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Corrigendum to Cleft Extensions of Hopf Algebroids

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Abstract Theorem 2.2 stated a monoidal isomorphism between the comodule categories of two bialgebroids in a Hopf algebroid. The proof of Theorem 2.2 was based on the journal version of Brzeziński (Ann Univ Ferrara Sez VII (NS) 51:15–27, 2005, Theorem 2.6), whose proof turned out to contain an unjustified step. Here we show that all other results in our paper remain valid if we drop unverified Theorem 2.2, and return to an earlier definition of a comodule of a Hopf algebroid that distinguishes between comodules of the two constituent bialgebroids.

Keywords Hopf algebroids · Comodules · Cleft extensions

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Throughout, $\mathscr{H}$ is a Hopf algebroid over base algebras L and R (over a commutative ring k), with structure maps denoted as in Section 2.2.

Although we are not aware of any counterexamples, the unjustified step found in the proof of the journal version of [7, Theorem 2.6] forces us to reformulate our paper without referring to [7, Theorem 2.6] and our derived Theorem 2.2. In this way we correct also the Examples 2.5(2) and 3.11 in [5], where cleft extensions by Hopf

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algebroids were first announced. The reader should be warned that there is a similar error in [3, Proposition 3.1], and that the gap in the proof of Theorem 2.2 seriously affects also the paper [1], see its Corrigendum.

Since there is no relation known any longer between comodule categories of two constituent bialgebroids in a Hopf algebroid, we return to a previous definition of a comodule for a Hopf algebroid in [3, Definition 3.2] and [2, Section 2.2]:

Definition 1 A *right comodule* of a Hopf algebroid $\mathscr{H}$ is a right L-module as well as a right R-module M, together with a right coaction $\rho_R : M \to M \otimes_R H$ of the constituent right bialgebroid $\mathscr{H}_R$ and a right coaction $\rho_L : M \to M \otimes_L H$ of the constituent left bialgebroid $\mathscr{H}_L$, such that ρ_R is an $\mathscr{H}_L$-comodule map and ρ_L is an $\mathscr{H}_R$-comodule map. Explicitly, ρ_R is right L-linear, ρ_L is right R-linear and

$$(M \otimes_R \gamma_L) \circ \rho_R = (\rho_R \otimes_L H) \circ \rho_L \qquad \text{and} \qquad (M \otimes_L \gamma_R) \circ \rho_L = (\rho_L \otimes_R H) \circ \rho_R.$$

Morphisms of $\mathscr{H}$-comodules are meant to be $\mathscr{H}_R$-comodule maps as well as $\mathscr{H}_L$-comodule maps. The category of right $\mathscr{H}$-comodules is denoted by $\mathbf{M}^{\mathscr{H}}$.

The category ${}^{\mathscr{H}}\mathbf{M}$ of left $\mathscr{H}$-comodules is defined symmetrically.

Note that since the right R- and L-actions on H commute, also any right-$\mathscr{H}$-comodule is a right $R \otimes_k L$-module.

Remark 2 The antipode S in a Hopf algebroid $\mathscr{H}$ defines a functor ${}^{\mathscr{H}}\mathbf{M} \to \mathbf{M}^{\mathscr{H}}$. Indeed, if M is a left $\mathscr{H}$-comodule, with $\mathscr{H}_R$-coaction $m \mapsto m^{[-1]} \otimes_R m^{[0]}$ and $\mathscr{H}_L$-coaction $m \mapsto m_{[-1]} \otimes_L m_{[0]}$, then it is a right $\mathscr{H}$-comodule with right R-action $mr := \pi_L(t_R(r))m$, right L-action $ml := \pi_R(t_L(l))m$ and respective coactions

$$m \mapsto m_{[0]} \otimes_R S\left(m_{[-1]}\right) \qquad \text{and} \qquad m \mapsto m^{[0]} \otimes_L S\left(m^{[-1]}\right). \tag{1}$$

Left $\mathscr{H}$-comodule maps are also right $\mathscr{H}$-comodule maps for these coactions.

A functor $\mathbf{M}^{\mathscr{H}} \to {}^{\mathscr{H}}\mathbf{M}$ is constructed symmetrically.

Proposition 3 *Let $\mathscr{H}$ be a Hopf algebroid and (M, ρ_L, ρ_R) be a right $\mathscr{H}$-comodule. Then any coinvariant of the $\mathscr{H}_R$-comodule (M, ρ_R) is coinvariant also for the $\mathscr{H}_L$-comodule (M, ρ_L).*

If moreover the antipode of $\mathscr{H}$ is bijective, then coinvariants of the $\mathscr{H}_R$-comodule (M, ρ_R) and the $\mathscr{H}_L$-comodule (M, ρ_L) coincide.

Proof For a right $\mathscr{H}$-comodule (M, ρ_L, ρ_R), consider the map

$$\Phi_M : M \otimes_R H \to M \otimes_L H, \qquad m \otimes_R h \mapsto \rho_L(m) S(h), \tag{2}$$

where H is a left L-module via the source map s_L and a left R-module via the target map t_R, and $M \otimes_L H$ is understood to be a right H-module via the second factor. Since $\Phi_M(\rho_R(m)) = m \otimes_L 1_H$ and $\Phi_M(m \otimes_R 1_H) = \rho_L(m)$, the first claim in Proposition 3 follows. In order to prove the second assertion, note that if S is an isomorphism, then so is Φ_M, with the inverse $\Phi_M^{-1}(m \otimes_L h) = S^{-1}(h) \rho_R(m)$, where $M \otimes_R H$ is understood to be a left H-module via the second factor. □

Although the functors U and V in Theorem 2.2 are not known to exist without further assumptions, they exist in all known examples of Hopf algebroids and they establish isomorphisms between the categories of $\mathscr{H}_L$-comodules, $\mathscr{H}_R$-comodules and $\mathscr{H}$-comodules. In the following theorem F_R and F_L denote the forgetful functors $\mathbf{M}^{\mathscr{H}_R} \to \mathbf{M}_k$ and $\mathbf{M}^{\mathscr{H}_L} \to \mathbf{M}_k$, respectively, while G_R and G_L denote the forgetful functors $\mathbf{M}^{\mathscr{H}} \to \mathbf{M}^{\mathscr{H}_R}$ and $\mathbf{M}^{\mathscr{H}} \to \mathbf{M}^{\mathscr{H}_L}$, respectively. H is regarded as an R-bimodule via right multiplication by s_R and t_R and an L-bimodule via left multiplication by s_L and t_L.

Theorem 4 *Consider a Hopf algebroid $\mathscr{H}$.*

(1) *If the equaliser*

$$M \xrightarrow{\ \rho_R\ } M\otimes_R H \underset{M\otimes_R\gamma_R}{\overset{\rho_R\otimes_R H}{\rightrightarrows}} M\otimes_R H\otimes_R H \tag{3}$$

in $\mathbf{M}_L$ is $H\otimes_L H$-pure, i.e. it is preserved by the functor $(-)\otimes_L H\otimes_L H : \mathbf{M}_L \to \mathbf{M}_L$, for any right $\mathscr{H}_R$-comodule (M, ρ_R), then there exists a functor $U : \mathbf{M}^{\mathscr{H}_R} \to \mathbf{M}^{\mathscr{H}_L}$, such that $F_L \circ U = F_R$ and $U \circ G_R = G_L$. In particular, G_R is full.

(2) *If the equaliser*

$$N \xrightarrow{\ \rho_L\ } N\otimes_L H \underset{N\otimes_L\gamma_L}{\overset{\rho_L\otimes_L H}{\rightrightarrows}} N\otimes_L H\otimes_L H$$

in $\mathbf{M}_R$ is $H\otimes_R H$-pure, i.e. it is preserved by the functor $(-)\otimes_R H\otimes_R H : \mathbf{M}_R \to \mathbf{M}_R$, for any right $\mathscr{H}_L$-comodule (N, ρ_L), then there exists a functor $V : \mathbf{M}^{\mathscr{H}_L} \to \mathbf{M}^{\mathscr{H}_R}$, such that $F_R \circ V = F_L$ and $V \circ G_L = G_R$. In particular, G_L is full.

(3) *If both purity assumptions in parts* (1) *and* (2) *hold, then the forgetful functors $G_R : \mathbf{M}^{\mathscr{H}} \to \mathbf{M}^{\mathscr{H}_R}$ and $G_L : \mathbf{M}^{\mathscr{H}} \to \mathbf{M}^{\mathscr{H}_L}$ are isomorphisms, hence U and V are inverse isomorphisms.*

We term a Hopf algebroid $\mathscr{H}$ satisfying the assumptions in Theorem 4(3) a *pure* Hopf algebroid.

Proof (1) Recall that (3) defines the $\mathscr{H}_R$-cotensor product $M\Box_{\mathscr{H}_R} H \simeq M$. By (2.7), H is an $\mathscr{H}_R$-$\mathscr{H}_L$ bicomodule, with left coaction γ_R and right coaction γ_L. Thus in light of [8, 22.3] and its Erratum, we can define a desired functor $U := (-)\Box_{\mathscr{H}_R} H$. Clearly, it satisfies $F_L \circ U = F_R$. For an $\mathscr{H}$-comodule (M, ρ_L, ρ_R), the coaction on the $\mathscr{H}_L$-comodule $U\big(G_R(M, \rho_L, \rho_R)\big) = U(M, \rho_R)$ is given by

$$M \xrightarrow{\rho_R} M\Box_{\mathscr{H}_R} H \xrightarrow{M\Box_{\mathscr{H}_R}\gamma_L} M\Box_{\mathscr{H}_R}(H\otimes_L H) \xrightarrow{\simeq} (M\Box_{\mathscr{H}_R} H)\otimes_L H \xrightarrow{M\otimes_R \pi_R\otimes_L H} M\otimes_L H, \tag{4}$$

where in the third step we used that since the equaliser (3) is $H\otimes_L H$-pure, it is in particular H-pure. Using that ρ_R is a right $\mathscr{H}_L$-comodule map and counitality of ρ_R, we conclude that (4) is equal to ρ_L. Hence $U \circ G_R = G_L$. This proves that for any two $\mathscr{H}$-comodules M and M', and any $\mathscr{H}_R$-comodule map $f : M \to M'$, $U(f) = f$ is an $\mathscr{H}_L$-comodule map hence an $\mathscr{H}$-comodule map, i.e. that G_R is full. Part (2) is proven symmetrically.

(3) For the functor U in part (1) and a right $\mathscr{H}_R$-comodule (M, ρ_R), denote $U(M, \rho_R) =: (M, \rho_L)$. With this notation, define a functor $\widehat{G}_R : \mathbf{M}^{\mathscr{H}_R} \to \mathbf{M}^{\mathscr{H}}$, with object map $(M, \rho_R) \mapsto (M, \rho_R, \rho_L)$, and acting on the morphisms as the identity map. Being coassociative, ρ_R is an $\mathscr{H}_R$-comodule map, so by part (1) it is an $\mathscr{H}_L$-comodule map. Symmetrically, by part (2) ρ_L is an $\mathscr{H}_R$-comodule map. So $\widehat{G}_R$ is a well defined functor. We claim that it is the inverse of G_R. Obviously, $G_R \circ \widehat{G}_R$ is the identity functor. In the opposite order, note that by construction $G_L \circ \widehat{G}_R = U$. Therefore, $G_L \circ \widehat{G}_R \circ G_R = U \circ G_R = G_L$, cf. part (1). That is, $\widehat{G}_R \circ G_R$ takes an $\mathscr{H}$-comodule (M, ρ_L, ρ_R) to the same $\mathscr{H}_L$-comodule (M, ρ_L). Since $\widehat{G}_R \circ G_R$ obviously takes (M, ρ_L, ρ_R) to the same $\mathscr{H}_R$-comodule (M, ρ_R) as well, we conclude that also $\widehat{G}_R \circ G_R$ is the identity functor.

In a symmetrical way, in terms of the functor $V(N, \rho_L) =: (N, \rho_R)$ in part (2), one constructs G_L^{-1} with object map $(N, \rho_L) \mapsto (N, \rho_L, \rho_R)$, and acting on the morphisms as the identity map. The identities $G_L \circ G_R^{-1} = U$ and $G_R \circ G_L^{-1} = V$ prove that U and V are mutually inverse isomorphisms, as stated. □

Example 5 We list some families of pure Hopf algebroids.

(1) All purity conditions in Theorem 4 hold if H is *flat* as a left L- and a left R-module. Indeed, in this case the functors $(-)\otimes_L H \otimes_L H : \mathbf{M}_L \to \mathbf{M}_L$ and $(-)\otimes_R H \otimes_R H : \mathbf{M}_R \to \mathbf{M}_R$ preserve any equaliser. In particular, Frobenius Hopf algebroids in [2] (being finitely generated and projective) are flat.

(2) Weak Hopf algebras, introduced in [6], determine Hopf algebroids over Frobenius-separable base algebras $L \cong R^{op}$, cf. [4, 4.1.2]. Recall that Frobenius separability of a k-algebra R means the existence of a k-module map $\psi : R \to k$ and an element $\sum_i e_i \otimes_k f_i \in R \otimes_k R$, such that

$$\sum_i \psi(re_i) f_i = r = \sum_i e_i \psi(f_i r), \quad \text{for all } r \in R, \qquad \text{and} \qquad \sum_i e_i f_i = 1_R.$$

Note that this implies that $\sum_i re_i \otimes_k f_i = \sum_i e_i \otimes_k f_i r$, for all $r \in R$ (hence the name *separable*). For a Frobenius-separable algebra R, any right R-module X and left R-module Y, the canonical epimorphism $X \otimes_k Y \to X \otimes_R Y$ is split by $x \otimes_R y \mapsto \sum_i x e_i \otimes_k f_i y$. For the base algebras of a weak Hopf algebra, the Frobenius-separability structure arises from the restriction of the counit and the image of the unit element 1_H under the coproduct.

Let H be a weak Hopf algebra with (weak) coproduct $\Delta : H \to H \otimes_k H$. Denote its left and right (or 'target' and 'source') subalgebras by L and R, respectively. (These serve as the base algebras of the corresponding Hopf algebroid, see [4].) Note that Δ is an R-L bimodule map.

Any right comodule (M, ρ) of a weak Hopf algebra H (as a coalgebra) can be equipped with a right R-action via $mr = m_{<0>} \varepsilon(m_{<1>} r)$, where $\rho(m) = m_{<0>} \otimes_k m_{<1>}$ and ε denotes the counit of H. Moreover, any right comodule (M, ρ) of H yields a right coaction of the constituent right bialgebroid, by composing ρ with the (split) epimorphism $p_M : M \otimes_k H \to M \otimes_R H$. Define a right L-module structure on M via (3).

For any left L-module N, consider the following diagram (in $\mathbf{M}_k$):

$$\begin{tikzcd}
M\otimes_L N \arrow[r, "p_M\circ\rho\otimes_L N"] \arrow[d] & M\otimes_R H\otimes_L N \arrow[r, shift left, "p_M\circ\rho\otimes_R H\otimes_L N"] \arrow[r, shift right, swap, "M\otimes_R p_H\circ\Delta\otimes_L N"] \arrow[d] & M\otimes_R H\otimes_R H\otimes_L N \arrow[d] \\
M\otimes_k N \arrow[r, swap, "\rho\otimes_k N"] & M\otimes_k H\otimes_k N \arrow[r, shift left, "\rho\otimes_k H\otimes_k N"] \arrow[r, shift right, swap, "M\otimes_k\Delta\otimes_k N"] & M\otimes_k H\otimes_k H\otimes_k N\,.
\end{tikzcd} \tag{5}$$

The R-actions on H are given by right multiplications by the source and target maps and the L-actions on H are given by left multiplications. The vertical arrows denote the sections of the canonical epimorphisms given by the Frobenius-separability structures of L and R. The diagram is easily checked to be serially commutative (meaning commutativity with either *simultaneous* choice of the upper or the lower ones of the parallel arrows). Clearly,

$$\begin{tikzcd}
M \arrow[r, "\rho"] & M\otimes_k H \arrow[r, shift left, "\rho\otimes_k H"] \arrow[r, shift right, swap, "M\otimes_k\Delta"] & M\otimes_k H\otimes_k H
\end{tikzcd}$$

is a split equaliser in $\mathbf{M}_k$ (with splitting provided by the counit of H), hence the bottom row of the diagram in (5) is an equaliser. This proves that also the top row is an equaliser, so in particular the purity conditions in Theorem 4 (1) hold. The conditions in Theorem 4 (2) are verified by a symmetrical reasoning.

(3) For any k-algebra L, the tensor product algebra $H := L\otimes_k L^{op}$ carries a Hopf algebroid structure, see [4, 4.1.3]. Since the left L-action on H is given by multiplication in the first factor, the functors $F((-)\otimes_L H\otimes_L H)$ and $F(-)\otimes_k L\otimes_k L : \mathbf{M}_L \to \mathbf{M}_k$ are naturally isomorphic, where $F : \mathbf{M}_L \to \mathbf{M}_k$ denotes the forgetful functor. The forgetful functor F has a left adjoint, hence it preserves any equaliser. The functor F takes (3) to a split equaliser (with splitting provided by π_R), which is then preserved by any functor. This proves that $F(-)\otimes_k L\otimes_k L$ and hence $F((-)\otimes_L H\otimes_L H)$ preserve (3). Since F also reflects equalisers, we conclude that (3) is preserved by $(-)\otimes_L H\otimes_L H : \mathbf{M}_L \to \mathbf{M}_L$. The purity conditions in Theorem 4 (2) are proven to hold similarly.

(4) In [1, Corrigendum], the purity conditions in Theorem 4 are proven to hold for a Hopf algebroid whose constituent R-coring (equivalently, the constituent L-coring) is coseparable.

Theorem 6 *For any Hopf algebroid $\mathscr{H}$, $\mathbf{M}^{\mathscr{H}}$ is a monoidal category. Moreover, there are strict monoidal forgetful functors rendering commutative the following diagram:*

$$\begin{tikzcd}
\mathbf{M}^{\mathscr{H}} \arrow[r, "G_R"] \arrow[d, swap, "G_L"] & \mathbf{M}^{\mathscr{H}_R} \arrow[d] \\
\mathbf{M}^{\mathscr{H}_L} \arrow[r] & {}_R\mathbf{M}_R\,.
\end{tikzcd}$$

Proof Commutativity of the diagram follows by comparing the unique R-actions that make R-bilinear the $\mathscr{H}_R$-coaction and the $\mathscr{H}_L$-coaction in an $\mathscr{H}$-comodule, respectively, (see (2.12) and (2.16), and the algebra isomorphism (2.10)). Strict monoidality of the functors on the right hand side and in the bottom row follows by [9, Theorem 5.6] (and its application to the opposite of the bialgebroid $\mathscr{H}_L$), cf. Section 2.3. In order to see strict monoidality of the remaining two functors G_R and G_L, recall that by [9, Theorem 5.6] (applied to $\mathscr{H}_R$ and the opposite of $\mathscr{H}_L$), the R-module tensor product of any two $\mathscr{H}$-comodules is an $\mathscr{H}_R$-comodule and an $\mathscr{H}_L$-comodule, via the diagonal coactions, cf. (2.14) and the second formula on page 437. It is straightforward to check compatibility of these coactions in the sense of Definition 1. Similarly, $R(\cong L^{op})$ is known to be an $\mathscr{H}_R$-comodule and an $\mathscr{H}_L$-comodule, and compatibility of the coactions is obvious. Finally, the R-module tensor product of $\mathscr{H}$-comodule maps is an $\mathscr{H}_R$-comodule map and an $\mathscr{H}_L$-comodule map by [9, Theorem 5.6]. Thus it is an $\mathscr{H}$-comodule map. By Theorem [9, Theorem 5.6] also the coherence natural transformations in ${}_R\mathbf{M}_R$ are $\mathscr{H}_R$- and $\mathscr{H}_L$-comodule maps, so $\mathscr{H}$-comodule maps, what completes the proof. □

Definition 7 A *right comodule algebra* of a Hopf algebroid $\mathscr{H}$ is a monoid in the monoidal category $\mathbf{M}^{\mathscr{H}}$ of right $\mathscr{H}$-comodules. Explicitly, an R-ring (A, μ, η), such that A is a right $\mathscr{H}$-comodule and $\eta : R \to A$ and $\mu : A \otimes_R A \to A$ are right $\mathscr{H}$-comodule maps. Using the notations $a \mapsto a^{[0]} \otimes_R a^{[1]}$ and $a \mapsto a_{[0]} \otimes_L a_{[1]}$ for the $\mathscr{H}_R$- and $\mathscr{H}_L$-coactions, respectively, $\mathscr{H}$-colinearity of η and μ means the identities, for all $a, a' \in A$,

$$1_A{}^{[0]} \otimes_R 1_A{}^{[1]} = 1_A \otimes_R 1_H, \qquad (aa')^{[0]} \otimes_R (aa')^{[1]} = a^{[0]} a'^{[0]} \otimes_R a^{[1]} a'^{[1]}$$

$$1_{A[0]} \otimes_L 1_{A[1]} = 1_A \otimes_L 1_H, \qquad (aa')_{[0]} \otimes_L (aa')_{[1]} = a_{[0]} a'_{[0]} \otimes_L a_{[1]} a'_{[1]}.$$

If A is a right comodule algebra of a Hopf algebroid $\mathscr{H}$, with $\mathscr{H}_R$-coinvariant subalgebra B, then we say that $B \subseteq A$ is a (right) *$\mathscr{H}$-extension*.

Symmetrically, a *left $\mathscr{H}$-comodule algebra* is a monoid in ${}^{\mathscr{H}}\mathbf{M}$ and a left $\mathscr{H}$-comodule algebra is said to be a left $\mathscr{H}$-extension of its $\mathscr{H}_L$-coinvariant subalgebra.

The functors in Remark 2 induced by the antipode are checked to be strictly anti-monoidal. Therefore, the opposite of a right $\mathscr{H}$-comodule algebra, with coactions in Remark 2, is a left $\mathscr{H}$-comodule algebra and conversely.

Above corrections make it necessary to make the following changes and supplements in the text of the paper.

- Throughout the paper, comodules of a Hopf algebroid $\mathscr{H}$ and $\mathscr{H}$-comodule maps have to be understood as in Definition 1. In particular, $\mathscr{H}$-colinearity of the cleaving map (cf. Definition 3.5) has to be defined in this way.
- A comodule algebra of a Hopf algebroid $\mathscr{H}$ and an $\mathscr{H}$-extension have to be meant in the sense of Definition 7.
- By coinvariants of a right comodule M of a Hopf algebroid $\mathscr{H}$, one should mean coinvariants of M as an $\mathscr{H}_R$-comodule (that implies coinvariance with respect to the $\mathscr{H}_L$-coaction, cf. Proposition 3).
- In Lemma 3.8, identities (3.9) and (3.10) are not known to be equivalent, in proving the lemma both of them need to be verified (by similar steps).

- In the last lines of page 448 and first line of page 449, in order to see that ν is right $\mathscr{H}$-colinear, left B-linearity of both ($\mathscr{H}_L$- and $\mathscr{H}_R$-) coactions has to be used, cf. Proposition 3.
- Theorem 5.3 holds without modification, but the proof needs to be corrected. Note that for any Hopf algebroid $\mathscr{H}$, the forgetful functor $\mathbf{M}^{\mathscr{H}} \to \mathbf{M}_L$ possesses a right adjoint $(-)\otimes_L H$. The unit of the adjunction is given by the $\mathscr{H}_L$-coaction $M \to M\otimes_L H$, for any right $\mathscr{H}$-comodule M. It is an $\mathscr{H}$-comodule map by definition. The counit is given by $N\otimes_L\pi_L : N\otimes_L H \to N$, for any right L-module N. Symmetrically, the forgetful functor ${}^{\mathscr{H}}\mathbf{M} \to {}_L\mathbf{M}$ possesses a right adjoint $H\otimes_L(-)$.
  Next observe that for a cleft extension $B \subseteq A$ of a Hopf algebroid $\mathscr{H}$ with a bijective antipode, the obvious inclusion $\mathrm{Hom}^{\mathscr{H},\mathscr{H}}(H, A\otimes_T A) \hookrightarrow \mathrm{Hom}^{\mathscr{H}_R,\mathscr{H}_R}(H, A\otimes_T A)$ is an isomorphism. Indeed, in terms of a cleaving map j and its convolution inverse j^{c}, for any $f \in \mathrm{Hom}^{\mathscr{H}_R,\mathscr{H}_R}(H, A\otimes_T A)$ and any $h \in H$,
$$\begin{aligned} f(h) &= f\big(h^{(1)}s_R(\pi_R(h^{(2)}))\big) = f(h^{(1)})\eta_R(\pi_R(h^{(2)})) = f(h^{(1)})j^{\mathsf{c}}(h^{(2)}{}_{(1)})j(h^{(2)}{}_{(2)}) \\ &= f(h_{(1)}{}^{(1)})j^{\mathsf{c}}(h_{(1)}{}^{(2)})j(h_{(2)}) = f(h_{(1)})^{[0]}j^{\mathsf{c}}(f(h_{(1)})^{[1]})j(h_{(2)}). \end{aligned} \tag{6}$$
  By Lemma 3.9, the left B-linearity of the right $\mathscr{H}_L$-coaction on A and the right $\mathscr{H}_L$-colinearity of j, this proves that f is right $\mathscr{H}_L$-colinear. Left $\mathscr{H}_L$-colinearity of f is proven symmetrically.
  Putting together the above observations, by Theorem 3.11 and Corollary 3.12 there is a chain of isomorphisms
$$\begin{aligned} \mathrm{Hom}^{\mathscr{H}_R,\mathscr{H}_R}\big(H, A\underset{T}{\otimes}A\big) &\simeq \mathrm{Hom}^{\mathscr{H},\mathscr{H}}\big(H, A\underset{T}{\otimes}A\big) \simeq \mathrm{Hom}^{\mathscr{H},\mathscr{H}}\big(H, H\underset{L}{\otimes}B\underset{T}{\otimes}B\underset{L}{\otimes}H\big) \\ &\simeq \mathrm{Hom}_{L,L}\big(H, B\underset{T}{\otimes}B\big). \end{aligned}$$
  The proof is then completed as in the paper.
- In Definition 5.4, a *left total T-integral* has to be defined as a left $\mathscr{H}$-comodule map, with respect to the left coactions on A that are related to the right coactions via the isomorphism in (1). (Left $\mathscr{H}_L$-colinearity of ϑ is needed to see that the range of the map (5.6) is in B, as stated.)
  If $B \subseteq A$ is a cleft extension by a Hopf algebroid $\mathscr{H}$ with a bijective antipode, then similarly to (6), any morphism $g \in \mathrm{Hom}^{\mathscr{H}_R,-}(H, A)$ is checked to satisfy, for $h \in H$, $g(h) = j^{\mathsf{c}}(h_{(1)})j\big(S^{-1}(g(h_{(2)})_{[1]})\big)g(h_{(2)})_{[0]}$. Therefore $\mathrm{Hom}^{\mathscr{H}_R,-}(H, A) = \mathrm{Hom}^{\mathscr{H},-}(H, A)$. In particular, in this situation a left total T-integral is the same as a left $\mathscr{H}_R$-colinear map $\vartheta : H \to A$ such that $\vartheta(H) \subseteq A^T$ and $\vartheta(1_H) = 1_A$.

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