

Lawvere completeness in Topology

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Abstract

It is known since 1973 that Lawvere's notion of (Cauchy-)complete enriched category is meaningful for metric spaces: it captures exactly Cauchy-complete metric spaces. In this paper we introduce the corresponding notion of Lawvere completeness for (\mathbb{T}, \mathbf{V}) -categories and show that it has an interesting meaning for topological spaces and quasi-uniform spaces: for the former ones means weak sobriety while for the latter means Cauchy completeness. Further, we show that \mathbf{V} has a canonical (\mathbb{T}, \mathbf{V}) -category structure which plays a key role: it is Lawvere-complete under reasonable conditions on the setting; permits us to define a Yoneda embedding in the realm of (\mathbb{T}, \mathbf{V}) -categories.

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0 Introduction

Lawvere in his 1973 paper *Metric spaces, generalized logic, and closed categories* formulates a notion of *complete* \mathbf{V} -category and shows that for (generalised) metric spaces it means Cauchy completeness. This notion of completeness deserved the attention of the categorical community, and the notion of *Cauchy-complete category*, or *Freyd-Karoubi complete category* is well-known, mostly in the context of \mathbf{Ab} -enriched categories. However, it never got the attention of the topological community. In this paper we interpret Lawvere's completeness in topological settings. We extend Lawvere's notion of complete \mathbf{V} -category to the (topological) setting of (\mathbb{T}, \mathbf{V}) -categories (for a symmetric and unital quantale \mathbf{V}), and show that it encompasses well-known notions in topological categories, meaning *weakly sober space* in the category of topological spaces and continuous maps, *weakly sober approach space* in the category of approach spaces and non-expansive maps, and *Cauchy-completeness* in the category of quasi-uniform spaces and uniformly continuous maps.

We present also a first step towards a possible construction of completion. Indeed, in the setting of \mathbf{V} -categories, it is well-known that the completion of a \mathbf{V} -category may be built out of the Yoneda embedding $X \rightarrow \mathbf{V}^{X^{\text{op}}}$. In the (\mathbb{T}, \mathbf{V}) -setting, we could prove that \mathbf{V} has a canonical (\mathbb{T}, \mathbf{V}) -categorical structure and that every (\mathbb{T}, \mathbf{V}) -category X has a canonical dual

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X^{op} . Using this structure and the free Eilenberg-Moore algebra structure $|X|$ on TX , we get two “Yoneda-like” morphisms

$$X \rightarrow \mathbf{V}^{X^{\text{op}}} \quad \text{and} \quad X \rightarrow \mathbf{V}^{|X|}.$$

For the latter one we prove a Yoneda Lemma (see 4.2).

Furthermore, we show that, under suitable conditions, \mathbf{V} is a Lawvere-complete (\mathbb{T}, \mathbf{V}) -category, a first step towards a completion construction which will be the subject of a forthcoming paper.

In order to make the presentation of this paper smoother, in Section 1 we recall the notions and properties of \mathbf{V} -categories we will generalize throughout. First we introduce \mathbf{V} -categories and \mathbf{V} -bimodules, and define Lawvere-complete \mathbf{V} -categories, for a commutative and unital quantale \mathbf{V} . \mathbf{V} is then naturally equipped with the \mathbf{V} -categorical structure hom . We give a direct proof of Lawvere completeness of the \mathbf{V} -category (\mathbf{V}, hom) .

In Section 2 we describe our basic setting for the study of (\mathbb{T}, \mathbf{V}) -categories and introduce them. We describe Kleisli composition in the category $\mathbf{V}\text{-Mat}$ of \mathbf{V} -valued matrices and define (\mathbb{T}, \mathbf{V}) -bimodule. Although (\mathbb{T}, \mathbf{V}) -bimodules do not compose in general, one can still formulate and study the notion of Lawvere-complete (\mathbb{T}, \mathbf{V}) -category.

Similarly to what was done in \mathbf{V} -categories, we define a canonical (\mathbb{T}, \mathbf{V}) -categorical structure on \mathbf{V} , as the composition of hom with the (canonical) \mathbb{T} -algebra structure on \mathbf{V} described by Manes in [21]. This is the subject of Section 3. In addition we also prove that, under some conditions, the (\mathbb{T}, \mathbf{V}) -category \mathbf{V} is Lawvere-complete.

In Section 1 we present the Yoneda embedding for \mathbf{V} -categories as a subproduct of the fact that a \mathbf{V} -matrix $\psi : X \multimap Y$ between \mathbf{V} -categories (X, a) and (Y, b) is a \mathbf{V} -bimodule if and only if, as a map $\psi : X^{\text{op}} \otimes Y \rightarrow \mathbf{V}$, is a \mathbf{V} -functor (Theorem 1.5); then the monoidal-closed structure of $\mathbf{V}\text{-Cat}$ gives us the *Yoneda Functor* $X \rightarrow \mathbf{V}^{X^{\text{op}}}$. In the (\mathbb{T}, \mathbf{V}) -setting this construction becomes more elaborated (see Theorem 3.3): a \mathbf{V} -matrix $\psi : TX \multimap Y$ is a (\mathbb{T}, \mathbf{V}) -bimodule $\psi : (X, a) \multimap (Y, b)$ if and only if both $\psi : |X| \otimes Y \rightarrow \mathbf{V}$ and $\psi : X^{\text{op}} \otimes Y \rightarrow \mathbf{V}$ are (\mathbb{T}, \mathbf{V}) -functors. Thus, given a (\mathbb{T}, \mathbf{V}) -category (X, a) , the (\mathbb{T}, \mathbf{V}) -bimodule $a : X \multimap X$ gives rise to two *Yoneda (\mathbb{T}, \mathbf{V}) -functors* $X \rightarrow \mathbf{V}^{X^{\text{op}}}$ and $X \rightarrow \mathbf{V}^{|X|}$.

In Section 5 we present the announced topological examples, with the exception of quasi-uniform spaces, which are presented in the Appendix, due to the fact that their presentation as lax algebras does not fit in the (\mathbb{T}, \mathbf{V}) -setting (as shown in [20]).

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1 The category of \mathbf{V} -categories

Although the material of this section can be found essentially on [17], we find that its inclusion here may enlighten the corresponding – but more technical – notions and results for (\mathbb{T}, \mathbf{V}) -

categories presented in the forthcoming sections.

1.1 V. Throughout \mathbf{V} is a (commutative and unital) *quantale*. In other words, \mathbf{V} is a complete lattice equipped with a symmetric and associative tensor product \otimes , with unit k , and with right adjoint hom ; that is, for each $u, v, w \in \mathbf{V}$,

$$u \otimes v \leq w \iff v \leq \text{hom}(u, w).$$

Considered as a (thin) category, \mathbf{V} is said to be *symmetric monoidal-closed*. If k is the bottom element \perp of \mathbf{V} , then $\mathbf{V} = 1$ is the trivial lattice. Throughout this paper we assume that \mathbf{V} is non-trivial, i.e. $k \neq \perp$.

Every non-trivial Heyting algebra – with $\otimes = \wedge$ and $k = \top$ the top element – is an example of such a lattice, in particular the two-element chain $2 = \{\text{false} \models \text{true}\}$, with the monoidal structure given by “&” (and) and “true”. The complete real half-line $\mathbf{P} = [0, \infty]$, with the categorical structure induced by the relation \geq (i.e., $a \rightarrow b$ means $a \geq b$), admits several interesting monoidal structures. First of all, with $\wedge = \max$ it is a Heyting algebra \mathbf{P}_{\max} . Another possible choice for \otimes is $+$; we denote \mathbf{P} equipped with this tensor by \mathbf{P}_+ . Note that in this example the right adjoint hom is given by truncated minus: $\text{hom}(u, v) = \max\{v - u, 0\}$.

1.2 V-Mat. The category $\mathbf{V}\text{-Mat}$ of *V-matrices* [3, 10] has sets as objects, and a morphism $r : X \multimap Y$ in $\mathbf{V}\text{-Mat}$ is a map $r : X \times Y \rightarrow \mathbf{V}$. Composition of \mathbf{V} -matrices $r : X \multimap Y$ and $s : Y \multimap Z$ is defined as matrix multiplication:

$$s \cdot r(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z).$$

The identity arrow $1_X : X \multimap X$ in $\mathbf{V}\text{-Mat}$ is the \mathbf{V} -matrix which sends all diagonal elements (x, x) to k and all other elements to the bottom element \perp of \mathbf{V} . In fact, each \mathbf{Set} -map $f : X \rightarrow Y$ can be interpreted as the \mathbf{V} -matrix

$$f : X \multimap Y, f(x, y) = \begin{cases} k & \text{if } f(x) = y, \\ \perp & \text{else.} \end{cases}$$

To keep notation simple, in the sequel we will write $f : X \rightarrow Y$ rather than $f : X \multimap Y$ for a \mathbf{V} -matrix induced by a map. The formula for matrix composition becomes considerably easier if one of the \mathbf{V} -matrices is a \mathbf{Set} -map:

$$s \cdot f(x, z) = s(f(x), z), \quad g \cdot r(x, z) = \bigvee_{y \in g^{-1}(z)} r(x, y)$$

for maps $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ and \mathbf{V} -matrices $r : X \multimap Y$ and $s : Y \multimap Z$.

The complete order on \mathbf{V} induces a complete order on $\mathbf{V}\text{-Mat}(X, Y) = \mathbf{V}^{X \times Y}$: for \mathbf{V} -matrices $r, r' : X \multimap Y$ we define

$$r \leq r' : \iff \forall x \in X \forall y \in Y r(x, y) \leq r'(x, y).$$

The *transpose* $r^\circ : Y \multimap X$ of a \mathbf{V} -matrix $r : X \multimap Y$ is defined by $r^\circ(y, x) = r(x, y)$. It is easy to see that $(\)^\circ : \mathbf{V}\text{-Mat}(X, Y) \rightarrow \mathbf{V}\text{-Mat}(Y, X)$ is order-preserving, and

$$1_X^\circ = 1_X, \quad (s \cdot r)^\circ = r^\circ \cdot s^\circ, \quad r^{\circ\circ} = r.$$

For each **Set**-map $f : X \rightarrow Y$ we have $1_X \leq f^\circ \cdot f$ and $f \cdot f^\circ \leq 1_Y$, i.e. f is left adjoint to f° and we write $f \dashv f^\circ$. In general, given \mathbf{V} -matrices $r : X \dashrightarrow Y$ and $s : Y \dashrightarrow X$, we say that r is left adjoint to s (and that s is right adjoint to r) if $1_X \leq s \cdot r$ and $1_Y \geq r \cdot s$.

Lemma. *Let \mathbf{V} be a quantale and $r, r' : X \dashrightarrow Y$ and $s, s' : Y \dashrightarrow X$ be \mathbf{V} -matrices such that $r \dashv s$ and $r' \dashv s'$. Then $r \leq r'$ if and only if $s' \leq s$. Consequently, if $r \leq r'$ and $s \leq s'$, then $r = r'$ and $s = s'$.*

As another consequence of the lemma above we have that left and right adjoints are uniquely determined by s (respectively r). Therefore we say that r is left adjoint if it has a right adjoint s , and likewise, s is right adjoint if it has a left adjoint r . In pointwise notation, we have $r \dashv s$ if and only if

$$\begin{aligned} \forall x \in X \quad \bigvee_{y \in Y} r(x, y) \otimes s(y, x) &\geq k, \\ \forall x \in X \quad \forall y, y' \in Y \quad s(y, x) \otimes r(x, y') &\leq \begin{cases} \perp & \text{if } y \neq y', \\ k & \text{if } y = y' \end{cases} \end{aligned}$$

which, by symmetry of \otimes , is equivalent to

$$\begin{aligned} \forall x \in X \quad \bigvee_{y \in Y} r(x, y) \otimes s(y, x) &= k, \\ \forall x \in X \quad \forall y, y' \in Y \quad (y \neq y' \Rightarrow s(y, x) \otimes r(x, y') &= \perp). \end{aligned}$$

Our next example shows that there exist indeed left adjoint \mathbf{V} -matrices which are not induced by **Set**-maps.

Example. Consider a set X and the Boolean algebra $\mathbf{V} = PX$ the powerset of X . Define a \mathbf{V} -matrix $r : 1 \dashrightarrow X$ by putting $r(\star, x) = \{x\}$ for $x \in X$. Then

$$r^\circ \cdot r(\star, \star) = \bigcup_{x \in X} \{x\} = X \quad \text{and} \quad r \cdot r^\circ(x, y) = \{x\} \cap \{y\} = \begin{cases} \emptyset & \text{if } x \neq y, \\ \{x\} & \text{if } x = y, \end{cases}$$

hence $r \dashv r^\circ$. But r is not a **Set**-map unless X has at most one element.

We wish to characterise those quantales \mathbf{V} where the class of left adjoint \mathbf{V} -matrices coincides with the class of **Set**-maps. In order to do so we introduce some notation. Let $u, v \in \mathbf{V}$. We say that v is a \otimes -complement of u if

$$u \vee v = k \quad \text{and} \quad u \otimes v = \perp.$$

Clearly, each $u \in \mathbf{V}$ has at most one \otimes -complement. Moreover, if u is \otimes -complemented (i.e. has a \otimes -complement v), then

$$u = u \otimes k = u \otimes (u \vee v) = (u \otimes u) \vee (u \otimes v) = u \otimes u,$$

that is, u is idempotent. Our next result generalises [14, 2.14].

Proposition. *Let \mathbf{V} be a quantale. Each left adjoint \mathbf{V} -matrix is a **Set**-map if and only if k and \perp are the only \otimes -complemented elements of \mathbf{V} and*

$$\forall u, v \in \mathbf{V} \ (u \otimes v = k \Rightarrow u = k = v).$$

Proof. Assume first that each left adjoint \mathbf{V} -matrix is a **Set**-map. Let $u, v \in \mathbf{V}$. If $u \otimes v = k$, then $u \dashv v$, and we have $u = v = k$. Suppose that $u \vee v = k$ and $u \otimes v = \perp$. Let $X = \{u, v\}$ and define $r : 1 \dashrightarrow X$ with $r(\star, u) = u$ and $r(\star, v) = v$. Then $r \dashv r^\circ$ and, by assumption, $u = k$ or $v = k$.

Let $r : X \dashrightarrow Y$ and $s : Y \dashrightarrow X$ be \mathbf{V} -matrices such that $r \dashv s$. Let $x \in X$. There is some $y \in Y$ such that $r(x, y) \otimes s(y, x) > \perp^1$. Then

$$k = (r(x, y) \otimes s(y, x)) \vee \bigvee_{y' \neq y} (r(x, y') \otimes s(y', x))$$

and

$$r(x, y) \otimes s(y, x) \otimes \bigvee_{y' \neq y} r(x, y') \otimes s(y', x) = \bigvee_{y' \neq y} r(x, y) \otimes s(y, x) \otimes r(x, y') \otimes s(y', x) = \perp.$$

Hence, by assumption, $r(x, y) = k = s(y, x)$ and $r(x, y') \otimes s(y', x) = \perp$ for all $y' \neq y$. We have shown that, for each $x \in X$, there exists exactly one $y \in Y$ with $r(x, y) = k = s(y, x)$. Consider now $f : X \rightarrow Y$ which assigns to x this unique y . Clearly, $f \leq r$, but also $f^\circ \leq s$ since

$$f^\circ(y, x) = k \Rightarrow f(x) = y \Rightarrow s(y, x) = k.$$

The assertion follows now from the previous lemma. \square

1.3 \mathbf{V} -categories. \mathbf{V} -enriched categories were introduced and studied in [11, 17] in the more general context of symmetric monoidal-closed categories. For a very nice presentation of this material we refer to [18]. In the next subsections we recall some well-known facts about \mathbf{V} -categories, which will serve as a guideline for our study of (\mathbb{T}, \mathbf{V}) -categories.

A \mathbf{V} -enriched category (or simply \mathbf{V} -category) is a pair (X, a) with X a set and $a : X \dashrightarrow X$ a \mathbf{V} -matrix such that

$$1_X \leq a \cdot a \quad \text{and} \quad a \cdot a \leq a;$$

equivalently, the map $a : X \times X \rightarrow \mathbf{V}$ satisfies the following conditions:

(R) for each $x \in X$, $k \leq a(x, x)$;

(T) for each $x, x', x'' \in X$, $a(x, x') \otimes a(x', x'') \leq a(x, x'')$.

Given \mathbf{V} -categories (X, a) and (Y, b) , a \mathbf{V} -functor $f : (X, a) \rightarrow (Y, b)$ is a map $f : X \rightarrow Y$ such that, for each $x, x' \in X$, $a(x, x') \leq b(f(x), f(x'))$. \mathbf{V} -categories and \mathbf{V} -functors are the objects and morphisms of the category $\mathbf{V}\text{-Cat}$. Finally, given a \mathbf{V} -category $X = (X, a)$, the *dual category* X^{op} of X is defined by $X^{\text{op}} = (X, a^\circ)$.

We remark that $\mathbf{V}\text{-Cat}$ is actually a *closed category* since the tensor product on \mathbf{V} can be naturally transported to $\mathbf{V}\text{-Cat}$. More precisely, for \mathbf{V} -categories $X = (X, a)$ and $Y = (Y, b)$, we

¹Since $\perp < k$. The assertion of the proposition is trivially true if $k = \perp$.

put $X \otimes Y = (X \times Y, a \otimes b)$ where $a \otimes b((x, y), (x', y')) = a(x, x') \otimes b(y, y')$ for all $x, x' \in X$ and $y, y' \in Y$. Then, for each \mathbf{V} -category $X = (X, a)$, the functor $X \otimes - : \mathbf{V}\text{-Cat} \rightarrow \mathbf{V}\text{-Cat}$ has a right adjoint $-^X$ defined by $Y^X = (\mathbf{V}\text{-Cat}(X, Y), d)$ with $d(f, g) = \bigwedge_{x \in X} a(f(x), g(x))$.

Being monoidal-closed, \mathbf{V} has a natural structure as \mathbf{V} -category:

$$\text{hom} : \mathbf{V} \rightarrow \mathbf{V}.$$

Indeed, for $u, v, w \in \mathbf{V}$,

$$k \otimes v = v \Rightarrow k \leq \text{hom}(v, v),$$

$$u \otimes (\text{hom}(u, v) \otimes \text{hom}(v, w)) \leq v \otimes \text{hom}(v, w) \leq w \Rightarrow \text{hom}(u, v) \otimes \text{hom}(v, w) \leq \text{hom}(u, w),$$

that is, $1_{\mathbf{V}} \leq \text{hom}$ and $\text{hom} \cdot \text{hom} \leq \text{hom}$.

For $\mathbf{V} = 2$, with the usual notation $x \leq x' : \iff a(x, x') = \text{true}$, axioms (R) and (T) read as

$$\forall x \in X \text{ true} \models x \leq x \quad \text{and} \quad \forall x, x', x'' \in X \ x \leq x' \ \& \ x' \leq x'' \models x \leq x'',$$

that is, (X, \leq) is an ordered set². A 2-functor is a map $f : (X, \leq) \rightarrow (Y, \leq)$ between ordered sets such that

$$\forall x, x' \in X \ x \leq x' \models f(x) \leq f(x');$$

that is, f is a monotone map. Hence 2-Cat is equivalent to the category \mathbf{Ord} of ordered sets and monotone maps.

A \mathbf{P}_+ -category is a set X endowed with a map $a : X \times X \rightarrow \mathbf{P}_+$ such that

$$\forall x \in X \ 0 \geq a(x, x) \quad \text{and} \quad \forall x, x', x'' \in X \ a(x, x') + a(x', x'') \geq a(x, x'');$$

that is, $a : X \times X \rightarrow \mathbf{P}_+$ is a (generalised) metric on X . A \mathbf{P}_+ -functor is a map $f : (X, a) \rightarrow (Y, b)$ between metric spaces satisfying the following inequality:

$$\forall x, x' \in X \ a(x, x') \geq b(f(x), f(x')),$$

which means precisely that f is a non-expansive map. Therefore the category $\mathbf{P}_+\text{-Cat}$ coincides with the category \mathbf{Met} of metric spaces and non-expansive maps. (For more details, see [18, 10].)

For $\mathbf{V} = \mathbf{P}_{\max}$, the transitivity axiom (T) reads as

$$\max\{a(x, x'), a(x', x'')\} \geq a(x, x''),$$

hence the category $\mathbf{P}_{\max}\text{-Cat}$ coincides with the category \mathbf{UMet} of (*generalised*) *ultrametric spaces and non-expansive maps*.

1.4 \mathbf{V} -bimodules. Given \mathbf{V} -categories (X, a) and (Y, b) , a *bimodule*³ $\psi : (X, a) \multimap (Y, b)$ is a \mathbf{V} -matrix $\psi : X \rightarrow Y$ such that $\psi \cdot a \leq \psi$ and $b \cdot \psi \leq \psi$; that is, for each $x, x' \in X$ and $y, y' \in Y$,

$$a(x, x') \otimes \psi(x', y) \leq \psi(x, y) \quad \text{and} \quad \psi(x, y') \otimes b(y', y) \leq \psi(x, y).$$

It is easy to verify that bimodules compose and that \mathbf{V} -categorical structures are themselves bimodules. In fact, they are the identities for the composition of bimodules, that is, for any bimodule $\psi : (X, a) \multimap (Y, b)$, $\psi \cdot a = \psi$ and $b \cdot \psi = \psi$. Therefore, \mathbf{V} -categories and \mathbf{V} -bimodules constitute a category, which we will denote by $\mathbf{V}\text{-Mod}$. The category $\mathbf{V}\text{-Mod}$ inherits the bicategorical structure of $\mathbf{V}\text{-Mat}$ via the forgetful functor $\mathbf{V}\text{-Mod} \rightarrow \mathbf{V}\text{-Mat}$.

²Note that we do *not* require \leq to be anti-symmetric.

³Also known as *profunctor* or *distributor* (see [2, 5, 26]).

1.5 V-functors as V-bimodules. Any V-functor $f : (X, a) \rightarrow (Y, b)$ defines a pair of matrices $f_* : (X, a) \multimap (Y, b)$ and $f^* : (Y, b) \multimap (X, a)$, with $f_* = b \cdot f$ and $f^* = f^\circ \cdot b$, that is $f_*(x, y) = b(f(x), y)$ and $f^*(y, x) = b(y, f(x))$, which are in fact bimodules: for every $x, x' \in X$ and $y, y' \in Y$,

$$\begin{aligned} a(x, x') \otimes f_*(x', y) &= a(x, x') \otimes b(f(x'), y) \leq b(f(x), f(x')) \otimes b(f(x'), y) \leq b(f(x), y), \\ f_*(x, y') \otimes b(y', y) &= b(f(x), y') \otimes b(y', y) \leq b(f(x), y), \end{aligned}$$

and similarly for f^* .

Moreover, the bimodules f_* and f^* form an adjunction, as we show next. We recall first that, given bimodules $\varphi : (X, a) \multimap (Y, b)$ and $\psi : (Y, b) \multimap (X, a)$, φ is left adjoint to ψ , $\varphi \dashv \psi$, if $1_{(X, a)} \leq \psi \cdot \varphi$ and $\varphi \cdot \psi \leq 1_{(Y, b)}$, i.e. $a \leq \psi \cdot \varphi$ and $\varphi \cdot \psi \leq b$. It is now straightforward to check that $f_* \dashv f^*$, since, for $x, x' \in X$ and $y, y' \in Y$, the inequality

$$a(x, x') \leq \bigvee_{y \in Y} f_*(x, y) \otimes f^*(y, x') = \bigvee_{y \in Y} b(f(x), y) \otimes b(y, f(x')) = b(f(x), f(x'))$$

follows from V-functoriality of f , while

$$\bigvee_{x \in X} f^*(y, x) \otimes f_*(x, y') = \bigvee_{x \in X} b(y, f(x)) \otimes b(f(x), y') \leq b(y, y')$$

follows from the associativity axiom for V-categories. A quite different connection between functors and bimodules offers the following

Theorem. For V-categories $X = (X, a)$ and $Y = (Y, b)$ and a V-matrix $\psi : X \multimap Y$, the following conditions are equivalent:

- (i) $\psi : X \multimap Y$ is a bimodule;
- (ii) $\psi : X^{\text{op}} \otimes Y \rightarrow V$ is a V-functor.

Proof. (i) \Rightarrow (ii): For $x, x' \in X$ and $y, y' \in Y$,

$$\begin{aligned} \psi(x, y) \otimes a^\circ(x, x') \otimes b(y, y') &= a(x', x) \otimes \psi(x, y) \otimes b(y, y') \\ &\leq \psi(x', y) \otimes b(y, y') \\ &\leq \psi(x', y'), \end{aligned}$$

hence

$$a^\circ(x, x') \otimes b(y, y') \leq \text{hom}(\psi(x, y), \psi(x', y')).$$

(ii) \Rightarrow (i): For $x, x' \in X$ and $y, y' \in Y$,

$$\begin{aligned} a(x, x') \otimes \psi(x', y) &\leq \psi(x', y) \otimes a^\circ(x', x) \otimes b(y, y) \\ &\leq \psi(x, y), \end{aligned}$$

that is $a \cdot \psi \leq \psi$, and

$$\begin{aligned} \psi(x, y') \otimes b(y', y) &\leq \psi(x, y') \otimes a^\circ(x, x) \otimes b(y', y) \\ &\leq \psi(x, y), \end{aligned}$$

that is $\psi \cdot b \leq \psi$. □

Corollary. *There is a \mathbf{V} -functor $\lceil a \rceil : X \rightarrow \mathbf{V}^{X^{\text{op}}}$. Moreover, for each $x \in X$ and $f \in \mathbf{V}^{X^{\text{op}}}$, we have*

$$d(a(-, x), f) = f(x).$$

Proof. Note that $d(a(-, x), f) = \bigwedge_y \text{hom}(a(y, x), f(y)) \leq f(x)$. On the other hand, for each $y \in Y$,

$$\begin{aligned} a(y, x) \leq \text{hom}(f(x), f(y)) &\iff f(x) \otimes a(y, x) \leq f(y) \\ &\iff f(x) \leq \text{hom}(a(y, x), f(y)). \end{aligned}$$

□

1.6 Lawvere-complete \mathbf{V} -categories.

Definition. A \mathbf{V} -category (X, a) is said to be *Lawvere-complete* if, for any \mathbf{V} -category (Y, b) , for every pair of adjoint bimodules

$$\begin{array}{ccc} & \varphi & \\ & \circ & \\ b \circ Y & \begin{array}{c} \xrightarrow{\varphi} \\ \perp \\ \xleftarrow{\psi} \end{array} & X \circ a \\ & \psi & \end{array}$$

φ is in the image of $(\)_* : \mathbf{V}\text{-Cat} \rightarrow \mathbf{V}\text{-Mod}$, i.e. there exists a \mathbf{V} -functor $f : (X, a) \rightarrow (Y, b)$ such that $f_* = \varphi$ and $f^* = \psi$.

It is interesting to notice that, in order to check Lawvere completeness, we can restrict to the case (Y, b) is the \mathbf{V} -category $(1, p)$, where $1 = \{\star\}$ is a singleton and $p(\star, \star) = k$.

Proposition. *For a \mathbf{V} -category (X, a) , the following conditions are equivalent:*

- (i) (X, a) is Lawvere-complete;
- (ii) *for each pair of adjoint bimodules $(\varphi : (1, p) \multimap (X, a)) \dashv (\psi : (X, a) \multimap (1, p))$, there exists a \mathbf{V} -functor $f : (1, p) \rightarrow (X, a)$ such that $\varphi = f_*$ and $\psi = f^*$ (in this situation we say that $f(\star)$ represents the adjunction $\varphi \dashv \psi$).*

Proof. It is a special case of Proposition 2.7. We omit the proof here because it follows, step by step, the proof of Proposition 2.7. □

Theorem. *The \mathbf{V} -category (\mathbf{V}, hom) is Lawvere-complete.*

Proof. Although this fact can be deduced from more general categorical results, we prefer to give here a direct proof, which provides guidance for the more general corresponding result for the (\mathbb{T}, \mathbf{V}) -categorical structure of \mathbf{V} we will study later.

Consider

$$\begin{array}{ccc} & \varphi & \\ & \circ & \\ p \circ 1 & \begin{array}{c} \xrightarrow{\varphi} \\ \perp \\ \xleftarrow{\psi} \end{array} & V \circ \text{hom} \\ & \psi & \end{array}$$

From the above theorem it follows that

$$\begin{aligned} (1) \quad \varphi \text{ is a bimodule} &\iff \varphi : (\mathbf{V}, \text{hom}) \rightarrow (\mathbf{V}, \text{hom}) \text{ is a } \mathbf{V}\text{-functor} \\ &\iff \forall u, v \in \mathbf{V} \text{ hom}(u, v) \leq \text{hom}(\varphi(u), \varphi(v)); \end{aligned}$$

$$\begin{aligned}
(2) \quad \psi \text{ is a bimodule} &\iff \psi : (\mathbf{V}, \text{hom}^{\text{op}}) \rightarrow (\mathbf{V}, \text{hom}) \text{ is a } \mathbf{V}\text{-functor} \\
&\iff \forall u, v \in \mathbf{V} \text{ hom}(u, v) \leq \text{hom}(\psi(v), \psi(u));
\end{aligned}$$

the conditions for the adjunction read as:

$$(3) \quad \varphi \dashv \psi \iff \forall u, v \in \mathbf{V} \quad \psi(u) \otimes \varphi(v) \leq \text{hom}(u, v) \quad \& \quad k \leq \bigvee_{u \in V} \varphi(u) \otimes \psi(u)$$

We will show that the adjunction $\varphi \dashv \psi$ is represented by $\psi(k)$, i.e. $\varphi(v) = \text{hom}(\psi(k), v)$ and $\psi(v) = \text{hom}(v, \psi(k))$, for every $v \in \mathbf{V}$. First we notice that from (3) it follows that $\psi(k) \otimes \varphi(v) \leq \text{hom}(k, v) = v$, hence $\varphi(v) \leq \text{hom}(\psi(k), v)$. Now the proof consists of checking three equalities:

$$(1\text{st}) \quad \psi(k) = \bigvee_{u \in V} \psi(u) \otimes u:$$

It is immediate that $\psi(k) = \psi(k) \otimes k \leq \bigvee_{u \in V} \psi(u) \otimes u$, and, moreover, for every $v \in \mathbf{V}$,

$$\begin{aligned}
\psi(u) \otimes u &= \psi(u) \otimes \text{hom}(k, u) \leq \psi(u) \otimes \text{hom}(\psi(u), \psi(k)) && (\text{by (2)}) \\
&\leq \psi(k).
\end{aligned}$$

$$(2\text{nd}) \quad \forall v \in \mathbf{V} \text{ hom}(v, \psi(k)) = \bigvee_{u \in V} \text{hom}(v, u) \otimes \psi(u):$$

To show “ \geq ” we just observe that

$$v \otimes (\text{hom}(v, u) \otimes \psi(u)) \leq u \otimes \psi(u) \leq \psi(k);$$

for “ \leq ”, we have

$$\begin{aligned}
\text{hom}(v, \psi(k)) &\leq \text{hom}(v, \psi(k)) \otimes \bigvee_{u \in V} \varphi(u) \otimes \psi(u) && (\text{by (3)}) \\
&= \bigvee_{u \in V} \text{hom}(v, \psi(k)) \otimes \varphi(u) \otimes \psi(u) \\
&\leq \bigvee_{u \in V} \text{hom}(v, \psi(k)) \otimes \text{hom}(\psi(k), u) \otimes \psi(u) && (\text{since } \varphi \leq \text{hom}(\psi(k), -)) \\
&\leq \bigvee_{u \in V} \text{hom}(v, u) \otimes \psi(u).
\end{aligned}$$

$$(3\text{rd}) \quad \text{Since } \psi = \psi \cdot \text{hom} \text{ we have } \forall v \in \mathbf{V} \psi(v) = \bigvee_{u \in V} \text{hom}(v, u) \otimes \psi(u). \quad \square$$

A new insight on Lawvere completeness for \mathbf{V} -categories may be found in [25].

2 Basic properties of (\mathbb{T}, \mathbf{V}) -categories

In the first part of this section we present the setting for the study of (\mathbb{T}, \mathbf{V}) -categories, or (Eilenberg-Moore) lax algebras, that can be studied in more detail in [7, 10, 8].

2.1 \mathbb{T} and its extension. Recall that a *monad* $\mathbb{T} = (T, e, m)$ on **Set** consists of a functor $T : \mathbf{Set} \rightarrow \mathbf{Set}$ together with natural transformations $e : \text{Id}_{\mathbf{Set}} \rightarrow T$ (unit) and $m : TT \rightarrow T$ (multiplication) such that

$$m \cdot Tm = m \cdot m_T \quad \text{and} \quad m \cdot Te = 1_T = m \cdot e_T.$$

There are two *trivial monads* on **Set**, one sending all sets X to the terminal set 1 , and the other with $T\emptyset = \emptyset$ and $TX = 1$ for $X \neq \emptyset$. Any other monad is called *non-trivial*.

By a *lax extension* of a **Set**-monad $\mathbb{T} = (T, e, m)$ to **V-Mat** we mean an extension of the endofunctor $T : \mathbf{Set} \rightarrow \mathbf{Set}$ to **V**-matrices acting on **Set**-maps as T and satisfying

(a) $(Ta)^\circ = T(a^\circ)$ (and we write Ta°),

(b) $Tb \cdot Ta \leq T(b \cdot a)$,

(c) $a \leq a' \Rightarrow Ta \leq Ta'$,

(d) $e_Y \cdot a \leq Ta \cdot e_X$,

(e) $m_Y \cdot T^2a \leq Ta \cdot m_X$,

for all $a, a' : X \multimap Y$ and $b : Y \multimap Z$ ⁴. Note that we have automatically equality in (b) if $a = f$ is a **Set**-map. A **Set**-monad $\mathbb{T} = (T, e, m)$ admitting a lax extension to **V-Mat** is called *V-admissible*. Although \mathbb{T} may have many lax extensions to **V-Mat**, in the sequel we usually have a fixed extension in mind when talking about a **V**-admissible monad. Trivially, the identity monad $\mathbb{1} = (\text{Id}, 1, 1)$ on **Set** can be extended to the identity monad on **V-Mat**. In [1] M. Barr shows how to extend **Set**-monads to **Rel** = **2-Mat**: first observe that each relation $r : X \multimap Y$ can be written as $r = p \cdot q^\circ$ where $q : R \rightarrow X$ and $p : R \rightarrow Y$ are the projection maps, then put $Tr = Tp \cdot Tq^\circ$. All conditions above but the second one are satisfied, and this extension satisfies (b) if and only if the **Set**-functor T has (BC) (that is, sends pullbacks to weak pullbacks). In [8] we showed how to make the step from **Rel** to **V-Mat**, provided that in addition **V** is *constructively completely distributive* (ccd)⁵. Given a monad $\mathbb{T} = (T, e, m)$ and a **V**-matrix $a : X \multimap Y$, we define relations $a_v : X \multimap Y$ ($v \in \mathbf{V}$) by $a_v(x, y) = \text{true} \iff a(x, y) \geq v$ and put, for $\mathfrak{x} \in TX$ and $\mathfrak{y} \in TY$,

$$Ta(\mathfrak{x}, \mathfrak{y}) = \bigvee \{v \in \mathbf{V} \mid Ta_v(\mathfrak{x}, \mathfrak{y}) = \text{true}\}.$$

Then the formula above defines an extension of $T : \mathbf{Set} \rightarrow \mathbf{Set}$ provided that either $k = \top$ or $T\emptyset = \emptyset$. Moreover, all five conditions above are satisfied. In addition we have

(f) $Tb \cdot Ta = T(b \cdot a)$ provided that $\otimes = \wedge$,

(g) $Tg \cdot Ta = T(g \cdot a)$,

for all **V**-matrices $a : X \multimap Y$ and $b : Y \multimap Z$ and all maps $g : Y \rightarrow Z$. In some occasions we will need that the (**Set**-based) natural transformation $m : TT \rightarrow T$ has (BC) (that is, each naturality square is a weak pullback); this guarantees that m is also a (strict) natural transformation for the extension of T to **V-Mat** described above.

⁴The conditions for our extension are stronger than Seal's in [24].

⁵Recall that a lattice Y is (ccd) if $\bigvee : 2^{Y^{\text{op}}} \rightarrow Y$ has a left adjoint; for more details see [27].

2.2 (\mathbb{T}, \mathbb{V}) -categories. Let $\mathbb{T} = (T, e, m)$ be a \mathbb{V} -admissible monad. A (\mathbb{T}, \mathbb{V}) -category is a pair (X, a) consisting of a set X and a \mathbb{V} -matrix $a : TX \multimap X$ such that:

$$1_X \leq a \cdot e_X \quad \text{and} \quad a \cdot Ta \leq a \cdot m_X;$$

that is, the map $a : TX \times X \rightarrow \mathbb{V}$ satisfies the conditions:

$$(R) \quad \text{for each } x \in X, \quad k \leq a(e_X(x), x);$$

$$(T) \quad \text{for each } \mathfrak{X} \in T^2X, \mathfrak{x} \in TX, x \in X, \quad Ta(\mathfrak{X}, \mathfrak{x}) \otimes a(\mathfrak{x}, x) \leq a(m_X(\mathfrak{X}), x).$$

Given (\mathbb{T}, \mathbb{V}) -categories (X, a) and (Y, b) , a (\mathbb{T}, \mathbb{V}) -functor $f : (X, a) \rightarrow (Y, b)$ is a map $f : X \rightarrow Y$ such that, for each $\mathfrak{x} \in TX$ and $x \in X$, $a(\mathfrak{x}, x) \leq b(Tf(\mathfrak{x}), f(x))$. (\mathbb{T}, \mathbb{V}) -categories and (\mathbb{T}, \mathbb{V}) -functors are the objects and morphisms of the category $(\mathbb{T}, \mathbb{V})\text{-Cat}$.

Note that each Eilenberg-Moore algebra for \mathbb{T} can be viewed as a (\mathbb{T}, \mathbb{V}) -category; in fact, we have an embedding

$$\text{Set}^{\mathbb{T}} \hookrightarrow (\mathbb{T}, \mathbb{V})\text{-Cat}.$$

In particular, for each set X we have the (\mathbb{T}, \mathbb{V}) -category (TX, m_X) which we denote by $|X|$.

Obviously, each \mathbb{V} -category is a (\mathbb{T}, \mathbb{V}) -category for $\mathbb{T} = \mathbb{1}$ the identity monad “identically” extended to $\mathbb{V}\text{-Mat}$. A further class of interesting examples involves the ultrafilter monad $\mathbb{U} = (U, e, m)$. The extension of $U : \text{Set} \rightarrow \text{Set}$ to $\mathbb{V}\text{-Mat}$ of 2.1 can be equivalently described by

$$Ur(\mathfrak{x}, \mathfrak{y}) = \bigwedge_{(A \in \mathfrak{x}, B \in \mathfrak{y})} \bigvee_{(x \in A, y \in B)} r(x, y),$$

for all $r : X \multimap Y$ in $\mathbb{V}\text{-Mat}$, $\mathfrak{x} \in TX$ and $\mathfrak{y} \in TY$. The main result of [1] states that $(\mathbb{U}, 2)\text{-Cat} \cong \text{Top}$. In [7] it is shown that $(\mathbb{U}, \mathbb{P}_+)\text{-Cat} \cong \text{App}$, the category of approach spaces and non-expansive maps (see [19] for details.)

2.3 The dual (\mathbb{T}, \mathbb{V}) -category. We have the canonical forgetful functor

$$\begin{aligned} E : (\mathbb{T}, \mathbb{V})\text{-Cat} &\rightarrow \mathbb{V}\text{-Cat}, \\ (X, a) &\mapsto (X, a \cdot e_X) \end{aligned}$$

with left adjoint

$$\begin{aligned} E^\circ : \mathbb{V}\text{-Cat} &\rightarrow (\mathbb{T}, \mathbb{V})\text{-Cat}, \\ (X, a) &\mapsto (X, e_X^\circ \cdot Ta) \end{aligned}$$

Furthermore, (the extension of) T induces an endofunctor

$$\begin{aligned} T : \mathbb{V}\text{-Cat} &\rightarrow \mathbb{V}\text{-Cat}, \\ (X, a) &\mapsto (TX, Ta) \end{aligned}$$

If m is a (strict) natural transformation, we can represent this functor as the composite

$$\begin{array}{ccc} & (\mathbb{T}, \mathbb{V})\text{-Cat} & \\ E^\circ \nearrow & & \searrow M^\circ \\ \mathbb{V}\text{-Cat} & \xrightarrow{T} & \mathbb{V}\text{-Cat}, \end{array}$$

where $M^\circ : (\mathbb{T}, \mathbb{V})\text{-Cat} \rightarrow \mathbb{V}\text{-Cat}$ is given by $(X, a) \mapsto (TX, Ta \cdot m_X^\circ)$. In fact, given a \mathbb{V} -category (X, a) , we have

$$T(e_X^\circ \cdot Ta) \cdot m_X^\circ = Te_X^\circ \cdot T^2a \cdot m_X^\circ = Te_X^\circ \cdot m_X^\circ \cdot Ta = Ta.$$

The functors M° and E° are the keys to define the *dual* (\mathbb{T}, \mathbb{V}) -category X^{op} of a (\mathbb{T}, \mathbb{V}) -category $X = (X, a)$: we put $X^{\text{op}} = E^\circ(M^\circ(X)^{\text{op}})$. We point out that if X is a \mathbb{V} -category interpreted as a (\mathbb{T}, \mathbb{V}) -category, i.e. $X = (X, e_X^\circ \cdot Ta)$ for a given \mathbb{V} -category structure $a : X \rightarrowtail X$, then

$$X^{\text{op}} = E^\circ(M^\circ(E^\circ(X, a))^{\text{op}}) = E^\circ((TX, Ta)^{\text{op}}),$$

that is, X^{op} is the dual – as a \mathbb{V} -category – of $T(X, a)$.

Our Theorem 3.3 shows that this is indeed a reasonable definition.

Finally, for later use we record the following

Lemma. *Let (X, a) be a \mathbb{V} -category and (X, α) be a \mathbb{T} -algebra. Then $(X, a \cdot \alpha)$ is a (\mathbb{T}, \mathbb{V}) -category if and only if $\alpha : (TX, Ta) \rightarrow (X, a)$ is a \mathbb{V} -functor.*

Proof. First we remark that from $1_X \leq a$ and $1_X = \alpha \cdot e_X$ it follows that $1_X \leq (a \cdot \alpha) \cdot e_X$, that is $a \cdot \alpha$ always fulfils the reflexivity axiom. Now, if α is a \mathbb{V} -functor, i.e. $\alpha \cdot Ta \leq a \cdot \alpha$, then

$$(a \cdot \alpha) \cdot T(a \cdot \alpha) = a \cdot \alpha \cdot Ta \cdot T\alpha \leq a \cdot a \cdot \alpha \cdot T\alpha \leq (a \cdot \alpha) \cdot m_X.$$

Conversely, if $a \cdot \alpha$ is a (\mathbb{T}, \mathbb{V}) -categorical structure, then

$$\alpha \cdot Ta = \alpha \cdot Ta \cdot T\alpha \cdot Te_X \leq a \cdot \alpha \cdot Ta \cdot T\alpha \cdot Te_X \leq a \cdot \alpha \cdot m_X \cdot Te_X = a \cdot \alpha. \quad \square$$

2.4 Kleisli composition. Many notions and techniques can be transported from $\mathbb{V}\text{-Cat}$ to $(\mathbb{T}, \mathbb{V})\text{-Cat}$ by formally replacing composition of \mathbb{V} -matrices by *Kleisli composition* (see [15]) defined as

$$b * a := b \cdot Ta \cdot m_X^\circ,$$

$$\begin{array}{ccc} \begin{array}{c} TX \\ \downarrow a \\ Y \end{array} & \begin{array}{c} TY \\ \downarrow b \\ Z \end{array} & \longmapsto \begin{array}{ccc} TX & \xrightarrow{m_X^\circ} & TTX \\ & \searrow b*a & \downarrow Ta \\ & & TY \\ & & \downarrow b \\ & & Z \end{array} \end{array}$$

for all $a : TX \rightarrowtail Y$ and $b : TY \rightarrowtail Z$ in $\mathbb{V}\text{-Mat}$. The matrix $e_X^\circ : TX \rightarrowtail X$ acts as a lax identity for this composition, in the following sense:

$$a * e_X^\circ = a \quad \text{and} \quad e_X^\circ * b \geq b,$$

for $a : TX \rightarrowtail Y$ and $b : TY \rightarrowtail X$. Moreover,

$$c * (b * a) \leq (c * b) * a$$

if $T : \mathbb{V}\text{-Mat} \rightarrow \mathbb{V}\text{-Mat}$ preserves composition, and

$$c * (b * a) \geq (c * b) * a$$

if $m : TT \rightarrow T$ is a (strict) natural transformation.

2.5 (\mathbb{T}, \mathbb{V}) -bimodules. Given (\mathbb{T}, \mathbb{V}) -categories (X, a) and (Y, b) , a (\mathbb{T}, \mathbb{V}) -bimodule (or simply a bimodule) $\psi : (X, a) \multimap (Y, b)$ is a \mathbb{V} -matrix $\psi : TX \multimap Y$ such that $\psi * a \leq \psi$ and $b * \psi \leq \psi$. This means that $\psi \cdot Ta \cdot m_X^\circ \leq \psi$ and $b \cdot T\psi \cdot m_X^\circ \leq \psi$; that is, for $\mathfrak{X} \in T^2X$, $\mathfrak{x} \in TX$, $\mathfrak{y} \in TY$ and $y \in Y$,

$$Ta(\mathfrak{X}, \mathfrak{x}) \otimes \psi(\mathfrak{x}, y) \leq \psi(m_X(\mathfrak{X}), y),$$

$$T\psi(\mathfrak{X}, \mathfrak{y}) \otimes b(\mathfrak{y}, y) \leq \psi(m_X(\mathfrak{X}), y).$$

Whenever the Kleisli composition is associative (in particular if $T : \mathbb{V}\text{-Mat} \rightarrow \mathbb{V}\text{-Mat}$ is a functor and m is a natural transformation: see [15]), bimodules compose. The identities for the composition law are again the (\mathbb{T}, \mathbb{V}) -categorical structures, and we can consider the category $(\mathbb{T}, \mathbb{V})\text{-Mod}$ of (\mathbb{T}, \mathbb{V}) -categories and (\mathbb{T}, \mathbb{V}) -bimodules.

2.6 (\mathbb{T}, \mathbb{V}) -functors as (\mathbb{T}, \mathbb{V}) -bimodules. Analogously to the situation in \mathbb{V} -categories, each (\mathbb{T}, \mathbb{V}) -functor $f : (X, a) \rightarrow (Y, b)$ defines a pair of bimodules $f_* : (X, a) \multimap (Y, b)$ and $f^* : (Y, b) \multimap (X, a)$ as indicated in the following diagram

$$\begin{array}{ccc} & f^* & f_* \\ & \curvearrowright & \curvearrowright \\ TX & \xrightarrow{Tf} & TY \\ \downarrow a & & \downarrow b \\ X & \xrightarrow{f} & Y \end{array}$$

that is, $f_* := b \cdot Tf$ and $f^* := f^\circ \cdot b$. In fact, the following assertions are equivalent for (\mathbb{T}, \mathbb{V}) -categories (X, a) and (Y, b) and a function $f : X \rightarrow Y$.

- (i) $f : (X, a) \rightarrow (Y, b)$ is a (\mathbb{T}, \mathbb{V}) -functor.
- (ii) $f_* : (X, a) \multimap (Y, b)$ is a (\mathbb{T}, \mathbb{V}) -bimodule.
- (iii) $f^* : (Y, b) \multimap (X, a)$ is a (\mathbb{T}, \mathbb{V}) -bimodule.

We point out that, although in general bimodules do not compose, if $f : (X, a) \rightarrow (Y, b)$ is a functor, then, for any bimodules $\varphi : (Y, b) \multimap (Z, c)$ and $\psi : (Z, c) \multimap (Y, b)$,

$$\begin{array}{ccccc} & & b & & \\ & & \circ & & \\ & f_* & \curvearrowright & \varphi & \\ a \circ X & \xrightarrow{\quad} & Y & \xrightarrow{\quad} & Z \circ c \\ & f & \circ & \psi & \\ & f^* & \curvearrowright & & \end{array}$$

$\varphi * f_*$ and $f^* * \psi$ are bimodules, as we show next. First note that

$$\varphi * f_* = \varphi \cdot Tf \quad \text{and} \quad f^* * \psi = f^\circ \cdot \psi.$$

The latter equality follows from

$$f^* * \psi = f^\circ \cdot b \cdot T\psi \cdot m_Z^\circ = f^\circ \cdot \psi,$$

and \mathbb{V} -functoriality of f implies

$$\begin{aligned}
\varphi * f_* &= \varphi * (b \cdot Tf) = \varphi \cdot T(b \cdot Tf) \cdot m_X^\circ \\
&\geq \varphi \cdot T(f \cdot a) \cdot m_X^\circ && \text{(by functoriality of } f) \\
&\geq \varphi \cdot Tf \cdot Ta \cdot m_X^\circ \\
&\geq \varphi \cdot Tf \cdot Ta \cdot Te_X && \text{(since } m_X^\circ \geq Te_X) \\
&\geq \varphi \cdot Tf, && \text{(since } a \cdot e_X \geq 1_X)
\end{aligned}$$

whereby φ bimodule gives us

$$\varphi * f_* = \varphi * (b \cdot Tf) = \varphi \cdot Tb \cdot T^2 f \cdot m_X^\circ \leq \varphi \cdot Tb \cdot m_Y^\circ \cdot Tf = \varphi \cdot Tf.$$

The bimodule properties of $\varphi * f_*$ and $f^* * \psi$ follow now from

$$\begin{aligned}
c * (\varphi * f_*) &= c * (\varphi \cdot Tf) \leq (c * \varphi) \cdot Tf = \varphi * f_*, \\
(\varphi * f_*) * a &= \varphi \cdot Tf \cdot Ta \cdot m_X^\circ \leq \varphi \cdot Tb \cdot T^2 f \cdot m_X^\circ \leq \varphi \cdot Tb \cdot m_Y^\circ \cdot Tf = \varphi * f_*, \\
a * (f^* * \psi) &= a \cdot T(f^\circ \cdot \psi) \cdot m_Z^\circ = a \cdot Tf^\circ \cdot T\psi \cdot m_Z^\circ \leq f^\circ \cdot b \cdot T\psi \cdot m_Z^\circ = f^\circ \cdot \psi = f^* * \psi, \\
(f^* * \psi) * c &= f^\circ \cdot \psi \cdot Tc \cdot m_Z^\circ = f^\circ \cdot (\psi * c) = f^\circ \cdot \psi = f^* * \psi.
\end{aligned}$$

Therefore we can define the “whiskering” functors

$$\begin{aligned}
- * f_* : (\mathbb{T}, \mathbb{V})\text{-Mod}(Y, Z) &\longrightarrow (\mathbb{T}, \mathbb{V})\text{-Mod}(X, Z), \text{ and} \\
\varphi &\longmapsto \varphi \cdot Tf \\
f^* * - : (\mathbb{T}, \mathbb{V})\text{-Mod}(Z, Y) &\longrightarrow (\mathbb{T}, \mathbb{V})\text{-Mod}(Z, X) \\
\psi &\longmapsto f^\circ \cdot \psi.
\end{aligned}$$

Moreover, given a pair of adjoint bimodules $(\varphi : (Y, b) \dashv \multimap (Z, c)) \dashv (\psi : (Z, c) \dashv \multimap (Y, b))$, we have

$$\varphi * f_* \dashv f^* * \psi,$$

provided that the diagram

$$\begin{array}{ccc}
T^2 X & \xrightarrow{m_X} & TX \\
T^2 f \downarrow & & \downarrow Tf \\
T^2 Y & \xrightarrow{m_Y} & TY
\end{array}$$

satisfies (BC): $(\varphi * f_*) * (f^* * \psi) \leq c$ is always true, since

$$(\varphi \cdot Tf) * (f^\circ \cdot \psi) = \varphi \cdot Tf \cdot Tf^\circ \cdot T\psi \cdot m_Z^\circ \leq \varphi \cdot T\psi \cdot m_Z^\circ = \varphi * \psi \leq c,$$

while to conclude that $a \leq (f^* * \psi) * (\varphi * f_*)$ we need the condition above:

$$a \leq f^\circ \cdot b \cdot Tf \leq f^\circ \cdot \psi \cdot T\varphi \cdot m_Y^\circ \cdot Tf = f^\circ \cdot \psi \cdot T\varphi \cdot T^2 f \cdot m_X^\circ = (f^\circ \cdot \psi) * (\varphi \cdot Tf).$$

2.7 Lawvere-complete (\mathbb{T}, \mathbb{V}) -categories.

Definition. A (\mathbb{T}, \mathbb{V}) -category (X, a) is called *Lawvere-complete* if, for each (\mathbb{T}, \mathbb{V}) -category (Y, b) and each pair of adjoint bimodules

$$\begin{array}{c} \varphi \\ \curvearrowright \\ Y \quad \perp \quad X \quad \curvearrowleft \\ \psi \end{array}$$

there exists a functor $f : (Y, b) \rightarrow (X, a)$ such that $f_* = \varphi$ and $f^* = \psi$.

Analogously to the \mathbb{V} -categorical case, Lawvere completeness is fully tested by left adjoint bimodules with domain $(1, p)$, where $p = e_1^\circ$, hence $p(\star, \star) = k$ and $p(\mathfrak{x}, \star) = \perp$ for $\mathfrak{x} \neq \star$ in $T1$.

Proposition. Assume that either $T1 = 1$ or that the (**Set**-based) natural transformation m satisfies (BC). Then, for a (\mathbb{T}, \mathbb{V}) -category (X, a) , the following conditions are equivalent:

(i) (X, a) is Lawvere-complete;

(ii) each pair of adjoint bimodules $(1, p) \begin{array}{c} \varphi \\ \curvearrowright \\ \perp \\ \curvearrowleft \\ \psi \end{array} (X, a)$ is induced by a functor $f : (1, p) \rightarrow (X, a)$;

(iii) each pair of adjoint bimodules $(1, p) \begin{array}{c} \varphi \\ \curvearrowright \\ \perp \\ \curvearrowleft \\ \psi \end{array} (X, a)$ is induced by a map $f : 1 \rightarrow X$ (so that $\varphi = a \cdot Tf$ and $\psi = f^\circ \cdot a$).

Proof. (iii) \Rightarrow (i): Let $(Y, b) \begin{array}{c} \varphi \\ \curvearrowright \\ \perp \\ \curvearrowleft \\ \psi \end{array} (X, a)$ be a pair of adjoint bimodules. For each $y \in Y$,

let $g_y : (1, p) \rightarrow (Y, b)$ be the functor that picks y . This functor induces a pair of adjoint bimodules $(g_y)_* \dashv (g_y)^*$, whence we have

$$\begin{array}{ccccc} p \curvearrowright 1 & \begin{array}{c} (g_y)^* \\ \curvearrowright \\ \perp \\ \curvearrowleft \\ (g_y)^* \end{array} & Y & \begin{array}{c} b \\ \curvearrowright \\ \varphi \\ \curvearrowright \\ \perp \\ \curvearrowleft \\ \psi \end{array} & X \curvearrowleft a \end{array}$$

If m satisfies (BC), we know already that $\varphi_y \dashv \psi_y$, where $\varphi_y = \varphi * (g_y)_* = \varphi \cdot Tg_y$ and $\psi_y = (g_y)^* * \psi = g_y^\circ \cdot \psi$. The same happens whenever $T1 = 1$, as it is easily checked. By hypothesis, there exists a map $f_y : 1 \rightarrow X$ such that $\varphi_y = b \cdot Tf_y$ and $\psi_y = f_y^\circ \cdot b$. Gluing together the maps $(f_y)_{y \in Y}$ we obtain a map $f : Y \rightarrow X$. Then, for $\mathfrak{x} \in TX$ and $y \in Y$,

$$\psi(\mathfrak{x}, y) = \psi_y(\mathfrak{x}, \star) = f_y^\circ \cdot a(\mathfrak{x}, \star) = a(\mathfrak{x}, f_y(\star)) = a(\mathfrak{x}, f(y)),$$

that is, $\psi = f^* = f^\circ \cdot a$. We can show now that f is necessarily a functor:

$$b \cdot Tf^\circ \leq b \cdot Tf^\circ \cdot Ta \cdot Te_X \leq b \cdot Tf^\circ \cdot Ta \cdot m_X^\circ = b * \psi \leq \psi = f^\circ \cdot a.$$

This concludes the proof since, by unicity of adjoints, φ is necessarily f_* . □

3 V as a (\mathbb{T}, \mathbb{V}) -category

3.1 The \mathbb{T} -algebra structure of \mathbb{V} . Our next goal is to explore the notions introduced in the previous section. In particular we are aiming for results which extend known facts about \mathbb{V} -categories (as Theorem 1.5 or Theorem 1.6). To do so, from now on *we will always assume that the extension $T : \mathbb{V}\text{-Mat} \rightarrow \mathbb{V}\text{-Mat}$ is constructed as in [8] and consequently we assume \mathbb{V} to be constructively completely distributive. Furthermore, we assume that $\mathbb{T} = (T, e, m)$ is non-trivial and that T and m satisfy (BC).*

Under these conditions, as Manes essentially showed in [21],

$$\begin{aligned} \xi : T\mathbb{V} &\longrightarrow \mathbb{V} \\ \mathfrak{x} &\longmapsto \bigvee \{v \in \mathbb{V} \mid \mathfrak{x} \in T(\uparrow v)\} \end{aligned}$$

is a \mathbb{T} -algebra structure on \mathbb{V} , where $\uparrow v = \{u \in \mathbb{V} \mid v \leq u\}$.

There is an interesting link between this \mathbb{T} -algebra structure and the image under the lax functor $T : \mathbb{V}\text{-Mat} \rightarrow \mathbb{V}\text{-Mat}$ of the identity $1_{\mathbb{V}} : \mathbb{V} \rightarrow \mathbb{V}$ considered as a matrix $i : 1 \multimap \mathbb{V}$, with $i(\star, v) = v$. Let us compute $Ti : T1 \multimap T\mathbb{V}$. We consider, for each $v \in \mathbb{V}$, the relation

$$\begin{aligned} i_v : 1 \times V &\longrightarrow 2 \\ (\star, u) &\longmapsto \begin{cases} \text{true} & \text{if } v \leq u, \\ \text{false} & \text{elsewhere,} \end{cases} \end{aligned}$$

hence the diagram

$$\begin{array}{ccc} 1 & \xrightarrow{i_v} & V, \\ & \searrow q_v^\circ & \nearrow p_v \\ & \uparrow v & \end{array}$$

commutes where p_v and q_v are the projections. Now, for each $\mathfrak{x} \in T1$ and $\mathfrak{y} \in T\mathbb{V}$,

$$\begin{aligned} Ti(\mathfrak{x}, \mathfrak{y}) &= \bigvee \{v \in \mathbb{V} \mid T(i_v)(\mathfrak{x}, \mathfrak{y}) = \text{true}\} \\ &= \bigvee \{v \in \mathbb{V} \mid Tp_v Tq_v^\circ(\mathfrak{x}, \mathfrak{y}) = \text{true}\} \\ &= \bigvee \{v \in \mathbb{V} \mid \exists \mathfrak{z} \in \uparrow v : Tq_v(\mathfrak{z}) = \mathfrak{x} \text{ and } Tp_v(\mathfrak{z}) = \mathfrak{y}\}, \end{aligned}$$

hence, since T preserves injections and considering Tp_v as an inclusion, we can write

$$Ti(\mathfrak{x}, \mathfrak{y}) = \bigvee \{v \in \mathbb{V} \mid \mathfrak{y} \in T(\uparrow v) \text{ and } Tq_v(\mathfrak{y}) = \mathfrak{x}\} \leq \xi(\mathfrak{y}),$$

by definition of ξ . In particular, if $\mathfrak{x} = Tq(\mathfrak{y})$, for $q : \mathbb{V} \rightarrow 1$, then $Ti(\mathfrak{x}, \mathfrak{y}) = \xi(\mathfrak{y})$. Whenever $T1 = 1$, $Tq(\mathfrak{y}) = \star$ for every $\mathfrak{y} \in T\mathbb{V}$, whence

$$Ti(\star, \mathfrak{y}) = \bigvee \{v \in \mathbb{V} \mid \mathfrak{y} \in T(\uparrow v)\} = \xi(\mathfrak{y}).$$

This link between the extension of T and the \mathbb{T} -algebra structure ξ is more general. Whenever necessary, in the sequel we denote the **Set**-endofunctor T by T_o , and keep T for its extension to $\mathbb{V}\text{-Mat}$. Each \mathbb{V} -matrix $r : X \multimap Y$ can be considered also as a map $r : X \times Y \rightarrow \mathbb{V}$. The interplay between $T_o r$ and $T r$ is stated in the following result, whose proof is straightforward.

Proposition. For any \mathbf{V} -matrix $r : X \rightarrowtail Y$, each $\mathfrak{x} \in TX$ and $\mathfrak{y} \in TY$,

$$Tr(\mathfrak{x}, \mathfrak{y}) = \bigvee_{\mathfrak{w} : \substack{T_o\pi_X(\mathfrak{w})=\mathfrak{x} \\ T_o\pi_Y(\mathfrak{w})=\mathfrak{y}}} \xi \cdot T_or(\mathfrak{w}),$$

that is the following diagram

$$\begin{array}{ccc} T(X \times Y) & \xrightarrow{T_or} & TV \\ \langle T_o\pi_X, T_o\pi_Y \rangle \uparrow & & \downarrow \xi \\ TX \times TY & \xrightarrow{Tr} & V \end{array}$$

commutes.

Remark. Besides being the structure map of an Eilenberg-Moore algebra, $\xi : TV \rightarrow V$ satisfies also the inequalities

$$\otimes \cdot \langle \xi \cdot T_o\pi_1, \xi \cdot T_o\pi_2 \rangle \leq \xi \cdot T_o(\otimes) \quad \text{and} \quad k \cdot ! \leq \xi \cdot T_ok.$$

$$\begin{array}{ccc} T(V \times V) & \xrightarrow{T_o(\otimes)} & TV \\ \langle \xi \cdot T_o\pi_1, \xi \cdot T_o\pi_2 \rangle \downarrow & \leq & \downarrow \xi \\ V \times V & \xrightarrow{\otimes} & V \end{array} \quad \begin{array}{ccc} T1 & \xrightarrow{T_ok} & TV \\ ! \downarrow & \leq & \downarrow \xi \\ 1 & \xrightarrow{k} & V \end{array}$$

Recall that we assume $Tf = T_of$ for each **Set**-map $f : X \rightarrow Y$; this condition requires and implies equality in the latter inequality (see [16]).

3.2 The canonical (\mathbb{T}, \mathbf{V}) -categorical structure of \mathbf{V} . The composition of the natural \mathbf{V} -categorical and \mathbb{T} -algebra structures of \mathbf{V} defines an interesting structure, hom_ξ , of a (\mathbb{T}, \mathbf{V}) -category on \mathbf{V}

$$TV \xrightarrow{\text{hom}_\xi} V = (TV \xrightarrow{\xi} V \xrightarrow{\text{hom}} V),$$

as we show next.

Proposition. $\xi : (TV, T\text{hom}) \rightarrow (V, \text{hom})$ is a \mathbf{V} -functor.

Proof. We have to show that $\xi \cdot T\text{hom} \leq \text{hom} \cdot \xi$, or, equivalently, $T\text{hom} \leq \xi^\circ \cdot \text{hom} \cdot \xi$. This means that, for $\mathfrak{x}, \mathfrak{y} \in TV$,

$$T\text{hom}(\mathfrak{x}, \mathfrak{y}) \leq \text{hom}(\xi(\mathfrak{x}), \xi(\mathfrak{y})).$$

We consider again the matrix $i : 1 \rightarrowtail V$, and compute $1 \xrightarrow{i} V \xrightarrow{\text{hom}} V$:

$$\text{hom} \cdot i(\star, v) = \bigvee_{u \in V} i(\star, u) \otimes \text{hom}(u, v) = \bigvee_{u \in V} u \otimes \text{hom}(u, v) \leq v;$$

that is $\text{hom} \cdot i \leq i$. Hence $T\text{hom} \cdot Ti \leq T(\text{hom} \cdot i) \leq Ti$, and so, for $\mathfrak{x}, \mathfrak{y} \in TV$ and $\mathfrak{z} = Tq(\mathfrak{x})$ as in Section 3.1, we have

$$\xi(\mathfrak{x}) \otimes T\text{hom}(\mathfrak{x}, \mathfrak{y}) = Ti(\mathfrak{z}, \mathfrak{x}) \otimes T\text{hom}(\mathfrak{x}, \mathfrak{y}) \leq Ti(\mathfrak{z}, \mathfrak{y}) \leq \xi(\mathfrak{y}),$$

and therefore

$$T\text{hom}(\mathfrak{x}, \mathfrak{y}) \leq \text{hom}(\xi(\mathfrak{x}), \xi(\mathfrak{y}))$$

as claimed. \square

Corollary. $(\mathbf{V}, \text{hom}_\varepsilon)$ is a (\mathbb{T}, \mathbf{V}) -category.

Proof. Follows from the proposition above and Lemma 2.2. \square

3.3 The tensor product. The tensor product in \mathbf{V} defines in a natural way a (not necessarily closed) product structure in $(\mathbb{T}, \mathbf{V})\text{-Cat}$. Given (\mathbb{T}, \mathbf{V}) -categories $X = (X, a)$ and $Y = (Y, b)$, we put $X \otimes Y = (X \times Y, a \otimes b)$ where $a \otimes b(\mathfrak{w}, (x, y)) = a(T\pi_X(\mathfrak{w}), x) \otimes b(T\pi_Y(\mathfrak{w}), y)$ for all $\mathfrak{w} \in T(X \times Y)$, $x \in X$ and $y \in Y$. One easily verifies reflexivity of $a \otimes b$, while transitivity holds if and only if $\otimes \cdot \langle \xi \cdot T_o\pi_1, \xi \cdot T_o\pi_2 \rangle = \xi \cdot T_o(\otimes)$ (see Remark 3.1 and [16]) *which we assume from now on*. We remark that this condition guarantees that \mathbb{T} is a (lax) Hopf monad on $\mathbf{V}\text{-Mat}$ (see [22]) where the tensor product in \mathbf{V} is naturally extended to $\mathbf{V}\text{-Mat}$. However, we will not develop this aspect here.

It is well-known that in general the functor $X \otimes _ : (\mathbb{T}, \mathbf{V})\text{-Cat} \rightarrow (\mathbb{T}, \mathbf{V})\text{-Cat}$ has no right adjoint as, for example, \mathbf{Top} is not Cartesian closed. The problem of characterising those (\mathbb{T}, \mathbf{V}) -categories $X = (X, a)$ such that tensoring with X has a right adjoint is studied in [16].

Theorem. Let m be a natural transformation. For (\mathbb{T}, \mathbf{V}) -categories (X, a) and (Y, b) and a \mathbf{V} -matrix $\psi : TX \rightarrowtail Y$, the following assertions are equivalent.

- (i) $\psi : (X, a) \rightarrowtail (Y, b)$ is a (\mathbb{T}, \mathbf{V}) -bimodule.
- (ii) Both $\psi : |X| \otimes Y \rightarrow \mathbf{V}$ and $\psi : X^{\text{op}} \otimes Y \rightarrow \mathbf{V}$ are (\mathbb{T}, \mathbf{V}) -functors.

Proof. Assume that $\psi : (X, a) \rightarrowtail (Y, b)$ is a (\mathbb{T}, \mathbf{V}) -bimodule. First observe that, for $\mathfrak{W} \in T(TX \times Y)$,

$$\xi \cdot T_o\psi(\mathfrak{W}) \leq T\psi(T_o\pi_{TX}(\mathfrak{W}), T_o\pi_Y(\mathfrak{W})).$$

Let $\mathfrak{W} \in T(TX \times Y)$, $\mathfrak{x} \in TX$ and $y \in Y$. To see that $\psi : |X| \otimes Y \rightarrow \mathbf{V}$ is a (\mathbb{T}, \mathbf{V}) -functor, note that the structure c on $|X| \otimes Y$ is given by

$$c(\mathfrak{W}, (\mathfrak{x}, y)) = \begin{cases} \perp & \text{if } \mathfrak{x} \neq m_X(T_o\pi_{TX}(\mathfrak{W})), \\ b(T\pi_Y(\mathfrak{W}), y) & \text{if } \mathfrak{x} = m_X(T_o\pi_{TX}(\mathfrak{W})). \end{cases}$$

Assume $\mathfrak{x} = m_X(T_o\pi_{TX}(\mathfrak{W}))$. Since

$$b(T_o\pi_Y(\mathfrak{W}), y) \leq \text{hom}(\xi \cdot T_o\psi(\mathfrak{W}), \psi(\mathfrak{x}, y))$$

is equivalent to

$$\xi \cdot T_o\psi(\mathfrak{W}) \otimes b(T_o\pi_Y(\mathfrak{W}), y) \leq \psi(\mathfrak{x}, y),$$

the assertion follows at once. We show now that $\psi : X^{\text{op}} \otimes Y \rightarrow \mathbf{V}$ is a (\mathbb{T}, \mathbf{V}) -functor. As above we have that (with $a^{\text{op}} = c_{TX}^{\circ} \cdot Tm_X \cdot T^2a^{\circ}$ the structure on X^{op})

$$a^{\text{op}}(T_o\pi_{TX}(\mathfrak{W}), \mathfrak{x}) \otimes b(T_o\pi_Y(\mathfrak{W}), y) \leq \text{hom}(\xi \cdot T_o\psi(\mathfrak{W}), \psi(\mathfrak{x}, y))$$

is equivalent to

$$\xi \cdot T_o\psi(\mathfrak{W}) \otimes a^{\text{op}}(T_o\pi_{TX}(\mathfrak{W}), \mathfrak{x}) \otimes b(T_o\pi_Y(\mathfrak{W}), y) \leq \psi(\mathfrak{x}, y).$$

Now

$$\begin{aligned}
& \xi \cdot T_o\psi(\mathfrak{W}) \otimes a^{\text{op}}(T_o\pi_{TX}(\mathfrak{W}), \mathfrak{x}) \otimes b(T_o\pi_Y(\mathfrak{W}), y) \\
& \leq T^2a \cdot Tm_X^\circ \cdot e_{TX}(\mathfrak{x}, T_o\pi_{TX}(\mathfrak{W})) \otimes T\psi(T_o\pi_{TX}(\mathfrak{W}), T_o\pi_Y(\mathfrak{W})) \otimes b(T_o\pi_Y(\mathfrak{W}), y) \\
& \leq b \cdot T\psi \cdot T^2a \cdot Tm_X^\circ \cdot m_X^\circ(\mathfrak{x}, y) \\
& \leq b \cdot T\psi \cdot m_X \cdot Ta \cdot m_X^\circ(\mathfrak{x}, y) \\
& = \psi \cdot Ta \cdot m_X^\circ(\mathfrak{x}, y) = \psi(\mathfrak{x}, y).
\end{aligned}$$

Now assume that $\psi : |X| \otimes Y \rightarrow \mathbf{V}$ and $\psi : X^{\text{op}} \otimes Y \rightarrow \mathbf{V}$ are (\mathbb{T}, \mathbf{V}) -functors. Functoriality of $\psi : |X| \otimes Y \rightarrow \mathbf{V}$ implies, for all $\mathfrak{x} \in TX$ and $y \in Y$,

$$\begin{aligned}
\psi(\mathfrak{x}, y) & \geq \bigvee_{\substack{\mathfrak{x} \in TTX: \\ m_X(\mathfrak{x}) = \mathfrak{x}; \\ \mathfrak{y} \in TY}} \bigvee \left\{ \xi \cdot T_o\psi(\mathfrak{W}) \otimes b(\mathfrak{y}, y) \mid \mathfrak{W} \in T(TX \times Y) : \mathfrak{W} \mapsto \mathfrak{x}, \mathfrak{W} \mapsto \mathfrak{y} \right\} \\
& = \bigvee_{\substack{\mathfrak{x} \in TTX: \\ m_X(\mathfrak{x}) = \mathfrak{x}; \\ \mathfrak{y} \in TY}} T\psi(\mathfrak{x}, \mathfrak{y}) \otimes b(\mathfrak{y}, y) \\
& = \bigvee_{\substack{\mathfrak{x} \in TTX: \\ m_X(\mathfrak{x}) = \mathfrak{x}}} b \cdot T\psi(\mathfrak{x}, y) \\
& = b \cdot T\psi \cdot m_X^\circ(\mathfrak{x}, y).
\end{aligned}$$

On the other hand, by functoriality of $\psi : X^{\text{op}} \otimes Y \rightarrow \mathbf{V}$, for all $\mathfrak{x} \in TX$ and $y \in Y$ we have

$$\begin{aligned}
\psi(\mathfrak{x}, y) & \geq \bigvee_{\substack{\mathfrak{x} \in TTX, \\ \mathfrak{y} \in TY}} \bigvee \left\{ \xi \cdot T_o\psi(\mathfrak{W}) \otimes b(\mathfrak{y}, y) \otimes a^{\text{op}}(\mathfrak{x}, \mathfrak{x}) \mid \mathfrak{W} \in T(TX \times Y) : \mathfrak{W} \mapsto \mathfrak{x}, \mathfrak{W} \mapsto \mathfrak{y} \right\} \\
& = \bigvee_{\substack{\mathfrak{x} \in TTX, \\ \mathfrak{y} \in TY}} T\psi(\mathfrak{x}, \mathfrak{y}) \otimes b(\mathfrak{y}, y) \otimes T^2a \cdot Tm_X^\circ \cdot e_{TX}(\mathfrak{x}, \mathfrak{x}) \\
& = b \cdot T\psi \cdot T^2a \cdot Tm_X^\circ \cdot e_{TX}(\mathfrak{x}, \mathfrak{x}) \\
& \geq b \cdot e_Y \cdot \psi \cdot Ta \cdot m_X^\circ(\mathfrak{x}, y) \\
& \geq \psi \cdot Ta \cdot m_X^\circ(\mathfrak{x}, y).
\end{aligned}$$

□

3.4 \mathbf{V} is Lawvere-complete.

Theorem. Assume that $T1 = 1$. Then $(\mathbf{V}, \text{hom}_\xi)$ is a Lawvere-complete (\mathbb{T}, \mathbf{V}) -category provided that $a := M^\circ(\text{hom}_\xi) = \xi^\circ \cdot \text{hom} \cdot \xi$ (i.e., for $\mathfrak{v}, \mathfrak{w} \in T\mathbf{V}$, $a(\mathfrak{v}, \mathfrak{w}) = \text{hom}(\xi(\mathfrak{v}), \xi(\mathfrak{w}))$).

Proof. Let

be a pair of adjoint bimodules. By the previous theorem we know that:

$$\begin{aligned}
(4) \quad \varphi \text{ bimodule} & \iff \varphi : (\mathbf{V}, \text{hom}_\xi) \rightarrow (\mathbf{V}, \text{hom}_\xi) \text{ is a } (\mathbb{T}, \mathbf{V})\text{-functor} \\
& \iff \forall \mathfrak{v} \in T\mathbf{V} \quad \forall v \in \mathbf{V} \quad \text{hom}(\xi(\mathfrak{v}), v) \leq \text{hom}(\xi \cdot T\varphi(\mathfrak{v}), \varphi(v)).
\end{aligned}$$

In particular, for every $\mathbf{v} \in TV$, $k \leq \text{hom}(\xi(\mathbf{v}), \xi(\mathbf{v})) \leq \text{hom}(\xi \cdot T\varphi(\mathbf{v}), \varphi \cdot \xi(\mathbf{v}))$, hence $\xi \cdot T\varphi(\mathbf{v}) \leq \varphi \cdot \xi(\mathbf{v})$.

$$(5) \quad \begin{aligned} \psi \text{ bimodule} &\iff \psi : (TV, a^\circ) \rightarrow (V, \text{hom}) \text{ is a } \mathbf{V}\text{-functor} \\ &\iff \forall \mathbf{v}, \mathbf{w} \in TV \ a(\mathbf{v}, \mathbf{w}) \leq \text{hom}(\psi(\mathbf{w}), \psi(\mathbf{v})). \end{aligned}$$

Finally,

$$(6) \quad \varphi \dashv \psi \iff \begin{cases} (a) & \varphi * \psi \leq \text{hom} \cdot \xi \iff \forall \mathbf{v} \in TV \forall v \in \mathbf{V} \ \psi(\mathbf{v}) \otimes \varphi(v) \leq \text{hom}(\xi(\mathbf{v}), v), \\ (b) & p \leq \psi * \varphi \iff k \leq \bigvee_{\mathbf{u} \in TV} \psi(\mathbf{u}) \otimes \xi(T_o\varphi(\mathbf{u})). \end{cases}$$

We will show that the adjunction $\varphi \dashv \psi$ is represented by $\psi(\dot{k})$, where $\dot{k} = e_V(k)$. Similarly to the proof of Theorem 1.6, we split our argument in three steps:

$$(1\text{st}) \quad \psi(\dot{k}) = \bigvee_{\mathbf{v} \in TV} \psi(\mathbf{v}) \otimes \xi(\mathbf{v}):$$

“ \leq ” is immediate; for “ \geq ” we argue as follows:

$$\begin{aligned} \psi(\mathbf{v}) \otimes \xi(\mathbf{v}) &= \psi(\mathbf{v}) \otimes \text{hom}(\xi(\dot{k}), \xi(\mathbf{v})) \\ &= \psi(\mathbf{v}) \otimes a(\dot{k}, \mathbf{v}) && \text{(by hypothesis)} \\ &\leq \psi(\mathbf{v}) \otimes \text{hom}(\psi(\mathbf{v}), \psi(\dot{k})) && \text{(by (5))} \\ &\leq \psi(\dot{k}). \end{aligned}$$

$$(2\text{nd}) \quad \forall \mathbf{v} \in TV \ \text{hom}_\xi(\mathbf{v}, \psi(\dot{k})) = \bigvee_{\mathbf{u} \in TV} \text{hom}(\xi(\mathbf{v}), \xi(\mathbf{u})) \otimes \psi(\mathbf{u}):$$

To check “ \geq ” we just observe that

$$\xi(\mathbf{v}) \otimes (\text{hom}(\xi(\mathbf{v}), \xi(\mathbf{u})) \otimes \psi(\mathbf{u})) \leq \xi(\mathbf{u}) \otimes \psi(\mathbf{u}) \leq \psi(\dot{k}).$$

For “ \leq ”, first note that

$$\psi(\dot{k}) \otimes \varphi(\xi(\mathbf{u})) \leq \text{hom}(\xi(\dot{k}), \xi(\mathbf{u})) = \text{hom}(k, \xi(\mathbf{u})) = \xi(\mathbf{u})$$

from which follows

$$(7) \quad \xi(T\varphi(\mathbf{u})) \leq \varphi(\xi(\mathbf{u})) \leq \text{hom}(\psi(\dot{k}), \xi(\mathbf{u})).$$

From that we conclude that

$$\begin{aligned} \text{hom}(\xi(\mathbf{v}), \psi(\dot{k})) &\leq \text{hom}(\xi(\mathbf{v}), \psi(\dot{k})) \otimes \bigvee_{\mathbf{u} \in TV} \psi(\mathbf{u}) \otimes \xi(T\varphi(\mathbf{u})) && \text{(by (6b))} \\ &= \bigvee_{\mathbf{u} \in TV} \text{hom}(\xi(\mathbf{v}), \psi(\dot{k})) \otimes \psi(\mathbf{u}) \otimes \xi(T\varphi(\mathbf{u})) \\ &\leq \bigvee_{\mathbf{u} \in TV} \text{hom}(\xi(\mathbf{v}), \psi(\dot{k})) \otimes \text{hom}(\psi(\dot{k}), \xi(\mathbf{u})) \otimes \psi(\mathbf{u}) && \text{(by (7))} \\ &\leq \bigvee_{\mathbf{u} \in TV} \text{hom}(\xi(\mathbf{v}), \xi(\mathbf{u})) \otimes \psi(\mathbf{u}). \end{aligned}$$

$$(3\text{rd}) \quad \forall \mathbf{v} \in TV \ \psi(\mathbf{v}) = \bigvee_{\mathbf{u} \in TV} a(\mathbf{v}, \mathbf{u}) \otimes \psi(\mathbf{u}):$$

For “ \leq ” take $\mathbf{u} = \mathbf{v}$; for “ \geq ” we use (5): $a^\circ(\mathbf{u}, \mathbf{v}) \otimes \psi(\mathbf{u}) \leq \text{hom}(\psi(\mathbf{u}), \psi(\mathbf{v})) \otimes \psi(\mathbf{u}) \leq \psi(\mathbf{v})$. \square

Lemma. Assume that $T1 = 1$. Then $T(\text{hom}_\xi) \cdot m_V^\circ = \xi^\circ \cdot \text{hom} \cdot \xi$ provided that

$$(8) \quad \xi \cdot T_o \text{hom}(u, _) \leq \text{hom}(u, _) \cdot \xi,$$

$$\begin{array}{ccc} T\mathbf{V} & \xrightarrow{T_o(\text{hom}(u, _))} & T\mathbf{V} \\ \xi \downarrow & \leq & \downarrow \xi \\ \mathbf{V} & \xrightarrow{\text{hom}(u, _)} & \mathbf{V} \end{array}$$

for each $u \in \mathbf{V}$. The inequality (8) is surely true if $\text{hom}(u, _)$ preserves non-empty suprema.

Proof. First observe that

$$\begin{aligned} T(\text{hom}_\xi) \cdot m_V^\circ &= T \text{hom} \cdot T\xi \cdot m_V^\circ \\ &\leq \xi^\circ \cdot \text{hom} \cdot \xi \cdot T\xi \cdot m_V^\circ \quad (\text{because } \xi \text{ is a } \mathbf{V}\text{-functor, by Proposition 3.2}) \\ &= \xi^\circ \cdot \text{hom} \cdot \xi \cdot m_V \cdot m_V^\circ \\ &\leq \xi^\circ \cdot \text{hom} \cdot \xi. \end{aligned}$$

On the other hand, for $\mathbf{u}, \mathbf{v} \in T\mathbf{V}$, we have

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &\geq T \text{hom}_\xi(\dot{\mathbf{u}}, \mathbf{v}) \\ &= T \text{hom}(T_o \xi(\dot{\mathbf{u}}), \mathbf{v}) \\ &= T \text{hom}(\xi(\dot{\mathbf{u}}), \mathbf{v}) \\ &= \xi \cdot T_o \text{hom} \cdot T_o \langle \xi(\mathbf{u}), 1_V \rangle(\mathbf{v}) \\ &\geq \text{hom}(\xi(\mathbf{u}), _) \cdot \xi(\mathbf{v}) = \text{hom}(\xi(\mathbf{u}), \xi(\mathbf{v})). \end{aligned} \quad (*)$$

To see (*), just observe that $T_o \langle \xi(\mathbf{u}), 1_V \rangle(\mathbf{v})$ is the only element of $T(\mathbf{V} \times \mathbf{V})$ which projects to both $\xi(\dot{\mathbf{u}})$ and \mathbf{v} . Assume now that $\text{hom}(u, _)$ preserves non-empty suprema and let $u \in \mathbf{V}$ and $\mathbf{u} \in T\mathbf{V}$. Then

$$\begin{aligned} \text{hom}(u, \xi(\mathbf{u})) &= \text{hom}(u, \bigvee \{v \in \mathbf{V} \mid \mathbf{u} \in T(\uparrow v)\}) \\ &= \bigvee \{\text{hom}(u, v) \mid v \in \mathbf{V}, \mathbf{u} \in T(\uparrow v)\} \\ &\leq \bigvee \{v \in \mathbf{V} \mid T_o \text{hom}(u, _)(\mathbf{u}) \in T(\uparrow v)\}. \end{aligned} \quad \square$$

4 A Yoneda Lemma for (\mathbb{T}, \mathbf{V}) -categories

4.1 Function spaces. In this section we wish to obtain the analogue result to Corollary 1.5 in the setting of (\mathbb{T}, \mathbf{V}) -categories. This in turn requires an understanding of the right adjoint to $X \otimes _ : (\mathbb{T}, \mathbf{V})\text{-Cat} \rightarrow (\mathbb{T}, \mathbf{V})\text{-Cat}$, a problem studied in [16]. From there we import the following result.

Proposition. Let $X = (X, a)$ be a (\mathbb{T}, \mathbf{V}) -category. Then $X \otimes _ : (\mathbb{T}, \mathbf{V})\text{-Cat} \rightarrow (\mathbb{T}, \mathbf{V})\text{-Cat}$ has a right adjoint $_ \overset{X}{\dashv}$ provided that $a \cdot Ta = a \cdot m_X$.

Certainly, each (Eilenberg-Moore) \mathbb{T} -algebra, considered as a (\mathbb{T}, \mathbf{V}) -category, satisfies the condition above. Moreover, the (\mathbb{T}, \mathbf{V}) -categorical structure (X, a) induced by any \mathbf{V} -category $X = (X, r)$ (see 2.3) satisfies this condition if $Te_X \cdot e_X = m_X^\circ \cdot e_X$.

Let $X = (X, a)$ and (Y, b) be (\mathbb{T}, \mathbb{V}) -categories, and assume that $a \cdot Ta = a \cdot m_X$. Then Y^X has as underlying set

$$\{h : (X, a) \otimes (1, p) \rightarrow (Y, b) \mid h \text{ is a } (\mathbb{T}, \mathbb{V})\text{-functor}\},$$

thanks to the bijection (with $P = (1, p)$)

$$\frac{X \otimes P \rightarrow Y}{P \rightarrow Y^X}.$$

The structure $\llbracket a, b \rrbracket$ on Y^X is the largest structure making the evaluation map

$$\text{ev} : X \otimes Y^X \rightarrow Y, (x, h) \mapsto h(x)$$

a (\mathbb{T}, \mathbb{V}) -functor: for $\mathfrak{p} \in T(Y^X)$ and $h \in Y^X$ we have

$$\llbracket a, b \rrbracket(\mathfrak{p}, h) = \bigvee \left\{ v \in \mathbb{V} \mid \forall \mathfrak{q} \in T\pi_{Y^X}^{-1}(\mathfrak{p}), x \in X \ a(T\pi_X(\mathfrak{q}), x) \otimes v \leq b(\text{TeV}(\mathfrak{q}), h(x)) \right\}.$$

4.2 The Yoneda Embedding. By Theorem 3.3, the bimodule $a : X \multimap X$ gives rise to (\mathbb{T}, \mathbb{V}) -functors $a : |X| \otimes X \rightarrow \mathbb{V}$ and $a : X^{\text{op}} \otimes X \rightarrow \mathbb{V}$. According to the considerations above, we obtain a (\mathbb{T}, \mathbb{V}) -functor $y = \ulcorner a \urcorner : X \rightarrow \mathbb{V}^{|X|}$. Our next result should be compared with Corollary 1.5.

Theorem (Yoneda). *Let $X = (X, a)$ be a (\mathbb{T}, \mathbb{V}) -category. Then the following assertions hold.*

- (a) *For all $\mathfrak{x} \in TX$ and $\varphi \in \mathbb{V}^{|X|}$, $\llbracket m_X, \text{hom}_\xi \rrbracket(Ty(\mathfrak{x}), \varphi) \leq \varphi(\mathfrak{x})$.*
- (b) *Let $\varphi \in \mathbb{V}^{|X|}$. Then*

$$\forall \mathfrak{x} \in TX \ \varphi(\mathfrak{x}) \leq \llbracket m_X, \text{hom}_\xi \rrbracket(Ty(\mathfrak{x}), \varphi) \iff \varphi : X^{\text{op}} \rightarrow \mathbb{V} \text{ is a } (\mathbb{T}, \mathbb{V})\text{-functor}.$$

Proof. Note that the diagrams

$$\begin{array}{ccc} & & \mathbb{V} \\ & \nearrow a & \uparrow \text{ev} \\ TX \times X & \xrightarrow{1_{TX} \times y} & TX \times \mathbb{V}^{|X|} \end{array} \qquad \begin{array}{ccc} TX \times X & \xrightarrow{1_{TX} \times y} & TX \times \mathbb{V}^{|X|} \\ \pi_2 \downarrow & & \downarrow \pi_2 \\ X & \xrightarrow{y} & \mathbb{V}^{|X|} \end{array}$$

commute, where the right-hand side diagram is even a pullback. Let $\mathfrak{x} \in TX$ and $\varphi \in \mathbb{V}^{|X|}$.

Hence

$$\begin{aligned}
& \llbracket m_X, \text{hom}_\xi \rrbracket (T_o y(\mathfrak{x}), \varphi) \\
&= \bigvee \{v \in \mathbf{V} \mid \forall \mathfrak{y} \in TX, \mathfrak{y} \in m_X^{-1}(\mathfrak{y}), \mathfrak{W} \in T(TX \times \mathbf{V}^{|X|}) \\
&\quad (T_o \pi_1(\mathfrak{W}) = \mathfrak{y} \ \& \ T_o \pi_2(\mathfrak{W}) = T_o y(\mathfrak{x})) \Rightarrow v \leq \text{hom}(\xi \cdot T_o \text{ev}(\mathfrak{W}), \varphi(\mathfrak{y}))\} \\
&= \bigvee \{v \in \mathbf{V} \mid \forall \mathfrak{y} \in TX, \mathfrak{y} \in m_X^{-1}(\mathfrak{y}), \mathfrak{W} \in T(TX \times X) \\
&\quad (T_o \pi_1(\mathfrak{W}) = \mathfrak{y} \ \& \ T_o \pi_2(\mathfrak{W}) = \mathfrak{x}) \Rightarrow v \leq \text{hom}(\xi \cdot T_o a(\mathfrak{W}), \varphi(\mathfrak{y}))\} \\
&= \bigvee \{v \in \mathbf{V} \mid \forall \mathfrak{y} \in TX, \mathfrak{y} \in m_X^{-1}(\mathfrak{y}) \ v \leq \bigwedge_{\substack{\mathfrak{W} \in T(TX \times X) \\ T_o \pi_1(\mathfrak{W}) = \mathfrak{y} \\ T_o \pi_2(\mathfrak{W}) = \mathfrak{x}}} \text{hom}(\xi \cdot T_o a(\mathfrak{W}), \varphi(\mathfrak{y}))\} \\
&= \bigvee \{v \in \mathbf{V} \mid \forall \mathfrak{y} \in TX, \mathfrak{y} \in m_X^{-1}(\mathfrak{y}) \ v \leq \text{hom}(\bigvee_{\substack{\mathfrak{W} \in T(TX \times X) \\ T_o \pi_1(\mathfrak{W}) = \mathfrak{y} \\ T_o \pi_2(\mathfrak{W}) = \mathfrak{x}}} \xi \cdot T_o a(\mathfrak{W}), \varphi(\mathfrak{y}))\} \\
&= \bigvee \{v \in \mathbf{V} \mid \forall \mathfrak{y} \in TX, \mathfrak{y} \in m_X^{-1}(\mathfrak{y}) \ v \leq \text{hom}(Ta(\mathfrak{y}, \mathfrak{x}), \varphi(\mathfrak{y}))\} \\
&= \bigvee \{v \in \mathbf{V} \mid \forall \mathfrak{y} \in TX \ Ta \cdot m_X^\circ(\mathfrak{y}, \mathfrak{x}) \otimes v \leq \varphi(\mathfrak{y})\}.
\end{aligned}$$

In particular we have

$$v = k \otimes v \leq Ta \cdot m_X^\circ(\mathfrak{x}, \mathfrak{x}) \otimes v \leq \varphi(\mathfrak{x}),$$

which proves (a). On the other hand, $\varphi : (TX, Ta \cdot m_X^\circ) \rightarrow (\mathbf{V}, \text{hom})$ is a \mathbf{V} -functor if and only if

$$Ta \cdot m_X^\circ(\mathfrak{y}, \mathfrak{x}) \otimes \varphi(\mathfrak{x}) \leq \varphi(\mathfrak{y})$$

for all $\mathfrak{y}, \mathfrak{x} \in TX$, from which follows (b). \square

We put $\hat{X} = (\hat{X}, \hat{a})$ where $\hat{X} := \{\varphi \in \mathbf{V}^{|X|} \mid \varphi : X^{\text{op}} \rightarrow \mathbf{V} \text{ is a } (\mathbb{T}, \mathbf{V})\text{-functor}\}$ considered as a subcategory of $\mathbf{V}^{|X|}$. Recall that $a : X^{\text{op}} \otimes X \rightarrow \mathbf{V}$ is a (\mathbb{T}, \mathbf{V}) -functor, and therefore $a(-, x) : X^{\text{op}} \otimes P \rightarrow \mathbf{V}$ is a (\mathbb{T}, \mathbf{V}) -functor for each $x \in X$. If $T1 = 1$, then $P = (1, p) = (1, k)$ is the neutral element for \otimes and we can restrict the Yoneda functor y to \hat{X} .

Corollary. *Assume $T1 = 1$. Then the Yoneda functor $y : X \rightarrow \hat{X}$ is full and faithful.*

If $Te_X \cdot e_X = m_X^\circ \cdot e_X$, we might also consider the transpose $y_0 = \lceil a \rceil : X \rightarrow \mathbf{V}^{X^{\text{op}}}$ of $a : X^{\text{op}} \otimes X \rightarrow \mathbf{V}$ as below. However, unlike the situation for \mathbf{V} -categories, in general we do not have $\hat{X} \cong \mathbf{V}^{X^{\text{op}}}$ (see example below).

Proposition (Yoneda II). *Assume that $Te_X \cdot e_X = m_X^\circ \cdot e_X$ and let $X = (X, a)$ be a (\mathbb{T}, \mathbf{V}) -category. Then the following assertions hold.*

- (a) *For all $\mathfrak{x} \in TX$ and $\varphi \in \mathbf{V}^{X^{\text{op}}}$, $\llbracket a^{\text{op}}, \text{hom}_\xi \rrbracket (Ty_0(\mathfrak{x}), \varphi) \geq \varphi(\mathfrak{x})$.*
- (b) *Let $\mathfrak{x} \in TX$ such that $Ta \cdot e_{TX}(\mathfrak{x}, \mathfrak{x}) \geq k$. Then $\llbracket a^{\text{op}}, \text{hom}_\xi \rrbracket (Ty_0(\mathfrak{x}), \varphi) \leq \varphi(\mathfrak{x})$.*

Proof. Let $\mathfrak{x} \in TX$ and $\varphi \in V^{X^{\text{op}}}$. As above, we obtain

$$\begin{aligned}
& \llbracket a^{\text{op}}, \text{hom}_{\xi} \rrbracket (T_0 y(\mathfrak{x}), \varphi) \\
&= \bigvee \{v \in V \mid \forall \mathfrak{y} \in TX, \mathfrak{y} \in T^2 X, \mathfrak{W} \in T(TX \times V^{X^{\text{op}}}) \\
&\quad (T_0 \pi_1(\mathfrak{W}) = \mathfrak{y} \ \& \ T_0 \pi_2(\mathfrak{W}) = T_0 y(\mathfrak{x})) \Rightarrow a^{\text{op}}(\mathfrak{y}, \mathfrak{x}) \otimes v \leq \text{hom}(\xi \cdot T_0 \text{ev}(\mathfrak{W}), \varphi(\mathfrak{y}))\} \\
&= \bigvee \{v \in V \mid \forall \mathfrak{y} \in TX, \mathfrak{y} \in T^2 Y, Ta(\mathfrak{y}, \mathfrak{x}) \otimes a^{\text{op}}(\mathfrak{y}, \mathfrak{y}) \otimes v \leq \varphi(\mathfrak{y})\} \\
&= \bigvee \{v \in V \mid \forall \mathfrak{y} \in TX \quad a^{\text{op}} \cdot Ta^{\circ}(\mathfrak{x}, \mathfrak{y}) \otimes v \leq \varphi(\mathfrak{y})\}.
\end{aligned}$$

Furthermore, we have

$$a^{\text{op}} \cdot Ta^{\circ} = e_{TX}^{\circ} \cdot Tm_X^{\circ} \cdot TTa^{\circ} \cdot Ta^{\circ} = e_{TX}^{\circ} \cdot Ta^{\circ} \leq m_X \cdot Ta^{\circ}.$$

Hence $\varphi(\mathfrak{x}) \leq \llbracket a^{\text{op}}, \text{hom}_{\xi} \rrbracket (T_0 y_0(\mathfrak{x}), \varphi)$ and, if $k \leq Ta \cdot e_{TX}(\mathfrak{x}, \mathfrak{x}) = a^{\text{op}} \cdot Ta^{\circ}(\mathfrak{x}, \mathfrak{x})$, we also have $\llbracket a^{\text{op}}, \text{hom}_{\xi} \rrbracket (T_0 y_0(\mathfrak{x}), \varphi) \leq \varphi(\mathfrak{x})$. \square

Example. Unlike y , the functor y_0 does not need to be full and faithful. In fact, consider $X = \mathbb{N}$ as a $(\mathbb{U}, 2)$ -category, i.e. a topological space, equipped with the discrete topology $a = e_{\mathbb{N}}^{\circ}$. Then \mathbb{N}^{op} is the discrete space $\mathbb{N}^{\text{op}} = (U\mathbb{N}, e_{U\mathbb{N}}^{\circ})$. Let \mathfrak{x} be a free ultrafilter on \mathbb{N} . Then, for each $\mathfrak{y} \in U\mathbb{N}$, $a^{\text{op}} \cdot Ua^{\circ}(\mathfrak{x}, \mathfrak{y}) = e_{\mathbb{N}}^{\circ} \cdot Ue_{\mathbb{N}}(\mathfrak{x}, \mathfrak{y}) = \text{false}$ and therefore $Uy_0(\mathfrak{x}) \rightarrow \varphi$ for each $\varphi \in 2^{\mathbb{N}^{\text{op}}}$. On the other hand, for $\varphi = a(_, x)$ (x any element of \mathbb{N}) we have $\varphi(\mathfrak{x}) = \text{false}$. In particular we see that $y_0 : \mathbb{N} \rightarrow 2^{\mathbb{N}^{\text{op}}}$ is not full and faithful.

5 Examples

5.1 Ordered sets. Recall that $2\text{-Cat} = \text{Ord}$. Given an ordered set $X = (X, \leq)$, by Theorem 1.5 we have that a bimodule $\phi : 1 \multimap X$ is an order-preserving map $\phi : X \rightarrow 2$, while a bimodule $\psi : X \multimap 1$ is an order-preserving map $X^{\text{op}} \rightarrow 2$. We can identify φ with the upclosed set $A = \varphi^{-1}(\text{true})$ and ψ with the downclosed set $B = \psi^{-1}(\text{true})$. Under this identification, $\varphi \dashv \psi$ translates to

$$A \cap B \neq \emptyset \quad \text{and} \quad \forall x \in A \ \forall y \in B \ y \leq x.$$

Then any $z \in A \cap B$ is simultaneously a smallest element of A and a largest element of B , therefore z represents $\varphi \dashv \psi$. Hence, by Proposition 1.6, each ordered set is Lawvere-complete. Note that the proof of Proposition 1.6 makes use of the Axiom of Choice; in fact, as pointed out in [6], here we have no choice.

Theorem. *The following assertions are equivalent.*

- (i) *Each ordered set is Lawvere-complete.*
- (ii) *The Axiom of Choice.*

Proof. To see (ii) \Rightarrow (i), let $f : X \rightarrow Y$ be a surjective map. We equip Y with the discrete order Δ_Y and X with the kernel relation of f ; then we have not only $f_* \dashv f^*$ but also $f^* \dashv f_*$. Hence there exists some $g : Y \rightarrow X$ which represents $f^* \dashv f_*$, and such g necessarily satisfies $f \cdot g = 1_Y$. \square

5.2 Metric spaces. For $\mathbf{V} = \mathbf{P}_+$ we have $\mathbf{P}_+\text{-Cat} \cong \mathbf{Met}$. Let $X = (X, d)$ be a metric space. A pair of adjoint bimodules $\varphi \dashv \psi$ corresponds to a pair of non-expansive maps $\varphi : X \rightarrow \mathbf{P}_+$ and $\psi : X^{\text{op}} \rightarrow \mathbf{P}_+$ which satisfy

$$\inf_{x \in X} \varphi(x) + \psi(x) = 0 \quad \text{and} \quad \forall x, y \in X \quad \psi(y) + \varphi(x) \geq d(y, x).$$

As observed in [18], pairs of adjoint bimodules on X correspond exactly to equivalence classes of Cauchy sequences. To see this, recall first that a sequence $s = (x_n)_{n \in \mathbb{N}}$ is called *Cauchy* if

$$\inf_{k \in \mathbb{N}} \sup_{n, n' \geq k} d(x_n, x_{n'}) = 0.$$

Given a Cauchy sequence $s = (x_n)_{n \in \mathbb{N}}$, we have

$$\inf_{m \in \mathbb{N}} \sup_{n \geq m} d(x_n, x) = \sup_{m \in \mathbb{N}} \inf_{n \geq m} d(x_n, x)$$

as well as

$$\inf_{m \in \mathbb{N}} \sup_{n \geq m} d(x, x_n) = \sup_{m \in \mathbb{N}} \inf_{n \geq m} d(x, x_n),$$

and s gives rise to non-expansive maps

$$\begin{aligned} \varphi_s : X &\rightarrow \mathbf{P}_+ & \text{and} & & \psi_s : X^{\text{op}} &\rightarrow \mathbf{P}_+ \\ x &\mapsto \sup_{m \in \mathbb{N}} \inf_{n \geq m} d(x_n, x) & & & x &\mapsto \sup_{m \in \mathbb{N}} \inf_{n \geq m} d(x, x_n) \end{aligned}$$

One sees easily that $\varphi_s \dashv \psi_s$; moreover, two equivalent Cauchy sequences induce the same maps.

On the other hand, given an adjunction $\varphi \dashv \psi$, we may define $s = (x_n)_{n \in \mathbb{N}}$ such that $\varphi(x_n) + \psi(x_n) \leq \frac{1}{n}$, hence $d(x_n, x_m) \leq \frac{1}{n} + \frac{1}{m}$, and therefore s is a Cauchy sequence. Any two such sequences are equivalent. Furthermore, $\varphi \leq \varphi_s$ as well as $\psi \leq \psi_s$, therefore, since $\varphi \dashv \psi$ and $\varphi_s \dashv \psi_s$, we have even equality. Starting with a Cauchy sequence $s = (x_n)_{n \in \mathbb{N}}$, then for any sequence $t = (y_n)_{n \in \mathbb{N}}$ chosen for $\varphi \dashv \psi$ as above we have

$$\inf_{m \in \mathbb{N}} \inf_{k \in \mathbb{N}} \sup_{n \geq k} d(x_n, y_m) = 0 \quad \text{and} \quad \inf_{m \in \mathbb{N}} \inf_{k \in \mathbb{N}} \sup_{n \geq k} d(y_m, x_n) = 0,$$

hence s and t are equivalent. Finally, $s = (x_n)_{n \in \mathbb{N}}$ converges to x (i.e. s is equivalent to $(x)_{n \in \mathbb{N}}$) if and only if $\varphi_s \dashv \psi_s$ is represented by x .

The same argumentation applies also to the case $\mathbf{V} = \mathbf{P}_{\max}$: pairs of adjoint bimodules $\varphi \dashv \psi : 1 \rightarrow X$ with X an ultrametric space correspond precisely to Cauchy sequences in X , and convergence to representability.

Remark. A notion of non-symmetric Cauchy-sequence was introduced and studied in [3].

5.3 Topological spaces. We consider now $\mathbb{T} = \mathbb{U} = (U, e, m)$ the *ultrafilter monad* and $\mathbf{V} = 2$. As already stated, Proposition 3.1 describes our extension U in terms of $U_o : \mathbf{Set} \rightarrow \mathbf{Set}$ (for a direct calculation of U , see [7, Example 6.4]). Then $(\mathbb{U}, 2)\text{-Cat} = \mathbf{Top}$, as it was shown by Barr [1]. By Theorem 3.3, a bimodule $\varphi : U1 \multimap X$ from the one-element space 1 to a topological space X is essentially a continuous map $\varphi : X \rightarrow 2$ from X into the Sierpinski space 2 , hence we can identify it with a closed subset $A \subseteq X$. A bimodule $\psi : UX \multimap 1$ is basically a map $\psi : UX \rightarrow 2$ such that $\mathcal{A} = \psi^{-1}(\text{true})$ is closed in $|X|$ as well as in X^{op} . The topology on $|X|$

is given by the Zariski closure, that is, $\mathfrak{x} \in UX$ is in the closure of $\mathcal{M} \subseteq UX$ if $\bigcap \mathcal{M} \subseteq \mathfrak{x}$. To understand the structure of X^{op} , observe first that the order on $M^\circ X$ is given by

$$\begin{aligned} \mathfrak{x} \leq \mathfrak{y} &\iff \exists \mathfrak{X} \in U^2 X \ m_X(\mathfrak{X}) = \mathfrak{x} \text{ and } \mathfrak{X} \rightarrow \mathfrak{y} \\ &\iff \forall A \in \mathfrak{x}, B \in \mathfrak{y} \ \exists \mathfrak{a} \in UA, y \in B \ \mathfrak{a} \rightarrow y \\ &\iff \forall A \in \mathfrak{x}, B \in \mathfrak{y} \ \overline{A} \cap B \neq \emptyset. \end{aligned}$$

Denoting the filter base $\{\overline{A} \mid A \in \mathfrak{x}\}$ by $\overline{\mathfrak{x}}$, we have

$$\mathfrak{x} \leq \mathfrak{y} \iff \overline{\mathfrak{x}} \subseteq \mathfrak{y}.$$

Hence bimodules $\psi : UX \multimap 1$ can be identified with subsets $\mathcal{A} \subseteq UX$ which are Zariski closed and down-closed for the order described above. Now $\varphi \dashv \psi$ translates to

$$\exists \mathfrak{x}_0 \in UX \ \mathfrak{x}_0 \in \mathcal{A} \ \& \ A \in \mathfrak{x}_0 \qquad \text{and} \qquad \forall \mathfrak{x} \in \mathcal{A}, x \in A \ \mathfrak{x} \rightarrow x.$$

Clearly, each $\mathfrak{x} \in \mathcal{A}$ converges to all points of A . On the other hand, for any $\mathfrak{x} \in UX$ with this property we have $\mathfrak{x} \leq \mathfrak{x}_0$ and therefore $\mathfrak{x} \in \mathcal{A}$. We conclude that

$$\mathcal{A} = \{\mathfrak{x} \in UX \mid \forall x \in A \ \mathfrak{x} \rightarrow x\}.$$

A closed subset $A \subseteq X$ admits an ultrafilter $\mathfrak{x}_0 \in UA$ which converges to all $x \in A$ if and only if $\{V \subseteq X \mid V \text{ open}, V \cap A \neq \emptyset\}$ is a filter base. In the language of closed sets this is expressed by saying that A is not the union of two proper closed subsets, i.e. A is *irreducible*. Finally, ψ (and hence φ) is representable if and only if \mathfrak{x}_0 can be chosen principal, that is, if and only if there exists some point $x_0 \in A$ which converges to all $x \in A$. In conclusion, we have

Theorem. *The following assertions are equivalent for a topological space X .*

- (i) X is Lawvere-complete.
- (ii) Each irreducible closed subset $A \subseteq X$ is of the form $A = \overline{\{x\}}$ for some $x \in A$, i.e. X is weakly sober.

5.4 Approach spaces. Recall that $\mathbf{App} = (\mathbb{U}, \mathbf{P}_+)$ -Cat is the category of approach spaces and non-expansive maps. We fix an approach space $X = (X, a)$. As above, a bimodule $\varphi : U1 \multimap X$ is a non-expansive map $\varphi : X \rightarrow \mathbf{P}_+$, by Theorem 3.3. There is a bijective correspondence between maps $\varphi : X \rightarrow \mathbf{P}_+$ and families $(A_v)_{v \in \mathbf{P}_+}$ of subsets $A_v \subseteq X$ satisfying

$$(9) \qquad A_v = \bigcap_{u > v} A_u,$$

where $\varphi \mapsto (\varphi^{-1}([0, v]))_{v \in \mathbf{P}_+}$ and a family $(A_v)_{v \in \mathbf{P}_+}$ defines the map $x \mapsto \inf\{v \in \mathbf{P}_+ \mid x \in A_v\}$. Under this bijection, non-expansive maps correspond precisely to those families $(A_v)_{v \in \mathbf{P}_+}$ which satisfy in addition

$$(10) \qquad \forall u, v \in \mathbf{P}_+ \ \forall x \in X \ (d(A_u, x) \leq v \Rightarrow x \in A_{u+v}),$$

where $d(A, x) = \inf\{a(\mathfrak{x}, x) \mid \mathfrak{x} \in UA\}$.

We may think of the family $A = (A_v)_{v \in \mathbf{P}_+}$ satisfying (9) as a *variable set*⁶; we call A *closed* if it satisfies (10). Now it is not difficult to see that a right adjoint $\psi : X \multimap 1$ to $\varphi : 1 \multimap X$ is determined by the variable set $\mathcal{A} = (\mathcal{A}_v)_{v \in \mathbf{P}_+}$ given by

$$\mathcal{A}_v = \{\mathbf{r} \in UX \mid \forall u \in \mathbf{P}_+ \ \forall x \in A_u \ a(\mathbf{r}, x) \leq u + v\},$$

for each $v \in \mathbf{P}_+$. Furthermore, given $\varphi : 1 \multimap X$, the variable set \mathcal{A} defined as above corresponds to a right adjoint of φ if and only if

$$(11) \quad \forall u \in \mathbf{P}_+ \ (u > 0 \Rightarrow UA_u \cap \mathcal{A}_u \neq \emptyset).$$

In analogy to the situation in **Top**, we call a variable set A *irreducible* if it satisfies (11). Finally, we remark that the bimodule $\varphi : 1 \multimap X$ is represented by $x \in X$ precisely if the corresponding variable set A is of the form

$$A_v = \{y \in X \mid d(x, y) \leq v\},$$

for each $v \in \mathbf{P}_+$. Naturally, we say that such a variable set is *representable* (by x).

Theorem. *The following assertions are equivalent for an approach space X .*

- (i) X is Lawvere-complete.
- (ii) Each irreducible closed variable set A is representable.

We point out that this setting satisfies the conditions of Theorem 3.4, therefore it assures that \mathbf{P}_+ is Lawvere-complete.

Remark. The notion of approach frame and its connection with approach spaces was recently studied by Christophe Van Olmen in his PhD thesis [23]. In particular, the concept of *sober approach space* as a fixed point of the dual adjunction between **App** and the category **AFrm** of approach frames and homomorphisms was introduced. As confirmed by the author of [23], these are precisely the approach spaces where each irreducible closed variable set is uniquely representable.

6 Appendix: Lawvere-complete quasi-uniform spaces

6.1 Cauchy-complete quasi-uniform spaces. We recall that a *quasi-uniformity* U on a set X is a set of binary relations on X such that:

$$\forall u \in U \ \Delta \subseteq u;$$

$$\forall u \in U \ \exists v \in U \ v \cdot v \subseteq u.$$

The pair (X, U) is called a *quasi-uniform space*; it is a *uniform space* when, for all $u \in U$, $u^{-1} \in U$. Given quasi-uniform spaces (X, U) and (Y, V) , a map $f : X \rightarrow Y$ is *uniformly continuous* if

$$\forall v \in V \ \exists u \in U \ \forall x, y \in X \ xuy \Rightarrow f(x)v f(y).$$

⁶In fact, we may consider $A : \mathbf{P}_+ \rightarrow \mathbf{Set}$ as a sheaf where, for each $u \in \mathbf{P}_+$, $\{v < u\}$ is a cover of u .

Definition. Let (X, U) be a quasi-uniform space.

1. A pair $(\mathfrak{f}, \mathfrak{g})$ is a *filter* in (X, U) if \mathfrak{f} and \mathfrak{g} are filters in X such that

$$\forall F \in \mathfrak{f} \ \forall G \in \mathfrak{g} \ F \cap G \neq \emptyset.$$

2. A filter $(\mathfrak{f}, \mathfrak{g})$ in (X, U) is a *Cauchy filter* if

$$\forall u \in U \ \exists F \in \mathfrak{f} \ \exists G \in \mathfrak{g} \ F \times G \subseteq X_u := \{(x, x') \mid x u x'\}.$$

3. A filter $(\mathfrak{f}, \mathfrak{g})$ in (X, U) *converges to* $x_0 \in X$ if

$$\forall u \in U \ \exists F \in \mathfrak{f} \ \exists G \in \mathfrak{g} \ F \times G \subseteq X_{-ux_0} \times X_{x_0u-},$$

where $X_{-ux_0} := \{x \in X \mid x u x_0\}$ and $X_{x_0u-} := \{x \in X \mid x_0 u x\}$.

Lemma. Given a quasi-uniformity U in X and $x_0 \in X$, the neighbourhood filter of x_0

$$(\{X_{-ux_0} \mid u \in U\}, \{X_{x_0u-} \mid u \in U\})$$

is a minimal Cauchy filter in (X, U) .

Proposition. For a quasi-uniform space (X, U) , the following conditions are equivalent.

- (i) Every Cauchy filter converges.
- (ii) Every minimal Cauchy filter is the neighbourhood filter of a point x_0 .

A quasi-uniform space is said to be *Cauchy-complete* if it satisfies any of the equivalent conditions of the Proposition.

For further information see [12] and [13].

6.2 Quasi-uniform spaces as lax algebras. In order to describe quasi-uniform spaces as lax algebras, we turn back to the setting described in [7] and substitute the bicategory $\mathbf{V}\text{-Mat}$ of 2.1 by the bicategory \mathbf{Y} having sets as objects and (possibly improper) filters in $\mathbf{Rel}(X, Y)$ as morphisms, where \mathbf{Rel} is the bicategory of relations. The composition of two filters $R : X \multimap Y$ and $S : Y \multimap Z$ is the filter obtained by pointwise composition of relations $R \cdot S = \{s \cdot r \mid s \in S \text{ and } r \in R\}$, while $R \leq R'$ whenever $R' \subseteq R$ (as sets).

We define a *lax algebra* now exactly like a \mathbf{V} -category: it is a \mathbf{Y} -morphism $A : X \multimap X$ such that

$$1_X \leq A \quad \text{and} \quad A \cdot A \leq A,$$

or, equivalently,

$$\forall x \in X \ \forall a \in A \ x a x \quad \text{and} \quad \forall a \in A \ \exists a' \in A \ a' \cdot a' \leq a.$$

A *lax morphism* $f : (X, A) \rightarrow (Y, B)$ between lax algebras is a map $f : X \rightarrow Y$ such that $f \cdot A \leq B \cdot f$, i.e.

$$\forall b \in B \ \exists a \in A \ f \cdot a \leq b \cdot f.$$

It was shown in [7, Theorem 3.6] that this category of lax algebras and lax morphisms is equivalent to the category of quasi-uniform spaces and uniformly continuous maps.

6.3 Adjoint pairs of bimodules in quasi-uniform spaces. A bimodule $\Psi : (X, A) \multimap (Y, B)$ between lax algebras is a Y -morphism $\Psi : X \multimap Y$ such that $\Psi \cdot A \leq \Psi$ and $B \cdot \Psi \leq \Psi$. As in the context of V -categories, A and B act as identities for the composition with bimodules, so that a pair of bimodules $(\Phi : (Y, B) \multimap (X, A), (\Psi : (X, A) \multimap (Y, B)))$ is an *adjoint pair*, with $\Phi \dashv \Psi$, if $B \leq \Psi \cdot \Phi$ and $\Phi \cdot \Psi \leq A$. As before, every lax morphism $f : (X, A) \rightarrow (Y, B)$ defines a pair of adjoint bimodules $(f_* = B \cdot f : (X, A) \multimap (Y, B), f^* = f^\circ \cdot A : (Y, B) \multimap (X, A))$. It is easy to check that Proposition 2.7 is still valid in this context.

Proposition. *For a lax algebra (X, A) , the following conditions are equivalent.*

- (i) *Each pair of adjoint bimodules $(\Phi : (Y, B) \multimap (X, A)) \dashv (\Psi : (X, A) \multimap (Y, B))$ is induced by a lax morphism $(Y, B) \rightarrow (X, A)$.*
- (ii) *Each pair of adjoint bimodules $(\Phi : 1 \multimap (X, A)) \dashv (\Psi : (X, A) \multimap 1)$ is induced by a lax morphism $1 \rightarrow (X, A)$ (or simply a map).*

Theorem. *For Y -morphisms $\Phi : 1 \multimap X$ and $\Psi : X \multimap 1$, the following conditions are equivalent.*

- (i) $\Phi \dashv \Psi$.
- (ii) $(\{X_{-\psi\star} \mid \psi \in \Psi\}, \{X_{\star\varphi-} \mid \varphi \in \Phi\})$ is a minimal Cauchy filter in (X, A) .

Proof. The conditions $1 \leq \Psi \cdot \Phi$ and $\Phi \cdot \Psi \leq A$ read as

$$\forall \psi \in \Psi \ \exists \varphi \in \Phi \ X_{\star\varphi-} \cap X_{-\psi\star} \neq \emptyset,$$

$$\forall a \in A \ \exists \varphi \in \Phi \ \exists \psi \in \Psi \ X_{-\psi\star} \times X_{\star\varphi-} \subseteq X_a,$$

where the former condition means that $(\{X_{-\psi\star} \mid \psi \in \Psi\}, \{X_{\star\varphi-} \mid \varphi \in \Phi\})$ is a filter, while the latter one means that it is Cauchy.

(i) \Rightarrow (ii): It remains to be shown that this Cauchy filter is minimal. Let $(\mathfrak{f}, \mathfrak{g})$ be a filter contained in it. If $\mathfrak{f} \subsetneq \{X_{-\psi\star} \mid \psi \in \Psi\}$, i.e. if there exists $\psi \in \Psi$ such that $X_{-\psi\star} \not\subseteq \mathfrak{f}$, then there exist $a \in A$ and $\psi' \in \Psi$ with $\psi' \cdot a = \psi$, because ψ is a bimodule, hence a and ψ' are such that

$\bigcup_{x' \in X_{-\psi'\star}} X_{-ax'} \not\subseteq \mathfrak{f}$. Therefore

$$\forall F \in \mathfrak{f} \ \exists x \in F \ \forall x' \in X_{-\psi'\star} \ (x, x') \notin X_a.$$

Moreover, since

$$\forall G \in \mathfrak{g} \ G \in \{X_{\star\varphi-} \mid \varphi \in \Phi\} \Rightarrow \forall G \in \mathfrak{g} \ \exists y \in X_{-\psi'\star} \cap G,$$

we obtain

$$\forall F \in \mathfrak{f} \ \forall G \in \mathfrak{g} \ \exists x \in F \ \exists y \in G \ (x, y) \notin X_a,$$

that is $(\mathfrak{f}, \mathfrak{g})$ is not a Cauchy filter.

(ii) \Rightarrow (i): Let $\Phi : 1 \multimap (X, A)$ and $\Psi : (X, A) \multimap 1$ be a pair of bimodules and consider $(\{X_{-\psi\star} \mid \psi \in \Psi\}, \{X_{\star\varphi-} \mid \varphi \in \Phi\})$. We concluded already that the adjunction conditions are

equivalent to this pair being a Cauchy filter. But we did not show yet that Φ and Ψ are bimodules. For any $a \in A$,

$$(\{ \bigcup_{x \in X_{-\psi\star}} X_{-ax} \mid \psi \in \Psi, a \in A \}, \{ \bigcup_{y \in X_{\star\varphi-}} X_{ya-} \mid \varphi \in \Phi, a \in A \})$$

is a Cauchy filter contained in the former one, as we show next. First,

$$\bigcup_{x \in X_{-\psi\star}} X_{-ax} \cap \bigcup_{y \in X_{\star\varphi-}} X_{ya-} \supseteq X_{-\psi\star} \cap X_{\star\varphi-} \neq \emptyset.$$

To prove the other condition, let $a \in A$, and consider $b \in A$ such that $b \cdot b \cdot b \leq a$. There exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that $X_{-\psi\star} \times X_{\star\varphi-} \subseteq X_b$, and this implies that

$$\bigcup_{x \in X_{-\psi\star}} X_{-bx} \times \bigcup_{y \in X_{\star\varphi-}} X_{yb-} \subseteq X_a,$$

since

$$\begin{aligned} x' \in \bigcup_{x \in X_{-\psi\star}} X_{-bx} &\Rightarrow \exists x \in X_{-\psi\star} \ (x', x) \in X_b, \\ y' \in \bigcup_{y \in X_{\star\varphi-}} X_{yb-} &\Rightarrow \exists y \in X_{\star\varphi-} \ (y, y') \in X_b; \end{aligned}$$

hence, since also $(x, y) \in X_b$, we conclude that $(x', y') \in X_a$ as claimed. \square

6.4 Lawvere-complete=Cauchy-complete. It is now straightforward to prove that the two notions of completeness coincide.

Theorem. *For a quasi-uniform space (X, A) the following conditions are equivalent.*

- (i) (X, A) is a Lawvere-complete lax algebra.
- (ii) (X, A) is a Cauchy-complete quasi-uniform space.

Proof. (i) \Rightarrow (ii): Each minimal Cauchy filter in (X, A) defines an adjoint pair of bimodules $(\Phi : 1 \multimap (X, A)) \dashv (\Psi : (X, A) \multimap 1)$, which, by (i), is induced by a map $f : 1 \rightarrow X$, $\star \mapsto x_0$. Hence $\Phi = \{\varphi_b = b \cdot f \mid b \in B\}$ and $\Psi = \{\psi_b = f^\circ \cdot b \mid b \in B\}$. Moreover, $x \in X_{\star\varphi_b-}$ exactly when $b(x_0, x) = \top$, that is $X_{\star\varphi_b-} = X_{x_0b-}$, and $x \in X_{-\psi_b\star}$ exactly when $b(x, x_0) = \top$, which means $X_{-\psi_b\star} = X_{-bx_0}$.

(ii) \Rightarrow (i): Given an adjoint pair of bimodules $(\Phi : 1 \multimap (X, A)) \dashv (\Psi : (X, A) \multimap 1)$, by (ii) the minimal Cauchy filter it induces is the neighbourhood filter of a point x_0 . It is straightforward to check that $\Phi = A \cdot f$ and $\Psi = f^\circ \cdot A$ for $f : 1 \rightarrow X$, $\star \mapsto x_0$. \square

Final remark. The results of this section can be investigated in the more general setting introduced in [9], i.e., in proalgebras; here, for simplicity, we decided to state them only at the level of quasi-uniform structures, which are proalgebras for the identity monad.

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