A Generalization of De Vries Duality Theorem^{*}

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Abstract

Generalizing Duality Theorem of H. de Vries, we define a category which is dually equivalent to the category of all locally compact Hausdorff spaces and all perfect maps between them.

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Introduction

According to the famous Stone Duality Theorem ([17]), the category of all zero-dimensional compact Hausdorff spaces and all continuous maps between them is dually equivalent to the category **Bool** of all Boolean algebras and all Boolean homomorphisms between them. In 1962, H. de Vries [6] introduced the notion of *compingent Boolean algebra* and proved that the category of all compact Hausdorff spaces and all continuous maps between them is dually equivalent to the category of all complete compingent Boolean algebras and appropriate morphisms between them. In 1997, Roeper [15] defined the notion of *region-based topology* as one of the possible formalizations of the ideas of De Laguna [5] and Whitehead [19] for a region-based theory of space. Following [18, 9], the region-based topologies of Roeper appear here as *local contact algebras* (briefly, LCAs), because the axioms which they satisfy almost coincide with the axioms of local proximities of Leader [13]. In his paper [15], Roeper proved the following theorem: there is a bijective correspondence between all (up to homeomorphism) locally compact Hausdorff spaces and all (up to isomorphism) complete LCAs. It

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generalizes the theorem of de Vries [6] that there exists a bijective correspondence between all (up to homeomorphism) compact Hausdorff spaces and all (up to isomorphism) complete compingent Boolean algebras. Here, using Roeper's Theorem and the results of de Vries [6], a category dually equivalent to the category of all locally compact Hausdorff spaces and all perfect maps between them is defined (see Theorem 2.10 bellow), generalizing in this way the Duality Theorem of H. de Vries.

Let us mention that, using de Vries Duality Theorem, V. V. Fedorchuk [11] showed that the category of all compact Hausdorff spaces and all quasiopen maps between them is dually equivalent to the category of all complete compingent Boolean algebras and all complete Boolean homomorphisms between them satisfying one simple condition, and that in [7, 8] some extensions of the Fedorchuk Duality Theorem ([11]) to some categories whose objects are all locally compact Hausdorff spaces are obtained.

We now fix the notations.

If \mathcal{C} denotes a category, we write $X \in |\mathcal{C}|$ if X is an object of \mathcal{C} , and $f \in \mathcal{C}(X, Y)$ if f is a morphism of \mathcal{C} with domain X and codomain Y.

All lattices are with top (= unit) and bottom (= zero) elements, denoted respectively by 1 and 0. We do not require the elements 0 and 1 to be distinct.

If (X, τ) is a topological space and M is a subset of X, we denote by $\operatorname{cl}_{(X,\tau)}(M)$ (or simply by $\operatorname{cl}(M)$ or $\operatorname{cl}_X(M)$) the closure of M in (X, τ) and by $\operatorname{int}_{(X,\tau)}(M)$ (or briefly by $\operatorname{int}(M)$ or $\operatorname{int}_X(M)$) the interior of M in (X, τ) . The Alexandroff compactification of a locally compact Hausdorff non-compact space X will be denoted by αX and the added point by ∞_X (i.e. $\alpha X = X \cup \{\infty_X\}$).

The closed maps between topological spaces are assumed to be continuous but are not assumed to be onto. Recall that a map is *perfect* if it is closed and compact (i.e. point inverses are compact sets).

1 Preliminaries

Definition 1.1 An algebraic system $\underline{B} = (B, 0, 1, \vee, \wedge, *, C)$ is called a *contact algebra* (abbreviated as CA) if $(B, 0, 1, \vee, \wedge, *)$ is a Boolean algebra (where the operation "complement" is denoted by "*") and C is a binary relation on B, satisfying the following axioms:

(C1) If $a \neq 0$ then aCa;

(C2) If aCb then $a \neq 0$ and $b \neq 0$;

- (C3) aCb implies bCa;
- (C4) $aC(b \lor c)$ iff aCb or aCc.

Usually, we shall simply write (B, C) for a contact algebra. The relation C is called a *contact relation*. When B is a complete Boolean algebra, we will

say that (B, C) is a complete contact algebra (abbreviated as CCA).

We will say that two CA's (B_1, C_1) and (B_2, C_2) are *CA-isomorphic* iff there exists a Boolean isomorphism $\varphi : B_1 \longrightarrow B_2$ such that, for each $a, b \in B_1, aC_1b$ iff $\varphi(a)C_2\varphi(b)$. Note that in this paper, by a "Boolean isomorphism" we understand an isomorphism in the category **Bool**.

A CA (B, C) is called *connected* if it satisfies the following axiom:

(CON) If $a \neq 0, 1$ then aCa^* .

A contact algebra (B, C) is called a *normal contact algebra* (abbreviated as NCA) ([6, 11]) if it satisfies the following axioms (we will write "-C" for "not C"):

(C5) If a(-C)b then a(-C)c and $b(-C)c^*$ for some $c \in B$;

(C6) If $a \neq 1$ then there exists $b \neq 0$ such that b(-C)a.

A normal CA is called a *complete normal contact algebra* (abbreviated as CNCA) if it is a CCA. The notion of normal contact algebra was introduced by Fedorchuk [11] under the name *Boolean* δ -algebra as an equivalent expression of the notion of compingent Boolean algebra of de Vries. We call such algebras "normal contact algebras" because they form a subclass of the class of contact algebras.

Note that if $0 \neq 1$ then the axiom (C2) follows from the axioms (C6) and (C4).

For any CA (B, C), we define a binary relation " \ll_C " on B (called *non-tangential inclusion*) by " $a \ll_C b \leftrightarrow a(-C)b^*$ ". Sometimes we will write simply " \ll " instead of " \ll_C ".

The relations C and \ll are inter-definable. For example, normal contact algebras could be equivalently defined (and exactly in this way they were defined (under the name of compingent Boolean algebras) by de Vries in [6]) as a pair of a Boolean algebra $B = (B, 0, 1, \lor, \land, *)$ and a binary relation \ll on B subject to the following axioms:

 $(\ll 1) \ a \ll b \text{ implies } a \le b;$

 $(\ll 2) \ 0 \ll 0;$

 $(\ll 3) \ a \le b \ll c \le t \text{ implies } a \ll t;$

 $(\ll 4) a \ll c \text{ and } b \ll c \text{ implies } a \lor b \ll c;$

 $(\ll 5)$ If $a \ll c$ then $a \ll b \ll c$ for some $b \in B$;

(\ll 6) If $a \neq 0$ then there exists $b \neq 0$ such that $b \ll a$;

 $(\ll 7) a \ll b$ implies $b^* \ll a^*$.

Note that if $0 \neq 1$ then the axiom ($\ll 2$) follows from the axioms ($\ll 3$), ($\ll 4$), ($\ll 6$) and ($\ll 7$).

Obviously, contact algebras could be equivalently defined as a pair of a Boolean algebra B and a binary relation \ll on B subject to the axioms $(\ll 1)$ - $(\ll 4)$ and $(\ll 7)$.

It is easy to see that axiom (C5) (resp., (C6)) can be stated equivalently in the form of (\ll 5) (resp., (\ll 6)).

Example 1.2 Let *B* be a Boolean algebra. Then there exist the largest and the smallest contact relations on *B*; the largest one, ρ_l , is defined by $a\rho_l b$ iff $a \neq 0$ and $b \neq 0$, and the smallest one, ρ_s , by $a\rho_s b$ iff $a \wedge b \neq 0$.

Note that, for $a, b \in B$, $a \ll_{\rho_s} b$ iff $a \leq b$; hence $a \ll_{\rho_s} a$, for any $a \in B$. Thus (B, ρ_s) is a normal contact algebra.

Example 1.3 Recall that a subset F of a topological space (X, τ) is called *regular closed* if F = cl(int(F)). Clearly, F is regular closed iff it is the closure of an open set.

For any topological space (X, τ) , the collection $RC(X, \tau)$ (we will often write simply RC(X)) of all regular closed subsets of (X, τ) becomes a complete Boolean algebra $(RC(X, \tau), 0, 1, \wedge, \vee, *)$ under the following operations:

$$1 = X, 0 = \emptyset, F^* = \operatorname{cl}(X \setminus F), F \lor G = F \cup G, F \land G = \operatorname{cl}(\operatorname{int}(F \cap G)).$$

The infinite operations are given by the following formulas: $\bigvee \{F_{\gamma} \mid \gamma \in \Gamma\} = \operatorname{cl}(\bigcup \{F_{\gamma} \mid \gamma \in \Gamma\}) (= \operatorname{cl}(\bigcup \{\operatorname{int}(F_{\gamma}) \mid \gamma \in \Gamma\})), \text{ and } \bigwedge \{F_{\gamma} \mid \gamma \in \Gamma\} = \operatorname{cl}(\operatorname{int}(\bigcap \{F_{\gamma} \mid \gamma \in \Gamma\})).$

It is easy to see that setting $F\rho_{(X,\tau)}G$ iff $F \cap G \neq \emptyset$, we define a contact relation $\rho_{(X,\tau)}$ on $RC(X,\tau)$; it is called a *standard contact relation*. So, $(RC(X,\tau),\rho_{(X,\tau)})$ is a CCA (it is called a *standard contact algebra*). We will often write simply ρ_X instead of $\rho_{(X,\tau)}$. Note that, for $F, G \in RC(X)$, $F \ll_{\rho_X} G$ iff $F \subseteq \operatorname{int}_X(G)$.

Clearly, if (X, τ) is a normal Hausdorff space then the standard contact algebra $(RC(X, \tau), \rho_{(X,\tau)})$ is a complete NCA.

A subset U of (X, τ) such that $U = \operatorname{int}(\operatorname{cl}(U))$ is said to be *regular* open. The set of all regular open subsets of (X, τ) will be denoted by $RO(X, \tau)$ (or briefly, by RO(X)). Define Boolean operations and contact δ_X in RO(X) as follows: $U \lor V = \operatorname{int}(\operatorname{cl}(U \cup V)), U \land V = U \cap V, U^* = \operatorname{int}(X \setminus U),$ $0 = \emptyset, 1 = X$ and $U\delta_X V$ iff $\operatorname{cl}(U) \cap \operatorname{cl}(V) \neq \emptyset$. Then $(RO(X), \delta_X)$ is a CA. This algebra is also complete, considering the infinite meet $\bigwedge \{U_i \mid i \in I\} = \operatorname{int}(\bigcap_{i \in I} U_i).$

Note that $(RO(X), \delta_X)$ and $(RC(X), \rho_X)$ are isomorphic CAs. The isomorphism f between them is defined by f(U) = cl(U), for every $U \in RO(X)$.

The following notion is a lattice-theoretical counterpart of the corresponding notion from the theory of proximity spaces (see [14]):

1.4 Let (B, C) be a CA. Then a non-empty subset σ of B is called a *cluster* in (B, C) if the following conditions are satisfied:

(K1) If $a, b \in \sigma$ then aCb;

(K2) If $a \lor b \in \sigma$ then $a \in \sigma$ or $b \in \sigma$;

(K3) If aCb for every $b \in \sigma$, then $a \in \sigma$.

The set of all clusters in (B, C) will be denoted denoted by Clust(B, C).

The next assertion can be proved exactly as Lemma 5.6 of [14]:

Fact 1.5 If σ_1, σ_2 are two clusters in a normal contact algebra (B, C) and $\sigma_1 \subseteq \sigma_2$ then $\sigma_1 = \sigma_2$.

Fact 1.6 ([3]) Let (X, τ) be a topological space. Then the standard contact algebra $(RC(X, \tau), \rho_{(X,\tau)})$ is connected iff the space (X, τ) is connected.

The following notion is a lattice-theoretical counterpart of the Leader's notion of *local proximity* ([13]):

Definition 1.7 ([15]) An algebraic system $\underline{B}_l = (B, 0, 1, \lor, \land, *, \rho, \mathbb{B})$ is called a *local contact algebra* (abbreviated as LCA) if $(B, 0, 1, \lor, \land, *)$ is a Boolean algebra, ρ is a binary relation on B such that (B, ρ) is a CA, and \mathbb{B} is an ideal (possibly non proper) of B, satisfying the following axioms: (BC1) If $a \in \mathbb{R}$ and $a \ll a$ then $a \ll b \ll a$ for some $b \in \mathbb{R}$ (see

(BC1) If $a \in \mathbb{B}$, $c \in B$ and $a \ll_{\rho} c$ then $a \ll_{\rho} b \ll_{\rho} c$ for some $b \in \mathbb{B}$ (see 1.1 for " \ll_{ρ} ");

(BC2) If $a\rho b$ then there exists an element c of \mathbb{B} such that $a\rho(c \wedge b)$;

(BC3) If $a \neq 0$ then there exists $b \in \mathbb{B} \setminus \{0\}$ such that $b \ll_{\rho} a$.

Usually, we shall simply write (B, ρ, \mathbb{B}) for a local contact algebra. We will say that the elements of \mathbb{B} are *bounded* and the elements of $B \setminus \mathbb{B}$ are *unbounded*. When B is a complete Boolean algebra, the LCA (B, ρ, \mathbb{B}) is called a *complete local contact algebra* (abbreviated as CLCA).

We will say that two local contact algebras (B, ρ, \mathbb{B}) and $(B_1, \rho_1, \mathbb{B}_1)$ are *LCA-isomorphic* iff there exists a Boolean isomorphism $\varphi : B \longrightarrow B_1$ such that, for $a, b \in B$, $a\rho b$ iff $\varphi(a)\rho_1\varphi(b)$, and $\varphi(a) \in \mathbb{B}_1$ iff $a \in \mathbb{B}$.

An LCA (B, ρ, \mathbb{B}) is called *connected* if the CA (B, ρ) is connected.

Remark 1.8 Note that if (B, ρ, \mathbb{B}) is a local contact algebra and $1 \in \mathbb{B}$ then (B, ρ) is a normal contact algebra. Conversely, any normal contact algebra (B, C) can be regarded as a local contact algebra of the form (B, C, B).

The following lemmas are lattice-theoretical counterparts of some theorems from Leader's paper [13].

Lemma 1.9 ([18]) Let (B, ρ, \mathbb{B}) be a local contact algebra. Define a binary relation " C_{ρ} " on B by

(1)
$$aC_{\rho}b \quad iff \quad a\rho b \quad or \quad a, b \notin \mathbb{B}$$

Then " C_{ρ} ", called the Alexandroff extension of ρ , is a normal contact relation on B and (B, C_{ρ}) is a normal contact algebra. **Lemma 1.10** ([18]) Let $\underline{B}_l = (B, \rho, \mathbb{B})$ be a local contact algebra and let $1 \notin \mathbb{B}$. Then $\sigma_{\infty}^{\underline{B}_l} = \{b \in B \mid b \notin \mathbb{B}\}$ is a cluster in (B, C_{ρ}) (see 1.9 for the notation " C_{ρ} "). (Sometimes we will simply write σ_{∞} instead of $\sigma_{\infty}^{\underline{B}_l}$.)

Definition 1.11 Let (B, ρ, \mathbb{B}) be a local contact algebra. A cluster σ in (B, C_{ρ}) (see 1.9) is called *bounded* if $\sigma \cap \mathbb{B} \neq \emptyset$. The set of all bounded clusters in (B, C_{ρ}) will be denoted by BClust (B, ρ, \mathbb{B}) .

Fact 1.12 Let (B, ρ, \mathbb{B}) be a local contact algebra and σ be a bounded cluster in (B, C_{ρ}) (see 1.9). Then there exists $b \in \mathbb{B}$ such that $b^* \notin \sigma$.

Proof. Let $b_0 \in \sigma \cap \mathbb{B}$. Since $b_0 \ll_{\rho} 1$, (BC1) implies that there exists $b \in \mathbb{B}$ such that $b_0 \ll_{\rho} b$. Then $b_0(-\rho)b^*$ and since $b_0 \in \mathbb{B}$, we obtain that $b_0(-C_{\rho})b^*$. Thus $b^* \notin \sigma$.

Notation 1.13 Let (X, τ) be a topological space. We denote by $CR(X, \tau)$ the family of all compact regular closed subsets of (X, τ) . We will often write CR(X) instead of $CR(X, \tau)$.

If $x \in X$ then we set:

(2)
$$\sigma_x = \{F \in RC(X) \mid x \in F\}$$

Fact 1.14 Let (X, τ) be a locally compact Hausdorff space. Then the triple

 $(RC(X,\tau),\rho_{(X,\tau)},CR(X,\tau))$

(see 1.3 for $\rho_{(X,\tau)}$) is a complete local contact algebra ([15]). It is called a standard local contact algebra.

For every $x \in X$, σ_x is a bounded cluster in $(RC(X), C_{\rho_X})$ (see (2) and (1) for the notations).

We will need a lemma from [4]:

Lemma 1.15 Let X be a dense subspace of a topological space Y. Then the functions $r_{X,Y} : RC(Y) \longrightarrow RC(X), F \longrightarrow F \cap X$, and $e_{X,Y} : RC(X) \longrightarrow$ $RC(Y), G \longrightarrow cl_Y(G)$, are Boolean isomorphisms between Boolean algebras RC(X) and RC(Y), and $e_{X,Y} \circ r_{X,Y} = id_{RC(Y)}, r_{X,Y} \circ e_{X,Y} = id_{RC(X)}$. (We will often write r_X, e_X instead of $r_{X,Y}, e_{X,Y}$, respectively.)

The next proposition is well known (see, e.g., [2]):

Proposition 1.16 Let $f : X \longrightarrow Y$ be a perfect map between two locally compact Hausdorff non-compact spaces. Then the map f has a continuous extension $\alpha(f) : \alpha X \longrightarrow \alpha Y$; moreover, $\alpha(f)(\infty_X) = \infty_Y$.

For all undefined here notions and notations see [1, 12, 10, 14, 16].

2 The Results

The next theorem was proved by Roeper [15]. We will give a sketch of its proof; it follows the plan of the proof presented in [18]. The notations and the facts stated here will be used later on.

Theorem 2.1 (P. Roeper [15]) There exists a bijective correspondence between the class of all (up to isomorphism) complete local contact algebras and the class of all (up to homeomorphism) locally compact Hausdorff spaces.

Sketch of the Proof. (A) Let (X, τ) be a locally compact Hausdorff space. We put

(3)
$$\Psi^t(X,\tau) = (RC(X,\tau), \rho_{(X,\tau)}, CR(X,\tau))$$

(see 1.14 and 1.13 for the notations).

(B) Let $\underline{B}_l = (B, \rho, \mathbb{B})$ be a complete local contact algebra. Let $C = C_{\rho}$ be the Alexandroff extension of ρ (see 1.9). Then, by 1.9, (B, C) is a complete normal contact algebra. Put X = Clust(B, C) and let \mathcal{T} be the topology on X having as a closed base the family $\{\lambda_{(B,C)}(a) \mid a \in B\}$ where, for every $a \in B$,

(4)
$$\lambda_{(B,C)}(a) = \{ \sigma \in X \mid a \in \sigma \}.$$

Sometimes we will write simply λ_B instead of $\lambda_{(B,C)}$.

It can be proved that (X, \mathfrak{T}) is a compact Hausdorff space and

(5)
$$\lambda_B : (B, C) \longrightarrow (RC(X), \rho_X)$$
 is a CA-isomorphism.

(B1) Let $1 \in \mathbb{B}$. Then $C = \rho$ and $\mathbb{B} = B$, so that $(B, \rho, \mathbb{B}) = (B, C, B) = (B, C)$ is a complete normal contact algebra (see 1.8), and we put

(6)
$$\Psi^a(B,\rho,\mathbb{B}) = \Psi^a(B,C,B) = \Psi^a(B,C) = (X,\mathfrak{T}).$$

(B2) Let $1 \notin \mathbb{B}$. Then, by Lemma 1.10, the set $\sigma_{\infty} = \{b \in B \mid b \notin \mathbb{B}\}$ is a cluster in (B, C) and, hence, $\sigma_{\infty} \in X$. Let $L = X \setminus \{\sigma_{\infty}\}$. Then

(7)
$$L = \mathrm{BClust}(B, \rho, \mathbb{B}),$$

i.e. L is the set of all bounded clusters of (B, C_{ρ}) (sometimes we will write $L_{\underline{B}_l}$ or L_B instead of L); let the topology $\tau(=\tau_{\underline{B}_l})$ on L be the subspace topology, i.e. $\tau = \mathcal{T}|_L$. Then (L, τ) is a locally compact Hausdorff space. We put

(8)
$$\Psi^a(B,\rho,\mathbb{B}) = (L,\tau).$$

Let $\lambda_B^l(a) = \lambda_B(a) \cap L$, for each $a \in B$. One can show that $X = \alpha L$ and

(9)
$$\lambda_B^l : (B, \rho, \mathbb{B}) \longrightarrow (RC(L), \rho_L, CR(L))$$
 is an LCA-isomorphism.

(C) For every CLCA (B, ρ, \mathbb{B}) and every $a \in B$, set

(10)
$$\lambda_B^g(a) = \lambda_B(a) \cap \Psi^a(B, \rho, \mathbb{B}).$$

Then, by (5) and (9), we get that

(11) $\lambda_B^g: (B, \rho, \mathbb{B}) \longrightarrow (\Psi^t \circ \Psi^a)(B, \rho, \mathbb{B})$ is an LCA-isomorphism.

(D) Let (Y, τ) be a locally compact Hausdorff space. It can be shown that the map

(12)
$$t_{(Y,\tau)}:(Y,\tau)\longrightarrow \Psi^a(\Psi^t(Y,\tau)),$$

defined by $t_{(Y,\tau)}(y) = \{F \in RC(Y,\tau) \mid y \in F\}(=\sigma_y)$, for every $y \in Y$, is a homeomorphism; we will often write simply t_Y instead of $t_{(Y,\tau)}$.

Therefore $\Psi^{a}(\Psi^{t}(Y,\tau))$ is homeomorphic to (Y,τ) and $\Psi^{t}(\Psi^{a}(B,\rho,\mathbb{B}))$ is LCA-isomorphic to (B,ρ,\mathbb{B}) .

Definition 2.2 (De Vries [6]) Let **HC** be the category of all compact Hausdorff spaces and all continuous maps between them.

Let **DVAL** be the category whose objects are all complete NCAs and whose morphisms are all functions $\varphi : (A, C) \longrightarrow (B, C')$ between the objects of **DVAL** satisfying the conditions:

(DVAL1) $\varphi(0) = 0;$

(DVAL2) $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$, for all $a, b \in A$;

(DVAL3) If $a, b \in A$ and $a \ll_C b$ then $(\varphi(a^*))^* \ll_{C'} \varphi(b)$;

(DVAL4) $\varphi(a) = \bigvee \{\varphi(b) \mid b \ll_C a\}$, for every $a \in A$,

and let the composition "*" of two morphisms $\varphi_1 : (A_1, C_1) \longrightarrow (A_2, C_2)$ and $\varphi_2 : (A_2, C_2) \longrightarrow (A_3, C_3)$ of **DVAL** be defined by the formula

(13)
$$\varphi_2 * \varphi_1 = (\varphi_2 \circ \varphi_1) \check{},$$

where, for every function $\psi : (A, C) \longrightarrow (B, C')$ between two objects of **DVAL**, $\psi^{\check{}} : (A, C) \longrightarrow (B, C')$ is defined as follows:

(14)
$$\psi(a) = \bigvee \{\psi(b) \mid b \ll_C a\},$$

for every $a \in A$.

De Vries [6] proved the following duality theorem:

Theorem 2.3 The categories **HC** and **DVAL** are dually equivalent. In more details, let Φ^t : **HC** \longrightarrow **DVAL** be the contravariant functor defined by $\Phi^t(X,\tau) = (RC(X,\tau),\rho_X)$, for every $X \in |\mathbf{HC}|$, and $\Phi^t(f)(G) =$ $\mathrm{cl}(f^{-1}(\mathrm{int}(G)))$, for every $f \in \mathbf{HC}(X,Y)$ and every $G \in RC(Y)$, and let $\Phi^a : \mathbf{DVAL} \longrightarrow \mathbf{HC}$ be the contravariant functor defined by $\Phi^a(A,C) =$ $\Psi^{a}(A, C)$, for every $(A, C) \in |\mathbf{DVAL}|$, and $\Phi^{a}(\varphi)(\sigma') = \{a \in A \mid ifb \ll_{C} a^{*}$ then $(\varphi(b))^{*} \in \sigma'\}$, for every $\varphi \in \mathbf{DVAL}((A, C), (B, C'))$ and for every $\sigma' \in \mathrm{Clust}(B, C')$; then $\lambda : Id_{\mathbf{DVAL}} \longrightarrow \Phi^{t} \circ \Phi^{a}$, where $\lambda(A, C) = \lambda_{(A,C)}$ (see (4) and (5) for the notation $\lambda_{(A,C)}$), for every $(A, C) \in |\mathbf{DVAL}|$, and $t : Id_{\mathbf{HC}} \longrightarrow \Phi^{a} \circ \Phi^{t}$, where $t(X) = t_{X}$ (see (12) for the notation t_{X}), for every $X \in |\mathbf{HC}|$, are natural isomorphisms.

In [6], de Vries uses the regular open sets instead of regular closed sets, as we do, so that we present here the translations of his definitions for the case of regular closed sets.

Definition 2.4 We will denote by **PLC** the category of all locally compact Hausdorff spaces and all perfect maps between them.

Let **PAL** be the category whose objects are all complete LCAs and whose morphisms are all functions $\varphi : (A, \rho, \mathbb{B}) \longrightarrow (B, \eta, \mathbb{B}')$ between the objects of **PAL** satisfying the conditions:

(PAL1) $\varphi(0) = 0;$

(PAL2) $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$, for all $a, b \in A$;

(PAL3) If $a \in \mathbb{B}$, $b \in A$ and $a \ll_{\rho} b$ then $(\varphi(a^*))^* \ll_{\eta} \varphi(b)$;

(PAL4) For every $b \in \mathbb{B}'$ there exists $a \in \mathbb{B}$ such that $b \leq \varphi(a)$;

(PAL5) If $a \in \mathbb{B}$ then $\varphi(a) \in \mathbb{B}'$;

(PAL6) $\varphi(a) = \bigvee \{\varphi(b) \mid b \ll_{C_{\rho}} a \}$, for every $a \in A$ (see (1) for C_{ρ});

let the composition " \diamond " of two morphisms $\varphi_1 : (A_1, \rho_1, \mathbb{B}_1) \longrightarrow (A_2, \rho_2, \mathbb{B}_2)$ and $\varphi_2 : (A_2, \rho_2, \mathbb{B}_2) \longrightarrow (A_3, \rho_3, \mathbb{B}_3)$ of **PAL** be defined by the formula

(15)
$$\varphi_2 \diamond \varphi_1 = (\varphi_2 \circ \varphi_1)^{\check{}},$$

where, for every function $\psi : (A, \rho, \mathbb{B}) \longrightarrow (B, \eta, \mathbb{B}')$ between two objects of **PAL**, $\psi^{\tilde{}} : (A, \rho, \mathbb{B}) \longrightarrow (B, \eta, \mathbb{B}')$ is defined as follows:

(16)
$$\psi^{\check{}}(a) = \bigvee \{ \psi(b) \mid b \ll_{C_{\rho}} a \},$$

for every $a \in A$.

By **NAL** we denote the full subcategory of **PAL** having as objects all CNCAs (i.e., those CLCAs (A, ρ, \mathbb{B}) for which $\mathbb{B} = A$).

Note that the categories **DVAL** and **NAL** are isomorphic (it can be even said that they are identical) because the axiom (PAL5) is trivially fulfilled in the category **DVAL** (indeed, all elements of its objects are bounded), the axiom (PAL4) follows immediately from the obvious fact that $\varphi(1) = 1$ for every **DVAL**-morphism φ , and the compositions are the same.

We will generalize the Duality Theorem of de Vries showing that the categories **PAL** and **PLC** are dually equivalent.

We will first show that **PAL** is indeed a category.

Lemma 2.5 Let us regard two functions $\varphi : (A, \rho, \mathbb{B}) \longrightarrow (B, \eta, \mathbb{B}')$ and $\psi : (B, \eta, \mathbb{B}') \longrightarrow (B_1, \eta_1, \mathbb{B}'_1)$ between CLCAs. Then:

(a) If φ satisfies condition (PAL2) then φ is an order preserving function; (b) If φ satisfies conditions (PAL1) and (PAL2) then $\varphi(a^*) \leq (\varphi(a))^*$, for every $a \in A$;

(c) Let φ satisfy conditions (PAL3) and (PAL5). If $a, b \in A$ and $a \ll_{C_{\rho}} b$ then $(\varphi(a^*))^* \ll_{C_{\eta}} \varphi(b)$. Hence, if φ satisfies in addition conditions (PAL1) and (PAL2) then $\varphi(a) \ll_{C_{\eta}} \varphi(b)$;

(d) If φ satisfies conditions (PAL1) and (PAL3) then $\varphi(1) = 1$;

(e) If φ satisfies condition (PAL2) then φ^{*} satisfies conditions (PAL2) and (PAL6) (see (16) for φ^{*});

(f) If φ satisfies condition (PAL6) then $\varphi = \varphi^{*}$;

(g) If φ satisfies condition (PAL2) then $(\varphi^{\check{}})^{\check{}} = \varphi^{\check{}}$;

(h) If φ and ψ satisfy condition (PAL2) and φ satisfies in addition conditions (PAL1), (PAL3) and (PAL5) then $(\psi \circ \varphi)^{\check{}} = (\psi^{\check{}} \circ \varphi^{\check{}})^{\check{}}$.

Proof. The properties (a), (b), (d) and (f) are clearly fulfilled, and (g) follows from (e) and (f).

(c) Let $a, b \in A$ and $a \ll_{C_{\rho}} b$. Then $a \ll_{\rho} b$ and at least one of the elements a and b^* is bounded.

Let $a \in \mathbb{B}$. Then (PAL3) implies that $(\varphi(a^*))^* \ll_{\eta} \varphi(b)$. By (BC1), there exists $c \in \mathbb{B}$ such that $a \ll_{\rho} c$. Hence, using again (PAL3), we get that $(\varphi(a^*))^* \ll_{\eta} \varphi(c)$. Since $\varphi(c) \in \mathbb{B}'$ (according to (PAL5)), we obtain that $(\varphi(a^*))^* \in \mathbb{B}'$. Therefore, $(\varphi(a^*))^* \ll_{C_{\eta}} \varphi(b)$.

Let now $b^* \in \mathbb{B}$. Since $b^* \ll_{C_{\rho}} a^*$, we get, by the previous case, that $(\varphi(b))^* \ll_{C_{\eta}} \varphi(a^*)$. Thus $(\varphi(a^*))^* \ll_{C_{\eta}} \varphi(b)$.

(e) By (a), for every $a \in A$, $\varphi^{\check{}}(a) \leq \varphi(a)$. Let $a \in A$. If $c \ll_{C_{\rho}} a$ then there exists $d_c \in A$ such that $c \ll_{C_{\rho}} d_c \ll_{C_{\rho}} a$; hence $\varphi(c) \leq \varphi^{\check{}}(d_c)$. Now, $\varphi^{\check{}}(a) = \bigvee \{\varphi(c) \mid c \ll_{C_{\rho}} a\} \leq \bigvee \{\varphi^{\check{}}(d_c) \mid c \ll_{C_{\rho}} a\} \leq \bigvee \{\varphi^{\check{}}(e) \mid e \ll_{C_{\rho}} a\}$ $a\} \leq \bigvee \{\varphi(e) \mid e \ll_{C_{\rho}} a\} = \varphi^{\check{}}(a)$. Thus, $\varphi^{\check{}}(a) = \bigvee \{\varphi^{\check{}}(e) \mid e \ll_{C_{\rho}} a\}$. So, $\varphi^{\check{}}$ satisfies (PAL6). Further, let $a, b \in A$. Then $\varphi^{\check{}}(a) \land \varphi^{\check{}}(b) = \bigvee \{\varphi(d) \land \varphi(e) \mid d \ll_{C_{\rho}} a, e \ll_{C_{\rho}} b\} = \bigvee \{\varphi(d \land e) \mid d \ll_{C_{\rho}} a, e \ll_{C_{\rho}} b\} = \bigvee \{\varphi(c) \mid c \ll_{C_{\rho}} a \land b\} = \varphi^{\check{}}(a \land b)$. So, (PAL2) is fulfilled.

(h) Since $\varphi(a) \leq \varphi(a)$ for every $a \in A$, and $\psi(b) \leq \psi(b)$ for every $b \in B$, we get that $\psi(\varphi(a)) \leq \psi(\varphi(a))$, for every $a \in A$. Hence, using (16), we obtain that $(\psi \circ \varphi)(a) \leq (\psi \circ \varphi)(a)$, for every $a \in A$. Further, by (16), for every $a \in A$, $(\psi \circ \varphi)(a) = \bigvee \{\psi(\varphi(e)) \mid e \ll_{C_{\rho}} a\}$ and $(\psi \circ \varphi)(a) = \bigvee \{\psi(\varphi(b)) \mid b \ll_{C_{\rho}} a\} = \bigvee \{\bigvee \{\psi(c) \mid c \ll_{C_{\eta}} \varphi(b)\} \mid b \ll_{C_{\rho}} a\}$. Let $a \in A$ and $e \ll_{C_{\rho}} a$. Then there exist $b, d \in A$ such that $e \ll_{C_{\rho}} d \ll_{C_{\rho}} b \ll_{C_{\rho}} a$. Set $c = \varphi(e)$. Then, by (c), $c \ll_{C_{\eta}} \varphi(d) \leq \varphi(b)$. Hence $\psi(\varphi(e)) = \psi(c) \leq (\psi \circ \varphi)(a)$. We conclude that $(\psi \circ \varphi)(a) \leq (\psi \circ \varphi)(a)$. Therefore the desired equality is proved. **Proposition 2.6** Let $\varphi_i : (A_i, \rho_i, \mathbb{B}_i) \longrightarrow (A_{i+1}, \rho_{i+1}, \mathbb{B}_{i+1})$, where i = 1, 2, be two functions between CLCAs and let φ_1 and φ_2 satisfy conditions (PAL1)-(PAL5). Then the function $\varphi_2 \circ \varphi_1$ satisfies conditions (PAL1)-(PAL5).

Proof. Let $a \in \mathbb{B}_1$, $b \in A_1$ and $a \ll_{\rho_1} b$. Then, by (BC1), there exists $c \in \mathbb{B}_1$ such that $a \ll_{\rho_1} c \ll_{\rho_1} b$. From (PAL3) we get that $(\varphi_1(a^*))^* \ll_{\rho_2} \varphi_1(c)$. Then, since $\varphi_1(c) \in \mathbb{B}_2$ (by (PAL5)), $(\varphi_1(a^*))^* \in \mathbb{B}_2$. Now, using twice (PAL3), we obtain that $(\varphi_1(a^*))^* \ll_{\rho_2} \varphi_1(b)$ and $(\varphi_2(\varphi_1(a^*)))^* \ll_{\rho_3} \varphi_2(\varphi_1(b))$. Hence, the function $\varphi_2 \circ \varphi_1$ satisfies condition (PAL3). The rest is obvious.

Proposition 2.7 Let $\varphi : (A, \rho, \mathbb{B}) \longrightarrow (B, \eta, \mathbb{B}')$ be a function between CLCAs and let φ satisfies conditions (PAL1)-(PAL5). Then the function $\varphi^{\tilde{}}$ (see (16)) satisfies conditions (PAL1)-(PAL6) (i.e., it is a **PAL**-morphism).

Proof. Obviously, for every $a \in A$, $\varphi^{\check{}}(a) \leq \varphi(a)$. Hence, $\varphi^{\check{}}(0) = 0$, i.e. (PAL1) is fulfilled. For (PAL2) and (PAL6) see 2.5(e). Let $a \in \mathbb{B}$, $b \in A$ and $a \ll_{\rho} b$. Then, by (BC1), there exist $c, d \in \mathbb{B}$ such that $a \ll_{\rho} c \ll_{\rho} d \ll_{\rho} b$. Thus $a \ll_{C_{\rho}} c \ll_{C_{\rho}} d \ll_{C_{\rho}} b$ and hence $c^* \ll_{C_{\rho}} a^*$. We obtain that $\varphi(d) \leq \varphi^{\check{}}(b)$ and $\varphi(c^*) \leq \varphi^{\check{}}(a^*)$. Hence $(\varphi^{\check{}}(a^*))^* \leq (\varphi(c^*))^* \ll_{\eta} \varphi(d) \leq \varphi^{\check{}}(b)$. Therefore, $(\varphi^{\check{}}(a^*))^* \ll_{\eta} \varphi^{\check{}}(b)$. So, (PAL3) is fulfilled. Finally, it is easy to verify (PAL4) and (PAL5).

Proposition 2.8 PAL is a category.

Proof. This follows immediately from 2.5(f), 2.5(h), 2.6 and 2.7.

Proposition 2.9 Let X be a locally compact Hausdorff space. Then the NCAs $(RC(X), C_{\rho_X})$ and $(RC(\alpha X), \rho_{\alpha X})$ are CA-isomorphic (see 1.9 and 1.14 for the notations) and the maps $e_{X,\alpha X}$, $r_{X,\alpha X}$ are CA-isomorphisms between them (see 1.15 for the notations).

Proof. By 1.15, we have only to show that $AC_{\rho_X}B$ iff $cl_{\alpha X}(A)\rho_{\alpha X}cl_{\alpha X}(B)$, for every $A, B \in RC(X)$. This follows easily from the respective definitions. Hence, the map $e_{X,\alpha X} : (RC(X), C_{\rho_X}) \longrightarrow (RC(\alpha X, \rho_{\alpha X}))$ is a CA-isomorphism. Thus the map $r_{X,\alpha X}$ is also a CA-isomorphism. \Box

Theorem 2.10 The categories **PLC** and **PAL** are dually equivalent.

Proof. We will define two contravariant functors

$$\Xi^a : \mathbf{PAL} \longrightarrow \mathbf{PLC} \text{ and } \Xi^t : \mathbf{PLC} \longrightarrow \mathbf{PAL}.$$

I. The definition of Ξ^t .

For every $(X, \tau) \in |\mathbf{PLC}|$, we let $\Xi^t(X, \tau) = \Psi^t(X, \tau)$ (see (3) for Ψ^t). Let $f: (X, \tau) \longrightarrow (Y, \tau') \in \mathbf{PLC}(X, Y)$. We set

(17)
$$\Xi^t(f): \Xi^t(Y,\tau') \longrightarrow \Xi^t(X,\tau), \quad \Xi^t(f)(F) = cl_X(f^{-1}(\operatorname{int}_Y(F))).$$

Put, for the sake of brevity, $\varphi_f = \Xi^t(f)$. We have to show that φ_f is a **PAL**-morphism. Obviously, (PAL1) is fulfilled. For verifying (PAL4), let $H \in CR(X)$. Then f(H) is compact. Since Y is locally compact, there exists $F \in CR(Y)$ such that $f(H) \subseteq \operatorname{int}(F)$. Now we obtain that $H \subseteq f^{-1}(\operatorname{int}(F)) \subseteq \operatorname{int}(\operatorname{cl}(f^{-1}(\operatorname{int}(F)))) = \operatorname{int}(\varphi_f(F))$, i.e. $H \ll_{\rho_X} \varphi_f(F)$. Hence (PAL4) is checked.

Let now $F \in CR(Y)$. Then $\varphi_f(F) = \operatorname{cl}(f^{-1}(\operatorname{int}(F))) \subseteq f^{-1}(F)$. Since $f^{-1}(F)$ is compact (because f is perfect), $\varphi_f(F) \in CR(X)$. Therefore, (PAL5) is fulfilled.

By 1.16, f has a continuous extension $\alpha(f) : \alpha X \longrightarrow \alpha Y$. Set $\varphi_{\alpha f} = \Phi^t(\alpha(f))$ (see Theorem 2.3 for Φ^t). Then, by Theorem 2.3, $\varphi_{\alpha f}$ is a **DVAL**-morphism. We will prove that

(18)
$$r_{X,\alpha X} \circ \varphi_{\alpha f} = \varphi_f \circ r_{Y,\alpha Y}$$

(see 1.15 for the notations), i.e. that, for every $G \in RC(Y)$, the following equality holds:

(19)
$$X \cap \varphi_{\alpha f}(\mathrm{cl}_{\alpha Y}(G)) = \varphi_f(G),$$

or, in other words, that

$$X \cap \operatorname{cl}_{\alpha X}((\alpha(f))^{-1}(\operatorname{int}_{\alpha Y}(\operatorname{cl}_{\alpha Y}(G)))) = \operatorname{cl}_X(f^{-1}(\operatorname{int}_Y(G))).$$

Since the last equality follows easily from the obvious inclusions $\operatorname{int}_Y(G) \cup \{\infty_Y\} \supseteq \operatorname{int}_{\alpha Y}(\operatorname{cl}_{\alpha Y}(G)) \supseteq \operatorname{int}_Y(G), (18) \text{ is proved. Therefore, } \varphi_f = r_{X,\alpha X} \circ \varphi_{\alpha f} \circ e_{Y,\alpha Y} \text{ (see 1.15). Since } \varphi_{\alpha f} \text{ satisfies (DVAL2), we obtain that } \varphi_f \text{ satisfies (PAL2).}$

For establishing (PAL3), let $F \in CR(Y), G \in RC(Y)$ and $F \ll_{\rho_Y} G$. Then $F \ll_{C_{\rho_Y}} G$ and hence, by 2.9, $F \ll_{\rho_{\alpha_Y}} \operatorname{cl}_{\alpha_Y}(G)$. Thus, (DVAL3) implies that

(20)
$$(\varphi_{\alpha f}(F^{*\alpha}))^{*\alpha} \ll_{\rho_{\alpha X}} \varphi_{\alpha f}(\mathrm{cl}_{\alpha Y}(G)),$$

where "**a" is used as a common notation of the complement in the Boolean algebras $RC(\alpha X)$ and $RC(\alpha Y)$. Since, for every $H \in RC(X)$, $X \cap (\operatorname{cl}_{\alpha X}(H))^{*\alpha} = r_{X,\alpha X}((\operatorname{cl}_{\alpha X}(H))^{*\alpha}) = (r_{X,\alpha X}(\operatorname{cl}_{\alpha X}(H))^{*} = H^{*}$, we get, using again 2.9, that $(X \cap \varphi_{\alpha f}(F^{*\alpha}))^{*} \ll_{C_{\rho_X}} (X \cap \varphi_{\alpha f}(\operatorname{cl}_{\alpha Y}(G)))$; then, applying twice (19), the equality $F^{*\alpha}(= (e_{Y,\alpha Y}(F))^{*\alpha}) = e_{Y,\alpha Y}(F^{*})$ and (18), we obtain that $(\varphi_f(F^{*}))^{*} \ll_{C_{\rho_X}} \varphi_f(G)$, i.e. (PAL3) is fulfilled. Now, we will verify (PAL6). Let $F \in RC(Y)$; then $cl_{\alpha Y}(F) \in RC(\alpha Y)$ and hence, by (DVAL4),

$$\varphi_{\alpha f}(\mathrm{cl}_{\alpha Y}(F)) = \bigvee \{ \varphi_{\alpha f}(\mathrm{cl}_{\alpha Y}(G)) \mid G \in RC(Y), \mathrm{cl}_{\alpha Y}(G) \ll_{\rho_{\alpha Y}} \mathrm{cl}_{\alpha Y}(F) \}.$$

Since $r_{X,\alpha X}$ is an isomorphism, we obtain that $r_{X,\alpha X}(\varphi_{\alpha f}(\operatorname{cl}_{\alpha Y}(F))) = \bigvee\{r_{X,\alpha X}(\varphi_{\alpha f}(\operatorname{cl}_{\alpha Y}(G))) \mid G \in RC(Y), \operatorname{cl}_{\alpha Y}(G) \ll_{\rho_{\alpha Y}} \operatorname{cl}_{\alpha Y}(F)\}$. Thus, (18) and 2.9 imply that $\varphi_f(F) = \bigvee\{\varphi_f(G) \mid G \in RC(Y), G \ll_{C_{\rho_Y}} F\}$. So, (PAL6) is fulfilled.

Therefore, φ_f is a **PAL**-morphism.

Let $f \in \mathbf{PLC}(X, Y)$ and $g \in \mathbf{PLC}(Y, Z)$. We will prove that $\Xi^t(g \circ f) = \Xi^t(f) \diamond \Xi^t(g)$. Put $h = g \circ f$, $\varphi_h = \Xi^t(h)$, $\varphi_f = \Xi^t(f)$ and $\varphi_g = \Xi^t(g)$. Let $\alpha(f) : \alpha X \longrightarrow \alpha Y$, $\alpha(g) : \alpha Y \longrightarrow \alpha Z$ and $\alpha(h) : \alpha X \longrightarrow \alpha Z$ be the continuous extensions of f, g and h, respectively (see 1.16). Then, obviously, $\alpha(h) = \alpha(g) \circ \alpha(f)$. Set $\varphi_{\alpha f} = \Phi^t(\alpha(f)), \varphi_{\alpha g} = \Phi^t(\alpha(g))$ and $\varphi_{\alpha h} = \Phi^t(\alpha(h))$ Then, by Theorem 2.3, $\varphi_{\alpha h} = (\varphi_{\alpha f} \circ \varphi_{\alpha g})^{\check{}}$. Now, using (18) and 1.15, we get that $e_X \circ \varphi_h \circ r_Z = \varphi_{\alpha h} = (e_X \circ \varphi_f \circ \varphi_g \circ r_Z)^{\check{}}$. Thus, for every $F \in RC(\alpha Z)$, we have that $\varphi_h(r_Z(F)) = \bigvee \{(\varphi_f \circ \varphi_g)(r_Z(G)) \mid G \ll_{\rho_{\alpha Z}} F\}$. Now, 1.15 and 2.9 imply that $\varphi_h = (\varphi_f \circ \varphi_g)^{\check{}}$, i.e. $\varphi_h = \varphi_f \diamond \varphi_g$.

So, $\Xi^t : \mathbf{PLC} \longrightarrow \mathbf{PAL}$ is a contravariant functor.

II. The definition of Ξ^a .

For every $(A, \rho, \mathbb{B}) \in |\mathbf{PAL}|$, we let $\Xi^a(A, \rho, \mathbb{B}) = \Psi^a(A, \rho, \mathbb{B})$ (see (6) and (8) for Ψ^a).

Let $\varphi \in \mathbf{PAL}((A, \rho, \mathbb{B}), (B, \eta, \mathbb{B}'))$. We define the map

$$\Xi^{a}(\varphi):\Xi^{a}(B,\eta,\mathbb{B}')\longrightarrow\Xi^{a}(A,\rho,\mathbb{B})$$

by the formula

(21)
$$\Xi^{a}(\varphi)(\sigma') = \{ a \in A \mid \text{ if } b \ll_{C_{\rho}} a^{*} \text{ then } (\varphi(b))^{*} \in \sigma' \},$$

for every bounded cluster σ' in (B, C_{η}) . Set, for the sake of brevity, $\Xi^{a}(\varphi) = f_{\varphi}, X = \Xi^{a}(A, \rho, \mathbb{B})$ and $Y = \Xi^{a}(B, \eta, \mathbb{B}')$. We will show that $f_{\varphi} : Y \longrightarrow X$ is well-defined and is a perfect map.

Let $\varphi_C : (A, C_{\rho}) \longrightarrow (B, C_{\eta})$ be defined by $\varphi_C(a) = \varphi(a)$, for every $a \in A$. Then φ_C is a **DVAL**-morphism. Indeed, (DVAL3) follows from 2.5(c), and the other three axioms are obviously fulfilled. Set $f_{\alpha} = \Phi^a(\varphi_C)$. Then $f_{\alpha} : \alpha Y \longrightarrow \alpha X$ (see Theorem 2.3 and (B1), (B2) in the proof of Theorem 2.1). The definitions of f_{φ} and f_{α} coincide on the bounded clusters of (B, C_{η}) (see (21) and Theorem 2.3); hence, the right side of the formula (21) defines a cluster in (A, C_{ρ}) and f_{α} is an extension of f_{φ} . Thus, if we show that $f_{\alpha}^{-1}(\infty_X) = \{\infty_Y\}$, the map f_{φ} will be well-defined and will be a perfect map. Let us prove that $f_{\alpha}(Y) \subseteq X$, i.e. that if σ' is a bounded

cluster in (B, C_{η}) then $\sigma = f_{\alpha}(\sigma') = f_{\varphi}(\sigma')$ is a bounded cluster in (A, C_{ρ}) . So, let σ' be a bounded cluster in (B, C_{η}) and $\sigma = f_{\alpha}(\sigma')$. Then 1.12 implies that there exists $b \in \mathbb{B}'$ such that $b^* \notin \sigma'$. By (PAL4), there exists $a \in \mathbb{B}$ such that $b \leq \varphi(a)$. Thus $(\varphi(a))^* \leq b^*$ and hence $(\varphi(a))^* \notin \sigma'$. By (BC1), there exists $a_1 \in \mathbb{B}$ such that $a \ll_{\rho} a_1$. Then $a \ll_{C_{\rho}} a_1$ and, by the definition of σ , $a_1^* \notin \sigma$. Therefore $a_1 \in \mathbb{B} \cap \sigma$, i.e. σ is a bounded cluster in (A, C_{ρ}) . Hence $f_{\varphi}(Y) = f_{\alpha}(Y) \subseteq X$. Further, we have (by 1.10) that $\infty_X = A \setminus \mathbb{B}$ and $\infty_Y = B \setminus \mathbb{B}'$. Let us show that $f_{\alpha}(\infty_Y) = \infty_X$. Set $\sigma' = \infty_Y$ and $\sigma = f_{\alpha}(\sigma')$. Let $a \in \sigma$. Suppose that $a \in \mathbb{B}$. Then, by (BC1), there exist $a_1, a_2 \in \mathbb{B}$ such that $a \ll_{\rho} a_1 \ll_{\rho} a_2$. Thus $a \ll_{C_{\rho}} a_1 \ll_{C_{\rho}} a_2$. Hence $a_1^* \ll_{C_{\rho}} a^*$. Since $a \in \sigma$, the definition of σ implies that $(\varphi(a_1^*))^* \in \sigma'$. By 2.5(c), we have that $(\varphi(a_1^*))^* \leq \varphi(a_2)$. Therefore, $\varphi(a_2) \in \sigma'$. Since $\varphi(a_2) \in \mathbb{B}'$ (by (PAL5)), we obtain a contradiction. Thus $\sigma \subseteq A \setminus \mathbb{B}$. Now, 1.10 and 1.5 imply that $\sigma = A \setminus \mathbb{B}$, i.e. $f_{\alpha}(\infty_Y) = \infty_X$. Hence $f_{\alpha}^{-1}(X) = Y$. This shows that f_{φ} is a perfect map (because f_{α} is such). So, we have proved that $f_{\varphi} \in \mathbf{PLC}(Y, X)$.

Let $\varphi_i \in \operatorname{PAL}((A_i, \rho_i, \mathbb{B}_i), (A_{i+1}, \rho_{i+1}, \mathbb{B}_{i+1}))$ and $f_i = \Xi^a(\varphi_i)$ for $i = 1, 2, 2, \varphi = \varphi_2 \diamond \varphi_1, f_{\varphi} = \Xi^a(\varphi)$ and $X_i = \Xi^a(A_i, \rho_i, \mathbb{B}_i)$ for i = 1, 2, 3. We will prove that $f_{\varphi} = f_1 \circ f_2$. Let $(\varphi_i)_C : (A_i, C_{\rho_i}) \longrightarrow (A_{i+1}, C_{\rho_{i+1}})$ be defined by $(\varphi_i)_C(a) = \varphi_i(a)$ for every $a \in A_i$, where i = 1, 2. Then, as we know, $(\varphi_i)_C$ is a **DVAL**-morphism, for i = 1, 2. Set $f_{i\alpha} = \Phi^a((\varphi_i)_C)$ for $i = 1, 2, \psi = (\varphi_2)_C * (\varphi_1)_C, f_{\psi} = \Phi^a(\psi)$. Let $\varphi_C : (A_1, C_{\rho_1}) \longrightarrow (A_3, C_{\rho_3})$ be defined by $\varphi_C(a) = \varphi(a)$ for every $a \in A_1$. From the respective definitions we obtain that, for every $a \in A_1, \psi(a) = ((\varphi_2)_C \circ (\varphi_1)_C)$ $(a) = (\varphi_2 \circ \varphi_1)$ $(a) = \varphi(a)$. Thus, $\psi = \varphi_C$. Hence $f_{\psi} = \Phi^a(\varphi_C)$. We know that $\Phi^a(A_i, C_{\rho_i}) = \alpha X_i$, for i = 1, 2, 3, and $f_{i\alpha}$ is a continuous extension of f_i , for i = 1, 2. The equality " $\psi = \varphi_C$ " implies that f_{ψ} is a continuous extension of f_{φ} . From Theorem 2.3 we get that $f_{\psi} = f_{1\alpha} \circ f_{2\alpha}$. Since $f_{1\alpha}^{-1}(X_1) = X_2$ and $f_{2\alpha}^{-1}(X_2) = X_3$, we conclude that $f_{\varphi} = f_1 \circ f_2$.

We have proved that $\Xi^a : \mathbf{PAL} \longrightarrow \mathbf{PLC}$ is a contravariant functor.

III. $\Xi^a \circ \Xi^t$ is naturally isomorphic to the identity functor Id_{PLC} .

Recall that, for every $X \in |\mathbf{PLC}|$, the map $t_X : X \longrightarrow (\Xi^a \circ \Xi^t)(X)$, where $t_X(x) = \sigma_x$ for every $x \in X$, is a homeomorphism (see (12)). We will show that $t^l : Id_{\mathbf{PLC}} \longrightarrow \Xi^a \circ \Xi^t$, where for every $X \in |\mathbf{PLC}|, t^l(X) = t_X$, is a natural isomorphism.

Let $f \in \mathbf{PLC}(X, Y)$ and $f' = (\Xi^a \circ \Xi^t)(f), X' = (\Xi^a \circ \Xi^t)(X), Y' = (\Xi^a \circ \Xi^t)(Y)$. We have to prove that $t_Y \circ f = f' \circ t_X$. Let $\alpha(f) : \alpha X \longrightarrow \alpha Y$ and $\alpha(f') : \alpha X' \longrightarrow \alpha Y'$ be the continuous extensions of f and f', respectively (see 1.16). Then, by Theorem 2.3, we have that $t_{\alpha Y} \circ \alpha(f) = \alpha(f') \circ t_{\alpha X}$. Obviously, $t_{\alpha X}(\infty_X) = \{ cl_{\alpha X}(F) \mid F \in RC(X), \infty_X \in cl_{\alpha X}(F) \} = \{ cl_{\alpha X}(F) \mid F \in RC(X) \setminus CR(X) \} = \sigma_{\infty}^{(RC(\alpha X), \rho_{\alpha X})} = \infty_{X'}$, and, analogously, $t_{\alpha Y}(\infty_Y) = \infty_{Y'}$. Using 1.15 and taking the restrictions on X, we obtain that $t_Y \circ f = f' \circ t_X$, i.e. $Id_{\mathbf{PLC}} \cong \Xi^a \circ \Xi^t$.

IV. $\Xi^t \circ \Xi^a$ is naturally isomorphic to the identity functor $Id_{\mathbf{PAL}}$. Recall that for every $(A, \rho, \mathbb{B}) \in |\mathbf{PAL}|$, the function

$$\lambda_A^g: (A, \rho, \mathbb{B}) \longrightarrow (\Xi^t \circ \Xi^a)(A, \rho, \mathbb{B})$$

is an LCA-isomorphism (see (11)). We will show that $\lambda^g : Id_{\mathbf{PAL}} \longrightarrow \Xi^t \circ \Xi^a$, where for every $(A, \rho, \mathbb{B}) \in |\mathbf{PAL}|, \lambda^g(A, \rho, \mathbb{B}) = \lambda^g_A$, is a natural isomorphism.

Let $\varphi \in \mathbf{PAL}((A, \rho, \mathbb{B}), (B, \eta, \mathbb{B}'))$ and $\varphi' = (\Xi^t \circ \Xi^a)(\varphi), X = \Xi^a(A, \rho, \mathbb{B}), Y = \Xi^a(B, \eta, \mathbb{B}')$. We have to prove that $\lambda_B^g \circ \varphi = \varphi' \circ \lambda_A^g$. According to (15) and (16), it is enough to show that $\lambda_B^g \circ \varphi = \varphi' \circ \lambda_A^g$. Set $f = \Xi^a(\varphi)$. Hence $\varphi' = \Xi^t(f)$. Let $\varphi_C : (A, C_\rho) \longrightarrow (B, C_\eta)$ be defined by $\varphi_C(a) = \varphi(a)$ for every $a \in A$, and let $(\varphi')_C$ be defined analogously. Then φ_C and $(\varphi')_C$ are **DVAL**-morphisms. Set $f_\alpha = \Phi^a(\varphi_C)$ and $(\varphi_C)' = \Phi^t(f_\alpha)$. We know that $f_\alpha : \alpha Y \longrightarrow \alpha X$ is a continuous extension of f. By the proof of Theorem 2.3, $\lambda_B \circ \varphi_C = (\varphi_C)' \circ \lambda_A$ (see (4) for λ_A and λ_B). Note that $\lambda_A : (A, C_\rho) \longrightarrow (RC(\alpha X), \rho_{\alpha X})$ and $\lambda_B : (B, C_\eta) \longrightarrow (RC(\alpha Y), \rho_{\alpha Y})$. Let $(\varphi')_C : (RC(X), C_{\rho_X}) \longrightarrow (RC(Y), C_{\rho_Y})$ be defined by $(\varphi')_C(F) = \varphi'(F)$, for every $F \in RC(X)$. Then, by (18), $(\varphi')_C \circ r_X = r_Y \circ (\varphi_C)'$. By (10), $r_X \circ \lambda_A = \lambda_A^g$ and $r_Y \circ \lambda_B = \lambda_B^g$. The last three equalities imply that $\lambda_B^g \circ \varphi = \varphi' \circ \lambda_A^g$. Thus $Id_{\mathbf{PAL}} \cong \Xi^t \circ \Xi^a$.

Theorem 2.11 Let φ be a **PAL**-morphism. Then φ is an injection iff $\Xi^{a}(\varphi)$ is a surjection.

Proof. Let $\varphi \in \mathbf{PAL}((A, \rho, \mathbb{B}), (B, \eta, \mathbb{B}))$ and let $\varphi_C : (A, C_\rho) \longrightarrow (B, C_\eta)$ be defined by the formula $\varphi_C(a) = \varphi(a)$, for every $a \in A$. Then φ_C is a **DVAL**-morphism. Setting $f = \Xi^a(\varphi)$, we obtain that $\alpha(f) = \Phi^a(\varphi_C)$ (see the proof of Theorem 2.10). Obviously, $\alpha(f)$ is a surjection iff f is a surjection. By a theorem of de Vries ([6, Theorem 1.7.1]), $\Phi^a(\varphi_C)$ is a surjection iff φ_C is an injection. Hence, f is a surjection iff φ is an injection.

It is clear that if we want to build **PAL** as a category dually equivalent to the category **PLC** then the axiom (PAL5) is indispensable for describing the morphisms of the category **PAL**. With the next simple example we show that the axiom (PAL4) cannot be dropped as well.

Example 2.12 Let (A, ρ, \mathbb{B}) be a CLCA and $\mathbb{B} \neq A$. Then (A, ρ_s, A) is also a CLCA (by 1.2). Obviously, the map $i : (A, \rho, \mathbb{B}) \longrightarrow (A, \rho_s, A)$, where i(a) = a, for every $a \in A$, satisfies the axioms (PAL1)-(PAL3), (PAL5), (PAL6) but it does not satisfy the axiom (PAL4). If we suppose that our duality theorem is true without the presence of the axiom (PAL4)

in the definition of the category **PAL** then we will obtain, by Theorem 2.11, that there exists a continuous map from a compact Hausdorff space onto a locally compact non-compact Hausdorff space, a contradiction.

Fact 2.13 For every LCA (A, ρ, \mathbb{B}) , the triple (A, ρ_s, \mathbb{B}) is also an LCA (see 1.2 for ρ_s); if (A, ρ, \mathbb{B}) is a CLCA then the map $i : (A, \rho, \mathbb{B}) \longrightarrow (A, \rho_s, \mathbb{B})$, where i(a) = a, for every $a \in A$, is a **PAL**-morphism.

Proof. Since $a \ll_{\rho_s} a$, for every $a \in A$, the axiom (BC1) of 1.7 is clearly fulfilled. Obviously, for every $a, b \in A$, $a \ll_{\rho} b$ implies $a \ll_{\rho_s} b$. This implies that the axiom (BC3) is also satisfied. For checking (BC2), let $a, b \in A$ and $a\rho_s b$. Then $a \wedge b \neq 0$. Since $b = \bigvee \{c \mid c \in \mathbb{B}, c \ll_{\rho} b\}$, we have that $b = \bigvee \{c \mid c \in \mathbb{B}, c \wedge b^* = 0\}$. Hence $a \wedge b = \bigvee \{a \wedge c \mid c \in \mathbb{B}, c \wedge b^* = 0\}$. Thus, there exists $c \in \mathbb{B}$ such that $c \wedge b^* = 0$ and $a \wedge c \neq 0$. Therefore, there exists $c \in \mathbb{B}$ such that $a\rho_s(c \wedge b)$. So, (A, ρ_s, \mathbb{B}) is an LCA. The rest is clear.

Recall that a topological space X is said to be *extremally disconnected* if for every open set $U \subseteq X$, the closure $cl_X(U)$ is open in X. Clearly, a topological space X is extremally disconnected iff RC(X) consists only of clopen sets.

Proposition 2.14 Let (A, ρ, \mathbb{B}) be a CLCA and $X = \Xi^a(A, \rho, \mathbb{B})$. Then: (a)([15]) X is a compact Hausdorff space iff $\mathbb{B} = A$; (b) X is an extremally disconnected locally compact Hausdorff space iff $\rho = \rho_s$ (see 1.2 for ρ_s).

Proof. The assertion (a) is obvious.

(b) Recall that, by (11), $\lambda_A^g : (A, \rho, \mathbb{B}) \longrightarrow (RC(X), \rho_X)$ is an LCA-isomorphism.

Let X be extremally disconnected. Then, for every $a, b \in A$, $\lambda_A^g(a \wedge b) = \lambda_A^g(a) \wedge \lambda_A^g(b) = \operatorname{cl}(\operatorname{int}(\lambda_A^g(a) \cap \lambda_A^g(b))) = \lambda_A^g(a) \cap \lambda_A^g(b)$. Hence $a \wedge b \neq 0$ iff $a\rho b$. Thus $\rho = \rho_s$ (see 1.2).

Conversely, let $\rho = \rho_s$. Then, for every $a \in A$, $a \ll_{\rho} a$. Since for every $a, b \in A$, $a \ll_{\rho} b$ iff $\lambda_A^g(a) \subseteq \operatorname{int}_X(\lambda_A^g(b))$, we get that for every $a \in A$, $\lambda_A^g(a) \subseteq \operatorname{int}_X(\lambda_A^g(a))$, i.e. $\lambda_A^g(a)$ is a clopen set. Therefore, X is extremally disconnected.

Note that from 2.14(b), 2.13 and Theorem 2.11, we obtain immediately an easy proof of the following well-known fact: every locally compact Hausdorff space X is a perfect image of an extremally disconnected locally compact Hausdorff space Y.

Theorem 2.15 Let X and Y be two locally compact Hausdorff spaces, $\Xi^t(X) = (A, \rho, \mathbb{B})$ and $\Xi^t(Y) = (B, \eta, \mathbb{B}')$. Then a map $f : X \longrightarrow Y$ is a closed embedding iff the map $\varphi = \Xi^t(f)$ satisfies the following two conditions:

(1) $\forall a, b \in A \text{ with } a \ll_{C_{\rho}} b \text{ there exists } c \in B \text{ such that } a \ll_{C_{\rho}} \varphi(c) \ll_{C_{\rho}} b;$ (2) $\forall a, b \in B, \varphi(a) \ll_{C_{\rho}} \varphi(b) \text{ iff there exist } a_1, b_1 \in B \text{ such that } a_1 \ll_{C_{\eta}} b_1$ and $\varphi(a_1) = \varphi(a), \varphi(b_1) = \varphi(b).$

Proof. Obviously, $f: X \longrightarrow Y$ is a closed embedding iff the map $\alpha(f)$: $\alpha X \longrightarrow \alpha Y$ is an embedding (note that every closed embedding is a perfect map and see 1.16 for $\alpha(f)$). De Vries proved (see [6, Theorem 1.7.3]) that $\alpha(f)$ is an embedding iff the following two conditions are satisfied: (a) for every $F, G \in RC(\alpha X)$ with $F \ll_{\rho_{\alpha X}} G$, there exists $H \in$ $\Phi^t(\alpha(f))(RC(\alpha Y))$ such that $F \ll_{\rho_{\alpha X}} H \ll_{\rho_{\alpha X}} G$, and (b) for every $F, G \in$ $RC(\alpha Y), \Phi^t(\alpha(f))(F) \ll_{\rho_{\alpha X}} \Phi^t(\alpha(f))(G)$ iff there exist $F_1, G_1 \in RC(\alpha Y)$ such that $F_1 \ll_{\rho_{\alpha Y}} G_1$ and $\Phi^t(\alpha(f))(F_1) = \Phi^t(\alpha(f))(F), \Phi^t(\alpha(f))(G_1) =$ $\Phi^t(\alpha(f))(G)$. Now, using 2.9 and (18), it is easy to obtain that f is a closed embedding iff φ satisfies conditions (1) and (2).

Notations 2.16 Let us denote by PLCC the full subcategory of the category PLC whose objects are all connected locally compact Hausdorff spaces. Let PALC be the full subcategory of the category PAL whose objects are all connected CLCAs.

Theorem 2.17 The categories **PLCC** and **PALC** are dually equivalent.

Proof. It follows immediately from Theorem 2.10 and Fact 1.6.

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