

# DOUBLE GROUPOIDS AND HOMOTOPY 2-TYPES

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**ABSTRACT.** This work contributes to clarifying several relationships between certain higher categorical structures and the homotopy types of their classifying spaces. Double categories (Ehresmann, 1963) have well-understood geometric realizations, and here we deal with homotopy types represented by double groupoids satisfying a natural ‘filling condition’. Any such double groupoid characteristically has associated to it ‘homotopy groups’, which are defined using only its algebraic structure. Thus arises the notion of ‘weak equivalence’ between such double groupoids, and a corresponding ‘homotopy category’ is defined. Our main result in the paper states that the geometric realization functor induces an equivalence between the homotopy category of double groupoids with filling condition and the category of homotopy 2-types (that is, the homotopy category of all topological spaces with the property that the  $n^{\text{th}}$  homotopy group at any base point vanishes for  $n \geq 3$ ). A quasi-inverse functor is explicitly given by means of a new ‘homotopy double groupoid’ construction for topological spaces.

*Mathematical Subject Classification:* 18D05, 20L05, 55Q05, 55U40.

## 1. INTRODUCTION AND SUMMARY.

Higher-dimensional categories provide a suitable setting for the treatment of an extensive list of subjects of recognized mathematical interest. The construction of nerves and classifying spaces of higher categorical structures discovers ways to transport categorical coherence to homotopic coherence, and it has shown its relevance as a tool in algebraic topology, algebraic geometry, algebraic  $K$ -theory, string field theory, conformal field theory, and in the study of geometric structures on low-dimensional manifolds.

*Double groupoids*, that is, groupoid objects in the category of groupoids, were introduced by Ehresmann [15, 16] in the late fifties and later studied by several people because of their connection with several areas of mathematics. Roughly, a double groupoid consists of *objects*, *horizontal* and *vertical morphisms*, and *squares*. Each square, say  $\alpha$ , has objects as vertices and morphisms as edges, as in

$$\begin{array}{ccc} & \leftarrow & \\ \uparrow & \alpha & \uparrow \\ & \leftarrow & \end{array},$$

together with two groupoid compositions- the *vertical* and *horizontal compositions*- of squares, and compatible groupoid compositions of the edges, obeying several conditions (see Section 3 for details). Any double groupoid  $\mathcal{G}$  has a *geometric realization*  $|\mathcal{G}|$ , which is the topological space defined by first taking the double nerve  $\mathbb{N}\mathcal{G}$ , which is a bisimplicial set, and then realizing

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*Key words and phrases.* Double groupoid, classifying space, bisimplicial set, Kan complex, geometric realization, homotopy type.

The first author acknowledge support from the DGI of Spain (Project: MTM2007-65431); Consejería de Innovación de J. de Andalucía (P06-FQM-1889); MEC de España, ‘Ingenio Mathematica(i-Math)’ No. CSD2006-00032 (consolider-Ingenio 2010).

The second author thanks support to University of Granada (Beca Plan Propio 2009).

The third author acknowledges support from DGI of Spain (Project: MTM2009-12081) and tanks the University of Granada for its support and kind hospitality.

the diagonal to obtain a space:  $|\mathcal{G}| = |\text{diag}\mathbb{N}\mathcal{G}|$ . In this paper, we address the homotopy types obtained in this way from double groupoids satisfying a natural *filling condition*: Any filling problem

$$\begin{array}{c} \cdot \leftarrow \cdot \\ \vdots \uparrow \exists? \uparrow \\ \cdot \leftarrow \cdot \end{array}$$

finds a solution in the double groupoid. This filling condition on double groupoids is often assumed in the case of double groupoids arising in different areas of mathematics, such as in differential geometry or in weak Hopf algebra theory (see the papers by Mackenzie [25] and Andruskiewitsch and Natale [1], for example), and it is satisfied for those double groupoids that have emerged with an interest in algebraic topology, mainly thanks to the work of Brown, Higgins, Spencer, *et al.*, where the connection of double groupoids with crossed modules and a higher Seifert-van Kampen Theory has been established (see, for instance, the survey paper [5] and references therein. Thus, the filling condition is easily proven for edge symmetric double groupoids (also called special double groupoids) with connections (see, for example [7, 10] or [6, 9, 8], for more recent instances), for double groupoid objects in the category of groups (also termed  $\text{cat}^2$ -groups, [23, 11, 28]), or, for example, for 2-groupoids (regarded as double groupoids where one of the side groupoids of morphisms is discrete [27],[19]).

When a double groupoid  $\mathcal{G}$  has the filling condition, then there are characteristically associated to it ‘homotopy groups’,  $\pi_i(\mathcal{G}, a)$ , which we define using only the algebraic structure of  $\mathcal{G}$ , and which are trivial for integers  $i \geq 3$ . A first major result states that:

*If  $\mathcal{G}$  is a double groupoid with filling condition, then, for each object  $a$ , there are natural isomorphisms  $\pi_i(\mathcal{G}, a) \cong \pi_i(|\mathcal{G}|, |a|)$ ,  $i \geq 0$ .*

The proof of this result requires a prior recognition of the significance of the filling condition on double groupoids in the homotopy theory of simplicial sets; namely, we prove that

*A double category  $\mathcal{C}$  is a double groupoid with filling condition if and only if the simplicial set  $\text{diag}\mathbb{N}\mathcal{C}$  is a Kan complex.*

This fact can be seen as a higher version of the well-known fact that the nerve of a category is a Kan complex if and only if the category is a groupoid (see [21], for example).

Once we have defined the homotopy category of double groupoids satisfying the filling condition  $\text{Ho}(\mathbf{DG}_{\text{fc}})$ , to be the localization of the category of these double groupoids, with respect to the class of *weak equivalences* or double functors  $F : \mathcal{G} \rightarrow \mathcal{G}'$  inducing isomorphisms  $\pi_i F : \pi_i(\mathcal{G}, a) \cong \pi_i(\mathcal{G}', Fa)$  on the homotopy groups, we then obtain an induced functor

$$|\cdot| : \text{Ho}(\mathbf{DG}_{\text{fc}}) \rightarrow \text{Ho}(\mathbf{Top}), \quad \mathcal{G} \mapsto |\mathcal{G}|,$$

where  $\text{Ho}(\mathbf{Top})$  is the localization of the category of topological spaces with respect to the class of weak equivalences. Furthermore, we show a new functorial construction of a *homotopy double groupoid*  $\mathbf{I}X$ , for any topological space  $X$ , that induces a functor

$$\text{Ho}(\mathbf{Top}) \rightarrow \text{Ho}(\mathbf{DG}_{\text{fc}}), \quad X \mapsto \mathbf{I}X.$$

A main goal in this paper is to prove the following result, whose proof is somewhat indirect since it is given through an explicit description of a left adjoint functor,  $\mathbf{P} \dashv \mathbf{N}$ , to the double nerve functor  $\mathcal{G} \mapsto \mathbb{N}\mathcal{G}$ :

*Both induced functors on the homotopy categories  $\mathcal{G} \mapsto |\mathcal{G}|$  and  $X \mapsto \mathbf{I}X$  restrict by giving mutually quasi-inverse equivalence of categories*

$$\text{Ho}(\mathbf{DG}_{\text{fc}}) \simeq \text{Ho}(\mathbf{2-types}),$$

where  $\mathbf{Ho}(\mathbf{2-types})$  is the full subcategory of the homotopy category of topological spaces given by those spaces  $X$  with  $\pi_i(X, a) = 0$  for any integer  $i > 2$  and any base point  $a$ . From the point of view of this fact, the use of double groupoids and their classifying spaces in homotopy theory goes back to Whitehead [32] and Mac Lane-Whitehead [24] since double groupoids where one of the side groupoids of morphisms is discrete with only one object (= strict 2-groups, in the terminology of Baez [3]) are the same as crossed modules (this observation is attributed to Verdier in [10]). In this context, we should mention the work by Brown-Higgins [7] and Moerdijk-Svensson [27] since crossed modules over groupoids are essentially the same thing as 2-groupoids and double groupoids where one of the side groupoids of morphisms is discrete. Along the same line, our result is also a natural 2-dimensional version of the well-known equivalence between the homotopy category of groupoids and the homotopy category of 1-types (for a useful survey of groupoids in topology, see [4]).

The plan of this paper is, briefly, as follows. After this introductory Section 1, the paper is organized in six sections. Section 2 aims to make this paper as self-contained as possible; hence, at the same time as fixing notations and terminology, we also review necessary aspects and results from the background of (bi)simplicial sets and their geometric realizations that will be used throughout the paper. However, the material in Section 2 is quite standard, so the expert reader may skip most of it. The most original part is in Subsection 2.2, related to the extension condition on bisimplicial sets. In Section 3, after recalling the notion of a double groupoid and fixing notations, we mainly introduce the homotopy groups  $\pi_i(\mathcal{G}, a)$ , at any object  $a$  of a double groupoid with filling condition  $\mathcal{G}$ . Section 4 is dedicated to showing in detail the construction of the homotopy double groupoid  $\mathbf{IIX}$ , characteristically associated to any topological space  $X$ . Here, we prove that a continuous map  $X \rightarrow Y$  is a weak homotopy 2-equivalence (i.e., it induces bijections on the homotopy groups  $\pi_i$  for  $i \leq 2$ ) if and only if the induced double functor  $\mathbf{IIX} \rightarrow \mathbf{IYY}$  is a weak equivalence. Next, in Section 5, we first address the issue of to have a manageable description for the bisimplices in  $\mathbb{N}\mathcal{G}$ , the double nerve of a double groupoid, and then we determine the homotopy type of the geometric realization  $|\mathcal{G}|$  of a double groupoid with filling condition. Specifically, we prove that the homotopy groups of  $|\mathcal{G}|$  are the same as those of  $\mathcal{G}$ . Our goal in Section 6 is to prove that the double nerve functor,  $\mathcal{G} \mapsto \mathbb{N}\mathcal{G}$ , embeds, as a reflexive subcategory, the category of double groupoids satisfying the filling condition into a certain category of bisimplicial sets. The reflector functor  $K \mapsto \mathbf{PK}$  works as a bisimplicial version of Brown's construction in [6, Theorem 2.1]. Furthermore, as we will prove, the resulting double groupoid  $\mathbf{PK}$  always represents the homotopy 2-type of the input bisimplicial set  $K$ , in the sense that there is a natural weak 2-equivalence  $|K| \rightarrow |\mathbf{PK}|$ . This result becomes crucial in the final Section 7 where, bringing into play all the previous work, the equivalence of categories  $\mathbf{Ho}(\mathbf{DG}_{fc}) \simeq \mathbf{Ho}(\mathbf{2-types})$  is achieved.

## 2. SOME PRELIMINARIES ON BISIMPLICIAL SETS.

This section aims to make this paper as self-contained as possible; Therefore, while fixing notations and terminology, we also review necessary aspects and results from the background of (bi)simplicial sets and their geometric realizations used throughout the paper. However, the material in this section is quite standard and, in general, we employ the standard symbolism and nomenclature to be found in texts on simplicial homotopy theory, mainly in [17] and [26], so the expert reader may skip most of it. The most original part is in Subsection 2.2, related to the extension condition and the bihomotopy relation on bisimplicial sets.

### 2.1. Kan complexes: Fundamental groupoids and homotopy groups.

We start by fixing some notations. In the simplicial category  $\Delta$ , the generating coface and codegeneracy maps are denoted by  $d^i: [n-1] \rightarrow [n]$  and  $s^i: [n+1] \rightarrow [n]$  respectively. However,

for  $L : \Delta^o \rightarrow \mathbf{Set}$  any simplicial set, we write  $d_i = L(d^i) : L_n \rightarrow L_{n-1}$  and  $s_i = L(s^i) : L_n \rightarrow L_{n+1}$  for its corresponding face and degeneracy maps.

The *standard  $n$ -simplex* is  $\Delta[n] = \Delta(-, [n])$  and, as is usual, we identify any simplicial map  $x : \Delta[n] \rightarrow L$  with the simplex  $x(\iota_n) \in L_n$ , the image by  $x$  of the basic simplex  $\iota_n = id : [n] \rightarrow [n]$  of  $\Delta[n]$ . Thus, for example, the  $i^{th}$ -face of  $\Delta[n]$  is  $d^i = \Delta(-, d^i) : \Delta[n-1] \rightarrow \Delta[n]$ , the simplicial map with  $d^i(\iota_{n-1}) = d_i(\iota_n)$ . Similarly,  $s^i = \Delta(-, s^i) : \Delta[n+1] \rightarrow \Delta[n]$  is the simplicial map that we identify with the degenerated simplex  $s_i(\iota_n)$  of  $\Delta[n]$ .

The *boundary*  $\partial\Delta[n] \subset \Delta[n]$  is the smallest simplicial subset containing all the faces  $d^i : \Delta[n-1] \rightarrow \Delta[n]$ ,  $0 \leq i \leq n$ , of  $\Delta[n]$ . Similarly, for any given  $k$  with  $0 \leq k \leq n$ , the  $k^{th}$ -horn,  $\Lambda^k[n] \subset \Delta[n]$ , is the smallest simplicial subset containing all the faces  $d^i : \Delta[n-1] \rightarrow \Delta[n]$  for  $0 \leq i \leq n$  and  $i \neq k$ . For a more geometric (and useful) description of these simplicial sets, recall that there are coequalizers

$$\bigsqcup_{0 \leq i < j \leq n} \Delta[n-2] \rightrightarrows \bigsqcup_{0 \leq i \leq n} \Delta[n-1] \rightarrow \partial\Delta[n],$$

and

$$(2.1) \quad \bigsqcup_{\substack{0 \leq i < j \leq n \\ i \neq k \neq j}} \Delta[n-2] \rightrightarrows \bigsqcup_{\substack{0 \leq i \leq n \\ i \neq k}} \Delta[n-1] \rightarrow \Lambda^k[n],$$

given by the relations  $d^j d^i = d^i d^{j-1}$  if  $i < j$ .

A simplicial set  $L$  is a *Kan complex* if it satisfies the so-called *extension condition*. Namely, for any simplicial diagram

$$\begin{array}{ccc} \Lambda^k[n] & \longrightarrow & L \\ \downarrow & \nearrow \text{dotted} & \\ \Delta[n] & & \end{array}$$

there is a map  $\Delta[n] \rightarrow L$  (the dotted arrow) making the diagram commute.

In a Kan complex  $L$ , two simplices  $x, x' : \Delta[n] \rightarrow L$  are said to be *homotopic* whenever they have the same faces and there is a *homotopy* from  $x$  to  $x'$ , that is, a simplex  $y : \Delta[n+1] \rightarrow L$  making this diagram commutative

$$\begin{array}{ccc} \partial\Delta[n+1] & \xrightarrow{(xd^0s^{n-1}, \dots, xd^{n-1}s^{n-1}, x, x')} & L \\ \downarrow & \nearrow y & \\ \Delta[n+1] & & \end{array}$$

Being homotopic establishes an equivalence relation on the simplices of  $L$ , and we write  $[x]$  for the homotopy class of a simplex  $x$ . A useful result is the follows:

**Fact 2.1.** *Let  $y, y' : \Delta[n+1] \rightarrow L$  be two simplices such that  $[yd^i] = [y'd^i]$  for all  $i \neq k$ ; then  $[yd^k] = [y'd^k]$ .*

The *fundamental groupoid* of  $L$ , denoted  $PL$ , also called its Poincaré groupoid, has as objects the vertices  $a : \Delta[0] \rightarrow L$ , and a morphism  $[x] : a \rightarrow b$  is the homotopy class of a simplex  $x : \Delta[1] \rightarrow L$  with  $xd^0 = a$  and  $xd^1 = b$ . The composition in  $PL$  is defined by

$$[x] \circ [x'] = [yd^1],$$

where  $y : \Delta[2] \rightarrow L$  is any simplex with  $yd^2 = x$  and  $yd^0 = x'$ , and the identities are  $Ia = [as^0]$ .

The set of *path components* of  $L$ , denoted as  $\pi_0 L$ , is the set of connected components of  $PL$ , so it consists of all homotopy classes of the 0-simplices of  $L$ . For any given vertex of the Kan

complex  $a : \Delta[0] \rightarrow L$ ,  $\pi_0(L, a)$  is the set  $\pi_0 L$ , pointed by  $[a]$ , the component of  $a$ . The group of automorphisms of  $a$  in the fundamental groupoid of  $L$  is  $\pi_1(L, a)$ , the *fundamental group* of  $L$  at  $a$ . Furthermore, denoting every composite map  $\Delta[n] \rightarrow \Delta[0] \xrightarrow{a} L$  by  $a$  as well, the  $n^{\text{th}}$  *homotopy group*  $\pi_n(L, a)$  of  $L$  at  $a$  consists of homotopy classes of simplices  $x : \Delta[n] \rightarrow L$  for all simplices  $x$  with faces  $xd^i = a$ , for  $0 \leq i \leq n$ . The multiplication in the (abelian, for  $n \geq 2$ ) group  $\pi_n(L, a)$  is given by

$$[x] \circ [x'] = [yd^n],$$

where  $y : \Delta[n+1] \rightarrow L$  is a (any) solution to the extension problem

$$\begin{array}{ccc} \Lambda^n[n+1] & \xrightarrow{(a, \dots, a, x', -, x)} & L \\ \downarrow & \nearrow y & \\ \Delta[n+1] & & \end{array}$$

The following fact is used several times throughout the paper:

**Fact 2.2.** *Let  $L$  be a Kan complex and  $n$  an integer such that the homotopy groups  $\pi_n(L, a)$  vanish for all base vertices  $a$ . Then, every extension problem*

$$\begin{array}{ccc} \partial\Delta[n+1] & \longrightarrow & L \\ \downarrow & \nearrow \exists? & \\ \Delta[n+1] & & \end{array}$$

*has a solution. In particular, any two  $n$ -simplices with the same faces are homotopic.*

We shall end this preliminary subsection by recalling that two simplicial maps  $f, g : L \rightarrow L'$  are *homotopic* whenever there is a map  $L \times \Delta[1] \rightarrow L'$  which is  $f$  on  $L \times 0$  and  $g$  on  $L \times 1$ . The resulting homotopy relation becomes a congruence on the category  $\mathbf{KC}$  of Kan complexes, and the corresponding quotient category is the *homotopy category of Kan complexes*,  $\text{Ho}(\mathbf{KC})$ . A map between Kan complexes is a *homotopy equivalence* if it induces an isomorphism in the homotopy category. There is an analogous result of Whitehead's theorem on CW-complexes to Kan complexes:

**Fact 2.3.** *A simplicial map between Kan complexes,  $L \rightarrow L'$ , is a homotopy equivalence if and only if it induces an isomorphism  $\pi_i(L, a) \cong \pi_i(L', fa)$  for all base vertex  $a$  of  $L$  and any integer  $i \geq 0$ .*

## 2.2. Bisimplicial sets: The extension condition and the bihomotopy relation.

It is often convenient to view a bisimplicial set  $K : \Delta^o \times \Delta^o \rightarrow \mathbf{Set}$  as a (horizontal) simplicial object in the category of (vertical) simplicial sets. For this case, we write  $d_i^h = K(d^i, id) : K_{p,q} \rightarrow K_{p-1,q}$  and  $s_i^h = K(s^i, id) : K_{p,q} \rightarrow K_{p+1,q}$  for the horizontal face and degeneracy maps, and, similarly  $d_j^v = K(id, d^j)$  and  $s_j^v = K(id, s^j)$  for the vertical ones.

For simplicial sets  $X$  and  $Y$ , let  $X \otimes Y$  be the bisimplicial set with  $(X \otimes Y)_{p,q} = X_p \times Y_q$ . The *standard  $(p, q)$ -bisimplex* is

$$\Delta[p, q] := \Delta \times \Delta(-, ([p], [q])) = \Delta[p] \otimes \Delta[q],$$

the bisimplicial set represented by the object  $([p], [q])$ , and usually we identify any bisimplicial map  $x : \Delta[p, q] \rightarrow K$  with the  $(p, q)$ -bisimplex  $x(\iota_p, \iota_q) \in K$ . The functor  $([p], [q]) \mapsto \Delta[p, q]$  is then a co-bisimplicial bisimplicial set, whose cofaces and codegeneracy operators are denoted

by  $d_h^i$ ,  $d_v^j$ , and so on, as in the diagram

$$\Delta[p-1, q] \xrightleftharpoons[s_h^i = s^i \otimes id]{d_h^i = d^i \otimes id} \Delta[p, q] \xrightleftharpoons[s_v^j = id \otimes s^j]{d_v^j = id \otimes d^j} \Delta[p, q-1].$$

The  $(k, l)^{\text{th}}$ -horn  $\Lambda^{k,l}[p, q]$ , for any integers  $0 \leq k \leq p$  and  $0 \leq l \leq q$ , is the bisimplicial subset of  $\Delta[p, q]$  generated by the horizontal and vertical faces  $\Delta[p-1, q] \xrightarrow{d_h^i} \Delta[p, q]$  and  $\Delta[p, q-1] \xrightarrow{d_v^j} \Delta[p, q]$  for all  $i \neq k$  and  $j \neq l$ . There is a natural pushout diagram

$$\begin{array}{ccc} \Lambda^k[p] \otimes \Lambda^l[q] & \hookrightarrow & \Delta[p] \otimes \Lambda^l[q] \\ \downarrow & & \downarrow \\ \Lambda^k[p] \otimes \Delta[q] & \hookrightarrow & \Lambda^{k,l}[p, q], \end{array}$$

which, taking into account the coequalizers (2.1), states that the system of data to define a bisimplicial map  $x : \Lambda^{k,l}[p, q] \rightarrow K$  consists of a list of bisimplices

$$x = (x_0, \dots, x_{k-1}, -, x_{k+1}, \dots, x_p; x'_0, \dots, x'_{l-1}, -, x'_{l+1}, \dots, x'_q),$$

where  $x_i : \Delta[p-1, q] \rightarrow K$  and  $x'_j : \Delta[p, q-1] \rightarrow K$ , such that the following compatibility conditions hold:

- $x_j d_h^i = x_i d_h^{j-1}$ , for all  $0 \leq i < j \leq p$  with  $i \neq k \neq j$ ,
- $x'_j d_v^i = x'_i d_v^{j-1}$ , for all  $0 \leq i < j \leq q$  with  $i \neq l \neq j$ ,
- $x'_j d_h^i = x_i d_v^j$ , for all  $0 \leq i \leq p$ ,  $0 \leq j \leq q$  with  $i \neq k$ ,  $j \neq l$ .

A bisimplicial set  $K$  satisfies the *extension condition* if all simplicial sets  $K_{p,*}$  and  $K_{*,q}$  are Kan complexes, that is, if any of the extension problems

$$\begin{array}{ccc} \Delta[p] \otimes \Lambda^l[q] & \longrightarrow & K \\ \downarrow & \nearrow \exists? & \\ \Delta[p, q] & & \end{array} \quad \begin{array}{ccc} \Lambda^k[p] \otimes \Delta[q] & \longrightarrow & K \\ \downarrow & \nearrow \exists? & \\ \Delta[p, q] & & \end{array}$$

has a solution, and, moreover, if there is also a solution for any extension problem of the form

$$\begin{array}{ccc} \Lambda^{k,l}[p, q] & \longrightarrow & K \\ \downarrow & \nearrow \exists? & \\ \Delta[p, q] & & \end{array}$$

When a bisimplicial set  $K$  satisfies the extension condition, then every bisimplex  $x : \Delta[p, q] \rightarrow K$ , which can be regarded both as a simplex of the vertical Kan complex  $K_{p,*}$  and as a simplex of the horizontal Kan complex  $K_{*,q}$ , defines both a *vertical homotopy class*, denoted by  $[x]_v$ , and a *horizontal homotopy class*, denoted by  $[x]_h$ . The following lemma is needed.

**Lemma 2.4.** *Let  $x, x' : \Delta[p, q] \rightarrow K$  be bisimplices of bisimplicial set  $K$ , which satisfies the extension condition. The following conditions are equivalent:*

- i) *There exists  $y : \Delta[p, q] \rightarrow K$  such that  $[x]_h = [y]_h$  and  $[y]_v = [x']_v$ ,*
- ii) *There exists  $z : \Delta[p, q] \rightarrow K$  such that  $[x]_v = [z]_v$  and  $[z]_h = [x']_h$ .*

*Proof.* We only prove that i) implies ii) since the proof for the other implication is similar. Let  $\alpha : \Delta[p+1, q] \rightarrow K$  be a horizontal homotopy (i.e., a homotopy in the Kan complex  $K_{*,q}$ ) from  $x$  to  $y$ , and let  $\beta : \Delta[p, q+1] \rightarrow K$  be a vertical homotopy from  $y$  to  $x'$ . Since  $K$  satisfies the

extension condition, a bisimplicial map  $\Gamma : \Delta[p+1, q+1] \rightarrow K$  can be found such that the diagram below commutes.

$$\begin{array}{ccc} \Lambda^{p,q+1}[p+1, q+1] & \xrightarrow{(\beta d_h^0 s_h^{p-1}, \dots, \beta d_h^{p-1} s_h^{p-1}, -, \beta; \alpha d_v^0 s_v^{q-1}, \dots, \alpha d_v^{q-1} s_v^{q-1}, \alpha, -)} & K \\ \downarrow & \searrow \Gamma & \\ \Delta[p+1, q+1] & & \end{array}$$

Then, by taking  $\alpha' = \Gamma d_v^{q+1} : \Delta[p+1, q] \rightarrow K$ ,  $\beta' = \Gamma d_h^p : \Delta[p, q+1] \rightarrow K$ , and  $z = \alpha' d_h^p = \beta' d_v^{q+1} : \Delta[p, q] \rightarrow K$ , one sees that  $\alpha'$  becomes a horizontal homotopy (i.e., a homotopy in  $K_{*,q}$ ) from  $z$  to  $x'$  and  $\beta'$  becomes a vertical homotopy from  $x$  to  $z$ . Therefore,  $[x]_v = [z]_v$  and  $[z]_h = [x']_h$ , as required.  $\square$

The two simplices  $x, x' : \Delta[p, q] \rightarrow K$  in the above Lemma 2.4 are said to be *bihomotopic* if the equivalent conditions i) and ii) hold.

**Lemma 2.5.** *If  $K$  is a bisimplicial set satisfying the extension condition, then ‘to be bihomotopic’ is an equivalence relation on the bisimplices of bidegree  $(p, q)$  of  $K$ , for any  $p, q \geq 0$ .*

*Proof.* The relation is obviously reflexive, and it is symmetric thanks to Lemma 2.4. For transitivity, suppose  $x, x', x'' : \Delta[p, q] \rightarrow K$  such that  $x$  and  $x'$  are bihomotopic as well as  $x'$  and  $x''$  are. Then, for some  $y, y' : \Delta[p, q] \rightarrow K$ , we have  $[x]_h = [y]_h$ ,  $[y]_v = [x']_v$ ,  $[x']_h = [y']_h$ , and  $[y']_v = [x'']_v$ . Also, again by Lemma 2.4, there is  $z : \Delta[p, q] \rightarrow K$  such that  $[y]_h = [z]_h$  and  $[z]_v = [y']_v$ . It follows that  $[x]_h = [z]_h$  and  $[z]_v = [x'']_v$ , whence  $x$  and  $x''$  are bihomotopic.  $\square$

We will write  $[[x]]$  for the bihomotopic class of a bisimplex  $x : \Delta[p, q] \rightarrow K$ .

**Lemma 2.6.** *Let  $K$  be any bisimplicial set satisfying the extension condition. There are four well-defined mappings such that  $[[x]] \mapsto [xd_h^i]_v$ ,  $[[x]] \mapsto [xd_v^j]_h$ ,  $[x]_h \mapsto [[xs_v^j]]$ , and  $[x]_v \mapsto [[xs_h^i]]$  respectively, for any  $x : \Delta[p, q] \rightarrow K$ ,  $0 \leq i \leq p$  and  $0 \leq j \leq q$ .*

*Proof.* Suppose that  $[[x]] = [[x']]$ . Then,  $[x]_h = [y]_h$  and  $[y]_v = [x']_v$ , for some  $y : \Delta[p, q] \rightarrow K$ . It follows that  $xd_h^i = yd_h^i$  and there is a vertical homotopy, say  $z : \Delta[p, q+1] \rightarrow K$ , from  $y$  to  $x'$ . As  $zd_h^i : \Delta[p-1, q+1] \rightarrow K$  is then a vertical homotopy from  $yd_h^i$  to  $x'd_h^i$ , we conclude that  $[xd_h^i]_v = [x'd_h^i]_v$ . The proof that  $[xd_v^j]_h = [x'd_v^j]_h$  is similar. For the third mapping, note that any horizontal homotopy  $y : \Delta[p+1, q] \rightarrow K$  from  $x$  to  $x'$  yields the horizontal homotopy  $ys_v^j : \Delta[p+1, q+1] \rightarrow K$  from  $xs_v^j$  to  $x's_v^j$ . Therefore,  $[xs_v^j]_h = [x's_v^j]_h$ , whence  $[[xs_v^j]] = [[x's_v^j]]$ , as required. Similarly, we see that  $[x]_v = [x']_v$  implies  $[[xs_h^i]] = [[x's_h^i]]$ .  $\square$

We shall end this subsection by remarking that any bisimplicial set  $K$ , satisfying the extension condition, has associated *horizontal fundamental groupoids*  $PK_{*,q}$ , one for each integer  $q \geq 0$ , whose objects are the bisimplices  $x : \Delta[0, q] \rightarrow K$  and morphisms  $[y]_h : x' \rightarrow x$  horizontal homotopy classes of bisimplices  $y : \Delta[1, q] \rightarrow K$  with  $yd_h^0 = x'$  and  $yd_h^1 = x$ . The composition in these groupoids  $PK_{*,q}$  is written using the symbol  $\circ_h$ , so the composite of  $[y]_h$  with  $[y']_h : x'' \rightarrow x'$  is

$$[y]_h \circ_h [y']_h = [\gamma d_h^1]_h,$$

where  $\gamma : \Delta[2, q] \rightarrow K$  is a (any) bisimplex with  $\gamma d_h^2 = y$  and  $\gamma d_h^0 = y'$ . The identities are denoted as  $I^h x$ , that is,  $I^h x = [xs_h^0]_h$ .

And similarly,  $K$  also has associated *vertical fundamental groupoids*  $PK_{p,*}$ ,  $p \geq 0$ , whose morphisms  $[z]_v : zd_v^0 \rightarrow zd_v^1$  are vertical homotopy classes of bisimplices  $z : \Delta[p, 1] \rightarrow K$ . For these, we use the symbol  $\circ_v$  for denoting the composition and  $I^v$  for identities.

### 2.3. Weak homotopy types: Some related constructions.

Let **Top** denote the category of spaces and continuous maps. A map  $X \rightarrow X'$  in **Top** is a *weak equivalence* if it induces an isomorphism  $\pi_i(X, a) \cong \pi_i(X', fa)$  for all base points  $a$  of  $X$  and  $i \geq 0$ . The *category of weak homotopy types* is defined as the localization of the category of spaces with respect to the class of weak equivalences [29, 20] and, for any given integer  $n$ , the *category of homotopy  $n$ -types* is its full subcategory given by those spaces  $X$  with  $\pi_i(X, a) = 0$  for any integer  $i > n$  and any base point  $a$ .

There are various constructions on (bi)simplicial sets that traditionally aid in the algebraic study of homotopy  $n$ -types. Below is a brief review of the constructions used in this work.

Segal's *geometric realization* functor [31], for simplicial spaces  $K : \Delta^o \rightarrow \mathbf{Top}$ , is denoted by  $K \mapsto |K|$ . Recall that it is defined as the left adjoint to the functor that associates to a space  $X$  the simplicial space  $[n] \mapsto X^{\Delta_n}$ , where

$$\Delta_n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum t_i = 1, 0 \leq t_i \leq 1\}$$

denotes the affine simplex having  $[n]$  as its set of vertices and  $X^{\Delta_n}$  is the the function space of continuous maps from  $\Delta_n$  to  $X$ , given the compact-open topology. The underlying simplicial set is the *singular complex* of  $X$ , denoted by  $SX$ .

For instance, by regarding a set as a discrete space, the (Milnor's) geometric realization of a simplicial set  $L : \Delta^o \rightarrow \mathbf{Set}$  is  $|L|$ , which is a CW-complex whose  $n$ -cells are in one-to-one correspondence with the  $n$ -simplices of  $L$  which are nondegenerate. The following six facts are well-known:

**Facts 2.7.** (1) *For any space  $X$ ,  $SX$  is a Kan complex.*

- (2) *For any Kan complex  $L$ , there are natural isomorphisms  $\pi_i(L, a) \cong \pi_i(|L|, |a|)$ , for all base vertices  $a : \Delta[0] \rightarrow L$  and  $n \geq 0$ .*
- (3) *A simplicial map between Kan complexes  $L \rightarrow L'$  is a homotopy equivalence if and only if the induced map on realizations  $|L| \rightarrow |L'|$  is a homotopy equivalence.*
- (4) *For any Kan complex  $L$ , the unit of the adjunction  $L \rightarrow S|L|$  is a homotopy equivalence.*
- (5) *A continuous map  $X \rightarrow Y$  is a weak homotopy equivalence if and only if the induced  $SX \rightarrow SY$  is a homotopy equivalence.*
- (6) *For any space  $X$ , the counit  $|SX| \rightarrow X$  is a weak homotopy equivalence.*

When a bisimplicial set  $K : \Delta^o \times \Delta^o \rightarrow \mathbf{Set}$  is regarded as a simplicial object in the simplicial set category and one takes geometric realizations, then one obtains a simplicial space  $\Delta^o \rightarrow \mathbf{Top}$ ,  $[p] \mapsto |K_{p,*}|$ , whose Segal realization is taken to be  $|K|$ , the geometric realization of  $K$ . As there are natural homeomorphisms [30, Lemma in page 86]

$$|[p]| \mapsto |K_{p,*}| \cong |\mathrm{diag} K| \cong |[q]| \mapsto |K_{*,q}|,$$

where  $\mathrm{diag} K$  is the simplicial set obtained by composing  $K$  with the diagonal functor  $\Delta \rightarrow \Delta \times \Delta$ ,  $[n] \mapsto ([n], [n])$ , one usually takes

$$|K| = |\mathrm{diag} K|.$$

Composing with the ordinal sum functor  $or : \Delta \times \Delta \rightarrow \Delta$ ,  $([p], [q]) \mapsto [p+1+q]$ , gives Illusie's *total Dec* functor,  $L \mapsto \mathrm{Dec} L$ , from simplicial to bisimplicial sets [21, VI, 1.5]. More specifically, for any simplicial set  $L$ ,  $\mathrm{Dec} L$  is the bisimplicial set whose bisimplices of bidegree  $(p, q)$  are the  $(p+1+q)$ -simplices of  $L$ ,  $x : \Delta[p+1+q] \rightarrow L$ , and whose simplicial operators are given by  $xd_h^i = xd^i$ ,  $xs_h^i = xs^i$ , for  $0 \leq i \leq p$ , and  $xd_v^j = d^{p+1+j}$ ,  $xs_v^j = xs^{p+1+j}$ , for  $0 \leq j \leq q$ . The functor  $\mathrm{Dec}$  has a right adjoint [14]

$$(2.2) \quad \mathrm{Dec} \dashv \overline{W},$$



often called the *codiagonal* functor, whose description is as follows [2, III]: for any bisimplicial set  $K$ , an  $n$ -simplex of  $\overline{W}K$  is a bisimplicial map

$$\bigsqcup_{p=0}^n \Delta[p, n-p] \xrightarrow{(x_0, \dots, x_n)} K$$

such that  $x_p d_v^0 = x_{p+1} d_h^{p+1}$ , for  $0 \leq p < n$ , whose faces and degeneracies are given by

$$\begin{aligned} (x_0, \dots, x_n) d^i &= (x_0 d_v^i, \dots, x_{i-1} d_v^1, x_{i+1} d_h^i, \dots, x_n d_h^i), \\ (x_0, \dots, x_n) s^i &= (x_0 s_v^i, \dots, x_i s_v^0, x_i s_h^i, \dots, x_n s_h^i). \end{aligned}$$

The unit and the counit of the adjunction,  $u : L \rightarrow \overline{W} \text{Dec} L$  and  $v : \text{Dec} \overline{W} K \rightarrow K$ , are respectively defined by

$$\begin{aligned} u(y) &= (ys^0, \dots, ys^n) & (y : \Delta[n] \rightarrow L) \\ v(x_0, \dots, x_{p+1+q}) &= x_{p+1} d_h^0 & ((x_0, \dots, x_{p+1+q}) : \Delta[p, q] \rightarrow \text{Dec} \overline{W} X). \end{aligned}$$

The following facts are used in our development below:

**Facts 2.8.** (1) *For each  $n \geq 0$ , there is a natural Alexander-Whitney type diagonal approximation*

$$\begin{aligned} \phi : \text{Dec} \Delta[n] &\rightarrow \Delta[n, n], \\ (\Delta[p+1+q] \xrightarrow{x} \Delta[n]) &\mapsto (\Delta[p] \xrightarrow{x(d^{p+1})^q} \Delta[n], \Delta[q] \xrightarrow{x(d^0)^{p+1}} \Delta[n]) \end{aligned}$$

*such that, for any bisimplicial set  $K$ , the induced simplicial map  $\phi^* : \text{diag} K \rightarrow \overline{W} K$  determines a homotopy equivalence*

$$|\text{diag} K| \simeq |\overline{W} K|$$

*on the corresponding geometric realizations [12, Theorem 1.1].*

- (2) *For any simplicial map  $f : L \rightarrow L'$ , the induced  $|f| : |L| \rightarrow |L'|$  is a homotopy equivalence if and only if the induced  $|\text{Dec} f| : |\text{Dec} L| \rightarrow |\text{Dec} L'|$  is a homotopy equivalence [12, Corollary 7.2].*
- (3) *For any simplicial set  $L$  and any bisimplicial set  $K$ , both induced maps  $|u| : |L| \rightarrow |\overline{W} \text{Dec} L|$  and  $|v| : |\text{Dec} \overline{W} K| \rightarrow |K|$  are homotopy equivalences [12, Proposition 7.1 and discussion below].*
- (4) *If  $K$  is any bisimplicial set satisfying the extension condition, then  $\overline{W} K$  is a Kan complex [13, Proposition 2].*
- (5) *If  $L$  is a Kan complex, then  $\text{Dec} L$  satisfies the extension condition (the proof is a straightforward application of [26, Lemma 7.4]) or [13, Lemma 1]).*

### 3. DOUBLE GROUPOIDS SATISFYING THE FILLING CONDITION: HOMOTOPY GROUPS.

A (small) double groupoid [15, 16, 10, 22] is a groupoid object in the category of small groupoids. In general, we employ the standard nomenclature concerning double categories but, for the sake of clarity, we shall fix some terminology and notations below.

A (small) category can be described as a system  $(M, O, s, t, I, \circ)$ , where  $M$  is the set of morphisms,  $O$  is the set of objects,  $s, t : M \rightarrow O$  are the source and target maps, respectively,  $I : O \rightarrow M$  is the identities map, and  $\circ : M \times_t M \rightarrow M$  is the composition map, subject to the usual associativity and identity axioms. Therefore, a *double category* provides us with the following data: a set  $O$  of *objects*, a set  $H$  of *horizontal morphisms*, a set  $V$  of *vertical morphisms*, and a set  $C$  of *squares*, together with four category structures, namely, the *category of horizontal morphisms*  $(H, O, s^h, t^h, I^h, \circ_h)$ , the *category of vertical morphisms*  $(V, O, s^v, t^v, I^v, \circ_v)$ , the *horizontal*

category of squares  $(C, V, s^h, t^h, I^h, \circ_h)$ , and the vertical category of squares  $(C, H, s^v, t^v, I^v, \circ_v)$ . These are subject to the following three axioms:

$$\mathbf{Axiom\ 1:} \begin{cases} \text{(i)} & s^h s^v = s^v s^h, \quad t^h t^v = t^v t^h, \quad s^h t^v = t^v s^h, \quad s^v t^h = t^h s^v, \\ \text{(ii)} & s^h I^v = I^v s^h, \quad t^h I^v = I^v t^h, \quad s^v I^h = I^h s^v, \quad t^v I^h = I^h t^v, \\ \text{(iii)} & I^h I^v = I^v I^h. \end{cases}$$

Equalities in **Axiom 1** allow a square  $\alpha \in C$  to be depicted in the form

$$(3.1) \quad \begin{array}{ccc} & d & \xleftarrow{g} b \\ w \uparrow & \alpha & \uparrow u \\ & c & \xleftarrow{f} a \end{array}$$

where  $s^h \alpha = u$ ,  $t^h \alpha = w$ ,  $s^v \alpha = f$  and  $t^v \alpha = g$ , and the four vertices of the square representing  $\alpha$  are  $s^h s^v \alpha = a$ ,  $t^h t^v \alpha = d$ ,  $s^h t^v \alpha = b$  and  $s^v t^h \alpha = c$ . Moreover, if we represent identity morphisms by the symbol  $\text{---}$ , then, for any horizontal morphism  $f$ , any vertical morphism  $u$ , and any object  $a$ , the associated identity squares  $I^v f$ ,  $I^h u$  and  $Ia := I^h I^v a = I^v I^h a$  are respectively given in the form

$$\begin{array}{ccc} \cdot & \xleftarrow{f} & \cdot \\ \text{---} & & \text{---} \\ \cdot & \xleftarrow{f} & \cdot \end{array} \quad \begin{array}{ccc} \cdot & \text{---} & \cdot \\ u \uparrow & & \uparrow u \\ \cdot & \text{---} & \cdot \end{array} \quad \begin{array}{ccc} \cdot & \text{---} & \cdot \\ \text{---} & & \text{---} \\ \cdot & \text{---} & \cdot \end{array}$$

The equalities in **Axiom 2** below show the squares are compatible with the boundaries, whereas **Axiom 3** establishes the necessary coherence between the two vertical and horizontal compositions of squares.

$$\mathbf{Axiom\ 2:} \begin{cases} \text{(i)} & s^v(\alpha \circ_h \beta) = s^v \alpha \circ_h s^v \beta, \quad t^v(\alpha \circ_h \beta) = t^v \alpha \circ_h t^v \beta, \\ \text{(ii)} & s^h(\alpha \circ_v \beta) = s^h \alpha \circ_v s^h \beta, \quad t^h(\alpha \circ_v \beta) = t^h \alpha \circ_v t^h \beta, \\ \text{(iii)} & I^v(f \circ_h f') = I^v f \circ_h I^v f', \quad I^h(u \circ_v u') = I^h u \circ_v I^h u'. \end{cases}$$

**Axiom 3:** In the situation

$$\begin{array}{ccccc} \cdot & \xleftarrow{\quad} & \cdot & \xleftarrow{\quad} & \cdot \\ \uparrow & \alpha & \uparrow & \beta & \uparrow \\ \cdot & \xleftarrow{\quad} & \cdot & \xleftarrow{\quad} & \cdot \\ \uparrow & \gamma & \uparrow & \delta & \uparrow \\ \cdot & \xleftarrow{\quad} & \cdot & \xleftarrow{\quad} & \cdot \end{array}$$

the interchange law holds, that is,  $(\alpha \circ_h \beta) \circ_v (\gamma \circ_h \delta) = (\alpha \circ_v \gamma) \circ_h (\beta \circ_v \delta)$ .

A *double groupoid* is a double category such that all the four component categories are groupoids. We shall use the following notation for inverses in a double groupoid:  $f^{-1_h}$  denotes the inverse of a horizontal morphism  $f$ , and  $u^{-1_v}$  denotes the inverse of a vertical morphism  $u$ . For any square  $\alpha$  as in (3.1), the first one of

$$\begin{array}{ccc} b & \xleftarrow{g^{-1_h}} & d \\ u \uparrow & \alpha^{-1_h} & \uparrow w \\ a & \xleftarrow{f^{-1_h}} & c, \end{array} \quad \begin{array}{ccc} c & \xleftarrow{f} & a \\ w^{-1_v} \uparrow & \alpha^{-1_v} & \uparrow u^{-1_v} \\ d & \xleftarrow{g} & b, \end{array} \quad \begin{array}{ccc} a & \xleftarrow{f^{-1_h}} & c \\ u^{-1_v} \uparrow & \alpha^{-1} & \uparrow w^{-1_v} \\ b & \xleftarrow{g^{-1_h}} & d, \end{array}$$

is the inverse of  $\alpha$  in the horizontal groupoid of squares, the second one denotes the inverse of  $\alpha$  in the vertical groupoid of squares, and the third one is the square  $(\alpha^{-1_h})^{-1_v} = (\alpha^{-1_v})^{-1_h}$ , which is denoted simply by  $\alpha^{-1}$ .

The double groupoids we are interested in satisfy the condition below.

**Filling condition:** *Any filling problem*

$$\begin{array}{ccc} \cdot & \xleftarrow{g} & \cdot \\ \uparrow \exists? & & \uparrow u \\ \cdot & \xleftarrow{\quad} & \cdot \end{array}$$

has a solution; that is, for any horizontal morphism  $g$  and any vertical morphism  $u$  such that  $s^h g = t^v u$ , there is a square  $\alpha$  with  $s^h \alpha = u$  and  $t^v \alpha = g$ .

As we recalled in the introduction, this filling condition on double groupoids is often satisfied for those double groupoids arising in algebraic topology. Further below, in Sections 4 and 6, we post two new homotopical double groupoid constructions that relevant to our deliberations: one,  $\mathbf{IIX}$ , for topological spaces  $X$ , and the other,  $\mathbf{PK}$ , for bisimplicial sets  $K$ , both yielding double groupoids satisfying the filling condition.

The remainder of this section is devoted to defining *homotopy groups*,  $\pi_i(\mathcal{G}, a)$ , for double groupoids  $\mathcal{G}$  satisfying the filling condition. The useful observation below is a direct consequence of [1, Lemma 1.12].

**Lemma 3.1.** *A double groupoid  $\mathcal{G}$  satisfies the filling condition if and only if any filling problem such as the one below has a solution.*

$$\begin{array}{ccc} \cdot & \xleftarrow{\quad} & \cdot \\ w \uparrow \exists? & & \uparrow \exists? \\ \cdot & \xleftarrow{f} & \cdot \end{array}, \quad \begin{array}{ccc} \cdot & \xleftarrow{\quad} & \cdot \\ \uparrow \exists? & & \uparrow u \\ \cdot & \xleftarrow{f} & \cdot \end{array}, \quad \begin{array}{ccc} \cdot & \xleftarrow{g} & \cdot \\ w \uparrow \exists? & & \uparrow \exists? \\ \cdot & \xleftarrow{\quad} & \cdot \end{array},$$

Hereafter, we assume  $\mathcal{G}$  is a double groupoid satisfying the filling condition.

### 3.1. The pointed sets $\pi_0(\mathcal{G}, a)$ .

We state that two objects  $a, b$  of  $\mathcal{G}$  are *connected* whenever there is a pair of morphisms  $(g, u)$  in  $\mathcal{G}$  of the form

$$\begin{array}{ccc} b & \xleftarrow{g} & \cdot \\ \uparrow u & & \\ a & & \end{array},$$

that is, where  $g$  is a horizontal morphism and  $u$  a vertical morphism such that  $s^h g = t^v u$ ,  $t^h g = b$ , and  $s^v u = a$ . Because of the filling condition, this is equivalent to saying that there is a square in  $\mathcal{G}$  of the form

$$\begin{array}{ccc} b & \xleftarrow{g} & \cdot \\ w \uparrow \alpha & & \uparrow u \\ \cdot & \xleftarrow{f} & a \end{array},$$

and it is also equivalent to saying that there is a pair of matching morphisms  $(w, f)$  as

$$\begin{array}{ccc} b & & \\ w \uparrow & & \\ \cdot & \xleftarrow{f} & a. \end{array}$$

If  $a$  and  $b$  are recognized as being connected by means of the pair of morphisms  $(g, u)$  as above, then the pair  $(u^{-1v}, g^{-1h})$  shows that  $b$  is connected to  $a$ . Hence, being connected is a symmetric relation on the set of objects of  $\mathcal{G}$ . This relation is clearly reflexive thanks to the identity morphisms  $(I^h a, I^v a)$ , and it is also transitive. Suppose  $a$  is connected with  $b$ , which

itself is connected with another object  $c$ . Then, we have morphisms  $u, f, v, g$  as in the diagram

$$\begin{array}{ccccc} c & \xleftarrow{g} & \cdot & \xleftarrow{g'} & \cdot \\ & \uparrow v & \beta & \uparrow u' & \\ & b & \xleftarrow{f} & \cdot & \\ & & \uparrow u & & \\ & & a & & \end{array}$$

where  $\beta$  is any square with  $t^h\beta = v$  and  $s^v\beta = f$ , and the dotted  $g'$  and  $u'$  are the other sides of  $\beta$ . Consequently, on considering the pair of composites  $(g \circ_h g', u' \circ_v u)$ , we see that  $a$  and  $c$  are connected.

Therefore, being connected establishes an equivalence relation on the objects of the double groupoid and, associated to  $\mathcal{G}$ , we take

$$(3.2) \quad \pi_0\mathcal{G} = \text{the set of connected classes of objects of } \mathcal{G},$$

and we write  $\pi_0(\mathcal{G}, a)$  for the set  $\pi_0\mathcal{G}$  pointed with the class  $[a]$  of an object  $a$  of  $\mathcal{G}$ .

### 3.2. The groups $\pi_1(\mathcal{G}, a)$ .

Let  $a$  be any given object of  $\mathcal{G}$ , and let

$$(3.3) \quad \mathcal{G}(a) = \left\{ \begin{array}{c} a \xleftarrow{g} x \\ \uparrow u \\ a \end{array} \right\}$$

be the set of all pairs of morphisms  $(g, u)$ , where  $g$  is a horizontal morphism and  $u$  a vertical morphism in  $\mathcal{G}$  such that  $t^h g = a = s^v u$  and  $s^h g = t^v u$ .

Define a relation  $\sim$  on  $\mathcal{G}(a)$  by the rule  $(g, u) \sim (g', u')$  if and only if there are two squares  $\alpha$  and  $\alpha'$  in  $\mathcal{G}$  of the form

$$\begin{array}{ccc} a \xleftarrow{g} \cdot & & a \xleftarrow{g'} \cdot \\ w \uparrow \alpha \uparrow u & & w \uparrow \alpha' \uparrow u' \\ \cdot \xleftarrow{f} a & & \cdot \xleftarrow{f} a \end{array}$$

that is, such that  $t^h\alpha = t^h\alpha'$ ,  $s^v\alpha = s^v\alpha'$ ,  $s^h\alpha = u$ ,  $s^h\alpha' = u'$ ,  $t^v\alpha = g$ , and  $t^v\alpha' = g'$ .

**Lemma 3.2.** *The relation  $\sim$  is an equivalence.*

*Proof.* Since  $\mathcal{G}$  satisfies the filling condition, the relation is clearly reflexive, and it is obviously symmetric. To prove transitivity, suppose  $(g, u) \sim (g', u') \sim (g'', u'')$ , so that there are squares  $\alpha, \alpha', \beta$  and  $\beta'$  as below.

$$\begin{array}{cccc} a \xleftarrow{g} \cdot & a \xleftarrow{g'} \cdot & a \xleftarrow{g'} \cdot & a \xleftarrow{g''} \cdot \\ w \uparrow \alpha \uparrow u & w \uparrow \alpha' \uparrow u' & w' \uparrow \beta \uparrow u' & w' \uparrow \beta' \uparrow u'' \\ \cdot \xleftarrow{f} a & \cdot \xleftarrow{f} a & \cdot \xleftarrow{f'} a & \cdot \xleftarrow{f'} a \end{array}$$

Then, we have the horizontally composable squares

$$\begin{array}{ccccc} a \xleftarrow{g'} \cdot & \xleftarrow{g'^{-1h}} a & \xleftarrow{g} \cdot & & \\ w' \uparrow \beta & \uparrow \alpha'^{-1h} & \uparrow \alpha & \uparrow u & \\ \cdot \xleftarrow{f'} a & \xleftarrow{f'^{-1h}} \cdot & \xleftarrow{f} a & & \end{array}$$

whose composition  $\beta \circ_h \alpha'^{-1h} \circ_h \alpha$  and  $\beta'$  show that  $(g, u) \sim (g'', u'')$ . □

We write  $[g, u]$  for the  $\sim$ -equivalence class of  $(g, u) \in \mathcal{G}(a)$ . Now we define a product on

$$(3.4) \quad \pi_1(\mathcal{G}, a) := \mathcal{G}(a) / \sim$$

as follows: given  $[g_1, u_1], [g_2, u_2] \in \pi_1(\mathcal{G}, a)$ , by the filling condition on  $\mathcal{G}$ , we can choose a square  $\gamma$  with  $s^v \gamma = g_2$  and  $t^h \gamma = u_1$  so that we have a configuration in  $\mathcal{G}$  of the form

$$\begin{array}{ccccc} a & \xleftarrow{g_1} & \cdot & \xleftarrow{g} & \cdot \\ & u_1 \uparrow & & \gamma & \uparrow u \\ & a & \xleftarrow{g_2} & \cdot & \\ & & & \uparrow u_2 & \\ & & & a & \end{array}$$

where  $g = t^v \gamma$  and  $u = s^h \gamma$ . Then we define

$$(3.5) \quad [g_1, u_1] \circ [g_2, u_2] = [g_1 \circ_h g, u \circ_v u_2]$$

**Lemma 3.3.** *The product is well defined.*

*Proof.* Let  $[g_1, u_1] = [g'_1, u'_1]$ ,  $[g_2, u_2] = [g'_2, u'_2]$  be elements of  $\pi_1(\mathcal{G}, a)$ . Then, there are squares

$$\begin{array}{cccc} \begin{array}{ccc} a & \xleftarrow{g_1} & \cdot \\ w_1 \uparrow & \alpha & \uparrow u_1 \\ \cdot & \xleftarrow{f_1} & a \end{array} & \begin{array}{ccc} a & \xleftarrow{g'_1} & \cdot \\ w_1 \uparrow & \alpha' & \uparrow u'_1 \\ \cdot & \xleftarrow{f_1} & a \end{array} & \begin{array}{ccc} a & \xleftarrow{g_2} & \cdot \\ w_2 \uparrow & \beta & \uparrow u_2 \\ \cdot & \xleftarrow{f_2} & a \end{array} & \begin{array}{ccc} a & \xleftarrow{g'_2} & \cdot \\ w_2 \uparrow & \beta' & \uparrow u'_2 \\ \cdot & \xleftarrow{f_2} & a \end{array} \end{array}$$

and choosing squares  $\gamma$  and  $\gamma'$  as in

$$\begin{array}{ccc} \cdot & \xleftarrow{g} & \cdot \\ u_1 \uparrow & & \gamma & \uparrow u \\ a & \xleftarrow{g_2} & \cdot \end{array} \quad \begin{array}{ccc} \cdot & \xleftarrow{g'} & \cdot \\ u'_1 \uparrow & & \gamma' & \uparrow u' \\ a & \xleftarrow{g'_2} & \cdot \end{array}$$

we have  $[g_1, u_1] \circ [g_2, u_2] = [g_1 \circ_h g, u \circ_v u_2]$  and  $[g'_1, u'_1] \circ [g'_2, u'_2] = [g'_1 \circ_h g', u' \circ_v u'_2]$ . Now, letting  $\theta$  be any square with  $t^v \theta = f_1$  and  $s^h \theta = w_2$ , we have squares as in

$$\begin{array}{ccc} \begin{array}{ccc} a & \xleftarrow{g_1} & \cdot \\ w_1 \uparrow & \alpha & \uparrow \gamma & \uparrow u \\ \cdot & \xleftarrow{a} & \cdot \\ \uparrow & \theta & \uparrow \beta & \uparrow u_2 \\ \cdot & \xleftarrow{f_2} & a \end{array} & \begin{array}{ccc} a & \xleftarrow{g'_1} & \cdot \\ w_1 \uparrow & \alpha' & \uparrow \gamma' & \uparrow u' \\ \cdot & \xleftarrow{a} & \cdot \\ \uparrow & \theta & \uparrow \beta' & \uparrow u'_2 \\ \cdot & \xleftarrow{f_2} & a \end{array} \end{array}$$

whose corresponding composites  $(\alpha \circ_h \gamma) \circ_v (\theta \circ_h \beta)$  and  $(\alpha' \circ_h \gamma') \circ_v (\theta \circ_h \beta')$  show that  $[g_1 \circ_h g, u \circ_v u_2] = [g'_1 \circ_h g', u' \circ_v u'_2]$ , as required.  $\square$

**Lemma 3.4.** *The given multiplication turns  $\pi_1(\mathcal{G}, a)$  into a group.*

*Proof.* To see the associativity, let  $[g_1, u_1], [g_2, u_2], [g_3, u_3] \in \pi_1(\mathcal{G}, a)$ , and choose  $\gamma, \gamma'$  and  $\gamma''$  any three squares as in the diagram (3.6) below. Then we have

$$([g_1, u_1] \circ [g_2, u_2]) \circ [g_3, u_3] = [g_1 \circ_h g \circ_h g', u \circ_v u' \circ_v u_3] = [g_1, u_1] \circ ([g_2, u_2] \circ [g_3, u_3]).$$

$$(3.6) \quad \begin{array}{c} a \xleftarrow{g_1} \cdot \xleftarrow{g} \cdot \xleftarrow{g'} \cdot \\ \uparrow u_1 \quad \gamma \quad \uparrow \gamma' \quad \uparrow u \\ a \xleftarrow{g_2} \cdot \xleftarrow{\gamma''} \cdot \\ \uparrow u_2 \quad \uparrow u' \\ a \xleftarrow{g_3} \cdot \\ \uparrow u_3 \\ a \end{array}$$

The identity of  $\pi_1(\mathcal{G}, a)$  is  $[\Gamma^h a, \Gamma^v a]$ . In effect, if  $[g, u] \in \pi_1(\mathcal{G}, a)$ , then the diagrams

$$\begin{array}{ccc} a \xleftarrow{g} x & \xleftarrow{\quad} x & a \xleftarrow{g} x \\ \uparrow u \quad \Gamma^h u \quad \uparrow u & & \uparrow \Gamma^v g \quad \uparrow \\ a \xleftarrow{\quad} a & & a \xleftarrow{g} x \\ \uparrow & & \uparrow u \\ a & & a \end{array}$$

show that

$$[g, u] \circ [\Gamma^h a, \Gamma^v a] = [g \circ_h \Gamma^h x, u \circ_v \Gamma^v a] = [g, u] = [\Gamma^h a \circ_h g, \Gamma^v x \circ_v u] = [\Gamma^h a, \Gamma^v a] \circ [g, u].$$

Finally, to see the existence of inverses, let  $[g, u] \in \pi_1(\mathcal{G}, a)$ . By choosing any square  $\alpha$  with  $t^h \alpha = u^{-1v}$  and  $s^v \alpha = g^{-1h}$ , that is, of the form

$$\begin{array}{ccc} a & \xleftarrow{f} & \cdot \\ \uparrow u^{-1v} \quad \alpha \quad \uparrow v & & \\ \cdot & \xleftarrow{g^{-1h}} & a \end{array}$$

we find  $[f, v] := [t^v \alpha, s^h \alpha] \in \pi_1(\mathcal{G}, a)$ . Since the diagrams

$$\begin{array}{ccc} a \xleftarrow{g} \cdot \xleftarrow{g^{-1h}} a & & a \xleftarrow{f} \cdot \xleftarrow{f^{-1h}} a \\ \uparrow u \quad \alpha^{-1v} \quad \uparrow v^{-1v} & & \uparrow v \quad \alpha^{-1h} \quad \uparrow u^{-1v} \\ a \xleftarrow{f} \cdot & & a \xleftarrow{g} \cdot \\ \uparrow v & & \uparrow u \\ a & & a \end{array}$$

show that  $[g, u] \circ [f, v] = [\Gamma^h a, \Gamma^v a] = [f, v] \circ [g, u]$ , we have  $[g, u]^{-1} = [f, v]$ .  $\square$

### 3.3. The abelian groups $\pi_i(\mathcal{G}, a)$ , $i \geq 2$ .

These are easier to define than the previous ones. For  $i = 2$ , as in [10, Section 2], we take

$$(3.7) \quad \pi_2(\mathcal{G}, a) = \left\{ \begin{array}{c} a \xleftarrow{\quad} a \\ \uparrow \alpha \quad \uparrow \\ a \xleftarrow{\quad} a \end{array} \right\}$$

the set of all squares  $\alpha \in \mathcal{G}$  whose boundary edges are  $s^h \alpha = t^h \alpha = \Gamma^v a$  and  $s^v \alpha = t^v \alpha = \Gamma^h a$ .

By the general Eckman-Hilton argument, it is a consequence of the interchange law that, on  $\pi_2(\mathcal{G}, a)$ , operations  $\circ_h$  and  $\circ_v$  coincide and are commutative. In effect, for  $\alpha, \beta \in \pi_2(\mathcal{G}, a)$ ,

$$\begin{aligned} \alpha \circ_h \beta &= (\alpha \circ_v \Gamma a) \circ_h (\Gamma a \circ_v \beta) = (\alpha \circ_h \Gamma a) \circ_v (\Gamma a \circ_h \beta) \\ &= \alpha \circ_v \beta \\ &= (\Gamma a \circ_h \alpha) \circ_v (\beta \circ_h \Gamma a) = (\Gamma a \circ_v \beta) \circ_h (\alpha \circ_v \Gamma a) \\ &= \beta \circ_h \alpha. \end{aligned}$$

Therefore,  $\pi_2(\mathcal{G}, a)$  is an abelian group with product

$$(3.8) \quad \alpha \circ_h \beta = \alpha \circ_v \beta,$$

identity  $Ia = I^v I^h a$ , and inverses  $\alpha^{-1h} = \alpha^{-1v}$ .

The higher homotopy groups of the double groupoid are defined to be trivial, that is,

$$(3.9) \quad \pi_i(\mathcal{G}, a) = 0 \quad \text{if } i \geq 3.$$

### 3.4. Weak equivalences.

A *double functor*  $F : \mathcal{G} \rightarrow \mathcal{G}'$  between double categories takes objects, horizontal and vertical morphisms, and squares in  $\mathcal{G}$  to objects, horizontal and vertical morphisms, and squares in  $\mathcal{G}'$ , respectively, in such a way that all the structure categories are preserved.

Clearly, each double functor  $F : \mathcal{G} \rightarrow \mathcal{G}'$ , between double groupoids satisfying the filling condition, induces maps (group homomorphisms if  $i > 0$ )

$$\pi_i F : \pi_i(\mathcal{G}, a) \rightarrow \pi_i(\mathcal{G}', Fa)$$

for  $i \geq 0$  and  $a$  any object of  $\mathcal{G}$ . Call such a double functor a *weak equivalence* if it induces isomorphisms  $\pi_i F$  for all integers  $i \geq 0$ .

## 4. A HOMOTOPY DOUBLE GROUPOID FOR TOPOLOGICAL SPACES.

Our aim here is to provide a new construction of a double groupoid for a topological space that, as we will see later, captures the homotopy 2-type of the space. For any given space  $X$ , the construction of this *homotopy double groupoid*, denoted by  $\mathbf{IIX}$ , is as follows:

The objects in  $\mathbf{IIX}$  are the paths in  $X$ , that is, the continuous maps  $u : I = [0, 1] \rightarrow X$ .

The groupoid of horizontal morphisms in  $\mathbf{IIX}$  is the category with a unique morphism between each pair  $(u', u)$  of paths in  $X$  such that  $u'(1) = u(1)$ , and, similarly, the groupoid of vertical morphisms in  $\mathbf{IIX}$  is the category having a unique morphism between each pair  $(v, u)$  of paths in  $X$  such that  $v(0) = u(0)$ .

A square in  $\mathbf{IIX}$ ,  $[\alpha]$ , with a boundary as in

$$(4.1) \quad \begin{array}{ccc} v' & \xleftarrow{\quad} & v \\ \uparrow [\alpha] & & \uparrow \\ u' & \xleftarrow{\quad} & u \end{array}$$

is the equivalence class,  $[\alpha]$ , of a map  $\alpha : I^2 \rightarrow X$  whose effect on the boundary  $\partial(I^2)$  is such that  $\alpha(x, 0) = u(x)$ ,  $\alpha(0, y) = v(y)$ ,  $\alpha(1, 1 - y) = u'(y)$ , and  $\alpha(1 - x, 1) = v'(x)$ , for  $x, y \in I$ . We call such an application a “square in  $X$ ” and draw it as

$$(4.2) \quad \begin{array}{ccc} & \xleftarrow{v'} & \\ v \uparrow \alpha & & \downarrow u' \\ & \xrightarrow{u} & \end{array}$$

Two such mappings  $\alpha, \alpha'$  are equivalent, and then represent the same square in  $\mathbf{IIX}$ , whenever they are related by a homotopy relative to the sides of the square, that is, if there exists a continuous map  $H : I^2 \times I \rightarrow X$  such that  $H(x, y, 0) = \alpha(x, y)$ ,  $H(x, y, 1) = \alpha'(x, y)$ ,  $H(x, 0, t) = u(x)$ ,  $H(0, y, t) = v(y)$ ,  $H(x, 1, t) = v'(1 - x)$  and  $H(1, y, t) = u'(1 - y)$ , for  $x, y, t \in I$ .

Given the squares in  $\mathbf{IIX}$

$$\begin{array}{ccccc} & & w' & \xleftarrow{\quad} & w \\ & & \uparrow [\beta] & & \uparrow \\ v'' & \xleftarrow{\quad} & v' & \xleftarrow{\quad} & v \\ \uparrow [\alpha'] & & \uparrow [\alpha] & & \uparrow \\ u'' & \xleftarrow{\quad} & u' & \xleftarrow{\quad} & u \end{array}$$

the corresponding composite squares

$$\begin{array}{ccc}
v'' & \xleftarrow{\quad} & v \\
\uparrow [\alpha'] \circ_h [\alpha] & & \uparrow \\
u'' & \xleftarrow{\quad} & u
\end{array}
\quad
\begin{array}{ccc}
w' & \xleftarrow{\quad} & w \\
\uparrow [\beta] \circ_v [\alpha] & & \uparrow \\
u' & \xleftarrow{\quad} & u
\end{array}$$

are defined to be those represented by the squares in  $X$

$$\begin{array}{ccc}
\cdot & \xleftarrow{v''} & \cdot \\
\uparrow v & \swarrow \alpha' & \downarrow u'' \\
\cdot & \cdot & \cdot \\
\downarrow \alpha & \searrow u' & \downarrow \\
\cdot & \xrightarrow{u} & \cdot
\end{array}
\quad
\begin{array}{ccc}
\cdot & \xleftarrow{w'} & \cdot \\
\uparrow w & \swarrow \beta & \downarrow u' \\
\cdot & \cdot & \cdot \\
\downarrow \beta & \searrow \alpha & \downarrow \\
\cdot & \xrightarrow{u} & \cdot
\end{array}$$

obtained, respectively, by pasting  $\alpha'$  with  $\alpha$ , and  $\beta$  with  $\alpha$ , along their common pair of sides. That is,

$$(4.3) \quad [\alpha'] \circ_h [\alpha] = [\alpha' \circ_h \alpha], \quad [\beta] \circ_v [\alpha] = [\beta \circ_v \alpha]$$

where  $(\alpha' \circ_h \alpha)(x, y) = \begin{cases} \alpha(2x, x+y) & \text{if } x \leq y, x+y \leq 1, \\ \alpha(x+y, 2y) & \text{if } x \geq y, x+y \leq 1, \\ \alpha'(x+y-1, 2y-1) & \text{if } x \leq y, x+y \geq 1, \\ \alpha'(2x-1, x+y-1) & \text{if } x \geq y, x+y \geq 1, \end{cases}$

$$\text{and } (\beta \circ_v \alpha)(x, y) = \begin{cases} \alpha(2x-1, 1-x+y) & \text{if } x \geq y, x+y \geq 1, \\ \alpha(x-y, 2y) & \text{if } x \geq y, x+y \leq 1, \\ \beta(1+x-y, 2y-1) & \text{if } x \leq y, x+y \geq 1, \\ \beta(2x, y-x) & \text{if } x \leq y, x+y \leq 1. \end{cases}$$

It is not hard to see that both the horizontal and vertical compositions of squares in  $\Pi X$  are well defined. For example, to prove that  $[\alpha] = [\alpha_1]$  and  $[\alpha'] = [\alpha'_1]$  imply  $[\alpha' \circ_h \alpha] = [\alpha'_1 \circ_h \alpha_1]$ , let  $H, H' : I^2 \times I \rightarrow X$  be homotopies (*rel*  $\partial(I^2)$ ) from  $\alpha$  to  $\alpha_1$  and from  $\alpha'$  to  $\alpha'_1$  respectively. Then, a homotopy  $F : I^2 \times I \rightarrow X$  is defined by

$$F(x, y, t) = \begin{cases} H(2x, x+y, t) & \text{if } x \leq y, x+y \leq 1, \\ H(x+y, 2y, t) & \text{if } x \geq y, x+y \leq 1, \\ H'(x+y-1, 2y-1, t) & \text{if } x \leq y, x+y \geq 1, \\ H'(2x-1, x+y-1, t) & \text{if } x \geq y, x+y \geq 1, \end{cases}$$

showing that  $\alpha' \circ_h \alpha$  and  $\alpha'_1 \circ_h \alpha_1$  represent the same square in  $\Pi X$ .

The horizontal identity square on a vertical morphism  $(v, u)$  is

$$\text{I}^h(v, u) = \begin{array}{ccc} v & \xrightarrow{\quad} & v \\ \uparrow [e^h] & & \uparrow \\ u & \xrightarrow{\quad} & u \end{array}$$

where  $e^h(x, y) = \begin{cases} v(y-x) & \text{if } x \leq y, \\ u(x-y) & \text{if } x \geq y, \end{cases}$  whereas, for any horizontal morphism  $(u', u)$ , its corresponding vertical identity square is

$$\text{I}^v(u', u) = \begin{array}{ccc} u' & \xleftarrow{\quad} & u \\ \parallel [e^v] & & \parallel \\ u' & \xleftarrow{\quad} & u \end{array}$$

where  $e^v(x, y) = \begin{cases} u(x+y) & \text{if } x+y \leq 1, \\ u'(2-x-y) & \text{if } x+y \geq 1. \end{cases}$

**Theorem 4.1.**  $\Pi X$  is a double groupoid satisfying the filling condition.



*Proof.* The horizontal composition of squares in  $\mathbf{IIX}$  is associative since, for any three composable squares, say  $\begin{array}{c} \cdot \longleftarrow \cdot \longleftarrow \cdot \longleftarrow \cdot \\ \uparrow [\alpha''] \uparrow [\alpha'] \uparrow [\alpha] \uparrow \\ \cdot \longleftarrow \cdot \longleftarrow \cdot \longleftarrow \cdot \end{array}$ , a relative homotopy  $(\alpha'' \circ_h \alpha') \circ_h \alpha \xrightarrow{H} \alpha'' \circ_h (\alpha' \circ_h \alpha)$  is given by the formula

$$H(x, y, t) =$$

$$\left\{ \begin{array}{ll} \alpha\left(\frac{4x}{2-t}, \frac{(2+t)x+(2-t)y}{2-t}\right) & \text{if } x \leq y, (2-t)(1-y) \geq (2+t)x, \\ \alpha\left(\frac{(2-t)x+(2+t)y}{2-t}, \frac{4y}{2-t}\right) & \text{if } x \geq y, (2-t)(1-x) \geq (2+t)y, \\ \alpha'\left(t(1+x-y)+2(x+y-1), x+3y-2+t(1+x-y)\right) & \text{if } x \leq y, (2-t)(1-y) \leq (2+t)x, (1+t)x \leq (3-t)(1-y), \\ \alpha'\left(3x+y-2+t(1-x+y), t(1-x+y)+2(x+y-1)\right) & \text{if } x \geq y, (2-t)(1-x) \leq (2+t)y, (1+t)y \leq (3-t)(1-x), \\ \alpha''\left(\frac{x+3y-3+t(1+x-y)}{1+t}, \frac{t-3+4y}{1+t}\right) & \text{if } x \leq y, (1+t)x \geq (1-y)(3-t), \\ \alpha''\left(\frac{t-3+4x}{1+t}, \frac{3x+y-3+t(1-x+y)}{1+t}\right) & \text{if } x \geq y, (1+t)y \geq (3-t)(1-x). \end{array} \right.$$

And, similarly, we prove the associativity for the vertical composition of squares in  $\mathbf{IIX}$ . For identities, let  $[\alpha]$  be any square in  $\mathbf{IIX}$  as in (4.1). Then, a relative homotopy between  $\alpha$  and  $\alpha \circ_h e^h$  is given by the map  $H : I^2 \times I \rightarrow X$  defined by

$$H(x, y, t) = \left\{ \begin{array}{ll} v(y-x) & \text{if } x \leq y, x \leq \frac{1}{2}(1-t)(1+x-y), \\ u(x-y) & \text{if } x \geq y, x \leq \frac{1}{2}(1-t)(1+x-y), \\ \alpha\left(\frac{x+y-1+t(1+x-y)}{1+t}, \frac{2y+t-1}{1+t}\right) & \text{if } \frac{1}{2}(1-t)(1+x-y) \leq x \leq y, \\ \alpha\left(\frac{2x+t-1}{1+t}, \frac{x+y-1+t(1-x+y)}{1+t}\right) & \text{if } \frac{1}{2}(1-t)(1-x-y) \leq y \leq x. \end{array} \right.$$

Therefore,  $[\alpha] \circ_h I^h(v, u) = [\alpha]$ ; and similarly we prove the remaining needed equalities:  $[\alpha] = I^h \circ_h [\alpha] = [\alpha] \circ_v I^v = I^v \circ_v [\alpha]$ .

Let us now describe inverse squares in  $\mathbf{IIX}$ . For any given square  $[\alpha]$  as in (4.1), its respective horizontal and vertical inverses

$$\begin{array}{ccc} v & \longleftarrow & v' \\ \uparrow [\alpha]^{-1h} & & \uparrow [\alpha]^{-1v} \\ u & \longleftarrow & u', \quad v' & \longleftarrow & v \end{array}$$

are represented by the squares in  $X$ ,  $\alpha^{-1h}, \alpha^{-1v} : I^2 \rightarrow X$ , defined respectively by the formulas

$$\alpha^{-1h}(x, y) = \alpha(1-y, 1-x), \quad \alpha^{-1v}(x, y) = \alpha(y, x).$$

The equality  $[\alpha^{-1h}] \circ_h [\alpha] = I^h(v, u)$  holds, thanks to the homotopy  $H : I^2 \times I \rightarrow X$  defined by

$$H(x, y, t) = \left\{ \begin{array}{ll} \alpha(2x(1-t), (1-2t)x+y) & \text{if } x \leq y, x+y \leq 1, \\ \alpha(x+(1-2t)y, 2y(1-t)) & \text{if } x \geq y, x+y \leq 1, \\ \alpha(2(ty-t-y+1), 2(ty-t+1)-x-y) & \text{if } x \leq y, x+y \geq 1, \\ \alpha((2x-2)t+2-x-y, (2x-2)t+2-2x) & \text{if } x \geq y, x+y \geq 1. \end{array} \right.$$

And, similarly, one sees the remaining equalities  $[\alpha] \circ_h [\alpha]^{-1h} = I^h$ ,  $[\alpha] \circ_v [\alpha]^{-1v} = I^v$  and  $[\alpha]^{-1v} \circ_v [\alpha] = I^v$ .

By construction of  $\mathbf{II}X$ , conditions (i) and (ii) in **Axiom 1** are clearly satisfied. For (iii) in **Axiom 1**, we need to prove that, for any path  $u : I \rightarrow X$ , the equality  $\Gamma^h(u, u) = \Gamma^v(u, u)$  holds. But this follows from the relative homotopy  $H : e^h \rightarrow e^v$  defined by

$$H(x, y, t) = \begin{cases} u(y-x) & \text{if } x \leq y, (1-t)(1-y) \geq (1+t)x, \\ u(2y-1+t(1+x-y)) & \text{if } x \leq y, x+y \leq 1, (1-t)(1-y) \leq (1+t)x, \\ u(x-y) & \text{if } x \geq y, (1-t)(1-x) \geq (1+t)y, \\ u(2x-1+t(1-x+y)) & \text{if } x \geq y, x+y \leq 1, (1-t)(1-x) \leq (1+t)y, \\ u(1-2x+t(1+x-y)) & \text{if } x \leq y, x+y \geq 1, (1+t)(1-y) \geq (1-t)x, \\ u(y-x) & \text{if } x \leq y, x+y \geq 1, (1+t)(1-y) \leq (1-t)x, \\ u(1-2y+t(1-x+y)) & \text{if } x \geq y, x+y \geq 1, (1+t)(1-x) \geq (1-t)y, \\ u(x-y) & \text{if } x \geq y, (1+t)(1-x) \leq (1-t)y. \end{cases}$$

The given definition of how squares in  $\mathbf{II}X$  compose makes the conditions (i) and (ii) in **Axiom 2** clear, and the remaining condition (iii) holds since, for any three paths  $u, v, w : I \rightarrow X$  with  $u(1) = v(1) = w(1)$ , there is a relative homotopy between  $e^v(w, v) \circ_h e^v(v, u)$  and  $e^v(w, u)$ , defined by

$$H(x, y, t) = \begin{cases} u(y-x+\frac{4x}{1+t}) & \text{if } x \leq y, (1+t)(1-y) \geq (3-t)x, \\ v(2-3x-y+t(1+x-y)) & \text{if } x \leq y, x+y \leq 1, (1+t)(1-y) \leq (3-t)x, \\ u(x-y+\frac{4y}{1+t}) & \text{if } x \geq y, (1+t)(1-x) \geq (3-t)y, \\ v(2-x-3y+t(1-x+y)) & \text{if } x \geq y, x+y \leq 1, (1+t)(1-x) \leq (3-t)y, \\ v(x+3y-2+t(1+x-y)) & \text{if } x \leq y, x+y \geq 1, (3-t)(1-y) \geq (1+t)x, \\ w(y-x+\frac{4(y-1)}{1+t}) & \text{if } x \leq y, (3-t)(1-y) \leq (1+t)x, \\ v(3x+y-2+t(1-x+y)) & \text{if } x \geq y, x+y \geq 1, (3-t)(1-x) \geq (1+t)y, \\ w(x-y+\frac{4(1-x)}{1+t}) & \text{if } x \geq y, (3-t)(1-x) \leq (1+t)y. \end{cases}$$

And, similarly, one proves the equality  $\Gamma^h(w, u) \circ_v \Gamma^h(w, u) = \Gamma^h(w, u)$ , for any three paths in  $X$ ,  $u, v, w : I \rightarrow X$  with  $u(0) = v(0) = w(0)$ .

Then, it only remains to prove the interchange law in **Axiom 3**. To do so, let

$$\begin{array}{ccccc} w'' & \leftarrow & w' & \leftarrow & w \\ \uparrow & [\delta] & \uparrow & [\beta] & \uparrow \\ v'' & \leftarrow & v' & \leftarrow & v \\ \uparrow & [\gamma] & \uparrow & [\alpha] & \uparrow \\ u'' & \leftarrow & u' & \leftarrow & u \end{array}$$

be squares in  $\mathbf{II}X$ . Then, the required equality follows from the existence of the relative homotopy  $(\delta \circ_h \beta) \circ_v (\gamma \circ_h \alpha) \rightarrow (\delta \circ_v \gamma) \circ_h (\beta \circ_v \alpha)$  defined by the map  $H : I^2 \times I \rightarrow X$  such that

$$H(x, y, t) =$$

- |   |   |
|---|---|
| ① $\alpha(x+y-2ty, 4y)$   | $1-x+2ty \geq 5y, x-3y \geq 2ty,$   |
| ② $\alpha(\frac{2(x-y)}{1+t}, \frac{2(x-tx+y+3ty)}{2+t-t^2})$                           | $2+t-t^2-6x+4tx+2y \geq 8ty, (3+2t)y \geq x \geq y,$  |
| ③ $\alpha(\frac{t^2-2t+2x-2y+4ty}{1-2t+2t^2}, \frac{3t^2-t(1+4x)+2(x+y)}{2-4t+4t^2})$   | $t^2+t(4x-3) \geq 2(x+y-1), t^2-2x+6y \geq t(4x+8y-3),$<br>$t^2+6x+t(8y-4x-1) \geq 2(1+y),$ |
| ④ $\alpha(\frac{t-2(x+y)}{t-2}, \frac{2t(x+3y-1)-8y}{t^2-t-2})$                         | $1 \geq x+y, x-1 \geq (2t-5)y, 2x-t^2-6y \geq t(3-4x-8y),$                                  |
| ⑤ $v'(3-t-4x)$  | $x \geq y, 1 \geq x+y, 2(x+y-1) \geq t^2+t(4x-3),$  |
| ⑥ $\gamma(4x-3, x+y-1-2t(x-1))$   | $5x+y-5 \geq 2t(x-1), 2t(x-1)+3x \geq y+2,$   |
| ⑦ $\gamma(\frac{6+t^2-8x+t(6x+2y-7)}{t^2-t-2}, \frac{2(x+y-1)}{2-t})$                   | $x+y \geq 1, 5+2t(x-1) \geq 5x+y,$<br>$9t+6x \geq 4+t^2+8tx+2y+4ty,$                        |
| ⑧ $\gamma(\frac{t+t^2-4ty+2(x+y-1)}{2-4t+4t^2}, \frac{1+t^2+4t(x-1)-2x+2y}{1-2t+2t^2})$ | $t+t^2+2(x+y-1) \geq 4ty, t^2+2(1+x-3y) \geq t(8x-y-3),$<br>$4+t^2-6x+2y \geq t(9-8x-4y),$  |
| ⑨ $\gamma(\frac{t^2-2(x+y-1)+t(3-6x+2y)}{t^2-t-2}, \frac{1+t-2x+2y}{1+t})$              | $8tx+6y-4ty \geq 2+3t+t^2+2x, x \geq y, 2+y \geq 2t(x-1)+3x,$                               |
| ⑩ $v'(4y-1-t)$  | $x \geq y, x+y \geq 1, 2+4ty \geq t+t^2+2x+2y,$   |
| ⑪ $\beta(4x, x+y-2tx)$  | $1+2tx \geq y+5x, y \geq 3x+2tx,$   |
| ⑫ $\beta(\frac{2t(y+3x-1)-8x}{t^2-t-2}, \frac{t-2(x+y)}{t-2})$                          | $1 \geq x+y, y+(5-2t)x \geq 1, 2y+t(3-4y-8x)-6x \geq t^2,$                                  |
| ⑬ $\beta(\frac{3t^2-t(1+4y)+2(x+y)}{2-4t+4t^2}, \frac{t^2-2t+2y-2x+4tx}{1-2t+2t^2})$    | $t^2+t(4y-3) \geq 2(x+y-1), t^2+t(3-4y-8x)+6x \geq 2y,$<br>$t^2+6y-2(1+x) \geq t(1+4y-8x),$ |
| ⑭ $\beta(\frac{2(y-ty+x+3tx)}{2+t-t^2}, \frac{2(y-x)}{1+t})$                            | $2+t+4ty+2x \geq t^2+6y+8tx, (3+2t)x \geq y \geq x,$  |
| ⑮ $v'(3-t-4y)$  | $y \geq x, 1 \geq x+y, 2(x+y-1) \geq t^2+t(3-4y),$  |
| ⑯ $\delta(2t(1-y)+x+y-1, 4y-3)$   | $5y+x \geq 5+2t(y-1), 2t(y-1)+3y \geq x+2,$   |
| ⑰ $\delta(\frac{2(x+y-1)}{2-t}, \frac{6+t^2-8y+t(6y+2x-7)}{t^2-t-2})$                   | $x+y \geq 1, 9t+6y-8ty-2x-4tx \geq 4+t^2,$<br>$5+2t(y-1) \geq 5y+x,$                        |
| ⑱ $\delta(\frac{1+t^2+4t(y-1)-2y+2x}{1-2t+2t^2}, \frac{t+t^2-4tx+2(x+y-1)}{2-4t+4t^2})$ | $t+t^2-4tx \geq 2(1-x-y), t^2+2(1+y-3x) \geq t(8y-4x-3),$<br>$4+t^2-6y+2x \geq t(9-8y-4x),$ |
| ⑲ $\delta(\frac{1+t-2y+2x}{1+t}, \frac{t^2-2(x+y-1)+t(3-6y+2x)}{t^2-2-t})$              | $8ty+6x-4tx \geq 2+3t+t^2+2y, y \geq x, 2-3y+x \geq 2t(y-1),$                               |
| ⑳ $v'(4y-1-t)$  | $y \geq x, x+y \geq 1, 2+4tx \geq t+t^2+2y+2x,$   |

where parts ① in the homotopy  $H(x, y, t)$  above correspond to the areas with  $(x, y) \in I^2$  shown in Figure 1 below.

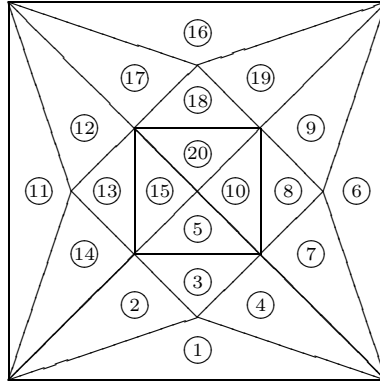


FIGURE 1.

Finally, we observe that  $\mathbf{IIX}$  satisfies the filling condition. Suppose a configuration of morphisms in  $\mathbf{IIX}$

$$\begin{array}{c} v' \leftarrow v \\ \uparrow \\ u \end{array}$$

is given. This means we have paths  $u, v, v' : I \rightarrow X$  with  $u(0) = v(0)$  and  $v(1) = v'(1)$ . Since the inclusion  $\partial I \hookrightarrow I$  is a cofibration, the map  $f : (\{0\} \times I) \cup (I \times \partial I) \rightarrow X$  with  $f(0, t) = v(t)$ ,  $f(t, 0) = u(t)$  and  $f(t, 1) = v'(1 - t)$  for  $0 \leq t \leq 1$ , has an extension to a map  $\alpha : I \times I \rightarrow X$ , which precisely represents a square in  $\mathbf{IIX}$  of the form

$$\begin{array}{ccc} v' & \leftarrow & v \\ \uparrow & [\alpha] & \uparrow \\ u' & \leftarrow & u \end{array}$$

where  $u' : I \rightarrow X$  is the path  $u'(t) = \alpha(1, 1 - t)$ . Hence,  $\mathbf{IIX}$  verifies the filling condition.  $\square$

In the previous Section 3 we introduced homotopy groups for double groupoids satisfying the filling condition. The next proposition provides greater specifics on the relationship between the homotopy groups of the associated homotopy double groupoid  $\mathbf{IIX}$  to a topological space  $X$  and the corresponding for  $X$ .

**Theorem 4.2.** *For any space  $X$ , any path  $u : I \rightarrow X$ , and  $0 \leq i \leq 2$ , there is an isomorphism*

$$\pi_i(\mathbf{IIX}, u) \cong \pi_i(X, u(0)).$$

*Proof.* For any two points  $x, y \in X$ , the constant paths  $c_x$  and  $c_y$  are in the same connected component of  $\mathbf{IIX}$  if and only if there is a pair of morphisms in  $\mathbf{IIX}$  of the form

$$\begin{array}{c} c_y \leftarrow u \\ \uparrow \\ c_x \end{array}$$

or, equivalently, if and only if there is a path  $u : I \rightarrow X$  in  $X$  such that  $u(1) = y$  and  $u(0) = x$ . Then, we have an injective map

$$\pi_0 X \rightarrow \pi_0 \mathbf{IIX}, \quad [x] \mapsto [c_x],$$

which is also surjective since, for any path  $u$  in  $X$ , we have a vertical morphism  $u \leftarrow c_{u(0)}$  in  $\mathbf{IIX}$ ; whence the announced bijection  $\pi_0 X \cong \pi_0 \mathbf{IIX}$ .

Next, we prove that there is an isomorphism  $\pi_1(\mathbf{IIX}, u) \cong \pi_1(X, u(0))$  for any given path  $u : I \rightarrow X$ . To do so, we shall use the fundamental groupoid  $\mathbf{IIX}$  of the space  $X$ ; that is, the groupoid whose objects are the points of  $X$  and whose morphisms are the (relative to  $\partial I$ ) homotopy classes  $[v]$  of paths  $v : I \rightarrow X$ . Simply by checking the construction, we see that an element  $[(u, v), (v, u)] \in \pi_1(\mathbf{IIX}, u)$  is determined by a path  $v : I \rightarrow X$ , with  $v(0) = u(0)$  and  $v(1) = u(1)$ . Moreover, for any other such  $v' : I \rightarrow X$ , it holds that  $[(u, v), (v, u)] = [(u, v'), (v', u)]$  in  $\pi_1(\mathbf{IIX}, u)$  if and only if there are squares in  $\mathbf{IIX}$  of the form

$$\begin{array}{ccc} u & \leftarrow & v \\ \uparrow & [\alpha] & \uparrow \\ w & \leftarrow & u \end{array} \quad \begin{array}{ccc} u & \leftarrow & v' \\ \uparrow & [\alpha'] & \uparrow \\ w & \leftarrow & u \end{array}$$

or, equivalently, if and only if there are squares in  $X$ ,  $\alpha, \alpha' : I^2 \rightarrow X$  with boundaries as in

$$\begin{array}{ccc} \cdot & \xleftarrow{u} & \cdot \\ v \uparrow & \alpha & \downarrow w \\ \cdot & \xrightarrow{u} & \cdot \end{array} \quad \begin{array}{ccc} \cdot & \xleftarrow{u} & \cdot \\ v' \uparrow & \alpha' & \downarrow w \\ \cdot & \xrightarrow{u} & \cdot \end{array}$$

Since this last condition simply means that, in the fundamental groupoid  $\Pi X$ , the equality  $[v] = [v']$  holds, we conclude with bijections

$$\begin{aligned} \pi_1(\Pi X, u) &\cong \text{Hom}_{\Pi X}(u(0), v(1)) \cong \pi_1(X, u(0)) \\ [(u, v), (v, u)] &\longmapsto [v] \longmapsto [u]^{-1} \circ [v] \end{aligned}$$

To see that the composite bijection  $\phi : [(u, v), (v, u)] \mapsto [u]^{-1} \circ [v]$  is actually an isomorphism, let  $v_1, v_2 : I \rightarrow X$  be paths in  $X$ , both from  $u(0)$  to  $u(1)$ . Then,  $[(u, v_1), (v_1, u)] \circ [(u, v_2), (v_2, u)] = [(u, v), (v, u)]$ , where  $v$  occurs in a configuration such as

$$\begin{array}{ccccc} u & \leftarrow & v_1 & \leftarrow & v \\ & & \uparrow & [\gamma] & \uparrow \\ & & u & \leftarrow & v_2 \\ & & & & \uparrow \\ & & & & u \end{array}$$

for some (any) square  $\gamma : I^2 \rightarrow X$  in  $X$  with boundary as below.

$$\begin{array}{ccc} & v_1 & \\ \cdot & \leftarrow & \cdot \\ v \uparrow & \gamma & \downarrow u \\ \cdot & \rightarrow & \cdot \\ & v_2 & \end{array}$$

It follows that, in  $\Pi X$ ,  $[v] = [v_1] \circ [u]^{-1} \circ [v_2]$  and therefore

$$\begin{aligned} \phi[(u, v_1), (v_1, u)] \circ \phi[(u, v_2), (v_2, u)] &= [u]^{-1} \circ [v_1] \circ [u]^{-1} \circ [v_2] = [u]^{-1} \circ [v] \\ &= \phi([(u, v_1), (v_1, u)] \circ [(u, v_2), (v_2, u)]). \end{aligned}$$

Finally, we consider the case  $i = 2$ . Let  $u : I \rightarrow X$  be any path with  $u(0) = x$ . Then, the mapping  $[\alpha] \mapsto \text{I}^h(c_x, u) \circ_v [\alpha] \circ_v \text{I}^h(u, c_x)$ , which carries a square  $[\alpha] \in \pi_2(\Pi X, u)$ , to the composite of

$$\begin{array}{ccc} c_x & \xrightarrow{\quad} & c_x \\ \uparrow [e^h] & & \uparrow \\ u & \xrightarrow{\quad} & u \\ \downarrow [\alpha] & & \downarrow \\ u & \xrightarrow{\quad} & u \\ \uparrow [e^h] & & \uparrow \\ c_x & \xrightarrow{\quad} & c_x \end{array}$$

establishes an isomorphism  $\pi_2(\Pi X, u) \cong \pi_2(\Pi X, c_x)$ . Now, it is clear that both  $\pi_2(\Pi X, c_x)$  and  $\pi_2(X, x)$  are the same abelian group of relative to  $\partial I^2$  homotopy classes of maps  $I^2 \rightarrow X$  which are constant  $x$  along the four sides of the square.  $\square$

The construction of the double groupoid  $\Pi X$  from a space  $X$  is easily seen to be functorial and, moreover, the isomorphisms in Theorem 4.2 above become natural. Then, we have the next corollary.

**Corollary 4.3.** *A continuous map  $f : X \rightarrow Y$  is a weak homotopy 2-equivalence if and only if the induced double functor  $\Pi f : \Pi X \rightarrow \Pi Y$  is a weak equivalence.*

## 5. THE GEOMETRIC REALIZATION OF A DOUBLE GROUPOID.

Hereafter, we shall regard each ordered set  $[n]$  as the category with exactly one arrow  $j \rightarrow i$  when  $0 \leq i \leq j \leq n$ . Then, a non-decreasing map  $[n] \rightarrow [m]$  is the same as a functor.

The geometric realization, or classifying space, of a category  $\mathcal{C}$ , [30], is  $|\mathcal{C}| := |\mathbf{NC}|$ , the geometric realization of its nerve [18]

$$\mathbf{NC} : \Delta^o \rightarrow \mathbf{Set}, \quad [n] \mapsto \text{Func}([n], \mathcal{C}),$$

that is, the simplicial set whose  $n$ -simplices are the functors  $F : [n] \rightarrow \mathcal{C}$ , or tuples of arrows in  $\mathcal{C}$

$$F = (F_i \xleftarrow{F_{i,j}} F_j)_{0 \leq i \leq j \leq n}$$

such that  $F_{i,j} \circ F_{j,k} = F_{i,k}$  and  $F_{i,i} = \text{Id}_{F_i}$ . If  $\mathcal{G}$  is a double category, then its geometric realization,  $|\mathcal{G}|$ , is defined by first taking the double nerve  $\mathbb{N}\mathcal{G}$ , which is a bisimplicial set, and then realizing to obtain a space

$$|\mathcal{G}| := |\mathbb{N}\mathcal{G}|.$$

To have a manageable description handle description for the bisimplices in  $\mathbb{N}\mathcal{G}$ , we can use the following construction: If  $\mathcal{A}$  and  $\mathcal{B}$  are categories, let  $\mathcal{A} \otimes \mathcal{B}$  be the double category whose objects are pairs  $(a, b)$ , where  $a$  is an object of  $\mathcal{A}$  and  $b$  is an object of  $\mathcal{B}$ ; horizontal morphisms are pairs  $(f, b) : (a, b) \rightarrow (c, b)$ , with  $f : a \rightarrow c$  a morphism in  $\mathcal{A}$ ; vertical morphisms are pairs  $(a, u) : (a, b) \rightarrow (a, d)$  with  $u : b \rightarrow d$  in  $\mathcal{B}$ ; and a square in  $\mathcal{A} \otimes \mathcal{B}$  is given by each morphism  $(f, u) : (a, b) \rightarrow (c, d)$  in the product category  $\mathcal{A} \times \mathcal{B}$ , by stating its boundary as in

$$\begin{array}{ccc} & (f, d) & \\ (c, d) & \xleftarrow{\quad} & (a, d) \\ (c, u) & \uparrow & (f, u) \quad \uparrow & (a, u) \\ & (c, b) & \xleftarrow{\quad} & (a, b) \\ & (f, b) & \end{array}$$

Compositions in  $\mathcal{A} \otimes \mathcal{B}$  are defined in the evident way.

Then, the double nerve  $\mathbb{N}\mathcal{G}$  of a double category  $\mathcal{G}$  is the bisimplicial set

$$\mathbb{N}\mathcal{G} : \Delta^o \times \Delta^o \rightarrow \mathbf{Set}, \quad ([p], [q]) \mapsto \text{DFunc}([p] \otimes [q], \mathcal{G}),$$

whose  $(p, q)$ -bisimplices are the double functors  $F : [p] \otimes [q] \rightarrow \mathcal{G}$  or configurations of squares in  $\mathcal{G}$  of the form

$$\left( \begin{array}{ccc} & F_{i,j}^r & \\ F_i^r & \xleftarrow{\quad} & F_j^r \\ F_i^{r,s} & \uparrow & F_{i,j}^{r,s} \quad \uparrow & F_j^{r,s} \\ & F_i^s & \xleftarrow{\quad} & F_j^s \\ & F_{i,j}^s & \end{array} \right)_{\substack{0 \leq i \leq j \leq p \\ 0 \leq r \leq s \leq q}},$$

such that  $F_{i,j}^{r,s} \circ_h F_{j,k}^{r,s} = F_{i,k}^{r,s}$ ,  $F_{i,j}^{r,s} \circ_v F_{i,j}^{s,t} = F_{i,j}^{r,t}$ ,  $F_{i,i}^{r,s} = \text{Id}_h F_i^{r,s}$ , and  $F_{i,j}^{r,r} = \text{Id}_v F_{i,j}^r$ .

But note that the double category  $[p] \otimes [q]$  is free on the bigraph

$$\left( \begin{array}{ccc} & (j-1, r-1) & \xleftarrow{\quad} & (j, r-1) \\ & \uparrow & & \uparrow \\ (j-1, r) & \xleftarrow{\quad} & (j, r) \end{array} \right)_{\substack{0 \leq i \leq j \leq p \\ 0 \leq r \leq s \leq q}},$$

and therefore, giving a double functor  $F : [p] \otimes [q] \rightarrow \mathcal{G}$  as above is equivalent to specifying the  $p \times q$  configuration of squares in  $\mathcal{G}$

$$\left( \begin{array}{ccc} & F_{j-1,j}^{r-1} & \\ F_{j-1}^{r-1} \swarrow & & \nwarrow F_j^{r-1} \\ & F_{j-1,j}^{r-1,r} & \\ F_{j-1}^{r-1,r} \uparrow & & \uparrow F_j^{r-1,r} \\ & F_{j-1}^r & \\ & F_{j-1,j}^r & \end{array} \right)_{\substack{1 \leq j \leq p \\ 1 \leq r \leq q}} .$$

Thus, each vertical simplicial set  $\mathbb{N}\mathcal{G}_{p,*}$  is the nerve of the “vertical” category having as objects strings of  $p$ -composable horizontal morphisms  $a_0 \leftarrow a_1 \leftarrow \cdots \leftarrow a_p$ , whose arrows consist of  $p$  horizontally composable squares as in

$$\begin{array}{ccccccc} b_0 & \leftarrow & b_1 & \leftarrow & \cdots & \leftarrow & b_p \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ a_0 & \leftarrow & a_1 & \leftarrow & \cdots & \leftarrow & a_p \end{array}$$

And, similarly, each horizontal simplicial set  $\mathbb{N}\mathcal{G}_{*,q}$  is the nerve of the “horizontal” category whose objects are the length  $q$  sequences of composable vertical morphisms of  $\mathcal{G}$ , with length  $q$  sequences of vertically composable squares as morphisms between them.

For instance, if  $\mathcal{A}$  and  $\mathcal{B}$  are categories, then  $\mathbb{N}(\mathcal{A} \otimes \mathcal{B}) = \mathbb{N}\mathcal{A} \otimes \mathbb{N}\mathcal{B}$ . In particular,

$$\mathbb{N}([p] \otimes [q]) = \Delta[p] \otimes \Delta[q] = \Delta[p, q],$$

is the standard  $(p, q)$ -bisimplex.

It is a well-known fact that the nerve  $\mathbb{N}\mathcal{C}$  of a category  $\mathcal{C}$  satisfies the Kan extension condition if and only if  $\mathcal{C}$  is a groupoid, and, in such a case, every  $(k, n)$ -horn  $\Lambda^k[n] \rightarrow \mathbb{N}\mathcal{C}$ , for  $n \geq 2$ , has a unique extension to an  $n$ -simplex of  $\mathbb{N}\mathcal{C}$

$$\begin{array}{ccc} \Lambda^k[n] & \rightarrow & \mathbb{N}\mathcal{C} \\ \downarrow & \nearrow \exists! & \\ \Delta[n] & & \end{array}$$

(see [21, Propositions 2.2.3 and 2.2.4], for example). For double categories  $\mathcal{G}$ , we have the following:

**Theorem 5.1.** *Let  $\mathcal{G}$  be a double category. The following statements are equivalent:*

- (i)  $\mathcal{G}$  is a double groupoid satisfying the filling condition.
- (ii) The bisimplicial set  $\mathbb{N}\mathcal{G}$  satisfies the extension condition.
- (iii) The simplicial set  $\text{diag}\mathbb{N}\mathcal{G}$  is a Kan complex.

*Proof.* (i)  $\Rightarrow$  (ii) Since  $\mathcal{G}$  is a double groupoid, all simplicial sets  $\mathbb{N}\mathcal{G}_{p,*}$  and  $\mathbb{N}\mathcal{G}_{*,q}$  are nerves of groupoids. Therefore, every extension problem of the form

$$\begin{array}{ccc} \Delta[p] \otimes \Lambda^l[q] & \rightarrow & \mathbb{N}\mathcal{G} \\ \downarrow & \nearrow \exists! & \\ \Delta[p, q] & & \end{array} \quad \text{or} \quad \begin{array}{ccc} \Lambda^k[p] \otimes \Delta[q] & \rightarrow & \mathbb{N}\mathcal{G} \\ \downarrow & \nearrow \exists! & \\ \Delta[p, q] & & \end{array}$$

has a solution and it is unique. Suppose then an extension problem of the form

$$(5.1) \quad \begin{array}{ccc} \Lambda^{k,l}[p, q] & \longrightarrow & \mathbb{N}\mathcal{G} \\ \downarrow & \nearrow & \\ \Delta[p, q] & & \end{array}$$

If  $p \geq 2$ , then the restricted map  $\Lambda^k[p] \otimes \Delta[q] \hookrightarrow \Lambda^{k,l}[p, q] \rightarrow \mathbb{N}\mathcal{G}$  has a unique extension to a bisimplex  $\Delta[p, q] \rightarrow \mathbb{N}\mathcal{G}$ , which is a solution to (5.1) (which in fact has a unique solution if  $p \geq 2$  or  $q \geq 2$ ). Hence, we reduce the proof to the case in which  $p = 1 = q$ , with the four possibilities  $k = 0, 1$  and  $l = 0, 1$ . But any such extension problem has a solution thanks to Lemma 3.1. For example, let us discuss the case  $k = 0 = l$ : A bisimplicial map  $\Lambda^{0,0}[1, 1] \xrightarrow{(-, w; -, g)} \mathbb{N}\mathcal{G}$  consists of two bisimplicial maps  $w : \Delta[0, 1] \rightarrow \mathbb{N}\mathcal{G}$  and  $g : \Delta[1, 0] \rightarrow \mathbb{N}\mathcal{G}$ , such that  $wd_v^1 = gd_h^1$ . That is, a vertical morphism  $w$  of  $\mathcal{G}$  and a horizontal morphism  $g$  of  $\mathcal{G}$ , such that both have the same target. By Lemma 3.1, there is a square  $\alpha$  in  $\mathcal{G}$  of the form

$$\begin{array}{ccc} \cdot & \xleftarrow{g} & \cdot \\ w \uparrow & \alpha & \uparrow g \\ \cdot & \xleftarrow{\dots} & \cdot \end{array}$$

which defines a bisimplicial map  $F : \Delta[1, 1] \rightarrow \mathbb{N}\mathcal{G}$  such that  $F_{0,1}^{0,1} = \alpha$ . Then  $Fd_h^1 = w$ ,  $Fd_v^1 = g$ , and the diagram below commutes, as required.

$$\begin{array}{ccc} \Lambda^{0,0}[1, 1] & \xrightarrow{(-, w; -, g)} & \mathbb{N}\mathcal{G} \\ \downarrow & \nearrow F & \\ \Delta[1, 1] & & \end{array}$$

(ii)  $\Rightarrow$  (i) The simplicial sets  $\mathbb{N}\mathcal{G}_{0,*}$ ,  $\mathbb{N}\mathcal{G}_{*,0}$ ,  $\mathbb{N}\mathcal{G}_{1,*}$ , and  $\mathbb{N}\mathcal{G}_{*,1}$  are respectively the nerves of the four component categories of the double category  $\mathcal{G}$ . Since all these simplicial sets satisfy the Kan extension condition, it follows that the four category structures involved are groupoids; that is,  $\mathcal{G}$  is a double groupoid. Furthermore, for any given filling problem in  $\mathcal{G}$ ,

$$\begin{array}{ccc} \cdot & \xleftarrow{g} & \cdot \\ \uparrow \exists? & & \uparrow u \\ \cdot & \xleftarrow{\dots} & \cdot \end{array}$$

we can solve the extension problem

$$\begin{array}{ccc} \Lambda^{1,0}[1, 1] & \xrightarrow{(u, -; -, g)} & \mathbb{N}\mathcal{G} \\ \downarrow & \nearrow F & \\ \Delta[1, 1] & & \end{array}$$

and the square  $F_{0,1}^{0,1}$  has  $u$  as horizontal source and  $g$  as vertical target. Thus  $\mathcal{G}$  satisfies the filling condition.

(i)  $\Rightarrow$  (iii) The higher dimensional part of the proof is in the following lemma, that we establish for future reference.



**Lemma 5.2.** *If  $\mathcal{G}$  is any double groupoid and  $n$  is any integer such that  $n \geq 3$ , then every extension problem*

$$\begin{array}{ccc} \Lambda^k[n] & \longrightarrow & \text{diag} \mathbb{N}\mathcal{G} \\ \downarrow & \nearrow \text{dotted} & \\ \Delta[n] & & \end{array}$$

*has a solution and it is unique.*

*Proof.* Let  $F = (F_{i,j}^{r,s}) : [n] \otimes [n] \rightarrow \mathcal{G}$  denote the double functor we are looking for solving the given extension problem. Recall that to give such an  $F$  is equivalent to specifying the  $n \times n$  configuration of squares

$$\left( \begin{array}{ccccc} & & F_{j-1,j}^{r-1} & & \\ & F_{j-1}^{r-1} & \xleftarrow{F_{j-1,j}^{r-1}} & F_j^{r-1} & \\ F_{j-1}^{r-1,r} \uparrow & & F_{j-1,j}^{r-1,r} & & \uparrow F_j^{r-1,r} \\ \left( F_{j-1}^{r-1,r} \right) & & & & \left( F_j^{r-1,r} \right) \\ F_{j-1}^r \xleftarrow{F_{j-1,j}^r} & & F_j^r & & \end{array} \right)_{\substack{1 \leq j \leq n \\ 1 \leq r \leq n}}.$$

We claim that  $F$  exists and, moreover, that it is completely showed from any three of its (known) faces  $[n-1] \otimes [n-1] \xrightarrow{d^m \otimes d^m} [n] \otimes [n] \xrightarrow{F} \mathcal{G}$ ,  $m \neq k$ ; therefore, by the input data  $\Lambda^k[n] \rightarrow \text{diag} \mathbb{N}\mathcal{G}$ . In effect, since each  $m^{\text{th}}$ -face consists of all squares  $F_{i,j}^{r,s}$  such that  $m \notin \{i, j, r, s\}$ , once we have selected any three integers  $m, p, q$  with  $m < p < q$  and  $k \notin \{m, p, q\}$ , we know explicitly all squares  $F_{i,j}^{r,s}$  except those in which  $m, p$  and  $q$  appear in the labels, that is:  $F_{q,j}^{m,p}$ ,  $F_{p,j}^{m,q}$ ,  $F_{m,q}^{j,p}$ , and so on. In the case where  $k \geq 3$ , if we take  $\{m, p, q\} = \{0, 1, 2\}$  then we have given all squares  $F_{i,j}^{r,s}$ , except those with  $\{0, 1, 2\} \subseteq \{r, s, i, j\}$ . In particular, we have all  $F_{i,i+1}^{r,r+1}$ , except four of them, namely,  $F_{2,3}^{0,1}$ ,  $F_{1,2}^{0,1}$ ,  $F_{0,1}^{1,2}$ , and  $F_{0,1}^{2,3}$ , which, however, are uniquely determined by the equations

$$F_{2,3}^{0,1} \circ_v F_{2,3}^{1,2} = F_{2,3}^{0,2}, \quad F_{0,1}^{2,3} \circ_h F_{1,2}^{2,3} = F_{0,2}^{2,3}, \quad F_{1,2}^{0,1} \circ_h F_{2,3}^{0,1} = F_{1,3}^{0,1}, \quad F_{0,1}^{1,2} \circ_v F_{0,1}^{2,3} = F_{0,1}^{1,3},$$

that is,  $F_{2,3}^{0,1} = F_{2,3}^{0,2} \circ_v (F_{2,3}^{1,2})^{-1_v}$ , and so on. The other possibilities for  $k$  are discussed in a similar way: If  $k = 2$ , then we select  $\{m, p, q\} = \{0, 1, n\}$  and determine  $F$  completely by taking into account the two equations  $F_{n-1,n}^{0,1} \circ_v F_{n-1,n}^{1,2} = F_{n-1,n}^{0,2}$ ,  $F_{0,1}^{n-1,n} \circ_h F_{1,2}^{n-1,n} = F_{0,2}^{n-1,n}$ .

If  $k = 1$ , then we take  $\{m, p, q\} = \{0, 2, 3\}$  and find the unknown squares  $F_{0,1}^{2,3}$  and  $F_{2,3}^{0,1}$  by the equations  $F_{0,1}^{1,2} \circ_v F_{0,1}^{2,3} = F_{0,1}^{1,3}$  and  $F_{1,2}^{0,1} \circ_h F_{2,3}^{0,1} = F_{1,3}^{0,1}$ , respectively.

Finally, in the case where  $k = 0$ , we take  $\{m, p, q\} = \{n-2, n-1, n\}$  and we determine the non-given four squares of the family  $(F_{i,i+1}^{r,r+1})$ , that is,  $F_{n-1,n}^{n-2,n-1}$ ,  $F_{n-2,n-1}^{n-1,n}$ ,  $F_{n-1,n}^{n-3,n-2}$ , and  $F_{n-3,n-2}^{n-1,n}$  by means of the four equations  $F_{n-3,n-1}^{n-3,n-2} \circ_h F_{n-1,n}^{n-3,n-2} = F_{n-3,n}^{n-3,n-2}$ ,  $F_{n-3,n-2}^{n-3,n-1} \circ_v F_{n-3,n-2}^{n-1,n} = F_{n-3,n-2}^{n-3,n}$ ,  $F_{n-1,n}^{n-3,n-2} \circ_v F_{n-1,n}^{n-2,n-1} = F_{n-1,n}^{n-3,n-1}$ , and  $F_{n-3,n-2}^{n-1,n} \circ_h F_{n-2,n-1}^{n-1,n} = F_{n-3,n-1}^{n-1,n}$ . This completes the proof of the lemma.  $\square$

We now return to the proof of (i)  $\Rightarrow$  (iii) in Theorem 5.1. After Lemma 5.2 above, it remains to prove that every extension problem

$$\begin{array}{ccc} \Lambda^k[2] & \longrightarrow & \text{diag} \mathbb{N}\mathcal{C} \\ \downarrow & \nearrow \text{dotted } \exists? & \\ \Delta[2] & & \end{array}$$

for  $k = 0, 1, 2$ , has a solution. In the case where  $k = 0$ , the data for a simplicial map  $(-, \tau, \sigma) : \Lambda^0[2] \rightarrow \text{diag}\mathbb{N}\mathcal{G}$  consists of a couple of squares in  $\mathcal{G}$  of the form

$$\begin{array}{ccc} a & \xleftarrow{\quad} & \cdot \\ \uparrow \sigma & & \uparrow \\ \cdot & \xleftarrow{\quad} & \cdot \end{array} \quad \begin{array}{ccc} a & \xleftarrow{\quad} & \cdot \\ \uparrow \tau & & \uparrow \\ \cdot & \xleftarrow{\quad} & \cdot \end{array}$$

and an extension solution  $\Delta[2] \dashrightarrow \text{diag}\mathbb{N}\mathcal{G}$  amounts to a diagram of squares as in

$$\begin{array}{ccccc} a & \xleftarrow{\quad} & \cdot & \xleftarrow{\quad} & \cdot \\ \uparrow \sigma & & \uparrow x & & \uparrow \\ \cdot & \xleftarrow{\quad} & \cdot & \xleftarrow{\quad} & \cdot \\ \uparrow y & & \uparrow z & & \uparrow \\ \cdot & \xleftarrow{\quad} & \cdot & \xleftarrow{\quad} & \cdot \end{array}$$

such that  $(\sigma \circ_{\text{h}} x) \circ_{\text{v}} (y \circ_{\text{h}} z) = \tau$ . To see that such squares  $x$ ,  $y$ , and  $z$  exist, we form the configuration (actually, a 3-simplex of  $\text{diag}\mathbb{N}\mathcal{G}$ )

$$\begin{array}{ccccc} a & \xleftarrow{\quad} & \cdot & \xleftarrow{\quad} & \cdot \\ \uparrow \sigma & & \uparrow \sigma^{-1}_{\text{h}} & & \uparrow \alpha^{-1}_{\text{v}} \\ \cdot & \xleftarrow{\quad} & \cdot & \xleftarrow{\quad} & \cdot \\ \uparrow \sigma^{-1}_{\text{v}} & & \uparrow \sigma^{-1} & & \uparrow \alpha \\ \cdot & \xleftarrow{\quad} & \cdot & \xleftarrow{\quad} & \cdot \\ \uparrow \beta^{-1}_{\text{h}} & & \uparrow \beta & & \uparrow \tau \\ \cdot & \xleftarrow{\quad} & \cdot & \xleftarrow{\quad} & \cdot \end{array}$$

where  $\alpha$  and  $\beta$  are any found thanks  $\mathcal{G}$  satisfies the filling condition. Then, we take  $x = \sigma^{-1}_{\text{h}} \circ_{\text{h}} \alpha^{-1}_{\text{v}}$ ,  $y = \sigma^{-1}_{\text{v}} \circ_{\text{v}} \beta^{-1}_{\text{h}}$ , and  $z = (\sigma^{-1} \circ_{\text{h}} \alpha) \circ_{\text{v}} (\beta \circ_{\text{h}} \tau)$ .

The case in which  $k = 2$  is dual of the case  $k = 0$  above, and the case when  $k = 1$  is easier: A simplicial map  $(\sigma, -, \tau) : \Lambda^1[2] \rightarrow \text{diag}\mathbb{N}\mathcal{G}$  amounts to a couple of squares in  $\mathcal{G}$  of the form

$$\begin{array}{ccc} a & \xleftarrow{\quad} & \cdot \\ \uparrow \sigma & & \uparrow \\ \cdot & \xleftarrow{\quad} & \cdot \end{array} \quad \begin{array}{ccc} \cdot & \xleftarrow{\quad} & \cdot \\ \uparrow \tau & & \uparrow \\ \cdot & \xleftarrow{\quad} & a \end{array}$$

and an extension solution  $\Delta[2] \dashrightarrow \text{diag}\mathbb{N}\mathcal{G}$  is given by any configuration of squares in  $\mathcal{G}$  of the form

$$\begin{array}{ccccc} \cdot & \xleftarrow{\quad} & \cdot & \xleftarrow{\quad} & \cdot \\ \uparrow \tau & & \uparrow x & & \uparrow \\ \cdot & \xleftarrow{\quad} & a & \xleftarrow{\quad} & \cdot \\ \uparrow y & & \uparrow \sigma & & \uparrow \\ \cdot & \xleftarrow{\quad} & \cdot & \xleftarrow{\quad} & \cdot \end{array}$$

Since  $\mathcal{G}$  satisfies the filling condition (recall Lemma 3.1), it is clear that filling squares  $x$  and  $y$  as above exist, and therefore the required extension map exists.

(iii)  $\Rightarrow$  (i) By [13, Theorem 8], all simplicial sets  $\mathbb{N}\mathcal{G}_{p,*}$  and  $\mathbb{N}\mathcal{G}_{*,q}$  satisfy the Kan extension condition. In particular, the nerves of the four component categories of the double category  $\mathcal{G}$ , that is, the simplicial sets  $\mathbb{N}\mathcal{G}_{0,*}$ ,  $\mathbb{N}\mathcal{G}_{*,0}$ ,  $\mathbb{N}\mathcal{G}_{1,*}$ , and  $\mathbb{N}\mathcal{G}_{*,1}$  are all Kan complexes. By [21, Propositions 2.2.3 and 2.2.4], it follows that the four category structures involved are groupoids, and so  $\mathcal{G}$  is a double groupoid.

To see that  $\mathcal{G}$  satisfies the filling condition, suppose that a filling problem

$$\begin{array}{ccc} \cdot & \xrightarrow{g} & \cdot \\ \uparrow \exists? & & \uparrow u \\ \cdot & \xleftarrow{\quad} & \cdot \end{array}$$

is given. Since the simplicial map  $\Lambda^1[2] \xrightarrow{(I^h u, -, I^v g)} \text{diag} \mathbb{N} \mathcal{C}$  has an extension to a 2-simplex  $\Delta[2] \dashrightarrow \text{diag} \mathbb{N} \mathcal{G}$ , we conclude the existence of a diagram of squares in  $\mathcal{G}$  of the form

$$\begin{array}{ccccc} & \xleftarrow{g} & & \xleftarrow{\cdots} & \\ \vdots & & \vdots & & \vdots \\ \vdots & \xleftarrow{I^v g} & \vdots & & \vdots \\ \vdots & \xleftarrow{g} & \vdots & & \vdots \\ \vdots & & \vdots & & \vdots \\ \vdots & \xleftarrow{\alpha} & \vdots & \xleftarrow{I^h u} & \vdots \\ \vdots & & \vdots & & \vdots \end{array}$$

and then, particularly, the existence of a square  $\alpha$  as is required.  $\square$

We now state our main result in this section.

**Theorem 5.3.** *Let  $\mathcal{G}$  be a double groupoid satisfying the filling condition. Then, for each object  $a$  of  $\mathcal{G}$ , there are natural isomorphisms*

$$(5.2) \quad \pi_i(\mathcal{G}, a) \cong \pi_i(|\mathcal{G}|, |a|), \quad i \geq 0.$$

*Proof.* By taking into account Fact 2.7 (1), we shall identify the homotopy groups of  $|\mathcal{G}|$  with those of the Kan complex (by Theorem 5.1)  $\text{diag} \mathbb{N} \mathcal{G}$ , which are defined, as we noted in the preliminary Section 2, using only its simplicial structure.

To compare the  $\pi_0$  sets, observe that the 0-simplices  $a \in \text{diag} \mathbb{N} \mathcal{G}_0 = \mathbb{N} \mathcal{G}_{0,0}$  are precisely the objects of  $\mathcal{G}$ . Furthermore, two 0-simplices  $a, b$  are in the same connected component of  $\text{diag} \mathbb{N} \mathcal{G}$  if and only if there is a square (i.e., a 1-simplex) of the form

$$\begin{array}{ccc} b & \xleftarrow{\cdots} & \cdot \\ \uparrow & \exists? & \uparrow \\ \cdot & \xleftarrow{\cdots} & a, \end{array}$$

that is, since  $\mathcal{G}$  satisfies the filling condition, if and only if  $a$  and  $b$  are connected in  $\mathcal{G}$  (see Subsection 3.1). Thus,  $\pi_0 |\mathcal{G}| = \pi_0 \mathcal{G}$ .

We now compare the  $\pi_1$  groups. An element  $[\alpha] \in \pi_1(|\mathcal{G}|, |a|)$  is the equivalence class of a square  $\alpha$  in  $\mathcal{G}$  of the form

$$\begin{array}{ccc} a & \xleftarrow{g} & \cdot \\ \uparrow & \alpha & \uparrow u \\ \cdot & \xleftarrow{\quad} & a \end{array}$$

and  $[\alpha] = [\alpha']$  if and only if there is a configuration of squares in  $\mathcal{G}$  of the form

$$\begin{array}{ccccccc} a & \xleftarrow{g} & \cdot & \xleftarrow{g'} & \cdot & & \\ \uparrow & \alpha & \uparrow u & x & \uparrow u' & & \\ \cdot & \xleftarrow{\quad} & a & \xleftarrow{\quad} & a & & \\ \uparrow & y & \uparrow & I a & \uparrow & & \\ \cdot & \xleftarrow{\quad} & a & \xleftarrow{\quad} & a & & \end{array}$$

such that  $(\alpha \circ_h x) \circ_v y = \alpha'$ . By recalling now the definition of the homotopy group  $\pi_1(\mathcal{G}, a)$ , we observe that, if  $[\alpha] = [\alpha']$  in  $\pi_1(|\mathcal{G}|, |a|)$ , then, by the existence of the squares  $\alpha$  and  $\alpha \circ_h x$ , we have  $[g, u] = [g \circ_h g', u']$  in  $\pi_1(\mathcal{G}, a)$ ; that is,  $[t^v \alpha, s^h \alpha] = [t^v \alpha', s^h \alpha']$ . It follows that there is a well-defined map

$$\begin{aligned} \Phi : \pi_1(|\mathcal{G}|, |a|) &\longrightarrow \pi_1(\mathcal{G}, a). \\ [\alpha] &\longmapsto [g, u] = [t^v \alpha, s^h \alpha] \end{aligned}$$

This map is actually a group homomorphism. To see that, let

$$\begin{array}{ccc} a & \xleftarrow{g_1} & \cdot \\ \uparrow \alpha_1 & \uparrow u_1 & \\ \cdot & \xleftarrow{\quad} & a \end{array} \quad \begin{array}{ccc} a & \xleftarrow{g_2} & \cdot \\ \uparrow \alpha_2 & \uparrow u_2 & \\ \cdot & \xleftarrow{\quad} & a \end{array}$$

be squares representing elements  $[\alpha_1], [\alpha_2] \in \pi_1(|\mathcal{G}|, |a|)$ . Then, its product in the homotopy group  $\pi_1(|\mathcal{G}|, |a|)$  is  $[\alpha_1] \circ [\alpha_2] = [(\alpha_1 \circ_h \beta) \circ_v (\gamma \circ_h \alpha_2)]$ , where  $\beta$  and  $\gamma$  are any squares in  $\mathcal{G}$  defining a configuration of the form (i.e., a 2-simplex of  $\text{diag}\mathbb{N}\mathcal{G}$ )

$$\begin{array}{ccccc} a & \xleftarrow{g_1} & \cdot & \xleftarrow{g} & \cdot \\ \uparrow \alpha_1 & \uparrow & \beta & \uparrow u & \\ \cdot & \xleftarrow{\quad} & a & \xleftarrow{\quad} & \cdot \\ \uparrow \gamma & \uparrow & \alpha_2 & \uparrow u_2 & \\ \cdot & \xleftarrow{\quad} & \cdot & \xleftarrow{\quad} & a \end{array}$$

Hence,

$$\Phi([\alpha_1] \circ [\alpha_2]) = [g_1 \circ_h g, u \circ_v u_2] = [g_1, u_1] \circ [g_2, u_2] = \Phi([\alpha_1]) \circ \Phi([\alpha_2]),$$

and therefore  $\Phi$  is a homomorphism.

From the filling condition on  $\mathcal{G}$ , it follows that  $\Phi$  is a surjective map. To prove that it is also injective, suppose  $\Phi[\alpha_1] = \Phi[\alpha_2]$ , where  $[\alpha_1], [\alpha_2] \in \pi_1(|\mathcal{G}|, a)$  are as above. This means that there are squares in  $\mathcal{G}$ , say  $x_1$  and  $x_2$ , of the form

$$\begin{array}{ccc} a & \xleftarrow{g_1} & \cdot \\ w \uparrow x_1 & \uparrow u_1 & \\ \cdot & \xleftarrow{f} & a \end{array} \quad \begin{array}{ccc} a & \xleftarrow{g_2} & \cdot \\ w \uparrow x_2 & \uparrow u_2 & \\ \cdot & \xleftarrow{f} & a \end{array}$$

with which we can form the following three 2-simplices of  $\text{diag}\mathbb{N}\mathcal{G}$

$$\begin{array}{ccc} \cdot & \xleftarrow{\quad} & \cdot \\ \uparrow & x_1 & \uparrow \text{I}^h u_1 \\ \cdot & \xleftarrow{\quad} & \cdot \\ \uparrow & x_1^{-1v} \circ_v \alpha_1 & \uparrow \text{I} a \\ \cdot & \xleftarrow{\quad} & \cdot \end{array} \quad \begin{array}{ccc} \cdot & \xleftarrow{\quad} & \cdot \\ \uparrow & x_2 & \uparrow \text{I}^h u_2 \\ \cdot & \xleftarrow{\quad} & \cdot \\ \uparrow & x_2^{-1v} \circ_v \alpha_2 & \uparrow \text{I} a \\ \cdot & \xleftarrow{\quad} & \cdot \end{array} \quad \begin{array}{ccc} \cdot & \xleftarrow{\quad} & \cdot \\ \uparrow & x_1 & \uparrow x_1^{-1h} \circ_h x_2 \\ \cdot & \xleftarrow{\quad} & \cdot \\ \uparrow & \text{I}^v f & \uparrow \text{I} a \\ \cdot & \xleftarrow{\quad} & \cdot \end{array}$$

The first one shows that  $[x_1] = [\alpha_1]$  in the group  $\pi_1(|\mathcal{G}|, a)$ , the second that  $[x_2] = [\alpha_2]$ , and the third that  $[x_1] = [x_2]$ . Whence  $[\alpha_1] = [\alpha_2]$ , as required.

Finally, we show the isomorphisms  $\pi_1(|\mathcal{G}|, |a|) \cong \pi_i(\mathcal{G}, a)$ , for  $i \geq 2$ . For  $i \geq 3$ , it follows from Lemma 5.2 that  $\pi_i(|\mathcal{G}|, a) = 0$ , and the result becomes obvious. For the case  $i = 2$ , it is also a consequence of the afore-mentioned Lemma 5.2 that the homotopy relation between 2-simplices in  $\text{diag}\mathbb{N}\mathcal{G}$  is trivial. Then, the group  $\pi_2(|\mathcal{G}|, |a|)$  consists of all 2-simplices in  $\text{diag}\mathbb{N}\mathcal{G}$  of the form

$$\begin{array}{|c|c|} \hline \text{I} a & \sigma \\ \hline \sigma^{-1} & \text{I} a \\ \hline \end{array}$$

for  $\sigma \in \pi_2(\mathcal{G}, a)$ , whence the isomorphism becomes clear.  $\square$

**Corollary 5.4.** *A double functor  $F: \mathcal{G} \rightarrow \mathcal{G}'$  is a weak equivalence if and only if the induced cellular map on realizations  $|F|: |\mathcal{G}| \rightarrow |\mathcal{G}'|$  is a homotopy equivalence.*

## 6. A LEFT ADJOINT TO THE DOUBLE NERVE FUNCTOR.

Recall from Theorem 5.1 (ii) that the double nerve  $\mathbb{N}\mathcal{G}$ , of any double groupoid satisfying the filling condition, satisfies the extension condition. Moreover, since both simplicial sets  $\mathbb{N}\mathcal{G}_{*,0}$  and  $\mathbb{N}\mathcal{G}_{0,*}$  are nerves of groupoids, all homotopy groups  $\pi_2(\mathbb{N}\mathcal{G}_{*,0}, a)$  and  $\pi_2(\mathbb{N}\mathcal{G}_{0,*}, a)$  vanish. Our goal in this section is to prove that the double nerve functor,  $\mathcal{G} \mapsto \mathbb{N}\mathcal{G}$ , embeds, as a reflexive subcategory, the category of double groupoids with filling condition into the category of those bisimplicial sets  $K$  that satisfy the extension condition and such that  $\pi_2(K_{*,0}, a) = 0 = \pi_2(K_{0,*}, a)$  for all vertices  $a \in K_{0,0}$ . That is, there is a reflector functor for such bisimplicial sets

$$K \mapsto \mathbf{P}K,$$

which works as a bisimplicial version of Brown's construction in [6, Theorem 2.1]. Furthermore, as we will prove, the resulting double groupoid  $\mathbf{P}K$  always represents the homotopy 2-type of the input bisimplicial set  $K$ , in the sense that there is a natural weak 2-equivalence  $|K| \rightarrow |\mathbf{P}K|$ .

For any given bisimplicial set  $K$ , under the assumption that it satisfies the extension condition and both the Kan complexes  $K_{*,0}$  and  $K_{0,*}$  have trivial groups  $\pi_2$ , the definition of the *homotopy double groupoid*  $\mathbf{P}K$  is as follows:

The objects of  $\mathbf{P}K$  are the vertices  $a : \Delta[0, 0] \rightarrow K$  of  $K$ .

The groupoid of horizontal morphisms is the horizontal fundamental groupoid  $\mathbf{P}K_{*,0}$ , and the groupoid of vertical morphisms is the vertical fundamental groupoid  $\mathbf{P}K_{0,*}$  (see the last part of Subsection 2.2). Thus, a horizontal morphism  $[f]_h : a \rightarrow b$  is the horizontal homotopy class of a bisimplex  $f : \Delta[1, 0] \rightarrow K$  with  $fd_h^0 = a$  and  $fd_h^1 = b$ , whereas a vertical morphism in  $\mathbf{P}K$ ,  $[u]_v : a \rightarrow b$ , is the vertical homotopy class of a bisimplex  $u : \Delta[0, 1] \rightarrow K$  with  $ud_v^0 = a$  and  $ud_v^1 = b$ .

A square of  $\mathbf{P}K$  is the bihomotopy class  $[[x]]$  of a bisimplex  $x : \Delta[1, 1] \rightarrow K$ , with boundary

$$\begin{array}{ccc} & [xd_v^1]_h & \\ [xd_h^1]_v \uparrow & \xleftarrow{\quad} & \uparrow [xd_h^0]_v \\ & [xd_v^0]_h & \end{array}$$

which is well defined thanks to Lemma 2.6.

The horizontal composition of squares in  $\mathbf{P}K$  is the only one making the correspondence

$$\left( xd_h^0 \xrightarrow{[x]_h} xd_h^1 \right) \mapsto \left( [xd_h^0]_v \xrightarrow{[[x]]} [xd_h^1]_v \right)$$

a surjective fibration of groupoids from the horizontal fundamental groupoid  $\mathbf{P}K_{*,1}$  to the horizontal groupoid of squares in  $\mathbf{P}K$ . To define this composition, we shall need the following:

**Lemma 6.1.** *Let  $x, y : \Delta[1, 1] \rightarrow K$  be bisimplices such that  $[xd_h^0]_v = [yd_h^1]_v$ . Then, there is a bisimplex  $x' : \Delta[1, 1] \rightarrow K$  such that  $[x']_v = [x]_v$  and  $x'd_h^0 = yd_h^1$ .*

*Proof.* Once any vertical homotopy from  $xd_h^0$  to  $yd_h^1$  is selected, say  $\alpha : \Delta[0, 2] \rightarrow K$ , let  $\beta : \Delta[1, 2] \rightarrow K$  be any bisimplex solving the extension problem

$$\begin{array}{ccc} \Lambda^{1,2}[1, 2] & \xrightarrow{(\alpha, -; xd_v^0 s_v^0, x, -)} & K. \\ \downarrow & \nearrow \beta & \\ \Delta[1, 2] & & \end{array}$$

Then, we take  $x' = \beta d_v^2 : \Delta[1, 1] \rightarrow K$ . Since  $\beta$  becomes a vertical homotopy from  $x$  to  $x'$ , we have  $[x]_v = [x']_v$ . Moreover,  $x' d_h^0 = \beta d_v^2 d_h^0 = \beta d_h^0 d_v^2 = \alpha d_v^2 = y d_h^1$ , as required.  $\square$

**Remark.** Note that, for any such bisimplex  $x'$  as in the lemma, we have  $[[x']] = [[x]]$  and  $x' d_v^i = x d_v^i$  for  $i = 0, 1$ .

Now define the horizontal composition of squares in  $\mathbf{PK}$  by

$$(6.1) \quad [[x]] \circ_h [[y]] = [[x']_h \circ_h [y]_h] \quad \text{if} \quad [x]_v = [x']_v \text{ and } x' d_h^0 = y d_h^1,$$

where  $[x']_h \circ_h [y]_h$  is the composite in the fundamental groupoid  $\mathbf{PK}_{*,1}$ , that is,

$$(6.2) \quad [[x]] \circ_h [[y]] = [[\gamma d_h^1]]$$

for  $\gamma : \Delta[2, 1] \rightarrow K$  any bisimplex with  $\gamma d_h^2 = x'$  and  $\gamma d_h^0 = y$ .

In view of Lemma 6.1, our product is given for all squares  $[[x]]$  and  $[[y]]$  with  $s^h[[x]] = t^h[[y]]$ . We also have the lemma below.

**Lemma 6.2.** *The horizontal composition of squares in  $\mathbf{PK}$  is well defined.*

*Proof.* We first prove that the square in (6.1) does not depend on the choice of  $x'$ . To do so, suppose  $x'' : \Delta[1, 1] \rightarrow K$  is another bisimplex such that  $[x]_v = [x'']_v$  and  $x'' d_h^0 = y d_h^1$ , and let  $\beta, \beta' : \Delta[1, 2] \rightarrow K$  be vertical homotopies from  $x$  to  $x'$  and from  $x$  to  $x''$  respectively. Then, both bisimplices  $\beta d_h^0 : \Delta[0, 2] \rightarrow K$  and  $\beta' d_h^0 : \Delta[0, 2] \rightarrow K$  have the same vertical faces. Since the 2<sup>nd</sup> homotopy groups of the Kan complex  $K_{0,*}$  vanish, it follows that  $\beta d_h^0$  and  $\beta' d_h^0$  are vertically homotopic (Fact 2.2). Choose  $\omega : \Delta[0, 3] \rightarrow K$  any vertical homotopy from  $\beta d_h^0$  to  $\beta' d_h^0$ , and then let  $\Gamma : \Delta[1, 3] \rightarrow K$  be a solution to the extension problem

$$\begin{array}{ccc} \Lambda^{1,3}[1, 3] & \xrightarrow{(\omega, -; x d_v^0 s_v^0 s_v^1, x s_v^1, \beta, -)} & K \\ \downarrow & \nearrow \Gamma & \\ \Delta[1, 3] & & \end{array}$$

Then, the bisimplex  $\tilde{\beta} = \Gamma d_v^3 : \Delta[1, 2] \rightarrow K$  has vertical faces

$$\begin{aligned} \tilde{\beta} d_v^0 &= \Gamma d_v^3 d_v^0 = \Gamma d_v^0 d_v^2 = x d_v^0 s_v^0 s_v^1 d_v^2 = x d_v^0 s_v^0, \\ \tilde{\beta} d_v^1 &= \Gamma d_v^3 d_v^1 = \Gamma d_v^1 d_v^2 = x s_v^1 d_v^2 = x, \\ \tilde{\beta} d_v^2 &= \Gamma d_v^3 d_v^2 = \Gamma d_v^2 d_v^2 = \beta d_v^2 = x', \end{aligned}$$

so that  $\tilde{\beta}$  is another vertical homotopy from  $x$  to  $x'$ , and moreover

$$\tilde{\beta} d_h^0 = \Gamma d_v^3 d_h^0 = \Gamma d_h^0 d_v^3 = \omega d_v^3 = \beta' d_h^0,$$

that is,  $\tilde{\beta}$  and  $\beta'$  have both the same horizontal 0-face, say  $\alpha$ . Now let  $\Phi : \Delta[1, 3] \rightarrow K$  and  $\theta : \Delta[2, 2] \rightarrow K$  be solutions to the following extension problems

$$\begin{array}{ccc} \Lambda^{1,3}[1, 3] & \xrightarrow{(\alpha s_v^1, -; x d_v^0 s_v^0 s_v^1, \tilde{\beta}, \beta', -)} & K \\ \downarrow & \nearrow \Phi & \\ \Delta[1, 3] & & \end{array} \quad \begin{array}{ccc} \Lambda^{1,2}[2, 2] & \xrightarrow{(y s_v^1, -; \Phi d_v^3; \gamma d_v^0 s_v^0, \gamma, -)} & K \\ \downarrow & \nearrow \theta & \\ \Delta[2, 2] & & \end{array}$$

where  $\gamma : \Delta[2, 1] \rightarrow K$  is any bisimplex such that  $\gamma d_h^2 = x'$  and  $\gamma d_h^0 = y$ . Then,  $\theta$  is actually a vertical homotopy from  $\gamma$  to  $\gamma' = \theta d_v^2$ , and this bisimplex  $\gamma'$  satisfies that

$$\begin{aligned} \gamma' d_h^2 &= \theta d_v^2 d_h^2 = \theta d_h^2 d_v^2 = \Phi d_v^3 d_v^2 = \Phi d_v^2 d_v^2 = \beta' d_v^2 = x'', \\ \gamma' d_h^0 &= \theta d_v^2 d_h^0 = \theta d_h^0 d_v^2 = y s_v^1 d_v^2 = y. \end{aligned}$$

Hence,  $[x']_h \circ_h [y]_h = [\gamma d_h^1]_h$  whereas  $[x'']_h \circ_h [y]_h = [\gamma' d_h^1]_h$ . Since the bisimplex  $\theta d_h^1 : \Delta[1, 2] \rightarrow K$  is a vertical homotopy from  $\gamma d_h^1$  to  $\gamma' d_h^1$ , we conclude that  $[[\gamma d_h^1]] = [[\gamma' d_h^1]]$ , that is,  $[[x']_h \circ_h [y]_h] = [[x'']_h \circ_h [y]_h]$ , as required.

Suppose now  $x_0, x_1, y : \Delta[1, 1] \rightarrow K$  bisimplices with  $[[x_0]] = [[x_1]]$  and  $[x_0 d_h^0]_v = [y d_h^1]_v$ . Then, for some  $x : \Delta[1, 1] \rightarrow K$ , we have  $[x_0]_v = [x]_v$  and  $[x]_h = [x_1]_h$ . Let  $x'_0 : \Delta[1, 1] \rightarrow K$  be any bisimplex with  $[x'_0]_v = [x]_v$  and  $x'_0 d_h^0 = y d_h^1$ . Since  $[x'_0]_v = [x_0]_v$ , we have

$$(6.3) \quad [[x_0]] \circ_h [[y]] = [[x'_0]_h \circ [y]_h].$$

Letting  $\beta : \Delta[1, 2] \rightarrow K$  be any vertical homotopy from  $x$  to  $x'_0$  and  $\delta : \Delta[2, 1] \rightarrow K$  be any horizontal homotopy from  $x_1$  to  $x$ , we can choose  $\theta : \Delta[2, 2] \rightarrow K$ , a bisimplex making commutative the diagram

$$\begin{array}{ccc} \Lambda^{1,2}[2, 2] & \xrightarrow{(\beta d_h^0 s_h^0, -, \beta; \delta d_v^0 s_v^0, \delta, -)} & K \\ \downarrow & \searrow \theta & \\ \Delta[2, 2] & & \end{array}$$

Then,  $\beta_1 = \theta d_h^1 : \Delta[1, 2] \rightarrow K$  is a vertical homotopy from  $x_1$  to  $x'_1 := \beta_1 d_v^2$ , and since

$$x'_1 d_h^0 = \beta_1 d_v^2 d_h^0 = \beta_1 d_h^0 d_v^2 = \theta d_h^1 d_h^0 d_v^2 = \theta d_h^0 d_h^0 d_v^2 = \beta d_h^0 d_v^2 = \beta d_v^2 d_h^0 = x'_0 d_h^0 = y d_h^1,$$

we have

$$(6.4) \quad [[x_1]] \circ_h [[y]] = [[x'_1]_h \circ [y]_h].$$

As  $\theta d_v^2 : \Delta[2, 1] \rightarrow K$  is a horizontal homotopy from  $x'_1$  to  $x'_0$ , we have  $[x'_0]_h = [x'_1]_h$ . Therefore, comparing (6.3) with (6.4), we obtain the desired conclusion, that is,

$$[[x_0]] \circ_h [[y]] = [[x_1]] \circ_h [[y]].$$

Finally, suppose  $x, y_0, y_1 : \Delta[1, 1] \rightarrow K$  with  $[[y_0]] = [[y_1]]$  and  $[x d_h^0]_v = [y_0 d_h^1]_v$ . Then,  $[y_0]_v = [y]_v$ ,  $[y]_h = [y_1]_h$ , for some  $y : \Delta[1, 1] \rightarrow K$ . Let  $x' : \Delta[1, 1] \rightarrow K$  be such that  $[x]_v = [x']_v$  and  $x' d_h^0 = y d_h^1$ . Since  $x' d_h^0 = y_1 d_h^1$ , we have

$$(6.5) \quad [[x]] \circ_h [[y_1]] = [[x']_h \circ [y_1]_h] = [[x']_h \circ [y]_h] = [[\gamma d_h^1]],$$

for  $\gamma : \Delta[2, 1] \rightarrow K$  any bisimplex with  $\gamma d_h^2 = x'$  and  $\gamma d_h^0 = y$ . Now, as  $[y_0]_v = [y]_v$ , we can select a vertical homotopy  $\delta : \Delta[1, 2] \rightarrow K$  from  $y$  to  $y_0$ , and then a bisimplex  $\beta_0 : \Delta[1, 2] \rightarrow K$  making commutative the diagram

$$\begin{array}{ccc} \Lambda^{1,2}[1, 2] & \xrightarrow{(\delta d_h^1, -; x' d_v^0 s_v^0, x', -)} & K \\ \downarrow & \searrow \beta_0 & \\ \Delta[1, 2] & & \end{array}$$

This bisimplex  $\beta_0$  becomes a vertical homotopy from  $x'$  to  $x'_0 := \beta_0 d_v^2$ , and this  $x'_0$  verifies that  $x'_0 d_h^0 = y_0 d_h^1$ . Hence,

$$(6.6) \quad [[x]] \circ_h [[y_0]] = [[x'_0]_h \circ [y_0]_h].$$

But, by taking  $\theta : \Delta[2, 2] \rightarrow K$  any bisimplex solving the extension problem

$$\begin{array}{ccc} \Lambda^{1,2}[2, 2] & \xrightarrow{(\delta, -, \beta_0; \gamma d_v^0 s_v^0, \gamma, -)} & K \\ \downarrow & \searrow \theta & \\ \Delta[2, 2] & & \end{array}$$

we obtain a bisimplex  $\gamma_0 := \theta d_v^2 : \Delta[2, 1] \rightarrow K$  satisfying that  $\gamma_0 d_h^0 = y_0$  and  $\gamma_0 d_h^2 = x'_0$ , whence

$$[[x]] \circ_h [[y_0]] = [[\gamma_0 d_h^1]].$$

As the bisimplex  $\theta d_h^1 : \Delta[1, 2] \rightarrow K$  is easily recognized to be a vertical homotopy from  $\gamma d_h^1$  to  $\gamma_0 d_h^1$ , we conclude  $[[\gamma d_h^1]] = [[\gamma_0 d_h^1]]$ . Consequently, the required equality

$$[[x]] \circ_h [[y_0]] = [[x]] \circ_h [[y_1]]$$

follows by comparing (6.5) with (6.6).  $\square$

Simply by exchanging the horizontal and vertical directions in the foregoing discussion, we also have a well-defined vertical composition of squares  $[[x]]$  and  $[[y]]$  in  $\mathbf{PK}$ , whenever  $[x d_v^0]_h = [y d_v^1]_h$ , which is given by

$$(6.7) \quad [[x]] \circ_v [[y]] = [[x']_v \circ_v [y]_v] \quad \text{if} \quad [x]_h = [x']_h \text{ and } x' d_v^0 = y d_v^1,$$

where  $[x']_v \circ_v [y]_v$  is the composite in the fundamental groupoid  $\mathbf{PK}_{1,*}$ , that is,

$$(6.8) \quad [[x]] \circ_v [[y]] = [[\gamma d_v^1]]$$

for  $\gamma : \Delta[1, 2] \rightarrow K$  any bisimplex with  $\gamma d_v^2 = x'$  and  $\gamma d_v^0 = y$ .

**Theorem 6.3.**  *$\mathbf{PK}$  is a double groupoid satisfying the filling condition.*

*Proof.* We first observe that, with both defined horizontal and vertical compositions, the squares in  $\mathbf{PK}$  form groupoids. The associativity for the horizontal composition of squares in  $\mathbf{PK}$  follows from the associativity of the composition of morphisms in the fundamental groupoid  $\mathbf{PK}_{*,1}$ . In effect, let  $[[x]]$ ,  $[[y]]$  and  $[[z]]$  be three horizontally composable squares in  $\mathbf{PK}$ . By changing representatives if necessary, we can assume that  $x d_h^0 = y d_h^1$  and  $y d_h^0 = z d_h^1$ . Then,

$$\begin{aligned} [[x]] \circ_h ([[y]] \circ_h [[z]]) &= [[x]] \circ_h [[y]_h \circ_h [z]_h] &= [[x]_h \circ_h ([y]_h \circ_h [z]_h)] \\ &= [[([x]_h \circ_h [y]_h) \circ_h [z]_h]] &= [[x]_h \circ_h [y]_h \circ_h [z]] \\ &= ([[x]] \circ_h [[y]]) \circ_h [[z]]. \end{aligned}$$

The horizontal identity square on the vertical morphism represented by a bisimplex  $u : \Delta[0, 1] \rightarrow K$  is

$$(6.9) \quad \Gamma^h[u]_v = [[u s_h^0]]$$

(recall Lemma 2.6), as can be easily deduced from the fact that  $[u s_h^0]_h$  is the identity morphism on  $u$  in the groupoid  $\mathbf{PK}_{*,1}$ . Thus, for example, for any  $x : \Delta[1, 1] \rightarrow K$ ,

$$[[x]] \circ_h \Gamma^h[x d_h^0]_v = [[x]_h \circ_h [x d_h^0 s_h^0]_h] = [[x]_h] = [[x]].$$

The horizontal inverse in  $\mathbf{PK}$  of a square  $[[x]]$  is  $[[x]]^{-1h} = [[x]_h^{-1}]$ , where  $[x]_h^{-1}$  is the inverse of  $[x]_h$  in  $\mathbf{PK}_{*,1}$ , as is easy to verify:

$$[[x]] \circ_h [[x]_h^{-1}] = [[x]_h \circ_h [x]_h^{-1}] = [[x d_h^1 s_h^0]] = \Gamma^h[x d_h^1]_v.$$

Similarly, we see that the associativity for the vertical composition of squares in  $\mathbf{PK}$  follows from the associativity of the composition in the fundamental groupoid  $\mathbf{PK}_{1,*}$ , that the vertical identity square on the horizontal morphism represented by a bisimplex  $f : \Delta[1, 0] \rightarrow K$  is  $\Gamma^v([f]_h) = [[f s_v^0]]$ , and that the vertical inverse in  $\mathbf{PK}$  of a square  $[[x]]$  is  $[[x]_v^{-1}]$ , where  $[x]_v^{-1}$  denotes the inverse of  $[x]_v$  in  $\mathbf{PK}_{1,*}$ .

We are now ready to prove that  $\mathbf{PK}$  is actually a double groupoid. **Axiom 1** is easily verified. Thus, for example, given any  $x : \Delta[1, 1] \rightarrow K$ ,

$$s^h s^v [[x]] = s^h [x d_v^0]_h = x d_v^0 d_h^0 = x d_h^0 d_v^0 = s^v [x d_h^0]_v = s^v s^h [[x]],$$



or, given any  $f : \Delta[1, 0] \rightarrow K$ ,

$$s^h \Gamma^v[f]_h = s^h [[f s_v^0]] = [f s_v^0 d_h^0]_v = [f d_h^0 s_v^0]_v = \Gamma^v f d_h^0 = \Gamma^v s^h[f]_h,$$

and so on. Also, for any  $a : \Delta[0, 0] \rightarrow K$ ,

$$\Gamma^h \Gamma^v a = \Gamma^h [a s_v^0]_v = [[a s_v^0 s_h^0]] = [[a s_h^0 s_v^0]] = \Gamma^v [a s_h^0]_h = \Gamma^v \Gamma^h a.$$

For **Axiom 2** (i), let  $[[x]]$  and  $[[y]]$  be two horizontally composable squares in  $\mathbf{PK}$ . We can assume that  $x d_h^0 = y d_h^1$ , and then  $[[x]] \circ_h [[y]] = [[\gamma d_h^1]]$ , for any  $\gamma : \Delta[2, 1] \rightarrow K$  with  $\gamma d_h^2 = x$  and  $\gamma d_h^0 = y$ . Hence,

$$\begin{aligned} s^v([[x]] \circ_h [[y]]) &= [\gamma d_h^1 d_v^0]_h = [\gamma d_v^0 d_h^1] = [\gamma d_v^0 d_h^2]_h \circ_h [\gamma d_v^0 d_h^0]_h \\ &= [\gamma d_h^2 d_v^0]_h \circ_h [\gamma d_h^0 d_v^0] = [x d_v^0]_h \circ_h [y d_v^0]_h = s^v[[x]] \circ_h s^v[[y]], \\ t^v([[x]] \circ_h [[y]]) &= [\gamma d_h^1 d_v^1]_h = [\gamma d_v^1 d_h^1]_h = [\gamma d_v^1 d_h^2]_h \circ_h [\gamma d_v^1 d_h^0]_h \\ &= [\gamma d_h^2 d_v^1]_h \circ_h [\gamma d_h^0 d_v^1]_h = [x d_v^1]_h \circ_h [y d_v^1]_h = t^v[[x]] \circ_h t^v[[y]]. \end{aligned}$$

**Axiom 2** (ii) is proved analogously, and for (iii), let  $f, f' : \Delta[1, 0] \rightarrow K$  be maps with  $f d_h^0 = f' d_h^1$ . Then,  $[f]_h \circ_h [f']_h = [\gamma d_h^1]_h$ , for  $\gamma : \Delta[2, 0] \rightarrow K$  any bisimplex with  $\gamma d_h^2 = f$  and  $\gamma d_h^0 = f'$ , and we have the equalities:

$$\Gamma^v([f]_h \circ_h [f']_h) = \Gamma^v[\gamma d_h^1]_h = [[\gamma d_h^1 s_v^0]] = [[\gamma s_v^0 d_h^1]] = [[\gamma s_v^0 d_h^2]] \circ_h [[\gamma s_v^0 d_h^0]] = \Gamma^v[f]_h \circ_h \Gamma^v[f']_h.$$

And similarly one sees that  $\Gamma^h([u]_v \circ_v [u']_v) = \Gamma^h[u]_v \circ_v \Gamma^h[u']_v$  for any  $u, u' : \Delta[0, 1] \rightarrow K$  with  $u d_v^0 = u' d_v^1$ .

To verify **Axiom 3**, that is, to prove that the interchange law holds in  $\mathbf{PK}$ , let

$$\begin{array}{ccccc} & \longleftarrow & & \longleftarrow & \\ \uparrow & & \uparrow & & \uparrow \\ & \longleftarrow & & \longleftarrow & \\ \uparrow & & \uparrow & & \uparrow \\ & \longleftarrow & & \longleftarrow & \end{array}$$

be squares in  $\mathbf{PK}$ . By an iterated use of Lemma 6.1 (and its corresponding version for vertical direction), we can assume that  $x d_h^0 = x' d_h^1$ ,  $x d_v^0 = y d_v^1$ ,  $x' d_v^0 = y' d_v^1$  and  $y d_h^0 = y' d_h^1$ . Let  $\alpha : \Delta[2, 1] \rightarrow K$  and  $\beta : \Delta[1, 2] \rightarrow K$  be bisimplicial maps such that  $\alpha d_h^2 = y$ ,  $\alpha d_h^0 = y'$ ,  $\beta d_v^2 = x'$  and  $\beta d_v^0 = y'$ ; therefore,  $[[y]] \circ_h [[y']] = [[\alpha d_h^1]]$  and  $[[x']] \circ_v [[y']] = [[\beta d_v^1]]$ . Now we select bisimplices  $\gamma : \Delta[1, 2] \rightarrow K$  and  $\delta : \Delta[2, 1] \rightarrow K$  as respective solutions to the following extension problems:

$$\begin{array}{ccc} \Lambda^{1,1}[1, 2] & \xrightarrow{(\beta d_h^1, -, y, -, x)} & K \\ \downarrow & \nearrow \gamma & \\ \Delta[1, 2] & & \end{array} \quad \begin{array}{ccc} \Lambda^{1,1}[2, 1] & \xrightarrow{(x', -, x; \alpha d_v^1, -)} & K \\ \downarrow & \nearrow \delta & \\ \Delta[2, 1] & & \end{array}$$

Then  $[[x]] \circ_v [[y]] = [[\gamma d_v^1]]$ ,  $[[x]] \circ_h [[x']] = [[\delta d_h^1]]$  and, moreover, we can find a bisimplex  $\theta : \Delta[2, 2] \rightarrow K$  making the triangle below commutative.

$$\begin{array}{ccc} \Lambda^{1,1}[2, 2] & \xrightarrow{(\beta, -, \gamma; \alpha, -, \delta)} & K \\ \downarrow & \nearrow \theta & \\ \Delta[2, 2] & & \end{array}$$

Letting  $\phi = \theta d_h^1 : \Delta[1, 2] \rightarrow K$  and  $\psi = \theta d_v^1 : \Delta[2, 1] \rightarrow K$ , we have the equalities:

$$\begin{aligned} \phi d_v^2 &= \theta d_v^2 d_h^1 = \delta d_h^1, & \phi d_v^0 &= \theta d_v^0 d_h^1 = \alpha d_h^1, \\ \psi d_h^2 &= \theta d_h^2 d_v^1 = \gamma d_v^1, & \psi d_h^0 &= \theta d_h^0 d_v^1 = \beta d_v^1, \end{aligned}$$

whence,

$$\begin{aligned} ([x] \circ_h [[x']]) \circ_v ([y] \circ_h [[y']]) &= [[\delta d_h^1]] \circ_v [[\alpha d_h^1]] = [[\phi d_v^1]], \\ ([x] \circ_v [[y]]) \circ_h ([x'] \circ_h [[y']]) &= [[\gamma d_v^1]] \circ_h [[\beta d_v^1]] = [[\psi d_h^1]]. \end{aligned}$$

Since  $\phi d_v^1 = \theta d_h^1 d_v^1 = \theta d_v^1 d_h^1 = \psi d_h^1$ , the interchange law follows.

Thus,  $\mathbf{P}K$  is a double groupoid and, moreover, it satisfies the filling condition: given morphisms

$$\begin{array}{c} \cdot \xleftarrow{[g]_h} \cdot \\ \uparrow [u]_v \\ \cdot \end{array}$$

represented by bisimplices  $u : \Delta[0, 1] \rightarrow K$  and  $g : \Delta[1, 0] \rightarrow K$  with  $gd_h^0 = ud_v^1$ , if  $x : \Delta[1, 1] \rightarrow K$  is any solution to the extension problem

$$\begin{array}{ccc} \Lambda^{0,1}[1, 1] & \xrightarrow{(-, g; u, -)} & K \\ \downarrow & \nearrow x & \\ \Delta[1, 1] & & \end{array}$$

then the bihomotopy class of  $x$  is a square in  $\mathbf{P}K$ ,  $\begin{array}{c} \cdot \xleftarrow{[g]_h} \cdot \\ \uparrow [[x]] \uparrow [u]_v \\ \cdot \xleftarrow{\quad} \cdot \end{array}$ , as required.  $\square$

The construction of the double groupoid  $\mathbf{P}K$  is clearly functorial on  $K$ , and we have the following:

**Theorem 6.4.** *The double nerve construction,  $\mathcal{G} \mapsto \mathbb{N}\mathcal{G}$ , embeds, as a reflexive subcategory, the category of double groupoids satisfying the filling condition into the category of those bisimplicial sets  $K$  that satisfy the extension condition and such that  $\pi_2(K_{*,0}, a) = 0 = \pi_2(K_{0,*}, a)$  for all vertices  $a \in K_{0,0}$ . The reflector functor for such bisimplicial sets is given by the above described homotopy double groupoid construction*

$$K \mapsto \mathbf{P}K.$$

Thus,  $\mathbf{P}\mathbb{N}\mathcal{G} = \mathcal{G}$ , and there are natural bisimplicial maps

$$(6.10) \quad \epsilon(K) : K \rightarrow \mathbf{P}K,$$

such that  $\mathbf{P}\epsilon = \text{id}$  and  $\epsilon\mathbb{N} = \text{id}$ .

*Proof.* From Theorem 5.1(ii), if  $\mathcal{G}$  is any double groupoid satisfying the filling condition, then its double nerve  $\mathbb{N}\mathcal{G}$  satisfies the extension condition and, since both simplicial sets  $\mathbb{N}\mathcal{G}_{*,0}$  and  $\mathbb{N}\mathcal{G}_{0,*}$  are nerves of groupoids, all homotopy groups  $\pi_2(\mathbb{N}\mathcal{G}_{*,0}, a)$  and  $\pi_2(\mathbb{N}\mathcal{G}_{0,*}, a)$  vanish. Moreover, since the bihomotopy relation is trivial on the bisimplices  $\Delta[p, q] \rightarrow \mathbb{N}\mathcal{G}$ , for  $p \geq 1$  or  $q \geq 1$ , it is easy to see that  $\mathbf{P}\mathbb{N}\mathcal{G} = \mathcal{G}$ .

For any bisimplicial set  $K$  in the hypothesis of the theorem, there is a natural bisimplicial map

$$\epsilon = \epsilon(K) : K \rightarrow \mathbf{P}K,$$

that takes a bisimplex  $x : \Delta[p, q] \rightarrow K$ , of  $K$ , to the bisimplex  $\epsilon x : [p] \otimes [q] \rightarrow \mathbf{P}K$ , of  $\mathbf{P}K$ , defined by the  $p \times q$  configuration of squares in  $\mathbf{P}K$

$$\left( \begin{array}{ccc} & \xleftarrow{\epsilon_{i,j}^r x} & \\ \epsilon_i^r x \uparrow & \epsilon_{i,j}^{r,s} x & \uparrow \epsilon_j^{r,s} x \\ & \xleftarrow{\epsilon_j^s x} & \end{array} \right)_{\substack{0 \leq i \leq j \leq p \\ 0 \leq r \leq s \leq q}},$$

where

$$\begin{aligned} \epsilon_{i,j}^{r,s} x &= [[x d_h^p \cdots d_h^{j+1} d_h^{j-1} \cdots d_h^{i+1} d_h^{i-1} \cdots d_h^0 d_v^q \cdots d_v^{s+1} d_v^{s-1} \cdots d_v^{r+1} d_v^{r-1} \cdots d_v^0]], \\ \epsilon_j^{r,s} x &= [x d_h^p \cdots d_h^{j+1} d_h^{j-1} \cdots d_h^0 d_v^q \cdots d_v^{s+1} d_v^{s-1} \cdots d_v^{r+1} d_v^{r-1} \cdots d_v^0]_v, \\ \epsilon_{i,j}^r x &= [x d_h^p \cdots d_h^{j+1} d_h^{j-1} \cdots d_h^{i+1} d_h^{i-1} \cdots d_h^0 d_v^q \cdots d_v^{r+1} d_v^{r-1} \cdots d_v^0]_h, \\ \epsilon_i^r x &= x d_h^p \cdots d_h^{i+1} d_h^{i-1} \cdots d_h^0 d_v^q \cdots d_v^{r+1} d_v^{r-1} \cdots d_v^0. \end{aligned}$$

Since a straightforward verification shows that  $\mathbf{P}\epsilon(K)$  is the identity map on  $\mathbf{P}K$ , for any  $K$ , and  $\epsilon(\mathbb{N}\mathcal{G})$  is the identity map on  $\mathbb{N}\mathcal{G}$ , for any double groupoid  $\mathcal{G}$ , it follows that  $\mathbb{N}$  is right adjoint to  $\mathbf{P}$ , with  $\epsilon$  and the identity being the unit and the counit of the adjunction respectively.  $\square$

With the next theorem we show that the double groupoid  $\mathbf{P}K$  represents the same homotopy 2-type as the bisimplicial set  $K$ .

**Theorem 6.5.** *Let  $K$  be any bisimplicial set satisfying the extension condition and such that  $\pi_2(K_{0,*}, a) = 0 = \pi_2(K_{*,0}, a)$  for all base vertices  $a$ . Then, the induced map by unit of the adjunction  $|\epsilon| : |K| \rightarrow |\mathbf{P}K| = |\mathbf{P}K|$  is a weak homotopy 2-equivalence.*

*Proof.* By Facts 2.8 (1) and (3) and Theorem 5.1, the map  $|\epsilon| : |K| \rightarrow |\mathbf{P}K|$  is, up to natural homotopy equivalences, induced by the simplicial map  $\overline{W}\epsilon : \overline{W}K \rightarrow \overline{W}\mathbf{P}K$ , where both  $\overline{W}K$  and  $\overline{W}\mathbf{P}K$  are Kan-complexes.

At dimension 0, we have the equalities  $\overline{W}K_0 = K_{0,0} = \overline{W}\mathbf{P}K_0$ , and the map  $\overline{W}\epsilon$  is the identity on 0-simplices. At dimension 1, the map

$$\overline{W}\epsilon : (x_{0,1}, x_{1,0}) \mapsto ([x_{0,1}]_v, [x_{1,0}]_h),$$

is clearly surjective, whence we conclude that the induced

$$\pi_0 \overline{W}\epsilon : \pi_0 \overline{W}K \rightarrow \pi_0 \overline{W}\mathbf{P}K \stackrel{(5.2)}{\cong} \pi_0 \mathbf{P}K$$

is a bijection and also that, for any vertex  $a \in K_{0,0}$ , that induced on the  $\pi_1$ -groups

$$\pi_1 \overline{W}\epsilon : \pi_1(\overline{W}K, a) \rightarrow \pi_1(\overline{W}\mathbf{P}K, a) \stackrel{(5.2)}{\cong} \pi_1(\mathbf{P}K, a)$$

is surjective. To see that  $\pi_1 \overline{W}\epsilon$  is actually an isomorphism, suppose that  $(x_{0,1}, x_{1,0}) \in \overline{W}K_1$ , with  $x_{0,1} d_v^1 = a = x_{1,0} d_h^0$ , represents an element in the kernel of  $\pi_1 \overline{W}\epsilon$ . This implies the existence of a bisimplex  $x : \Delta[1, 1] \rightarrow K$  whose bihomotopy class is a square in  $\mathbf{P}K$  with boundary as in

$$\begin{array}{ccc} & a \xleftarrow{[as_h^0]_h} a & \\ [x_{0,1}]_v \uparrow & [[x]] & \parallel [as_v^0]_v \\ & \cdot \xleftarrow{a} a & \\ & [x_{1,0}]_h & \end{array}$$

Using Lemma 6.1 twice (one in each direction), we can find a bisimplex  $x_{1,1} : \Delta[1, 1] \rightarrow K$ , such that  $[[x_{1,1}]] = [[x]]$ ,  $x_{1,1}d_v^1 = as_h^0$ , and  $x_{1,1}d_h^0 = as_v^0$ . Moreover, since  $[x_{1,1}d_v^0]_h = [x_{1,0}]_h$  and  $[x_{1,1}d_h^1]_v = [x_{0,1}]_v$ , there are bisimplices  $x_{2,0} : \Delta[2, 0] \rightarrow K$  and  $x_{0,2} : \Delta[0, 2] \rightarrow K$ , with faces as in the picture

$$\begin{array}{ccccc}
 & a & \xleftarrow{as_v^0} & a & \xleftarrow{as_h^0} & a \\
 & \swarrow x_{0,2} & \uparrow x_{0,1} & \uparrow x_{1,1} & \uparrow as_v^0 & \\
 & & & a & \xleftarrow{as_h^0} & a \\
 & & & \swarrow x_{2,0} & \uparrow x_{1,0} & \\
 & & & & a &
 \end{array}$$

This amounts to saying that the triplet  $(x_{0,2}, x_{1,1}, x_{2,0})$  is a 2-simplex of  $\overline{WK}$  which is a homotopy from  $(x_{0,1}, x_{1,0})$  to  $(as_v^0, as_h^0)$ . Then,  $(x_{0,1}, x_{1,0})$  represents the identity element of the group  $\pi_1(\overline{WK}, a)$ . This proves that  $\pi_1 \overline{W}\epsilon$  is an isomorphism.

Let us now analyze the homomorphism

$$\pi_2 \overline{W}\epsilon : \pi_2(\overline{WK}, a) \rightarrow \pi_2(\overline{W}\mathbf{NPK}, a) \stackrel{(5.2)}{\cong} \pi_2(\mathbf{PK}, a).$$

An element of  $\pi_2(\mathbf{PK}, a)$  is a square in  $\mathbf{PK}$  of the form

$$\begin{array}{ccc}
 & a & \xleftarrow{[as_h^0]_h} a \\
 [as_v^0]_v \uparrow & [[x]] & \uparrow [as_v^0]_v \\
 & a & \xleftarrow{[as_h^0]_h} a
 \end{array}$$

and the homomorphism  $\pi_2 \overline{W}\epsilon$  is induced by the mapping

$$\begin{array}{ccccc}
 & a & \xleftarrow{as_v^0} & a & \xleftarrow{as_h^0} & a \\
 & \swarrow x_{0,2} & \uparrow x_{0,1} & \uparrow x_{1,1} & \uparrow as_v^0 & \\
 & & & a & \xleftarrow{as_h^0} & a \\
 & & & \swarrow x_{2,0} & \uparrow x_{1,0} & \\
 & & & & a &
 \end{array} \mapsto [[x_{1,1}]].$$

That  $\pi_2 \overline{W}\epsilon$  is surjective is proven using a parallel argument to that given previously for proving that  $\pi_1 \overline{W}\epsilon$  is injective (given  $[[x]]$ , using Lemma 6.1 twice, we can find  $x_{1,1} : \Delta[1, 1] \rightarrow K$ , etc.). To prove that  $\pi_2 \overline{W}\epsilon$  is also injective, suppose  $(x_{0,2}, x_{1,1}, x_{2,0})$  as above, representing an element of  $\pi_2(\overline{WK}, a)$  into the kernel of  $\pi_2 \overline{W}\epsilon$ , that is, such that  $[[x_{1,1}]] = [[as_h^0 s_v^0]]$ . Then, there is a bisimplex  $y : \Delta[1, 1] \rightarrow K$  such that  $[x_{1,1}]_v = [y]_v$  and  $[y]_h = [as_h^0 s_v^0]_h$ , whence we can find bisimplices  $\alpha' : \Delta[1, 2] \rightarrow K$  and  $\beta' : \Delta[2, 1] \rightarrow K$  such that

$$\alpha'd_v^0 = yd_v^0 s_v^0, \quad \alpha'd_v^1 = y, \quad \alpha'd_v^2 = x_{1,1}, \quad \beta'd_h^0 = as_h^0 s_v^0, \quad \beta'd_h^1 = as_h^0 s_v^0, \quad \beta'd_h^2 = y.$$

Let us now choose  $\theta : \Delta[2, 2] \rightarrow K$  and  $\theta' : \Delta[1, 3] \rightarrow K$  as respective solutions to the following extension problems

$$\begin{array}{ccc}
 \Lambda^{2,0}[2, 2] & \xrightarrow{(as_h^0 s_v^0 s_v^0, as_h^0 s_v^0 s_v^0, -; -, \beta'd_v^1 s_v^0, \beta')} & K \\
 \downarrow & \searrow \theta & \downarrow \\
 \Delta[2, 2] & & K
 \end{array}
 \quad
 \begin{array}{ccc}
 \Delta[1] \otimes \Lambda^2[3] & \xrightarrow{(\theta d_h^2 d_v^0 s_v^0, \theta d_h^2, -, \alpha')} & K \\
 \downarrow & \searrow \theta' & \downarrow \\
 \Delta[1, 3] & & K
 \end{array}$$

Then, for  $\alpha = \theta' d_v^2 : \Delta[1, 2] \rightarrow K$  and  $\beta = \theta d_v^0 : \Delta[2, 1] \rightarrow K$ , we have the equalities

$$(6.11) \quad \alpha d_v^0 = \beta d_h^2, \quad \alpha d_v^1 = a s_h^0 s_v^0, \quad \alpha d_v^2 = x_{1,1}, \quad \beta d_h^0 = a s_h^0 s_v^0, \quad \beta d_h^1 = a s_h^0 s_v^0.$$

By Lemma 2.2, as the 2<sup>nd</sup> homotopy groups of  $K_{0,*}$  vanish and both bisimplices  $\alpha d_h^0$  and  $a s_v^0 s_v^0$  have the same vertical faces, there is a vertical homotopy  $\omega : \Delta[0, 3] \rightarrow K$  from  $a s_v^0 s_v^0$  to  $\alpha d_h^0$ . And similarly, since  $\beta d_v^1$  and  $x_{2,0}$  have the same horizontal faces and the 2<sup>nd</sup> homotopy groups of  $K_{*,0}$  are all trivial, there is a horizontal homotopy, say  $\omega' : \Delta[3, 0] \rightarrow K$ , from  $\beta d_v^1$  to  $x_{2,0}$ . Now, let  $\Gamma : \Delta[1, 3] \rightarrow K$  and  $\Gamma' : \Delta[3, 1] \rightarrow K$  be bisimplices solving, respectively, the extension problems

$$\begin{array}{ccc} \Lambda^{1,2}[1, 3] & \xrightarrow{(\omega, -; \alpha d_v^0 s_v^1, a s_v^0 s_v^0 s_h^0, -, \alpha)} & K \\ \downarrow & \nearrow \Gamma & \\ \Delta[1, 3] & & \end{array} \quad \begin{array}{ccc} \Lambda^{3,0}[3, 1] & \xrightarrow{(a s_v^0 s_h^0 s_h^0, a s_v^0 s_h^0 s_h^0, \beta, -, -, \omega')} & K \\ \downarrow & \nearrow \Gamma' & \\ \Delta[3, 1] & & \end{array}$$

and take  $x_{1,2} = \Gamma d_v^2 : \Delta[1, 2] \rightarrow K$  and  $x_{2,1} = \Gamma' d_h^3 : \Delta[2, 1] \rightarrow K$ . Then, the same equalities as in (6.11) hold for  $x_{1,2}$  instead of  $\alpha$  and  $x_{2,1}$  instead of  $\beta$ , and moreover  $x_{1,2} d_h^0 = a s_v^0 s_v^0$  and  $x_{2,1} d_v^1 = x_{2,0}$ . Finally, by taking  $x_{0,3} : \Delta[0, 3] \rightarrow K$  any bisimplex with  $x_{0,3} d_v^0 = x_{1,2} d_h^1$ ,  $x_{0,3} d_v^1 = a s_v^0 s_v^0$ ,  $x_{0,3} d_v^2 = a s_v^0 s_v^0$  and  $x_{0,3} d_v^3 = x_{0,2}$ , and  $x_{3,0} : \Delta[3, 0] \rightarrow K$  any horizontal homotopy from  $a s_h^0 s_h^0$  to  $x_{2,1} d_v^0$  (which exist thanks to Lemma 2.2), we have the 3-simplex  $(x_{0,3}, x_{1,2}, x_{2,1}, x_{3,0})$  of  $\overline{WK}$ , which is easily recognized as a homotopy from  $(a s_v^0 s_v^0, a s_h^0 s_h^0, a s_h^0 s_h^0)$  to  $(x_{0,2}, x_{1,1}, x_{2,0})$ . Consequently,  $(x_{0,2}, x_{1,1}, x_{2,0})$  represents the identity of the group  $\pi_2(\overline{WK}, a)$ . Therefore,  $\pi_2 \overline{W}\epsilon$  is an isomorphism, and the proof is complete  $\square$

## 7. THE EQUIVALENCE OF HOMOTOPY CATEGORIES

Recall that the category of weak homotopy types is defined to be the localization of the category of topological spaces with respect to the class of weak equivalences, and the *category of homotopy 2-types*, hereafter denoted by  $\text{Ho}(\mathbf{2}\text{-types})$ , is its full subcategory given by those spaces  $X$  with  $\pi_i(X, a) = 0$  for any integer  $i > 2$  and any base point  $a$ .

We now define the *homotopy category of double groupoids satisfying the filling condition*, denoted by  $\text{Ho}(\mathbf{DG}_{\text{fc}})$ , to be the localization of the category  $\mathbf{DG}_{\text{fc}}$ , of these double groupoids, with respect to the class of weak equivalences, as defined in Subsection 3.4.

By Corollaries 5.4 and 4.3, both the geometric realization functor  $\mathcal{G} \mapsto |\mathcal{G}|$  and the homotopy double groupoid functor  $X \mapsto \Pi X$  induce equally denoted functors

$$(7.1) \quad | \cdot | : \text{Ho}(\mathbf{DG}_{\text{fc}}) \rightarrow \text{Ho}(\mathbf{2}\text{-types}),$$

$$(7.2) \quad \Pi : \text{Ho}(\mathbf{2}\text{-types}) \rightarrow \text{Ho}(\mathbf{DG}_{\text{fc}}).$$

One of the main goals in this section is to prove the following:

**Theorem 7.1.** *Both induced functors (7.1) and (7.2) are mutually quasi-inverse, establishing an equivalence of categories*

$$\text{Ho}(\mathbf{DG}_{\text{fc}}) \simeq \text{Ho}(\mathbf{2}\text{-types}).$$

The proof of this Theorem 7.1 is somewhat indirect. Previously, we shall establish the following result, where  $\mathbf{KC}$  is the category of Kan complexes and

$$\text{Ho}(L \in \mathbf{KC} \mid \pi_i L = 0, i > 2)$$

is the full subcategory of the homotopy category of Kan complexes given by those  $L$  such that  $\pi_i(L, a) = 0$  for all  $i > 2$  and base vertex  $a \in L_0$ :

**Theorem 7.2.** *There are adjoint functors,  $\overline{\mathbf{W}}\mathbf{N} : \mathbf{DG}_{\text{fc}} \rightarrow \mathbf{KC}$ , the right adjoint, and  $\mathbf{P}\text{Dec} : \mathbf{KC} \rightarrow \mathbf{DG}_{\text{fc}}$ , the left adjoint, that induce an equivalence of categories*

$$\text{Ho}(\mathbf{DG}_{\text{fc}}) \simeq \text{Ho}(L \in \mathbf{KC} \mid \pi_i L = 0, i > 2).$$

*Proof.* The pair of adjoint functors  $\mathbf{P}\text{Dec} \dashv \overline{\mathbf{W}}\mathbf{N}$  is obtained by composition of the pair of adjoint functors  $\text{Dec} \dashv \overline{\mathbf{W}}$ , recalled in (2.2), with the pair of adjoint functors  $\mathbf{P} \dashv \mathbf{N}$ , stated in Theorem 6.4. For any double groupoid  $\mathcal{G} \in \mathbf{DG}_{\text{fc}}$ , its double nerve  $\mathbf{N}\mathcal{G}$  satisfies the extension condition, by Theorem 5.1, and therefore, by Fact 2.8 (4), the simplicial set  $\overline{\mathbf{W}}\mathbf{N}\mathcal{G}$  is a Kan complex. Conversely, if  $L$  is any Kan complex, then the bisimplicial set  $\text{Dec}L$  satisfies the extension condition by Fact 2.8 (5) and, moreover,  $\pi_2(\text{Dec}L_{*,0}, a) = 0 = \pi_2(\text{Dec}L_{0,*}, a)$  for all vertices  $a$ , since both augmented simplicial sets  $\text{Dec}L_{*,0} \xrightarrow{d_0} L_0$  and  $\text{Dec}L_{0,*} \xrightarrow{d_1} L_0$  have simplicial contractions, given respectively by the families of degeneracies  $(s_p : L_p \rightarrow L_{p+1})_{p \geq 0}$  and  $(s_0 : L_q \rightarrow L_{q+1})_{q \geq 0}$ . Therefore, in accordance with Theorem 6.4, the composite functor  $L \mapsto \mathbf{P}\text{Dec}L$  is well defined on Kan complexes.

By Fact 2.7 (3), the homotopy equivalences in Fact 2.8 (1), and Corollary 5.4, it follows that a double functor  $F : \mathcal{G} \rightarrow \mathcal{G}'$ , in  $\mathbf{DG}_{\text{fc}}$ , is a weak equivalence if and only if the induced simplicial map  $\overline{\mathbf{W}}\mathbf{N}F : \overline{\mathbf{W}}\mathbf{N}\mathcal{G} \rightarrow \overline{\mathbf{W}}\mathbf{N}\mathcal{G}'$  is a homotopy equivalence.

By Facts 2.7 (3) and 2.8 (2), Theorem 6.5, and Corollary 5.4, if  $f : L \rightarrow L'$  is any simplicial map between Kan complexes  $L, L'$  such that  $\pi_i(L, a) = 0 = \pi_i(L', a')$  for all  $i \geq 3$  and base vertices  $a \in L_0, a' \in L'_0$ , then  $f$  is a homotopy equivalence if and only if the induced  $\mathbf{P}\text{Dec}f : \mathbf{P}\text{Dec}L \rightarrow \mathbf{P}\text{Dec}L'$  is a weak equivalence of double groupoids.

If  $L$  is any Kan complex such that  $\pi_i(L, a) = 0$  for all  $i \geq 3$  and all base vertices  $a \in L_0$ , then the unit of the adjunction  $L \rightarrow \overline{\mathbf{W}}\mathbf{N}\mathbf{P}\text{Dec}L$  is a homotopy equivalence since it is the composition of the simplicial maps

$$L \xrightarrow{u} \overline{\mathbf{W}}\text{Dec}L \xrightarrow{\overline{\mathbf{W}}\epsilon(\text{Dec}L)} \overline{\mathbf{W}}\mathbf{N}\mathbf{P}\text{Dec}L,$$

where  $u$  is a homotopy equivalence by Fact 2.8(3) and Fact 2.7 (3), and then  $\overline{\mathbf{W}}\epsilon(\text{Dec}L)$  is also a homotopy equivalence by Theorem 6.5 and Fact 2.7 (3).

Finally, the counit  $\mathbf{P}\mathbf{v}(\mathbf{N}\mathcal{G}) : \mathbf{P}\text{Dec}\overline{\mathbf{W}}\mathbf{N}\mathcal{G} \rightarrow \mathbf{P}\mathbf{N}\mathcal{G} = \mathcal{G}$ , at any double groupoid  $\mathcal{G}$ , is a weak equivalence, thanks to Fact 2.8(3), Theorem 6.5, and Corollary 5.4. This makes the proof complete.  $\square$

Since, by Facts 2.7, the adjoint pair of functors  $\mid \mid \dashv \mathbf{S} : \mathbf{Top} \rightleftarrows \mathbf{KC}$  induces mutually quasi-inverse equivalences of categories

$$\text{Ho}(\mathbf{2-types}) \simeq \text{Ho}(L \in \mathbf{KC} \mid \pi_i L = 0, i > 2),$$

the following follows from Theorem 7.2 above, and Fact 2.8(1):

**Theorem 7.3.** *The induced functor (7.1),  $\mid \mid : \text{Ho}(\mathbf{DG}_{\text{fc}}) \rightarrow \text{Ho}(\mathbf{2-types})$ , is an equivalence of categories with a quasi-inverse the induced by the functor  $X \mapsto \mathbf{P}\text{Dec}SX$ .*

Theorem 7.3 gives half of Theorem 7.1. The remaining part, that is, that the induced functor (7.2) is a quasi-inverse equivalence of (7.1), follows from the proposition below.

**Proposition 7.4.** *The two induced functors  $\mathbf{II}, \mathbf{P}\text{Dec}\mathbf{S} : \text{Ho}(\mathbf{2-types}) \rightarrow \text{Ho}(\mathbf{DG}_{\text{fc}})$  are naturally equivalent.*

*Proof.* The proof consists in displaying a natural double functor

$$\eta : \mathbf{P}\text{Dec}SX \rightarrow \mathbf{II}X,$$

which is a weak equivalence for any topological space  $X$ . This is as follows:

On objects of  $\mathbf{PDecSX}$ , the double functor  $\eta$  carries a continuous map  $u : \Delta_1 \rightarrow X$  to the path  $\eta_u : I \rightarrow X$  given by  $\eta_u(x) = u(1 - x, x)$ .

On horizontal morphisms of  $\mathbf{PDecSX}$ ,  $\eta$  acts by

$$(gd^0 \xrightarrow{[g]_h} gd^1) \xrightarrow{\eta} (\eta_{gd^0} \rightarrow \eta_{gd^1}),$$

the unique horizontal morphism in  $\mathbf{II}X$  from the path  $\eta_{gd^0}$  to the path  $\eta_{gd^1}$ , for any continuous map  $g : \Delta_2 \rightarrow X$ . This correspondence is well defined since  $\eta_{gd^0}(1) = gd^0(0, 1) = g(0, 0, 1) = gd^1(0, 1) = \eta_{gd^1}(1)$ , and, moreover, if  $[g]_h = [g']_h$  in  $\mathbf{DecSX}$ , then  $gd^i = g'd^i$  for  $i = 0, 1$ . And, similarly, on vertical morphisms,  $\eta$  is given by

$$(gd^1 \xrightarrow{[g]_v} gd^2) \xrightarrow{\eta} (\eta_{gd^1} \rightarrow \eta_{gd^2}).$$

On squares in  $\mathbf{PDecSX}$ ,  $\eta$  is defined by

$$\begin{array}{ccc} \begin{array}{c} \cdot \xleftarrow{[\alpha d^3]_h} \cdot \\ [\alpha d^1]_v \uparrow \quad [[\alpha]] \uparrow \quad [\alpha d^0]_v \uparrow \\ \cdot \xleftarrow{[\alpha d^2]_h} \cdot \end{array} & \xrightarrow{\eta} & \begin{array}{ccc} \eta_{\alpha d^1 d^2} & \xleftarrow{\quad} & \eta_{\alpha d^0 d^2} \\ \uparrow & [\eta_\alpha] & \uparrow \\ \eta_{\alpha d^1 d^1} & \xleftarrow{\quad} & \eta_{\alpha d^0 d^1} \end{array} \end{array}$$

where, for any continuous map  $\alpha : \Delta_3 \rightarrow X$ , the map  $\eta_\alpha : I \times I \rightarrow X$  is given by the formula

$$\eta_\alpha(x, y) = \alpha(xy, (1 - x)(1 - y), (1 - x)y, x(1 - y)).$$

To see that  $\eta$  is well defined on squares in  $\mathbf{PDecSX}$ , suppose  $[[\alpha_1]] = [[\alpha_2]]$ . This means that  $[\alpha_1]_h = [\alpha]_h$  and  $[\alpha_2]_v = [\alpha]_v$ , for some  $\alpha : \Delta_3 \rightarrow X$ , in the bisimplicial set  $\mathbf{DecSX}$ . Then, there are maps  $\beta, \gamma : \Delta_4 \rightarrow X$  such that the following equalities hold:

$$\beta d^0 = \alpha_1 d^0 s^0, \quad \beta d^1 = \alpha_1, \quad \beta d^2 = \alpha = \gamma d^3, \quad \gamma d^4 = \alpha_2, \quad \gamma d^2 = \alpha_2 d^2 s^2;$$

whence the equalities of squares in  $\mathbf{II}X$ ,  $[\eta_{\alpha_1}] = [\eta_\alpha] = [\eta_{\alpha_2}]$ , follow from the homotopies  $F_1, F_2 : I^2 \times I \rightarrow X$ , respectively defined by the formulas

$$\begin{aligned} F_1(x, y, t) &= \beta(xy, t(1 - x)(1 - y), (1 - t)(1 - x)(1 - y), (1 - x)y, x(1 - y)), \\ F_2(x, y, t) &= \gamma(xy, (1 - x)(1 - y), (1 - x)y, tx(1 - y), (1 - t)x(1 - y)). \end{aligned}$$

Most of the details to confirm  $\eta$  is actually a double functor are routine and easily verifiable. We leave them to the reader since the only ones with any difficulty are those a) and b) proven below.

a) For  $\omega : \Delta_4 \rightarrow X$ ,  $[\eta_{\omega d^1}] = [\eta_{\omega d^2}] \circ_h [\eta_{\omega d^0}]$  and  $[\eta_{\omega d^3}] = [\eta_{\omega d^4}] \circ_v [\eta_{\omega d^2}]$ .

b) For  $g : \Delta_2 \rightarrow X$ ,  $[\eta_{gs^2}] = \Gamma^v(\eta_{gd^1}, \eta_{gd^0})$  and  $[\eta_{gs^0}] = \Gamma^h(\eta_{gd^2}, \eta_{gd^1})$ .

However, all these equalities in a) and b) hold thanks to the relative homotopies

$$\begin{aligned} H_1 : \eta_{\omega d^1} &\rightarrow \eta_{\omega d^2} \circ_h \eta_{\omega d^0}, & H_2 : \eta_{\omega d^3} &\rightarrow \eta_{\omega d^4} \circ_v \eta_{\omega d^2}, \\ H_3 : \eta_{gs^2} &\rightarrow e^v, & H_4 : \eta_{gs^0} &\rightarrow e^h, \end{aligned}$$

which are, respectively, defined by the maps  $H_i : I^2 \times I \rightarrow X$  such that

$$\begin{aligned}
H_1(x, y, t) &= \begin{cases} \omega((1-t)xy, 2tx(x+y), (1-x)(1-y)+tx(2x-2+y), y(1-x)+tx(1-2x-y), x(1-y)+tx(1-2x-y)) \\ \text{if } x+y \leq 1, x \leq y, \\ \omega((1-t)xy, 2ty(x+y), (1-x)(1-y)+ty(2y-2+x), y(1-x)+ty(1-x-2y), x(1-y)+ty(1-x-2y)) \\ \text{if } x+y \leq 1, x \geq y, \\ \omega(xy+t(1-y)(1-x-2y), 2t(1-y)(2-x-y), (1-t)(1-x)(1-y), y(1-x)-t(1-y)(2-x-2y), (1-y)(x-t(2-x-2y))) \\ \text{if } x+y \geq 1, x \leq y, \\ \omega(xy+t(1-x)(1-2x-y), 2t(1-x)(2-x-y), (1-t)(1-x)(1-y), (1-x)(y-t(2-2x-y)), x(1-y)-t(1-x)(2-2x-y)) \\ \text{if } x+y \geq 1, x \geq y, \end{cases} \\
H_2(x, y, t) &= \begin{cases} \omega(t(1-x)(2x-1-y)+xy, (1-x)(1-y)+t(1-x)(2x-1-y), (1-t)(1-x)y, 2t(1-x)(1-x+y), x(1-y)+t(1-x)(y-2x)) \\ \text{if } x+y \geq 1, x \geq y, \\ \omega(xy+ty(x-2y), (1-x)(1-y)+ty(x-2y), (1-t)(1-x)y, 2ty(1-x+y), x(1-y)+ty(2y-1-x)) \\ \text{if } x+y \leq 1, x \geq y, \\ \omega(xy+t(1-y)(2y-x-1), (1-x)(1-y)+t(1-y)(2y-1-x), (1-x)y+t(1-y)(x-2y), 2t(1-y)(1+x-y), (1-t)x(1-y)) \\ \text{if } x+y \geq 1, x \leq y, \\ \omega(xy+tx(y-2x), (1-x)(1-y)+tx(y-2x), y(1-x)+tx(2x-1-y), 2tx(1+x-y), (1-t)x(1-y)) \\ \text{if } x+y \leq 1, x \leq y, \end{cases} \\
H_3(x, y, t) &= \begin{cases} g((1-t)xy, (1-x)(1-y)-txy, x+y+2xy(t-1)) & \text{if } x+y \leq 1, \\ g(xy+t(x+y-1-xy), (1-t)(1-x)(1-y), x+y-2xy+2t(1-x)(1-y)) & \text{if } x+y \geq 1, \end{cases} \\
H_4(x, y, t) &= \begin{cases} g(1-x-y+2xy+2tx(1-y), (1-x)y+tx(y-1), (1-t)x(1-y)) & \text{if } x \leq y, \\ g(1-x-y+2xy+2ty(1-x), (1-t)(1-x)y, x(1-y)+ty(x-1)) & \text{if } x \geq y. \end{cases}
\end{aligned}$$

This double functor,  $\eta : \mathbf{P} \text{Dec SX} \rightarrow \mathbf{IIX}$ , which is clearly natural on the topological space  $X$ , is actually a weak equivalence since, for any 1-simplex  $u : \Delta_1 \rightarrow X$  and integer  $i \geq 0$ , the induced map  $\pi_i \eta : \pi_i(\mathbf{P} \text{Dec SX}, u) \rightarrow \pi_i(\mathbf{IIX}, \eta_u)$  occurs in this commutative diagram

$$\begin{array}{ccccc}
\pi_i(\mathbf{P} \text{Dec SX}, u) & \xleftarrow{\text{Th.5.3}} & \pi_i(|\mathbf{P} \text{Dec SX}|, u) & \xleftarrow{\text{Th.6.5}} & \pi_i(|\text{Dec SX}|, u) \\
\downarrow \pi_i \eta & & & & \downarrow \cong \text{Fact2.8(1)} \\
& & & & \pi_i(|\overline{\mathbf{W}} \text{Dec SX}|, u) \\
& & & & \downarrow \cong \text{Fact2.8(3)} \\
\pi_i(\mathbf{IIX}, \eta_u) & \xrightarrow{\text{Th.4.2}} & \pi_i(X, u(1, 0)) & \xleftarrow{\text{Fact2.7(6)}} & \pi_i(|\text{SX}|, u(1, 0))
\end{array}$$

in which all other maps are bijections (group isomorphisms for  $i \geq 1$ ) by the references in the labels.  $\square$

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