On dynamical adjoint functor

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Abstract

We give an explicit formula relating the dynamical adjoint functor and dynamical twist over nonalbelian base to the invariant pairing on parabolic Verma modules. As an illustration, we give explicit $U\big(\mathfrak{sl}(n)\big)$- and $U_\hbar\big(\mathfrak{sl}(n)\big)$-invariant star product on projective spaces.

1 Introduction

In this paper, we clarify certain points arising in the theory of dynamical Yang-Baxter equation (DYBE), namely, we give an explicit expression of the dynamical adjoint functor through the invariant pairing on parabolic Verma modules (PVM).

The DYBE appeared as a generalization of the ordinary Yang-Baxter equation in the mathematical physics literature [1, 2, 3] and was actively studied in the 90-s, see e.g. [4, 5]. It was later realized that it is related to quantization of homogeneous Poisson-Lie manifolds, [6]. Namely, the star-product can be obtained by a reduction of dynamical twist, which is a left-invariant differential tri-operator on the group. In return, this finding has extended the framework of the DYBE, which had been originally formulated over a commutative cocommutative base Hopf algebra, [4], to a general nonabelian base.

A recipe of constructing dynamical twist was suggested in [6], through dynamical adjoint functor (DAF) between certain module categories associated with a Levi subalgebra H in a reductive (classical or quantum) universal enveloping algebra U. One of them is a certain subcategory of finite dimensional H-modules, while the other is the parabolic O-category, both equipped with tensor multiplication by finite dimensional U-modules.

Remark that quantization of function algebras on homogeneous space involve only scalar PVM. General PVM appear in quantization of associated vector bundles as projective modules over function algebras, as discussed in [6]. This is where the nonabelian base really plays a role, along with the corresponding dynamical twist and DAF.

An alternative approach to quantization was employed in [7], where the star product on semisimple coadjoint orbits of simple complex Lie groups was constructed directly from the Shapovalov form on scalar PVM. It was clear that the methods of [6] and [7] were close and based on similar underlying ideas. Relation of the Shapovalov form on Verma modules with the dynamical twist was already indicated in earlier works on DYBE in the special case of Cartan base, [5]. This construction had motivated the generalization for the nonabelian base, which was given in [6], however, without straight use of the Shapovalov form. A sort of "nonabelian paring" associated with the triangular factorization of (quantized) universal enveloping algebras, which is equivalent to Shapovalov form in representations, was employed in [8] for construction of the dynamical twist. It was done directly, bypassing DAF. Thus, the explicit relation of the Shapovalov form to DAF over general Levi subalgebra, which is a more fundamental object than dynamical twist, has not been given much attention in the literature. In the present work we do it in a most elementary way.

We would like to mention the following two papers in connection with the present work. In [9], the dynamical twist is constructed with the use of the ABRR equation, [10]. The DAF is also present there, but with no explicit connection with the Shapovalov form. Another paper of interest, [11], directly generalizes the ideas of [7]. Remarkably, the approach of [11] can be suitable for certain conjugacy classes with non-Levi isotropy subgroups, which drop from the framework of the DYBE in its present version, but still can be quantized in a similar way, [12].

As an illustration, we give the star product on the homogeneous space $GL(n+1)/GL(n) \times GL(1)$ that is equviariant under the action of either classical or quantum group $GL(n+1)$. In this simple case the Shapovalov form can be calculated explicitly. We show that its $U(\mathfrak{g})$-invariant limit coincides with the star product on the projective space obtained in [13] by a different approach.

Unfortunately, the journal version contains a few typos and mistakes, which are corrected here. The erratum is added after the list of references.

2   Parabolic Verma modules

Let $\mathfrak{g}$ be a simple complex Lie algebra. Fix a Cartan subalgebra and denote by $\mathfrak{b}^\pm \subset \mathfrak{g}$ the Borel subalgebras relative to $\mathfrak{h}$. Consider a Levi subalgebra $\mathfrak{l} \subset \mathfrak{g}$ containg $\mathfrak{h}$ and denote by $\mathfrak{p}^\pm = \mathfrak{l} + \mathfrak{b}^\pm$ the corresponding parabolic subalgebras. The nil-radicals $\mathfrak{n}^\pm \subset \mathfrak{p}^\pm$ complement $\mathfrak{l}$ in the triangular decomposition $\mathfrak{n}^- \oplus \mathfrak{l} \oplus \mathfrak{n}^+$.

Denote by $U_\hbar(\mathfrak{g})$, $U_\hbar(\mathfrak{l})$, and $U_\hbar(\mathfrak{p}^\pm)$ the quantum universal enveloping algebras of the total Lie algebra $\mathfrak{g}$, the Levi subalgebra $\mathfrak{l}$, and the parabolic subalgebras $\mathfrak{p}^\pm$. They are Hopf $\mathbb{C}[[\hbar]]$-algebras, with the inclusions $U_\hbar(\mathfrak{l}) \subset U_\hbar(\mathfrak{p}^\pm) \subset U_\hbar(\mathfrak{g})$ being Hopf algebra homomorphisms. There are subalgebras $U_\hbar(\mathfrak{n}^\pm) \subset U_\hbar(\mathfrak{p}^\pm)$, which are deformations of the classical universal enveloping algebras $U(\mathfrak{n}^\pm)$ and which facilitate the triangular factorization

$$U_\hbar(\mathfrak{g}) = U_\hbar(\mathfrak{n}^-)U_\hbar(\mathfrak{l})U_\hbar(\mathfrak{n}^+). \tag{2.1}$$

This factorization makes $U_\hbar(\mathfrak{g})$ a free $U_\hbar(\mathfrak{n}^-) - U_\hbar(\mathfrak{n}^+)$-bimodule generated by $U_\hbar(\mathfrak{l})$. Accordingly, the parabolic subalgebras admit free factorizations

$$U_\hbar(\mathfrak{p}^-) = U_\hbar(\mathfrak{n}^-)U_\hbar(\mathfrak{l}), \quad U_\hbar(\mathfrak{p}^+) = U_\hbar(\mathfrak{l})U_\hbar(\mathfrak{n}^+), \tag{2.2}$$

giving rise to (2.1). Factorization (2.2) has the structure of smash product, as $U_\hbar(\mathfrak{n}^\pm)$ are invariant under the adjoint action of $U_\hbar(\mathfrak{l})$ on $U_\hbar(\mathfrak{g})$. Note that $U_\hbar(\mathfrak{n}^\pm)$ are neither Hopf algebras nor coideals over $U_\hbar(\mathfrak{l})$.

Let A be a finite dimensional $U_\hbar(\mathfrak{l})$-module (finite and free over $\mathbb{C}[[\hbar]]$). It becomes a $U_\hbar(\mathfrak{p}^\pm)$-module with the trivial action of $U_\hbar(\mathfrak{n}^\pm)$. The parabolic Verma modules $M_A^\pm$ over $U_\hbar(\mathfrak{g})$ are defined as induced from A:

$$M_A^\pm = U_\hbar(\mathfrak{g}) \otimes_{U_\hbar(\mathfrak{p}^\pm)} A \simeq U_\hbar(\mathfrak{n}^\mp)A.$$

The last isomorphism indicates that $M_A^\pm$ are free $U_\hbar(\mathfrak{n}^\mp)$-modules generated by A. Consider $M_A^\pm$ as a module over $U_\hbar(\mathfrak{p}^\mp)$ by restriction. In what follows, we use sections of the action map $U_\hbar(\mathfrak{p}^\mp) \otimes A \to M_A^\pm$. We call the image of $M_A^\pm$ in $U_\hbar(\mathfrak{p}^\mp) \otimes A$ under such a section *a lift* of $M_A^\pm$. The presentation $M_A^\pm \simeq U_\hbar(\mathfrak{n}^\mp)A$ is an example of lift.

For any finite dimensional $U_\hbar(\mathfrak{l})$-module A let A^* denote the (left) dual to A. The dual action of $U_\hbar(\mathfrak{l})$ on A^* is given by $(h\alpha)(a) = (\alpha)(\gamma(h)a)$, for $\alpha \in A^*$, $a \in A$ and $h \in U_\hbar(\mathfrak{l})$.

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The triangular factorization of $U_\hbar(\mathfrak{g})$ relative to $U_\hbar(\mathfrak{l})$ defines a projection $U_\hbar(\mathfrak{g}) \to U_\hbar(\mathfrak{l})$, $u \mapsto [u]_\mathfrak{l}$, which is $U_\hbar(\mathfrak{l})$-invariant with respect to the left and right regular action. The pairing

$$U_\hbar(\mathfrak{g}) \otimes A^* \otimes U_\hbar(\mathfrak{g}) \otimes A \to \mathbb{C}[[\hbar]], \quad \langle u \otimes \alpha, v \otimes a \rangle = \langle \alpha, [\gamma(u)v]_\mathfrak{l} a \rangle$$

is $U_\hbar(\mathfrak{g})$-invariant by construction and factors through the pairing between $M_{A^*}^-$ and M_A^+. It is non-degenerate if and only if the modules $M_{A^*}^-$ and M_A^+ are irreducible. This pairing is $U_\hbar(\mathfrak{g})$-invariant and equivalent to the contravariant Shapovalov form on M_A^+, [14].

In the sequel, we need the following fact about induced modules of Hopf algebras.

Lemma 2.1. *Suppose U is a Hopf algebra and H is a Hopf subalgebra in U. Let A be an H-module and V be a U-module regarded as an H module by restriction. Then there are natural isomorphisms*

$$\mathrm{Ind}_H^U A \otimes V \simeq \mathrm{Ind}_H^U(A \otimes V), \quad V \otimes \mathrm{Ind}_H^U A \simeq \mathrm{Ind}_H^U(V \otimes A),$$

where Ind_H^U designates induction from an H-module to a U-module.

The proof is a straightforward use of Hopf algebra yoga involving coproducts, antipode and counit. The isomorphisms are identical on $A \otimes V$ and $V \otimes A$, respectively.

Proposition 2.2. *The tensor product $M_A^+ \otimes M_B^-$ is isomorphic to the induced module $U_\hbar(\mathfrak{g}) \otimes_{U_\hbar(\mathfrak{l})} (A \otimes B)$.*

Proof. Based on Lemma 2.1,

$$
\begin{aligned}
M_A^+ \otimes M_B^- &\simeq \mathrm{Ind}_{U_\hbar(\mathfrak{p}^+)}^{U_\hbar(\mathfrak{g})}(A \otimes M_B^-) \simeq \mathrm{Ind}_{U_\hbar(\mathfrak{p}^+)}^{U_\hbar(\mathfrak{g})}\left(A \otimes \mathrm{Ind}_{U_\hbar(\mathfrak{l})}^{U_\hbar(\mathfrak{p}^+)}(B)\right) \\
&\simeq \mathrm{Ind}_{U_\hbar(\mathfrak{p}^+)}^{U_\hbar(\mathfrak{g})}\mathrm{Ind}_{U_\hbar(\mathfrak{l})}^{U_\hbar(\mathfrak{p}^+)}(A \otimes B) \simeq \mathrm{Ind}_{U_\hbar(\mathfrak{l})}^{U_\hbar(\mathfrak{g})}(A \otimes B), \quad\quad (2.3)
\end{aligned}
$$

as required. $\square$

3 Dynamical adjoint functor

Let us recall the definition of dynamical twist over general base [6]. For simplicity, we take for the base a Hopf algebra H assuming it to be a Hopf subalgebra in the total Hopf algebra U. A dynamical twist is an invertible element $F \in H \otimes U \otimes U$ subject to the cocycle identity

$$(\mathrm{id}_H \otimes \mathrm{id}_U \otimes \Delta)(F)(\delta \otimes \mathrm{id}_U \otimes \mathrm{id}_U)(F) = (\mathrm{id}_H \otimes \Delta \otimes \mathrm{id}_U)(F)(F \otimes 1_U)$$

and the normalization condition $(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(F) = 1 \otimes 1 \otimes 1 = (\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon)(F)$. In practice, the base Hopf algebra H needs to be replaced by a certain "localization", which is already not a coalgebra but only a right coideal, see [9, 8].

In representation-theoretic terms, the dynamical twist is a family of operators

$$F_{A,V,W} \colon (A \otimes V) \otimes W \to A \otimes (V \otimes W),$$

where V, W are U-modules and A is an H-module. The above mentioned localization of H means that not all A are admissible. The cocycle identity turns into

$$F_{A,Z,V\otimes W} F_{A\otimes Z,V,W} = F_{A,Z\otimes V,W}(F_{A,Z,V} \otimes \mathrm{id}_W).$$

Here Z is another U-module, and the second factor in the left-hand side regards $A \otimes Z$ as an H-module through the coaction.

We recall a general construction of DAF. This functor was introduced in [6], where it was used for construction of the dynamical twist. The name "dynamical adjoint" should be taken as a single term, as the functor of concern is rather "dynamizaion" of the adjoint functor to the restriction functor than dualization of anything.

Suppose $\mathcal{M}$ is a monoidal category and $\mathcal{B}, \mathcal{B}'$ are two (right) module categories over $\mathcal{M}$. A functor J from $\mathcal{B}$ to $\mathcal{B}'$ is called dynamical adjoint if for all $A, B \in \mathcal{B}$ and $V \in \mathcal{M}$ there is an isomorphism

$$\Theta \colon \mathrm{Hom}_{\mathcal{B}}(B, A \otimes V) \simeq \mathrm{Hom}_{\mathcal{B}'}(J(B), J(A) \otimes V).$$

Here $\otimes$ stands for the actions of $\mathcal{M}$ on $\mathcal{B}$ and $\mathcal{B}'$. Given such functor, the dynamical twist is a morphism $F \colon A \otimes V \otimes W \to A \otimes V \otimes W$ whose J-image is the composition

$$J(A \otimes V \otimes W) \xrightarrow{\Theta(\mathrm{id}_{A\otimes V\otimes W})} J(A \otimes V) \otimes W \xrightarrow{\Theta(\mathrm{id}_{A\otimes V})\otimes \mathrm{id}_W} J(A) \otimes V \otimes W.$$

In our special case, $\mathcal{M}$ is the category of finite dimensional representations of $U_\hbar(\mathfrak{g})$, $\mathcal{B}$ is a certain subcategory of finite dimensional representations of $U_\hbar(\mathfrak{l})$, and $\mathcal{B}'$ is the category of integrable modules over $U_\hbar(\mathfrak{g})$. The functor J is the parabolic induction, so $J(A) = M_A^+$ on objects. The category $\mathcal{B}$ is determined by the requirements that M_A^+ is irreducible once A is irreducible and $\mathcal{B}$ is invariant under tensoring with objects from $\mathcal{M}$ regarded as $U_\hbar(\mathfrak{l})$-modules.

That the functor is DAF follows from Proposition 2.2. Indeed, if the module M_A^+ is irreducible, then $M_{A^*}^-$ is its (restricted) dual, and

$$\mathrm{Hom}_{\mathfrak{g}}(M_B^+, M_A^+ \otimes V) \simeq \mathrm{Hom}_{\mathfrak{g}}(M_{A^*}^- \otimes M_B^+, V) \simeq \mathrm{Hom}_{\mathfrak{l}}(A^* \otimes B, V) \simeq \mathrm{Hom}_{\mathfrak{l}}(B, A \otimes V),$$

as required. The isomorphism in the middle is the Frobenius reciprocity facilitated by Proposition 2.2.

Assuming M_A^+ irreducible, denote by $S_{A,A^*} \in M_A^+ \otimes M_{A^*}^-$ the $U_\hbar(\mathfrak{g})$-invariant canonical element of the pairing between $M_{A^*}^-$ and M_A^+. For the dual bases $\{x_i\} \subset M_A^+$, $\{x^i\} \subset M_{A^*}^-$, it equals $S_{A,A^*} = \sum_i x_i \otimes x^i$. We will use the following Sweedler-like notation for the presentation

$$S_{A,A^*} = S_1 \otimes S_2 = S_- S_A \otimes S_+ S_{A^*}.$$

where $S_- \otimes S_A \otimes S_+ \otimes S_{A^*} \in U_\hbar(\mathfrak{p}^-) \otimes A \otimes U_\hbar(\mathfrak{p}^+) \otimes A^*$ symbolizes a lift of S_{A,A^*}.

We have the following obvious property of the family $\{S_{A,A^*}\}$. Suppose B is an $U_\hbar(\mathfrak{l})$-submodule in A. It is separable as a direct summand, so the dual module B^* is separable as a direct summand in A^*. Let $\pi\colon A \to B$ and $\psi\colon A^* \to B^*$ be the intertwining projections. Let $\hat{\pi}\colon M_A^+ \to M_B^+$ and $\hat{\psi}\colon M_{A^*}^- \to M_{B^*}^-$ be the induced $U_\hbar(\mathfrak{g})$-morphisms. Then the projection $(\hat{\pi} \otimes \hat{\psi})(S_{A,A^*}) = S_- \pi(S_B) \otimes S_+ \psi(S_{B^*})$ is equal to the canonical element S_{B,B^*}. Thus, a lift of S_{B,B^*} can be obtained from a lift of S_{A,A^*} in a natural way.

Every $U_\hbar(\mathfrak{g})$-module V is at the same time a $U_\hbar(\mathfrak{l})$-module by restriction. It becomes an $U_\hbar(\mathfrak{p}^\pm)$-module when extended trivially to $U_\hbar(\mathfrak{n}^\pm)$. This representation of $U_\hbar(\mathfrak{p}^\pm)$ is different as compared to restricted from $U_\hbar(\mathfrak{g})$. To distinguish it from the representation ρ restricted from $U_\hbar(\mathfrak{g})$, we use the notation $\rho_\pm$.

Fix a $U_\hbar(\mathfrak{g})$-module V and assign to every element of $M_{A^*}^-$ a linear operator $A \otimes V \to V$ by

$$u\alpha\colon a \otimes v \mapsto \left(\alpha \otimes (\rho(u^{(1)}))\right)\left(\rho_+\left(\gamma(u^{(2)})\right)(a \otimes v)\right), \tag{3.4}$$

where $u \otimes \alpha \in U_\hbar(\mathfrak{p}^+) \otimes A^*$ is a lift of $u\alpha$, and $v \in V$. The map is correctly defined. Indeed, regarding $M_{A^*}^-$ as a $U_\hbar(\mathfrak{p}^+)$-module, it is isomorphic to $\mathrm{Ind}_\mathfrak{l}^{\mathfrak{p}^+} A^*$. The assignment (3.4) coincides on $(ux)\alpha$ and $u(x\alpha)$ for every x from $U_\hbar(\mathfrak{l})$, because $\rho_+\left(\gamma(x^{(2)})\right) = \rho\left(\gamma(x^{(2)})\right)$. One can check that the constructed map $M_{A^*}^- \to \mathrm{End}(A \otimes V, V)$ is $U_\hbar(\mathfrak{l})$-invariant. Note that for the classical universal enveloping algebras formula (3.4) simplifies to

$$u\alpha\colon a \otimes v \mapsto \alpha(a)\rho(u)(v),$$

because the lift u can be taken in the Hopf subalgebra $U(\mathfrak{n}^+) \subset U(\mathfrak{g})$, which is annihilated by ρ_+.

DAF gives rise to the collection of intertwining operators

$$M_{A \otimes V}^+ \overset{\Theta(\mathrm{id}_{A \otimes V})}{\longrightarrow} M_A^+ \otimes V.$$

Proposition 3.1. *The restriction to $A \otimes V \subset M^+_{A \otimes V}$ of the intertwining operator $\Theta(\mathrm{id}_{A \otimes V})$ acts by the assignment*

$$\Theta(\mathrm{id}_{A \otimes V})\colon a \otimes v \mapsto S_1 \otimes S_2(a \otimes v),$$

where $S_{A,A^*} = S_1 \otimes S_2 \in M^+_A \otimes M^-_{A^*}$

Proof. The operator $\Theta(\mathrm{id}_{A \otimes V})$ factorizes into the composition

$$M^+_{A \otimes V} \to M^+_A \otimes M^-_{A^*} \otimes M^+_{A \otimes V} \simeq M^+_A \otimes \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{l}}(A^* \otimes A \otimes V) \to M^+_A \otimes V,$$

where the left arrow is the coevaluation $1 \mapsto S_{A,A^*}$, while right arrow is the induced extension from the evaluation map $A^* \otimes A \otimes V \to V$. Note that the isomorphism in the middle is identical on $A^* \otimes A \otimes V$. Applying this composition to an element $a \otimes v \in A \otimes V \subset M^+_{A \otimes V}$ gives the result immediately. $\qquad\square$

Remark that for the classical universal enveloping algebras we can write the simple formula

$$\Theta(\mathrm{id}_{A \otimes V})\colon a \otimes v \mapsto S_1 \otimes S_{A^*}(a)S_+(v).$$

Here the factor S_+ acts on $v \in V$ as an element of $U(\mathfrak{g})$.

To relate the dynamical twist to the invariant pairing, consider the projection $P\colon M^+_A \to A$ defined as the composition

$$M^+_A \to A \otimes A^* \otimes M^+_A \to A,$$

where the left arrow is induced by coevaluation $\mathbb{C}[[\hbar]] \to A \otimes A^*$, and the right map is the invariant pairing between $A^* \subset M^-_{A^*}$ and M^+_A. By construction, P is $U_\hbar(\mathfrak{l})$-invariant and identical on $A \subset M^+_A$. Moreover, one can check that it is $U_\hbar(\mathfrak{p}^-)$-invariant.

The operator P implements the isomorphism

$$\mathrm{Hom}_{\mathfrak{g}}(M^+_B, M^+_A \otimes V) \to \mathrm{Hom}_{\mathfrak{l}}(B, A \otimes V) \tag{3.5}$$

that is inverse to Θ. Namely, given a $U_\hbar(\mathfrak{g})$-invariant operator $M^+_B \to M^+_A \otimes V$, restrict it to B and compose with $P \otimes \mathrm{id}_V$, to get a $U_\hbar(\mathfrak{l})$-operator $B \to A \otimes V$. Therefore the dynamical twist F factorizes to the chain

$$A \otimes V \otimes W \hookrightarrow M_{A \otimes V \otimes W} \to M_{A \otimes V} \otimes W \to M_A \otimes V \otimes W \to A \otimes V \otimes W.$$

Proposition 3.2. *The dynamical twist $F_{A,V,W}$ is expressed through the canonical element $S_{A \otimes V, V^* \otimes A^*} = S_- S_{A \otimes V} \otimes S_+ S_{V^* \otimes A^*} \in M^+_{A \otimes V} \otimes M^-_{V^* \otimes A^*}$ by the formula*

$$F(a \otimes v \otimes w) = (\rho_- \otimes \rho) \circ \Delta(S_-)(S_{A \otimes V}) \otimes (S_+ S_{V^* \otimes A^*})(a \otimes v \otimes w). \tag{3.6}$$

Proof. Immediate consequence of (3.5) and Proposition 3.1. $\qquad\square$

Remark that for the classical universal enveloping algebras the natural lift gives $S_\pm \in U(\mathfrak{n}^\pm)$ killed by $\rho_\pm$. The formula (3.6) takes the simple form

$$F(a \otimes v \otimes w) = (\mathrm{id}_A \otimes S_-)(S_{A \otimes V}) \otimes S_+(w) S_{V^* \otimes A^*}(a \otimes v),$$

where $S_\pm$ act on V, W as $U(\mathfrak{g})$-modules. In the quantum case, the subalgebra $U_\hbar(\mathfrak{n}^-)$ can be taken a left coideal, and the formula can also be simplified by replacing $(\rho_- \otimes \rho) \circ \Delta(S_-)$ with $(\mathrm{id} \otimes \rho)(S_-)$.

Now suppose that $A = \mathbb{C}_\lambda$ is a scalar $U_\hbar(\mathfrak{l})$-module corresponding to weight λ. Denote by $S^\lambda = \lambda(S_-^{(1)}) S_-^{(2)} \otimes S_+^{(1)} \lambda^*(S_+^{(2)})$, where $S_- \otimes S_+$ is a lift of $S_{\mathbb{C}_\lambda, \mathbb{C}^{\lambda^*}} \in M_\lambda^+ \otimes M_{\lambda^*}^-$ to $U_\hbar(\mathfrak{p}^-) \otimes U_\hbar(\mathfrak{p}^+)$.

Proposition 3.3. *The operators F^λ and S^λ coincide on invariants $V^\mathfrak{l} \otimes W^\mathfrak{l} \subset V \otimes W$:*

$$F^\lambda(v \otimes w) = S^\lambda(v \otimes w), \quad v \in V^\mathfrak{l}, \quad w \in W^\mathfrak{l}.$$

Proof. Apply the projections $\mathbb{C}_\lambda \otimes V \to \mathbb{C}_\lambda \otimes V^\mathfrak{l}$, $\mathbb{C}_\lambda \otimes W \to \mathbb{C}_\lambda \otimes W^\mathfrak{l}$ to (3.6). $\qquad\square$

The formula for S^λ can be simplified to $S^\lambda = S_- \otimes S_+$ by an appropriate choice of basis.

4 Quantum algebra $U_\hbar(\mathfrak{sl}(n))$

Further we apply the above theory to construct the star-product on the homogeneous space space $GL(n+1)/GL(n) \times GL(1)$, for which case the invariant pairing on PVM can be explicitly calculated. We will focus on the situation of quantum groups, because the classical case can be readily obtained from that by a certain limit procedure.

Recall the definition of the quantized universal enveloping algebra $U_\hbar(\mathfrak{sl}(n))$, see [15]. Let R and R^+ denote respectively the root system and the set of positive roots of the Lie algebra $\mathfrak{sl}(n)$. The set $\Pi_+ = (\alpha_1, \alpha_1, \ldots, \alpha_{n-1})$ of simple positive roots is equipped with the natural ordering.

The quantum group $U_q(\mathfrak{sl}(n))$ is generated by $e_\alpha, f_\alpha, h_\alpha$, $\alpha \in \Pi_+$, subject to the relations

$$[h_\alpha, e_\beta] = (\alpha, \beta) e_\alpha, \quad [h_\alpha, f_\beta] = -(\alpha, \beta) e_\alpha, \quad [e_\alpha, f_\beta] = \delta_{\alpha, \beta} \frac{q^{h_\alpha} - q^{-h_\alpha}}{q - q^{-1}},$$

where $(.,.)$ is the inner product on $\mathfrak{h}^* = \mathrm{Span}(R)$. Here $q = e^\hbar$ with $\hbar$ being the deformation parameter.

Also, the Chevalley generators e_α, f_α satisfy the Serre relations

$$e_\alpha^2 e_\beta - (q + q^{-1}) e_\alpha e_\beta e_\alpha + e_\beta e_\alpha^2 = 0, \quad f_\alpha^2 f_\beta - (q + q^{-1}) f_\alpha f_\beta f_\alpha + f_\beta f_\alpha^2 = 0.$$

if $(\alpha, \beta) = -1$ and $[e_\alpha, e_\beta] = 0 = [f_\alpha, f_\beta]$ if $(\alpha, \beta) = 0$.

The comultiplication Δ and antipode γ are defined on the generators by

$$\Delta(h_\alpha) = h_\alpha \otimes 1 + 1 \otimes h_\alpha, \quad \gamma(h_\alpha) = -h_\alpha,$$

$$\Delta(e_\alpha) = e_\alpha \otimes 1 + q^{h_\alpha} \otimes e_\alpha, \quad \gamma(e_\alpha) = -q^{-h_\alpha} e_\alpha,$$

$$\Delta(f_\alpha) = f_\alpha \otimes q^{-h_\alpha} + 1 \otimes f_\alpha, \quad \gamma(f_\alpha) = -f_\alpha q^{h_\alpha}.$$

The counit homomorphism ε annihilates e_α, f_α, h_α.

The elements e_α, h_α generate the positive Borel subalgebra $U_\hbar(\mathfrak{b}^+)$ in $U_q(\mathfrak{sl}(n))$. Similarly, f_α, h_α generate the negative Borel subalgebra $U_\hbar(\mathfrak{b}^-)$. They are deformations of the classical Borel subalgebras whose Poincaré-Birgoff-Witt basis is generated by the Cartan generators constructed as follows.

Every positive root has the form $\mu = \alpha_k + \alpha_{k+1} + \ldots + \alpha_m$, for some $m > k$; the integer $m - k + 1$ is called height of μ. Put

$$e_\mu = [e_k, [e_{k+1}, \ldots [e_{m-1}, e_m]_q \ldots]_q]_q, \quad \tilde{e}_\mu = q^{2(m-k)}[e_k, [e_{k+1}, \ldots [e_{m-1}, e_m]_{\bar{q}} \ldots]_{\bar{q}}]_{\bar{q}},$$

where $e_k = e_{\alpha_k}$. The roots can be written in an orthogonal basis $\{\varepsilon_i\}_{i=1}^n$ as $\varepsilon_i - \varepsilon_j$, $i, j = 1, \ldots, n$, $i \neq j$. Lexicographically ordered pairs (i, j) induce an ordering on positive roots $\varepsilon_i - \varepsilon_j$, $i < j$, consistent with the ordered the basis $(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) \subset \mathfrak{h}^*$. The ordered monomials in $e_\mu, \mu \in R^+$, form a BPW basis in $U_\hbar(\mathfrak{b}^+)$ over the Cartan subalgebra in $U_\hbar(\mathfrak{h})$ generated by h_α.

As all $\alpha_i \in \Pi_+$ enter positive roots at most once, we may regard elements of R^+ as sets of simple roots. This makes sense of writing $\mu \subset \nu$ and $\nu - \mu$ for some pairs of roots $\mu, \nu \in R^+$. Also, $\alpha \cap \beta$ determines a root unless it is empty.

Lemma 4.1. *For all positive roots μ, the vectors e_μ and $\tilde{e}_\mu$ are related by the antipode, $\gamma(\tilde{e}_\mu) = -q^{-h_\mu} e_\mu$.*

Proof. If $\alpha < \beta$ are adjacent simple roots, then the following calculation

$$\gamma(q^2[e_\alpha, e_\beta]_{\bar{q}}) = q^2[q^{-h_\beta} e_\beta, q^{-h_\alpha} e_\alpha]_{\bar{q}} = q^{-h_{\alpha+\beta}} q[e_\beta, e_\alpha]_{\bar{q}} = -q^{-h_{\alpha+\beta}}[e_\alpha, e_\beta]_q$$

proves the statement for roots of height 2. The general case is processed by induction on the height of the root. $\qquad \square$

In the classical limit mod $\hbar$, the elements e_μ turn into positive root vectors of the Borel subalgebra $\mathfrak{b}^+ \subset \mathfrak{g}$ and modulo $\hbar$ they coincide with $\tilde{e}_\mu$. For our purposes, we need both e_μ and $\tilde{e}_\mu$.

The commutation relations among the root vectors e_μ are described below.

Proposition 4.2. *Let μ and ν be positive roots such that $\mu < \nu$.*

1. *Suppose that $\mu + \nu \in R^+$. Then*

$$[e_\mu, e_\nu]_q \;=\; e_{\mu+\nu}. \tag{4.7}$$

2. *Suppose that $\mu + \nu \notin R^+$. Then*

$$[e_\nu, e_\mu]_{\bar{q}} \;=\; 0, \quad \mu \cap \nu = \nu, \quad \nu < \mu - \nu \in R^+, \tag{4.8}$$

$$[e_\mu, e_\nu]_{\bar{q}} \;=\; 0, \quad \mu \cap \nu = \nu, \quad \nu > \mu - \nu \in R^+, \tag{4.9}$$

$$[e_\mu, e_\nu] \;=\; 0, \quad \mu \cap \nu = \nu, \quad \mu - \nu \notin R^+ \quad \text{or } \nu \cap \mu = \emptyset, \tag{4.10}$$

$$[e_\mu, e_\nu] \;=\; -(q - q^{-1}) e_{\mu \cup \nu} e_{\mu \cap \nu}$$
$$\;=\; -(q - q^{-1}) e_{\mu \cap \nu} e_{\mu \cup \nu}, \qquad \nu \neq \mu \cap \nu \in \mathbb{R}^+. \tag{4.11}$$

Proof. Formula (4.7) is just the definition of $e_{\mu+\nu}$ if the height of μ is 1. If $\mu = \alpha_i + \ldots + \alpha_{k-1}$ with $k > i$, put $\mu' = \alpha_i + \ldots + \alpha_{k-1}$; then Then

$$[e_\mu, e_\nu]_q = [[e'_\mu, e_{k-1}]_q, e_\nu]_q = [e'_\mu, [e_{k-1}, e_\nu]_q]_q = e_{\mu+\nu}.$$

by induction on the height of μ.

Commutation relations (4.8) and (4.9) are generalizations of the Serre relations and follow from Lemma 7.3 by induction on heights of μ and ν. The case $\mu + \nu, \mu - \nu \notin R^+$ falls either in situation (4.10) or (4.11). Let us check (4.10). The case $\mu \cap \nu = \emptyset$ is clear. The alternative is $\mu = \nu' + \nu + \nu''$, where $\nu', \nu'' \in R^+$ and $\nu' < \nu' < \nu''$. Then (4.10) follows from (7.25), where we put $x = e_{\nu'}$, $y = e_\nu$, $z = e_{\nu''}$.

The second equality in (4.11) follows from (4.10). To prove the first equality, put, under the assumption of (4.11), $\mu = \mu' + \mu \cap \nu$ for some $\mu', \nu'' \in R^+$ such that $\mu' < \mu \cap \nu < \nu''$. To check the formula (4.11), pull the root vectors $e_{\mu'}$ and $e_{\mu \cap \nu}$ to the right in $(e_{\mu'} e_{\mu \cap \nu} - q e_{\mu \cap \nu} e_{\mu'}) e_\nu$ using already proved (4.7), (4.8), (4.9), and (4.10). This gives (4.11). $\square$

The elements $\tilde{e}_\mu$ satisfy similar identities, which can be readily derived from these by applying the antipode anti-isomorphism.

Root vectors of the negative Borel subalgebra are defined as follows. Let ω be the involutive automorphism of $U_\hbar(\mathfrak{sl}(n))$ defined on the simple root vectors by $e_\alpha \leftrightarrow -f_\alpha, h \to -h$. Introduce f_μ as $-(-q)^{k-1}\omega(e_\mu)$, where k is the height of μ. Explicitly, for $\mu = \alpha_k + \alpha_{k+1} + \ldots + \alpha_m$, $k < m$, put

$$f_\mu = [f_m, [f_{m-1}, \ldots [f_{k+1}, f_k]_{\bar{q}} \ldots]_{\bar{q}}]_{\bar{q}},$$

where $f_i = f_{\alpha_i}$. The elements f_μ, $\mu \in R^+$ generate a PBW basis of $U_\hbar(\mathfrak{n}^-)$, which can be obtained by the isomorphism $\omega \colon U_\hbar(\mathfrak{n}^+) \to U_\hbar(\mathfrak{n}^-)$. The commutation relations on f_μ can be derived from Proposition 4.2 by applying the automorphism ω.

Further we need more commutation relations among the elements of $U_\hbar(\mathfrak{sl}(n))$.

Proposition 4.3. *For every positive roots γ*

$$[e_\gamma, f_\gamma] = \frac{q^{h_\gamma} - q^{-h_\gamma}}{q - q^{-1}}.$$

If $\mu < \gamma$ and $\mu + \gamma$ is a root, then

$$[e_\gamma, f_{\gamma+\mu}] = -q^{-1} f_\mu q^{-h_\gamma}, \quad [e_\gamma, f_{\mu+\gamma}] = f_\mu q^{h_\gamma}, \quad [f_\gamma, e_{\gamma+\mu}] = f_\mu q^{h_\gamma}, \quad [f_\gamma, e_{\mu+\gamma}] = -q e_\mu q^{-h_\gamma},$$

$$[f_\gamma, e_{\mu+\gamma+\nu}] = 0, \quad [e_\gamma, f_{\mu+\gamma+\nu}] = 0,$$

$$[e_{\mu+\gamma}, f_{\gamma+\nu}] = (q - q^{-1}) f_\nu e_\mu q^{-h_\gamma}.$$

The proof of these formulas is given in Appendix.

Corollary 4.4. *For every positive roots ν and μ such that $\nu + \mu \in R^+$, and positive integer k,*

$$[e_\mu, f_{\mu+\nu}^k] = -q^{-1} \frac{q^{2k} - 1}{q^2 - 1} f_{\mu+\nu}^{k-1} f_\nu q^{-h_\mu},$$

$$[e_\nu, f_{\mu+\nu}^k] = q^{-k+1} \frac{q^{2k} - 1}{q^2 - 1} f_\mu f_{\mu+\nu}^{k-1} q^{h_\nu}.$$

$$[e_\nu, f_\nu^k] = f_\nu^{k-1} \left(q^{h_\nu+1} \frac{1 - q^{-2k}}{(q - q^{-1})^2} + q^{-h_\nu-1} \frac{1 - q^{2k}}{(q - q^{-1})^2} \right),$$

5   The invariant form in simple case

In this section we calculate the invariant form on scalar PVM assuming $\mathfrak{l} = \mathfrak{gl}(n) \oplus \mathfrak{gl}(1) \subset \mathfrak{gl}(n+1) = \mathfrak{g}$. The positive root system of $\mathfrak{l}$ is generated by the simple roots $\alpha_2, \ldots \alpha_n$.

The nil-radical Lie algebras $\mathfrak{n}^{\pm}$ are spanned by the root vectors corresponding to the roots $\pm\alpha_1, \pm(\alpha_1 + \alpha_2), \ldots, \pm(\alpha_1 + \ldots + \alpha_n)$.

Introduce generators

$$x_1 = e_{\alpha_1}, \quad x_2 = e_{\alpha_1+\alpha_2}, \quad \ldots, \quad x_n = e_{\alpha_1+\ldots+\alpha_n},$$

$$\tilde{x}_1 = \tilde{e}_{\alpha_1}, \quad \tilde{x}_2 = \tilde{e}_{\alpha_1+\alpha_2}, \quad \ldots, \quad \tilde{x}_n = \tilde{e}_{\alpha_1+\ldots+\alpha_n},$$

$$y_1 = f_{\alpha_1}, \quad y_2 = f_{\alpha_1+\alpha_2}, \quad \ldots, \quad y_n = f_{\alpha_1+\ldots+\alpha_n}.$$

The subalgebras $U_\hbar(\mathfrak{n}^+)$ and $U_\hbar(\mathfrak{n}^-)$ are generated, respectively, by $\{x_i\}$, and $\{y_i\}$.

Let v_λ be the generator of the PVM, M_λ^+, induced from a character $\lambda \in \mathfrak{h}^*$ of the Levi subalgebra $\mathfrak{l}$. The weight λ is proportional to the basis weight ε_1 in the standard orthogonal basis of the $\mathfrak{gl}(n+1)$ weight space. With abuse of notation, we will use the same symbol for the coefficient $\lambda \sim \lambda\varepsilon_1$. This should not cause any confusion in the context.

The generator of the dual module $M_{-\lambda}^-$ will be denoted by $v_{-\lambda}$. The monomials

$$y_n^{m_n} y_{n-1}^{m_{n-1}} \ldots y_1^{m_1} v_\lambda \in M_\lambda^+, \quad \tilde{x}_n^{m_n} \tilde{x}_{n-1}^{m_{n-1}} \ldots \tilde{x}_1^{m_1} v_\lambda \in M_{-\lambda}^-,$$

where m_i are non-negative integers, form a basis in M_λ^+ and, respectively, in $M_{-\lambda}^-$.

Lemma 5.1. *The matrix coefficient* $\langle \tilde{x}_n^{k_n} \tilde{x}_{n-1}^{k_{n-1}} \ldots \tilde{x}_1^{k_1} v_{-\lambda}, y_n^{m_n} y_{n-1}^{m_{n-1}} \ldots y_1^{m_1} v_\lambda \rangle$ *vanishes unless* $k_i = m_i$, *for all* $i = 1, \ldots, n$.

Proof. Since $\gamma(\tilde{x}_i) = q^{-h_\mu} x_\mu$, this matrix coefficient is proportional to

$$\langle v_{-\lambda}, x_1^{k_1} \ldots x_{n-1}^{k_{n-1}} x_n^{k_n} y_n^{m_n} y_{n-1}^{m_{n-1}} \ldots y_1^{m_1} v_\lambda \rangle.$$

Suppose first that $k_n > m_n$. Commutation of x_n with $y_n^{m_n}$ reduces the degree m_n by one and produces a factor from $U_\hbar(\mathfrak{h})$. Pushing it further to the right we get a e_{α_n}-factor by commutation with y_{n-1}. This factor commutes with all elements y_i, $i = 1, \ldots n-1$, and can be placed to the rightmost position, where it annihilates v_λ. Similar effect will be produced by commutation of x_n with other y_i. Thus, only the term from commutation with $y_n^{m_n}$ survives on the way of x_n to the right. Repeating this for other x_n-factors, we see that the matrix coefficient vanishes if $k_n > m_n$. If $k_n < m_n$, similar arguments can be used when pushing y_n to the left till they meet $v_{-\lambda}$.

Thus, the only possibility for the matrix coefficient to survive is $k_n = m_n$. In this case, commutation of $x_n^{m_n}$ with $y_n^{m_n}$ produces an element from $U_\hbar(\mathfrak{h})$, which in its turn gives rise to a scalar factor. This reduces the consideration to the case to $m_n = k_n = 0$, and one can repeat the above reasoning. The obvious induction on n completes the proof. □

Further we calculate the matrix coefficients explicitly. Let $\mathbb{Z}_+$ denote set of non-negative integers. Fix an n-tuple $\boldsymbol{m} = (m_1, \ldots, m_n) \in \mathbb{Z}_+^n$ and put $|\boldsymbol{m}| = \sum_{i=1}^n m_i$. It is convenient to pass to "quantum integers" $\hat{k} = q^{k-1}[k]_q = \sum_{i=0}^{k-1} q^{2i}$, for non-negative k.

Proposition 5.2. *The non-vanishing matrix coefficients of the invariant paring are given by*

$$\langle \tilde{x}_n^{m_n} \tilde{x}_{n-1}^{m_{n-1}} \ldots \tilde{x}_1^{m_1} v_{-\lambda}, y_n^{m_n} y_{n-1}^{m_{n-1}} \ldots y_1^{m_1} v_\lambda \rangle = (-1)^{|\boldsymbol{m}|} q^{-\psi(\boldsymbol{m})} \prod_{i=1}^n \hat{m}_i! \prod_{j=0}^{|\boldsymbol{m}|-1} [\lambda - j]_q. \quad (5.12)$$

where $\psi(\boldsymbol{m}) = (\lambda + \frac{1}{2})|\boldsymbol{m}| - \frac{1}{2}|\boldsymbol{m}|^2$.

Proof. Applying Lemma 4.1 we get for the matrix coefficient

$$(-1)^{\sum_{i=1}^n m_i} \langle v_{-\lambda}, (q^{-h_1} x_1)^{m_1} \ldots (q^{-h_n} x_n)^{m_n} y_n^{m_n} \ldots y_1^{m_1} v_\lambda \rangle =$$

$$= (-1)^{\sum_{i=1}^n m_i} q^{-\lambda \sum_{i=1}^n m_i + \frac{1}{2}(\sum_{i=1}^n m_i)(\sum_{i=1}^n m_i - 1) + \frac{1}{2} \sum_{i=1}^n m_i(m_i-1)} \times$$

$$\times \langle v_{-\lambda}, x_1^{m_1} \ldots x_n^{m_n} y_n^{m_n} \ldots y_1^{m_1} v_\lambda \rangle.$$

The numeric coefficient is equal to $(-1)^{|\boldsymbol{m}|} q^{-\lambda|\boldsymbol{m}| - |\boldsymbol{m}| + \frac{1}{2}|\boldsymbol{m}|^2 + \frac{1}{2} \sum_{i=1}^n m_i^2}$. For every non-negative integer m and a complex parameter z denote

$$Z_q[m, z] = \frac{1}{q - q^{-1}} (q^z \frac{1 - q^{-2m}}{1 - q^{-2}} - q^{-z} \frac{1 - q^{2m}}{1 - q^2}) = [m]_q [z - m + 1]_q.$$

The matrix coefficient $\langle v_{-\lambda}, x_1^{m_1} \ldots x_n^{m_n} y_n^{m_n} \ldots y_1^{m_1} v_\lambda \rangle$ is found to be

$$\frac{1}{q - q^{-1}} (q^{\lambda - \sum_{i=1}^{n-1} m_i} \frac{1 - q^{-2m_n}}{1 - q^{-2}} - q^{-\lambda + \sum_{i=1}^{n-1} m_i} \frac{1 - q^{2m_n}}{1 - q^2}) \times \ldots = Z_q[m_n, \lambda - \sum_{l=1}^{n-1} m_i] \times \ldots$$

In the product omitted on the right, we go down from m_n to 1 in the first argument of Z_q. Then repeat the procedure as though the dimension of the vector $\boldsymbol{m}$ were $n - 1$ rather then n, and proceed until we get to $n = 0$. The result for the matrix coefficient will be

$$\prod_{i=1}^n \prod_{j=1}^{m_i} Z_q[j, \lambda - \sum_{l=1}^{i-1} m_i] = \prod_{i=1}^n [m_i]_q! \prod_{j=0}^{|\boldsymbol{m}|-1} [\lambda - j]_q.$$

To complete the proof, one should pass to the q-integers $\hat{m}_i$. $\qquad \square$

Let $\mathbb{C}_\hbar[G]$ denote the affine coordinate ring on the quantum group $GL_q(n+1)$ (the Hopf dual to the quantized universal enveloping algebra $U_\hbar(\mathfrak{gl}(n+1))$.) Denote by $\mathbb{C}_\hbar[G]^{\mathfrak{l}}$ the subalgebra of $U_\hbar(\mathfrak{l})$-invariants in $\mathbb{C}_\hbar[G]$ under the left co-regular action. This space naturally

inherits the right co-regular action of $U_\hbar(\mathfrak{g})$ compatible with the multiplication $\cdot_\hbar$. It is known that $\cdot_\hbar$ is a star product, [16].

Notice that the tensors $y_i \otimes \tilde{x}_i$ commute with each other, as follows from Proposition 4.2. For the reasons that will be clear later, we would like to modify y_i to $\tilde{y}_i$ in such a way that the tensors $D_i = \tilde{y}_i \otimes \tilde{x}_i$, $i = 1, \ldots, n$ satisfy the quantum plane relations

$$D_j D_i = q^2 D_i D_j, \quad j < i. \tag{5.13}$$

Another condition on this transformation is to leave the matrix coefficients of the invariant paring untouched. This can be achieved by means of the replacement

$$y \mapsto \tilde{y}_i = q^{-(\eta_i, \lambda)} y_i q^{-\eta_i}, \quad i = 1, \ldots, n, \tag{5.14}$$

where $\eta_i \in \mathfrak{h}$ are to be determined.

Proposition 5.3. *There exists a unique sequence $\eta_i \in \mathfrak{h}$, $i = 1, \ldots, n$, such that $D_i = \tilde{y}_i \otimes \tilde{x}_i$ satisfy the quantum plane relations and*

$$\langle \tilde{x}_n^{m_n} \tilde{x}_{n-1}^{m_{n-1}} \ldots \tilde{x}_1^{m_1} v_{-\lambda}, \tilde{y}_n^{m_n} \tilde{y}_{n-1}^{m_{n-1}} \ldots \tilde{y}_1^{m_1} v_\lambda \rangle = (-1)^{|\boldsymbol{m}|} q^{-\psi(\boldsymbol{m})} \prod_{i=1}^{n} \hat{m}_i! \prod_{j=0}^{|\boldsymbol{m}|-1} [\lambda - j]_q. \tag{5.15}$$

All other matrix coefficients are zero.

Proof. Introduce a new basis $\{\beta_i\}_{i=1}^n$ in $\mathfrak{h}$ setting $\beta_i = \alpha_1 + \ldots + \alpha_i$. Note that the vectors $\tilde{x}_i$, $\tilde{y}_i$ carry weights $\pm\beta_i$. The Gram matrix (β_i, β_j) and its inverse are, respectively

$$\begin{pmatrix} 2 & 1 & 1 & \ldots & 1 \\ 1 & 2 & 1 & \ldots & 1 \\ & \ldots & & & \\ 1 & 1 & 1 & \ldots & 2 \end{pmatrix}, \quad \begin{pmatrix} \frac{n}{n+1} & \frac{-1}{n+1} & \frac{-1}{n+1} & \cdots & \frac{-1}{n+1} \\ \frac{-1}{n+1} & \frac{n}{n+1} & \frac{-1}{n+1} & \cdots & \frac{-1}{n+1} \\ & \ldots & & & \\ \frac{-1}{n+1} & \frac{-1}{n+1} & \frac{-1}{n+1} & \cdots & \frac{n}{n+1} \end{pmatrix}.$$

Define $\eta_i = \sum_{k=1}^n B_{ik} \beta_k$ through the system of equations

$$(\eta_j, \beta_i) = -2 + (\eta_i, \beta_j), \quad j < i, \qquad (\eta_i, \beta_j) = 0, \quad i \geqslant j. \tag{5.16}$$

The transition matrix is uniquely defined and equal to

$$B = \begin{pmatrix} 0 & -2 & -2 & \ldots & -2 \\ 0 & 0 & -2 & \ldots & -2 \\ & & \ldots & & \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \begin{pmatrix} \frac{n}{n+1} & \frac{-1}{n+1} & \frac{-1}{n+1} & \cdots & \frac{-1}{n+1} \\ \frac{-1}{n+1} & \frac{n}{n+1} & \frac{-1}{n+1} & \cdots & \frac{-1}{n+1} \\ & & \ldots & & \\ \frac{-1}{n+1} & \frac{-1}{n+1} & \frac{-1}{n+1} & \cdots & \frac{n}{n+1} \end{pmatrix}.$$

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Now we can complete the proof. Notice that the left group of equations (5.16) facilitates the quantum plane relations (5.13). Further, it is clear that the matrix coefficients in the new basis involving $\tilde{y}_i$ are proportional to the old ones. For the non-vanishing matrix coefficient (5.3) is equal to

$$q^{\sum_{k=1}^n \frac{m_k(m_k-1)}{2}(\eta_k,\beta_k)+\sum_{i>j}(\eta_i,\beta_j)m_im_j}\langle \tilde{x}_n^{m_n}\tilde{x}_{n-1}^{m_{n-1}}\ldots\tilde{x}_1^{m_1}v_{-\lambda},y_n^{m_n}y_{n-1}^{m_{n-1}}\ldots y_1^{m_1}v_\lambda\rangle.$$

The scalar multiplier disappears due to the left condition in (5.16). This completes the proof. □

6 Star product on complex projective spaces

In this section we apply the results of the preceding considerations to construction of $U_\hbar\big(\mathfrak{sl}(n+1)\big)$-invariant star product on the homogeneous space $GL(n+1)/GL(n)\times GL(1)$. We start with the following well known fact.

Lemma 6.1. *If $D_1,\ldots,D_n$ satisfy the quantum plain relations (5.13), then*

$$(D_1+\ldots+D_n)^m = \sum_{m_1+\ldots+m_n=m} \frac{\hat{m}!}{\hat{m}_1!\ldots\hat{m}_n!}D_n^{m_n}\ldots D_1^{m_1}, \tag{6.17}$$

for all non-negative integers m.

This lemma can be easily proved by induction on n. Now we can construct the star product. Define the tensor $\tilde{\boldsymbol{y}}\otimes\tilde{\boldsymbol{x}}=\sum_{i=1}^n\tilde{y}_i\otimes\tilde{x}_i$.

Theorem 6.2. *The element*

$$\sum_{m=0}^\infty \frac{(-1)^m q^{(\lambda+1)m-m^2}}{[m]_q!\prod_{j=0}^{m-1}[\lambda-j]_q}(\tilde{\boldsymbol{y}}\otimes\tilde{\boldsymbol{x}})^m, \tag{6.18}$$

is a lift of the inverse invariant form on $M_{-\lambda}\otimes M_\lambda$.

Proof. An immediate consequence of Proposition 5.3 and Lemma 6.17. □

The operator (6.18) is not quasiclassical modulo $\hbar$. To make it a star product deformation of the ordinary multiplication in $\mathbb{C}[G]^\mathfrak{l}$, we need to extend the ring of scalars by Laurent series and consider the module $M_{\frac{\lambda}{\hbar}}^+$.

Corollary 6.3. *For $f, g \in \mathbb{C}_\hbar[G]^{\mathfrak{l}}$, the multiplication*

$$f *_\hbar g := \sum_{\boldsymbol{m} \in \mathbb{Z}_+^n} \frac{(-1)^{|\boldsymbol{m}|} q^{(\lambda + \frac{1}{2})m - \frac{1}{2}m^2}}{\prod_{i=1}^n \hat{m}_i! \prod_{j=0}^{|\boldsymbol{m}|-1} [\frac{\lambda}{\hbar} - j]_q} \left(\tilde{y}_n^{m_n} \dots \tilde{y}_1^{m_1} f \right) \cdot_\hbar \left(\tilde{x}_n^{m_n} \dots \tilde{x}_1^{m_1} g \right),$$

is a $U_\hbar\big(\mathfrak{gl}(n+1)\big)$-invariant star product on $\mathbb{C}_\hbar[G]^{\mathfrak{l}}$ under the right co-regular action.

Remark that the star product given by this formula involves the star product in $\mathbb{C}_\hbar[G]$, whose explicit expression through the classical multiplication in $\mathbb{C}[G]^{\mathfrak{l}}$ is unknown. Therefore it cannot be regarded as perfectly explicit.

Let us reserve the same notation for the classical limits of the root vectors x_i, y_i. Recall that x_i and $\tilde{x}_i$ have the same classical limits.

Corollary 6.4. *For $f, g \in \mathbb{C}[G]^{\mathfrak{l}}$, the multiplication*

$$f *_t g := \sum_{\boldsymbol{m} \in \mathbb{Z}_+^n} \frac{(-t)^{|\boldsymbol{m}|}}{\prod_{i=1}^n m_i! \prod_{j=0}^{|\boldsymbol{m}|-1} (\lambda - jt)} \left(x_n^{m_n} \dots x_1^{m_1} f \right) \cdot \left(y_n^{m_n} \dots y_1^{m_1} g \right), \tag{6.19}$$

is a $U\big(\mathfrak{gl}(n+1)\big)$-invariant star-product on $\mathbb{C}[G]^{\mathfrak{l}}$ under the right co-regular action.

This multiplication is obtained from (6.18) in two steps: taking limit $\hbar \to 0$ and subsequent replacement of λ by $\frac{\lambda}{t}$.

In classical universal enveloping algebra setting the (scalar reduced) dynamical twist F^λ takes the form

$$F^\lambda = \sum_{m=0}^{\infty} \frac{(-t)^m}{m! \prod_{j=0}^{m-1} (\lambda - jt)} (\boldsymbol{x} \otimes \boldsymbol{y})^m. \tag{6.20}$$

The tensor $\boldsymbol{x} \otimes \boldsymbol{y} = \sum_{i=1}^n x_i \otimes y_i$ is the $U_\hbar(\mathfrak{l})$-invariant element of $\mathfrak{n}_{\mathfrak{l}}^+ \otimes \mathfrak{n}_{\mathfrak{l}}^-$.

7 Comparison with earlier results

In the present section we compare the star-product on $GL(n+1)/GL(n) \times GL(1)$ with that on the complexified projective space $\mathbb{C}P^n$ regarded as a real manifold. This star product was obtained in [13] by completely different methods. In both cases they form a one parameter family. In our setting, it corresponds to the highest weight of the module M_λ, while in [13] to the radius of $\mathbb{C}P^n$. We prove that the two star products coincide up to a shift of this parameter.

Let us rewrite the star product (6.19) in local coordinates. Introduce a parametrization of a neighborhood of the identity in $GL(n+1)$:

$$(\boldsymbol{\zeta}, \boldsymbol{\omega}, h) \mapsto \begin{pmatrix} 1 & 0 \\ \boldsymbol{\omega} & 1_n \end{pmatrix} \begin{pmatrix} 1 & \boldsymbol{\zeta} \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix},$$

where $\boldsymbol{\zeta}$ and $\boldsymbol{\omega}$ are, respectively, n-dimensional row and column, $1_n \in GL(n)$ is the unit matrix, and $h = \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix} \in H = GL(1) \times GL(n)$. A function $\varphi \in \mathbb{C}[G]$ is H-invariant if and only if it is independent of h in these local coordinates, $\varphi(\boldsymbol{\zeta}, \boldsymbol{\omega}, h) = \varphi(\boldsymbol{\zeta}, \boldsymbol{\omega}, 1_{n+1})$. Then we shall write simply $\varphi(\boldsymbol{\zeta}, \boldsymbol{\omega})$.

It is obvious that the left-invariant vector field x_i is represented by the partial derivative $\frac{\partial}{\partial \zeta_i}$. To evaluate the left-invariant vector field y_i at the point $(\boldsymbol{\zeta}, \boldsymbol{\omega}, 1)$, consider the the right shift of $(\boldsymbol{\zeta}, \boldsymbol{\omega}, 1)$ by $e^{t\boldsymbol{\omega}_1}$, which in the local chart reads

$$\begin{pmatrix} 1 & 0 \\ \boldsymbol{\omega} + \frac{\boldsymbol{\omega}_1 t}{1 + (\boldsymbol{\zeta}, \boldsymbol{\omega}_1)t} & 1_n \end{pmatrix} \begin{pmatrix} 1 & \boldsymbol{\zeta}(1 + (\boldsymbol{\zeta}, \boldsymbol{\omega}_1)t) \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} 1 + (\boldsymbol{\zeta}, \boldsymbol{\omega}_1)t & 0 \\ 0 & \frac{1}{1 + (\boldsymbol{\zeta}, \boldsymbol{\omega}_1)t} 1_n \end{pmatrix}.$$

Assuming φ an H-invariant function, we find $(e^{t\boldsymbol{\omega}_1}\varphi)(\boldsymbol{\zeta}, \boldsymbol{\omega}) = \varphi\left(\boldsymbol{\zeta} + \boldsymbol{\zeta}(\boldsymbol{\zeta}, \boldsymbol{\omega}_1)t, \boldsymbol{\omega} + \frac{\boldsymbol{\omega}_1 t}{1 + (\boldsymbol{\zeta}, \boldsymbol{\omega}_1)t}\right)$, hence

$$y_i \varphi(\boldsymbol{\zeta}, \boldsymbol{\omega}) = \frac{\partial}{\partial \omega_i} \varphi(\boldsymbol{\zeta}, \boldsymbol{\omega}) + \zeta_i \sum_{k=1}^{n} \zeta_k \frac{\partial}{\partial \zeta_k} \varphi(\boldsymbol{\zeta}, \boldsymbol{\omega}),$$

$$x_i \varphi(\boldsymbol{\zeta}, \boldsymbol{\omega}) = \frac{\partial}{\partial \zeta_i} \varphi(\boldsymbol{\zeta}, \boldsymbol{\omega}).$$

A version of star-product on $\mathbb{C}P^n$ (regarded as a real manifold) was constructed in [13] as a homogeneous (delation-invariant) star product on $V = \mathbb{C}^{n+1}$. To compare it with our result, consider its complexified version on $V \oplus V^*$:

$$\phi * \psi = \phi\psi + \sum_{r=1}^{\infty} \left(-\frac{t}{2\mu}\right)^k \sum_{s=1}^{r} \sum_{k=1}^{s} \frac{(-1)^{r-k} k^{r-1}}{s!(s-k)!(k-1)!} (z, w)^s \cdot \left(\frac{\partial}{\partial z} \otimes \frac{\partial}{\partial w}\right)^s (\phi \otimes \psi), \quad (7.21)$$

where $\frac{\partial}{\partial z} \otimes \frac{\partial}{\partial w} = \sum_{i=1}^{n+1} \frac{\partial}{\partial z_i} \otimes \frac{\partial}{\partial w_i}$, and the dot means the classical multiplication.

The vector space $V \oplus V^*$ carries the natural representation of $GL(n+1)$, which is extended by the group of delations $GL(1)$. The multiplication (7.21) is invariant with respect the direct product $GL(n+1) \times GL(1)$. In particular, it restricts to homogeneous functions of zero degree, regarded as functions on the projective space $\mathbb{C}P^{2n+1}$.

Let $\{e_i\} \subset V$ be the standard basis and $\{e^i\} \subset V^*$ be its dual. Consider the $GL(n+1)$-orbit O passing through $o = e_1 \oplus e^1$. The isotropy subgroup of this point is $1 \times GL(n)$,

so this orbit is isomorphic to the coset space $GL(n+1)/1 \times GL(n)$. We extend the above parametrization of $GL(n+1)/GL(1) \times GL(n)$ to a parametrization of O:

$$(\boldsymbol{\zeta}, \boldsymbol{\omega}, a) \mapsto \begin{pmatrix} 1 & 0 \\ \boldsymbol{\omega} & 1 \end{pmatrix} \begin{pmatrix} 1 & \boldsymbol{\zeta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1}1_n \end{pmatrix} \quad o = (a, a\boldsymbol{\omega}) \oplus (a^{-1} + a^{-1}(\boldsymbol{\omega}, \boldsymbol{\zeta}), -a^{-1}\boldsymbol{\zeta}).$$

It can be extended to a local parametrization

$$a = z_0, \quad b = (\boldsymbol{z}, \boldsymbol{w}), \quad \zeta_i = -z_0 w_i, \quad \omega_i = \frac{z_i}{z_0}, \quad i = 1, \ldots, n.$$

of $V \oplus V^*$ near o. In this chart, the basic vector fields are represented as follows:

$$\frac{\partial}{\partial z_0} \mapsto \frac{\partial}{\partial a} + \frac{1}{a}\big(b + (\boldsymbol{\zeta}, \boldsymbol{\omega})\big)\frac{\partial}{\partial b} + \frac{1}{a}\sum_{i=1}^n(\zeta_i\frac{\partial}{\partial \zeta_i} - \omega_i\frac{\partial}{\partial \omega_i}), \quad \frac{\partial}{\partial w_0} \mapsto a\frac{\partial}{\partial b},$$

$$\frac{\partial}{\partial z_i} \mapsto -\frac{\zeta_i}{a}\frac{\partial}{\partial b} + \frac{1}{a}\frac{\partial}{\partial \omega_i}, \quad \frac{\partial}{\partial w_i} \mapsto \omega_i a\frac{\partial}{\partial b} + a\frac{\partial}{\partial \zeta_i}.$$

In the new coordinates, a function ϕ on $V \oplus V^*$ is homogeneous if and only if

$$\phi(\lambda a, \lambda^2 b, \lambda^2 \boldsymbol{\zeta}, \boldsymbol{\omega}) = \phi(a, b, \boldsymbol{\zeta}, \boldsymbol{\omega}).$$

It is $GL(1) \times 1_n$-invariant if it is independent of a. The correspondence between functions on $GL(n+1)/GL(1) \times GL(n)$ and homogeneous $GL(1) \times 1_n$-invariant functions on $V \oplus V^*$ is $\phi(\boldsymbol{\zeta}, \boldsymbol{\omega}) \mapsto \phi(b^{-1}\boldsymbol{\zeta}, \boldsymbol{\omega})$, with the reverse correspondence being the specialization at $b = 1$. We rewrite the action of the basic vector fields on such functions to find

$$\frac{\partial}{\partial z_0} \mapsto -\sum_{i=1}^n\big((\boldsymbol{\zeta}, \boldsymbol{\omega})\zeta_i\frac{\partial}{\partial \zeta_i} + \omega_i\frac{\partial}{\partial \omega_i}\big), \quad \frac{\partial}{\partial w_0} \mapsto \sum_{i=1}^n \zeta_i\frac{\partial}{\partial \zeta_i},$$

$$\frac{\partial}{\partial z_i} \mapsto \zeta_i\sum_{k=1}^n \zeta_k\frac{\partial}{\partial \zeta_k} + \frac{\partial}{\partial \omega_i}, \quad \frac{\partial}{\partial w_i} \mapsto \omega_i\sum_{i=1}^n \zeta_i\frac{\partial}{\partial \zeta_i} + \frac{\partial}{\partial \zeta_i}.$$

Now it is easy to check the equality

$$\sum_{i=1}^n \frac{\partial}{\partial z_i} \otimes \frac{\partial}{\partial w_i} = \sum_{i=1}^n \frac{\partial}{\partial \omega_i} \otimes \frac{\partial}{\partial \zeta_i} + \sum_{i,j=1}^n \zeta_i\zeta_j\frac{\partial}{\partial \zeta_i} \otimes \frac{\partial}{\partial \zeta_j} = \boldsymbol{x} \otimes \boldsymbol{y}.$$

The bidifferential operator from (7.21), when restricted to $(GL(1) \times 1_n) \times GL(1)$-invariant functions, reads

$$\mathrm{id} \otimes \mathrm{id} + \sum_{r=1}^{\infty}\Big(-\frac{t}{2\mu}\Big)^r \sum_{s=1}^r \sum_{k=1}^s \frac{(-1)^{r-k}k^{r-1}}{s!(s-k)!(k-1)!}(\boldsymbol{x} \otimes \boldsymbol{y})^s. \tag{7.22}$$

It is therefore a series in the same operator $\boldsymbol{x} \otimes \boldsymbol{y}$ as (6.20). This reduces comparison of the two star products to a comparison of power series in two variables, t and $\boldsymbol{x} \otimes \boldsymbol{y}$.

Proposition 7.1. *Operators (6.20) and (7.22) coincide upon the identification $2\mu - t = \lambda$.*

Proof. The proof is based on the following formula

$$\sum_{k_1+\ldots+k_m=k} a_1^{k_1} a_2^{k_2} \ldots a_m^{k_m} = \sum_{i=1}^{m} \frac{a_i^{k+m-1}}{\prod_{j\neq i} (a_i - a_j)},$$

which holds true for any communing variables a_i. Applying this formula for $a_i = i$, we get

$$\sum_{k_1+\ldots+k_m=r-m} 1^{k_1} 2^{k_2} \ldots m^{k_m} = \sum_{k=1}^{m} \frac{k^{r-1}}{\prod_{j<k} (a_k - a_j) \prod_{j>k} (a_k - a_j)} = \sum_{k=1}^{m} \frac{(-1)^{m-k} k^{r-1}}{(k-1)!(m-k)!}.$$

Put $\theta = \frac{t}{2\mu}$ and rearrange the series in (7.22) as

$$\mathrm{id} \otimes \mathrm{id} + \sum_{r=1}^{\infty} \sum_{m=1}^{r} \Big(\sum_{k_1+\ldots+k_m=r-m} (1\theta)^{k_1} (2\theta)^{k_2} \ldots (m\theta)^{k_m} \Big) \frac{(-\theta)^m (\boldsymbol{x} \otimes \boldsymbol{y})^m}{m!},$$

and further as

$$\mathrm{id} \otimes \mathrm{id} + \sum_{m=1}^{\infty} \Big(\sum_{r=m}^{\infty} \sum_{k_1+\ldots+k_m=r-m} (1\theta)^{k_1} (2\theta)^{k_2} \ldots (m\theta)^{k_m} \Big) \frac{(-\theta)^m (\boldsymbol{x} \otimes \boldsymbol{y})^m}{m!}.$$

The summation $\sum_{r=m}^{\infty} \sum_{k_1+\ldots+k_m=r-m}$ is rearranged to summation $\sum_{k_1,\ldots,k_m=0}^{\infty}$. It contracts the sum in the brackets to $\frac{1}{(1-\theta)\ldots(1-m\theta)}$. This immediately implies the statement. $\square$

Appendix

Below we collect some useful auxiliary algebraic material about the properties of "commutator" $[x, y]_a = xy - ayx$ defined in any associative algebra for some scalar a. Next is a sort of "Jacobi identity" for such quasi-commutators.

Lemma 7.2. *For any three elements x, y, z of an associative algebra and any three scalars a, b, c*

$$[x, [y, z]_a]_b = [[x, y]_c, z]_{\frac{ab}{c}} + c[y, [x, z]_{\frac{b}{c}}]_{\frac{a}{c}}. \tag{7.23}$$

The proof of this statement is elementary. Next state a useful fact, which is a sort of Serre relation for "adjacent root vectors" of higher weights.

Lemma 7.3. *Suppose some elements y, z, x of an associative algebra satisfy the identities*

$$[y, [y, z]_b]_{b^{-1}} = 0, \quad [x, [x, y]_a]_{a^{-1}} = 0, \quad [x, z] = 0. \tag{7.24}$$

for some invertible scalars b, a. Then $[[x, y]_a, [[x, y]_a, z]_b]_{b^{-1}} = 0$.

Proof. Put $A = (b + b^{-1})$ and $B = (a + a^{-1})$. The relations (7.24) imply the equalities

$$
\begin{aligned}
0 &= B(xy)^2 z - (yx)(xy)z - AB(xy)z(xy) + A(yx)z(xy) + Bz(xy)^2 - z(yx)(xy), \\
0 &= B(yx)^2 z - (yx)(xy)z - AB(yx)z(yx) + A(yx)z(xy) + Bz(yx)^2 - z(yx)(xy).
\end{aligned}
$$

The first line is a result of multiplication of the left identity (7.24) by x^2 on the left and using the second and third identities (7.24). The second line is produced in a similar way, multiplying the left equality by x^2 on the right.

Multiply the second line by a^2 and add to the first line. The resulting equation will take the form

$$
0 = B[[x, y]_a, [[x, y]_a, z]_b]_{b^{-1}},
$$

as required. $\qquad\square$

Remark that the hypothesis of the lemma is symmetric with respect to replacement of a by a^{-1}, as well as b by b^{-1}. Therefore, this replacement can be made arbitrarily in the statement.

It follows that if pairs x, y and y, z commute as adjacent root vectors and x, z as distant root vectors, then the $[x, y]_a$ commutes with z as with y, as thought z does not notice the presence of x in $[x, y]_a$.

Lemma 7.4. *Suppose x, y, z satisfy the relations*

$$
[y, [y, x]_q]_{q^{-1}} = 0, \quad [y, [y, z]_q]_{q^{-1}} = 0,
$$

and x commutes with z. Then

$$
[y, [x, [y, z]_q]_q] = 0. \tag{7.25}
$$

Proof. Using the "Jacobi identity" 7.23 with $a = c = q$, $b = 1$ we get

$$
[y, [x, [y, z]_q]_q] = [[y, x]_q, [y, z]_q] + [x, [y, [y, z]_q]_{q^{-1}}].
$$

The right-hand side is zero: the first term vanishes as proved, the second due to the assumption. This proves (7.25). $\qquad\square$

Remark that q can be replaced by q^{-1} in the two assumption equalities arbitrarily, as the double commutator $[y, [y, x]_q]_{q^{-1}}$ is stable under this transformation.

Proof of Proposition 4.3. Let $\gamma = \alpha + \mu$, where α is a simple positive root.

$$
\begin{aligned}
[e_\gamma, f_\gamma] &= [[e_\alpha, e_\mu]_q, [f_\mu, f_\alpha]_{\bar q}] = [[[e_\alpha, e_\mu]_q, f_\mu], f_\alpha]_{\bar q} + [f_\mu, [[e_\alpha, e_\mu]_q, f_\alpha]]_{\bar q} \\
&= [[e_\alpha, [e_\mu, f_\mu]]_q, f_\alpha]_{\bar q} + [f_\mu, [[e_\alpha, f_\alpha], e_\mu]_q]_{\bar q} \\
&= [[e_\alpha, \frac{q^{h_\mu} - q^{-h_\mu}}{q - q^{-1}}]_q, f_\alpha]_{\bar q} + [f_\mu, [\frac{q^{h_\alpha} - q^{-h_\alpha}}{q - q^{-1}}, e_\mu]_q]_{\bar q} \\
&= \frac{-q^{-1}(1 - q^2)}{q - q^{-1}}[e_\alpha, f_\alpha]q^{-h_\mu} - \frac{q - q^{-1}}{q - q^{-1}}[f_\mu, e_\mu]q^{h_\alpha} \\
&= \frac{q^{h_\alpha} - q^{-h_\alpha}}{q - q^{-1}}q^{-h_\mu} + \frac{q^{h_\mu} - q^{-h_\mu}}{q - q^{-1}}q^{h_\alpha} = \frac{q^{h_\alpha + h_\mu} - q^{-h_\alpha - h_\mu}}{q - q^{-1}},
\end{aligned}
$$

$$
[e_\mu, f_{\mu+\gamma}] = [e_\mu, [f_\gamma, f_\mu]_{\bar q}] = [f_\gamma, \frac{q^{h_\mu} - q^{-h_\mu}}{q - q^{-1}}]_{\bar q} = -\frac{1 - q^{-2}}{q - q^{-1}}f_\gamma q^{-h_\mu} = -q^{-1}f_\gamma q^{-h_\mu},
$$

$$
[e_\gamma, f_{\mu+\gamma}] = [e_\gamma, [f_\gamma, f_\mu]_{\bar q}] = [\frac{q^{h_\gamma} - q^{-h_\gamma}}{q - q^{-1}}, f_\mu]_{\bar q} = \frac{q - q^{-1}}{q - q^{-1}}f_\mu q^{h_\gamma} = f_\mu q^{h_\gamma},
$$

$$
\begin{aligned}
[e_\gamma, f_{\mu+\gamma+\nu}] &= [e_\gamma, [f_{\gamma+\nu}, f_\mu]_{\bar q}] = -q^{-1}[f_\nu q^{-h_\gamma}, f_\mu]_{\bar q} = -q^{-1}(f_\nu q^{-h_\gamma}f_\mu - \bar q f_\mu f_\nu q^{-h_\gamma}) = \\
&= -q^{-1}(\bar q f_\nu f_\mu q^{-h_\gamma} - \bar q f_\mu f_\nu q^{-h_\gamma}) = 0,
\end{aligned}
$$

$$
\begin{aligned}
[e_{\mu+\gamma}, f_{\gamma+\nu}] &= [[e_\mu, e_\gamma]_q, [f_\nu, f_\gamma]_{\bar q}] = [f_\nu, [[e_\mu, e_\gamma]_q, f_\gamma]]_{\bar q} = [f_\nu, [e_\mu, [e_\gamma, f_\gamma]]_q]_{\bar q} \\
&= [f_\nu, [e_\mu, \frac{q^{h_\gamma} - q^{-h_\gamma}}{q - q^{-1}}]_q]_{\bar q} = -\frac{1 - q^2}{q - q^{-1}}(1 - \bar q^2)f_\nu e_\mu q^{-h_\gamma} \\
&= (q - q^{-1})f_\nu e_\mu q^{-h_\gamma},
\end{aligned}
$$

$\square$

Proof of Corollary 4.4.

$$
\begin{aligned}
[e_\mu, f^k_{\mu+\gamma}] &= -q^{-1}f^{k-1}_{\mu+\gamma}f_\gamma q^{-h_\mu} - q^{-1}f^{k-2}_{\mu+\gamma}f_\gamma q^{-h_\mu}f_{\mu+\gamma} + \dots \\
&= -q^{-1}f^{k-1}_{\mu+\gamma}f_\gamma q^{-h_\mu} - q^{-1+1}f^{k-1}_{\mu+\gamma}f_\gamma q^{-h_\mu} + \dots = -q^{-1}\frac{q^{2k} - 1}{q^2 - 1}f^{k-1}_{\mu+\gamma}f_\gamma q^{-h_\mu},
\end{aligned}
$$

$$
\begin{aligned}
[e_\gamma, f^k_{\mu+\gamma}] &= f_\mu q^{h_\gamma}f^{k-1}_{\mu+\gamma} + f_{\mu+\gamma}f_\mu q^{h_\gamma}f^{k-2}_{\mu+\gamma} + \dots \\
&= q^{-k+1}f_\mu f^{k-1}_{\mu+\gamma}q^{h_\gamma} + q^{-k+3}f_\mu f^{k-1}_{\mu+\gamma}q^{h_\gamma} + \dots = q^{-k+1}\frac{q^{2k} - 1}{q^2 - 1}f_\mu f^{k-1}_{\mu+\gamma}q^{h_\gamma},
\end{aligned}
$$

$$
\begin{aligned}
[e_\gamma, f^k_\gamma] &= f^{k-1}_\gamma \frac{q^{h_\gamma} - q^{-h_\gamma}}{q - q^{-1}} + f^{k-2}_\gamma \frac{q^{h_\gamma} - q^{-h_\gamma}}{q - q^{-1}}f_\gamma + \dots \\
&= f^{k-1}_\gamma \Big(\frac{q^{h_\gamma} - q^{-h_\gamma}}{q - q^{-1}} + \frac{q^{h_\gamma-2} - q^{-h_\gamma+2}}{q - q^{-1}} + \dots \Big) \\
&= f^{k-1}_\gamma \Big(q^{h_\gamma+1}\frac{1 - q^{-2k}}{(q - q^{-1})^2} + q^{-h_\gamma-1}\frac{1 - q^{2k}}{(q - q^{-1})^2}\Big).
\end{aligned}
$$

□

References

[1] Gervais, J.-L., Neveu, A.: Novel triangle relation and absence of tachyons in Liouville string theory. Nucl. Phys. B 238, 125–141 (1984).

[2] Alekseev, A., Faddeev, L.: $(T^*G)_t$: a toy model for conformal field theory. Comm. Math. Phys. 141, 413–422 (1991).

[3] G. Felder: *Conformal field theories and integrable models associated to elliptic curves*, Proc. ICM Zurich (1994) 1247–1255.

[4] Etingof, P., Schiffmann, O.: Lectures on the dynamical Yang-Baxter equation. Quantum Groups and Lie theory. London Math. Soc. Lecture Note 290, 89–129 (2001).

[5] Etingof, P., Varchenko, A.: Exchange dynamical quantum groups, Commun. Math. Phys. 205, 19–52 (1999).

[6] Donin, J., Mudrov, A.: Dynamical Yang-Baxter equation and quantum vector bundles. Commun. Math. Phys. 254, 719–760 (2005).

[7] Alekseev, A., Lachowska, A.: Invariant ∗-product on coadjoint orbits and the Shapovalov pairing. Comment. Math. Helv. 80, 795-810 (2005).

[8] Enriquez, B., Etingof, P., Marshall, I.: Quantization of some Poisson-Lie dynamical r-matrices and Poisson homogeneous spaces, Contemp. Math. 433, 135–176 (2007).

[9] Enriquez, B., Etingof, P.: Quantization of classical dynamical r-matrices with nonabelian base. Commun. Math. Phys. 254, 603-650 (2005).

[10] Arnaudon, D., Buffenoir, E., Ragoucy, E., Roche, Ph.: Universal Solutions of Quantum Dynamical Yang-Baxter Equations. Lett. Math. Phys. 44, 201–214 (1998).

[11] Karolinsky, E., Stolin, A., Tarasov, V. : Irreducible highest weight modules and equivariant quantization. Adv. Math. 211, 266–283 (2007).

[12] Mudrov, A.: Quantum sphere $\mathbb{S}^4$, arXive.math.QA/0336411.

[13] Bordemann, M., Brischle, M., Emmrich, C., Waldmann S.: Phase Space Reduction for Star-Products: An Explicit Construction for $\mathbb{C}P^n$, Lett. Math. Phys., **36**, 357–371 (1996).

[14] Jantzen, J. C.: Lectures on quantum groups. Grad. Stud. in Math. 6. AMS, Providence, RI, 1996.

[15] Drinfeld, V.: Quantum Groups. In: Gleason, A. V. (ed) Proc. Int. Congress of Mathematicians, Berkeley, 1986, pp 798–820, AMS, Providence (1987).

[16] Takhtajan, L. A., Introduction to quantum groups. Lecture Notes in Phys. 370, 3–28 (1990).

Erratum to the journal version

1. Definition of S^λ before Proposition 3.3 should be as

   Denote by $S^\lambda = \lambda(S_-^{(1)})S_-^{(2)} \otimes S_+^{(1)} \lambda^*(S_+^{(2)})$, where $S_- \otimes S_+$ is a lift of $S_{\mathbb{C}^\lambda, \mathbb{C}^{\lambda^*}} \in M_\lambda^+ \otimes M_{\lambda^*}^-$ to $U_\hbar(\mathfrak{p}^-) \otimes U_\hbar(\mathfrak{p}^+)$.

   It turns to the journal version if the lift is appropriate.

2. Page 9, the ordering on roots (after definition of e_μ and $\tilde{e}_\mu$):

   The roots can be written in an orthogonal basis $\{\varepsilon_i\}_{i=1}^n$ of weights of the natural representation as $\varepsilon_i - \varepsilon_j$, $i, j = 1, \ldots, n$, $i \neq j$. The lexicographical ordering on pairs (i, j) induce an ordering on positive roots $\varepsilon_i - \varepsilon_j$, $i < j$, consistent with the ordered basis $(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) \subset \mathfrak{h}^*$.

3. Formula (5.14): the scalar factor should be $q^{(\eta_i, \lambda)}$ rather than $q^{-(\eta_i, \lambda)}$.

4. In Theorem 6.3 and Corollary 6.4: x and y should be interchanged. Another way to fix this error is to understand by $\cdot_\hbar$ the opposite multiplication in the RTT dual in Corollary 6.3; then $U_\hbar$ should be taken with the opposite comultiplication.

   The star product of Corollary 6.5 is correct because the classical multiplication in $\mathbb{C}[G]$ is commutative.