, so that $\mathcal{V}_{L_2} =$ CPSpec(M). Consequently CPSpec(M) is not the union of two *proper* closed subsets, i.e. it is irreducible.■

Lemma 3.14. 1. *M is subdirectly irreducible if and only if the intersection of any two non-empty closed subsets of* CPSpec(M) *is non-empty.*

2. *If M is subdirectly irreducible, then* CPSpec(M) *is connected. If* CPSpec(M) *is connected and* CPSpec$(M) = \mathcal{S}(M)$, *then M is subdirectly irreducible.*

Proof. 1. Assume that M is subdirectly irreducible with unique simple $(\mathcal{D}, \mathcal{C})$-subbicomodule $0 \neq S \subseteq M$. If $\mathcal{V}_{L_1}, \mathcal{V}_{L_2} \subseteq$ CPSpec(M) are any two non-empty closed subsets, then $L_1 \neq 0 \neq L_2$ and so $\mathcal{V}_{L_1} \cap \mathcal{V}_{L_2} = \mathcal{V}_{L_1 \cap L_2} \neq \varnothing$, since $S \subseteq L_1 \cap L_2 \neq 0$. On the otherhand, assume that the intersection of any two non-empty closed subsets of CPSpec(M) is non-empty. Let $0 \neq L_1, L_2 \subseteq M$ be any non-zero $(\mathcal{D}, \mathcal{C})$-subbicomodules, so that $\mathcal{V}_{L_1} \neq \varnothing \neq \mathcal{V}_{L_2}$. By assumption $\mathcal{V}_{L_1 \cap L_2} = \mathcal{V}_{L_1} \cap \mathcal{V}_{L_2} \neq \varnothing$, hence $L_1 \cap L_2 \neq 0$ and it follows by 2.4 that M is subdirectly irreducible.

2. If M is subdirectly irreducible, then CPSpec(M) is connected by "1". On the otherhand, if CPSpec$(M) = \mathcal{S}(M)$, then $\mathbf{Z}_M$ is discrete by Theorem 3.6 and so M is subdirectly irreducible (since a discrete topological space is connected if and only if it has only one point).■

Proposition 3.15. 1. *If $K \in$ CPSpec(M), then $\mathcal{V}_K \subseteq$ CPSpec(M) is irreducible.*

2. *If $\mathcal{V}_L$ is an irreducible component of $\mathbf{Z}_M$, then L is a maximal fully M-coprime $(\mathcal{D}, \mathcal{C})$-subbicomodule.*

Proof. 1. Let $K \in$ CPSpec(M) and suppose $\mathcal{V}_K = A \cup B = (\mathcal{V}_K \cap \mathcal{V}_X) \cup (\mathcal{V}_K \cap \mathcal{V}_Y)$ for two $(\mathcal{D}, \mathcal{C})$-subbicomodules $X, Y \subseteq M$ (so that $A, B \subseteq \mathcal{V}_K$ are closed subsets w.r.t. the relative topology on $\mathcal{V}_K \hookrightarrow$ CPSpec(M)). It follows then that $\mathcal{V}_K = (\mathcal{V}_{K \cap X}) \cup (\mathcal{V}_{K \cap Y}) = \mathcal{V}_{(K \cap X :_M^{(\mathcal{D},\mathcal{C})} K \cap Y)}$ and so $K \subseteq (K \cap X :_M^{(\mathcal{D},\mathcal{C})} K \cap Y)$, hence $K \subseteq X$ so that $\mathcal{V}_K = A$; or $K \subseteq Y$, so that $\mathcal{V}_K = B$. Consequently $\mathcal{V}_K$ is irreducible.

2. Assume $\mathcal{V}_L$ is an irreducible component of CPSpec(M) for some $0 \neq L \in \mathcal{L}(M)$. If $L \subseteq K$ for some $K \in$ CPSpec(M), then $\mathcal{V}_L \subseteq \mathcal{V}_K$ and it follows then that $L = K$ (since $\mathcal{V}_K \subseteq$ CPSpec(M) is irreducible by "1"). We conclude then that L is fully M-coprime and is moreover maximal in CPSpec(M).■

Lemma 3.16. *If $n \geq 2$ and $\mathcal{A} = \{K_1, ..., K_n\} \subseteq \mathrm{CPSpec}(M)$ is a connected subset, then for every $i \in \{1, ..., n\}$, there exists $j \in \{1, ..., n\} \backslash \{i\}$ such that $K_i \subseteq K_j$ or $K_j \subseteq K_i$.*

Proof. Without loss of generality, suppose $K_1 \nsubseteq K_j$ and $K_j \nsubseteq K_1$ for all $2 \leq j \leq n$ and set $F := \sum_{i=2}^{n} K_i$, $W_1 := \mathcal{A} \cap \mathcal{X}_{K_1} = \{K_2, ..., K_n$ and $W_2 := \mathcal{A} \cap \mathcal{X}_F = \{K_1\}$ (if $n = 2$, then clearly $W_2 = \{K_1\}$; if $n > 2$ and $K_1 \notin W_2$, then $K_1 \subseteq \sum_{i=2}^{n} K_i \subseteq (K_2 :_M^{(\mathcal{D},\mathcal{C})} \sum_{i=3}^{n} K_i)$ and it follows that $K_1 \subseteq \sum_{i=3}^{n} K_i$; by induction one shows that $K_1 \subseteq K_n$, a contradiction). So $\mathcal{A} = W_1 \cup W_2$, a disjoint union of proper non-empty open subsets (a contradiction).■

Notation. For $\mathcal{A} \subseteq \mathrm{CPSpec}(M)$ set $\varphi(\mathcal{A}) := \sum_{K \in \mathcal{A}} K$ ($:= 0$, iff $\mathcal{A} = \varnothing$). Moreover, set

$$\mathbf{CL}(\mathbf{Z}_M) := \{\mathcal{A} \subseteq \mathrm{CPSpec}(M) \mid \mathcal{A} = \overline{\mathcal{A}}\} \text{ and } \mathcal{E}(M) := \{L \in \mathcal{L}(M) \mid \mathrm{CPcorad}(L) = L\}.$$

Lemma 3.17. *The closure of any subset $\mathcal{A} \subseteq \mathrm{CPSpec}(M)$ is $\overline{\mathcal{A}} = \mathcal{V}_{\varphi(\mathcal{A})}$.*

Proof. Let $\mathcal{A} \subseteq \mathrm{CPSpec}(M)$. Since $\mathcal{A} \subseteq \mathcal{V}_{\varphi(\mathcal{A})}$ and $\mathcal{V}_{\varphi(\mathcal{A})}$ is a closed set, we have $\overline{\mathcal{A}} \subseteq \mathcal{V}_{\varphi(\mathcal{A})}$. On the other hand, suppose $H \in \mathcal{V}_{\varphi(\mathcal{A})} \backslash \mathcal{A}$ and let $\mathcal{X}_L$ be a neighbourhood of H, so that $H \nsubseteq L$. Then there exists $W \in \mathcal{A}$ with $W \nsubseteq L$ (otherwise $H \subseteq \varphi(\mathcal{A}) \subseteq L$, a contradiction), i.e. $W \in \mathcal{X}_L \cap (\mathcal{A} \backslash \{H\}) \neq \varnothing$ and so K is a cluster point of $\mathcal{A}$. Consequently, $\overline{\mathcal{A}} = \mathcal{V}_{\varphi(\mathcal{A})}$.■

Theorem 3.18. *We have a bijection $\mathbf{CL}(\mathbf{Z}_M) \longleftrightarrow \mathcal{E}(M)$. If M is self-cogenerator and ${}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$ is right Noetherian, then there is a bijection $\mathbf{CL}(\mathbf{Z}_M) \backslash \{\varnothing\} \longleftrightarrow \mathrm{CSPSpec}(M)$.*

Proof. For $L \in \mathcal{E}(M)$, set $\psi(L) := \mathcal{V}_L$. Then for $L \in \mathcal{E}(M)$ and $\mathcal{A} \in \mathbf{CL}(\mathbf{Z}_M)$ we have $\varphi(\psi(L)) = \varphi(\mathcal{V}_L) = L \cap \mathrm{CPcorad}(M) = \mathrm{CPcorad}(L) = L$ and $\psi(\varphi(\mathcal{A})) = \mathcal{V}_{\varphi(\mathcal{A})} = \overline{\mathcal{A}} = \mathcal{A}$. If M is self-cogenerator and ${}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$ is right Noetherian, then $\mathrm{CSPSpec}(M) = \mathcal{E}(M) \backslash \{0\}$ by Proposition 2.14 and we are done.■

4 Applications and Examples

In this section we give some applications and examples. First of all we remark that taking $\mathcal{D} := R$ ($\mathcal{C} := R$), considered with the trivial coring structure, our results on the Zariski topology for bicomodules in the third section can be reformulated for Zariski topology on the fully coprime spectrum of right $\mathcal{C}$-comodules (left $\mathcal{D}$-comodules). However, our main application will be to the Zariski topology on the fully coprime spectrum of non-zero corings, considered as *duo bicomodules* in the canonical way.

Throughout this section, $\mathcal{C}$ is a non-zero A-coring with ${}_A\mathcal{C}$ and $\mathcal{C}_A$ flat.

4.1. The (A, A)-bimodule ${}^*\mathcal{C}^* := \mathrm{Hom}_{(A,A)}(\mathcal{C}, A) := {}^*\mathcal{C} \cap \mathcal{C}^*$ is an A^{op}-ring with multiplication $(f * g)(c) = \sum f(c_1)g(c_2)$ for all $f, g \in {}^*\mathcal{C}^*$ and unit $\varepsilon_{\mathcal{C}}$; hence every $(\mathcal{C}, \mathcal{C})$-bicomodule M is a $({}^*\mathcal{C}^*, {}^*\mathcal{C}^*)$-bimodule and the *centralizer*

$$\mathbf{C}(M) := \{f \in {}^*\mathcal{C}^* \mid f \rightharpoonup m = m \leftharpoonup f \text{ for all } m \in M\}$$

is an R-algebra. If M is faithful as a left (right) ${}^*\mathcal{C}^*$-module, then $\mathbf{C}(M) \subseteq Z({}^*\mathcal{C}^*)$.

4.2. Considering $\mathcal{C}$ as a $(\mathcal{C},\mathcal{C})$-bicomodule in the natural way, $\mathcal{C}$ is a $({}^*\mathcal{C}^*, {}^*\mathcal{C}^*)$-bimodule that is faithful as a left (right) ${}^*\mathcal{C}^*$-module, hence the centralizer

$$\mathbf{C}(\mathcal{C}) := \{f \in {}^*\mathcal{C}^* \mid f \rightharpoonup c = c \leftharpoonup f \text{ for every } c \in \mathcal{C}\}$$

embeds in the center of ${}^*\mathcal{C}^*$ as an R-subalgebra, i.e. $\mathbf{C}(\mathcal{C}) \hookrightarrow Z({}^*\mathcal{C}^*)$. If $ac = ca$ for all $a \in A$, then we have a morphism of R-algebras $\eta : Z(A) \to \mathbf{C}(\mathcal{C})$, $a \mapsto [\varepsilon_{\mathcal{C}}(a-) = \varepsilon_{\mathcal{C}}(-a)]$.

Remark 4.3. Notice that $\mathbf{C}(\mathcal{C}) \subseteq Z({}^*\mathcal{C}) \subseteq Z({}^*\mathcal{C}^*)$ and $\mathbf{C}(\mathcal{C}) \subseteq Z(\mathcal{C}^*) \subseteq Z({}^*\mathcal{C}^*)$ (compare [BW2003, 17.8. (4)]). If ${}_A\mathcal{C}$ $(\mathcal{C}_A)$ is A-cogenerated, then it follows by [BW2003, 19.10 (3)] that $Z({}^*\mathcal{C}) = \mathbf{C}(\mathcal{C}) \subseteq Z(\mathcal{C}^*)$ $(Z(\mathcal{C}^*) = \mathbf{C}(\mathcal{C}) \subseteq Z({}^*\mathcal{C}))$. If ${}_A\mathcal{C}_A$ is A-cogenerated, then $Z({}^*\mathcal{C}^*) \subseteq \mathbf{C}(\mathcal{C})$ (e.g. [BW2003, 19.10 (4)]), whence $Z({}^*\mathcal{C}) = Z({}^*\mathcal{C}^*) = Z(\mathcal{C}^*)$.

Lemma 4.4. *For every $(\mathcal{C},\mathcal{C})$-bicomodule M we have a morphism of R-algebras*

$$\phi_M : \mathbf{C}(M) \to {}^{\mathcal{C}}\mathrm{End}^{\mathcal{C}}(M)^{op},\ f \mapsto [m \mapsto f \rightharpoonup m = m \leftharpoonup f]\ (\textit{with } \mathrm{Im}(\phi_M) \subseteq Z({}^{\mathcal{C}}\mathrm{E}^{\mathcal{C}}_M)). \tag{5}$$

Proof. First of all we prove that ϕ_M is well-defined: for $f \in \mathbf{C}(M)$ and $m \in M$ we have

$$\begin{array}{rcl}\sum(\phi_M(f)(m))_{<0>} \otimes_A (\phi_M(f)(m))_{<1>} & = & \sum(m \leftharpoonup f)_{<0>} \otimes_A (m \leftharpoonup f)_{<1>} \\ & = & \sum f(m_{<-1>})m_{<0><0>} \otimes_A m_{<0><1>} \\ & = & \sum f(m_{<0><-1>})m_{<0><0>} \otimes_A m_{<1>} \\ & = & \sum(m_{<0>} \leftharpoonup f) \otimes_A m_{<1>} \\ & = & \sum \phi_M(f)(m_{<0>}) \otimes_A m_{<1>},\end{array}$$

and

$$\begin{array}{rcl}\sum(\phi_M(f)(m))_{<-1>} \otimes_A (\phi_M(f)(m))_{<0>} & = & \sum(f \rightharpoonup m)_{<-1>} \otimes_A (f \rightharpoonup m)_{<0>} \\ & = & \sum m_{<0><-1>} \otimes_A m_{<0><0>} f(m_{<1>}) \\ & = & \sum m_{<-1>} \otimes_A m_{<0><0>} f(m_{<0><1>}) \\ & = & \sum m_{<-1>} \otimes_A (f \rightharpoonup m_{<0>}) \\ & = & \sum m_{<-1>} \otimes_A \phi_M(f)(m_{<0>}),\end{array}$$

i.e. $\phi_M(f) : M \to M$ is $(\mathcal{C},\mathcal{C})$-bicolinear. Obviously, $\phi_M(f * g) = \phi_M(f) \circ^{op} \phi_M(g)$ for all $f, g \in \mathbf{C}(M)$, i.e. ϕ_M is a morphism of R-algebras. Moreover, since every $g \in {}^{\mathcal{C}}\mathrm{E}^{\mathcal{C}}_M$ is $({}^*\mathcal{C}^*, {}^*\mathcal{C}^*)$-bilinear, we have $g(f \rightharpoonup m) = f \rightharpoonup g(m)$ for every $f \in {}^*\mathcal{C}^*$ and $m \in M$, i.e. $\mathrm{Im}(\phi_M) \subseteq Z({}^{\mathcal{C}}\mathrm{E}^{\mathcal{C}}_M)$.■

Lemma 4.5. *We have an isomorphism of R-algebras $\mathbf{C}(\mathcal{C}) \overset{\phi_{\mathcal{C}}}{\simeq} {}^{\mathcal{C}}\mathrm{End}^{\mathcal{C}}(\mathcal{C})$, with inverse $\psi_{\mathcal{C}} : g \mapsto \varepsilon_{\mathcal{C}} \circ g$. In particular, $({}^{\mathcal{C}}\mathrm{End}^{\mathcal{C}}(\mathcal{C}), \circ)$ is commutative and $\mathcal{C} \in {}^{\mathcal{C}}\mathbb{M}^{\mathcal{C}}$ is duo.*

Proof. First of all we prove that ψ is well-defined: for $g \in {}^{\mathcal{C}}\mathrm{End}^{\mathcal{C}}(\mathcal{C})$ and $c \in \mathcal{C}$ we have

$$\begin{array}{rcccccl}\psi_{\mathcal{C}}(g) \rightharpoonup c & = & \sum c_1\psi(g)(c_2) & = & \sum c_1\varepsilon_{\mathcal{C}}(g(c_2)) & = & \sum g(c)_1\varepsilon_{\mathcal{C}}(g(c)_2) \\ & = & g(c) & = & \sum \varepsilon_{\mathcal{C}}(g(c)_1)g(c)_2 & = & \sum \varepsilon_{\mathcal{C}}(g(c_1))c_2 \\ & = & \sum \psi(g)(c_1)c_2 & = & c \leftharpoonup \psi(g), & & \end{array}$$

i.e. $\psi_{\mathcal{C}}(g) \in \mathbf{C}(\mathcal{C})$. For any $f \in \mathbf{C}(\mathcal{C})$, $g \in {}^{\mathcal{C}}\mathrm{End}^{\mathcal{C}}(\mathcal{C})$ and $c \in \mathcal{C}$ we have $((\psi_{\mathcal{C}} \circ \phi_{\mathcal{C}})(f))(c) = \varepsilon_{\mathcal{C}}(\phi_{\mathcal{C}}(f)(c)) = \varepsilon_{\mathcal{C}}(f \rightharpoonup c) = f(c)$ and $((\phi_{\mathcal{C}} \circ \psi_{\mathcal{C}})(g))(c) = \sum c_1\psi_{\mathcal{C}}(g)(c_2) = \sum c_1\varepsilon_{\mathcal{C}}(g(c_2)) = \sum g(c)_1\varepsilon_{\mathcal{C}}(g(c)_2) = g(c)$.■

Zariski topologies for corings

Definition 4.6. A right (left) $\mathcal{C}$-subcomodule $K \subseteq \mathcal{C}$ is called a *right (left) $\mathcal{C}$-coideal*. A $(\mathcal{C},\mathcal{C})$-subbicomodule of M is called a *$\mathcal{C}$-bicoideal*.

Notation. With $\mathcal{B}(\mathcal{C})$ we denote the class of $\mathcal{C}$-bicoideals and with $\mathcal{L}(\mathcal{C}^r)$ (resp. $\mathcal{L}(\mathcal{C}^l)$) the class of right (left) $\mathcal{C}$-coideals. For a $\mathcal{C}$-bicoideal $K \in \mathcal{B}(\mathcal{C})$, K^r (K^l) indicates that we consider K as a right (left) $\mathcal{C}$-comodule, rather than a $(\mathcal{C},\mathcal{C})$-bicomodule. We also set

$$\begin{array}{lllllll} \mathrm{CPSpec}(\mathcal{C}) & := & \{K \in \mathcal{B}(\mathcal{C}) \mid K \text{ is fully } \mathcal{C}\text{-coprime}\}; & \tau_{\mathcal{C}} & := & \{\mathcal{X}_L \mid L \in \mathcal{B}(\mathcal{C})\}; \\ \mathrm{CPSpec}(\mathcal{C}^r) & := & \{K \in \mathcal{B}(\mathcal{C}) \mid K^r \text{ is fully } \mathcal{C}^r\text{-coprime}\}; & \tau_{\mathcal{C}^r} & := & \{\mathcal{X}_L \mid L \in \mathcal{L}(\mathcal{C}^r)\}; \\ \mathrm{CPSpec}(\mathcal{C}^l) & := & \{K \in \mathcal{B}(\mathcal{C}) \mid K^l \text{ is fully } \mathcal{C}^l\text{-coprime}\}; & \tau_{\mathcal{C}^l} & := & \{\mathcal{X}_L \mid L \in \mathcal{L}(\mathcal{C}^l)\}. \end{array}$$

In what follows we announce only the main result on the Zariski topologies for corings, leaving to the interested reader the restatement of the other results of the third section.

Theorem 4.7. *1.* $\mathbf{Z}_{\mathcal{C}} := (\mathrm{CPSpec}(\mathcal{C}), \tau_{\mathcal{C}})$ *is a topological space.*

2. $\mathbf{Z}^{f.i.}_{\mathcal{C}^r} := (\mathrm{CPSpec}(C^r), \tau^{f.i.}_{\mathcal{C}^r})$ *and* $\mathbf{Z}^{f.i.}_{\mathcal{C}^l} := (\mathrm{CPSpec}(C^l), \tau^{f.i.}_{\mathcal{C}^l})$ *are topological spaces.*

Proposition 4.8. *Let $\theta : \mathcal{C} \to \mathcal{C}'$ be a morphism of non-zero A-corings with ${}_A\mathcal{C}$, ${}_A\mathcal{C}'$ flat, $\mathcal{C}^r$ intrinsically injective self-cogenerator and $\mathcal{C}'^r$ self-cogenerator.*

1. *If θ is injective and $\mathcal{C}'^r$ is self-injective, or if $\mathcal{C}^*$ is right-duo, then we have a map $\widetilde{\theta}$: $\mathrm{CPSpec}(\mathcal{C}^r) \to \mathrm{CPSpec}(\mathcal{C}'^r)$, $K \mapsto \theta(K)$ (and so $\theta(\mathrm{CPcorad}(\mathcal{C}^r)) \subseteq \mathrm{CPcorad}(\mathcal{C}'^r)$).*
2. *If $\mathcal{C}^r$, $\mathcal{C}'^r$ are duo, then the induced map $\boldsymbol{\theta} : \mathbf{Z}_{\mathcal{C}^r} \to \mathbf{Z}_{\mathcal{C}'^r}$ is continuous.*
3. *If every $K \in \mathrm{CPSpec}(\mathcal{C}^r)$ is inverse image of a $K' \in \mathrm{CPSpec}(\mathcal{C}'^r)$, then $\widetilde{\theta}$ is injective.*
4. *If θ is injective and $\mathcal{C}'^r$ is self-injective, then $\boldsymbol{\theta} : \mathbf{Z}^{f.i.}_{\mathcal{C}^r} \to \mathbf{Z}^{f.i.}_{\mathcal{C}'^r}$ is continuous. If moreover, $\widetilde{\theta} : \mathrm{CPSpec}(\mathcal{C}^r) \to \mathrm{CPSpec}(\mathcal{C}'^r)$ is surjective, then $\boldsymbol{\theta}$ is open and closed.*
5. *If $\mathcal{C} \overset{\theta}{\simeq} \mathcal{C}'$, then $\mathbf{Z}^{f.i.}_{\mathcal{C}^r} \overset{\boldsymbol{\theta}}{\approx} \mathbf{Z}^{f.i.}_{\mathcal{C}'^r}$ (homeomorphic spaces).*

Proof. First of all notice for every $K \in \mathcal{L}(\mathcal{C}^r)$ $(K \in \mathcal{B}(\mathcal{C}))$, we have $\theta(K) \in \mathcal{L}(\mathcal{C}'^r)$ $(\theta(K) \in \mathcal{B}(\mathcal{C}'))$ and for every $K' \in \mathcal{L}(\mathcal{C}'^r)$ $(K' \in \mathcal{B}(\mathcal{C}'))$, $\theta^{-1}(K') \in \mathcal{L}(\mathcal{C}^r)$ $(\theta^{-1}(K') \in \mathcal{B}(\mathcal{C}))$.

1. If θ is injective and $\mathcal{C}'^r$ is self-injective, then $\mathrm{CPSpec}(\mathcal{C}^r) = \mathcal{B}(\mathcal{C}) \cap \mathrm{CPSpec}(\mathcal{C}'^r)$ by [Abu2006, Proposition 4.7.]. Assume now that $\mathcal{C}^*$ is right-duo. Since θ is a morphism of A-corings, the canonical map $\theta^* : \mathcal{C}'^* \to \mathcal{C}^*$ is a morphism of A^{op}-rings. If $K \in \mathrm{CPSpec}(\mathcal{C}^r)$, then $\mathrm{ann}_{\mathcal{C}^*}(K) \lhd \mathcal{C}^*$ is a prime ideal by [Abu2006, Proposition 4.10.], whence completely prime since $\mathcal{C}^*$ is right-duo. It follows then that $\mathrm{ann}_{\mathcal{C}'^*}(\theta(K)) = \theta(K)^{\perp \mathcal{C}'^*} = (\theta^*)^{-1}(K^{\perp \mathcal{C}^*}) = (\theta^*)^{-1}(\mathrm{ann}_{\mathcal{C}^*}(K))$ is a prime ideal, whence $\theta(K) \in \mathrm{CPSpec}(\mathcal{C}'^r)$ by [Abu2006, Proposition 4.10.]. It is obvious then that $\theta(\mathrm{CPcorad}(\mathcal{C}^r)) \subseteq \mathrm{CPcorad}(\mathcal{C}'^r)$.
2. Since $\mathcal{C}^r \in \mathbb{M}^{\mathcal{C}}$, $\mathcal{C}'^r \in \mathbb{M}^{\mathcal{C}'}$ are duo, $\mathbf{Z}_{\mathcal{C}^r} := \mathbf{Z}^{f.i.}_{\mathcal{C}^r}$ and $\mathbf{Z}_{\mathcal{C}'^r} := \mathbf{Z}^{f.i.}_{\mathcal{C}'^r}$ are topological spaces. Since $\mathcal{C}^r$ is intrinsically injective, $\mathcal{C}^*$ is right-due and by "1" $\widetilde{\theta} : \mathrm{CPSpec}(\mathcal{C}^r) \to \mathrm{CPSpec}(\mathcal{C}'^r)$ is well-defined. For $L' \in \mathcal{L}(\mathcal{C}'^r)$, $\widetilde{\theta}^{-1}(\mathcal{X}_{L'}) = \mathcal{X}_{\theta^{-1}(L')}$, i.e. $\boldsymbol{\theta}$ is continuous.

3. Suppose $\widetilde{\theta}(K_1) = \widetilde{\theta}(K_2)$ for some $K_1, K_2 \in \mathrm{CPSpec}(\mathcal{C}^r)$ with $K_1 = \theta^{-1}(K_1')$, $K_2 = \theta^{-1}(K_2')$ where $K_1', K_2' \in \mathrm{CPSpec}(\mathcal{C}'^r)$. Then $K_1 = \theta^{-1}(K_1') = \theta^{-1}(\theta(\theta^{-1}(K_1'))) = \theta^{-1}(\theta(\theta^{-1}(K_1'))) = \theta^{-1}(\theta(\theta^{-1}(K_2'))) = \theta^{-1}(K_2') = K_2$.

4. By [Abu2006, Proposition 4.7.] $\mathrm{CPSpec}(\mathcal{C}^r) = \mathcal{B}(\mathcal{C}^r) \cap \mathrm{CPSpec}(\mathcal{C}'^r)$, hence for $L \in \mathcal{L}(\mathcal{C}^r)$ and $L' \in \mathcal{L}(\mathcal{C}'^r)$ we have $\boldsymbol{\theta}^{-1}(\mathcal{V}_{L'}) = \mathcal{V}_{\theta^{-1}(L')}$, $\boldsymbol{\theta}(\mathcal{V}_L) = \mathcal{V}_{\theta(L)}$ and $\boldsymbol{\theta}(\mathcal{X}_L) = \mathcal{X}_{\theta(L)}$.

5. Since θ is an isomorphism, $\widetilde{\theta}$ is bijective by [Abu2006, Proposition 4.5.]. In this case $\boldsymbol{\theta}$ and $\boldsymbol{\theta}^{-1}$ are obviously continuous (see "4").■

Example 4.9. ([NT2001, Example 1.1.]) Let k be a field and $C := k[X]$ be the cocommutative k-coalgebra with $\Delta(X^n) := X^n \otimes_k X^n$ and $\varepsilon(X^n) := 1$ for all $n \geq 0$. For each $n \geq 0$, set $C_n := kX^n$. Then $\mathrm{CPSpec}(C) = \mathcal{S}(C) = \{C_n \mid n \geq 0\}$. Notice that

1. $\mathbf{Z}_C$ is discrete by Theorem 3.6, hence $\mathbf{Z}_C$ is Lindelof (but not compact) by Theorem 3.9.

2. $\mathrm{CPSpec}(C)$ is not connected: $\mathrm{CPSpec}(C) = \{C_n \mid n \geq 1\} \cup \{C_0\} = \mathcal{X}_{\{k\}} \cup \mathcal{X}_{<X,X^2,\ldots>}$ (notice that $\mathrm{CPSpec}(C)$ is not subdirectly irreducible, compare with Lemma 3.14.

Example 4.10. ([NT2001, Example 1.2.]) Let k be a field and $C := k[X]$ be the cocommutative k-coalgebra with $\Delta(X^n) := \sum_{j=1}^{n} X^j \otimes_k X^{n-j}$ and $\varepsilon(X^n) := \delta_{n,0}$ for all $n \geq 0$. For each $n \geq 0$ set $C_n :=< 1, ..., X^n >$. For each $n \geq 1$, $C_n \subseteq (C_{n-1} :_C < kX^n >)$, hence not fully C-coprime and it follows that $\mathrm{CPSpec}(C) = \{k, C\}$ (since k is simple, whence fully C-coprime and $C^* \simeq k[[X]]$ is an integral domain, whence C is fully coprime). Notice that

1. C is subdirectly irreducible with unique simple subcoalgebra $C_0 = k$;

2. the converse of Remark 3.5 "3(d)" does not hold in general: $\mathrm{Corad}(C) = k \subseteq C_1$ while $\mathcal{X}_{C_1} = \{C\} \neq \varnothing$ (compare Proposition 3.7 "2").

3. $\mathrm{CPSpec}(C)$ is connected, although $\mathcal{S}(C) \subsetneqq \mathrm{CPSpec}(C)$ (see Lemma 3.14 "2").

4. $\mathbf{Z}_C$ is not T_1 by Theorem 3.6, since $C \in \mathrm{CPSpec}(C) \backslash \mathcal{S}(C)$: in fact, if $C \in \mathcal{X}_{L_1}$ and $C_0 \in \mathcal{X}_{L_2}$ for some C-subcoalgebras $L_1, L_2 \subseteq C$, then $L_2 = 0$ (since C is subdirectly irreducible with unique simple subcoalgebra C_0); hence $\mathcal{X}_{L_2} = \{C_0, C\} = \mathrm{CPSpec}(C)$ and $\mathcal{X}_{L_1} \cap \mathcal{X}_{L_2} = \mathcal{X}_{L_1} \neq \varnothing$.

Remark 4.11. As this paper extends results of [NT2001], several proofs and ideas are along the lines of the original ones. However, our results are much more general (as [NT2001] is restricted to coalgebras over fields). Moreover, we should warn the reader that in addition to the fact that several results in that paper are redundant or repeated, several other results are even *absurd*, e.g. Proposition 2.8., Corollary 2.4. and Theorem 2.4. (as noticed by Chen Hui-Xiang in his review; Zbl 1012.16041) in addition to [NT2001, Lemma 2.6.] as we clarified in Remark 3.8. We corrected the statement of some of these results (e.g. Proposition 3.7 corrects [NT2001, Lemma 2.6.]; while Proposition 4.8 suggests a correction of [NT2001, Theorem 2.4.] which does not hold in general as the counterexample [Abu2006, 5.20.] shows). Moreover, we improved some other results (e.g. applying Theorem 3.6 to coalgebras over base fields improves and puts together several scattered results of [NT2001]).

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