Classical Invariants for Global Actions and Groupoid Atlases

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Abstract

A global action is the algebraic analogue of a topological manifold. This construction was introduced in first place by A. Bak as a combinatorial approach to K-Theory and the concept was later generalized by Bak, Brown, Minian and Porter to the notion of groupoid atlas. In this paper we define and investigate homotopy invariants of global actions and groupoid atlases, such as the strong fundamental groupoid, the weak and strong nerves, classifying spaces and homology groups. We relate all these new invariants to classical constructions in topological spaces, simplicial complexes and simplicial sets. This way we obtain new combinatorial formulations of classical and non classical results in terms of groupoid atlases.

1 Introduction

The *global action* construction of K-theory, introduced by A.Bak [2, 3], associates to a ring A an algebraic object, namely the global action $GL(A)$ which constitutes the algebraic analogue of the standard topological construction. The underlying set of the global action $GL(A)$ consists of the points of the general linear group of A and the action consists of the virtual triangular subgroups of the general linear group acting on the general linear group by left multiplication.

This new approach introduced by Bak has the advantage that the solutions of algebraic problems can be followed algebraically step by step. The notion of global action gives algebraic objects such as groups, structures which allow one to develop homotopy theory similarly to the classical way, by defining paths and deformations of morphisms. In [12, 13], the second named author developed an axiomatic homotopy theory for categories with natural cylinders, which can be applied to global actions.

Recently, A. Bak, R. Brown, E.G. Minian and T. Porter generalized ideas and constructions of global actions and introduced *groupoid atlases* [4]. As it was pointed out in [4], there were many advantages in formulating the concept of global action in terms of groupoids instead of group actions, so that it becomes part of a wider notion, namely the concept of a groupoid atlas. This was done using the well know transition of group actions to groupoids. A groupoid atlas can be regarded as an *algebraic manifold*, where the local groupoids play the role of the charts.

In this paper we define and investigate homotopy invariants of global actions and groupoid atlases. We study the strong fundamental groupoid, the weak and strong nerves, classifying

spaces and homology theory of groupoid atlases. We also relate all these new invariants to classical constructions in topological spaces, simplicial complexes and simplicial sets and obtain this way new combinatorial formulations of classical and non classical results in terms of groupoid atlases.

The rest of the paper is organized as follows.

In section 2 we recall the basic definitions, examples and results on global actions and groupoid atlases. Nothing is very new in this section with the exception of a couple of new examples. One of these examples appears naturally when the global action $A(G, \mathcal{H})$ (defined in [4]) acts on a G- set X. This induces a new and interesting global action. In the particular case that the general linear group acts on the quadratic forms, this construction is related to hermitian K-Theory.

It is important to remark that the model for the line L that we introduce here is not exactly the same one that is used in [2] or [4]. Both models are isomorphic at the weak level, but the original notion of the line becomes quite rigid when one works with morphisms that also preserve information of the local actions. This change is essential when defining the strong fundamental group of a groupoid atlas.

We explain below the reasons of this change. We shall also prove that both models are *equivalent* groupoid atlases. Moreover, our model is the *regularization* of the original one.

In section 3 we introduce some of the fundamental concepts of this work, such as the notion of *equivalence* between maps of groupoid atlases and the notions of *irreducible* and *regular* atlases.

In section 4 we study the strong fundamental group of a groupoid atlas. We use first a *geometric* approach and later we prove that it can also be defined and computed with a more algebraic approach, related to the vertex group of the colimit groupoid of the atlas. We relate the strong fundamental groupoid with the weak one (introduced in [4]) and we also show how to compute the fundamental groupoid of a topological space using groupoid atlases, via the Van Kampen Theorem. New formulations of classical and non classical results on the fundamental group of an open covering are also discussed.

The last section of the paper is devoted to simplicial methods. To each groupoid atlas we associate a simplicial set, which we call the *(strong) nerve* for historical reasons and also because it generalizes in some sense the nerve of a groupoid. We also introduce a weak version of the nerve (compare with [4]). We obtain this way another definition for the fundamental group of a groupoid atlas. This independently defined simplicial version is proved to coincide with the other two defined in the previous section.

We finish the paper with an introduction to the homology theory of groupoid atlases. We compute some easy but clarifying examples. The article ends with a result which relates the local homology groups with the homology of the whole atlas.

2   Preliminaries: Global Actions and Groupoid Atlases

Before recalling the basic definitions of *global actions*, let us begin with an easy example, which was already exhibited in [4].

Let G be a group and let $\mathcal{H} = \{H_\alpha\}_{\alpha \in \phi}$ be a family of subgroups of G, acting on G by left multiplication. This will induce a global action, denoted by $A(G, \mathcal{H})$.

If $\mathcal{H}$ consists of a single group H, then this global action is simply the set of orbits of the action, but when $\mathcal{H}$ consists of more than one subgroup of G, then the different orbits of the actions interact. We will see later that this interaction is crucial from the homotopical point of view. For example, consider the case $A = A(D_3, \mathcal{H})$, where D_3 denotes the dihedral group of order 6. Take $\phi = \{a, b\}$, $H_a = \langle r \rangle$ with r a rotation of order 3 and $H_b = \langle s \rangle$ with s a symmetry. The actions of each H_α divides G in orbits which determine the following covering of G.

We denote by $\mathcal{U}$ this covering. Its nerve $N\mathcal{U}$ is the simplicial complex corresponding to the following diagram

In section 3 we shall study the strong fundamental group of global actions. As we can guess, in this case the fundamental group will be isomorphic to $\mathbb{Z} * \mathbb{Z}$.

2.1   Definitions and examples on global actions

Definition 2.1.1. A *global action* A consists of a set X_A together with a family of group actions $\{G_\alpha \curvearrowright X_\alpha \mid \alpha \in \phi_A\}$ on subsets $X_\alpha \subseteq X_A$. These actions are related by certain morphisms which glue them together coherently. More exactly,

$$A = (X_A, \phi_A, \{X_\alpha\}_\alpha, \{G_\alpha\}_\alpha)$$

where

  (a) X_A is a set,

  (b) $\phi_A = (\phi_A, \leqslant)$ is an index set equipped with a reflexive relation,

  (c) for each $\alpha \in \phi_A$, there is a subset $X_\alpha \subset X_A$ and a *local* group G_α acting on X_α and

(d) if $\alpha \leqslant \beta$, there exists a homomorphism $\phi_\alpha^\beta : G_\alpha \to G_\beta$ such that $g.x = \phi_\alpha^\beta(g).x$ for every x in $X_\alpha \cap X_\beta$, $g \in G_\alpha$.

Moreover, $\phi_\alpha^\alpha = id_{G_\alpha}$ and $\phi_\gamma^\beta \circ \phi_\alpha^\gamma = \phi_\alpha^\beta$, whenever these compositions have sense.

Note that $G_\alpha(X_\alpha \cap X_\beta) \subset X_\alpha \cap X_\beta$, i.e. the α-orbits of each element in the intersection are included in the intersection.

We call A a *single domain global action* if $X_A = X_\alpha$ for each α. If $X_A = \bigcup_\alpha X_\alpha$, we say that A is *covered*.

The global action $A = A(G, \mathcal{H})$ of above is an example of a single domain global action. In this case, $X_A = G$, the local actions $H_\alpha \curvearrowright X_\alpha = G$ are the subgroups of the family acting by left multiplication, and any two indices α and β satisfy $\alpha \leqslant \beta$ if and only if $H_\alpha \subset H_\beta$. The associated group homomorphisms ϕ_α^β are the inclusions.

Another interesting example of a single domain global action is the *general linear global action* $GL(n, R)$, where R is an associative ring with unit. This example was already studied in [2] and [4], but we recall it here briefly, since it is one of the motivating examples for this theory. The homotopy groups of this global action coincide with the K-theory groups of the ring R.

Let $n \in \mathbb{N}$. A subset α of $\Lambda = \{(i, j) \mid i \neq j, \ 1 \leqslant i, j \leqslant n\}$ is called *closed* if every time that it contains the pairs (i, j) and (j, k), then the pair (i, k) is also in α.

Consider the poset $\phi = \{\alpha \subset \Lambda \mid \alpha \ closed\}$ partially ordered by inclusion.

Let $G_\alpha = GL(n, R)_\alpha$ be the subgroup of $GL(n, R)$ generated by the matrices

$$\{E_{ij}(r) \mid r \in R, (i, j) \in \alpha\},$$

where $E_{ij}(r)$ is the matrix containing 1 in the diagonal, r in the position (ij) and 0 elsewhere.

It is not difficult to verify that a matrix A belongs to $GL(n, R)_\alpha$ if and only if $A_{ij} = 1$ for $i = j$ and $A_{ij} = 0$ if $(i, j) \notin \alpha$.

For $\alpha \subset \beta$, we denote ϕ_α^β the inclusion $GL(n, R)_\alpha \to GL(n, R)_\beta$. Now let $X_\alpha = GL(n, R)$. The subgroup $GL(n, R)_\alpha$ acts on $GL(n, R)$ by left multiplication.

Note that the general linear global action is a particular example of the actions $A(G, \mathcal{H})$ introduced before. If $A(G, \mathcal{H})$ is considered as an atlas with discrete index set (see [4]), this is not longer true since in the general linear global action situation we are making some identifications between local actions.

Example 2.1.2. Let $A = A(G, \mathcal{H})$ and let $G \curvearrowright X$ be a G-set. The global action $A \curvearrowright X$ is an extension of the action of G on X and it is defined as follows. Take $X_{A \curvearrowright X} = X_\alpha = X$ for each $\alpha \in \phi_{A \curvearrowright X} = \phi_A$, with the action induced by the action of G.

As a particular case, consider the general lineal group $GL(n, \mathbb{R})$ acting over the quadratic forms in $\mathbb{R}^n$ by base change, i.e.

$$(C \cdot Q)(x) = Q(C.x), \ C \in GL_n(\mathbb{R}), \ Q \text{ a quadratic form, } x \in \mathbb{R}^n.$$

This action restricts well to the subset X of positive defined quadratic forms. If A is the matrix of Q, then $C^t A C$ is the matrix of $C \cdot Q$.

2.2   Path components and the weak fundamental group

Consider again our first example $A = A(D_3, \mathcal{H})$, where D_3 denotes the dihedral group of order 6, and take the element $1 \in D_3$. When different elements of the local groups H_a and H_b act consecutively on it, we obtain a path like the following

$$1 \xrightarrow{s\cdot} s \xrightarrow{r\cdot} rs \xrightarrow{s\cdot} srs \xrightarrow{r\cdot} rsrs = 1.$$

In general, the various actions of the local groups G_α interact on the intersections of the local sets X_α and this induces a global dynamics in A. The elements of X_A move along X_A through the actions of the different local groups. This idea suggests definitions for paths and loops in A and therefore notions for path connectedness and simply connectedness.

Let $A = (X_A, \phi_A, \{X_\alpha\}_\alpha, \{G_\alpha\}_\alpha)$ be a global action and let $\alpha \in \phi_A$. An α-*frame* is a finite subset $\{x_0, \ldots, x_n\} \subset X_A$ such that for each i there exists $g_i \in G_\alpha$ with $g_i \cdot x_{i-1} = x_i$. A (weak) *path* is a finite sequence $x_0, \ldots, x_n$ such that for each i the set $\{x_{i-1}, x_i\}$ is a *local frame*, i.e. an α-frame for some index α.
Given two elements x, y of X_A, we say that they are in the same *connected component* of A if there exists a path joining both points. As usual, we denoted by $\pi_0(A)$ the set of (path) components of A.

Let us compute $\pi_0(A)$ in the examples of above.
Consider first the case $A = GL(n, R)$. If x and y are in the same component, then there is a finite sequence of matrices $E_i \in G_i = GL(n, R)_{\alpha_i}$ such that

$$E_n E_{n-1} \ldots E_1 x = y$$

which implies that they determine the same class in the quotient $GL(n, R)/E(n, R)$. Here $E(n, R)$ denotes the subgroup of elementary matrices. Since every elementary matrix can be factored as a finite products of $E_i \in GL(n, R)_{\alpha_i}$, we obtain that

$$\pi_0(GL(n, R)) = GL(n, R)/E(n, R) = K_1(n, R).$$

For more details, see [2, 3, 5, 4, 11, 18].
In the case $A = A(G, \mathcal{H})$, a similar argument shows that $x, y \in A$ are in the same component if and only if there exists a sequence $h_{\alpha_i} \in H_{\alpha_i}$ such that $h_{\alpha_n} \ldots h_{\alpha_1} x = y$. If we denote $\langle \mathcal{H} \rangle = \langle H_i \mid i \in \phi \rangle$ the subgroup of G generated by all H_i, then we obtain

$$\pi_0(A(G, \mathcal{H})) = G/\langle \mathcal{H} \rangle.$$

Recall that a *weak morphism* $f : A \to B$ is a set theoretic function $f : X_A \to X_B$ which preserves local frames. A path is a particular example of a weak morphism between global actions. More exactly, let L be the global action with underlying set $X_L = \mathbb{Z}$, with the index set $\phi_L \subset \mathcal{P}(\mathbb{Z})$ the family of subset of $\mathbb{Z}$ of the form $\{n\}$ and $\{n, n+1\}$, and whose local actions are the free and transitive actions of the trivial group and G_2 respectively. A path in A is simply a morphism $L \to A$ that *stabilizes* in both directions (i.e. $\exists N$ such that $f(n) = f(n+1)$ for $|n| > N$).
If a weak morphism $f : L \to A$ does not stabilize, we call it a *weak curve*. A path with the same initial and final point is a *weak loop*.

Remark 2.2.1. As we pointed out in the introduction, the model for the line L that we introduce here is not the same one used in [2] or [4]. Both models are isomorphic at the weak level, but the original notion becomes very rigid when one works with morphisms that preserve the information of the local actions.

Recall that the product of global actions is defined as follows. Given two global actions A and B, the *product* $A \times B$ is the global action with underlying set $X_A \times X_B$ and index set $\phi_A \times \phi_B$, equipped with the product relation. The local action of $A \times B$ indexed by (α, β) is the product action between $G_\alpha \curvearrowright (X_A)_\alpha$ and $G_\beta \curvearrowright (X_B)_\beta$.
The product $A \times B$, defined as above, satisfies the universal property of the categorical product in the category of global actions.

A homotopy between paths ω and ω' is defined as a weak morphism

$$H : L \times L \to A$$

for which there exist integers N_0, N_1 such that $H(-, N_0) = \omega$ and $H(-, N_1) = \omega'$, and that stabilizes in an appropriate sense (see [2]). In particular, the local frames of the product $L \times L$ are the subsets S satisfying

$$S \subset \{(n, m), (n, m + 1), (n + 1, m), (n + 1, m + 1)\}$$

for some $n, m \in \mathbb{Z}$.

Definition 2.2.2. Let ω and ω' be loops based on $x \in X_A$. A *homotopy* between ω and ω' is a function $H : \mathbb{Z} \times \mathbb{Z} \to X_A$ such that

- for all m, n, the sets $\{H(n, m), H(n, m + 1), H(n + 1, m), H(n + 1, m + 1)\}$ are local frames of A, and

- there exist $N_0 < N_1, M_0 < M_1 \in \mathbb{Z}$ such that $H(-, M_0) = \omega$, $H(-, M_1) = \omega'$ and

$$H(n, m) = \begin{cases} H(N_0, m) & if \ n < N_0 \\ H(N_1, m) & if \ n > N_1 \\ H(n, M_0) & if \ m < M_0 \\ H(n, M_1) & if \ m > M_1 \end{cases}$$

We can thus define the weak fundamental group of a global action X_A with base point $a \in X_A$ as follows.

Definition 2.2.3. The *weak fundamental group* $\pi_1^w(A, x)$ is the set of homotopy classes of loops at x. The multiplication is defined via concatenation of paths.

Let us compute $\pi_1^w(A, x)$ in the case $A = A(G, \mathcal{H})$. Without loss of generality, we may suppose $x = e$ the neutral element of G. We denote with $\coprod_{\cap} H_\alpha$ the amalgamated product of H_α, i.e. the colimit in the category of groups over the diagram $\{H_\alpha \cap H_\beta, H_\alpha\}$.

Proposition 2.2.4. *The weak fundamental group of $A(G, \mathcal{H})$ can be computed as the kernel of the canonical map $\coprod_{\cap} H_\alpha \to G$.*

Proof. The group $\pi_1^w(A, e)$ is isomorphic to the fundamental group of the nerve of the cover of G by H-coclasses, with $H \in \mathcal{H}$ (see below). Let $\widetilde{G} = \coprod_{\cap} H_\alpha$ and write $j(H)$ for the image of $H \to \widetilde{G}$. Let $N\widetilde{G}$ be the nerve of the cover of $\widetilde{G}$ by $j(H)$-coclasses. The canonical group homomorphism $\widetilde{G} \to G$ induces the universal covering $N\widetilde{G} \to NG$ between the nerves (see [1],[4]). Given $j(H)$ a vertex of $N\widetilde{G}$, a Deck transformation φ is determined by $\varphi(j(H))$, which could be any element of $\{g.j(H)|g \in Ker(\widetilde{G} \to G)\}$, the fiber over H. Multiplication by g gives a simplicial map on $N\widetilde{G}$ for $g \in \widetilde{G}$, hence it gives a Deck transformation for $g \in Ker(\widetilde{G} \to G)$. We conclude that the group of Deck transformations is exactly $Ker(\widetilde{G} \to G)$. $\qquad\square$

Note. Here, by a *covering* of simplicial complexes, we mean a simplicial map with the unique lifting property of simplices. Observe that a simplicial map $K \to L$ is a covering if and only if $|K| \to |L|$ is a covering of topological spaces.

At the weak level, a global action is the same as a set equipped with a cover. Let A be a global action, X_A the underlying set and $\mathcal{U}_A$ the covering determined by the local orbits. Let $V(\mathcal{U}_A)$ be the *Vietoris complex* of $\mathcal{U}_A$, whose simplices are the finite subsets of X_A that are included in some element of $\mathcal{U}_A$. Let $N(\mathcal{U}_A)$ be the *nerve* of this covering, whose simplices are the finite subsets of $\mathcal{U}_A$ with non trivial intersection. Since local frames in A are just simplices in $V\mathcal{U}_A$, we have

$$\pi_0(A) \cong \pi_0(V\mathcal{U}_A) \cong \pi_0(N\mathcal{U}_A)$$

and

$$\pi_1^w(A) \cong \pi_1(V\mathcal{U}_A) \cong \pi_1(N\mathcal{U}_A).$$

Dowker's theorem, proved in [4], relates both simplicial complexes $N\mathcal{U}_A$ and $V\mathcal{U}_A$. In general, one has the following result.

Proposition 2.2.5. *The functor $A \mapsto VA$ is an equivalence between the category of (covered) global actions with weak morphism and the category of simplicial complex.*

An inverse can be obtained by giving to a simplicial complex K a global action

$$\{S(s) \curvearrowright s \mid s \ simplex \ of \ K\},$$

with the simplices of K as indices and whose underlying set is the set of vertices of K. Here $S(X)$ means the group of bijections of the set X.

There exists a stronger notion of morphisms of global actions, which originally were called regular in [2].
A *regular morphism* $f : A \to B$ is a triple (X_f, ϕ_f, G_f) satisfying

(a) $\phi_f : \phi_A \to \phi_B$ is a relation preserving map,

(b) $G_f(\alpha) : G_\alpha \to G_{\phi_f(\alpha)}$ is a group morphism such that if $\alpha \leqslant \beta$ the diagram

$$
\begin{array}{ccc}
G_\alpha & \longrightarrow & G_{\phi_f(\alpha)} \\
\downarrow & & \downarrow \\
G_\beta & \longrightarrow & G_{\phi_f(\beta)}
\end{array}
$$

   commutes,

(c) $X_f : X_A \to X_B$ is a set function such that $X_f(X_\alpha) \subset X_{\phi_f(\alpha)}$ and

(d) for each $\alpha \in \phi_A$, $(G_f, X_f) : G_\alpha \curvearrowright X_\alpha \to G_{\phi_f(\alpha)} \curvearrowright X_{\phi_f(\alpha)}$ is a morphism of actions, i.e. $G_f(\alpha)(g) \cdot X_f(x) = X_f(g \cdot x)\ \forall\ g \in G_\alpha,\ x \in X_\alpha$.

In order to generalize the constructions of above, we must be very careful about what a strong path or a strong homotopy is. Regular morphisms, that a priori allow us to work with curves, paths and loops in a strong sense, are very restrictive. To give an idea, let us consider a global action such that all its local groups are finite of odd order. Because a regular morphism contains group morphisms as part of its data, there are no regular morphisms $L \to A$ which are not constant. The regular maps $L \to A$ do not measure, in general, the dynamics of A.

Important Note. In this paper, the word **regular** will mean a different concept (see 3.4.1). Since regular morphisms of global actions (in the sense of [2]) are not used in this article, there should be no confusion.

2.3   Groupoid atlases

Giving a group G acting on a set X, one can associate to the group action $G \curvearrowright X$ a groupoid $\mathcal{G} = G \ltimes X$. The objects of $\mathcal{G}$ are the elements of X and for any $x, y \in X$, the arrows from x to y are the pairs (g, x) with $g \in G$ such that $g \cdot x = y$. Composition is defined in the obvious way.

Applying this construction to each local orbit of a global action A, we obtain a groupoid atlas. The concept of a groupoid atlas was introduced in [4].

Definition 2.3.1. A *groupoid atlas* $A = (X_A, \phi_A, \mathcal{G}_A)$ consists of a set X_A, an index set ϕ_A equipped with a reflexive relation, and for each $\alpha \in \phi$, a groupoid $\mathcal{G}_\alpha$ such that

   - $X_\alpha = Obj\ \mathcal{G}_\alpha \subset X_A$,

   - if $\alpha \leqslant \beta$ in ϕ_A, $X_\alpha \cap X_\beta$ is union of components of $\mathcal{G}_\alpha$, and there is a functor $\phi_\alpha^\beta : \mathcal{G}_\alpha|_{X_\alpha \cap X_\beta} \to \mathcal{G}_\beta|_{X_\alpha \cap X_\beta}$ which restricts to the identity in objects.

The structural functor ϕ_α^β is the identity when $\alpha = \beta$, and it commutes with the compositions. The groupoids $\mathcal{G}_\alpha$ are called the *local groupoids* of A.

Notation. Let $\mathcal{G}$ be a groupoid and let $X \subset Obj\ \mathcal{G}$. We denote by $\mathcal{G}|_X$ the full subgroupoid of $\mathcal{G}$ on X.

Example 2.3.2. As we mentioned above, every global action induces a groupoid atlas. In particular, we endow $A(G, \mathcal{H})$ and $GL(n, R)$ with a structure of global action. Although not every groupoid atlas comes from a global action. For some examples, see [14].

Example 2.3.3. Any groupoid $\mathcal{G}$ can be viewed as a groupoid atlas with trivial index set. This groupoid atlas is denoted by $a(\mathcal{G})$. This construction induces a fully faithful functor into any of the categories of groupoid atlases that we discuss later.

Example 2.3.4. Let K be a simplicial complex. We define a groupoid atlas $a(K)$ as follows: $X_{aK} = V_K$, the set of vertices of K; $\phi_{aK} = S_K$, the set of simplices of K ordered by inclusion; for each simplex s, the local groupoid $\mathcal{G}_s$ is the simply connected groupoid (tree) with object set s; the structural functors ϕ_s^t are the inclusions.
As a particular case, the *n-sphere* is defined to be the groupoid atlas $a(\partial\Delta[n])$, where $\partial\Delta[n]$ is the simplicial complex with vertices $\{0, ..., n\}$ and simplices all the nonempty proper subsets of $\{0, ..., n\}$.

This functorial construction $K \mapsto aK$ associates to every simplicial complex K a groupoid atlas aK satisfying some extra properties that we will discuss later. For example, aK is irreducible, regular and infimum. Also, this construction induces a fully faithful functor into the category $[\mathcal{G}pd\mathcal{A}tl]$ defined in the next section. This way one might view groupoid atlases as generalized simplicial complexes, for which there are many others local models than the (homotopy trivial) simplices.

Another interesting examples of groupoid atlases arise from the fundamental groupoids of topological spaces.

Example 2.3.5. Let X be a topological space and $\mathcal{U}$ an open cover of X. The groupoid atlas $A(X, \mathcal{U})$ is defined as follows. The underlying set is X and the index set ϕ is the poset $(\mathcal{U}, \subset)$. For each $U \in \mathcal{U}$, the local groupoid $\mathcal{G}_U$ is the fundamental groupoid $\pi_1(U)$ and the morphisms ϕ_U^V are induced by the inclusions $U \hookrightarrow V$.

In section 4 we relate both the weak and the strong version of the fundamental group of $A(X, \mathcal{U})$ with $\pi_1(X)$.

Weak morphisms between groupoid atlases are, as one might suppose, functions between the underlying sets which preserve local frames. In this context, a *local frame* is a finite subset of some connected component of a local groupoid. This category, with groupoid atlases as objects and weak morphism as arrows, is canonically equivalent to those of global actions and simplicial complexes. We call it the *weak category* of groupoid atlases.

The notion of (strong) morphism of groupoid atlases is not as rigid as the notion of regular morphism of global actions.

Definition 2.3.6. A *morphism* $f : A \to B$ between groupoid atlases is a triple $(X_f, \phi_f, \mathcal{G}_f)$ satisfying

- $X_f : X_A \to X_B$ is a set-theoretic function,

- $\phi_f : \phi_A \to \phi_B$ is a function which preserves the relation $\leq$,

- $\mathcal{G}_f : \mathcal{G}_A \to \mathcal{G}_B$ is a (generalized) natural transformation of groupoid diagrams over the function ϕ_f, which restricts to X_f in the objects.

  In other words, for each α a functor $\mathcal{G}_f(\alpha) : \mathcal{G}_\alpha \to \mathcal{G}_{\phi_f(\alpha)}$ is given in such a way that $Obj\ \mathcal{G}_f(\alpha) = X_f|_{X_\alpha}$ and, if $\alpha \leqslant \beta$, the diagram

$$
\begin{array}{ccc}
\mathcal{G}_\alpha & \longrightarrow & \mathcal{G}_{\phi_f(\alpha)} \\
\downarrow & & \downarrow \\
\mathcal{G}_\beta & \longrightarrow & \mathcal{G}_{\phi_f(\beta)}
\end{array}
$$

  commutes.

We denote by $\mathcal{GpdAtl}$ the category of groupoid atlases with (strong) morphisms.

The atlas A is *covered* if every element of X_A is an object of some local groupoid. All the groupoid atlases that we consider are assumed to be covered.
Note that, if A is covered and $f : A \to B$ is any morphism, then the function X_f is locally determined by the values of the functors $\{\mathcal{G}_f(\alpha)\}_\alpha$ in objects. However, X_f must be part of the data.
To illustrate this, consider the following example. Suppose A is the groupoid atlas with $X_A = *$, $\phi_A = \{1, 2\}$ with the discrete order and $X_1 = X_2 = *$ and let B be any groupoid atlas. Suppose also that we are given a map $\phi_f : \phi_A \to \phi_B$ and a generalized natural transformation $\mathcal{G}_f : \mathcal{G}_A \to \mathcal{G}_B$. Note that the map $\phi_f : \phi_A \to \phi_B$ picks two indices $\phi_{f(1)}$ and $\phi_{f(2)}$ of ϕ_B, and $\mathcal{G}_f : \mathcal{G}_A \to \mathcal{G}_B$ determines two objects x and y of $\mathcal{G}_{\phi_{f(1)}}$ and $\mathcal{G}_{\phi_{f(2)}}$, respectively. If we take $x \neq y$, then $(\phi_f, \mathcal{G}_f)$ does not come from a morphism of atlases.

Under certain condition on the atlas A, a relation preserving map $\phi_f : \phi_A \to \phi_B$ and a natural family of functors $\mathcal{G}_\alpha \to \mathcal{G}_{\phi_f(\alpha)}$ do determine a map $A \to B$. We introduce the concept of a *good* atlas, which solves this problem and plays a fundamental role in the next sections. This concept is weaker than (but intimately related with) the notion of *infimum*, introduced in [2] and [4] (cf. 4.4).

Definition 2.3.7. Let A be a groupoid atlas. We say that A is *good* if for every $x \in X_A$ the set

$$
\phi_x = \{\alpha \in \phi_A \mid x \in X_\alpha\}
$$

has an initial element, i.e. there exists $\alpha_x \in \phi_x$ such that $\alpha_x \leqslant \beta$ for all $\beta \in \phi_x$.

Remark 2.3.8. Suppose A is a good atlas. Given a relation preserving map $\phi_f : \phi_A \to \phi_B$ and $\mathcal{G}_f : \mathcal{G}_A \to \mathcal{G}_B$ a natural family of functors over ϕ_f, the various local functions $\{Obj\ \mathcal{G}_f(\alpha)\}_{\alpha \in \phi_A}$ agree in the intersections because $Obj\ \mathcal{G}_f(\alpha)(x)$ must be $Obj\ \mathcal{G}_f(\alpha_x)(x)$ for all α, and therefore there is a well defined function $X_f : X_A \to X_B$. In this case, a morphism $f : A \to B$ can be regarded as a pair $(\phi_f, \mathcal{G}_f)$.

3 Equivalences of Maps and Irreducible Atlases

In order to define the right notion of strong fundamental group we need to change first a little bit our notion of morphism. Many maps $A \to B$ seem to carry the same information, and they just differ in the indices. For example, if A is such that $\phi_A = \{0 < 1\}$ with $\mathcal{G}_0$ a subgroupoid of $\mathcal{G}_1$ and ϕ_0^1 is the inclusion, this atlas does not contain more information than the one coming from $\mathcal{G}_1$. Moreover, given $a \in X_A$, it is necessary to identify the various identities of this element viewed as an object of the different local groupoids. Also, we should understand $\phi_\alpha^\beta(g)$ as an *extension* of the *movement g*.

This induces the notion of equivalence between morphisms.

3.1 Morphisms modulo equivalences

Definition 3.1.1. Given $f, f' : A \to B$, we say that f is a *corestriction* of f' if $\phi_f(\alpha) \leqslant \phi_{f'}(\alpha)$ for all $\alpha \in \phi_A$ and all the diagrams

$$
\begin{array}{ccc}
 & \mathcal{G}_\alpha & \\
\mathcal{G}_{f(\alpha)} \swarrow & & \searrow \mathcal{G}_{f'(\alpha)} \\
\mathcal{G}_{\phi_f(\alpha)} & \xrightarrow[\phi_{\phi_f(\alpha)}^{\phi_{f'(\alpha)}}]{} & \mathcal{G}_{\phi_{f'(\alpha)}}
\end{array}
$$

are commutative. We write $f \lhd f'$.

The morphisms $f, f' : A \to B$ are *equivalent* if they are in the same class of the equivalence relation generated by the corestrictions. We write $f \sim f'$.

Remark 3.1.2. It is easy to see that $f \sim f'$ implies $X_f = X_{f'}$.

Note that the relation $\lhd$ is transitive. Moreover, if $f \lhd f'$, then $g \circ f \lhd g \circ f'$ and $f \circ h \lhd f' \circ h$ for any g, h. It follows that, if $f, f' : A \to B$ and $g, g' : B \to C$ are such that $f \sim f'$ and $g \sim g'$, then $g \circ f \sim g' \circ f'$. Therefore, we can define a new category $[\mathcal{G}pd\mathcal{A}tl]$, with groupoid atlases as objects and the classes of (strong) morphisms as maps. We denote the class of f by $[f]$.

The notion of equivalence between morphisms loosen the dependence of A from the index set ϕ_A. Let us go back to the example at the beginning of this section. The information of the atlas A seems to lie just in $\mathcal{G}_1$. If we call $B = a\mathcal{G}_1$ the atlas with this only local groupoid, the canonical arrows $f : A \to B$ and $g : B \to A$ satisfy $gf \sim id_A$ and $fg = id_B$, so they are isomorphic in $[\mathcal{G}pd\mathcal{A}tl]$.

Definition 3.1.3. We say that $f : A \to B$ is an *equivalence* if there exists $g : B \to A$ such that $gf \sim id_A$ and $fg \sim id_B$, i.e. $[f]$ is an isomorphism in $[\mathcal{G}pd\mathcal{A}tl]$. Given atlases A and B, we say that they are *equivalent* if there is an equivalence between them.

3.2   Irreducible atlases

Recall that the set ϕ_A of a groupoid atlas $(X_A, \phi_A, \mathcal{G}_A)$ is equipped with a reflexive relation. If $\alpha, \beta \in \phi_A$ are such that $\alpha \leqslant \beta$, by definition $X_\alpha \cap X_\beta$ can be written as union of connected components of the groupoid $\mathcal{G}_\alpha$. If in addition $\mathcal{G}_\alpha$ is connected, it follows that $X_\alpha \subset X_\alpha \cap X_\beta$ and the structural functor $\phi_\alpha^\beta : \mathcal{G}_\alpha|_{X_\alpha \cap X_\beta} \to \mathcal{G}_\beta|_{X_\alpha \cap X_\beta}$ can be regarded simply as a functor between $\mathcal{G}_\alpha$ and $\mathcal{G}_\beta$. So, when each local groupoid is connected, a groupoid atlas is nothing but a particular diagram of groupoids.

Definition 3.2.1. We will say that a groupoid atlas A is *irreducible* if $\mathcal{G}_\alpha$ is a connected groupoid for every index α in ϕ_A.

Given any atlas A, there is a natural way to associate to it an irreducible atlas $i(A)$, considering each component of any local groupoid as an individual local groupoid. The underlying set of $i(A)$ is the same of A, so $X_{i(A)} = X_A$. The index set of $i(A)$ must repeat each index of A as many times as many components of the corresponding groupoid. Explicitly,

$$\phi_{i(A)} = \{ \, (\alpha, X) \mid \alpha \in \phi_A, \ X = Obj \, \mathcal{G}' \subset X_A, \ \mathcal{G}' \text{ component of } \mathcal{G}_\alpha \}$$

and the relation is induced from ϕ_A:

$$(\alpha, X) \leqslant (\beta, Y) \iff \alpha \leqslant \beta \text{ and } X \subset Y.$$

Finally, the local groupoid $\mathcal{G}_{(\alpha,X)}$ is the component of $\mathcal{G}_\alpha$ with object set X, and if $(\alpha, X) \leqslant (\beta, Y)$, the morphism $\phi_{(\alpha,X)}^{(\beta,Y)}$ equals ϕ_α^β. Like in any irreducible atlas, the structural maps of $i(A)$ are defined over the whole local groupoid.

This construction is functorial, since every map $f = (X_f, \phi_f, \mathcal{G}_f) : A \to B$ induces a new map $if : iA \to iB$ given by $X_{if} = X_f$, $\phi_{if}(\alpha, X) = (\phi_f(\alpha), Y)$ where Y is the component of $\mathcal{G}_{\phi_f(\alpha)}$ that contains $\mathcal{G}_f(\alpha)(X)$, and $\mathcal{G}_{if}(\alpha, X) : \mathcal{G}_{(\alpha,X)} \to \mathcal{G}_{\phi_{if}(\alpha,X)}$ is the restriction of the functor $\mathcal{G}_f(\alpha) : \mathcal{G}_\alpha \to \mathcal{G}_{\phi_f(\alpha)}$.

The canonical map $\varphi_A : iA \to A$ that forgets the second coordinate of the indices ($\phi_{\varphi_A}(\alpha, X) = \alpha$) and such that $\mathcal{G}_{\varphi_A}(\alpha, X) : \mathcal{G}_\alpha|_X \to \mathcal{G}_\alpha$ is the inclusion of each component, is natural in A. It satisfies also the following universal property, whose proof is straightforward.

Proposition 3.2.2. *Let $g : B \to A$ a map of groupoid atlases, and suppose that B is irreducible. Then, there exists a unique map $h : B \to iA$ such that $g = \varphi_A \circ h$.*

It is clear that $iA = A$ when the atlas A is already irreducible. The functor i is a right adjoint for the inclusion of the full subcategory of irreducible atlases into the category of atlases.

The map $\phi_A : iA \to A$ is a weak isomorphism, but it is not an isomorphism or an equivalence in general. However, the groupoid atlases A and iA are intimately related: they share all the invariants that we will study, such as the nerve, the homology groups and even the fundamental group.

3.3   A new approach

Given an irreducible atlas A and indices α, β and γ in ϕ_A such that $\alpha \leqslant \beta \leqslant \gamma$, then $X_\alpha \subset X_\beta \subset X_\gamma$ and the composition $\phi_\beta^\gamma \circ \phi_\alpha^\beta$ is defined and it agrees with ϕ_α^γ if $\alpha \leqslant \gamma$ in ϕ. Therefore, in an irreducible atlas we can assume that the relation $\leqslant$ is transitive without loss of generality.

Proposition 3.3.1. *Given an irreducible atlas A, the relation $\leqslant$ defined in its set of indices ϕ_A is a partial order "up to equivalence", i.e. there exists an atlas B whose set of indices is partially ordered by $\leqslant$, and an equivalence $A \to B$.*

Proof. Recall that the relation $\leqslant$ defined in ϕ_A is always reflexive, and since A is irreducible, it can also be taken transitive. It only remains to make $\leqslant$ antisymmetric.
The atlas B will be obtained from A by deleting some local groupoids. We say that two indices $\alpha, \beta \in \phi_A$ are *paired* if $\alpha \leqslant \beta$ and $\beta \leqslant \alpha$. Clearly, been paired is an equivalence relation. We denote by $[\alpha]$ the paired class of α. If α and β are paired indices, then $X_\alpha = X_\beta$ and the functor ϕ_α^β must be an isomorphism. Let c be a selector function that assigns to each paired class an element of itself. We define B as follows,

- $X_B = X_A$,

- $\phi_B = \{c[\alpha] \mid \alpha \in \phi_A\}$, with the relation induced by the inclusion $\phi_B \hookrightarrow \phi_A$,

- $\mathcal{G}_B$ is the restriction of $\mathcal{G}_A$ to ϕ_B.

The map $f : B \to A$ is the canonical inclusion. The inverse, $g : B \to A$ is given by

- $X_g = id : X_A \to X_B$,

- $\phi_g : \phi_A \to \phi_B$ is the map $\alpha \mapsto c[\alpha]$. If $\alpha \leqslant \beta$, because α and $c[\alpha]$ are paired, β and $c[\beta]$ are paired and $\leqslant$ is transitive, we have $c[\alpha] \leqslant c[\beta]$,

- the functor $\mathcal{G}_g(\alpha) : \mathcal{G}_\alpha \to \mathcal{G}_{c[\alpha]}$ equals $\phi_\alpha^{c[\alpha]}$ the structural functor of A. The square

$$\begin{array}{ccc} \mathcal{G}_\alpha & \xrightarrow{\phi_\alpha^{c[\alpha]}} & \mathcal{G}_{c[\alpha]} \\ {\scriptstyle \phi_\alpha^\beta}\downarrow & & \downarrow{\scriptstyle \phi_{c[\alpha]}^{c[\beta]}} \\ \mathcal{G}_\beta & \xrightarrow[\phi_\beta^{c[\beta]}]{} & \mathcal{G}_{c[\beta]} \end{array}$$

is commutative because of the naturality of $\mathcal{G}_A$.

The composition $g \circ f$ is the identity of B, and $f \circ g$ is a corestriction of id_A. It follows that A and B are equivalent. □

The next definition, which is motivated by the last proposition, gives another approach to irreducible groupoid atlases. The proof of the equivalence between both definitions is omitted.

Alternative Definition 3.3.2. Given a partially ordered set ϕ, an *irreducible groupoid atlas* A with index set ϕ is a diagram in the category of groupoids $\mathcal{G}_A : \phi \to \mathcal{G}pd$ such that for each α the groupoid $\mathcal{G}_A(\alpha) = \mathcal{G}_\alpha$ is connected and if $\alpha \leqslant \beta$ then the induced functor ϕ_α^β, is an inclusion on the objects.

With this definition, the underlying set X_A is the union $\bigcup Obj\ \mathcal{G}_\alpha$.
A morphism $f : A \to B$ is a pair $(\phi_f, \mathcal{G}_f)$, with $\phi_f : \phi_A \to \phi_B$ a map of posets and $\mathcal{G}_f$ a natural transformation between $\mathcal{G}_A$ and $\mathcal{G}_B \circ \phi_f$ that induces a function X_f.

$$\phi_A \xrightarrow{\ \ \phi_f\ \ } \phi_B$$
$$\mathcal{G}_A \searrow \quad \swarrow \mathcal{G}_B$$
$$\mathcal{G}pd$$

In this context, the corestrictions arise naturally. Given $f, f' : A \to B$, the relation $f \lhd f'$ is equivalent to the existence of a natural transformation $\eta : \phi_f \Rightarrow \phi'_f : \phi_A \to \phi_B$ such that $\mathcal{G}_A \xrightarrow{\mathcal{G}'_f} \mathcal{G}_B \circ \phi_{f'}$ equals the composition $\mathcal{G}_A \xrightarrow{\mathcal{G}_f} \mathcal{G}_B \circ \phi_f \xrightarrow{\mathcal{G}_B \circ \eta} \mathcal{G}_B \circ \phi_{f'}$,

$$\phi_A \ \substack{\xrightarrow{\phi_{f'}} \\ \Uparrow \\ \xrightarrow[\phi_f]{}} \ \phi_B$$
$$\mathcal{G}_A \searrow \quad \swarrow \mathcal{G}_B$$
$$\mathcal{G}pd$$

3.4   Regular atlases

We may think a regular atlas as a *well pointed* atlas. Our model for the line is regular, and this is very convenient because of the existence of sections for the projection $A \times L \to A$ (they are necessary for the homotopies).

Definition 3.4.1. An atlas A is called *regular* if it is good and $\mathcal{G}_{\alpha_x} = *$ for every $x \in X_A$, where α_x is the minimum of ϕ_x (cf. 2.3.7).

Remark 3.4.2. Given $x \in X_A$, the index α_x in the definition of regular atlas is not unique, but two of such indices must be mutually related by $\leqslant$, and therefore the uniqueness can be assumed up to equivalence (proposition 3.3.1) when A is irreducible.

There is a regularization functor $A \to r(A)$ defined as follows.

$$X_{r(A)} = X_A$$

and

$$\phi_{r(A)} = \phi_A \sqcup \{\alpha_x\}_{x \in X_A}, \quad \text{with } \alpha_x \leqslant \alpha \text{ if and only if } x \in X_\alpha.$$

The local groupoids $\mathcal{G}_{\alpha_x}$ are singletons and the $\mathcal{G}_\alpha$'s are as before.

Example 3.4.3. The line L is a regular groupoid atlas. As we pointed out above, it is the regularization of the one used in [2] and [4].

Sometimes one need to replace a groupoid atlas A by its regularization rA, in order to work with paths, homotopies and morphisms in a right way. The following proposition asserts that, when A is good, this replacement does not change the equivalence class of the atlas. Then, every good atlas can be supposed to be regular up to equivalence.

Proposition 3.4.4. *Let A be a good atlas. The canonical inclusion $A \to rA$ is an equivalence.*

Proof. Let f be the inclusion $A \to rA$, we define $g : rA \to A$ by sending $\phi_g(\alpha_x)$ to the initial element of ϕ_x. It is easy to see that g is a morphism, $gf = id_A$ and $id_{rA} \sim fg$ (in fact, $id_{rA} \lhd fg$). $\square$

We will say that a morphism $f : A \to B$ between regular atlases is *regular* if $\phi_f(\alpha_x) = \alpha_{X_f(x)}$ for all α. The following three results will be used to characterize the (strong) fundamental group of groupoid atlases.

Proposition 3.4.5. *Let $f : A \to B$ be a morphism between regular atlases. Then there exists $f' : A \to B$ regular such that $f \sim f'$.*

Proof. Let us define f' and prove that it is equivalent to f. Take $X_{f'}$ equals X_f. For each α there is at least one $x \in X_A$ such that $\alpha = \alpha_x$ is the initial element of ϕ_x. Define $\phi_{f'}$ as the function that sends the initial element of ϕ_x to the initial element of $\phi_{X_f(x)}$, and the other indices α to $\phi_f(\alpha)$. The family of functors $\mathcal{G}_{f'}(\alpha) : \mathcal{G}_\alpha \to \mathcal{G}_{f'(\alpha)}$ is defined as follows: if there is an x such that $\alpha = \alpha_x$, $\mathcal{G}_{f'}(\alpha)$ is uniquely determined by X_f; otherwise, $\mathcal{G}_{f'}(\alpha)$ is just $\mathcal{G}_f(\alpha)$. Naturality follows from the naturality of $\mathcal{G}_f$, and since $f' \lhd f$ the result follows. $\square$

Proposition 3.4.6. *Let $f : A \to B$. If B is good, then there exists $g : r(A) \to B$ that extends f. Moreover, if g' is another extension, then $g \sim g'$.*

Proof. It is sufficient to extend ϕ_f to the indices $\{\alpha_x \mid x \in X_A\}$ of ϕ_{rA} that are not in ϕ_A, and this can be done sending each α_x to the minimum of $\{\alpha \in \phi_B \mid X_f(x) \in X_\alpha\}$. The morphism g obtained this way is a corestriction of any other extension. $\square$

When A is good and $f = id_A : A \to A$, the map $g : rA \to A$ is exactly the one constructed in 3.4.4. Proposition 3.4.6 proves that the regularization functor is right adjoint to the inclusion of the full subcategory of $[\mathcal{G}pd\mathcal{A}tl]$ formed by the good atlases.

Proposition 3.4.7. *Let $f, g : A \to B$ with A regular and B good. If $\phi_f(\alpha) = \phi_g(\alpha)$ and $\mathcal{G}_f(\alpha) = \mathcal{G}_g(\alpha)$ for all $\alpha \in \phi_A$ except perhaps for the indices $\{\alpha_x\}$, then $f \sim g$.*

Proof. Follows from the previous result. $\square$

4   The (Strong) Fundamental Group

Along this section, we work with good groupoid atlases.

4.1   Global points and global arrows

From the notion of equivalence, we will obtain a definition for points and arrows of an atlas A in a global sense, i.e. independent of the local groupoids.
We denote with $*$ the singleton groupoid (a single object and a single arrow), and with $\mathcal{I}$ the 2-point simply connected groupoid. Recall that $a : \mathcal{G}pd \to \mathcal{G}pd\mathcal{A}tl$ is the functor that maps each groupoid $\mathcal{G}$ into the atlas whose unique local groupoid is $\mathcal{G}$.

A morphism $a(*) \to A$ is a pair (α, x), where $\alpha \in \phi_A$ and $x \in X_\alpha$ are the images of the unique index and the unique object respectively. A same element $x \in X_A$ gives rise to many morphisms $a(*) \to A$, one for each α such that $x \in X_\alpha$. On the other hand, if $\alpha \leqslant \beta$, the two corresponding morphisms are equivalent.

Definition 4.1.1. A *global point* in A is the equivalence class of an arrow $a(*) \to A$.

$$\{points \ of \ A\} = Hom_{[\mathcal{G}pd\mathcal{A}tl]}(a(*), A)$$

Remark 4.1.2. The canonical function $\{points \ of \ A\} \to X_A$ that maps the class of $p : a(*) \to A$ in $X_p(*)$ is bijective when the atlas A is good. In fact, it is surjective because A is covered and injective because each ϕ_x has initial element.

As we pointed out before, we think $\phi_\alpha^\beta(g) \in \mathcal{G}_\beta$ as an extension of the *movement* $g \in \mathcal{G}_\alpha$. Let us consider the case of $A(G, \mathcal{H})$, with the family $\mathcal{H}$ closed under (finite) intersections. The arrows of a local groupoid $H \ltimes G$ come from the action of the subgroup H in G by left multiplication. Recall that (h, g) is the arrow with source g and target hg. These arrows are identified with the corresponding elements of G, without considering which H does actually contain them: if h belongs to H and H', since $\mathcal{H}$ is closed under intersections we might think $(h, g) \in H \ltimes G$ and $(h, g) \in H' \ltimes G$ as extensions of $(h, g) \in (H \cap H') \ltimes G$. Thus, the arrows of A are

$$\{(h, g)| \ g \in G, \ h \in H \ for \ some \ H \in \mathcal{H}\}$$

For a general atlas A, we propose the following definition.

Definition 4.1.3. A *global arrow* $[g]$ of A is the class of some local arrow $g \in \mathcal{G}_\alpha$ by the relation generated by $g \sim \phi_\alpha^\beta(g)$. Equivalently, a global arrow is the equivalence class of a morphism $a(\mathcal{I}) \to A$.

$$\{arrows \ of \ A\} = Hom_{[\mathcal{G}pd\mathcal{A}tl]}(a(\mathcal{I}), A)$$

Since $g \sim g'$ implies $X_g = X_{g'}$, there are well defined *source* and *target* of a global arrow $[g]$. They are $X_g(0)$ and $X_g(1)$, respectively.

Let A be a good atlas, and $x \in X_A$. If $x \in X_\alpha \cap X_\beta$, then $id_x \in \mathcal{G}_\alpha \sim id_x \in \mathcal{G}_\beta$ because we can reach one from the other by functors ϕ_α^β. We denote this global arrow as id_x. Observe

that $[g] = id_x$ does not imply that g is the identity of x in some local groupoid, because the structural functors ϕ_α^β may not be faithful.

Finally, recall that the *standard n-simplex* $\Delta[n]$ in the category of groupoids is the simply connected groupoid with object set $\{0, ..., n\}$. In particular, we have $* = \Delta[0]$ and $\mathcal{I} = \Delta[1]$. These classical definitions give us models for the simplices in $\mathcal{G}pd\mathcal{A}tl$, composing with a. As a generalization of what we did with points and arrows, they will be used in section 5 to define the nerve of a groupoid atlas.

4.2 Curves, paths and loops

In this subsection we will interpret a curve $L \to A$ as a sequence of global arrows. This leads to a very nice result: the fundamental group of a groupoid atlas equals the fundamental group of the colimit groupoid over the set of indices.

In the category $\mathcal{G}pd\mathcal{A}tl$, we consider the inclusions $i_n : a(\mathcal{I}) \to L$ given by $\phi_{i_n}(*) = \{n, n+1\}$ and $id = \mathcal{G}_{i_n}(*) : \mathcal{I} \to \mathcal{G}_{\{n,n+1\}}$. A morphism $\lambda : L \to A$ gives rise to a sequence $(\lambda_n)_{n \in \mathbb{Z}}$ defining $\lambda_n : a(\mathcal{I}) \to A$, $\lambda_n = \lambda \circ i_n$.

Given $\lambda : L \to A$, in the triple $(X_\lambda, \phi_\lambda, \mathcal{G}_\lambda)$ the function X_λ is determinated by the others, and any natural family of functors $(\phi_\lambda, \mathcal{G}_\lambda)$ induces a map $\lambda : L \to A$ since L is regular. Moreover, the local groupoids of L of the form $\mathcal{G}_{\{n\}}$ are singletons so each functor $\mathcal{G}_\lambda(\{n\})$ must be trivial, and each functor $\mathcal{G}_\lambda(\{n, n+1\})$ is given by an arrow in $\mathcal{G}_{\phi(\{n,n+1\})}$(the image of the arrow $\{n \to n+1\}$). Thus, a morphism λ determines and is determined by the following data.

- $(\alpha_n)_n \subset \phi_A$ where $\alpha_n = \phi_\lambda(\{n\})$, and

- $(\lambda_n)_n$, where $\lambda_n = \lambda \circ i_n : \mathcal{I} \to A$ satisfies $s(\lambda_{n+1}) = t(\lambda_n)$ and $\alpha_n, \alpha_{n+1} \leqslant \phi_{\lambda_n}(*)$.

Remark 4.2.1. A map $L \to A$ gives rise to a curve with a *framing* β in the vocabulary of [4]. Here, β is the function $n \to \alpha_n$. Conversely, one can get a map $L \to A$ from a weak curve w with a framing β, specifying local arrows g_n which make w a curve.

Definition 4.2.2. A *curve l* in A is an equivalence class of a morphism $\lambda : L \to A$.

If $(\lambda_n)_n$ is a sequence of local arrows such that $s(\lambda_{n+1}) = t(\lambda_n)$, we can construct a morphism $\lambda : L \to A$ taking for each n an index $\alpha_n \leqslant \phi_{\lambda_{n-1}}(*), \phi_{\lambda_n}(*)$, which exists when A is good. Here $\phi_{\lambda_n}(*)$ is the index of the groupoid that contains the local arrow λ_n. Since L is good, by proposition 3.4.7, the class of this λ does not depend on the choice of the α_n. Hence, we have established the following (partial) correspondence:

$$(\lambda_n : a(\mathcal{I}) \to A)_n / s\lambda_{n+1} = t\lambda_n \Rightarrow [\lambda] \text{ curve}$$
$$([\lambda_n])_n \subset \{arrows\ of\ A\} \Leftarrow \lambda : L \to A$$

If $\lambda \sim \lambda'$ then $[\lambda_n] = [\lambda] \circ [i_n] = [\lambda'] \circ [i_n] = [\lambda'_n]$ for all n. We investigate now under which hypothesis the converse is also true, namely: $\lambda_n \sim \lambda'_n \ \forall n \ \Rightarrow \ [\lambda] = [\lambda']$.

Proposition 4.2.3. *Let $\lambda, \lambda' : L \to A$ be such that $\lambda_n = \lambda'_n \ \forall n \neq n_0$ and $\lambda_{n_0} \lhd \lambda'_{n_0}$. Then, $\lambda \sim \lambda'$.*

Proof. For each n, let $\alpha_n \in \phi_A$ be less or equal than $\phi_\lambda(\{n\})$ and $\phi_{\lambda'}(\{n\})$. Let $h : L \to A$ be given for the sequences $(\alpha_n)_n$ and $(\lambda_n)_n$. It is easy to check that $h \lhd \lambda$ and $h \lhd \lambda'$.   □

Corollary 4.2.4. *Let $\lambda, \lambda' : L \to A$. Suppose that there exists a finite subset $J \subset \mathbb{Z}$ with $\lambda_n = \lambda'_n$ for $n \notin J$ and $\lambda_n \sim \lambda'_n$ for $n \in J$, then $\lambda \sim \lambda'$.*

Proof. It suffices to construct a finite sequence $\lambda = \lambda^0, \lambda^1, \ldots, \lambda^k = \lambda'$ such that λ^i and λ^{i+1} are in the conditions of the proposition.   □

We say that $\lambda : L \to A$ *stabilizes* to the left (resp. to the right) if there are $x \in X_A$ and $\alpha \in \phi_A$ such that $\lambda_n = id_x \in \mathcal{G}_\alpha$ for small (resp. big) enough values of n.

Proposition 4.2.5. *If $\lambda, \lambda' : L \to A$ stabilize in both directions and $\lambda_n \sim \lambda_{n'}$ for every n, then $\lambda \sim \lambda'$.*

Proof. Let (N_0, N_1) be a stabilization pair for λ and λ'. We have $\lambda_n = id_{x_0} \in \mathcal{G}_{\alpha_0}$ for $n \leqslant N_0$ and some α_0 in ϕ_A, and $\lambda_n = id_{x_1} \in \mathcal{G}_{\alpha_1}$ for $n \geqslant N_1$ and some α_1. The map λ gives an infinit sequence $(\phi_\lambda(\{n\}))_n \in \phi_A$ that not necessary stablize in any direction. However, we may suppose that $\phi_\lambda(\{n\}) = \alpha_0$ for $n \leqslant N_0$ and $\phi_\lambda(\{n\}) = \alpha_1$ for $n \geqslant N_1$ without change the equivalence class of λ (actually, λ is a corestriction of a map satisfying this).
Similarly, let α'_0 and α'_1 be the indices where λ' stabilizes.
Choose γ_0, γ_1 in ϕ_A such that $\gamma_i \leqslant \alpha_i, \alpha'_i$, $i = 0, 1$. Define $(\delta_n)_n \subset \phi_A$ as

$$
\delta_n = \begin{cases} \gamma_0 & \text{if } n \leqslant N_0 \\ \phi_\lambda(\{n\}) & \text{if } N_0 < n < N_1 \\ \gamma_1 & \text{if } n \geqslant N_1 \end{cases} .
$$

Define $g_n : \mathcal{I} \to A$ by

$$
g_n = \begin{cases} id_{x_0} \in \mathcal{G}_{\gamma_0} & \text{si } n \leqslant N_0 \\ \lambda_n & \text{si } N_0 < n < N_1 \\ id_{x_1} \in \mathcal{G}_{\gamma_1} & \text{si } n \geqslant N_1 \end{cases} .
$$

Denote $\widetilde{\lambda} : L \to A$ the morphism induced by the sequences $(\delta_n)_n$ and $(g_n)_n$. Clearly, $\widetilde{\lambda} \lhd \lambda$. Analogously, define $\widetilde{\lambda'} : L \to A$ such that $\widetilde{\lambda'} \lhd \lambda'$. By the propostion of above and since $\widetilde{\lambda}_n = \widetilde{\lambda'}_n$ for $n < N_0$ or $n > N_1$ and $\widetilde{\lambda}_n = \lambda_n \sim \lambda'_n = \widetilde{\lambda'}_n$ for $N_0 \leqslant n \leqslant N_1$, it follows that $\widetilde{\lambda} \sim \widetilde{\lambda'}$. Therefore, $\lambda \sim \lambda$.   □

If l is a curve, we say that l *stabilizes* to the left (resp. to the right) if there exists $\lambda : L \to A$ such that $[\lambda] = l$ and λ stabilizes to the left (resp. to the right).

Definition 4.2.6. A *path* in A is a curve which stabilizes in both directions.

Note that if l is a path in A, then there exists $\lambda : L \to A$, $N_0, N_1 \in \mathbb{Z}$, $x_0, x_1 \in X_A$ and $\alpha_0, \alpha_1 \in \phi_A$ such that $l = [\lambda]$, $\lambda_n = id_{x_0} \in \mathcal{G}_{\alpha_0}$ for $n < N_0$ and $\lambda_n = id_{x_1} \in \mathcal{G}_{\alpha_1}$ for $n > N_1$.

Consider again the correspondence between curves and sequences of global arrows. A path $l = [\lambda]$ is associated to the sequence $([\lambda_n])_n \subset \{arrows\ of\ A\}$ which stabilizes in identities at both sides.

Conversely, given a sequence $(g_n)_n \subset \{arrows\ of\ A\}$ such that $s(g_{n+1}) = t(g_n)$, $g_n = id_{x_0}$ for $n < N_0$ and $g_n = id_{x_1}$ for $n > N_1$, one can take local arrows $(\lambda_n)_n$ such that $[\lambda_n] = g_n$ and construct the curve associated to this sequence. Note that this curve does not have to be a path. This can be solved choosing α_0, α_1 such that $x_0 \in X_{\alpha_0}$, $x_1 \in X_{\alpha_1}$ and taking $\lambda_n = id_{x_0} \in \mathcal{G}_{\alpha_0}$ and $\lambda_n = id_{x_1} \in \mathcal{G}_{\alpha_1}$ for n small or big enough. This new curve is a path, and is uniquely determinated by proposition 4.2.5. Thus we have obtained the following result.

Proposition 4.2.7. *The constructions of above are mutually inverse. They define a bijection between the set of paths of an atlas A and the sequences $(g_n)_n \subset \{arrows\ of\ A\}$ that stablizes on identities and satisfies $s(g_{n+1}) = t(g_n)$ for all n.*

This correspondence preserves sources and targets, and concatenations of paths up to homotopy. This leads to a simplicial computation of the fundamental group. In fact, we will prove in the next section that the fundamental group of an atlas A is the fundamental group of its *nerve NA*.

4.3 The fundamental group

In this subsection we introduce the (strong) fundamental group of a good groupoid atlas A with a fixed base point. Later we will generalize this construction to any groupoid atlas. The definitions and results introduced here admit formulations in terms of the fundamental groupoid.

Recall that L is regular. Its points can be represented by the inclusions

$$j_n : * \to L, \ \phi_{j_n}(*) = \{n\}, \ id = \mathcal{G}_{j_n}(*) : * \to \mathcal{G}_{\{n\}}.$$

The projections $p_1, p_2 : L \times L \to L$ admit sections $id_L \times j_n, j_n \times id_L$ respectively. We will simply write j_n instead of $id_L \times j_n$, the inclusion of L in the n-th row.

Definition 4.3.1. A *homotopy* between two paths l and l' is a map

$$H : L \times L \to A$$

such that there exist integers N_0, N_1, M_0 and M_1 satisfying

- $H \circ j_{N_0} = \lambda$ with $[\lambda] = l$,

- $H \circ j_{N_1} = \lambda'$ with $[\lambda'] = l'$,

- there are $\alpha_0, \alpha_1 \in \phi_A$ such that for all local arrow f of $L \times L$, $\mathcal{G}_H(f) = id_{x_0} \in \mathcal{G}_{\alpha_0}$ if the first coordinate of $s(f)$ is less than M_0 and $\mathcal{G}_H(f) = id_{x_1} \in \mathcal{G}_{\alpha_1}$ if the first coordinate of $t(f)$ is greater than M_1.

It is clear that homotopy of paths is an equivalence relation.

Definition 4.3.2. Let A be a good atlas and $x \in X_A$. The fundamental group $\pi_1(A, x)$ is the set of homotopy classes of loops at x, with the operation induced by the concatenation of paths.

Proposition 4.3.3. *If $f : A \xrightarrow{\sim} B$ is an equivalence, then $\pi_1(A, x_0) \cong \pi_1(B, X_f(x_0))$.*

Proof. It is a consequence of the definition of paths and homotopies. The fundamental group is actually a well defined invariant in the category $[\mathcal{G}pd\mathcal{A}tl]$. For an alternative proof, see section 5. □

Important remark: We extend the definition of the fundamental group to non necessarily good atlases, taking the fundamental group of their regularizations. By the last proposition, this new definition agree with the original one on good atlases.

A homotopy $H : l \cong l'$ can be decomposed into more elementary homotopies, considering the various ways to reach the upper-right point from the lower-left one in a lattice (say $[0, N] \times [0, N] \subset X_{L \times L}$), that gives rise to a sequence of paths $..., \lambda_i, \lambda_{i+1}, ...$ where two consecutive paths just differ in a single square. Note that all these paths become loops when H is applied.
These elementary homotopies relate a path $([g_{M_0}], \ldots, [g_{M_1}])$, viewed as a sequence of global arrows, with another $([h_{M_0}], \ldots, [h_{M_1}])$, where $h_{i+1} \circ h_i = g_{i+1} \circ g_i \in \mathcal{G}_\alpha$ for some i and some α, and $h_j = g_j \in \mathcal{G}_{\alpha_j}$ for $j \neq i, i+1$. Inserting the loop $([g_{M_0}], \ldots, [g_i \circ g_{i-1}], \ldots, [g_{M_1}])$ between them, we can observe that the homotopies in A define the same equivalence relation that the one used to represent the morphisms in a colimit on the category of categories (cf. [8],[9]).

This proves the following result.

Theorem 4.3.4. *Let A be an irreducible groupoid atlas and let $x \in X_A$. Consider $\mathcal{G}(A) = colim \, \mathcal{G}_\alpha$ the colimit over the diagram ϕ_A. Then the fundamental group $\pi_1(A, x)$ equals the fundamental group of the groupoid $\pi_1(\mathcal{G}(A), x)$.*

Remark 4.3.5. The hypotesis of irreducible on A is necessary to make $\{\mathcal{G}_\alpha\}_\alpha$ a groupoid diagram. Otherwise, the functors ϕ_α^β are partially defined and the colimit does not have sense.

Last theorem shows another way to define the group $\pi_1(A, x_0)$ when A is irreducible. When A is not irreducible, the group $\pi_1(iA, x_0)$ defined in this way is isomorphic to the fundamental group defined as before. This approach will be study in section 5.

Example 4.3.6. We return to example 2.3.5. By definition, the local groupoid $\mathcal{G}_U = \pi_1(U)$ is connected when the open subset U is path connected. So, the atlas $A(X,\mathcal{U})$ is irreducible if and only if U is path connected for all $U \in \mathcal{U}$. If in addition the family $\mathcal{U}$ is closed under finite intersections, then

$$\pi_1(A(X,\mathcal{U})) = \operatorname*{colim}_{U \in \mathcal{U}} \pi_1(U) = \pi_1(X)$$

as a consequence of the last theorem and (the groupoid formulation) of Van Kampen's theorem (cf. [6], [10]).

4.4   Some examples

The fundamental group $\pi_1(A, x)$ appears as a natural invariant for groupoid atlases in many ways: from paths and homotopies of paths (4.3.2), by simplicial calculation (5.2.7) and even as a purely algebraic invariant (4.3.4, 5.3). In some special cases coincides with the weaker one $\pi_1^w(A, x)$. The group $\pi_1^w(A, x)$ describes some of the algebra of the global nontrivial loops, while $\pi_1(A, x)$ detects also the local ones. It is clear that these groups are not equal in general.

The following examples are the simplest cases where the differences between these two groups are noticed. In a proposition of below we will prove that these are essentially the unique kinds of differences they could have.

Given a groupoid atlas A and $x \in X_A$, recall that the group $\pi_1^w(A, x)$ is the fundamental group of the simplicial complex $V\mathcal{U}_A$. We denote this complex by VA.

Example 4.4.1. Let $\mathcal{G}$ be a groupoid and let x be an object of $\mathcal{G}$. The groups $\pi_1(a(\mathcal{G}), x)$ and $\pi_1^w(a(\mathcal{G}), x)$ depends only of the component of $\mathcal{G}$ that contains x, so $\mathcal{G}$ can be supposed connected. The Vietoris complex $V(a(\mathcal{G}))$ is the simplex spanned by $O = Obj\ \mathcal{G}$, so it is contractible. Hence, $\pi_1^w(a(\mathcal{G}), x) = \pi_1(V(a(\mathcal{G})), x) = 0$. On the other hand, $\pi_1(a(\mathcal{G}), x) = Hom_{\mathcal{G}}(x, x)$, which is not trivial in general.

Example 4.4.2. Let A be a groupoid atlas with discrete set of indices $\phi_A = \{\alpha, \beta\}$ and such that $X_A = X_\alpha = X_\beta$ and the local groupoids $\mathcal{G}_\alpha$ and $\mathcal{G}_\beta$ are simply connected. Every finite subset of X_A is a local frame, so as in the last example, VA is contractible and $\pi_1^w(A, x) = 0$ for any x. Let $y \xrightarrow{i} z$ be the unique arrow in $\mathcal{G}_i$ with source y and target z. The (strong) loop whose associated sequence of global arrows is

$$([x \xrightarrow{\alpha} y], [y \xrightarrow{\beta} x])$$

is a non trivial element of $\pi_1(A, x)$ for any $y \in X_A$, $y \neq x$ (in fact, $\pi_1(A, x)$ is the free group with one generator $([x \xrightarrow{\alpha} y], [y \xrightarrow{\beta} x])$ for each y). Therefore, the groups π_1 and π_1^w are distinct if A has other points than x.

There is a canonical group morphism

$$p : \pi_1(A, x) \to \pi_1^w(A, x)$$

which sends a loop l to the weak loop X_λ, where λ is a representative of l that stabilizes in both directions. The map p can also be defined using the simplicial information: mapping a path of global arrows $(g_1, ..., g_N)$ to the path $(\{s(g_1), t(g_1)\}, ..., \{s(g_N), t(g_N)\})$ of edges of VA. This map is defined up to homotopy. Note that p is the group morphism induced by the simplicial map p of 5.4.

The last examples show that p is not an isomorphism in general, but it is so in many interesting cases.

Consider for example the groupoid atlas $A = A(G, \mathcal{H})$, with G a group and $\mathcal{H}$ a family of subgroups closed under finite intersections. Let x be an element of G. In order to compute the group $\pi_1(A, x)$, note that $\{arrows\ of\ A\} = \{(h, g)|\ g \in G,\ h \in H\ for\ some\ H \in \mathcal{H}\}$, as one can easily check from definition 4.1.3 and the paragraph above it. Then, we have the isomorphism

$$\pi_1(A, x) \xrightarrow{\sim} \pi_1^w(A, x)$$

since (the groupoid form of) p maps bijectively global arrows into edges and elementary homotopy triangles into three-elements local frames. This result is not true if the family $\mathcal{H}$ is not closed under intersections.

The general linear global action is a particular case of an $A(G, \mathcal{H})$ where $\mathcal{H}$ is closed under intersections. To see that, recall that for $A = GL(n, R)$ the index set ϕ_A consists of the closed subsets of $\{(i, j)\ /\ i \neq j,\ 1 \leqslant i, j \leqslant n\}$ partially ordered by inclusion. If α and β are in ϕ_A, it is clear that $\alpha \cap \beta$ is also closed, and hence it is in ϕ_A. The subgroup $GL(n, R)_{\alpha \cap \beta}$ of the linear group equals $GL(n, R)_\alpha \cap GL(n, R)_\beta$. Therefore, the natural map between both fundamental groups of $GL(n, R)$ is an isomorphism.

The homotopy of $A(G, \mathcal{H})$ is locally trivial, in the sense that the local groupoids of A are simply connected groupoids. This leads us to a more general family of atlases in which the map relating both groups $\pi_1^w(A, x)$ and $\pi_1(A, x)$ is an isomorphism.

Recall the definition of infimum from [2], [3]. A groupoid atlas A is *infimum* if for every local frame s the set

$$\phi_s = \{\alpha \in \phi_A \mid s \subset X_\alpha\}$$

has an initial element. Note that an infimum groupoid atlas is, in particular, a good atlas. This condition, when all local groupoids are simply connected, implies that both groups $\pi_1(A, x)$ and $\pi_1^w(A, x)$ are isomorphic by p.

Theorem 4.4.3. *If A is an infimum groupoid atlas such that every local groupoid $\mathcal{G}_\alpha$ of A is simply connected, then the canonical map $p : \pi_1(A, x) \to \pi_1^w(A, x)$ is an isomorphism.*

Proof. Given a two elements local frame $\{y, z\} \subset X_A$, since $\phi_{\{x,y\}}$ has an initial element, there is a unique global arrow with source y and target z. Then, p maps bijectively the paths of global arrows into the paths of edges. The 2-simplices of VA arise from the three elements local frames. Under this bijection, any three edges which are the faces of a 2-simplex in VA correspond to a homotopy triangle. $\square$

Remark 4.4.4. We propose a stronger version of the definition of infimum atlases (the definition of above is the original introduced in [2]) that seems to fit better from the strong point of view. We might say that A is *infimum* if ϕ_s has an initial element for every simplex s of NA. When A is such that all its local groupoids are simply connected, the original and the strong definition of infimum agree, since, in this case, a simplex is essentially determined by its underlying local frame.

Remark 4.4.5. Theorem 4.4.3 remains true when all the local groupoids are simply connected and A satisfies the following condition: the sets ϕ_s are filtered for all local frame s. Note that this condition is weaker than the infimum condition, but it is sufficient to prove the result.

We finish this section considering the atlas $A = A(X, \mathcal{U})$, where X is a topological space and $\mathcal{U}$ is an open cover of X. The atlas A is infimum if $\mathcal{U}$ is closed under intersections, and we have seen in 4.3.6 that A is irreducible if U is path connected for all $U \in \mathcal{U}$. When these conditions hold, the strong fundamental group of A equals $\pi_1(X)$ (cf. 4.3.4) and the weak one is, by Dowker's theorem, the fundamental group of the nerve of the cover.
By last proposition, these groups are isomorphic when each U is simply connected. By the remark of above, it is sufficient for $\mathcal{U}$ to be closed under finite intersections. Thus, we obtain as a corollary of 4.4.3 an alternative proof of the following result.

Corollary 4.4.6. *Let X be a topological space and $\mathcal{U}$ an open cover by simply connected open subsets. If $\mathcal{U}$ is closed under finite intersections, then the group $\pi_1(X, x)$ equals the fundamental group of the nerve of the cover $N\mathcal{U}$.*

5 Nerves for Atlases

Let $\mathcal{SSet}$ the category of simplicial sets and simplicial morphisms. We introduce two functorial constructions $\mathcal{GpdAtl} \rightarrow \mathcal{SSet}$ for the *nerve* of a groupoid atlas A. The first one, $N^w A$, is based on the cover by components of local groupoids. The second one, NA, preserves more information about the local groupoids. Using these constructions, we can define homology theory of groupoid atlases and also a weak and a strong version of the classifying space of a groupoid atlas.

5.1 The weak nerve $N^w(A)$

Definition 5.1.1. The *weak nerve* $N^w A$ of a groupoid atlas is the simplicial set whose n-simplices are the sequences of $n+1$ elements of the same component of a local groupoid,

$$(N^w A)_n = \{(x_0, \ldots, x_n) \mid \{x_0, \ldots, x_n\} \text{ is an } \alpha\text{-frame for some } \alpha\}$$

equipped with usual faces and degeneracies: the face d_i erases the i-th element and the degeneracy s_j repeats the j-th.

Given a weak map of groupoid atlases $f : A \to B$, we have $f_* : N^w A \to N^w B$ defined by $(x_0, \ldots, x_k) \mapsto (f(x_0), \ldots, f(x_k))$. It is well defined since f preserves local frames. Note that this construction is functorial: $id_* = id$ and $(f \circ g)_* = f_* \circ g_*$.

Consider $a(\Delta[n])$ the n-simplex in the category $\mathcal{G}pd\mathcal{A}tl$. Since it has only one index, the entire set $X_{a(\Delta[n])} = \{0, ..., n\}$ is a local frame. A weak map $a(\Delta[n]) \to A$ is a function $\{0, \ldots, n\} \to X_A$ such that its image is a local frame. Thus, a simplex $s \in (N^w A)_n$ can be viewed as a weak map $a(\Delta[n]) \to A$. Faces and degeneracies are, under this correspondence, compositions with the canonical inclusions $a(\Delta[n-1]) \to a(\Delta[n])$ and projections $a(\Delta[n+1]) \to a(\Delta[n])$, respectively. Hence,

$$(N^w A)_n = Hom_{weak}(a(\Delta[n]), A).$$

The weak nerve of A is the simplicial set that naturally arises from the complex VA (cf. [17]). The geometric realization $|N^w A|$ has the homotopy type of the polyhedron induced by VA, so it has the same fundamental group and the same homology groups.

5.2   The (strong) nerve $N(A)$

The weak nerve is constructed from the covering of X_A by the components of each $\mathcal{G}_\alpha$. It has no more information about the groupoid structure than that. The strong version of the nerve appears naturally when one looks for a construction that preserves the local information.

Definition 5.2.1. Let A be a regular atlas. The *nerve NA* is the simplicial set whose n-simplices are

$$NA_n = \{x_0 \xrightarrow{g_1} x_1 \xrightarrow{g_2} ... \xrightarrow{g_n} x_n \mid g_1, ..., g_n \in \mathcal{G}_\alpha \text{ for some } \alpha\}/ \sim$$

where we identify $(g_1, ..., g_k)$ with $(\phi_\alpha^\beta(g_1), ..., \phi_\alpha^\beta(g_n))$, its image through the structure functors ϕ_α^β. The face and degeneracy maps are defined as it is usual for nerves of categories: a face composes two arrows, a degeneracy inserts an identity (cf. [16]). They pass to the quotient since ϕ_α^β are functors.

Remark 5.2.2. Given a groupoid $\mathcal{G}$, the nerve $N(a\mathcal{G})$ coincides with the nerve $N\mathcal{G}_\alpha$ defined as usual for categories.

Note that the 0-simplices of an atlas A are the points and the 1-simplices are the global arrows. In general, the set NA_k is a quotient of the disjoint union of the k-simplices of the nerves of the local groupoids $(N\mathcal{G}_\alpha)_k$.

Given $f : A \to B$ a map of groupoid atlases, it determines a map between the nerves $f_* : NA \to NB$ by the formula

$$f_*[(g_1, ..., g_k)] = [(\mathcal{G}_f(\alpha)(g_1), ..., \mathcal{G}_f(\alpha)(g_k))]$$

for arrows $g_1, ..., g_k$ in $\mathcal{G}_\alpha$. This definition does not depend on the representative of the class $[(g_1, ..., g_k)] \in NA_k$.

Proposition 5.2.3. *If $f \lhd f'$ then $f_* = f'_*$. Therefore, equivalent atlases have isomorphic nerves.*

Proof. The simplex $[(g_1, ..., g_k)]$ is mapped by f_* and f'_* into $[(\mathcal{G}_f(\alpha)(g_1), ..., \mathcal{G}_f(\alpha)(g_k))]$ and $[(\mathcal{G}_{f'}(\alpha)(g_1), ..., \mathcal{G}_{f'}(\alpha)(g_k))]$, respectively. Since $f \lhd f'$, we have $f(\alpha) \leqslant f'(\alpha)$ and $\phi_{\phi_f(\alpha)}^{\phi_{f'}(\alpha)} \circ \mathcal{G}_f(\alpha) = \mathcal{G}_{f'}(\alpha)$.

$$
\begin{array}{ccc}
 & \mathcal{G}_\alpha & \\
\overset{\mathcal{G}_f(\alpha)}{\swarrow} & & \overset{\mathcal{G}_{f'}(\alpha)}{\searrow} \\
\mathcal{G}_{\phi_f(\alpha)} & \xrightarrow{\phi_{\phi_f(\alpha)}^{\phi_{f'}(\alpha)}} & \mathcal{G}_{\phi_{f'}(\alpha)}
\end{array}
$$

We conclude that $(\mathcal{G}_f(\alpha)(g_1), ..., \mathcal{G}_f(\alpha)(g_k)) \sim (\mathcal{G}_{f'}(\alpha)(g_1), ..., \mathcal{G}_{f'}(\alpha)(g_k))$. $\square$

The definition of the nerve NA can be extended to non regular atlases, defining NA as the nerve of its regularization:

$$NA = N(rA)$$

By the last proposition, this identity remains true even when A is already regular, since in this case, $A \sim rA$ (cf. 3.4.4).

Recall that the k-simplices of the nerve NC of a small category C can be presented as

$$NC_k = Hom_{\mathcal{C}at}(\{0 \to 1 \to ... \to k\}, C),$$

and when C is a groupoid, by the universal property of localization, we have

$$NC_k = Hom_{\mathcal{G}pd}(\Delta[k], C).$$

It is clear that $Hom_{\mathcal{G}pd\mathcal{A}tl}(a(\Delta[k]), A)$ equals the disjoint union of the k-simplices of $N\mathcal{G}_\alpha$, since a map $a(\Delta[k]) \to A$ picks an index α of ϕ_A and gives a functor $\Delta[k] \to \mathcal{G}_\alpha$. Note that two maps $a(\Delta[k]) \to A$ are equivalent if and only if the simplices that they define are identified in NA. Thus,

$$NA_k = Hom_{\mathcal{G}pd\mathcal{A}tl}(a(\Delta[k]), A)/ \sim = Hom_{[\mathcal{G}pd\mathcal{A}tl]}(a(\Delta[k]), A).$$

Therefore, as it happens with the weak nerve, the functor $N : [\mathcal{G}pd\mathcal{A}tl] \to \mathcal{S}\mathcal{S}et$ is an example of *singular functor* in the vocabulary of [7]. It means that there exists a functor $\theta : \Delta \to [\mathcal{G}pd\mathcal{A}tl]$ from the category of finite ordinal numbers such that $NA_k = Hom_{[\mathcal{G}pd\mathcal{A}tl]}(\theta(\{0 < ... < k\}), A)$, and the faces and degeneracies are given by composition with $\theta(x)$, where x is an elementary injection or surjection in Δ. In this case, the functor θ is $\{0 < ... < k\} \mapsto a(\Delta[k])$.

Remark 5.2.4. The nerve of a groupoid atlas NA is not in general a Kan complex. It might happen that two 1-simplices cannot be extended to another 1-simplex: take local groupoids $\mathcal{G}_\alpha$ and $\mathcal{G}_\beta$ and let x, y be objects of $\mathcal{G}_\alpha$ and $\mathcal{G}_\beta$, respectively. Let z be an object shared by these two groupoids. Suppose that $\{x, y, z\}$ is not a local frame. If $g : x \to z$ and $g' : z \to y$ are arrows of $\mathcal{G}_\alpha$ and $\mathcal{G}_\beta$ respectively, then there is not a 2-simplex s in NA such that $d_0(s) = g'$ and $d_2(s) = g$.

Example 5.2.5. In the nerve of the groupoid atlas corresponding to the global action $\mathcal{A}(D_3, \mathcal{H})$ discussed in the introduction, there is no $s \in NA_2$ satisfying $d_0(s) = 1 \to r$ and $d_2(s) = r \to s \cdot r$.

If the atlas A is irreducible, the set NA_k is the colimit over ϕ_A of the sets $(N\mathcal{G}_\alpha)_k$, since it is the largest quotient of the disjoint union that makes the diagrams

$$(N\mathcal{G}_\alpha)_k \xrightarrow{\phi_\alpha^\beta} (N\mathcal{G}_\beta)_k$$
$$(\coprod (N\mathcal{G}_\alpha)_k)/\sim$$

commutative. Since limits and colimits in $\mathcal{SS}et$ can be computed coordinatewise, we have

$$NA = \operatorname*{colim}_{\phi_A} N\mathcal{G}_\alpha.$$

When A is not irreducible, there is not a well defined diagram in the category of sets since the functions $N\mathcal{G}_\alpha \to N\mathcal{G}_\beta$ are partially defined.

Remark 5.2.6. The nerve NA of an atlas A is equal to the nerve $N(iA)$ of the irreducible atlas iA, since both atlases have the same simplices by the universal property of iA. Then, the nerve of an arbitrary atlas can be computed as the nerve of an irreducible atlas. In the rest of this section we will assume, without loss of generality, that A is irreducible.

Proposition 5.2.7. *The fundamental group of the atlas A is equal to $\pi_1(NA, x)$.*

Proof. We have seen that the fundamental group of A is the set of paths of global arrows modulo the relations generated by $(g, h) \sim (hg)$. Note that g, h and hg are the three faces of the 2-simplex (g, h), so the group $\pi_1(A, x)$ is the set of paths of 1-simplices that start and end in x modulo the simplicial homotopies, i.e. $\pi_1(A, x)$ equals $\pi_1(NA, x)$. $\square$

Corollary 5.2.8. *Let A be a groupoid atlas and $x \in X_A$. The map $iA \to A$ induces an isomorphism $\pi_1(iA, x) \to \pi_1(A, x)$.*

Last proposition also gives an alternative proof of 4.3.3.

5.3   The colimit groupoid $\mathcal{G}(A)$

In theorem 4.3.4 we introduced another algebraic object related to A, namely the *colimit groupoid* over the diagram $\mathcal{G}_A$, which we denoted $\mathcal{G}(A)$.

$$\mathcal{G}(A) = \underset{\phi_A}{colim}\ \mathcal{G}_\alpha$$

Let $f = (\phi_f, \mathcal{G}_f) : A \to B$ be a map in $\mathcal{G}pd\mathcal{A}tl$. The family $\{\mathcal{G}_\alpha \to \mathcal{G}_{\phi_f(\alpha)} \to \mathcal{G}(B)\}_{\alpha \in \phi_A}$ commutes with the structural functors ϕ_α^β, hence induces a functor $f_* : \mathcal{G}(A) \to \mathcal{G}(B)$ by the universal property of $\mathcal{G}(A)$. With this definition on arrows, $\mathcal{G} : \mathcal{G}pd\mathcal{A}tl \to \mathcal{G}pd$ becomes a left adjoint for the functor $a : \mathcal{G}pd \to \mathcal{G}pd\mathcal{A}tl$.

$$Hom_{\mathcal{G}pd}(\mathcal{G}(A), G) \equiv Hom_{\mathcal{G}pd\mathcal{A}tl}(A, aG)$$

The functor $\mathcal{G}$ factors through $[\mathcal{G}pd\mathcal{A}tl]$, since a map and a corestriction of it give rise to the same family $\{\mathcal{G}_\alpha \to \mathcal{G}_{\phi_f(\alpha)} \to \mathcal{G}(B)\}_{\alpha \in \phi_A}$.

It is easy to see that $Obj\ \mathcal{G}(A) = X_A$. Recall that an arrow of $\mathcal{G}(A)$ is a path of colimit arrows modulo the smallest equivalence class that contains the local identities and compositions. Here, a *colimit arrow* $x \to y$ means an element of $colim_{\phi_A}\ Hom_{\mathcal{G}_\alpha}(x, y)$. In other words, it is an arrow of the colimit graph of the underlying graphs of $\mathcal{G}_\alpha$.

Remark 5.3.1. Note that a global arrow of A is the same as a colimit arrow. They are the arrows of $\mathcal{G}(A)$ that are in the image of some $\mathcal{G}_\alpha \to \mathcal{G}(A)$.

The functors $i_\alpha : \mathcal{G}_\alpha \to \mathcal{G}(A)$ give rise to simplicial maps $Ni_\alpha : N\mathcal{G}_\alpha \to N\mathcal{G}(A)$. Last remark can be generalized to the following result.

Proposition 5.3.2. *The nerve NA is the subsimplicial set of $N\mathcal{G}(A)$ that consists of those simplices that are in the image of Ni_α, for some α.*

Proof. Let S be the union $\bigcup Ni_\alpha(N\mathcal{G}_\alpha)$. With the faces and degeneracies of $N\mathcal{G}(A)$, S results a simplicial set. Recall that an n-simplex s of NA is the class of a functor

$$\Delta[n] \xrightarrow{s} \mathcal{G}_\alpha$$

for some α, under the quotient map that identifies s with $\phi_\alpha^\beta \circ s$ for all $\alpha \leqslant \beta$. We assign to s the composition $i_\alpha \circ s : \Delta[n] \to \mathcal{G}(A)$. This factors through the quotient, so there is a function $NA_n \to S_n$. Since this function preserves faces and degeneracies, it gives rise to a simplicial map $NA \to S$. It is surjective by definition of S. It is not difficult to prove that it is also injective. $\qquad\square$

Remark 5.3.3. It is well known that the nerve of a groupoid is a Kan complex, therefore the nerve NA is not equal in general to $N\mathcal{G}(A)$ (see 5.2.4).

5.4 The classifying space of a groupoid atlas

Given an atlas A, we can associate to it a topological space BA via its nerve. In this subsection we introduce this construction and relate it with other spaces associated to A.

Definition 5.4.1. The *classifying space BA* of an atlas A is the geometric realization of its nerve NA.

Like the geometric realization of any simplicial set, BA is a CW-complex with a cell for each non-degenerated simplex of NA. As a consequence of our results on the nerve of a groupoid atlas, we obtain the following propositions.

Proposition 5.4.2. *Let A be a groupoid atlas. Then the canonical map $\varphi_A : iA \to A$ induces a homeomorphism $B(\varphi_A) : BiA \to BA$.*

Proposition 5.4.3. *Let $f, g : A \to B$ be two equivalent maps. The induced continuous functions Bf and Bg are equal. In particular, equivalent atlases have homeomorphic classifying spaces.*

Remark 5.4.4. Given a simplicial complex K, its associated groupoid atlas $a(K)$ is infimum and all its local groupoids are simply connected. Then, by proposition 5.2.6 its weak nerve equals its strong nerve. It follows that $B(aK) = B^w(aK)$. Since $B^w(aK)$ is homotopy equivalent to the polyhedron K, we conclude that any *interesting homotopy type* can be obtained as the classifying space of a groupoid atlas.

Remark 5.4.5. An algebraic loop in A induces a topological one in BA, since $BL \cong \mathbb{R}$ and a map of groupoid atlases $L \to A$ which stabilizes can be restricted to a closed real interval. This assignation preserves homotopy classes and induces an isomorphism $\pi_1(A, x) \to \pi_1(BA, x)$, which is just the composition $\pi_1(A, x) \xrightarrow{\sim} \pi_1(NA, x) \xrightarrow{\sim} \pi_1(BA, x)$.

Relation between the weak and the strong nerves

Given a groupoid atlas A, there is a canonical projection $p : NA \to N^w A$ given by

$$(x_0 \xrightarrow{g_1} x_1 \to ... \xrightarrow{g_n} x_n) \mapsto (x_0, ..., x_n)$$

Every simplex of $N^w A$ is in the image of p, thus $p : (N^w A)_k \to NA_k$ is onto for all k.

Proposition 5.4.6. *If A is infimum and all its local groupoids are simply connected, then $p : NA \to N^w A$ is an isomorphism.*

Proof. The argument is analogous to the one used in 4.4.3. It is sufficient to prove that over each simplex of $N^w A$ there is one and only one simplex of NA. $\square$

In general, this is not longer true. As we can see in the following example, p might have no inverse, even no section.

Example 5.4.7. Let $X_A = \{a, b, c, d\}$, $\phi_A = \{1, 2\}$ discrete. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the simply connected groupoids with objects $\{a, b, c\}$ and $\{a, b, d\}$ respectively.

Consider $(a, b) \in (N^w A)_1$. Suppose that there exists a section $i : N^w A \to NA$ for p. Then $p(i(a, b)) = (a, b)$, and $i(a, b) = a \xrightarrow{1} b$ or $a \xrightarrow{2} b$, arrows of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Since $i(a, b, c) = a \xrightarrow{1} b \xrightarrow{1} c$, $i(a, b, d) = a \xrightarrow{2} b \xrightarrow{2} d$ and since i commutes with the face maps then we have

$$a \xrightarrow{1} b = d_2 \circ i(a, b, c) = i \circ d_2(a, b, c) = i(a, b) = i \circ d_2(a, b, d) = d_2 \circ i(a, b, d) = a \xrightarrow{2} b$$

which is a contradiction.

Compare the spaces BA and $B^w A$.

In BA one can notice the existence of a non trivial algebraic loop $a \xrightarrow{1} b \xrightarrow{2} a$, while the space $B^w A$ is simply connected.

5.5   A little of homology

Definition 5.5.1. Let A be a groupoid atlas and let R be a commutative ring. The *homology of A with coefficients in R* is the homology of the associated simplicial set NA.

$$H_n(A, R) = H_n(NA, R)$$

Explicitly, for each n we put $C_n(A, R) = R[NA_n]$, the free R-module with basis NA_n, and the boundary map $d : C_n \to C_{n-1}$ is defined, as usual, by

$$d(x) = \sum_i (-1)^i d_i(x)$$

in the basis elements. We will simply denote $C_n(A)$ and $H_n(A)$ when $R = \mathbb{Z}$.

Generalities 5.5.2.

(a) Two equivalent atlases have the same homology groups, since their nerves are iso-morphics (cf. 5.2.3).

(b) Similarly, the homologies of A and iA are the same. Thus, we can restrict our attention to the homology of irreducible atlases.

(c) If ϕ_A has a single element α, the homology groups of A are the homology groups of the groupoid $\mathcal{G}_\alpha$, which are the direct sum of the homologies of the vertex groups of each component of $\mathcal{G}_\alpha$.

(d) If $n = 0$, the group $H_0(A, R)$ is free with one generator for each component of A. Thus, we have an isomorphism

$$H_0(A, R) = R[\pi_0(A)].$$

(e) When $R = \mathbb{Z}$, the groups $\pi_1(A)$ and $H_1(A)$ are related by a quotient map $\pi_1(A) \twoheadrightarrow H_1(A)$ with kernel the commutator of the fundamental group (the first one coincides with $\pi_1(NA)$ and the second one with $H_1(NA)$). Thus, the classical equation

$$H_1(A) = \pi_1(A, a)/[\pi_1(A, a), \pi_1(A, a)]$$

remains true in this context.

Example 5.5.3. Recall that a simplicial set and its geometrical realization share all their homology groups. We can compute the homology of the atlas A of 5.4.7 from its classifying space, since it has the homotopy type of S^1.

$$H_n(A) = \begin{cases} 0 & \text{if } n \neq 0, 1 \\ \mathbb{Z} & \text{if } n = 0, 1 \end{cases}$$

Given A a groupoid atlas, note that the chain complex $C_*(A, R)$ is the colimit of the complexes $C_*(\mathcal{G}_\alpha, R)$: the free functor $R[-]$ is left adjoint to the forgetful functor, hence it preserves colimits.

$$C_*(A, R) = R[NA] = colim\ R[N\mathcal{G}_\alpha] = colim\ C_*(\mathcal{G}_\alpha, R)$$

Example 5.5.4. In the case of $A(G, \mathcal{H})$, the chain complex $C_*(A)$ coincides with the chain complex $\beta(\mathcal{H})$, defined in [1] as the colimit of the $\mathbb{Z}[G]$-complexes induced by the non normalized homogeneous bar resolutions $\beta(H_\alpha)$ of the groups H_α. The homology groups of A are the homology groups of the nerve of the cover of G by the H-orbits, $H \in \mathcal{H}$.

Remark 5.5.5. The nerve NA is a simplicial subset of $N\mathcal{G}(A)$. They may differ (5.3.3) as the functor $N : \mathcal{G}pd \rightarrow \mathcal{SS}et$ does not preserve colimits. When A is a connected atlas, $H_*(\mathcal{G}(A))$ equals the homology groups of its fundamental group. We deduce that $H_*(A) \not\cong H_*(\mathcal{G}(A))$ in general, since BA can reach the homotopy type of any CW-complex, and the fundamental group of a CW-complex does not determine all its homology groups.

5.6 Relation between HA and the local homology groups $H\mathcal{G}_\alpha$

Consider the map $p = \coprod i_\alpha : \coprod_{\alpha \in \phi} N\mathcal{G}_\alpha \to NA$. It induces an epimorphism of chain complexes p_*, with kernel Ker_*. The short exact sequence

$$0 \to Ker_* \to \bigoplus_{\alpha \in \phi} C_*(\mathcal{G}_\alpha) \xrightarrow{p_*} C_*(A) \to 0$$

induces a long exact sequence relating the homologies of the complex Ker_*, the local groupoids $\mathcal{G}_\alpha$ and the groupoid atlas A.

$$... \xrightarrow{\partial} H_n(Ker) \to \bigoplus H_n(\mathcal{G}_\alpha) \to H_n(A) \xrightarrow{\partial} H_{n-1}(Ker) \to ...$$

In some cases, the complex Ker_* can be written as a direct sum of some of the complexes $C_*(\mathcal{G}_\alpha)$. In these cases, the homology of A can be computed from the homology of the local groupoids.

Example 5.6.1. Let A be the *1-sphere*, defined in 2.3.4. The set X_A is $\{0, 1, 2\}$, the set of indices ϕ_A consists of the proper non empty subsets of X_A, ordered by inclusion, and for every s, the local groupoid $\mathcal{G}_s$ is the tree over s. Since it is infimum and all its local groupoids are simply connected, by (5.4.6) we have

$$NA_n = N^w A_n = \{0, 1\}^{n+1} \cup \{0, 2\}^{n+1} \cup \{1, 2\}^{n+1}$$

For each s, the nerve of the local groupoid $\mathcal{G}_s$ is $(N\mathcal{G}_s)_n = s^{n+1}$. Then, there is a short exact sequence of complexes

$$0 \to (C_*(\mathcal{G}_{\{0\}}) \oplus C_*(\mathcal{G}_{\{1\}}) \oplus C_*(\mathcal{G}_{\{2\}}))^2 \xrightarrow{i_*} \bigoplus_{s \in \phi_A} C_*(\mathcal{G}_s) \xrightarrow{p_*} C_*(A) \to 0$$

where

$$i_*(\sum c_i\{i\}^{n+1}, \sum c_i'\{i\}^{n+1})_s = \begin{cases} c_i + c_i' & if \ s = \{i\} \\ -c_i - c_j' & if \ s = \{i, j\}, \ j \equiv i + 1 \bmod 3 \end{cases}$$

Since the groupoids $\mathcal{G}_s$ are simply connected, the non trivial part of the long exact sequence relating the homology groups is, in this case,

$$0 \to H_1(A) \to \mathbb{Z}^6 \to \mathbb{Z}^6 \to H_0(A) \to 0.$$

The group $H_0(A)$ is isomorphic to $\mathbb{Z}$ because A is connected, and so $H_1(A) \cong \mathbb{Z}$: it is free of rank 1 since $H_1(A) \hookrightarrow \mathbb{Z}^6$ and the Euler characteristic of the last complex is 0.

Remark 5.6.2. Of course, the homology of $a(\partial\Delta[1])$ can be easily computed noting that its classifying space is homeomorphic to S^1, but we proceeded in this way to illustrate the general idea (see below).

Remark 5.6.3. All the local groupoids are, in this example, simply connected. Then, by the long exact sequence, the homology groups $H_n(A)$ should be trivial for $n \geqslant 2$. This does not happen, for example, with the n-sphere $(n \geqslant 2)$, whose local groupoids are also simply connected. Therefore, for many examples we cannot find any expression of Ker_* as a sum of complexes $C_*(\mathcal{G}_\alpha)$.

In the example of above, we use the usual presentation of the colimit $C_*(A)$ as the cokernel of the map

$$\bigoplus_{\alpha<\beta\in\phi} C_*(\mathcal{G}_\alpha) \xrightarrow{j} \bigoplus_{\alpha\in\phi} C_*(\mathcal{G}_\alpha)$$

which is defined on the basis elements by $j(s, \alpha < \beta) = (s, \alpha) - (\phi_\alpha^\beta(s), \beta)$. Here (s, i) denotes the element of the factor i given by the simplex s. In general j is not a monomorphism, then one cannot identify Ker_* with the sum $\bigoplus_{\alpha<\beta\in\phi} C_*(\mathcal{G}_\alpha)$.

Proposition 5.6.4. *The map j is mono if and only if ϕ_s has no cycles (viewed as a graph) for all simplices s.*

Proof. Note that $Ker(j) = \bigoplus Ker(j_n)$, where j_n is the n-th component of j.
Let $k : \bigoplus_{\alpha\in\phi} C_n(\mathcal{G}_\alpha) \to \bigoplus_{\alpha\in\phi} C_n(A)$ be the map defined in the basis elements by $(s, \alpha) \mapsto (\overline{s}, \alpha)$. Here $\overline{s}$ denotes the class of the simplex s in NA. Let $\sum_i c_i(s_i, \alpha_i < \beta_i)$ be an element of $Ker(j_n)$. We have

$$\begin{aligned}
0 &= k(j(\sum_i c_i(s_i, \alpha_i < \beta_i))) \\
&= k(\sum_i c_i j(s_i, \alpha_i < \beta_i)) \\
&= k(\sum_i c_i(s_i, \alpha_i) - c_i(\phi_\alpha^\beta(s_i), \beta_i)) \\
&= \sum_i c_i k(s_i, \alpha_i) - c_i k(\phi_\alpha^\beta(s_i), \beta_i) \\
&= \sum_i c_i(\overline{s_i}, \alpha_i) - c_i(\overline{s_i}, \beta_i)
\end{aligned}$$

Then, for all $s \in NA_n$ we have $0 = \sum_{\overline{s_i}=s} c_i(s, \alpha_i) - c_i(s, \beta_i)$ and $d(\sum_{\overline{s_i}=s} c_i(\alpha_i, \beta_i)) = 0$, where d is the boundary map of the simplicial chain complex of ϕ_s. This way, a non trivial element of $Ker(j_n)$ gives a cycle in some ϕ_s. The converse is clear. $\square$

Remarks 5.6.5.

- If there exist $\alpha, \beta, \gamma \in \phi_A$ such that $\alpha < \beta < \gamma$, then j is not a monomorphism.

- j could be a monomorphism even if ϕ_A has non trivial cycles, e.g. A the 1-sphere.

We expose a simple aplication of last result. Let A be a groupoid atlas such that ϕ_A is discrete ($\alpha \leqslant \beta \Rightarrow \alpha = \beta$). Recall that, since A is not good, $NA = NrA$ by definition, and note that $(\phi_{rA})(s)$ is discrete for s of dimension ≥ 1 and is a star for s a point. Thus, we have

Corollary 5.6.6. *If ϕ_A is discrete, then $H_n(A) = \bigoplus_\alpha H_n(\mathcal{G}_\alpha)$ for $n \geqslant 2$. For $n = 1$ we have $H_1(A) = (\bigoplus_\alpha H_1(\mathcal{G}_\alpha)) \oplus F$ with F a free abelian group.*

The procedure of above can be emulated in other contexts, considering a cofinal subset of local groupoids $\{\mathcal{G}_\alpha\}_{\alpha \in S}$ to make the canonical map $p : \bigoplus_{\alpha \in S} C_*(\mathcal{G}_\alpha) \to C_*(A)$ an epimorphism (when ϕ_A is finite, then S could be the set of maximal objects), and then construct $Ker_*(p)$ as the sum of the chain complexes associated to certain local groupoids.

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