

# ON SOLID AND RIGID MONOIDS IN MONOIDAL CATEGORIES

JAVIER J. GUTIÉRREZ

ABSTRACT. We introduce the notion of solid monoid and rigid monoid in monoidal categories and study the formal properties of these objects in this framework. We show that there is a one to one correspondence between solid monoids, smashing localizations and mapping colocalizations, and prove that rigid monoids appear as localizations of the unit of the monoidal structure. As an application, we study solid and rigid ring spectra in the stable homotopy category and characterize connective solid ring spectra as Moore spectra of subrings of the rationals.

## 1. INTRODUCTION

A *solid ring* in the sense of Bousfield–Kan [5] is a ring  $R$  with unit whose core  $cR$  is  $R$  itself, where the *core* of  $R$  is defined as

$$cR = \{x \in R \mid 1_R \otimes_{\mathbb{Z}} x = x \otimes_{\mathbb{Z}} 1_R\}.$$

They were also called  $T$ -rings in [3] and  $\mathbb{Z}$ -epimorphs in [9]. Indeed, the unit  $\mathbb{Z} \rightarrow R$  is an epimorphism of rings if and only if  $R$  is solid.

For example  $\mathbb{Z}$ ,  $\mathbb{Q}$  and  $\mathbb{Z}/n$  are solid, but the  $p$ -adic integers  $\widehat{\mathbb{Z}}_p$  and  $\mathbb{R}$  are not. Solid rings are completely classified and all of them are commutative and countable [5, Proposition 3.1]. From this classification it turns out, for example, that the only torsion-free solid rings are the subrings of the rationals.

A *rigid ring*  $R$  is a ring with unit such that the evaluation at the unit morphism

$$\mathrm{Hom}_{\mathbb{Z}}(R, R) \longrightarrow R$$

that sends  $\varphi$  to  $\varphi(1_R)$  is an isomorphism. The terminology of rigid rings was first used in [8] in order to describe the class of rings appearing as localizations of the circle  $S^1$  in the category of topological spaces, and as localizations of the integers in the category of groups (see [8, Theorem 5.9]). This class of rings had been previously studied under the name of  $E$ -rings [16]. All of them are commutative and they have been classified in the torsion-free finite rank case [15].

Examples of rigid rings are  $\mathbb{Z}/n$ , subrings of  $\mathbb{Q}$ , and  $\widehat{\mathbb{Z}}_p$  for any prime  $p$ . There are many other examples such as all solid rings, and the products  $\prod_{p \in P} \mathbb{Z}/p$  and  $\prod_{p \in P} \widehat{\mathbb{Z}}_p$ , where  $P$  is any set of primes. However, there are groups such as the Prüfer group  $\mathbb{Z}/p^\infty$  or the  $p$ -adic field  $\widehat{\mathbb{Q}}_p$  that do not admit a rigid ring structure.

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There exist rigid rings of arbitrarily large cardinality [10], and this fact was used in [8] and [6] to prove that there is a proper class of non-equivalent  $f$ -localizations in the category of topological spaces and in the category of spectra, respectively.

In this paper, we define the concept of solid monoid and rigid monoid in monoidal categories, similarly to their algebraic counterparts, and describe the formal properties they satisfy in this framework (see Section 3). Thus, if  $(\mathcal{E}, \otimes, I, \text{Hom}_{\mathcal{E}})$  is a closed symmetric monoidal category, then a monoid  $(R, \mu, \eta)$  is a *solid monoid* if  $\mu$  is an isomorphism, and  $R$  is a *rigid monoid* if the induced morphism

$$\eta^* : \text{Hom}_{\mathcal{E}}(R, R) \longrightarrow \text{Hom}_{\mathcal{E}}(I, R)$$

is an isomorphism in  $\mathcal{E}$ .

Solid and rigid monoids are closely related with localization and colocalization functors in  $\mathcal{E}$ , and in Section 2 we recall the basics of this theory in the setting of enriched monoidal categories. If  $R$  is a solid monoid in  $\mathcal{E}$ , then the functors  $X \mapsto X \otimes R$  and  $X \mapsto \text{Hom}_{\mathcal{E}}(R, X)$  are idempotent. We will show that this property characterizes solid monoids. In fact, we will prove that there is a bijection between the following classes:

- (i) Solid monoids.
- (ii) Smashing localizations.
- (iii) Mapping colocalizations.

Here, a *smashing localization functor* means a localization functor of the form  $L_A X = X \otimes A$  for a fixed  $A$ , and a *mapping colocalization functor* is one of the form  $C_A X = \text{Hom}_{\mathcal{E}}(A, X)$  for a fixed  $A$ . Moreover, we show that if  $R$  is a solid monoid, then the following categories are equivalent: the category of  $R$ -modules, the category of  $L_R$ -local objects, the category of  $C_R$ -colocal objects and the category of  $\eta$ -local objects, where  $\eta: I \rightarrow R$  is the unit of  $R$ .

For an arbitrary localization functor  $L$ , we prove that the object  $LI$  has a rigid monoid structure, where  $I$  denotes the unit of the monoidal structure, and that all rigid monoids appear this way.

Finally, in Section 4 we particularize our results to the stable homotopy category and prove that if  $(R, \mu, \eta)$  is a solid ring spectrum, then the  $\eta$ -localization functor  $L_{\eta}$  always commutes with the suspension, and that the only connective solid ring spectra are Moore spectra  $MA$ , with  $A$  a subring of the rationals.

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## 2. LOCALIZATIONS AND COLOCALIZATIONS IN ENRICHED CATEGORIES

Throughout the paper  $\mathcal{E}$  will denote a cocomplete closed symmetric monoidal category, with tensor product  $\otimes$ , unit  $I$  and internal hom  $\text{Hom}_{\mathcal{E}}(-, -)$ . A functor  $F: \mathcal{E} \rightarrow \mathcal{E}'$  between symmetric monoidal categories is called *monoidal* if it is equipped with a binatural transformation  $F(-) \otimes_{\mathcal{E}'} F(-) \rightarrow F(- \otimes_{\mathcal{E}} -)$  and a unit  $I_{\mathcal{E}'} \rightarrow F(I_{\mathcal{E}})$  satisfying the usual associativity, symmetry and unit conditions. A symmetric monoidal functor is called *strong* if the structure maps are isomorphisms.

Let  $\mathcal{V}$  be a symmetric monoidal category. Recall that a *closed symmetric monoidal  $\mathcal{V}$ -category* is a closed symmetric monoidal category  $\mathcal{E}$  together with an adjunction  $i: \mathcal{V} \rightleftarrows \mathcal{E}: r$ , where the left adjoint  $i$  is strong monoidal. (Note that this automatically implies that  $r$  is monoidal.)

Any closed symmetric monoidal  $\mathcal{V}$ -category  $\mathcal{E}$  is enriched, tensored and cotensored over  $\mathcal{V}$ . Indeed, if  $A$  is any object of  $\mathcal{V}$  and  $X, Y$  are objects of  $\mathcal{E}$ , then we define  $\text{Hom}(X, Y) = r(\text{Hom}_{\mathcal{E}}(X, Y))$ ,  $A \otimes X = i(A) \otimes_{\mathcal{E}} X$  and  $X^A = \text{Hom}_{\mathcal{E}}(i(A), X)$ . Thus, we have natural isomorphisms

$$\mathcal{E}(A \otimes X, Y) \cong \mathcal{V}(A, \text{Hom}(X, Y)) \cong \mathcal{E}(X, Y^A).$$

Conversely, any closed symmetric monoidal category  $\mathcal{E}$  that is enriched, tensored and cotensored over  $\mathcal{V}$  is a closed symmetric monoidal  $\mathcal{V}$ -category, where

$$i(A) = A \otimes I_{\mathcal{E}} \quad \text{and} \quad r = \text{Hom}(I_{\mathcal{E}}, X).$$

Let  $\mathcal{C}$  be any category. A functor  $L: \mathcal{C} \rightarrow \mathcal{C}$  is called *coaugmented* if it is equipped with a natural transformation  $l: \text{Id} \rightarrow L$ . A coaugmented functor is *idempotent* if  $l_{LX} = Ll_X$  and  $l_{LX}$  is an isomorphism for every  $X$  in  $\mathcal{C}$ .

Dually, a functor  $C: \mathcal{C} \rightarrow \mathcal{C}$  is called *augmented* if it is equipped with a natural transformation  $c: C \rightarrow \text{Id}$ . An augmented functor is *idempotent* if  $c_{CX} = Cc_X$  and  $c_{CX}$  is an isomorphism for every  $X$  in  $\mathcal{C}$ .

**Definition 2.1.** A coaugmented idempotent functor  $(L, l)$  in  $\mathcal{C}$  is called a *localization functor*. Dually, an augmented idempotent functor  $(C, c)$  in  $\mathcal{C}$  is called a *colocalization functor*.

The closure under isomorphisms of the objects in the image of  $L$  are called  *$L$ -local objects* and the closure under isomorphisms of the objects in the image of  $C$  are called  *$C$ -colocal objects*. A map  $f$  is called an  *$L$ -local equivalence* if  $L(f)$  is an isomorphism, and it is called a  *$C$ -colocal equivalence* if  $C(f)$  is an isomorphism.

Therefore, for every  $X$  in  $\mathcal{C}$ , the morphism  $l_X: X \rightarrow LX$  is an  $L$ -local equivalence, and it has the following universal property:  $l_X$  is initial among all morphisms from  $X$  to objects isomorphic to  $LY$  for some  $Y$ , that is, the induced map

$$(2.1) \quad l_X^*: \mathcal{C}(LX, LY) \longrightarrow \mathcal{C}(X, LY)$$

is a bijection for every  $X$  and  $Y$ . In fact,  $L$ -local objects and  $L$ -local equivalences are *orthogonal*, that is, if  $Z$  is  $L$ -local and  $f: X \rightarrow Y$  is an  $L$ -local equivalence, then the induced map

$$(2.2) \quad f^*: \mathcal{C}(Y, Z) \longrightarrow \mathcal{C}(X, Z)$$

is a bijection. Moreover, an object is  $L$ -local if and only if it is orthogonal to all  $L$ -local equivalences, and a morphism is an  $L$ -local equivalence if and only if it is orthogonal to all  $L$ -local objects.

Dually, in the case of colocalization functors, the morphism  $c_X$  is terminal among all morphism from  $C$ -colocal objects to  $X$ , and an orthogonality relation similar to (2.2) between  $C$ -colocal objects and  $C$ -colocal equivalences holds.

Thus, if we want to define localization and colocalization functors in closed monoidal categories or  $\mathcal{V}$ -enriched categories, it makes sense to replace the set  $\mathcal{C}(-, -)$  in (2.1) and (2.2), and in the corresponding formulas for colocalizations, by the internal hom or the  $\mathcal{V}$ -enrichment.

**Definition 2.2.** Let  $\mathcal{E}$  be a closed symmetric monoidal  $\mathcal{V}$ -category with associated adjunction  $i: \mathcal{V} \rightleftarrows \mathcal{E}: r$  and let  $\text{Hom}(-, -) = r \text{Hom}_{\mathcal{E}}(-, -)$ .

- (i) An  $(i, r)$ -*localization functor* is a localization functor  $(L, l)$  in  $\mathcal{E}$  such that for every  $L$ -local equivalence  $f: X \rightarrow Y$  and every  $L$ -local object  $Z$  in  $\mathcal{E}$ , the induced map

$$(2.3) \quad f^*: \text{Hom}(Y, Z) \longrightarrow \text{Hom}(X, Z)$$

is an isomorphism in  $\mathcal{V}$ .

- (ii) An  $(i, r)$ -*colocalization functor* is a colocalization functor  $(C, c)$  in  $\mathcal{E}$  such that for every  $C$ -colocal equivalence  $f: X \rightarrow Y$  and every  $C$ -colocal object  $Z$  in  $\mathcal{E}$ , the induced map

$$(2.4) \quad f_*: \text{Hom}(Z, X) \longrightarrow \text{Hom}(Z, Y)$$

is an isomorphism in  $\mathcal{V}$ .

Note that in order to check if an object is  $L$ -local, it is enough to see that condition (2.3) holds for every  $L$ -equivalence of the form  $l_X: X \rightarrow LX$ . In the same way, an object  $Z$  is  $C$ -colocal if and only if (2.4) holds for every  $C$ -colocal equivalence of the form  $c_X: CX \rightarrow X$ .

Let  $\mathcal{E}$  be a closed symmetric monoidal category and consider the strict monoidal functor  $i: \mathcal{S}\text{ets} \rightarrow \mathcal{E}$  defined by  $i(A) = \coprod_{a \in A} I$ , where  $\mathcal{S}\text{ets}$  is closed monoidal with the cartesian product. This functor has a right adjoint  $r$ , namely  $r(X) = \mathcal{E}(I, X)$ . In this case, the enriched orthogonality condition (2.3) for an  $(i, r)$ -localization reduces to condition (2.2).

If we view  $\mathcal{E}$  itself as a closed symmetric monoidal  $\mathcal{E}$ -category, just by taking  $i$  and  $r$  to be the identity functors, then an  $(\text{Id}, \text{Id})$ -localization functor in  $\mathcal{E}$  is a *closed localization functor* following the terminology used in [7, Definition 3.3]. In the same way, we define *closed colocalizations* in  $\mathcal{E}$  as  $(\text{Id}, \text{Id})$ -colocalizations in  $\mathcal{E}$ .

Observe that any  $(i, r)$ -localization functor in  $\mathcal{E}$  satisfies orthogonality condition (2.2) for any  $i$  and  $r$ . Indeed, by adjointness

$$\mathcal{V}(I_{\mathcal{V}}, \text{Hom}(X, Y)) = \mathcal{V}(I_{\mathcal{V}}, r \text{Hom}_{\mathcal{E}}(X, Y)) \cong \mathcal{E}(I_{\mathcal{E}}, \text{Hom}_{\mathcal{E}}(X, Y)) \cong \mathcal{E}(X, Y),$$

and thus, by applying the functor  $\mathcal{V}(I, -)$  to (2.3), we get a bijection

$$\mathcal{E}(Y, Z) \longrightarrow \mathcal{E}(X, Z).$$

Similarly, one gets that any  $(i, r)$ -colocalization functor satisfies the corresponding orthogonality condition at the level of sets of morphisms, for any  $i$  and  $r$ .

Observe that the class of  $L$ -local objects and the class of  $C$ -colocal objects are closed under retracts. The following lemma gathers some closure properties of local and colocal objects and equivalences with respect to the tensor product and the internal hom.

**Lemma 2.3.** *Let  $(L, l)$  be an  $(i, r)$ -localization functor and  $(C, c)$  be an  $(i, r)$ -colocalization functor in a closed symmetric monoidal  $\mathcal{V}$ -category  $\mathcal{E}$ . Let  $A, B$  in  $\mathcal{V}$  and  $X, Y$  in  $\mathcal{E}$ .*

- (i) *If  $h$  is an  $L$ -local equivalence, then so is the tensor product  $i(A) \otimes h$ . If  $h$  is a  $C$ -colocal equivalence, then so is  $\text{Hom}_{\mathcal{E}}(i(A), h)$ .*
- (ii) *If  $f: i(A) \rightarrow Y$  and  $g: X \rightarrow i(B)$  are  $L$ -local equivalences, then so is the tensor product  $f \otimes g$ .*

- (iii) If  $X$  is  $L$ -local, then so is  $\mathrm{Hom}_{\mathcal{E}}(i(A), X)$ . If  $X$  is  $C$ -colocal, then so is the tensor product  $i(A) \otimes X$ .

*Proof.* We only give proofs for the statements for the case of localizations. The dual case is proved by using similar arguments. By the enriched Yoneda lemma, to prove (i) it is enough to check that

$$\mathcal{V}(W, r(\mathrm{Hom}_{\mathcal{E}}(i(A) \otimes X, Z))) \longrightarrow \mathcal{V}(W, r(\mathrm{Hom}_{\mathcal{E}}(i(A) \otimes Y, Z)))$$

is an isomorphism for all  $L$ -local objects  $Z$  in  $\mathcal{E}$  and all  $W$  in  $\mathcal{V}$ . By adjointness and the fact that  $i$  is strong monoidal we have that

$$\begin{aligned} \mathcal{V}(W, r(\mathrm{Hom}_{\mathcal{E}}(i(A) \otimes X, Z))) &\cong \mathcal{E}(i(W) \otimes i(A) \otimes X, Z) \\ &\cong \mathcal{E}(i(W \otimes A), \mathrm{Hom}_{\mathcal{E}}(X, Z)) \cong \mathcal{V}(W \otimes A, r(\mathrm{Hom}_{\mathcal{E}}(X, Z))) \\ &\cong \mathcal{V}(W \otimes A, r(\mathrm{Hom}_{\mathcal{E}}(Y, Z))) \cong \mathcal{V}(W, r(\mathrm{Hom}_{\mathcal{E}}(i(A) \otimes Y, Z))). \end{aligned}$$

Part (ii) follows from (i) since  $f \otimes g = (f \otimes i(B)) \circ (i(A) \otimes g)$  and the composition of  $L$ -equivalences is an  $L$ -equivalence.

To prove (iii), let  $h: W \rightarrow Z$  be any  $L$ -equivalence. Then

$$\begin{aligned} \mathrm{Hom}(Z, \mathrm{Hom}_{\mathcal{E}}(i(A), X)) &\cong r(\mathrm{Hom}_{\mathcal{E}}(Z, \mathrm{Hom}_{\mathcal{E}}(i(A), X))) \cong r(\mathrm{Hom}_{\mathcal{E}}(Z \otimes i(A), X)) \\ &\cong r(\mathrm{Hom}_{\mathcal{E}}(W \otimes i(A), X)) \cong \mathrm{Hom}(W, \mathrm{Hom}_{\mathcal{E}}(i(A), X)), \end{aligned}$$

where the third isomorphism follows since  $h \otimes i(A)$  is an  $L$ -equivalence by (i).  $\square$

**Corollary 2.4.** *If  $(L, l)$  is a closed localization in  $\mathcal{E}$ , then the tensor product of an  $L$ -local equivalence with any object is an  $L$ -local equivalence, and if  $Z$  is  $L$ -local, then  $\mathrm{Hom}_{\mathcal{E}}(W, Z)$  is  $L$ -local for any  $W$ . Dually, if  $(C, c)$  is a closed colocalization on  $\mathcal{E}$ , then the tensor product of a  $C$ -colocal object with any object is again  $C$ -colocal, and if  $h$  is a  $C$ -colocal equivalence, then so is  $\mathrm{Hom}_{\mathcal{E}}(W, h)$  for any  $W$ .  $\square$*

**Definition 2.5.** Let  $\mathcal{E}$  be a closed symmetric monoidal category. A localization functor  $(L, l)$  in  $\mathcal{E}$  is called *smashing*, if there is an object  $A$  such that  $LX = X \otimes A$  for every  $X$ . A colocalization functor  $(C, c)$  is called *mapping* if there is an object  $A$  in  $\mathcal{E}$  such that  $CX = \mathrm{Hom}_{\mathcal{E}}(A, X)$  for every  $X$ .

Note that if  $(L, l)$  is smashing, then  $(L, l)$  is canonically isomorphic to  $(L, \mathrm{Id} \otimes l_I)$ , that is, there is a canonical natural isomorphism  $\phi: L \rightarrow L$  such that  $\phi \circ l = \mathrm{Id} \otimes l_I$ . Moreover,  $A \cong LI$  and  $X \otimes LY \cong L(X \otimes Y)$  for all  $X$  and  $Y$  in  $\mathcal{E}$ . Similarly, if  $C$  is mapping, then  $C(\mathrm{Hom}_{\mathcal{E}}(X, Y)) \cong \mathrm{Hom}_{\mathcal{E}}(X, CY)$ .

**Proposition 2.6.** *Let  $(L, l)$  be a smashing localization functor and let  $(C, c)$  be a mapping colocalization functor in  $\mathcal{E}$ . Then, for every closed symmetric monoidal  $\mathcal{V}$ -structure  $(i, r)$  on  $\mathcal{E}$ , we have the following:*

- (i) *The localization functor  $(L, l)$  is an  $(i, r)$ -localization.*
- (ii) *The colocalization functor  $(C, c)$  is an  $(i, r)$ -colocalization.*

*Proof.* To prove (i) we need to check that the induced map

$$l_X^*: \mathrm{Hom}(LX, LY) \longrightarrow \mathrm{Hom}(X, LY)$$

is an isomorphism in  $\mathcal{V}$  for all  $X$  and  $Y$ . By the enriched Yoneda lemma it is enough to prove that

$$\mathcal{V}(W, \mathrm{Hom}(LX, LY)) \longrightarrow \mathcal{V}(W, \mathrm{Hom}(X, LY))$$

is an isomorphism for all  $W$  in  $\mathcal{V}$ . This follows directly, since by adjointness

$$\begin{aligned} \mathcal{V}(W, \text{Hom}(LX, LY)) &\cong \mathcal{E}(i(W), \text{Hom}_{\mathcal{E}}(X \otimes A, LY)) \\ &\cong \mathcal{E}(i(W) \otimes X \otimes A, LY) \cong \mathcal{E}(L(i(W) \otimes X), LY) \end{aligned}$$

and using the fact that  $\mathcal{E}(L(i(W) \otimes X), LY) \cong \mathcal{E}(i(W) \otimes X, LY)$ . Part (ii) is proved by a similar argument.  $\square$

We recall now an important source of examples of localization functors and colocalization functors, namely localization with respect to morphisms and colocalization with respect to objects.

**2.1. Localization with respect to morphisms.** Let  $\mathcal{E}$  be a closed symmetric monoidal  $\mathcal{V}$ -category with adjunction  $(i, r)$  and let  $\mathcal{L}$  be a class of morphisms in  $\mathcal{E}$ . Recall that  $\text{Hom}(-, -) = r \text{Hom}_{\mathcal{E}}(-, -)$ .

- (i) An object  $Z$  in  $\mathcal{E}$  is  $\mathcal{L}$ -local if for every  $f: X \rightarrow Y$  in  $\mathcal{L}$  the induced map

$$f^*: \text{Hom}(Y, Z) \longrightarrow \text{Hom}(X, Z)$$

is an isomorphism in  $\mathcal{V}$ .

- (ii) A morphism  $g: U \rightarrow W$  is called an  $\mathcal{L}$ -local equivalence if the induced map

$$g^*: \text{Hom}(W, Z) \longrightarrow \text{Hom}(U, Z)$$

is an isomorphism in  $\mathcal{V}$  for every  $\mathcal{L}$ -local object  $Z$ .

**Definition 2.7.** An  $(i, r)$ - $\mathcal{L}$ -localization functor in  $\mathcal{E}$  is an  $(i, r)$ -localization functor  $(L, l)$  such that the class of  $L$ -local objects coincides with the class of  $\mathcal{L}$ -local objects and the class of  $L$ -local equivalences coincides with the class of  $\mathcal{L}$ -local equivalences. We denote this localization functor by  $L_{\mathcal{L}}$ .

Now, consider  $\mathcal{E}$  as an  $\mathcal{E}$ -category and let  $E$  be any object in  $\mathcal{E}$ .

- (i) A morphism  $f$  in  $\mathcal{E}$  is called an  $E$ -equivalence if  $E \otimes f$  is an isomorphism  
(ii) An object  $Z$  in  $\mathcal{E}$  is called  $E$ -local if for every  $E$ -equivalence  $f: X \rightarrow Y$ , the induced map

$$f^*: \text{Hom}_{\mathcal{E}}(Y, Z) \longrightarrow \text{Hom}_{\mathcal{E}}(X, Z)$$

is an isomorphism in  $\mathcal{E}$ .

An  $E$ -localization functor is an  $\mathcal{L}$ -localization functor, where the class of  $\mathcal{L}$ -local equivalences equals the class of  $E$ -equivalences. We will denote this localization functor by  $L_E$ . Every smashing localization  $L$  is of this type, namely  $L = L_{LI}$ .

**2.2. Colocalization with respect to objects.** Now, let  $\mathcal{K}$  be a class of objects of  $\mathcal{E}$ .

- (i) A morphism  $g: U \rightarrow W$  is called a  $\mathcal{K}$ -colocal equivalence if the induced map

$$g_*: \text{Hom}(Z, U) \longrightarrow \text{Hom}(Z, W)$$

is an isomorphism in  $\mathcal{V}$  for every  $Z$  in  $\mathcal{K}$ .

- (ii) An object  $Z$  in  $\mathcal{E}$  is  $\mathcal{K}$ -colocal if for every  $\mathcal{K}$ -colocal equivalence  $f: X \rightarrow Y$  the induced map

$$f_*: \text{Hom}(Z, X) \longrightarrow \text{Hom}(Z, Y)$$

is an isomorphism in  $\mathcal{V}$ .

**Definition 2.8.** An  $(i, r)$ - $\mathcal{K}$ -colocalization functor in  $\mathcal{E}$  is an  $(i, r)$ -colocalization functor  $(C, c)$  such that the class of  $C$ -colocal objects coincides with the class of  $\mathcal{K}$ -local objects and the class of  $C$ -colocal equivalences coincides with the class of  $\mathcal{K}$ -colocal equivalences. We denote this colocalization functor by  $C_{\mathcal{K}}$ .

### 3. SOLID MONOIDS AND RIGID MONOIDS

In this section we define solid monoid and rigid monoid in (enriched) monoidal categories as a generalization of the algebraic notion of solid ring [3, 5, 9] and rigid ring [8, 16]. These special classes of monoids are closely related to localization and colocalization functors. In fact, as we will show, they all appear as suitable localizations of the unit of the monoidal structure.

Throughout this section, we will implicitly assume that all localization and colocalization functors with respect to morphisms and objects that are mentioned exist.

Let  $\mathcal{E}$  be a closed symmetric monoidal category. Recall that a *monoid*  $(R, \mu, \eta)$  in  $\mathcal{E}$  is an object  $R$  equipped with two morphisms  $\mu: R \otimes R \rightarrow R$  and  $\eta: I \rightarrow R$  such that the following diagrams commute:

$$(3.1) \quad \begin{array}{ccc} R \otimes R \otimes R & \xrightarrow{1 \otimes \mu} & R \otimes R \\ \mu \otimes 1 \downarrow & & \downarrow \mu \\ R \otimes R & \xrightarrow{\mu} & R \end{array} \quad \begin{array}{ccc} I \otimes R & \xrightarrow{\eta \otimes 1} & R \otimes R \xleftarrow{1 \otimes \eta} R \otimes I \\ & \searrow \lambda & \downarrow \mu \\ & & R, \end{array} \quad \begin{array}{c} \nearrow \rho \end{array}$$

where  $\lambda$  and  $\rho$  are the natural isomorphisms coming from the fact that  $I$  is a left and right identity for the tensor product. A *map of monoids* between  $(R, \mu, \eta)$  and  $(R', \mu', \eta')$  is a map  $f: R \rightarrow R'$  compatible with the structure maps  $\mu, \eta, \mu'$  and  $\eta'$ , that is, such that  $f \circ \mu = \mu' \circ (f \otimes f)$  and  $f \circ \eta = \eta'$ .

**Definition 3.1.** A monoid  $(R, \mu, \eta)$  in  $\mathcal{E}$  is called a *solid monoid* if the multiplication map  $\mu$  is an isomorphism. A *solid comonoid* is a solid monoid in the opposite category  $\mathcal{E}^{\text{op}}$ .

Note that if  $R$  is a solid monoid, then by the commutativity of the second diagram in (3.1), the morphisms  $\eta \otimes 1$  and  $1 \otimes \eta$  are isomorphisms. In fact, this property characterizes solid monoids.

**Proposition 3.2.** *An object  $R$  in  $\mathcal{E}$  is a solid monoid if and only if there exist a morphism  $\eta: I \rightarrow R$  such that both  $\eta \otimes 1$  and  $1 \otimes \eta$  are isomorphisms.*

*Proof.* One implication is clear from the previous remark. For the converse, assume that we have an object  $R$  in  $\mathcal{E}$  and a morphism  $\eta: I \rightarrow R$  such that  $\eta \otimes 1$  and  $1 \otimes \eta$  are isomorphisms. First note that if  $f: R \rightarrow R$  is any map such that  $f \circ \eta = \eta$ , then  $f = 1$ . Indeed, we have a commutative diagram

$$\begin{array}{ccc} R \otimes I & \xrightarrow{f \otimes 1_I} & R \otimes I \\ 1 \otimes \eta \downarrow & & \downarrow 1 \otimes \eta \\ R \otimes R & \xrightarrow{f \otimes 1} & R \otimes R, \end{array}$$

where the vertical arrows are isomorphisms. Therefore

$$f \otimes 1_I = (1 \otimes \eta)^{-1} \circ (f \otimes 1) \circ (1 \otimes \eta).$$

But if  $f \circ \eta = \eta$ , then  $(f \otimes 1) \circ (\eta \otimes 1) = \eta \otimes 1$ . Since  $\eta \otimes 1$  is an isomorphism, this forces  $f \otimes 1 = 1 \otimes 1$ , and thus  $f \otimes 1_I = 1 \otimes 1_I = 1$ . Consider now the following isomorphism  $g: R \rightarrow R$  defined as the composite

$$R \xrightarrow{\lambda^{-1}} I \otimes R \xrightarrow{\eta \otimes 1} R \otimes R \xrightarrow{(1 \otimes \eta)^{-1}} R \otimes I \xrightarrow{\rho} R.$$

Since  $((\eta \otimes 1) \circ \lambda^{-1}) \circ \eta = (\eta \otimes \eta) \circ \lambda^{-1} = (\eta \otimes \eta) \circ \rho^{-1} = ((1 \otimes \eta) \circ \rho^{-1}) \circ \eta$ , we have that  $g \circ \eta = \eta$  and so  $g = 1$ . This implies that the following diagram of isomorphisms commute

$$\begin{array}{ccccc} I \otimes R & \xrightarrow{\eta \otimes 1} & R \otimes R & \xleftarrow{1 \otimes \eta} & R \otimes I \\ & \searrow \lambda^{-1} & R & \nearrow \rho^{-1} & \\ & & & & \end{array}$$

Hence, there exists a unique isomorphism  $\mu: R \otimes R \rightarrow R$  rendering the two triangles commutative. It is straightforward to check that  $(R, \mu, \eta)$  is a solid monoid.  $\square$

For any category  $\mathcal{C}$ , the category of endofunctors  $\text{Fun}(\mathcal{C}, \mathcal{C})$  admits a monoidal structure, where the tensor product is given by composition and the unit is the identity functor. If  $\mathcal{E}$  is a closed symmetric monoidal category, then we can define a functor  $F: \mathcal{E} \rightarrow \text{Fun}(\mathcal{E}, \mathcal{E})$  by setting  $F(X)(-) = - \otimes X$  and another functor  $G: \mathcal{E}^{\text{op}} \rightarrow \text{Fun}(\mathcal{E}, \mathcal{E})$  by setting  $G(X)(-) = \text{Hom}_{\mathcal{E}}(X, -)$ . One can check that both functors are faithful and reflect isomorphisms.

Moreover,  $F$  preserves solid monoids and the functor  $G$  sends solid monoids to solid comonoids. Indeed, if  $(R, \mu, \eta)$  is a solid monoid, then  $(F(R), \text{Id} \otimes \eta)$  is a localization functor in  $\mathcal{E}$ , that is, a solid monoid in  $\text{Fun}(\mathcal{E}, \mathcal{E})$ . In fact,  $F(R) = L_R$  is the  $R$ -localization defined in Section 2.1, since both functors have the same class of local equivalences. Similarly,  $(G(R), \text{Hom}_{\mathcal{E}}(\eta, \text{Id}))$  is a colocalization functor in  $\mathcal{E}$ , that is, a solid comonoid in  $\text{Fun}(\mathcal{E}, \mathcal{E})$ . The functor  $G(R)$  is precisely the colocalization functor  $C_R$  of Section 2.2.

**Theorem 3.3.** *Let  $\mathcal{E}$  be a closed symmetric monoidal category. Then, there is a one to one correspondence between the following classes:*

- (i) *Solid monoids.*
- (ii) *Smashing localization functors.*
- (iii) *Mapping colocalization functors.*

*Proof.* Let  $(R, \mu, \eta)$  be a solid monoid in  $\mathcal{E}$ . Then the functor  $(F(R), \text{Id} \otimes \eta)$  defined above is a localization functor in  $\mathcal{E}$  that is also smashing and  $F(R)(I) = R$ . Conversely, let  $(L, l)$  be a smashing localization functor. Then, the morphisms

$$\begin{aligned} LI &\cong I \otimes LI \xrightarrow{l_I \otimes 1} LI \otimes LI = L(LI), \\ LI &\cong LI \otimes I \xrightarrow{1 \otimes l_I} LI \otimes LI = L(LI) \end{aligned}$$

are isomorphisms. Thus, by Proposition 3.2  $LI$  is a solid monoid.

Similarly, the augmented functor  $(G(R), \text{Hom}_{\mathcal{E}}(\eta, \text{Id}))$  is a mapping colocalization functor and  $G(R)(I) = \text{Hom}_{\mathcal{E}}(R, I)$ . And conversely, if  $(C, c)$  is a mapping colocalization functor, say  $CX = \text{Hom}_{\mathcal{E}}(A, X)$ , then  $c$  gives natural transformation

$$\text{Hom}_{\mathcal{E}}(A, -) = C \longrightarrow \text{Id} = \text{Hom}_{\mathcal{E}}(I, -).$$

By the Yoneda lemma this corresponds to a morphism  $\eta: I \rightarrow A$ . But this morphism satisfies that  $\eta \otimes 1$  and  $1 \otimes \eta$  are isomorphisms, by using the Yoneda lemma and the fact that  $CCX \cong CX$ . Again, by Proposition 3.2 we infer that  $A$  is a solid monoid.  $\square$

*Remark 3.4.* The previous result tells us that for a monoid  $(R, \mu, \eta)$  in  $\mathcal{E}$  the following assertions are equivalent:

- (i)  $R$  is solid.
- (ii) The functor  $(L, l)$  defined by  $LX = X \otimes R$  and  $l = \text{Id} \otimes \eta$  is a localization functor and  $R \cong LI$ .
- (iii) The functor  $(C, c)$  defined by  $CX = \text{Hom}_{\mathcal{E}}(R, X)$  and  $c = \text{Hom}_{\mathcal{E}}(\eta, \text{Id})$  is a colocalization functor.

**Proposition 3.5.** *Let  $\mathcal{E}$  be a closed symmetric monoidal category. If  $L$  is a smashing localization functor in  $\mathcal{E}$ , then  $L$  is equivalent to the closed localization functor  $L_{l_I}$ , where  $l_I: I \rightarrow LI$  denotes the localization of the unit.*

*Proof.* By Theorem 3.3 and Remark 3.4, if  $L$  is smashing, then  $L = L_R$ , where  $R \cong LI$ . The morphism  $l_I$  is an  $R$ -equivalence, since  $R$  is a solid monoid. Thus, every  $L$ -local equivalence is an  $R$ -equivalence. Conversely, let  $f: X \rightarrow Y$  be an  $R$ -equivalence,  $Z$  any  $L$ -local object and consider the following commutative diagram

$$\begin{array}{ccc} \text{Hom}_{\mathcal{E}}(Y, Z) & \longrightarrow & \text{Hom}_{\mathcal{E}}(X, Z) \\ \uparrow & & \uparrow \\ \text{Hom}_{\mathcal{E}}(Y \otimes LI, Z) & \longrightarrow & \text{Hom}_{\mathcal{E}}(X \otimes LI, Z) \end{array}$$

The vertical arrows are isomorphisms since tensoring  $l_I$  with any object is an  $l_I$ -local equivalence by Corollary 2.4. The bottom arrow is also an isomorphism since  $X \rightarrow Y$  is an  $R$ -equivalence, hence  $X \otimes LI \rightarrow Y \otimes LI$  is an isomorphism. Therefore, the top arrow is also an isomorphism.  $\square$

**3.1. Modules over solid monoids.** Recall that if  $(R, \mu, \eta)$  is a monoid in a closed symmetric monoidal category  $\mathcal{E}$ , then an  $R$ -module consists of an object  $X$  together with a morphism  $m: R \otimes X \rightarrow X$  such that the following diagrams commute:

$$\begin{array}{ccc} R \otimes R \otimes X & \xrightarrow{\mu \otimes 1} & R \otimes X \\ 1 \otimes m \downarrow & & \downarrow m \\ R \otimes X & \xrightarrow{m} & X \end{array} \quad \begin{array}{ccc} I \otimes X & \xrightarrow{\eta \otimes 1} & R \otimes X & \xleftarrow{1 \otimes \eta} & X \otimes I \\ & \searrow \lambda & \downarrow m & \swarrow \rho & \\ & & X & & \end{array}$$

Morphisms of  $R$ -modules are those compatible with the module structure. We will denote by  $R\text{-mod}$  the category of  $R$ -modules.

**Theorem 3.6.** *Let  $\mathcal{E}$  be a closed monoidal category and let  $(R, \mu, \eta)$  be a solid monoid in  $\mathcal{E}$ . We denote by  $L_R\text{-loc}$  the full subcategory of  $L_R$ -local objects and by  $C_R\text{-coloc}$  the full subcategory of  $C_R$ -colocal objects. Then there is an equivalence of categories*

$$L_R\text{-loc} \cong R\text{-mod} \cong C_R\text{-coloc}.$$

*Proof.* If  $X$  is an  $R$ -module, then  $X$  is a retract of  $X \otimes R$  and also a retract of  $\text{Hom}_{\mathcal{E}}(R, X)$ . But, by Remark 3.4,  $LX = X \otimes R$  and  $CX = \text{Hom}_{\mathcal{E}}(R, X)$  define a localization and a colocalization functor, namely  $L_R$  and  $C_R$ , respectively. Thus,  $X$  is a retract of an  $L_R$ -local object and also a retract of a  $C_R$ -colocal object. Since local and colocal objects are closed under retracts, the natural maps

$$X \cong X \otimes I \xrightarrow{1 \otimes \eta} X \otimes R \quad \text{and} \quad \text{Hom}_{\mathcal{E}}(R, X) \xrightarrow{\eta^*} \text{Hom}_{\mathcal{E}}(I, X) \cong X$$

are isomorphisms.  $\square$

Note that this theorem together with Proposition 3.5 imply that if  $(R, \mu, \eta)$  is a solid monoid, then the categories  $R$ -mod and the full subcategory of  $\eta$ -local objects are also equivalent, where  $\eta: I \rightarrow R$  is the unit of the ring.

**3.2. Rigid monoids in enriched categories.** Let  $\mathcal{E}$  be a closed symmetric monoidal  $\mathcal{V}$ -category, with associated adjunction  $i: \mathcal{V} \rightleftarrows \mathcal{E}: r$ .

**Definition 3.7.** A monoid  $(R, \mu, \eta)$  in  $\mathcal{E}$  is called an  $(i, r)$ -rigid monoid if the induced morphism

$$\eta^*: \text{Hom}(R, R) \longrightarrow \text{Hom}(I, R)$$

is an isomorphism in  $\mathcal{V}$ .

Observe that if  $\mathcal{E}$  is a  $\mathcal{V}$ -category, then solid monoids in  $\mathcal{E}$  are defined using the tensor product in  $\mathcal{E}$ , while rigid monoids in  $\mathcal{E}$  are defined in terms of the  $\mathcal{V}$ -enrichment, which depends on the adjunction  $(i, r)$ . However, as we will prove, the class of solid monoids is contained in the class of rigid monoids.

**Proposition 3.8.** *Let  $\mathcal{E}$  be a closed symmetric monoidal  $\mathcal{V}$ -category, with associated adjunction  $i: \mathcal{V} \rightleftarrows \mathcal{E}: r$ . If  $(R, \mu, \eta)$  is a solid monoid in  $\mathcal{E}$ , then  $R$  is an  $(i, r)$ -rigid monoid.*

*Proof.* By Theorem 3.3 the functor  $LX = X \otimes R$  is a smashing localization functor in  $\mathcal{E}$ . Now, Proposition 2.6 implies that  $L$  is an  $(i, r)$ -localization functor. Hence, for every  $L$ -local  $Z$  we have an isomorphism

$$\eta_I^*: \text{Hom}(LI, Z) \longrightarrow \text{Hom}(I, Z).$$

In particular, taking  $Z = LI \cong R$  we obtain that  $R$  is an  $(i, r)$ -rigid monoid.  $\square$

**Proposition 3.9.** *Let  $(L, \eta)$  be an  $(i, r)$ -localization functor in a closed symmetric monoidal  $\mathcal{V}$ -category  $\mathcal{E}$ . If  $(i(A), \mu, \eta)$  is a (commutative) monoid in  $\mathcal{E}$  and  $Li(A) \cong i(B)$  for some  $A$  and  $B$  in  $\mathcal{V}$ , then  $Li(A)$  admits a unique (commutative) monoid structure such that the localization map  $\eta_{i(A)}: i(A) \rightarrow Li(A)$  is a morphism of monoids.*

*Proof.* The unit map  $\bar{\eta}$  of  $Li(A)$  is  $\eta_{i(A)} \circ \eta$ . The product map  $\bar{\mu}$  is defined by using the universal property of the localization and adjointness. Indeed, we have natural bijections

$$\begin{aligned} \mathcal{E}(i(A) \otimes i(A), Li(A)) &\cong \mathcal{E}(i(A), \text{Hom}_{\mathcal{E}}(i(A), Li(A))) \\ &\cong \mathcal{V}(A, r \text{Hom}_{\mathcal{E}}(i(A), Li(A))) \cong \mathcal{V}(A, r \text{Hom}_{\mathcal{E}}(Li(A), Li(A))) \\ &\cong \mathcal{E}(i(A) \otimes i(B), Li(A)) \cong \mathcal{V}(B, r \text{Hom}_{\mathcal{E}}(i(A), Li(A))) \\ &\cong \mathcal{E}(i(B), \text{Hom}_{\mathcal{E}}(Li(A), Li(A))) \cong \mathcal{E}(Li(A) \otimes Li(A), Li(A)). \end{aligned}$$

Hence, the product  $\mu$  extends to a unique map  $\bar{\mu}: Li(A) \otimes Li(A) \rightarrow Li(A)$  rendering commutative the diagram

$$\begin{array}{ccc} i(A) \otimes i(A) & \xrightarrow{\mu} & i(A) \\ \eta_{i(A)} \otimes \eta_{i(A)} \downarrow & & \downarrow \eta_{i(A)} \\ Li(A) \otimes Li(A) & \xrightarrow{\bar{\mu}} & Li(A). \end{array}$$

The associativity of  $\bar{\mu}$  and its compatibility with  $\bar{\eta}$  follows from the commutativity of the diagrams for  $\mu$  and  $\eta$  and the universal property of  $L$  (using Lemma 2.3).

In the same way one can prove that  $Li(A)$  is commutative when  $i(A)$  is commutative.  $\square$

**Theorem 3.10.** *Let  $\mathcal{E}$  be a closed monoidal  $\mathcal{V}$ -category with associated adjunction  $(i, r)$  and let  $(L, \eta)$  be an  $(i, r)$ -localization functor in  $\mathcal{E}$ . If  $LI \cong i(B)$  for some  $B$  in  $\mathcal{V}$ , then  $LI$  is an  $(i, r)$ -rigid monoid in  $\mathcal{E}$ . In fact, all  $(i, r)$ -rigid monoids appear as  $LI$ , for some  $(i, r)$ -localization functor  $L$  in  $\mathcal{E}$ .*

*Proof.* Since  $i(I_{\mathcal{V}}) = I$  and  $LI \cong i(B)$  by assumption, we may apply Proposition 3.9 to infer that  $LI$  is again a monoid in  $\mathcal{E}$ . Moreover,  $LI$  is  $L$ -local and  $\eta_I: I \rightarrow LI$  is an  $L$ -equivalence. Thus

$$\mathrm{Hom}(LI, LI) \longrightarrow \mathrm{Hom}(I, LI)$$

is an isomorphism, and hence  $LI$  is an  $(i, r)$ -rigid monoid.

Conversely, suppose that  $(R, \mu, \eta)$  is an  $(i, r)$ -rigid monoid and let  $\eta: I \rightarrow R$  be its unit. Then the  $(i, r)$ - $\eta$ -localization (see Section 2.1) satisfies  $L_{\eta}I \cong R$ . Indeed,  $\eta$  is an  $\eta$ -equivalence and  $R$  is  $\eta$ -local, since

$$\eta^*: \mathrm{Hom}(R, R) \longrightarrow \mathrm{Hom}(I, R)$$

is an isomorphism, because  $R$  is an  $(i, r)$ -rigid monoid.  $\square$

**Corollary 3.11.** *All  $(i, r)$ -rigid monoids of the form  $i(B)$  for some  $B$  in  $\mathcal{V}$  are commutative.*

*Proof.* If  $i(B)$  is  $(i, r)$ -rigid, then we know by Theorem 3.10 that  $i(B) = LI$  for some  $(i, r)$ -localization functor  $L$ . But now, Proposition 3.9 implies that  $LI$  is a commutative monoid since  $I = i(I_{\mathcal{V}})$ .  $\square$

#### 4. SOLID RING SPECTRA AND RIGID RING SPECTRA

We will apply now the results of the previous section to the stable homotopy category of spectra  $\mathcal{S}p$ ; see [1, 13]. This is a triangulated category equipped with a compatible closed symmetric monoidal structure. We denote by  $\wedge$  the smash product, by  $S$  the sphere spectrum, by  $\Sigma$  the suspension operator, and by  $F(-, -)$  the internal function spectrum. We write  $[X, Y]$  for the abelian group of morphisms between two spectra  $X$  and  $Y$  in  $\mathcal{S}p$  and we say that a spectrum  $X$  is *connective* if  $\pi_k(X) = [\Sigma^k S, X] = 0$  for  $k < 0$ . We denote the full subcategory of connective spectra by  $\mathrm{conn}(\mathcal{S}p)$ . There is a *connective cover functor*  $(-)^c$  that assigns to every spectrum  $X$  a connective spectrum  $X^c$  and a natural map

$$c_X: X^c \longrightarrow X$$

such that  $\pi_k(c_X)$  is an isomorphism for all  $k \geq 0$ .

We will be interested in defining localization and colocalization functors in  $\mathcal{S}p$  coming from the following two situations (see Definition 2.2 for notation):

- (I)  $\mathcal{V} = \mathcal{E} = \mathcal{S}p$  and  $i = r = \text{Id}$ . In this case,  $\text{Hom}(-, -) = F(-, -)$
- (II)  $\mathcal{V} = \text{conn}(\mathcal{S}p)$ ,  $\mathcal{E} = \mathcal{S}p$ ,  $i$  is the inclusion and  $r$  is the connective cover functor. In this case,  $\text{Hom}(-, -) = F^c(-, -)$  is the connective cover of the function spectrum.

We will refer to the localizations and colocalizations in (I) as *stable*. (These were called *closed* in Section 2.) This terminology reflects the fact that the localizations and colocalizations coming from (I) always commute with the suspension operator—that is, they are exact or triangulated functors in  $\mathcal{S}p$ —while the ones coming from (II) do not necessarily have this property.

Examples of stable localizations are given by *homological localizations*; see [4]. Using the notation of Section 2.1, these correspond to the  $E$ -localizations functors. Given a spectrum  $E$ , a homological localization functor with respect to  $E$  is a localization functor  $L_E$  on  $\mathcal{S}p$  that turns  $E$ -homology equivalences into homotopy equivalences in a universal way. Recall that each spectrum  $E$  gives rise to a homology theory defined as  $E_k(X) = \pi_k(E \wedge X)$  for every spectrum  $X$  and every  $k \in \mathbb{Z}$ . A map of spectra  $f: X \rightarrow Y$  is an  $E$ -equivalence if the induced map

$$f_*: E_k(X) \longrightarrow E_k(Y)$$

is an isomorphism for all  $k \in \mathbb{Z}$ . A spectrum  $Z$  is  $E$ -local if each  $E$ -equivalence  $X \rightarrow Y$  induces a homotopy equivalence  $[Y, Z] \cong [X, Z]$ . Given a homological localization functor  $L_E$ , there is an associated stable colocalization functor  $A_E$  constructed by taking the fiber of the localization map. Thus, for every spectrum  $X$  we have an exact triangle

$$A_E X \longrightarrow X \longrightarrow L_E X \longrightarrow \Sigma A_E X.$$

The functor  $A_E$  is called the  $E_*$ -acyclization functor in [4]. Miller's *finite localizations* [14] are smashing, and therefore homological localizations. Other smashing localizations include localizations at sets of primes, and homological localization with respect to the spectrum  $K$  of (complex)  $K$ -theory or the Johnson–Wilson spectrum  $E(n)$  for any  $n$ .

As examples of localizations and colocalizations of type (II), we have  $k$ th *Postnikov sections*  $P_{\Sigma^k S}$  and  $k$ th *connective covers*  $\text{Cell}_{\Sigma^{k+1} S}$ . In the notation of Sections 2.1 and 2.2 (with  $i$  the inclusion and  $r$  the connective cover), they correspond to the functors  $P_{\Sigma^k S} = L_{\Sigma^k S \rightarrow 0}$  and  $\text{Cell}_{\Sigma^{k+1} S} = C_{\Sigma^{k+1} S}$ , respectively. For any spectrum  $X$  we have that

$$\pi_n(\text{Cell}_{\Sigma^{k+1} S} X) = \begin{cases} 0 & \text{if } n \leq k, \\ \pi_n(X) & \text{if } n > k, \end{cases} \quad \pi_n(P_{\Sigma^k S} X) \cong \begin{cases} 0 & \text{if } n \geq k, \\ \pi_n(X) & \text{if } n < k. \end{cases}$$

Neither  $\text{Cell}_{\Sigma^{k+1} S}$  nor  $P_{\Sigma^k S}$  commute with suspension. Note that the connective cover functor  $(-)^c$  is precisely  $\text{Cell}_S$ . More generally,  $f$ -localizations [6] and  $E$ -cellularizations [12] in  $\mathcal{S}p$  are also functors of type (II).

A ring spectrum is *solid* if the multiplication map is a homotopy equivalence, and a ring spectrum  $R$  is called a *rigid ring spectrum* if it is an  $(i, r)$ -rigid monoid with  $i$  and  $r$  as in (II), that is, if the connective cover of the evaluation map  $F^c(R, R) \rightarrow R^c$  is a homotopy equivalence. A ring spectrum is a *stable rigid ring spectrum* if it is an  $(i, r)$ -rigid monoid with  $i = r = \text{Id}$  as in (I), that is, if the evaluation map  $F(R, R) \rightarrow R$  is a homotopy equivalence.

Given an abelian group  $A$ , we denote by  $HA$  its corresponding *Eilenberg–Mac Lane spectrum* and by  $MA$  its corresponding *Moore spectrum*. The former is characterized by the property that  $\pi_k(HA) = A$  if  $k = 0$  and it is zero if  $k \neq 0$ , and the latter is characterized by the property that it is connective,  $(H\mathbb{Z})_k(MA) = 0$  if  $k \neq 0$ , and  $(H\mathbb{Z})_0(MA) = \pi_0(MA) = A$ .

By Proposition 3.8, every solid ring spectrum is a rigid ring spectrum and a stable rigid ring spectrum, but the converse does not hold in general. For instance, the ring spectrum  $H\widehat{\mathbb{Z}}_p$ , where  $\widehat{\mathbb{Z}}_p$  are the  $p$ -adic integers is rigid but not solid (neither stable rigid). If it were solid, then  $H\widehat{\mathbb{Z}}_p \wedge H\widehat{\mathbb{Z}}_p \cong H\widehat{\mathbb{Z}}_p$  and this would imply that  $\widehat{\mathbb{Z}}_p \otimes \widehat{\mathbb{Z}}_p \cong \widehat{\mathbb{Z}}_p$ . However  $F^c(H\widehat{\mathbb{Z}}_p, H\widehat{\mathbb{Z}}_p) \cong H\widehat{\mathbb{Z}}_p$ , since  $[S, H\widehat{\mathbb{Z}}_p] \cong [H\widehat{\mathbb{Z}}_p, H\widehat{\mathbb{Z}}_p]$  and  $[\Sigma^k H\widehat{\mathbb{Z}}_p, H\widehat{\mathbb{Z}}_p] = 0$  for all  $k \geq 1$ .

Applying Theorems 3.3 and 3.10 to the category  $\mathcal{S}p$  readily implies

**Theorem 4.1.** *Let  $L$  be a localization functor in  $\mathcal{S}p$  and let  $S$  be the sphere spectrum. Then we have the following:*

- (i) *If  $L$  is smashing (hence stable and homological), then  $LS$  is a solid ring spectrum, and all solid ring spectra appear as smashing localizations of the sphere spectrum.*
- (ii) *If  $L$  is any localization functor and  $LS$  is connective, then  $LS$  is a rigid ring spectrum and all rigid ring spectra appear as localizations of the sphere spectrum.*
- (iii) *If  $L$  is a stable localization functor, then  $LS$  is a stable rigid ring spectrum and all stable rigid ring spectra appear as stable localizations of the sphere spectrum.  $\square$*

More explicitly, if  $R$  is a solid ring spectrum, then  $R \cong L_R S$ ; and if  $R$  is a rigid ring spectrum, then  $R \cong L_\eta S$ , where  $\eta: S \rightarrow R$  is the unit of the ring spectrum  $R$ . If  $R$  is a stable rigid ring spectrum, then  $R \cong L_{\Sigma^* \eta} S$ , where  $\Sigma^* \eta = \{\Sigma^k \eta \mid k \in \mathbb{Z}\}$ .

For any solid ring spectrum  $R$ , the colocalization functor  $C_R$  is precisely stable  $R$ -cellularization  $\text{Cell}_R$ , and the localization functor  $L_R$  is homological localization with respect to  $R$ . If we denote by  $L_R \mathcal{S}p$  the full subcategory of  $R$ -local spectra and by  $\text{Cell}_R \mathcal{S}p$  the full subcategory of  $R$ -cellular spectra, then Theorem 3.6 gives

**Proposition 4.2.** *If  $R$  is a solid ring spectrum, e.g.,  $R = H\mathbb{Q}$ ,  $L_K S$  or  $L_{E(n)} S$ , then there is an equivalence of categories  $L_R \mathcal{S}p \cong R\text{-mod} \cong \text{Cell}_R \mathcal{S}p$ .  $\square$*

Observe that Proposition 3.5 implies that any of the categories of Proposition 4.2 is also equivalent to the category of  $\eta$ -local objects, where  $\eta: S \rightarrow R$  is the unit of  $R$ . This has the following consequence:

**Corollary 4.3.** *If  $(R, \mu, \eta)$  is a solid ring spectrum, then the  $R$ -homological localization functor  $L_R$  is equivalent to the stable localization  $L_\eta$ .*

*Remark 4.4.* Proposition 4.2 is a particular instance of a more general result in the context of stable model categories [11, Theorem 2.7] and, in fact, the above equivalence is induced by a Quillen equivalence at the level of the corresponding model categories. Also, as proved in [2, Corollary 4.14], Corollary 4.3 also holds at the level of the localized model structures, that is, the left Bousfield localizations  $L_R$  and  $L_\eta$  coincide.

The following result (see [6, Theorems 5.12 and 5.14]) relates  $f$ -localizations of the integral Eilenberg-Mac Lane spectrum  $H\mathbb{Z}$  with algebraic rigid rings (that is, rigid monoids in the category of abelian groups).

**Theorem 4.5.** *Let  $L_f$  be any  $f$ -localization functor in  $\mathcal{S}p$ . Then,  $L_f H\mathbb{Z} \cong HA$  for some rigid ring  $A$  and all (algebraic) rigid rings appear this way. If  $L_f$  is smashing, then  $A$  is a subring of the rationals.  $\square$*

Moreover, the only connective solid ring spectra are Moore spectra of subrings of the rationals.

**Theorem 4.6.** *Let  $L$  be any localization functor in  $\mathcal{S}p$  and assume that  $LS$  is connective. Then  $LS$  is a solid ring spectrum if and only if  $LS \cong MA$ , where  $A$  is a subring of the rationals.*

*Proof.* Suppose that  $LS$  is a solid ring. Then, by Theorem 4.1(i), we have that  $LS \cong L'S$ , where  $L'$  is a smashing localization functor (in fact,  $L' = L_{LS}$ ). Now, it follows from [6, Theorem 5.14] that  $(H\mathbb{Z})_k(L'S) = 0$  if  $k \neq 0$  and  $(H\mathbb{Z})_0(L'S) \cong R$  a subring of the rationals. Thus,  $LS \cong MA$ , since it is connective.

The converse holds since if  $R$  is a subring of the rationals, then the multiplication map  $MA \wedge MA \rightarrow MA$  is an isomorphism.  $\square$

**Corollary 4.7.** *If  $R$  is a connective solid ring spectrum, then  $R \cong MA$  for some subring of the rationals  $A$ .  $\square$*

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RADBOUD UNIVERSITEIT NIJMEGEN, INSTITUTE FOR MATHEMATICS, ASTROPHYSICS, AND PARTICLE PHYSICS, HEYENDAALSEWEG 135, 6525 AJ NIJMEGEN, THE NETHERLANDS

*E-mail address:* `j.gutierrez@math.ru.nl`