# A Generalization of De Vries Duality Theorem* 

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#### Abstract

Generalizing Duality Theorem of H. de Vries, we define a category which is dually equivalent to the category of all locally compact Hausdorff spaces and all perfect maps between them.


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## Introduction

According to the famous Stone Duality Theorem ([17]), the category of all zero-dimensional compact Hausdorff spaces and all continuous maps between them is dually equivalent to the category Bool of all Boolean algebras and all Boolean homomorphisms between them. In 1962, H. de Vries [6] introduced the notion of compingent Boolean algebra and proved that the category of all compact Hausdorff spaces and all continuous maps between them is dually equivalent to the category of all complete compingent Boolean algebras and appropriate morphisms between them. In 1997, Roeper [15] defined the notion of region-based topology as one of the possible formalizations of the ideas of De Laguna [5] and Whitehead [19] for a region-based theory of space. Following [18, 9], the region-based topologies of Roeper appear here as local contact algebras (briefly, LCAs), because the axioms which they satisfy almost coincide with the axioms of local proximities of Leader [13]. In his paper [15], Roeper proved the following theorem: there is a bijective correspondence between all (up to homeomorphism) locally compact Hausdorff spaces and all (up to isomorphism) complete LCAs. It

[^0]generalizes the theorem of de Vries [6] that there exists a bijective correspondence between all (up to homeomorphism) compact Hausdorff spaces and all (up to isomorphism) complete compingent Boolean algebras. Here, using Roeper's Theorem and the results of de Vries [6], a category dually equivalent to the category of all locally compact Hausdorff spaces and all perfect maps between them is defined (see Theorem 2.10 bellow), generalizing in this way the Duality Theorem of H. de Vries.

Let us mention that, using de Vries Duality Theorem, V. V. Fedorchuk [11] showed that the category of all compact Hausdorff spaces and all quasiopen maps between them is dually equivalent to the category of all complete compingent Boolean algebras and all complete Boolean homomorphisms between them satisfying one simple condition, and that in $[7,8]$ some extensions of the Fedorchuk Duality Theorem ([11]) to some categories whose objects are all locally compact Hausdorff spaces are obtained.

We now fix the notations.
If $\mathcal{C}$ denotes a category, we write $X \in|\mathcal{C}|$ if $X$ is an object of $\mathcal{C}$, and $f \in \mathcal{C}(X, Y)$ if $f$ is a morphism of $\mathcal{C}$ with domain $X$ and codomain $Y$.

All lattices are with top (= unit) and bottom (= zero) elements, denoted respectively by 1 and 0 . We do not require the elements 0 and 1 to be distinct.

If $(X, \tau)$ is a topological space and $M$ is a subset of $X$, we denote by $\operatorname{cl}_{(X, \tau)}(M)$ (or simply by $\operatorname{cl}(M)$ or $\left.\mathrm{cl}_{X}(M)\right)$ the closure of $M$ in $(X, \tau)$
 $(X, \tau)$. The Alexandroff compactification of a locally compact Hausdorff non-compact space $X$ will be denoted by $\alpha X$ and the added point by $\infty_{X}$ (i.e. $\alpha X=X \cup\left\{\infty_{X}\right\}$ ).

The closed maps between topological spaces are assumed to be continuous but are not assumed to be onto. Recall that a map is perfect if it is closed and compact (i.e. point inverses are compact sets).

## 1 Preliminaries

Definition 1.1 An algebraic system $\underline{B}=\left(B, 0,1, \vee, \wedge,{ }^{*}, C\right)$ is called a contact algebra (abbreviated as CA) if $\left(B, 0,1, \vee, \wedge,{ }^{*}\right)$ is a Boolean algebra (where the operation "complement" is denoted by "*") and $C$ is a binary relation on $B$, satisfying the following axioms:
(C1) If $a \neq 0$ then $a C a$;
(C2) If $a C b$ then $a \neq 0$ and $b \neq 0$;
(C3) $a C b$ implies $b C a$;
(C4) $a C(b \vee c)$ iff $a C b$ or $a C c$.
Usually, we shall simply write $(B, C)$ for a contact algebra. The relation $C$ is called a contact relation. When $B$ is a complete Boolean algebra, we will
say that $(B, C)$ is a complete contact algebra (abbreviated as CCA).
We will say that two CA's $\left(B_{1}, C_{1}\right)$ and $\left(B_{2}, C_{2}\right)$ are CA-isomorphic iff there exists a Boolean isomorphism $\varphi: B_{1} \longrightarrow B_{2}$ such that, for each $a, b \in B_{1}, a C_{1} b$ iff $\varphi(a) C_{2} \varphi(b)$. Note that in this paper, by a "Boolean isomorphism" we understand an isomorphism in the category Bool.

A CA $(B, C)$ is called connected if it satisfies the following axiom: (CON) If $a \neq 0,1$ then $a C a^{*}$.

A contact algebra $(B, C)$ is called a normal contact algebra (abbreviated as NCA) ([6, 11]) if it satisfies the following axioms (we will write " $-C$ " for "not $C$ "):
(C5) If $a(-C) b$ then $a(-C) c$ and $b(-C) c^{*}$ for some $c \in B$;
(C6) If $a \neq 1$ then there exists $b \neq 0$ such that $b(-C) a$.
A normal CA is called a complete normal contact algebra (abbreviated as CNCA) if it is a CCA. The notion of normal contact algebra was introduced by Fedorchuk [11] under the name Boolean $\delta$-algebra as an equivalent expression of the notion of compingent Boolean algebra of de Vries. We call such algebras "normal contact algebras" because they form a subclass of the class of contact algebras.

Note that if $0 \neq 1$ then the axiom (C2) follows from the axioms (C6) and (C4).

For any CA $(B, C)$, we define a binary relation " $<_{C}$ " on $B$ (called non-tangential inclusion) by " $a<_{C} b \leftrightarrow a(-C) b^{*}$ ". Sometimes we will write simply " $\ll$ " instead of " $<_{C}$ ".

The relations $C$ and $\ll$ are inter-definable. For example, normal contact algebras could be equivalently defined (and exactly in this way they were defined (under the name of compingent Boolean algebras) by de Vries in [6]) as a pair of a Boolean algebra $B=\left(B, 0,1, \vee, \wedge,{ }^{*}\right)$ and a binary relation $\ll$ on $B$ subject to the following axioms:
$(\ll 1) a \ll b$ implies $a \leq b ;$
$(\ll 2) 0 \ll 0 ;$
$(\ll 3) a \leq b \ll c \leq t$ implies $a \ll t$;
$(\ll 4) a \ll c$ and $b \ll c$ implies $a \vee b \ll c$;
$(\ll 5)$ If $a \ll c$ then $a \ll b \ll c$ for some $b \in B$;
$(\ll 6)$ If $a \neq 0$ then there exists $b \neq 0$ such that $b \ll a$;
$(\ll 7) a \ll b$ implies $b^{*} \ll a^{*}$.
Note that if $0 \neq 1$ then the axiom $(\ll 2)$ follows from the axioms $(\ll 3)$, $(\ll 4)$, $(\ll 6)$ and $(\ll 7)$.

Obviously, contact algebras could be equivalently defined as a pair of a Boolean algebra $B$ and a binary relation $\ll$ on $B$ subject to the axioms $(\ll 1)-(\ll 4)$ and $(\ll 7)$.

It is easy to see that axiom (C5) (resp., (C6)) can be stated equivalently in the form of ( $\ll 5$ ) (resp., $(\ll 6)$ ).

Example 1.2 Let $B$ be a Boolean algebra. Then there exist the largest and the smallest contact relations on $B$; the largest one, $\rho_{l}$, is defined by $a \rho_{l} b$ iff $a \neq 0$ and $b \neq 0$, and the smallest one, $\rho_{s}$, by $a \rho_{s} b$ iff $a \wedge b \neq 0$.

Note that, for $a, b \in B, a \ll_{\rho_{s}} b$ iff $a \leq b$; hence $a \ll_{\rho_{s}} a$, for any $a \in B$. Thus ( $B, \rho_{s}$ ) is a normal contact algebra.

Example 1.3 Recall that a subset $F$ of a topological space $(X, \tau)$ is called regular closed if $F=\operatorname{cl}(\operatorname{int}(F))$. Clearly, $F$ is regular closed iff it is the closure of an open set.

For any topological space $(X, \tau)$, the collection $R C(X, \tau)$ (we will often write simply $R C(X)$ ) of all regular closed subsets of $(X, \tau)$ becomes a complete Boolean algebra ( $\left.R C(X, \tau), 0,1, \wedge, \vee,{ }^{*}\right)$ under the following operations:

$$
1=X, 0=\emptyset, F^{*}=\operatorname{cl}(X \backslash F), F \vee G=F \cup G, F \wedge G=\operatorname{cl}(\operatorname{int}(F \cap G))
$$

The infinite operations are given by the following formulas: $\bigvee\left\{F_{\gamma} \mid \gamma \in\right.$ $\Gamma\}=\operatorname{cl}\left(\bigcup\left\{F_{\gamma} \mid \gamma \in \Gamma\right\}\right)\left(=\operatorname{cl}\left(\bigcup\left\{\operatorname{int}\left(F_{\gamma}\right) \mid \gamma \in \Gamma\right\}\right)\right)$, and $\bigwedge\left\{F_{\gamma} \mid \gamma \in \Gamma\right\}=$ $\operatorname{cl}\left(\operatorname{int}\left(\bigcap\left\{F_{\gamma} \mid \gamma \in \Gamma\right\}\right)\right)$.

It is easy to see that setting $F \rho_{(X, \tau)} G$ iff $F \cap G \neq \emptyset$, we define a contact relation $\rho_{(X, \tau)}$ on $R C(X, \tau)$; it is called a standard contact relation. So, $\left(R C(X, \tau), \rho_{(X, \tau)}\right)$ is a CCA (it is called a standard contact algebra). We will often write simply $\rho_{X}$ instead of $\rho_{(X, \tau)}$. Note that, for $F, G \in R C(X)$, $F \ll_{\rho_{X}} G$ iff $F \subseteq \operatorname{int}_{X}(G)$.

Clearly, if $(X, \tau)$ is a normal Hausdorff space then the standard contact algebra $\left(R C(X, \tau), \rho_{(X, \tau)}\right)$ is a complete NCA.

A subset $U$ of $(X, \tau)$ such that $U=\operatorname{int}(\operatorname{cl}(U))$ is said to be regular open. The set of all regular open subsets of $(X, \tau)$ will be denoted by $R O(X, \tau)$ (or briefly, by $R O(X)$ ). Define Boolean operations and contact $\delta_{X}$ in $R O(X)$ as follows: $U \vee V=\operatorname{int}(\operatorname{cl}(U \cup V)), U \wedge V=U \cap V, U^{*}=\operatorname{int}(X \backslash U)$, $0=\emptyset, 1=X$ and $U \delta_{X} V$ iff $\operatorname{cl}(U) \cap \operatorname{cl}(V) \neq \emptyset$. Then $\left(R O(X), \delta_{X}\right)$ is a CA. This algebra is also complete, considering the infinite meet $\bigwedge\left\{U_{i} \mid i \in I\right\}=$ $\operatorname{int}\left(\bigcap_{i \in I} U_{i}\right)$.

Note that $\left(R O(X), \delta_{X}\right)$ and $\left(R C(X), \rho_{X}\right)$ are isomorphic CAs. The isomorphism $f$ between them is defined by $f(U)=\operatorname{cl}(U)$, for every $U \in$ $R O(X)$.

The following notion is a lattice-theoretical counterpart of the corresponding notion from the theory of proximity spaces (see [14]):
1.4 Let $(B, C)$ be a CA. Then a non-empty subset $\sigma$ of $B$ is called a cluster in $(B, C)$ if the following conditions are satisfied:
(K1) If $a, b \in \sigma$ then $a C b$;
(K2) If $a \vee b \in \sigma$ then $a \in \sigma$ or $b \in \sigma$;
(K3) If $a C b$ for every $b \in \sigma$, then $a \in \sigma$.
The set of all clusters in $(B, C)$ will be denoted denoted by $\operatorname{Clust}(B, C)$.
The next assertion can be proved exactly as Lemma 5.6 of [14]:
Fact 1.5 If $\sigma_{1}, \sigma_{2}$ are two clusters in a normal contact algebra $(B, C)$ and $\sigma_{1} \subseteq \sigma_{2}$ then $\sigma_{1}=\sigma_{2}$.

Fact $1.6([3])$ Let $(X, \tau)$ be a topological space. Then the standard contact algebra $\left(R C(X, \tau), \rho_{(X, \tau)}\right)$ is connected iff the space $(X, \tau)$ is connected.

The following notion is a lattice-theoretical counterpart of the Leader's notion of local proximity ([13]):

Definition 1.7 ([15]) An algebraic system $\underline{B}_{l}=\left(B, 0,1, \vee, \wedge,{ }^{*}, \rho, \mathbb{B}\right)$ is called a local contact algebra (abbreviated as LCA) if $\left(B, 0,1, \vee, \wedge,{ }^{*}\right)$ is a Boolean algebra, $\rho$ is a binary relation on $B$ such that $(B, \rho)$ is a CA, and $\mathbb{B}$ is an ideal (possibly non proper) of $B$, satisfying the following axioms:
(BC1) If $a \in \mathbb{B}, c \in B$ and $a \ll_{\rho} c$ then $a \ll_{\rho} b<_{\rho} c$ for some $b \in \mathbb{B}$ (see 1.1 for " $<_{\rho}$ ");
(BC2) If $a \rho b$ then there exists an element $c$ of $\mathbb{B}$ such that $a \rho(c \wedge b)$;
(BC3) If $a \neq 0$ then there exists $b \in \mathbb{B} \backslash\{0\}$ such that $b<_{\rho} a$.
Usually, we shall simply write $(B, \rho, \mathbb{B})$ for a local contact algebra. We will say that the elements of $\mathbb{B}$ are bounded and the elements of $B \backslash \mathbb{B}$ are unbounded. When $B$ is a complete Boolean algebra, the LCA $(B, \rho, \mathbb{B})$ is called a complete local contact algebra (abbreviated as CLCA).

We will say that two local contact algebras $(B, \rho, \mathbb{B})$ and $\left(B_{1}, \rho_{1}, \mathbb{B}_{1}\right)$ are $L C A$-isomorphic iff there exists a Boolean isomorphism $\varphi: B \longrightarrow B_{1}$ such that, for $a, b \in B, a \rho b$ iff $\varphi(a) \rho_{1} \varphi(b)$, and $\varphi(a) \in \mathbb{B}_{1}$ iff $a \in \mathbb{B}$.

An LCA $(B, \rho, \mathbb{B})$ is called connected if the $\mathrm{CA}(B, \rho)$ is connected.
Remark 1.8 Note that if $(B, \rho, \mathbb{B})$ is a local contact algebra and $1 \in \mathbb{B}$ then $(B, \rho)$ is a normal contact algebra. Conversely, any normal contact algebra $(B, C)$ can be regarded as a local contact algebra of the form ( $B, C, B$ ).

The following lemmas are lattice-theoretical counterparts of some theorems from Leader's paper [13].
Lemma 1.9 ([18]) Let $(B, \rho, \mathbb{B})$ be a local contact algebra. Define a binary relation " $C_{\rho}$ " on $B$ by

$$
\begin{equation*}
a C_{\rho} b \text { iff } a \rho b \text { or } a, b \notin \mathbb{B} . \tag{1}
\end{equation*}
$$

Then " $C_{\rho}$ ", called the Alexandroff extension of $\rho$, is a normal contact relation on $B$ and $\left(B, C_{\rho}\right)$ is a normal contact algebra.

Lemma $1.10([18])$ Let $\underline{B}_{l}=(B, \rho, \mathbb{B})$ be a local contact algebra and let $1 \notin \mathbb{B}$. Then $\sigma_{\infty}^{B_{l}}=\{b \in B \mid b \notin \mathbb{B}\}$ is a cluster in $\left(B, C_{\rho}\right)$ (see 1.9 for the notation " $C_{\rho}$ "). (Sometimes we will simply write $\sigma_{\infty}$ instead of $\sigma_{\infty}^{\underline{B}}$.)

Definition 1.11 Let $(B, \rho, \mathbb{B})$ be a local contact algebra. A cluster $\sigma$ in $\left(B, C_{\rho}\right)$ (see 1.9) is called bounded if $\sigma \cap \mathbb{B} \neq \emptyset$. The set of all bounded clusters in $\left(B, C_{\rho}\right)$ will be denoted by $\operatorname{BClust}(B, \rho, \mathbb{B})$.

Fact 1.12 Let $(B, \rho, \mathbb{B})$ be a local contact algebra and $\sigma$ be a bounded cluster in $\left(B, C_{\rho}\right)$ (see 1.9). Then there exists $b \in \mathbb{B}$ such that $b^{*} \notin \sigma$.

Proof. Let $b_{0} \in \sigma \cap \mathbb{B}$. Since $b_{0}<_{\rho} 1$, (BC1) implies that there exists $b \in \mathbb{B}$ such that $b_{0}<_{\rho} b$. Then $b_{0}(-\rho) b^{*}$ and since $b_{0} \in \mathbb{B}$, we obtain that $b_{0}\left(-C_{\rho}\right) b^{*}$. Thus $b^{*} \notin \sigma$.

Notation 1.13 Let $(X, \tau)$ be a topological space. We denote by $C R(X, \tau)$ the family of all compact regular closed subsets of $(X, \tau)$. We will often write $C R(X)$ instead of $C R(X, \tau)$.

If $x \in X$ then we set:

$$
\begin{equation*}
\sigma_{x}=\{F \in R C(X) \mid x \in F\} \tag{2}
\end{equation*}
$$

Fact 1.14 Let $(X, \tau)$ be a locally compact Hausdorff space. Then the triple

$$
\left(R C(X, \tau), \rho_{(X, \tau)}, C R(X, \tau)\right)
$$

(see 1.3 for $\rho_{(X, \tau)}$ ) is a complete local contact algebra ([15]). It is called a standard local contact algebra.

For every $x \in X, \sigma_{x}$ is a bounded cluster in $\left(R C(X), C_{\rho_{X}}\right)$ (see (2) and (1) for the notations).

We will need a lemma from [4]:
Lemma 1.15 Let $X$ be a dense subspace of a topological space $Y$. Then the functions $r_{X, Y}: R C(Y) \longrightarrow R C(X), F \longrightarrow F \cap X$, and $e_{X, Y}: R C(X) \longrightarrow$ $R C(Y), G \longrightarrow \mathrm{cl}_{Y}(G)$, are Boolean isomorphisms between Boolean algebras $R C(X)$ and $R C(Y)$, and $e_{X, Y} \circ r_{X, Y}=i d_{R C(Y)}, r_{X, Y} \circ e_{X, Y}=i d_{R C(X)} .(W e$ will often write $r_{X}, e_{X}$ instead of $r_{X, Y}, e_{X, Y}$, respectively.)

The next proposition is well known (see, e.g., [2]):
Proposition 1.16 Let $f: X \longrightarrow Y$ be a perfect map between two locally compact Hausdorff non-compact spaces. Then the map $f$ has a continuous extension $\alpha(f): \alpha X \longrightarrow \alpha Y$; moreover, $\alpha(f)\left(\infty_{X}\right)=\infty_{Y}$.

For all undefined here notions and notations see $[1,12,10,14,16]$.

## 2 The Results

The next theorem was proved by Roeper [15]. We will give a sketch of its proof; it follows the plan of the proof presented in [18]. The notations and the facts stated here will be used later on.

Theorem 2.1 (P. Roeper [15]) There exists a bijective correspondence between the class of all (up to isomorphism) complete local contact algebras and the class of all (up to homeomorphism) locally compact Hausdorff spaces.

Sketch of the Proof. (A) Let $(X, \tau)$ be a locally compact Hausdorff space. We put

$$
\begin{equation*}
\Psi^{t}(X, \tau)=\left(R C(X, \tau), \rho_{(X, \tau)}, C R(X, \tau)\right) \tag{3}
\end{equation*}
$$

(see 1.14 and 1.13 for the notations).
(B) Let $\underline{B}_{l}=(B, \rho, \mathbb{B})$ be a complete local contact algebra. Let $C=C_{\rho}$ be the Alexandroff extension of $\rho$ (see 1.9). Then, by $1.9,(B, C)$ is a complete normal contact algebra. Put $X=\operatorname{Clust}(B, C)$ and let $\mathcal{T}$ be the topology on $X$ having as a closed base the family $\left\{\lambda_{(B, C)}(a) \mid a \in B\right\}$ where, for every $a \in B$,

$$
\begin{equation*}
\lambda_{(B, C)}(a)=\{\sigma \in X \mid a \in \sigma\} . \tag{4}
\end{equation*}
$$

Sometimes we will write simply $\lambda_{B}$ instead of $\lambda_{(B, C)}$.
It can be proved that $(X, \mathcal{T})$ is a compact Hausdorff space and

$$
\begin{equation*}
\lambda_{B}:(B, C) \longrightarrow\left(R C(X), \rho_{X}\right) \text { is a CA-isomorphism. } \tag{5}
\end{equation*}
$$

(B1) Let $1 \in \mathbb{B}$. Then $C=\rho$ and $\mathbb{B}=B$, so that $(B, \rho, \mathbb{B})=(B, C, B)=$ $(B, C)$ is a complete normal contact algebra (see 1.8), and we put

$$
\begin{equation*}
\Psi^{a}(B, \rho, \mathbb{B})=\Psi^{a}(B, C, B)=\Psi^{a}(B, C)=(X, \mathcal{T}) \tag{6}
\end{equation*}
$$

(B2) Let $1 \notin \mathbb{B}$. Then, by Lemma 1.10, the set $\sigma_{\infty}=\{b \in B \mid b \notin \mathbb{B}\}$ is a cluster in $(B, C)$ and, hence, $\sigma_{\infty} \in X$. Let $L=X \backslash\left\{\sigma_{\infty}\right\}$. Then

$$
\begin{equation*}
L=\operatorname{BClust}(B, \rho, \mathbb{B}), \tag{7}
\end{equation*}
$$

i.e. $L$ is the set of all bounded clusters of $\left(B, C_{\rho}\right)$ (sometimes we will write $L_{\underline{B}_{l}}$ or $L_{B}$ instead of $\left.L\right)$; let the topology $\tau\left(=\tau_{\underline{B}_{l}}\right)$ on $L$ be the subspace topology, i.e. $\tau=\left.\mathcal{T}\right|_{L}$. Then $(L, \tau)$ is a locally compact Hausdorff space. We put

$$
\begin{equation*}
\Psi^{a}(B, \rho, \mathbb{B})=(L, \tau) \tag{8}
\end{equation*}
$$

Let $\lambda_{B}^{l}(a)=\lambda_{B}(a) \cap L$, for each $a \in B$. One can show that $X=\alpha L$ and

$$
\begin{equation*}
\lambda_{B}^{l}:(B, \rho, \mathbb{B}) \longrightarrow\left(R C(L), \rho_{L}, C R(L)\right) \text { is an LCA-isomorphism. } \tag{9}
\end{equation*}
$$

(C) For every CLCA $(B, \rho, \mathbb{B})$ and every $a \in B$, set

$$
\begin{equation*}
\lambda_{B}^{g}(a)=\lambda_{B}(a) \cap \Psi^{a}(B, \rho, \mathbb{B}) \tag{10}
\end{equation*}
$$

Then, by (5) and (9), we get that

$$
\begin{equation*}
\lambda_{B}^{g}:(B, \rho, \mathbb{B}) \longrightarrow\left(\Psi^{t} \circ \Psi^{a}\right)(B, \rho, \mathbb{B}) \text { is an LCA-isomorphism. } \tag{11}
\end{equation*}
$$

(D) Let $(Y, \tau)$ be a locally compact Hausdorff space. It can be shown that the map

$$
\begin{equation*}
t_{(Y, \tau)}:(Y, \tau) \longrightarrow \Psi^{a}\left(\Psi^{t}(Y, \tau)\right), \tag{12}
\end{equation*}
$$

defined by $t_{(Y, \tau)}(y)=\{F \in R C(Y, \tau) \mid y \in F\}\left(=\sigma_{y}\right)$, for every $y \in Y$, is a homeomorphism; we will often write simply $t_{Y}$ instead of $t_{(Y, \tau)}$.

Therefore $\Psi^{a}\left(\Psi^{t}(Y, \tau)\right)$ is homeomorphic to $(Y, \tau)$ and $\Psi^{t}\left(\Psi^{a}(B, \rho, \mathbb{B})\right)$ is LCA-isomorphic to $(B, \rho, \mathbb{B})$.

Definition 2.2 (De Vries [6]) Let HC be the category of all compact Hausdorff spaces and all continuous maps between them.

Let DVAL be the category whose objects are all complete NCAs and whose morphisms are all functions $\varphi:(A, C) \longrightarrow\left(B, C^{\prime}\right)$ between the objects of DVAL satisfying the conditions:
(DVAL1) $\varphi(0)=0$;
(DVAL2) $\varphi(a \wedge b)=\varphi(a) \wedge \varphi(b)$, for all $a, b \in A$;
(DVAL3) If $a, b \in A$ and $a<_{C} b$ then $\left(\varphi\left(a^{*}\right)\right)^{*}<_{C^{\prime}} \varphi(b)$;
(DVAL4) $\varphi(a)=\bigvee\left\{\varphi(b) \mid b<_{C} a\right\}$, for every $a \in A$,
and let the composition "*" of two morphisms $\varphi_{1}:\left(A_{1}, C_{1}\right) \longrightarrow\left(A_{2}, C_{2}\right)$ and $\varphi_{2}:\left(A_{2}, C_{2}\right) \longrightarrow\left(A_{3}, C_{3}\right)$ of DVAL be defined by the formula

$$
\begin{equation*}
\varphi_{2} * \varphi_{1}=\left(\varphi_{2} \circ \varphi_{1}\right)^{\swarrow}, \tag{13}
\end{equation*}
$$

where, for every function $\psi:(A, C) \longrightarrow\left(B, C^{\prime}\right)$ between two objects of DVAL, $\psi^{\checkmark}:(A, C) \longrightarrow\left(B, C^{\prime}\right)$ is defined as follows:

$$
\begin{equation*}
\psi^{\breve{ }}(a)=\bigvee\left\{\psi(b) \mid b<_{C} a\right\} \tag{14}
\end{equation*}
$$

for every $a \in A$.
De Vries [6] proved the following duality theorem:
Theorem 2.3 The categories HC and DVAL are dually equivalent. In more details, let $\Phi^{t}: \mathbf{H C} \longrightarrow \mathbf{D V A L}$ be the contravariant functor defined by $\Phi^{t}(X, \tau)=\left(R C(X, \tau), \rho_{X}\right)$, for every $X \in|\mathbf{H C}|$, and $\Phi^{t}(f)(G)=$ $\operatorname{cl}\left(f^{-1}(\operatorname{int}(G))\right)$, for every $f \in \mathbf{H C}(X, Y)$ and every $G \in R C(Y)$, and let $\Phi^{a}:$ DVAL $\longrightarrow \mathbf{H C}$ be the contravariant functor defined by $\Phi^{a}(A, C)=$
$\Psi^{a}(A, C)$, for every $(A, C) \in \mid$ DVAL $\mid$, and $\Phi^{a}(\varphi)\left(\sigma^{\prime}\right)=\left\{a \in A \mid\right.$ if $b<_{C} a^{*}$ then $\left.(\varphi(b))^{*} \in \sigma^{\prime}\right\}$, for every $\varphi \in \operatorname{DVAL}\left((A, C),\left(B, C^{\prime}\right)\right)$ and for every $\sigma^{\prime} \in \operatorname{Clust}\left(B, C^{\prime}\right)$; then $\lambda: I d_{\text {DVAL }} \longrightarrow \Phi^{t} \circ \Phi^{a}$, where $\lambda(A, C)=\lambda_{(A, C)}$ (see (4) and (5) for the notation $\left.\lambda_{(A, C)}\right)$, for every $(A, C) \in|\mathbf{D V A L}|$, and $t: I d_{\mathbf{H C}} \longrightarrow \Phi^{a} \circ \Phi^{t}$, where $t(X)=t_{X}$ (see (12) for the notation $t_{X}$ ), for every $X \in|\mathbf{H C}|$, are natural isomorphisms.

In [6], de Vries uses the regular open sets instead of regular closed sets, as we do, so that we present here the translations of his definitions for the case of regular closed sets.

Definition 2.4 We will denote by PLC the category of all locally compact Hausdorff spaces and all perfect maps between them.

Let PAL be the category whose objects are all complete LCAs and whose morphisms are all functions $\varphi:(A, \rho, \mathbb{B}) \longrightarrow\left(B, \eta, \mathbb{B}^{\prime}\right)$ between the objects of PAL satisfying the conditions:
(PAL1) $\varphi(0)=0$;
(PAL2) $\varphi(a \wedge b)=\varphi(a) \wedge \varphi(b)$, for all $a, b \in A$;
(PAL3) If $a \in \mathbb{B}, b \in A$ and $a<_{\rho} b$ then $\left(\varphi\left(a^{*}\right)\right)^{*}<_{\eta} \varphi(b)$;
(PAL4) For every $b \in \mathbb{B}^{\prime}$ there exists $a \in \mathbb{B}$ such that $b \leq \varphi(a)$;
(PAL5) If $a \in \mathbb{B}$ then $\varphi(a) \in \mathbb{B}^{\prime}$;
(PAL6) $\varphi(a)=\bigvee\left\{\varphi(b) \mid b<_{C_{\rho}} a\right\}$, for every $a \in A$ (see (1) for $C_{\rho}$ );
let the composition " $\diamond$ " of two morphisms $\varphi_{1}:\left(A_{1}, \rho_{1}, \mathbb{B}_{1}\right) \longrightarrow\left(A_{2}, \rho_{2}, \mathbb{B}_{2}\right)$ and $\varphi_{2}:\left(A_{2}, \rho_{2}, \mathbb{B}_{2}\right) \longrightarrow\left(A_{3}, \rho_{3}, \mathbb{B}_{3}\right)$ of PAL be defined by the formula

$$
\begin{equation*}
\varphi_{2} \diamond \varphi_{1}=\left(\varphi_{2} \circ \varphi_{1}\right)^{2} \tag{15}
\end{equation*}
$$

where, for every function $\psi:(A, \rho, \mathbb{B}) \longrightarrow\left(B, \eta, \mathbb{B}^{\prime}\right)$ between two objects of PAL, $\psi^{\prime}:(A, \rho, \mathbb{B}) \longrightarrow\left(B, \eta, \mathbb{B}^{\prime}\right)$ is defined as follows:

$$
\begin{equation*}
\psi^{\ulcorner }(a)=\bigvee\left\{\psi(b) \mid b<_{C_{\rho}} a\right\} \tag{16}
\end{equation*}
$$

for every $a \in A$.
By NAL we denote the full subcategory of PAL having as objects all CNCAs (i.e., those CLCAs $(A, \rho, \mathbb{B})$ for which $\mathbb{B}=A$ ).

Note that the categories DVAL and NAL are isomorphic (it can be even said that they are identical) because the axiom (PAL5) is trivially fulfilled in the category DVAL (indeed, all elements of its objects are bounded), the axiom (PAL4) follows immediately from the obvious fact that $\varphi(1)=1$ for every DVAL-morphism $\varphi$, and the compositions are the same.

We will generalize the Duality Theorem of de Vries showing that the categories PAL and PLC are dually equivalent.

We will first show that PAL is indeed a category.

Lemma 2.5 Let us regard two functions $\varphi:(A, \rho, \mathbb{B}) \longrightarrow\left(B, \eta, \mathbb{B}^{\prime}\right)$ and $\psi:\left(B, \eta, \mathbb{B}^{\prime}\right) \longrightarrow\left(B_{1}, \eta_{1}, \mathbb{B}_{1}^{\prime}\right)$ between CLCAs. Then:
(a) If $\varphi$ satisfies condition (PAL2) then $\varphi$ is an order preserving function;
(b) If $\varphi$ satisfies conditions (PAL1) and (PAL2) then $\varphi\left(a^{*}\right) \leq(\varphi(a))^{*}$, for every $a \in A$;
(c) Let $\varphi$ satisfy conditions (PAL3) and (PAL5). If $a, b \in A$ and $a \ll_{C_{\rho}} b$ then $\left(\varphi\left(a^{*}\right)\right)^{*}<_{C_{\eta}} \varphi(b)$. Hence, if $\varphi$ satisfies in addition conditions (PAL1) and (PAL2) then $\varphi(a)<_{C_{\eta}} \varphi(b)$;
(d) If $\varphi$ satisfies conditions (PAL1) and (PAL3) then $\varphi(1)=1$;
(e) If $\varphi$ satisfies condition (PAL2) then $\varphi^{*}$ satisfies conditions (PAL2) and (PAL6) (see (16) for $\varphi^{\vee}$ );
(f) If $\varphi$ satisfies condition (PAL6) then $\varphi=\varphi^{2}$;
(g) If $\varphi$ satisfies condition (PAL2) then $\left(\varphi^{\vee}\right)^{\nu}=\varphi^{2}$;
(h) If $\varphi$ and $\psi$ satisfy condition (PAL2) and $\varphi$ satisfies in addition conditions (PAL1), (PAL3) and (PAL5) then $(\psi \circ \varphi)^{2}=\left(\psi^{2} \circ \varphi^{\vee}\right)^{2}$.

Proof. The properties (a), (b), (d) and (f) are clearly fulfilled, and (g) follows from (e) and (f).
(c) Let $a, b \in A$ and $a<_{C_{\rho}} b$. Then $a<_{\rho} b$ and at least one of the elements $a$ and $b^{*}$ is bounded.

Let $a \in \mathbb{B}$. Then (PAL3) implies that $\left(\varphi\left(a^{*}\right)\right)^{*}<_{\eta} \varphi(b)$. By (BC1), there exists $c \in \mathbb{B}$ such that $a<_{\rho} c$. Hence, using again (PAL3), we get that $\left(\varphi\left(a^{*}\right)\right)^{*}<_{\eta} \varphi(c)$. Since $\varphi(c) \in \mathbb{B}^{\prime}$ (according to (PAL5)), we obtain that $\left(\varphi\left(a^{*}\right)\right)^{*} \in \mathbb{B}^{\prime}$. Therefore, $\left(\varphi\left(a^{*}\right)\right)^{*}<_{C_{\eta}} \varphi(b)$.

Let now $b^{*} \in \mathbb{B}$. Since $b^{*}<_{C_{\rho}} a^{*}$, we get, by the previous case, that $(\varphi(b))^{*}<_{C_{\eta}} \varphi\left(a^{*}\right)$. Thus $\left(\varphi\left(a^{*}\right)\right)^{*}<_{C_{\eta}} \varphi(b)$.
(e) By (a), for every $a \in A, \varphi^{\curlyvee}(a) \leq \varphi(a)$. Let $a \in A$. If $c<_{C_{\rho}} a$ then there exists $d_{c} \in A$ such that $c<_{C_{\rho}} d_{c}<_{C_{\rho}} a$; hence $\varphi(c) \leq \varphi^{2}\left(d_{c}\right)$. Now, $\varphi^{\circ}(a)=\bigvee\left\{\varphi(c) \mid c<_{C_{\rho}} a\right\} \leq \bigvee\left\{\varphi^{\circ}\left(d_{c}\right) \mid c<_{C_{\rho}} a\right\} \leq \bigvee\left\{\varphi^{\circ}(e) \mid e<_{C_{\rho}}\right.$ $a\} \leq \bigvee\left\{\varphi(e) \mid e<_{C_{\rho}} a\right\}=\varphi^{\curlyvee}(a)$. Thus, $\varphi^{\curlyvee}(a)=\bigvee\left\{\varphi^{\circ}(e) \mid e<_{C_{\rho}} a\right\}$. So, $\varphi^{\wedge}$ satisfies (PAL6). Further, let $a, b \in A$. Then $\varphi^{\wedge}(a) \wedge \varphi^{\wedge}(b)=$ $\bigvee\left\{\varphi(d) \wedge \varphi(e) \mid d<_{C_{\rho}} a, e<_{C_{\rho}} b\right\}=\bigvee\left\{\varphi(d \wedge e) \mid d<_{C_{\rho}} a, e<_{C_{\rho}}\right.$ $b\}=\bigvee\left\{\varphi(c) \mid c<_{C_{\rho}} a \wedge b\right\}=\varphi^{\wedge}(a \wedge b)$. So, (PAL2) is fulfilled.
(h) Since $\varphi^{\smile}(a) \leq \varphi(a)$ for every $a \in A$, and $\psi^{\sim}(b) \leq \psi(b)$ for every $b \in B$, we get that $\psi^{\nu}\left(\varphi^{\nu}(a)\right) \leq \psi(\varphi(a))$, for every $a \in A$. Hence, using (16), we obtain that $\left(\psi^{\sim} \circ \varphi^{\vee}\right)^{\check{ }}(a) \leq(\psi \circ \varphi)^{\wedge}(a)$, for every $a \in A$. Further, by (16),

 and $e<_{C_{\rho}} a$. Then there exist $b, d \in A$ such that $e<_{C_{\rho}} d<_{C_{\rho}} b \ll C_{\rho} a$. Set $c=\varphi(e)$. Then, by $(c), c<_{C_{\eta}} \varphi(d) \leq \varphi^{\curlyvee}(b)$. Hence $\psi(\varphi(e))=\psi(c) \leq$ $\left(\psi^{\sim} \circ \varphi^{\vee}\right)^{\wedge}(a)$. We conclude that $(\psi \circ \varphi)^{\wedge}(a) \leq\left(\psi^{\vee} \circ \varphi^{\vee}\right)^{\wedge}(a)$. Therefore the desired equality is proved.

Proposition 2.6 Let $\varphi_{i}:\left(A_{i}, \rho_{i}, \mathbb{B}_{i}\right) \longrightarrow\left(A_{i+1}, \rho_{i+1}, \mathbb{B}_{i+1}\right)$, where $i=$ 1,2 , be two functions between $C L C A s$ and let $\varphi_{1}$ and $\varphi_{2}$ satisfy conditions (PAL1)-(PAL5). Then the function $\varphi_{2} \circ \varphi_{1}$ satisfies conditions (PAL1)(PAL5).

Proof. Let $a \in \mathbb{B}_{1}, b \in A_{1}$ and $a<_{\rho_{1}} b$. Then, by ( BC 1 ), there exists $c \in \mathbb{B}_{1}$ such that $a<_{\rho_{1}} c<_{\rho_{1}} b$. From (PAL3) we get that $\left(\varphi_{1}\left(a^{*}\right)\right)^{*}<_{\rho_{2}}$ $\varphi_{1}(c)$. Then, since $\varphi_{1}(c) \in \mathbb{B}_{2}$ (by (PAL5)), $\left(\varphi_{1}\left(a^{*}\right)\right)^{*} \in \mathbb{B}_{2}$. Now, using twice (PAL3), we obtain that $\left(\varphi_{1}\left(a^{*}\right)\right)^{*} \ll \rho_{2} \varphi_{1}(b)$ and $\left(\varphi_{2}\left(\varphi_{1}\left(a^{*}\right)\right)\right)^{*}<_{\rho_{3}}$ $\varphi_{2}\left(\varphi_{1}(b)\right)$. Hence, the function $\varphi_{2} \circ \varphi_{1}$ satisfies condition (PAL3). The rest is obvious.

Proposition 2.7 Let $\varphi:(A, \rho, \mathbb{B}) \longrightarrow\left(B, \eta, \mathbb{B}^{\prime}\right)$ be a function between $C L C A s$ and let $\varphi$ satisfies conditions (PAL1)-(PAL5). Then the function $\varphi^{-}$ (see (16)) satisfies conditions (PAL1)-(PAL6) (i.e., it is a PAL-morphism).

Proof. Obviously, for every $a \in A, \varphi^{\imath}(a) \leq \varphi(a)$. Hence, $\varphi^{\imath}(0)=0$, i.e. (PAL1) is fulfilled. For (PAL2) and (PAL6) see 2.5(e). Let $a \in \mathbb{B}, b \in A$ and $a<_{\rho} b$. Then, by (BC1), there exist $c, d \in \mathbb{B}$ such that $a<_{\rho} c<_{\rho} d<_{\rho} b$. Thus $a \ll_{C_{\rho}} c<_{C_{\rho}} d<_{C_{\rho}} b$ and hence $c^{*}<_{C_{\rho}} a^{*}$. We obtain that $\varphi(d) \leq$ $\varphi^{\curlyvee}(b)$ and $\varphi\left(c^{*}\right) \leq \varphi^{\curlyvee}\left(a^{*}\right)$. Hence $\left(\varphi^{\curlyvee}\left(a^{*}\right)\right)^{*} \leq\left(\varphi\left(c^{*}\right)\right)^{*}<_{\eta} \varphi(d) \leq \varphi^{\curlyvee}(b)$. Therefore, $\left(\varphi^{\vee}\left(a^{*}\right)\right)^{*}<_{\eta} \varphi^{\circ}(b)$. So, (PAL3) is fulfilled. Finally, it is easy to verify (PAL4) and (PAL5).

## Proposition 2.8 PAL is a category.

Proof. This follows immediately from 2.5(f), 2.5(h), 2.6 and 2.7.
Proposition 2.9 Let $X$ be a locally compact Hausdorff space. Then the NCAs $\left(R C(X), C_{\rho_{X}}\right)$ and $\left(R C(\alpha X), \rho_{\alpha X}\right)$ are $C A$-isomorphic (see 1.9 and 1.14 for the notations) and the maps $e_{X, \alpha X}, r_{X, \alpha X}$ are CA-isomorphisms between them (see 1.15 for the notations).

Proof. By 1.15, we have only to show that $A C_{\rho_{X}} B$ iff $\mathrm{cl}_{\alpha X}(A) \rho_{\alpha X} \mathrm{cl}_{\alpha X}(B)$, for every $A, B \in R C(X)$. This follows easily from the respective definitions. Hence, the map $e_{X, \alpha X}:\left(R C(X), C_{\rho_{X}}\right) \longrightarrow\left(R C\left(\alpha X, \rho_{\alpha X}\right)\right.$ is a CA-isomorphism. Thus the map $r_{X, \alpha X}$ is also a CA-isomorphism.

Theorem 2.10 The categories PLC and PAL are dually equivalent.
Proof. We will define two contravariant functors

$$
\Xi^{a}: \mathbf{P A L} \longrightarrow \mathbf{P L C} \text { and } \Xi^{t}: \mathbf{P L C} \longrightarrow \mathbf{P A L}
$$

$I$. The definition of $\Xi^{t}$.
For every $(X, \tau) \in|\mathbf{P L C}|$, we let $\Xi^{t}(X, \tau)=\Psi^{t}(X, \tau)$ (see (3) for $\Psi^{t}$ ). Let $f:(X, \tau) \longrightarrow\left(Y, \tau^{\prime}\right) \in \mathbf{P L C}(X, Y)$. We set

$$
\begin{equation*}
\Xi^{t}(f): \Xi^{t}\left(Y, \tau^{\prime}\right) \longrightarrow \Xi^{t}(X, \tau), \quad \Xi^{t}(f)(F)=c l_{X}\left(f^{-1}\left(\operatorname{int}_{Y}(F)\right)\right) \tag{17}
\end{equation*}
$$

Put, for the sake of brevity, $\varphi_{f}=\Xi^{t}(f)$. We have to show that $\varphi_{f}$ is a PAL-morphism. Obviously, (PAL1) is fulfilled. For verifying (PAL4), let $H \in C R(X)$. Then $f(H)$ is compact. Since $Y$ is locally compact, there exists $F \in C R(Y)$ such that $f(H) \subseteq \operatorname{int}(F)$. Now we obtain that $H \subseteq f^{-1}(\operatorname{int}(F)) \subseteq \operatorname{int}\left(\operatorname{cl}\left(f^{-1}(\operatorname{int}(F))\right)\right)=\operatorname{int}\left(\varphi_{f}(F)\right)$, i.e. $H \ll \rho_{X} \varphi_{f}(F)$. Hence (PAL4) is checked.

Let now $F \in C R(Y)$. Then $\varphi_{f}(F)=\operatorname{cl}\left(f^{-1}(\operatorname{int}(F))\right) \subseteq f^{-1}(F)$. Since $f^{-1}(F)$ is compact (because $f$ is perfect), $\varphi_{f}(F) \in C R(X)$. Therefore, (PAL5) is fulfilled.

By 1.16, $f$ has a continuous extension $\alpha(f): \alpha X \longrightarrow \alpha Y$. Set $\varphi_{\alpha f}=$ $\Phi^{t}(\alpha(f))$ (see Theorem 2.3 for $\Phi^{t}$ ). Then, by Theorem 2.3, $\varphi_{\alpha f}$ is a DVALmorphism. We will prove that

$$
\begin{equation*}
r_{X, \alpha X} \circ \varphi_{\alpha f}=\varphi_{f} \circ r_{Y, \alpha Y} \tag{18}
\end{equation*}
$$

(see 1.15 for the notations), i.e. that, for every $G \in R C(Y)$, the following equality holds:

$$
\begin{equation*}
X \cap \varphi_{\alpha f}\left(\mathrm{cl}_{\alpha Y}(G)\right)=\varphi_{f}(G) \tag{19}
\end{equation*}
$$

or, in other words, that

$$
X \cap \operatorname{cl}_{\alpha X}\left((\alpha(f))^{-1}\left(\operatorname{int}_{\alpha Y}\left(\operatorname{cl}_{\alpha Y}(G)\right)\right)\right)=\operatorname{cl}_{X}\left(f^{-1}\left(\operatorname{int}_{Y}(G)\right)\right)
$$

Since the last equality follows easily from the obvious inclusions $\operatorname{int}_{Y}(G) \cup$ $\left\{\infty_{Y}\right\} \supseteq \operatorname{int}_{\alpha Y}\left(\mathrm{cl}_{\alpha Y}(G)\right) \supseteq \operatorname{int}_{Y}(G),(18)$ is proved. Therefore, $\varphi_{f}=r_{X, \alpha X} \circ$ $\varphi_{\alpha f} \circ e_{Y, \alpha Y}$ (see 1.15). Since $\varphi_{\alpha f}$ satisfies (DVAL2), we obtain that $\varphi_{f}$ satisfies (PAL2).

For establishing (PAL3), let $F \in C R(Y), G \in R C(Y)$ and $F \ll_{\rho_{Y}} G$. Then $F \ll_{\rho_{\rho_{Y}}} G$ and hence, by 2.9, $F \ll_{\rho_{\alpha Y}} \mathrm{cl}_{\alpha Y}(G)$. Thus, (DVAL3) implies that

$$
\begin{equation*}
\left(\varphi_{\alpha f}\left(F^{* \alpha}\right)\right)^{* \alpha} \ll_{\rho_{\alpha X}} \varphi_{\alpha f}\left(\mathrm{cl}_{\alpha Y}(G)\right) \tag{20}
\end{equation*}
$$

where "* ${ }^{\text {" }}$ is used as a common notation of the complement in the Boolean algebras $R C(\alpha X)$ and $R C(\alpha Y)$. Since, for every $H \in R C(X), X \cap$ $\left(\mathrm{cl}_{\alpha X}(H)\right)^{* \alpha}=r_{X, \alpha X}\left(\left(\mathrm{cl}_{\alpha X}(H)\right)^{* \alpha}\right)=\left(r_{X, \alpha X}\left(\mathrm{cl}_{\alpha X}(H)\right)^{*}=H^{*}\right.$, we get, using again 2.9, that $\left(X \cap \varphi_{\alpha f}\left(F^{* \alpha}\right)\right)^{*}<_{C_{\rho_{X}}}\left(X \cap \varphi_{\alpha f}\left(\mathrm{cl}_{\alpha Y}(G)\right)\right)$; then, applying twice (19), the equality $F^{* \alpha}\left(=\left(e_{Y, \alpha Y}(F)\right)^{* \alpha}\right)=e_{Y, \alpha Y}\left(F^{*}\right)$ and (18), we obtain that $\left(\varphi_{f}\left(F^{*}\right)\right)^{*}<_{C_{\rho_{X}}} \varphi_{f}(G)$, i.e. (PAL3) is fulfilled.

Now, we will verify (PAL6). Let $F \in R C(Y)$; then $\operatorname{cl}_{\alpha Y}(F) \in R C(\alpha Y)$ and hence, by (DVAL4),

$$
\varphi_{\alpha f}\left(\mathrm{cl}_{\alpha Y}(F)\right)=\bigvee\left\{\varphi_{\alpha f}\left(\operatorname{cl}_{\alpha Y}(G)\right) \mid G \in R C(Y), \mathrm{cl}_{\alpha Y}(G) \ll \rho_{\alpha Y} \operatorname{cl}_{\alpha Y}(F)\right\}
$$

Since $r_{X, \alpha X}$ is an isomorphism, we obtain that $r_{X, \alpha X}\left(\varphi_{\alpha f}\left(\mathrm{cl}_{\alpha Y}(F)\right)\right)=$ $\bigvee\left\{r_{X, \alpha X}\left(\varphi_{\alpha f}\left(\mathrm{cl}_{\alpha Y}(G)\right)\right) \mid G \in R C(Y), \operatorname{cl}_{\alpha Y}(G) \ll \rho_{\alpha_{X}} \operatorname{cl}_{\alpha Y}(F)\right\}$. Thus, (18) and 2.9 imply that $\varphi_{f}(F)=\bigvee\left\{\varphi_{f}(G) \mid G \in R C(Y), G<_{C_{\rho_{Y}}} F\right\}$. So, (PAL6) is fulfilled.

Therefore, $\varphi_{f}$ is a PAL-morphism.
Let $f \in \mathbf{P L C}(X, Y)$ and $g \in \mathbf{P L C}(Y, Z)$. We will prove that $\Xi^{t}(g \circ$ $f)=\Xi^{t}(f) \diamond \Xi^{t}(g)$. Put $h=g \circ f, \varphi_{h}=\Xi^{t}(h), \varphi_{f}=\Xi^{t}(f)$ and $\varphi_{g}=\Xi^{t}(g)$. Let $\alpha(f): \alpha X \longrightarrow \alpha Y, \alpha(g): \alpha Y \longrightarrow \alpha Z$ and $\alpha(h): \alpha X \longrightarrow \alpha Z$ be the continuous extensions of $f, g$ and $h$, respectively (see 1.16). Then, obviously, $\alpha(h)=\alpha(g) \circ \alpha(f)$. Set $\varphi_{\alpha f}=\Phi^{t}(\alpha(f)), \varphi_{\alpha g}=\Phi^{t}(\alpha(g))$ and $\varphi_{\alpha h}=\Phi^{t}(\alpha(h))$ Then, by Theorem 2.3, $\varphi_{\alpha h}=\left(\varphi_{\alpha f} \circ \varphi_{\alpha g}\right)^{\iota}$. Now, using (18) and 1.15, we get that $e_{X} \circ \varphi_{h} \circ r_{Z}=\varphi_{\alpha h}=\left(e_{X} \circ \varphi_{f} \circ \varphi_{g} \circ r_{Z}\right)^{\llcorner }$. Thus, for every $F \in R C(\alpha Z)$, we have that $\varphi_{h}\left(r_{Z}(F)\right)=\bigvee\left\{\left(\varphi_{f} \circ \varphi_{g}\right)\left(r_{Z}(G)\right) \mid G \ll \rho_{\alpha Z} F\right\}$. Now, 1.15 and 2.9 imply that $\varphi_{h}=\left(\varphi_{f} \circ \varphi_{g}\right)^{2}$, i.e. $\varphi_{h}=\varphi_{f} \diamond \varphi_{g}$.

So, $\Xi^{t}: \mathbf{P L C} \longrightarrow \mathbf{P A L}$ is a contravariant functor.
II. The definition of $\Xi^{a}$.

For every $(A, \rho, \mathbb{B}) \in|\mathbf{P A L}|$, we let $\Xi^{a}(A, \rho, \mathbb{B})=\Psi^{a}(A, \rho, \mathbb{B})$ (see (6) and (8) for $\Psi^{a}$ ).

Let $\varphi \in \mathbf{P A L}\left((A, \rho, \mathbb{B}),\left(B, \eta, \mathbb{B}^{\prime}\right)\right)$. We define the map

$$
\Xi^{a}(\varphi): \Xi^{a}\left(B, \eta, \mathbb{B}^{\prime}\right) \longrightarrow \Xi^{a}(A, \rho, \mathbb{B})
$$

by the formula

$$
\begin{equation*}
\Xi^{a}(\varphi)\left(\sigma^{\prime}\right)=\left\{a \in A \mid \text { if } b<_{C_{\rho}} a^{*} \text { then }(\varphi(b))^{*} \in \sigma^{\prime}\right\} \tag{21}
\end{equation*}
$$

for every bounded cluster $\sigma^{\prime}$ in $\left(B, C_{\eta}\right)$. Set, for the sake of brevity, $\Xi^{a}(\varphi)=$ $f_{\varphi}, X=\Xi^{a}(A, \rho, \mathbb{B})$ and $Y=\Xi^{a}\left(B, \eta, \mathbb{B}^{\prime}\right)$. We will show that $f_{\varphi}: Y \longrightarrow X$ is well-defined and is a perfect map.

Let $\varphi_{C}:\left(A, C_{\rho}\right) \longrightarrow\left(B, C_{\eta}\right)$ be defined by $\varphi_{C}(a)=\varphi(a)$, for every $a \in A$. Then $\varphi_{C}$ is a DVAL-morphism. Indeed, (DVAL3) follows from $2.5(\mathrm{c})$, and the other three axioms are obviously fulfilled. Set $f_{\alpha}=\Phi^{a}\left(\varphi_{C}\right)$. Then $f_{\alpha}: \alpha Y \longrightarrow \alpha X$ (see Theorem 2.3 and (B1), (B2) in the proof of Theorem 2.1). The definitions of $f_{\varphi}$ and $f_{\alpha}$ coincide on the bounded clusters of ( $B, C_{\eta}$ ) (see (21) and Theorem 2.3); hence, the right side of the formula (21) defines a cluster in $\left(A, C_{\rho}\right)$ and $f_{\alpha}$ is an extension of $f_{\varphi}$. Thus, if we show that $f_{\alpha}^{-1}\left(\infty_{X}\right)=\left\{\infty_{Y}\right\}$, the map $f_{\varphi}$ will be well-defined and will be a perfect map. Let us prove that $f_{\alpha}(Y) \subseteq X$, i.e. that if $\sigma^{\prime}$ is a bounded
cluster in $\left(B, C_{\eta}\right)$ then $\sigma=f_{\alpha}\left(\sigma^{\prime}\right)=f_{\varphi}\left(\sigma^{\prime}\right)$ is a bounded cluster in $\left(A, C_{\rho}\right)$. So, let $\sigma^{\prime}$ be a bounded cluster in $\left(B, C_{\eta}\right)$ and $\sigma=f_{\alpha}\left(\sigma^{\prime}\right)$. Then 1.12 implies that there exists $b \in \mathbb{B}^{\prime}$ such that $b^{*} \notin \sigma^{\prime}$. By (PAL4), there exists $a \in \mathbb{B}$ such that $b \leq \varphi(a)$. Thus $(\varphi(a))^{*} \leq b^{*}$ and hence $(\varphi(a))^{*} \notin \sigma^{\prime}$. By (BC1), there exists $a_{1} \in \mathbb{B}$ such that $a<_{\rho} a_{1}$. Then $a<_{C_{\rho}} a_{1}$ and, by the definition of $\sigma, a_{1}^{*} \notin \sigma$. Therefore $a_{1} \in \mathbb{B} \cap \sigma$, i.e. $\sigma$ is a bounded cluster in $\left(A, C_{\rho}\right)$. Hence $f_{\varphi}(Y)=f_{\alpha}(Y) \subseteq X$. Further, we have (by 1.10) that $\infty_{X}=A \backslash \mathbb{B}$ and $\infty_{Y}=B \backslash \mathbb{B}^{\prime}$. Let us show that $f_{\alpha}\left(\infty_{Y}\right)=\infty_{X}$. Set $\sigma^{\prime}=\infty_{Y}$ and $\sigma=f_{\alpha}\left(\sigma^{\prime}\right)$. Let $a \in \sigma$. Suppose that $a \in \mathbb{B}$. Then, by ( BC 1 ), there exist $a_{1}, a_{2} \in \mathbb{B}$ such that $a<_{\rho} a_{1}<_{\rho} a_{2}$. Thus $a<_{C_{\rho}} a_{1}<_{C_{\rho}} a_{2}$. Hence $a_{1}^{*}<_{C_{\rho}} a^{*}$. Since $a \in \sigma$, the definition of $\sigma$ implies that $\left(\varphi\left(a_{1}^{*}\right)\right)^{*} \in \sigma^{\prime}$. By 2.5(c), we have that $\left(\varphi\left(a_{1}^{*}\right)\right)^{*} \leq \varphi\left(a_{2}\right)$. Therefore, $\varphi\left(a_{2}\right) \in \sigma^{\prime}$. Since $\varphi\left(a_{2}\right) \in \mathbb{B}^{\prime}$ (by (PAL5)), we obtain a contradiction. Thus $\sigma \subseteq A \backslash \mathbb{B}$. Now, 1.10 and 1.5 imply that $\sigma=A \backslash \mathbb{B}$, i.e. $f_{\alpha}\left(\infty_{Y}\right)=\infty_{X}$. Hence $f_{\alpha}^{-1}(X)=Y$. This shows that $f_{\varphi}$ is a perfect map (because $f_{\alpha}$ is such). So, we have proved that $f_{\varphi} \in \mathbf{P L C}(Y, X)$.

Let $\varphi_{i} \in \mathbf{P A L}\left(\left(A_{i}, \rho_{i}, \mathbb{B}_{i}\right),\left(A_{i+1}, \rho_{i+1}, \mathbb{B}_{i+1}\right)\right)$ and $f_{i}=\Xi^{a}\left(\varphi_{i}\right)$ for $i=$ $1,2, \varphi=\varphi_{2} \diamond \varphi_{1}, f_{\varphi}=\Xi^{a}(\varphi)$ and $X_{i}=\Xi^{a}\left(A_{i}, \rho_{i}, \mathbb{B}_{i}\right)$ for $i=1,2,3$. We will prove that $f_{\varphi}=f_{1} \circ f_{2}$. Let $\left(\varphi_{i}\right)_{C}:\left(A_{i}, C_{\rho_{i}}\right) \longrightarrow\left(A_{i+1}, C_{\rho_{i+1}}\right)$ be defined by $\left(\varphi_{i}\right)_{C}(a)=\varphi_{i}(a)$ for every $a \in A_{i}$, where $i=1,2$. Then, as we know, $\left(\varphi_{i}\right)_{C}$ is a DVAL-morphism, for $i=1,2$. Set $f_{i \alpha}=\Phi^{a}\left(\left(\varphi_{i}\right)_{C}\right)$ for $i=1,2$, $\psi=\left(\varphi_{2}\right)_{C} *\left(\varphi_{1}\right)_{C}, f_{\psi}=\Phi^{a}(\psi)$. Let $\varphi_{C}:\left(A_{1}, C_{\rho_{1}}\right) \longrightarrow\left(A_{3}, C_{\rho_{3}}\right)$ be defined by $\varphi_{C}(a)=\varphi(a)$ for every $a \in A_{1}$. From the respective definitions we obtain that, for every $a \in A_{1}, \psi(a)=\left(\left(\varphi_{2}\right)_{C} \circ\left(\varphi_{1}\right)_{C}\right)^{\wedge}(a)=\left(\varphi_{2} \circ \varphi_{1}\right)^{\wedge}(a)=\varphi(a)$. Thus, $\psi=\varphi_{C}$. Hence $f_{\psi}=\Phi^{a}\left(\varphi_{C}\right)$. We know that $\Phi^{a}\left(A_{i}, C_{\rho_{i}}\right)=\alpha X_{i}$, for $i=1,2,3$, and $f_{i \alpha}$ is a continuous extension of $f_{i}$, for $i=1,2$. The equality " $\psi=\varphi_{C}$ " implies that $f_{\psi}$ is a continuous extension of $f_{\varphi}$. From Theorem 2.3 we get that $f_{\psi}=f_{1 \alpha} \circ f_{2 \alpha}$. Since $f_{1 \alpha}^{-1}\left(X_{1}\right)=X_{2}$ and $f_{2 \alpha}^{-1}\left(X_{2}\right)=X_{3}$, we conclude that $f_{\varphi}=f_{1} \circ f_{2}$.

We have proved that $\Xi^{a}: \mathbf{P A L} \longrightarrow \mathbf{P L C}$ is a contravariant functor.
III. $\Xi^{a} \circ \Xi^{t}$ is naturally isomorphic to the identity functor $I d_{\mathbf{P L C}}$.

Recall that, for every $X \in|\mathbf{P L C}|$, the map $t_{X}: X \longrightarrow\left(\Xi^{a} \circ \Xi^{t}\right)(X)$, where $t_{X}(x)=\sigma_{x}$ for every $x \in X$, is a homeomorphism (see (12)). We will show that $t^{l}: I d_{\mathbf{P L C}} \longrightarrow \Xi^{a} \circ \Xi^{t}$, where for every $X \in|\mathbf{P L C}|, t^{l}(X)=t_{X}$, is a natural isomorphism.

Let $f \in \mathbf{P L C}(X, Y)$ and $f^{\prime}=\left(\Xi^{a} \circ \Xi^{t}\right)(f), X^{\prime}=\left(\Xi^{a} \circ \Xi^{t}\right)(X), Y^{\prime}=$ $\left(\Xi^{a} \circ \Xi^{t}\right)(Y)$. We have to prove that $t_{Y} \circ f=f^{\prime} \circ t_{X}$. Let $\alpha(f): \alpha X \longrightarrow \alpha Y$ and $\alpha\left(f^{\prime}\right): \alpha X^{\prime} \longrightarrow \alpha Y^{\prime}$ be the continuous extensions of $f$ and $f^{\prime}$, respectively (see 1.16). Then, by Theorem 2.3, we have that $t_{\alpha Y} \circ \alpha(f)=\alpha\left(f^{\prime}\right) \circ$ $t_{\alpha X}$. Obviously, $t_{\alpha X}\left(\infty_{X}\right)=\left\{\mathrm{cl}_{\alpha X}(F) \mid F \in R C(X), \infty_{X} \in \mathrm{cl}_{\alpha X}(F)\right\}=$ $\left\{\operatorname{cl}_{\alpha X}(F) \mid F \in R C(X) \backslash C R(X)\right\}=\sigma_{\infty}^{\left(R C(\alpha X), \rho_{\alpha X}\right)}=\infty_{X^{\prime}}$, and, analogously, $t_{\alpha Y}\left(\infty_{Y}\right)=\infty_{Y^{\prime}}$. Using 1.15 and taking the restrictions on $X$, we
obtain that $t_{Y} \circ f=f^{\prime} \circ t_{X}$, i.e. $I d_{\mathbf{P L C}} \cong \Xi^{a} \circ \Xi^{t}$.
$I V . \Xi^{t} \circ \Xi^{a}$ is naturally isomorphic to the identity functor $I d_{\mathbf{P A L}}$. Recall that for every $(A, \rho, \mathbb{B}) \in|\mathbf{P A L}|$, the function

$$
\lambda_{A}^{g}:(A, \rho, \mathbb{B}) \longrightarrow\left(\Xi^{t} \circ \Xi^{a}\right)(A, \rho, \mathbb{B})
$$

is an LCA-isomorphism (see (11)). We will show that $\lambda^{g}: I d_{\text {PAL }} \longrightarrow$ $\Xi^{t} \circ \Xi^{a}$, where for every $(A, \rho, \mathbb{B}) \in|\mathbf{P A L}|, \lambda^{g}(A, \rho, \mathbb{B})=\lambda_{A}^{g}$, is a natural isomorphism.

Let $\varphi \in \operatorname{PAL}\left((A, \rho, \mathbb{B}),\left(B, \eta, \mathbb{B}^{\prime}\right)\right)$ and $\varphi^{\prime}=\left(\Xi^{t} \circ \Xi^{a}\right)(\varphi), X=$ $\Xi^{a}(A, \rho, \mathbb{B}), Y=\Xi^{a}\left(B, \eta, \mathbb{B}^{\prime}\right)$. We have to prove that $\lambda_{B}^{g} \diamond \varphi=\varphi^{\prime} \diamond \lambda_{A}^{g}$. According to (15) and (16), it is enough to show that $\lambda_{B}^{g} \circ \varphi=\varphi^{\prime} \circ \lambda_{A}^{g}$. Set $f=\Xi^{a}(\varphi)$. Hence $\varphi^{\prime}=\Xi^{t}(f)$. Let $\varphi_{C}:\left(A, C_{\rho}\right) \longrightarrow\left(B, C_{\eta}\right)$ be defined by $\varphi_{C}(a)=\varphi(a)$ for every $a \in A$, and let $\left(\varphi^{\prime}\right)_{C}$ be defined analogously. Then $\varphi_{C}$ and $\left(\varphi^{\prime}\right)_{C}$ are DVAL-morphisms. Set $f_{\alpha}=\Phi^{a}\left(\varphi_{C}\right)$ and $\left(\varphi_{C}\right)^{\prime}=\Phi^{t}\left(f_{\alpha}\right)$. We know that $f_{\alpha}: \alpha Y \longrightarrow \alpha X$ is a continuous extension of $f$. By the proof of Theorem 2.3, $\lambda_{B} \circ \varphi_{C}=\left(\varphi_{C}\right)^{\prime} \circ \lambda_{A}$ (see (4) for $\lambda_{A}$ and $\lambda_{B}$ ). Note that $\lambda_{A}:\left(A, C_{\rho}\right) \longrightarrow\left(R C(\alpha X), \rho_{\alpha X}\right)$ and $\lambda_{B}:\left(B, C_{\eta}\right) \longrightarrow\left(R C(\alpha Y), \rho_{\alpha Y}\right)$. Let $\left(\varphi^{\prime}\right)_{C}:\left(R C(X), C_{\rho_{X}}\right) \longrightarrow\left(R C(Y), C_{\rho_{Y}}\right)$ be defined by $\left(\varphi^{\prime}\right)_{C}(F)=\varphi^{\prime}(F)$, for every $F \in R C(X)$. Then, by (18), $\left(\varphi^{\prime}\right)_{C} \circ r_{X}=r_{Y} \circ\left(\varphi_{C}\right)^{\prime}$. By (10), $r_{X} \circ \lambda_{A}=\lambda_{A}^{g}$ and $r_{Y} \circ \lambda_{B}=\lambda_{B}^{g}$. The last three equalities imply that $\lambda_{B}^{g} \circ \varphi=\varphi^{\prime} \circ \lambda_{A}^{g}$. Thus $I d_{\mathbf{P A L}} \cong \Xi^{t} \circ \Xi^{a}$.

Theorem 2.11 Let $\varphi$ be a PAL-morphism. Then $\varphi$ is an injection iff $\Xi^{a}(\varphi)$ is a surjection.

Proof. Let $\varphi \in \mathbf{P A L}((A, \rho, \mathbb{B}),(B, \eta, \mathbb{B}))$ and let $\varphi_{C}:\left(A, C_{\rho}\right) \longrightarrow\left(B, C_{\eta}\right)$ be defined by the formula $\varphi_{C}(a)=\varphi(a)$, for every $a \in A$. Then $\varphi_{C}$ is a DVAL-morphism. Setting $f=\Xi^{a}(\varphi)$, we obtain that $\alpha(f)=\Phi^{a}\left(\varphi_{C}\right)$ (see the proof of Theorem 2.10). Obviously, $\alpha(f)$ is a surjection iff $f$ is a surjection. By a theorem of de Vries $\left(\left[6\right.\right.$, Theorem 1.7.1]), $\Phi^{a}\left(\varphi_{C}\right)$ is a surjection iff $\varphi_{C}$ is an injection. Hence, $f$ is a surjection iff $\varphi$ is an injection.

It is clear that if we want to build PAL as a category dually equivalent to the category PLC then the axiom (PAL5) is indispensable for describing the morphisms of the category PAL. With the next simple example we show that the axiom (PAL4) cannot be dropped as well.

Example 2.12 Let $(A, \rho, \mathbb{B})$ be a CLCA and $\mathbb{B} \neq A$. Then $\left(A, \rho_{s}, A\right)$ is also a CLCA (by 1.2). Obviously, the map $i:(A, \rho, \mathbb{B}) \longrightarrow\left(A, \rho_{s}, A\right)$, where $i(a)=a$, for every $a \in A$, satisfies the axioms (PAL1)-(PAL3), (PAL5), (PAL6) but it does not satisfy the axiom (PAL4). If we suppose that our duality theorem is true without the presence of the axiom (PAL4)
in the definition of the category PAL then we will obtain, by Theorem 2.11, that there exists a continuous map from a compact Hausdorff space onto a locally compact non-compact Hausdorff space, a contradiction.

Fact 2.13 For every $L C A(A, \rho, \mathbb{B})$, the triple $\left(A, \rho_{s}, \mathbb{B}\right)$ is also an $L C A$ (see 1.2 for $\left.\rho_{s}\right)$; if $(A, \rho, \mathbb{B})$ is a $C L C A$ then the map $i:(A, \rho, \mathbb{B}) \longrightarrow$ $\left(A, \rho_{s}, \mathbb{B}\right)$, where $i(a)=a$, for every $a \in A$, is a PAL-morphism.

Proof. Since $a<_{\rho_{s}} a$, for every $a \in A$, the axiom (BC1) of 1.7 is clearly fulfilled. Obviously, for every $a, b \in A, a<_{\rho} b$ implies $a<\rho_{\rho_{s}} b$. This implies that the axiom (BC3) is also satisfied. For checking (BC2), let $a, b \in A$ and $a \rho_{s} b$. Then $a \wedge b \neq 0$. Since $b=\bigvee\left\{c \mid c \in \mathbb{B}, c<_{\rho} b\right\}$, we have that $b=\bigvee\left\{c \mid c \in \mathbb{B}, c \wedge b^{*}=0\right\}$. Hence $a \wedge b=\bigvee\left\{a \wedge c \mid c \in \mathbb{B}, c \wedge b^{*}=0\right\}$. Thus, there exists $c \in \mathbb{B}$ such that $c \wedge b^{*}=0$ and $a \wedge c \neq 0$. Therefore, there exists $c \in \mathbb{B}$ such that $a \rho_{s}(c \wedge b)$. So, $\left(A, \rho_{s}, \mathbb{B}\right)$ is an LCA. The rest is clear.

Recall that a topological space $X$ is said to be extremally disconnected if for every open set $U \subseteq X$, the closure $\operatorname{cl}_{X}(U)$ is open in $X$. Clearly, a topological space $X$ is extremally disconnected iff $R C(X)$ consists only of clopen sets.

Proposition 2.14 Let $(A, \rho, \mathbb{B})$ be a $C L C A$ and $X=\Xi^{a}(A, \rho, \mathbb{B})$. Then: (a)([15]) $X$ is a compact Hausdorff space iff $\mathbb{B}=A$;
(b) $X$ is an extremally disconnected locally compact Hausdorff space iff $\rho=$ $\rho_{s}\left(\right.$ see 1.2 for $\left.\rho_{s}\right)$.

Proof. The assertion (a) is obvious.
(b) Recall that, by $(11), \lambda_{A}^{g}:(A, \rho, \mathbb{B}) \longrightarrow\left(R C(X), \rho_{X}\right)$ is an LCAisomorphism.

Let $X$ be extremally disconnected. Then, for every $a, b \in A, \lambda_{A}^{g}(a \wedge$ $b)=\lambda_{A}^{g}(a) \wedge \lambda_{A}^{g}(b)=\operatorname{cl}\left(\operatorname{int}\left(\lambda_{A}^{g}(a) \cap \lambda_{A}^{g}(b)\right)\right)=\lambda_{A}^{g}(a) \cap \lambda_{A}^{g}(b)$. Hence $a \wedge b \neq 0$ iff $a \rho b$. Thus $\rho=\rho_{s}$ (see 1.2).

Conversely, let $\rho=\rho_{s}$. Then, for every $a \in A, a<_{\rho} a$. Since for every $a, b \in A, a<_{\rho} b$ iff $\lambda_{A}^{g}(a) \subseteq \operatorname{int}_{X}\left(\lambda_{A}^{g}(b)\right)$, we get that for every $a \in A$, $\lambda_{A}^{g}(a) \subseteq \operatorname{int}_{X}\left(\lambda_{A}^{g}(a)\right)$, i.e. $\lambda_{A}^{g}(a)$ is a clopen set. Therefore, $X$ is extremally disconnected.

Note that from 2.14(b), 2.13 and Theorem 2.11, we obtain immediately an easy proof of the following well-known fact: every locally compact Hausdorff space $X$ is a perfect image of an extremally disconnected locally compact Hausdorff space $Y$.

Theorem 2.15 Let $X$ and $Y$ be two locally compact Hausdorff spaces, $\Xi^{t}(X)=(A, \rho, \mathbb{B})$ and $\Xi^{t}(Y)=\left(B, \eta, \mathbb{B}^{\prime}\right)$. Then a map $f: X \longrightarrow Y$
is a closed embedding iff the map $\varphi=\Xi^{t}(f)$ satisfies the following two conditions:
(1) $\forall a, b \in A$ with $a<_{C_{\rho}} b$ there exists $c \in B$ such that $a \ll_{C_{\rho}} \varphi(c)<_{C_{\rho}} b$;
(2) $\forall a, b \in B, \varphi(a)<_{C_{\rho}} \varphi(b)$ iff there exist $a_{1}, b_{1} \in B$ such that $a_{1}<_{C_{\eta}} b_{1}$ and $\varphi\left(a_{1}\right)=\varphi(a), \varphi\left(b_{1}\right)=\varphi(b)$.

Proof. Obviously, $f: X \longrightarrow Y$ is a closed embedding iff the map $\alpha(f)$ : $\alpha X \longrightarrow \alpha Y$ is an embedding (note that every closed embedding is a perfect map and see 1.16 for $\alpha(f)$ ). De Vries proved (see [6, Theorem 1.7.3]) that $\alpha(f)$ is an embedding iff the following two conditions are satisfied: (a) for every $F, G \in R C(\alpha X)$ with $F \ll_{\rho_{\alpha X}} G$, there exists $H \in$ $\Phi^{t}(\alpha(f))(R C(\alpha Y))$ such that $F \ll \rho_{\alpha X} H \ll \rho_{\alpha X} G$, and (b) for every $F, G \in$ $R C(\alpha Y), \Phi^{t}(\alpha(f))(F) \ll \rho_{\alpha X} \Phi^{t}(\alpha(f))(G)$ iff there exist $F_{1}, G_{1} \in R C(\alpha Y)$ such that $F_{1}<\rho_{\rho_{\alpha Y}} G_{1}$ and $\Phi^{t}(\alpha(f))\left(F_{1}\right)=\Phi^{t}(\alpha(f))(F), \Phi^{t}(\alpha(f))\left(G_{1}\right)=$ $\Phi^{t}(\alpha(f))(G)$. Now, using 2.9 and (18), it is easy to obtain that $f$ is a closed embedding iff $\varphi$ satisfies conditions (1) and (2).

Notations 2.16 Let us denote by PLCC the full subcategory of the category PLC whose objects are all connected locally compact Hausdorff spaces. Let PALC be the full subcategory of the category PAL whose objects are all connected CLCAs.

Theorem 2.17 The categories PLCC and PALC are dually equivalent.
Proof. It follows immediately from Theorem 2.10 and Fact 1.6.

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