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Regular Behaviours with Names

On Rational Fixpoints of Endofunctors on Nominal Sets

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In fond memory of our colleague and mentor Horst Herrlich

Preliminary Version

Abstract Nominal sets provide a framework to study key notions of syntax and semantics such as fresh names, variable binding and α-equivalence on a conveniently abstract categorical level. Coalgebras for endofunctors on nominal sets model, e.g., various forms of automata with names as well as infinite terms with variable binding operators (such as λ-abstraction). Here, we first study the behaviour of orbit-finite coalgebras for functors $\bar{F}$ on nominal sets that lift some finitary set functor F. We provide sufficient conditions under which the rational fixpoint of $\bar{F}$, i.e. the collection of all behaviours of orbit-finite $\bar{F}$-coalgebras, is the lifting of the rational fixpoint of F. Second, we describe the rational fixpoint of the quotient functors: we introduce the notion of a sub-strength of an endofunctor on nominal sets, and we prove that for a functor G with a sub-strength the rational fixpoint of each quotient of G is a canonical quotient of the rational fixpoint of G. As applications, we obtain a concrete description of the rational fixpoint for functors arising from so-called binding signatures with exponentiation, such as those arising in coalgebraic models of infinitary λ-terms and various flavours of automata.

Keywords Nominal sets · final coalgebras · rational fixpoints · lifted functors

1 Introduction

Nominal sets (or sets with atoms) were introduced by Mostowski and Fraenkel in the 1920s and 1930s as a permutation model for set theory. They are sets equipped with an action of the group of finite permutations on a given fixed set $\mathcal{V}$ of atoms (playing the roles of names or variables in applications). Gabbay and Pitts [12] coined the term *nominal sets* for such sets, and use them as a convenient framework for dealing with binding operators, name abstraction and structural induction. The

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notion of support of a nominal set allows one to define the notions of "free" and "bound" names abstractly (we recall this in Section 2.2). For example, in order to deal with variable binding in the λ-calculus one considers the functor

$$L_\alpha X = \mathcal{V} + [\mathcal{V}]X + X \times X$$

on Nom, the category of nominal sets, expressing the type of term constructors (note that the abstraction functor $[\mathcal{V}]X$ is a quotient of $\mathcal{V} \times X$ modulo renaming "bound" variables). Gabbay and Pitts proved that the initial algebra for L_α is formed by all λ-terms modulo α-equivalence. This implies that in lieu of having to deal syntactically with the subtle issues arising in the presence of free and bound variables in inductive definitions on terms, one can simply use initiality as a definition principle.

Recently, Kurz et al. [22] have characterized the final coalgebra for L_α (and more generally, for functors arising from so-called binding signatures): it is carried by the set of all infinitary λ-terms (i.e. finite or infinite λ-trees) with finitely many free variables modulo α-equivalence. This then allows defining operations on infinitary λ-terms by coinduction, for example substitution and operations that assign to an infinitary λ-term its normal form computations (e.g. the Böhm, Levy-Longo, and Berarducci trees of a given infinitary λ-term).

But while the final coalgebra of a functor F collects the behaviour of *all* coalgebras, one is often interested only in behaviours of coalgebras whose carrier admits a *finite* representation; in the case of nominal sets this means that the carrier is *orbit-finite*. In general, for a finitary endofunctor F on a locally finitely presentable category, the behaviour of F-coalgebras with a finitely presentable carrier is captured by the notion of *rational fixpoint* for F (see [4, 26]). This fixpoint lies between the initial algebra and the final coalgebra for F; as a coalgebra, it is characterized as the final *locally finitely presentable* coalgebra[1]. Examples of rational fixpoints include the sets of regular languages, of eventually periodic and rational streams, respectively, and of rational formal power-series. For a polynomial endofunctor F_Σ on sets associated to the signature Σ, the rational fixpoint consists of Elgot's *regular* Σ-trees [10], i.e. those (finite and infinite) Σ-trees that have only finitely many different subtrees (up to isomorphism). Recently, Milius and Wißmann [27] gave a description of the rational fixpoint of L_α on Nom; it is formed by all rational λ-trees modulo α-equivalence.

In this paper we extend the latter result to a description of the rational fixpoint for an axiomatically defined class of functors. This class (properly) includes all *binding functors*, i.e. functors arising from binding signatures, but also the finite power-set functor and exponentiation by orbit-finite strong nominal sets, and our class is closed under coproducts, finite products, composition and quotients of functors. Unsurprisingly, in the special case of a functor for a binding signature, the rational fixpoint is formed by the rational trees over the given binding signature modulo α-equivalence. However, the proof of our more general result is surprisingly non-trivial, and not related to the one given in [27] for the special case L_α. Instead we take a fresh approach and first consider endofunctors $\bar{F}$ on Nom that are a

[1] A coalgebra is locally finitely presentable (lfp) if every state in it generates a finitely presentable subcoalgebra. We are aware of the terminological clash with locally finitely presentable *categories* but it seems contrived to call coalgebras with the mentioned property by any other name.

lifting of some finitary endofunctor F on sets. In general, it is easy to see that for such functors $\bar{F}$ the initial algebra is a lifting of the inital F-algebra. However, the final coalgebra does not lift in general; for example, for the Nom-functor $LX = \mathcal{V} + \mathcal{V} \times X + X \times X$, the final coalgebra of the underlying Set-functor consists of all λ-trees, and the final L-coalgebra in Nom consists of all λ-trees with finitely many variables. The rational fixpoint of L, on the other hand, does, by our results, lift from Set to Nom; however, this does not hold for arbitrary liftings of finitary functors. We introduce the notion of a *localizable* lifting (Definition 3.12) and we prove that the rational fixpoint of a localizable lifting $\bar{F}$ on Nom is a lifting of the rational fixpoint of F on sets (Theorem 4.14).

In order to characterize the rational fixpoint of functors that make use of the nominal structure, like L_α, we then turn our attention to quotients of a functor G on Nom. In fact, we introduce the notion of a *sub-strength* (Definition 5.4) of an endofunctor on Nom, and we prove that whenever G is equipped with a sub-strength then the rational fixpoint of any quotient of G is a canonical quotient coalgebra of the rational fixpoint of G (Corollary 5.14).

We will then see that the combination of Theorem 4.14 and Corollary 5.14 allows us to obtain the desired description of the rational fixpoint of a functor arising from a binding signature in combination with exponentiation by an orbit-finite strong nominal set. As a special case we obtain that ϱL_α is formed by those α-equivalence classes of λ-trees which contain at least one rational λ-tree.

The fact that our results cover exponentiation, which occurs prominently in functors for various automata models, is based on a construction that identifies exponentiation by an orbit-finite strong exponent as a quotient of a polynomial functor.

2 Preliminaries

We summarize the requisite background on permutations, nominal sets, and rational fixpoints of functors. We assume that readers are familiar with basic notions of category theory and with algebras and coalgebras for an endofunctor, but start with a terse review of the latter.

Recall that a *coalgebra* for an endofunctor $F : \mathcal{C} \to \mathcal{C}$ is a pair (C, c) consisting of an object C of $\mathcal{C}$ and a morphism $c : X \to FX$ called the *structure* of the coalgebra. A coalgebra *homomorphism* from (C, c) to (D, d) is a $\mathcal{C}$-morphism $f : C \to D$ such that $d \cdot f = Ff \cdot c$. A very important concept is that of a *final* coalgebra, i. e. an F-coalgebra $t : \nu F \to F(\nu F)$ such that for every F-coalgebra (C, c) there exists a unique homomorphism $c^\dagger : (C, c) \to (\nu F, t)$. Final coalgebras exist under mild assumptions on $\mathcal{C}$ and F, e.g. whenever $\mathcal{C}$ is locally presentable and F is accessible [25].

Intuitively, an F-coalgebra (C, c) can be thought of as a dynamic system with an object C of states and with observations about the states (e.g. output, next states etc.) given by c. The type of observations that can be made about a dynamic systems is described by the functor F. We denote by $\mathsf{Coalg}\, F$ the category of F-coalgebras and their homomorphisms. For more intuition and concrete examples we refer the reader to introductory texts on coalgebras [32, 16, 1].

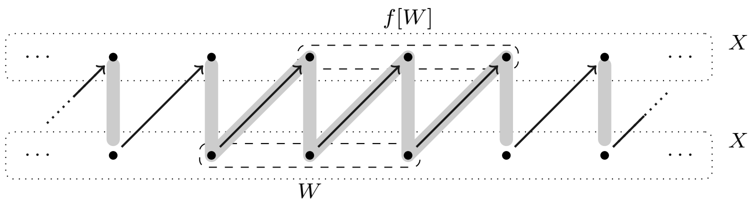

Fig. 1 Diagrammatic illustration of (2.1). The thin black arrows describe f and the thick grey arrows the second case in the definition of $f|_W$.

Assumption 2.1 Throughout the paper, all Set-functors F are w.l.o.g. assumed to preserve monos [6]; for convenience of notation, we will in fact sometimes assume that F preserves subset inclusions.

2.1 Permutations

We first need a few basic observations about permutations, in particular that every permutation on an infinite set X can be restricted to each finite subset of X.

Definition 2.2 For a (not necessarily finite) permutation $f : X \to X$ and a finite subset $W \subseteq X$, we define the *restriction* of f to W as

$$f|_W(v) = \begin{cases} f(v) & v \in W \\ f^{-n}(v) & n \geq 0 \text{ minimal s.t. } f^{-n}(v) \notin f[W]. \end{cases} \tag{2.1}$$

Intuitively, the second case of $f|_W$ searches backwards along f for some value that is not used by the first case; this is visualized in Figure 1.

Lemma 2.3 *For any permutation $f : X \to X$ and finite $W \subseteq X$, $f|_W$ is a finite permutation.*

Proof We first show that $f|_W(v)$ is indeed defined for all v: assume that the second case in (2.1) does not apply, i.e. $v \notin W$ with $f^{-n}(v) \in f[W]$ for all $n \geq 0$. By finiteness of $f[W]$, we then have $f^{-m}(v) = v$ for some $m \geq 1$ and therefore $f(v) = f(f^{-m}(v)) \in f[W]$, which implies that $v \in W$, so the first case in (2.1) applies.

For injectivity, let $f|_W(u) = f|_W(v)$ for $u, v \in \mathcal{V}$ and distinguish the following cases:

- For $u, v \in W$, $f(u) = f(v)$ and so $u = v$ as required.
- For $u \in W$, $v \notin W$, we have $f(u) = f^{-n}(v) \notin f[W]$, contradiction.
- For $u, v \notin W$, we have $f^{-n}(u) = f^{-m}(v)$, with n, m minimal.
  - If $n = m$, then $u = v$ as required.
  - If $n \neq m$, w.l.o.g. $n > m$, then $f(v) = f^{-(n-m-1)}(u) \in f[W]$ by minimality of n, since $n - m - 1 \geq 0$. This implies $v \in W$, contradiction.

For surjectivity, let $v \in X$.

- If $v \in f[W]$, then $f^{-1}(v) \in W$ and thus $f|_W(f^{-1}(v)) = v$.

– If $v \notin f[W]$, then let $k \geq 0$ be minimal such that $f^k(v) \notin W$. Such a k exists because W is finite and $v \notin f[W]$. So we have $f|_W(f^k(v)) = f^{-k}(f^k(v)) = v$ because firstly $f^k(v) \notin W$, and secondly for all $n < k$,

$$f^{-n}(f^k(v)) = f^{k-n}(v) \in f[W]. \qquad \square$$

This shows that $f|_W$ is a permutation. To see that $f|_W$ is finite, note that $f|_W(v) = v$ for $v \notin W \cup f[W]$, which is a finite set.

Remark 2.4 In summary, we have: 1. $f|_W[W] = f[W]$, 2. $f|_W[f[W] \setminus W] = W \setminus f[W]$, and 3. $f|_W$ fixes every element that is not contained in W or $f[W]$. Moreover, the permutation

$$g := f|_W^{-1} \cdot f$$

maps any $v \in W$ to $g(v) = f|_W^{-1}(f(v)) = v$ by (2.1). Since $f|_W \cdot g = f$, this means that we can factor any permutation f into a finite permutation $f|_W$ and a permutation g that fixes W.

Remark 2.5 Restriction is compatible with composition in the following sense: for permutations f, h and finite $W \subseteq X$, we have

$$(f \cdot h)|_W(v) = f \cdot h(v) = f|_{h[W]} \cdot h(v) = f|_{h[W]} \cdot h|_W(v) \quad \text{for all } v \in W$$

using (2.1) multiple times.

2.2 Nominal Sets

We now briefly recall the key definitions in the theory of nominal sets; see [30] for a detailed introduction.

Recall that given a monoid (or more specifically a group) M, an *M-set* is a set X equipped with a left action of M, which we denote by mere juxtaposition or by the infix operator $\cdot$. The M-sets are the Eilenberg-Moore algebras of the monad $M \times (-)$, which has the unit $\eta(x) = (e, x)$ and multiplication $\mu(n, (m, x)) = (nm, x)$ where e is the unit of M. Given M-sets $(X, \cdot)$ and $(Y, *)$, a map $f : X \to Y$ is *equivariant* if $\pi * f(x) = f(\pi \cdot x)$ for all $\pi \in M$, $x \in X$. M-sets and equivariant maps form a category, M-set.

We fix a set $\mathcal{V}$ of (variable) *names* (or *atoms*). As usual, the symmetric group $\mathfrak{S}(\mathcal{V})$ is the group of all permutations of $\mathcal{V}$; we denote by $\mathfrak{S}_\mathsf{f}(\mathcal{V})$ the subgroup of finite permutations of $\mathcal{V}$, i.e. the subgroup of $\mathfrak{S}(\mathcal{V})$ generated by the transpositions. We have an obvious left action of $\mathfrak{S}_\mathsf{f}(\mathcal{V})$ on $\mathcal{V}$ given by $\pi \cdot v = \pi(v)$ for any $\pi \in \mathfrak{S}_\mathsf{f}(\mathcal{V})$ and $v \in \mathcal{V}$. Given a $\mathfrak{S}_\mathsf{f}(\mathcal{V})$-set X, we define

$$\mathsf{fix}(x) = \{\pi \in \mathfrak{S}_\mathsf{f}(\mathcal{V}) \mid \pi \cdot x = x\} \text{ and } \mathsf{Fix}(A) = \{\pi \in \mathfrak{S}_\mathsf{f}(\mathcal{V}) \mid \pi \cdot x = x \text{ for all } x \in A\}$$

for $x \in X$ and $A \subseteq X$. We say that a set $A \subseteq \mathcal{V}$ is a *support* of $x \in X$ or that A *supports* x if

$$\mathsf{Fix}(A) \subseteq \mathsf{fix}(x),$$

i.e. if any permutation that fixes all names in A also fixes x. Moreover, $x \in X$ is *finitely supported* if there exists a finite set of names that supports x. In this case, it can be shown (see e.g. [30]) that x has a least support, denoted $\mathsf{supp}(x)$ and called the *support of* x. We say that $v \in \mathcal{V}$ is *fresh* for x, and write $v \,\#\, x$, if $v \in \mathcal{V} \setminus \mathsf{supp}(x)$.

A *nominal set* is a $\mathfrak{S}_\mathsf{f}(\mathcal{V})$-set $(X,\cdot)$ (or just X) such that all elements of X are finitely supported. We denote by Nom the full subcategory of $\mathfrak{S}_\mathsf{f}(\mathcal{V})$-set spanned by the nominal sets. We have forgetful functors $V:\mathfrak{S}_\mathsf{f}(\mathcal{V})\text{-set}\to\mathsf{Set}$ and $U:\mathsf{Nom}\to\mathsf{Set}$. Note that for each nominal set X, the function $\mathsf{supp}:X\to\mathcal{P}_\mathsf{f}(\mathcal{V})$ mapping each element to its (finite) support is an equivariant map.

Example 2.6 (1) The set $\mathcal{V}$ of names with the group action $\pi\cdot v=\pi(v)$ is a nominal set; for each $v\in\mathcal{V}$ the singleton $\{v\}$ supports v.
(2) Every ordinary set X can be made into a nominal set DX (D for *discrete*) by equipping it with the *trivial group action* (also called the *trivial* or *discrete nominal structure*) $\pi\cdot x=x$ for all $x\in X$ and $\pi\in\mathfrak{S}_\mathsf{f}(\mathcal{V})$. So each $x\in DX$ has empty support.
(3) The finite λ-terms form a nominal set with the group action given by renaming of (free as well as bound!) variables [11]. The support of a λ-term is the set of all variables that occur in it. In contrast, the set of all (potentially infinite) λ-trees is not nominal since λ-trees with infinitely many variables do not have finite support. However, the set of all λ-trees with finitely many variables is nominal.
(4) Given a nominal set X, the set $\mathcal{P}_\mathsf{f}(X)$ of finite subsets of X equipped with the point-wise action of $\mathfrak{S}_\mathsf{f}(\mathcal{V})$ is a nominal set. The support $\mathsf{supp}(Y)$ of $Y\in\mathcal{P}_\mathsf{f}(X)$ is the union $\bigcup_{x\in Y}\mathsf{supp}(x)$. In particular, the support of each finite $W\in\mathcal{P}_\mathsf{f}(\mathcal{V})$ is W itself. Note that $\mathcal{P}(\mathcal{V})$ with the point-wise action is not a nominal set because any subset of $\mathcal{V}$ that is neither finite nor cofinite fails to be finitely supported. However, the set $\mathcal{P}_\mathsf{fs}(X)\subseteq\mathcal{P}(X)$ of finitely supported subsets of X is a nominal set.

Remark 2.7 (1) For an equivariant map $f:X\to Y$ between nominal sets, we have $\mathsf{supp}(f(x))\subseteq\mathsf{supp}(x)$ for any $x\in X$. To see this, let $\pi\in\mathsf{Fix}(\mathsf{supp}(x))$. Then $\pi\cdot f(x)=f(\pi\cdot x)=f(x)$, so $\mathsf{supp}(x)$ also supports $f(x)$ and thus $\mathsf{supp}(f(x))\subseteq\mathsf{supp}(x)$.
(2) For $\pi\in\mathfrak{S}_\mathsf{f}(\mathcal{V})$ and a $\mathfrak{S}_\mathsf{f}(\mathcal{V})$-set X we denote by π_X the bijection $X\to X$ defined by $x\mapsto\pi\cdot x$. Note that π_X fails to be equivariant unless X is discrete. However, π commutes with all equivariant maps $f:X\to Y$ in the sense that $f\pi_X=\pi_Y f$; in other words: $\pi:U\to U$ is a natural isomorphism.
(3) Every nominal set X can be uniquely extended to a $\mathfrak{S}(\mathcal{V})$-set [14]. By the discussion in Section 2.1, the $\mathfrak{S}(\mathcal{V})$-action can be defined as $\pi\cdot x=\pi|_{\mathsf{supp}(x)}\cdot x$ for $\pi\in\mathfrak{S}(\mathcal{V})$. By the first item, maps f that are equivariant w.r.t. the action of $\mathfrak{S}_\mathsf{f}(\mathcal{V})$ are equivariant also w.r.t. the extended action: For $\pi\in\mathfrak{S}(\mathcal{V})$, we have $f(\pi\cdot x)=f(\pi|_{\mathsf{supp}(x)}\cdot x)=\pi|_{\mathsf{supp}(x)}\cdot f(x)=\pi|_{\mathsf{supp}(f(x))}\cdot f(x)=\pi\cdot f(x)$, using in the second-to-last step that $\pi|_{\mathsf{supp}(x)}$ and $\pi|_{\mathsf{supp}(f(x))}$ agree on $\mathsf{supp}(f(x))$.

The category of nominal sets is (equivalent to) a Grothendieck topos (the so-called Schanuel topos), and so it has rich categorical structure [14]. In the following we recall the structural properties needed in the current paper.

Monomorphisms and epimorphisms in Nom are precisely the injective and surjective equivariant maps, respectively. It is not difficult to see that every epimorphism in Nom is strong, i.e., it has the unique diagonalization property w.r.t. any monomorphism: given an epimorphism $e:A\twoheadrightarrow B$, a monomorphism $m:C\hookrightarrow D$ and $f:A\to C$, $g:B\to D$ such that $g\cdot e=m\cdot f$, there exists a unique diagonal $d:B\to C$ with $d\circ e=f$ and $m\circ d=g$.

Furthermore, Nom has image-factorizations; this means that every equivariant map $f : A \to C$ factorizes into an epimorphism e followed by a monomorphism m:

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[rr, bend left, "f"] & B \arrow[r, hook, "m"] & C \end{tikzcd}$$

Note that the intermediate object B is (isomorphic to) the image $f[A]$ in B with the induced action. For an endofunctor F on Nom preserving monos, this factorization system lifts to $\mathsf{Coalg}\, F$: every F-coalgebra homomorphism f has a factorization $f = m \cdot e$ where e and m are F-coalgebra homomorphisms that are epimorphic and monomorphic in Nom, respectively.

Being a Grothendieck topos, Nom is complete and cocomplete. Moreover, colimits and finite limits are formed as in Set, and in fact the forgetful functor $U : \mathsf{Nom} \to \mathsf{Set}$ creates all colimits and all finite limits [29]. Furthermore, Nom is a locally finitely presentable category [13, 5]. Recall that a locally finitely presentable category is a cocomplete category $\mathcal{C}$ having a set $\mathcal{A}$ of finitely presentable objects such that every object of $\mathcal{C}$ is a filtered colimit of objects from $\mathcal{A}$. Petrişan [28, Proposition 2.3.7] shows that the finitely presentable objects of Nom are precisely the orbit-finite nominal sets:

Definition 2.8 Given a nominal set X and $x \in X$, the set $\{\pi \cdot x \mid \pi \in \mathfrak{S}_{\mathsf{f}}(\mathcal{V})\}$ is called the *orbit* of x. A nominal set $(X, \cdot)$ is said to be *orbit-finite* if it has only finitely many orbits.

The notion of orbit-finiteness plays a central role in our paper since the rational fixpoint of an endofunctor F on Nom can be constructed as the filtered colimit of all F-coalgebras with orbit-finite carrier.

We now collect a few easy properties of orbit-finite sets that we are going to need. First of all, orbit-finite sets are closed under finite products and subobjects; hence, under all finite limits (see [30, Chapter 5]). And they are clearly closed under finite coproducts (a well known property of finitely presentable objects) and quotient objects (since the codomain of a surjective equivariant map clearly has fewer orbits); hence under all finite colimits.

Generally, for the value of $\pi \cdot x$, it matters only what π does on the atoms in $\mathsf{supp}(x)$:

Lemma 2.9 *For $x \in (X, \cdot)$ and any $\pi, \sigma \in \mathfrak{S}_{\mathsf{f}}(\mathcal{V})$ with $\pi(v) = \sigma(v)$ for all $v \in \mathsf{supp}(x)$, we have $\pi \cdot x = \sigma \cdot x$.*

Proof Under the given assumptions, $\pi^{-1}\sigma \in \mathsf{Fix}(\mathsf{supp}(x)) \subseteq \mathsf{fix}(x)$. ◻

Lemma 2.10 *For any $x_1, x_2 \in X$ in the same orbit, we have $|\mathsf{supp}(x_1)| = |\mathsf{supp}(x_2)|$.*

Proof By equivariance of supp, π induces a bijection between $\mathsf{supp}(x)$ and $\mathsf{supp}(\pi \cdot x) = \pi \cdot \mathsf{supp}(x) = \pi[\mathsf{supp}(x)]$. ◻

Lemma 2.11 *For an element x of a nominal set X, there are at most $|\mathsf{supp}(x)|!$ many elements with support $\mathsf{supp}(x)$ in the orbit of x.*

Proof Let $\pi \in \mathfrak{S}_\mathsf{f}(\mathcal{V})$ such that $\mathsf{supp}(x) = \mathsf{supp}(\pi \cdot x)$. Then

$$\pi[\mathsf{supp}(x)] = \pi \cdot \mathsf{supp}(x) = \mathsf{supp}(\pi \cdot x) = \mathsf{supp}(x),$$

which shows that π restricts to a permutation of $\mathsf{supp}(x)$.

If σ is also such that $\mathsf{supp}(x) = \mathsf{supp}(\sigma \cdot x)$ and restricts to the same permutation on $\mathsf{supp}(x)$ as π, then $\pi \cdot x = \sigma \cdot x$ by Lemma 2.9. Therefore the number of elements in question is at most the number of permutations of $\mathsf{supp}(x)$, i.e. at most $|\,\mathsf{supp}(x)|!$. □

One of the properties that make nominal sets interesting for applications in computer science is that one can think of an element x of a nominal set as an abstract term and of $\mathsf{supp}(x)$ as the set of free variables of x. It is then possible to speak about α-equivalence on a nominal set, and this leads to Gabbay and Pitts' *abstraction functor* [12, Lemma 5.1]:

Definition 2.12 Let X be a nominal set. We define α-equivalence $\sim_\alpha$ as the relation on $\mathcal{V} \times X$ defined by

$$(v_1, x_1) \sim_\alpha (v_2, x_2) \;\text{ if } (v_1\, z)x_1 = (v_2\, z)x_2 \text{ for } z \# \{v_1, v_2, x_1, x_2\},$$

where the definition of $z \# M$ spelled out for the case of a finite set M means that z is fresh for every element of M. The $\sim_\alpha$-equivalence class of (v, x) is denoted by $\langle v \rangle x$. The abstraction $[\mathcal{V}]X$ of X is the quotient $(\mathcal{V} \times X)/{\sim_\alpha}$ with the group action defined by

$$\pi \cdot \langle v \rangle x = \langle \pi(v) \rangle (\pi \cdot x).$$

For an equivariant map $f : X \to Y$, $[\mathcal{V}]f : [\mathcal{V}]X \to [\mathcal{V}]Y$ is defined by $\langle v \rangle x \mapsto \langle v \rangle (f(x))$.

2.3 The Rational Fixpoint

Recall that by Lambek's Lemma [24], the structure maps of the initial algebra and the final coalgebra for a functor F are isomorphisms, so both yield *fixpoints* of F. Here we shall be interested in a third fixpoint that lies between the initial algebra and the final coalgebra, the *rational fixpoint* of F. The rational fixpoint can be characterized either as the initial iterative algebra for F [4] or as the final locally finitely presentable coalgebra for F [26]. We will need only the latter description here.

The rational fixpoint can be defined for any *finitary* endofunctor F on a locally finitely presentable category $\mathcal{C}$, i.e. F is an endofunctor on $\mathcal{C}$ that preserves filtered colimits. Examples of locally finitely presentable categories are Set, the categories of posets and of graphs, every finitary variety of algebras (such as groups, rings, and vector spaces) and every Grothendieck topos (such as Nom). The finitely presentable objects in these categories are: all finite sets, posets or graphs, algebras presented by finitely many generators and relations, and, as we mentioned before, the orbit-finite nominal sets.

Now let $F : \mathcal{C} \to \mathcal{C}$ be finitary on the locally finitely presentable category $\mathcal{C}$ and consider the full subcategory $\mathsf{Coalg}_\mathsf{f}\, F$ of $\mathsf{Coalg}\, F$ given by all F-coalgebras with finitely presentable carrier. The *locally finitely presentable* F-coalgebras are characterized as precisely those coalgebras that arise as a colimit of a filtered diagram of coalgebras from $\mathsf{Coalg_f}\,F$ [26]. It follows that the *final* locally finitely presentable coalgebra can be constructed as the colimit of *all* coalgebras from $\mathsf{Coalg_f}\,F$. More precisely, one defines a coalgebra $r : \varrho F \to F(\varrho F)$ as the colimit of the inclusion functor of $\mathsf{Coalg_f}\,F$: $(\varrho F, r) := \mathrm{colim}(\mathsf{Coalg_f}\,F \hookrightarrow \mathsf{Coalg}\,F)$. Note that since the forgetful functor $\mathsf{Coalg}\,F \to \mathcal{C}$ creates all colimits, this colimit is actually formed on the level of $\mathcal{C}$. The colimit ϱF then carries a uniquely determined coalgebra structure r making it the colimit above.

As shown in [4], ϱF is a fixpoint for F, i.e. its coalgebra structure r is an isomorphism. From [26] we obtain that local finite presentability of a coalgebra (C, c) has the following concrete characterizations: (1) for $\mathcal{C} = \mathsf{Set}$ local finiteness, i.e. every element of C is contained in a finite subcoalgebra of C; (2) for $\mathcal{C} = \mathsf{Nom}$, local orbit-finiteness, i.e. every element of C is contained in an orbit-finite subcoalgebra of C; (3) for $\mathcal{C}$ the category of vector spaces over a field K, local finite dimensionality, i.e., every element of C is contained in a subcoalgebra of C carried by a finite dimensional subspace of C.

Example 2.13 We list a few examples of rational fixpoints; for more see [4, 26, 8].

(1) Consider the functor $FX = 2 \times X^A$ on Set where A is an input alphabet and $2 = \{0, 1\}$. The F-coalgebras are precisely the deterministic automata over A (without initial states). The final coalgebra is carried by the set $\mathcal{P}(A^*)$ of all formal languages, and the rational fixpoint is its subcoalgebra of regular languages over A.
(2) For $FX = \mathbb{R} \times X$ on Set, the final coalgebra is carried by the set $\mathbb{R}^\omega$ of all real streams, and the rational fixpoint is its subcoalgebra of all eventually periodic streams, i.e. streams $uvvv\cdots$ with $u, v \in \mathbb{R}^*$. Taking the same functor on the category of real vector spaces, we obtain the same final coalgebra $\mathbb{R}^\omega$ with the componentwise vector space structure, but this time the rational fixpoint is formed by all rational streams (see [33, 26]).
(3) Recall that in general algebra a finitary signature Σ of operation symbols with prescribed arity is a sequence $(\Sigma_n)_{n<\omega}$ of sets. This give rise to an associated polynomial endofunctor F_Σ on Set given by $F_\Sigma X = \coprod_{n<\omega} \Sigma_n \times X^n$. Its initial algebra is formed by all Σ-terms and its final coalgebra by all (finite and infinite) Σ-trees, i.e. rooted and ordered trees such that every node with n children is labelled by an n-ary operation symbol. And the rational fixpoint consists precisely of all *rational* Σ-trees [10, 9], i.e. those Σ-trees that have only finitely many different subtrees up to isomorphism [15].
(4) For the finite powerset functor $\mathcal{P}_\mathsf{f}$, the initial algebra is the ω-th step of the cumulative hierarchy of sets, i.e. $\bigcup_{n<\omega} \mathcal{P}_\mathsf{f}{}^n(\emptyset)$. An isomorphic description is as the set of all finite extensional trees, where a tree is called *extensional* if distinct children of any vertex define non-isomorphic subtrees. A final $\mathcal{P}_\mathsf{f}$-coalgebra is carried by the set of all strongly-extensional finitely branching trees, where a tree t is called *strongly-extensional* if for any node x of t no two subtrees of t rooted at x are *tree-bisimilar*; for further explanation and details see [35] or [2, Corollary 3.19]. And the rational fixpoint of $\mathcal{P}_\mathsf{f}$ is given by all rational strongly-extensional trees.
(5) The bag functor $\mathcal{B} : \mathsf{Set} \to \mathsf{Set}$ assigns to every set X the set of all finite multisets on X, i.e. the free commutative monoid over X. Here we consider trees where children of a vertex are not ordered (in constrast to Σ-trees in

item (3)), i.e. the usual graph theoretic notion of tree. Then the initial algebra for $\mathcal{B}$ is given by all finite trees, the final coalgebra by all finitely branching ones and the rational fixpoint by all rational ones. This follows from the results in [3].

Note that in all the above examples, the rational fixpoint ϱF is a subcoalgebra of the final coalgebra νF. This need not be the case in general (see [8, Example 3.15] for a counterexample). However, we do have the following result:

Proposition 2.14 [8, Proposition 3.12] *Suppose that in $\mathcal{C}$, finitely presentable objects are closed under strong quotients and that F is finitary and preserves monomorphisms. Then the rational fixpoint ϱF is the subcoalgebra of νF given by the union of the images of all coalgebra homomorphisms $c^\dagger : (C, c) \to (\nu F, t)$ where (C, c) ranges over $\mathsf{Coalg_f}\, F$.*[2]

In particular, for a finitary functor F on Set or Nom, respectively, that preserves monomorphisms, the rational fixpoint is the union of the images in νF of all finite (or orbit-finite resp.) coalgebras; in symbols:

$$\varrho F = \bigcup_{(C,c) \text{ in } \mathsf{Coalg_f}\, F} c^\dagger[C] \ \subseteq\ \nu F.$$

Note that it is sufficient to let (C, c) range over those coalgebras in $\mathsf{Coalg_f}\, F$ where $c^\dagger$ is injective, because for an arbitrary (orbit-)finite (C, c) in $\mathsf{Coalg_f} F$, its image $c^\dagger[C]$ is again an (orbit-)finite F-coalgebra, and has an injective structure map.

3 Liftings of Finitary Functors

We now direct our attention to liftings of finitary Set-functors, with a view to investigating their rational fixpoints. We fix some terminology:

Definition 3.1 A *lifting* (or, for distinction, a Nom-*lifting*) of a functor $F : \mathsf{Set} \to \mathsf{Set}$ is a functor $\bar{F} : \mathsf{Nom} \to \mathsf{Nom}$ such that $U\bar{F} = FU$. Further, an $\mathfrak{S}_\mathsf{f}(\mathcal{V})$-*set lifting* of F is a functor $\hat{F} : \mathfrak{S}_\mathsf{f}(\mathcal{V})$-set $\to \mathfrak{S}_\mathsf{f}(\mathcal{V})$-set such that $V\hat{F} = FV$. We say that a functor $G : \mathsf{Nom} \to \mathsf{Nom}$ is a *lifting* if it is a Nom-lifting of some Set-functor F.

Notation 3.2 Throughout this work, we will use the bar notation in the above definition to denote functors on Nom that are liftings of Set-endofunctors.

Lemma 3.3 *A functor $G : \mathsf{Nom} \to \mathsf{Nom}$ is a lifting iff G is a lifting of UGD, and in fact if G is a lifting of F then $F = UGD$.*

Proof In the first claim, ‘if’ is trivial; we prove ‘only if’ in conjunction with the second claim. So let $UG = FU$ for some Set-functor F; then $UGD = FUD = F$. $\square$

Definition 3.4 Let $\hat{F}$ be a $\mathfrak{S}_\mathsf{f}(\mathcal{V})$-set lifting of the functor $F : \mathsf{Set} \to \mathsf{Set}$. We say that $\hat{F}$ is Nom-*restricting* if it preserves nominal sets, i.e. $\hat{F}$ restricts to a functor $\bar{F} : \mathsf{Nom} \to \mathsf{Nom}$ (which is, then, a Nom-lifting of F).

[2] In a general locally finitely presentable category the *image* of $c^\dagger$ is obtained by taking a (strong epi,mono)-factorization of $c^\dagger$, and the *union* is then obtained as a directed colimit of the resulting subobjects of $(\nu F, t)$.

Recall that a *monad-over-functor distributive law* between a monad T and a functor F on a category $\mathcal{C}$ is a natural transformation $\lambda : TF \to FT$ such that the diagrams

$$\begin{tikzcd} & F \arrow[dl, "\eta F"'] \arrow[dr, "F\eta"] & \\ TF \arrow[rr, "\lambda"'] & & FT \end{tikzcd} \tag{3.1}$$

and

$$\begin{tikzcd} T^2F \arrow[r, "T\lambda"] \arrow[d, "\mu F"'] & TFT \arrow[r, "\lambda T"] & FT^2 \arrow[d, "F\mu"] \\ TF \arrow[rr, "\lambda"'] & & FT \end{tikzcd} \tag{3.2}$$

commute. Such distributive laws are in bijective correspondence with liftings of F to the Eilenberg-Moore category of T [17]. In one direction of this correspondence, we obtain from a distribute law λ the lifting $\bar{F}$ that maps an Eilenberg-Moore algebra $TA \xrightarrow{a} A$ to the algebra

$$TFA \xrightarrow{\lambda_A} FTA \xrightarrow{Fa} A.$$

In particular, $\mathfrak{S}_\mathsf{f}(\mathcal{V})$-set-liftings of a Set-functor F are in bijection with distributive laws

$$\lambda : \mathfrak{S}_\mathsf{f}(\mathcal{V}) \times F \to F(\mathfrak{S}_\mathsf{f}(\mathcal{V}) \times (-)\,) \tag{3.3}$$

of the monad $\mathfrak{S}_\mathsf{f}(\mathcal{V}) \times (-)$ over F. We will be interested exclusively in Nom-liftings that arise by restricting $\mathfrak{S}_\mathsf{f}(\mathcal{V})$-set-liftings, i.e. come from a distributive law (3.3); explicitly:

Definition 3.5 A distributive law λ of $\mathfrak{S}_\mathsf{f}(\mathcal{V}) \times (-)$ over F is Nom*-restricting* if the corresponding $\mathfrak{S}_\mathsf{f}(\mathcal{V})$-set lifting of F is Nom-restricting.

One class of Nom-restricting liftings are *canonical* liftings, introduced next. After that, we introduce the bigger class of *localizable* liftings, and in the next section we study their rational fixpoints.

Recall that every Set-functor F comes with a *(tensorial) strength*, i.e. a transformation

$$s_{X,Y} : X \times FY \to F(X \times Y)$$

natural in X and Y, making the diagrams

$$\begin{tikzcd} 1 \times FY \arrow[r, "s_{1,Y}"] \arrow[dr, "\iota_{FY}"'] & F(1 \times Y) \arrow[d, "F\iota_Y"] \\ & FY \end{tikzcd} \tag{3.4}$$

and

$$\begin{tikzcd} (X \times Z) \times FY \arrow[r, "s_{X\times Z,Y}"] \arrow[d, "\alpha_{X,Z,FY}"'] & F((X \times Z) \times Y) \arrow[drr, "F\alpha_{X,Z,Y}"] & & \\ X \times (Z \times FY) \arrow[r, "X \times s_{Z,Y}"'] & X \times F(Z \times Y) \arrow[rr, "s_{X,Z\times Y}"'] & & F(X \times (Z \times Y)) \end{tikzcd} \tag{3.5}$$

commute, where $\iota : 1 \times \mathrm{Id} \to \mathrm{Id}$ and $\alpha : (\mathrm{Id} \times \mathrm{Id}) \times \mathrm{Id} \to \mathrm{Id} \times (\mathrm{Id} \times \mathrm{Id})$ are the left unitor and the associator, respectively, of the Cartesian monoidal structure [20]. We remark that using ι and α, we can rephrase the definition of the monad $M \times (-)$ for a monoid (M, m, e): the unit is

$$\eta_X \equiv \left(X \xrightarrow{\iota_X^{-1}} 1 \times X \xrightarrow{e \times X} M \times X \right)$$

considering the unit element as a morphism $e : 1 \to M$, and the multiplication is

$$\mu_X \equiv \left(M \times (M \times X) \xrightarrow{\alpha^{-1}_{M,M,X}} (M \times M) \times X \xrightarrow{m \times X} M \times X \right).$$

Lemma 3.6 *Given a monoid (M, m, e) and a* Set*-functor F with strength s, the natural transformation*

$$s_{M,X} : M \times FX \to F(M \times X)$$

is a distributive law of the monad $M \times (-)$ over the functor F.

Proof Using $(F\iota_X)^{-1} = F\iota_X^{-1}$, the commutatitivity of the following verifies (3.1):

$$\begin{tikzcd}
 & FX \arrow[dl, "\iota^{-1}_{FX}"'] \arrow[dr, "F\iota^{-1}_X"] \arrow[d, phantom, "(3.4)"] \arrow[ddl, bend right=60, "\eta_{FX}"'] \arrow[ddr, bend left=60, "F\eta_X"] & \\
1 \times FX \arrow[rr, "s_{1,X}"] \arrow[d, "m \times FX"'] \arrow[drr, phantom, "\text{Naturality}"] & & F(1 \times X) \arrow[d, "F(m \times X)"] \\
M \times FX \arrow[rr, "s_{M,X}"] & & F(M \times X)
\end{tikzcd}$$

Also using $F(\alpha^{-1}_{M,M,X}) = (F\alpha_{M,M,X})^{-1}$, note that

$$\begin{tikzcd}
M \times (M \times FX) \arrow[r, "M + s_{M,FX}"] \arrow[d, "\alpha^{-1}_{M,M,FX}"'] \arrow[dd, bend right=60, "\mu_{FX}"'] & M \times F(M \times X) \arrow[r, "s_{M,F(M+X)}"] \arrow[d, phantom, "(3.5)"] & F(M \times (M \times X)) \arrow[d, "F\alpha^{-1}_{M,M,X}"] \arrow[dd, bend left=60, "F\mu_X"] \\
(M \times M) \times FX \arrow[rr, "s_{M \times M,X}"] \arrow[d, "m \times FX"'] \arrow[drr, phantom, "\text{Naturality}"] & & F((M \times M) \times X) \arrow[d, "F(m \times X)"] \\
M \times FX \arrow[rr, "s_{M,X}"'] & & F(M \times X)
\end{tikzcd}$$

commutes, so $s_{M,\_}$ satisfies (3.2), and hence is a distributive law. □

Definition 3.7 For $M = \mathfrak{S}_{\mathsf{f}}(\mathcal{V})$, we refer to the distributive law described in Lemma 3.6 as the *canonical distributive law* of $\mathfrak{S}_{\mathsf{f}}(\mathcal{V}) \times (-)$ over F, and to the arising $\mathfrak{S}_{\mathsf{f}}(\mathcal{V})$-set lifting of F as the *canonical $\mathfrak{S}_{\mathsf{f}}(\mathcal{V})$-set lifting of F.*

Lemma 3.8 *The canonical $\mathfrak{S}_{\mathsf{f}}(\mathcal{V})$-set lifting of a finitary* Set*-functor is* Nom*-restricting.*

Proof Let $F : \mathsf{Set} \to \mathsf{Set}$ be finitary, let $\bar{F}$ denote the canonical $\mathfrak{S}_\mathsf{f}(\mathcal{V})$-set lifting of F, let X be a nominal set with nominal structure $\alpha : \mathfrak{S}_\mathsf{f}(\mathcal{V}) \times X \to X$, and let $x \in FX$. We have to show that x has finite support in $\bar{F}X$. Since F is finitary, $x : 1 \to FX$ factors through some Fi with i a subset inclusion $S \hookrightarrow X$ of a finite subset S.

Then $W = \mathsf{supp}(S)$ supports x: Put $G = \mathfrak{S}_\mathsf{f}(\mathcal{V} \setminus W)$, let $m : G \to \mathfrak{S}_\mathsf{f}(\mathcal{V})$ be the evident subgroup inclusion, and let $\pi \in G$. Since W supports S and S is finite, the elements of G fix S pointwise, i.e.

$$\begin{tikzcd} G \times S \arrow[r, "m \times i"] \arrow[rr, bend left, "i \circ \mathsf{outr}"] & \mathfrak{S}_\mathsf{f}(\mathcal{V}) \times X \arrow[r, "\alpha"] & X \end{tikzcd} \tag{3.6}$$

commutes, where outr denotes the right-hand product projection. With β denoting the nominal structure on FX and s the strength (so $s_{\mathfrak{S}_\mathsf{f}(\mathcal{V}),\_}$ is the canonical distributive law), we have that

$$\begin{tikzcd}
1 \cong 1 \times 1 \arrow[r, "1 \times x"] \arrow[rrddd, bend right, "\pi \cdot x"'] \arrow[rrddd, bend left=60, "x"] & 1 \times FS \arrow[r, "s_{1,S}"] \arrow[d, "\pi \times FS"'] \arrow[dr, phantom, "\text{Naturality of } s"] & F(1 \times S) \arrow[d, "F(\pi \times S)"] \\
& G \times FS \arrow[r, "s_{G,S}"] \arrow[d, "m \times Fi"'] \arrow[dr, phantom, "\text{Naturality of } s"] & F(G \times S) \arrow[d, "F(m \times i)"] \arrow[dd, bend left, "F(i \circ \mathsf{outr})"] \\
& \mathfrak{S}_\mathsf{f}(\mathcal{V}) \times FX \arrow[r, "s_{\mathfrak{S}_\mathsf{f}(\mathcal{V}),X}"] \arrow[dr, "\beta"', "\text{Def.}"] & F(\mathfrak{S}_\mathsf{f}(\mathcal{V}) \times X) \arrow[d, "F\alpha"] \\
& & FX
\end{tikzcd}$$

commutes, where the unlabelled triangle commutes by Diagram (3.6) and the decomposition of x in the upper right hand part is by the strength law (3.4). This shows that $\pi \cdot x = x$, as required. □

Definition 3.9 We refer to the lifting of a Set-functor F to Nom arising from Lemma 3.8 as the *canonical lifting of* F. Moreover, a lifting $G : \mathsf{Nom} \to \mathsf{Nom}$ is *canonical* if it is a canonical lifting of some functor (i.e. a canonical lifting of UGD).

The canonical lifting is the expected lifting for many Set-functors:

Example 3.10 (1) For a polynomial functor F_Σ on Set (see Example 2.13(3)) the canonical lifting $\bar{F}_\Sigma$ maps a nominal set $(X, \cdot)$ to the expected coproduct of finite products in Nom where each Σ_n is equipped with the trivial nominal structure.

(2) The canonical lifting of the finite powerset functor $\mathcal{P}_\mathsf{f}$ maps a nominal set $(X, \cdot)$ to $\mathcal{P}_\mathsf{f}(X)$ equipped with the usual nominal structure, which is given by $\pi \cdot Y = \{\pi \cdot y \mid y \in Y\}$ for $Y \in \mathcal{P}_\mathsf{f}(X)$.

(3) The canonical lifting of the bag functor $\bar{\mathcal{B}}$ maps a nominal set $(X, \cdot)$ to $\mathcal{B}(X)$ equipped with the nominal structure that acts elementwise as in the previous item.

(4) An interesting more general class of examples are Joyal's *analytic functors* [18, 19]. An endofunctor F on Set is *analytic* if it is the left Kan extension of a functor from the category B of natural numbers and bijections to Set along the inclusion. These are described explicitly as follows. For a subgroup G of $\mathfrak{S}(n)$, $n < \omega$, the *symmetrized representable functor* maps a set X to the set X^n/G of orbits under the action of G on X^n by coordinate interchange, i.e., X^n/G is the quotient of X^n modulo the equivalence $\sim_G$ with $(x_0, \dots, x_{n-1}) \sim_G (y_0, \dots, y_{n-1})$ iff $(x_{\pi(0)}, \dots, x_{\pi(n-1)}) = (y_0, \dots, y_{n-1})$ for some $\pi \in G$. It is not difficult to prove that an endofunctor on Set is analytic iff it is a coproduct of symmetrized representables. So every analytic functor H can be written in the form

$$FX = \coprod_{\substack{n<\omega \\ G \le \mathfrak{S}_\mathsf{f}(n)}} A_{n,G} \times X^n/G \ . \tag{3.7}$$

Clearly every analytic functor is finitary, and Joyal proved in [18,19] that a finitary endofunctor on Set is analytic iff it weakly preserves wide pullbacks. The canonical lifting of an analytic functor F is given by equipping for any nominal set $(X, \cdot)$ the quotients X^n/G with the obvious group action:

$$\pi \cdot [(x_0, \dots, x_{n-1}]_{\sim_G} = [\pi \cdot x_0, \dots, \pi \cdot x_{n-1}]_{\sim_G}.$$

Note that the bag functor from the previous item is the special case where we take $A_{n,G} = 1$ for $G = \mathfrak{S}_\mathsf{f}(n)$ and 0 else, for every n. The finite power-set functor is not analytic.

(5) Another interesting analytic functor is the *cyclic shift functor* $\mathcal{Z}$ that maps a set X to the set of all assignments of elements of X to the corners of any regular polygon (modulo rotation of the polygon). In fact, this is the analytic functor obtained by putting $A_{n,G} = 1$ for G generated by the cyclic right shift $\pi(i) = (i{+}1) \mod n$ for $i = 0, \dots, n{-}1$, and $A_{n,G} = 0$ otherwise. The canonical lifing of $\mathcal{Z}$ is as expected: given a nominal set Y, the nominal structure on $\bar{\mathcal{Z}}Y$ acts by applying the original action on Y to the elements labelling the corners of a regular polygon.

However, many important functors on Nom are liftings but not canonical liftings.

Example 3.11 (1) The simplest examples are constant functors $\bar{K}X = (Y, \cdot\,)$ for a nontrivial nominal set $(Y, \cdot\,)$. The functor $\bar{K}$ is clearly a lifting of the constant functor $KX = Y$ on Set but the canonical lifting of K is constantly DY.

(2) Similarly, more interesting composite functors such as the functor $\bar{L}X = \mathcal{V} + X \times X + \mathcal{V} \times X$ mentioned in the introduction with the standard action on $\mathcal{V}$, i.e. $\pi{\cdot}v = \pi(v)$ for $v \in \mathcal{V}$, or the functors $\bar{\mathcal{B}}(-){+}\mathcal{V}$ or $\bar{\mathcal{Z}}(-){+}\mathcal{V}$ are non-canonical liftings.

We therefore identify a property of distributive laws that (in combination with restriction of liftings to Nom) suffices to enable our main result on rational fixpoints of liftings:

Definition 3.12 Let $F : \mathsf{Set} \to \mathsf{Set}$ be a functor. A monad-over-functor distributive law $\lambda : \mathfrak{S}_\mathsf{f}(\mathcal{V}) \times F(-) \to F(\mathfrak{S}_\mathsf{f}(\mathcal{V}) \times -)$ is *localizable* if for any X and any $W \subseteq \mathcal{V}$, λ restricts to a natural transformation $\lambda^W : \mathfrak{S}_\mathsf{f}(W) \times FX \to F(\mathfrak{S}_\mathsf{f}(W) \times (-)\,)$, i.e. we have $\lambda_X \cdot (m_W \times \mathrm{id}_{FX}) = F(m_W \times \mathrm{id}_X) \cdot \lambda^W_X$ where $m_W : \mathfrak{S}_\mathsf{f}(W) \to \mathfrak{S}_\mathsf{f}(\mathcal{V})$ is the evident subgroup inclusion. A lifting $G : \mathsf{Nom} \to \mathsf{Nom}$ is *localizable* if it is induced by a Nom-restricting localizable distributive law.

Lemma 3.13 *Canonical liftings are localizable.*

Proof The equation $\lambda_X \cdot (m_W \times \mathrm{id}_{FX}) = F(m_W \times \mathrm{id}_X) \cdot \lambda_X^W$ postulated in Definition 3.12 is an instance of naturality of the strength in the left argument. ◻

Example 3.14 By the previous lemma, in particular the identity functor on Nom is a localizable lifting. Moreover, all constant functors on Nom are trivially localizable.

Lemma 3.15 *The class of finitary and mono-preserving localizable liftings is closed under finite products, arbitrary coproducts, and functor composition.*

Proof Finitarity and preservation of monos are clear. The lifting property is immediate from creation of finite products and coproducts by $U : \mathsf{Nom} \to \mathsf{Set}$. Since moreover finite products and coproducts in Nom are formed as in $\mathfrak{S}_\mathsf{f}(\mathcal{V})$-set, it is clear that they preserve the property of being induced by a Nom-restricting distributive law. It remains to show preservation of localizability by the mentioned constructions. Let $W \subseteq \mathcal{V}$.

Finite products: Since the terminal functor is constant, it suffices to consider binary products $G \times H$ of functors G, H on Nom induced by Nom-restricting localizable distributive laws λ_G, λ_H. Indeed, $G \times H$ is induced by the distributive law

$$(\lambda_{G\times H})(\pi, (x, y)) = ((\lambda_G)_X(\pi, x), (\lambda_H)_X(\pi, y)),$$

and if $\pi \in \mathfrak{S}_\mathsf{f}(W)$ then the right-hand side is in $(G \times H)(\mathfrak{S}_\mathsf{f}(W) \times X)$.

Coproducts: For $i \in I$, let the functors G_i on Nom be induced by Nom-restricting localizable distributive laws λ_{G_i}. Then $G = \coprod_{i\in I} G_i$ is induced by the distributive laws

$$(\lambda_G)_X(\pi, \mathsf{in}_i(x)) = \mathsf{in}_i((\lambda_{G_i})_X(\pi, x)),$$

where in_i denotes the i-th coproduct injection, and if $\pi \in \mathfrak{S}_\mathsf{f}(W)$ then the right-hand side is in $\coprod G_i(\mathfrak{S}_\mathsf{f}(W) \times X)$.

Functor composition: Let G, H be functors on Nom induced by Nom-restricting localizable distributive laws λ_G, λ_H. Then GH is induced by the distributive law

$$(\lambda_{GH})_X(\pi, x) = G(\lambda_H)_X((\lambda_G)_{HX}(\pi, x)),$$

and if $\pi \in \mathfrak{S}_\mathsf{f}(W)$ then the right-hand side is in $GH(\mathfrak{S}_\mathsf{f}(W) \times X)$. ◻

Definition 3.16 Recall that the class of *polynomial functors* is the smallest class of endofunctors on Nom that contains all constant functors and the identity functor and is closed under coproducts and finite products.

Note in particular that the functors in Example 3.11 are polynomial. By Example 3.14 and Lemma 3.15, we have

Lemma 3.17 *All polynomial functors are finitary localizable liftings and preserve monos.*

We see next that there are liftings that fail to be localizable, and indeed our example does not allow the desired lifting of rational fixpoints from Set to Nom:

Example 3.18 Consider the functor $FX = \mathcal{V} \times X$ with the lifting $\tilde{F}(X, \cdot) = (\mathcal{V}, \cdot) \times (X, \star)$, where $\cdot$ is the usual action on $\mathcal{V}$ and $\pi \star x$ is defined as $g \cdot \pi \cdot g^{-1} \cdot x$ for some fixed permutation $g : \mathcal{V} \to \mathcal{V}$ such that there is a name $v_0 \in \mathcal{V}$ for which the names $g^n \cdot v_0 =: v_n$ are pairwise distinct for $n \in \mathbb{Z}$. This is well-defined because by Remark 2.7, $(X, \cdot)$ uniquely extends to a $\mathfrak{S}(\mathcal{V})$-set. This lifting corresponds to the distributive law defined by

$$\lambda_X : \mathfrak{S}_\mathsf{f}(\mathcal{V}) \times \mathcal{V} \times X \to \mathcal{V} \times \mathfrak{S}_\mathsf{f}(\mathcal{V}) \times X, \quad (\pi, v, x) \mapsto (\pi(v), g \cdot \pi \cdot g^{-1}, x).$$

This distributive law does not satisfy locality; to see this, consider $W = \{v_0, v_1\}$ and $\pi = (v_0\, v_1) \in \mathfrak{S}_\mathsf{f}(W)$; then $g \cdot \pi \cdot g^{-1} = (v_1\, v_2)$ is not in $\mathfrak{S}_\mathsf{f}(W)$ (qua subgroup of $\mathfrak{S}_\mathsf{f}(\mathcal{V})$), since $v_2 \notin W$.

We have rational fixpoints $\varrho\tilde{F}$ in Nom and ϱF in Set; we show that (1) neither is $U\varrho\tilde{F}$ a subcoalgebra of ϱF, and (2) nor does ϱF lift to an $\tilde{F}$-coalgebra:

(1) The rational fixpoint $\varrho\tilde{F}$ contains behaviours that are not lfp in Set, so that $U(\varrho\tilde{F})$ is not a subcoalgebra of ϱF: consider the coalgebra $c : (\mathcal{V}, \cdot) \to \tilde{F}(\mathcal{V}, \cdot) = (\mathcal{V}, \cdot) \times (\mathcal{V}, \star)$ defined by $c(v) = (v, g(v))$. This coalgebra structure is equivariant, because

$$c(\pi \cdot v) = (\pi \cdot v, g \cdot \pi \cdot v)) = (\pi \cdot v, g \cdot \pi \cdot g^{-1} \cdot g \cdot v)) = (\pi \cdot v, \pi \star g(v)).$$

Since $\mathcal{V}$ is orbit-finite, c is lfp. Moreover, c is a subcoalgebra of $\varrho\tilde{F}$, i.e. the coalgebra homomorphism $c^\dagger : (\mathcal{V}, c) \to \varrho(\tilde{F}, r)$ is monic, because v can be recovered from $c^\dagger(v)$ via $v = \mathsf{outl} \circ \tilde{F}c^\dagger \circ c(v) = \mathsf{outl} \circ r \circ c^\dagger(v)$.
However, $(\mathcal{V}, c)$ considered as an F-coalgebra in Set is not lfp, because the smallest subcoalgebra containing v_0 is $\{g^n \cdot v_0 \mid n \geq 0\}$, which is infinite by the choice of g and v_0.

(2) The coalgebra $d : 1 \to F1$ defined by $d(*) = (v_0, *) \in \mathcal{V} \times 1$ (where $1 = \{*\}$) is, trivially, lfp and a subcoalgebra of ϱF. The unique coalgebra homomorphism $1 \to \varrho F$ defines an element $d^\dagger \in \varrho F$.
Assuming some $\mathfrak{S}_\mathsf{f}(\mathcal{V})$-set structure $\cdot$ on ϱF such that $r : (\varrho F, \cdot) \to \tilde{F}(\varrho F, \cdot)$ is equivariant, the support of $d^\dagger$ must contain v_0 and the support of $d^\dagger$ in $(\varrho F, \star)$, which is the support of $g^{-1} \cdot d^\dagger$ in $(\varrho F, \cdot)$. Iterating this observation we find that the support of $d^\dagger$ contains $g^{-n} \cdot v_0$ for all $n \in \mathbb{N}$, hence is infinite.

Remark 3.19 The functor $\tilde{F}$ from the previous example is naturally isomorphic to the harmless (in fact, polynomial) functor

$$\bar{F}(X, \cdot) = (\mathcal{V}, \cdot) \times (X, \cdot),$$

where the isomorphism $\tau : \bar{F} \to \tilde{F}$ is given by

$$\tau_X(v, x) = (v, g \cdot x)$$

using the fact that the action of $\mathfrak{S}_\mathsf{f}(\mathcal{V})$ on the nominal set X extends uniquely to an action of $\mathfrak{S}(\mathcal{V})$ (Remark 2.7.3). In fact, τ_X is clearly bijective. We have to show that τ_X is equivariant:

$$\tau_X(\pi \cdot v, \pi \cdot x) = \big(\pi \cdot v, g \cdot \pi \cdot x\big) = \big(\pi \cdot v, g \cdot \pi \cdot g^{-1} \cdot g \cdot x\big) = \big(\pi \cdot v, \pi \star (g \cdot x)\big).$$

Finally, we compute the naturality square for an equivariant map $f : X \to Y$:

$$\tau_Y \cdot (\mathrm{id}_\mathcal{V} \times f)(v, x) = (v, g \cdot f(x)) = (v, f(g \cdot x)) = (\mathrm{id}_\mathcal{V} \times f) \cdot \tau_X(v, x).$$

This fact may be slightly surprising, and shows that lifting of the rational fixpoint is a representation-dependent property of functors on Nom rather than an intrinsic one; it serves only as a technical tool in the computation of rational fixpoints. The isomorphism $\tilde{F} \cong \bar{F}$ implies that the counterexample to localizability is not only somewhat contrived but can also be circumvented; that is, instead of calculating the rational fixpoint of $\tilde{F}$ we can calculate that of $\bar{F}$, which is perfectly amenable to our methods. In fact we have no example of a lifting that is not isomorphic to a localizable one.

In Example 3.18, we have seen that Nom-restricting distributive laws need not be localizable. As the following simple example shows, localizability and Nom-restriction are in fact independent, i.e. localizable distributive laws also need not be Nom-restricting:

Example 3.20 Let $(Y, \cdot)$ be some non-nominal $\mathfrak{S}_\mathsf{f}(\mathcal{V})$-set. Consider the constant functor $KX = Y$ with the distributive law $\lambda_X : \mathfrak{S}_\mathsf{f}(\mathcal{V}) \times KX \to K(\mathfrak{S}_\mathsf{f}(\mathcal{V}) \times X)$ defined by

$$\lambda_X(\pi, y) = \pi \cdot y \qquad (y \in Y).$$

Since K is constant, λ is trivially localizable. However, λ induces, as its $\mathfrak{S}_\mathsf{f}(\mathcal{V})$-set lifting, the constant functor $\bar{K} = (Y, \cdot)$, and hence fails to be Nom-restricting.

4 Rational Fixpoints of Localizable Liftings

We proceed to analyse rational fixpoints of liftings $\bar{F}$ in relation to rational fixpoints of the underlying functor F. We have seen in Example 3.18 that even when F is finitary, the rational fixpoint of $\bar{F}$ in general need not be a lifting of the rational fixpoint of F (in contrast to the situation with initial algebras). Our main result (Theorem 4.14) establishes that for *localizable* liftings, the rational fixpoint of F does lift to the rational fixpoint of $\bar{F}$. As a consequence, we also obtain concrete descriptions of the rational fixpoint for functors on Nom that are quotients of lifted functors (but not themselves liftings of Set-functors) (Section 5), e.g. functors associated to a binding signature (Section 6.1).

Assumption 4.1 In this section, assume that $\bar{F} : \mathsf{Nom} \to \mathsf{Nom}$ is a localizable lifting of a finitary functor $F : \mathsf{Set} \to \mathsf{Set}$.

Lemma 4.2 *If for a coalgebra $c : C \to \bar{F}C$ the underlying coalgebra $c : C \to FC$ is lfp in* Set*, then $c : C \to \bar{F}C$ is lfp in* Nom*.*

Proof Let $x \in C$, and let O be the orbit of x; we have to construct an orbit-finite subcoalgebra Q of C containing x. The lfp property of (C, c) in Set provides us with a finite subcoalgebra (P, p) with $x \in P$. We take $Q \subseteq C$ to be the union of the orbits of the elements of P, i.e. the closure of P in C under the $\mathfrak{S}_\mathsf{f}(\mathcal{V})$-action. Then Q is a nominal set; applying $\bar{F}$ to the equivariant inclusion $Q \to C$, we obtain that FQ is closed under the $\mathfrak{S}_\mathsf{f}(\mathcal{V})$-action in FC. Note also that Q is orbit-finite since P is finite. We are done once show that Q is closed under the coalgebra structure c. Let $y \in P$ and $\pi \in \mathfrak{S}_\mathsf{f}(\mathcal{V})$, so that $\pi \cdot y \in Q$. Since P is a subcoalgebra of (C, c) in Set, we have $p(y) = c(y)$ and hence

$$c(\pi \cdot y) = \pi \cdot c(y) = \pi \cdot p(y).$$

Since $p(y) \in FP \subseteq FQ$ and FQ is closed under the $\mathfrak{S}_\mathsf{f}(\mathcal{V})$-action in FC, it follows that $c(\pi \cdot y) \in FQ$. ◻

Lemma 4.3 *If $c : C \to \bar{F}C$ is an orbit-finite coalgebra in* Nom*, then $c : C \to FC$ is lfp in* Set*.*

Proof First define the following closure operator on subsets X of C:

$$\mathscr{C}\!\ell(X) = \{y \in C \mid \mathsf{supp}(y) \subseteq \textstyle\bigcup_{x \in X} \mathsf{supp}(x)\}.$$

By Lemma 2.11, $\mathscr{C}\!\ell$ preserves finite sets. Now let $x \in C$. Pick a subset $O \subseteq C$ that contains precisely one element from each orbit of C, and put $P = \mathscr{C}\!\ell(\{x\} \cup O)$, with in_P denoting the embedding $P \rightarrowtail C$. Since O is finite, P is finite, and since F is finitary, there exists a finite set $Q \subseteq C$ such that $c \cdot \mathsf{in}_P$ factorizes through $F\mathsf{in}_Q : FQ \to FC$. The subset $W = \mathscr{C}\!\ell(P \cup Q) \subseteq C$ is finite as well, and we have:

$$\begin{tikzcd}
C \arrow[r, "c"] & FC \\
P \arrow[u, tail, "\mathsf{in}_P"] \arrow[r, dashed, "c_P"] \arrow[dr, dashed] & FW \arrow[u, tail, "F\mathsf{in}_W"'] \\
 & FQ \arrow[u, tail] \arrow[uu, tail, bend right=60, "F\mathsf{in}_Q"']
\end{tikzcd}$$

Now let G be the finite subgroup of $\mathfrak{S}_\mathsf{f}(\mathcal{V})$ given by the permutations of $\mathsf{supp}(W)$ and note that $\mathsf{supp}(W) = \mathsf{supp}(P \cup Q)$. Then for any $\pi \in G$ and $z \in P$, $\pi \cdot z$ is in W because

$$\mathsf{supp}(\pi \cdot z) = \pi \cdot \mathsf{supp}(z) \subseteq \pi \cdot \mathsf{supp}(W) = \mathsf{supp}(W) = \mathsf{supp}(P \cup Q),$$

where the second-to-last equation holds because $\pi \in G$ and the inclusion holds because $z \in P \subseteq W$. This means we have a commutative diagram

$$\begin{tikzcd}
G \times P \arrow[r, tail] \arrow[d, two heads, "\alpha'"'] & \mathfrak{S}_\mathsf{f}(\mathcal{V}) \times C \arrow[d, "\alpha"] \\
W \arrow[r, tail, "i"'] & C
\end{tikzcd}$$

where α denotes the nominal structure on C. We will now prove that the left-hand map α' is surjective. To see this, let $y \in W$. Then there are $z \in O \subseteq P$ and $\pi \in \mathfrak{S}_\mathsf{f}(\mathcal{V})$ such that $\pi \cdot z = y$, because O contains precisely one element from each orbit. Consider the factorization of π into $\pi = \pi|_{\mathsf{supp}(z)} \cdot g$ as in Remark 2.4. Then g fixes every element of $\mathsf{supp}(z)$, so $g \cdot z = z$, and $\pi|_{\mathsf{supp}(z)}$ fixes every element not contained in $\mathsf{supp}(z) \cup \pi \cdot \mathsf{supp}(z)$. Since $\pi \cdot z = y$ we have $\pi \cdot \mathsf{supp}(z) = \mathsf{supp}(y)$, and because $y \in W$ and $z \in P \subseteq W$ we know that $\mathsf{supp}(z) \cup \mathsf{supp}(y) \subseteq \mathsf{supp}(W)$. Thus $\pi|_{\mathsf{supp}(z)}$ fixes every element not contained in $\mathsf{supp}(W)$ and therefore $\pi|_{\mathsf{supp}(z)}$ lies in G. It follows that

$$\alpha'(\pi|_{\mathsf{supp}(z)}, z) = \pi|_{\mathsf{supp}(z)} \cdot z = \pi|_{\mathsf{supp}(z)} \cdot g \cdot z = \pi \cdot z = y,$$

showing α' to be surjective as desired.

Now fix a splitting $d : W \rightarrowtail G \times P$ of α', i.e. we have $\alpha' \cdot d = \mathrm{id}_W$. Denote by $\alpha'' : G \times W \to C$ the restriction of α. Let $\beta = F\alpha \cdot \lambda_C$ be the nominal structure on FC and $\beta' : G \times FW \to FC$ its restriction. Now consider the diagram below:

$$
\begin{tikzcd}
C \arrow[r, "c"] & FC \arrow[r, equal] & FC \arrow[rr, equal] & & FC \\
 & & & F(G \times W) \arrow[ul, "F\alpha''"'] & \\
G \times P \arrow[u, "i \cdot \alpha'"] \arrow[r, "G \times c_P"'] \arrow[ur, phantom, "(1)"] & G \times FW \arrow[u, "\beta'"] \arrow[r, "\lambda^G_W"'] \arrow[ur, phantom, "(2)"] & F(G \times W) \arrow[u, "F\alpha''"] \arrow[ur, "\mathrm{id}"] \arrow[r, tail, "F(G+d)"'] & F(G^2 \times P) \arrow[u, "F(G \times \alpha')"'] \arrow[r, "F(\mu + P)"'] \arrow[ul, phantom, "(3)"] \arrow[ur, phantom, "(4)"] & F(G \times P) \arrow[uu, "F(i \cdot \alpha')"']
\end{tikzcd}
$$

The middle triangle trivially commutes, and so do the other parts:

(1) commutes because c is equivariant.
(2) commutes using the definition of β, naturality of λ and Assumption 4.1 (denote by $j : G \to \mathfrak{S}_{\mathsf{f}}(\mathcal{V})$ the inclusion of the subgroup G):

$$
\begin{tikzcd}
FC \arrow[r, equal] & FC \\
F(\mathfrak{S}_{\mathsf{f}}(\mathcal{V}) \times C) \arrow[u, "F\alpha"] & \\
\mathfrak{S}_{\mathsf{f}}(\mathcal{V}) \times FC \arrow[u, "\lambda_C"] \arrow[r, "\lambda_C"'] & F(\mathfrak{S}_{\mathsf{f}}(\mathcal{V}) \times C) \arrow[uu, "F\alpha"'] \\
\mathfrak{S}_{\mathsf{f}}(\mathcal{V}) \times FW \arrow[u, tail, "\mathrm{id} \times Fi"] \arrow[r, "\lambda_W"'] & F(\mathfrak{S}_{\mathsf{f}}(\mathcal{V}) \times W) \arrow[u, tail, "F(\mathrm{id} \times i)"'] \\
G \times FW \arrow[u, "j \times \mathrm{id}"] \arrow[r, "\lambda^G_W"'] \arrow[uuuu, bend left=60, "\beta'"] & F(G \times W) \arrow[u, "F(j \times \mathrm{id})"'] \arrow[uuuu, bend right=60, "F\alpha''"']
\end{tikzcd}
$$

(3) commutes since $\alpha' \cdot d = \mathrm{id}_W$.
(4) commutes using the axioms of the group action α; here μ denotes the multiplication of the group G and we also use that G is a subgroup of $\mathfrak{S}_{\mathsf{f}}(\mathcal{V})$.

Thus, we see that $G \times P$ is a finite coalgebra and $i \cdot \alpha' : G \times P \to C$ a coalgebra homomorphism with $\alpha'(\mathrm{id}, x) = x$. Therefore $x \in C$ is contained in a finite subcoalgebra and we conclude that (C, c) is lfp. $\square$

Because $U : \mathsf{Nom} \to \mathsf{Set}$ creates, and lfp coalgebras are closed under, filtered colimits, we can immediately generalize Lemma 4.3 to lfp coalgebras:

Corollary 4.4 *If $c : C \to \bar{F}C$ is lfp in Nom, then $c : C \to FC$ is lfp in Set.*

Combining this with Lemma 4.2 we obtain:

Corollary 4.5 *A coalgebra $c : C \to \bar{F}C$ in Nom is lfp if and only if the underlying coalgebra is lfp in Set.*

In order to lift the rational fixpoint $(\varrho F, r)$ from Set to Nom, we need to equip it with nominal structure:

Lemma 4.6 *The rational fixpoint $(\varrho F, r)$ carries a canonical group action making r equivariant.*

Proof We define the desired group action by coinduction. To this end we consider the F-coalgebra

$$\mathfrak{S}_\mathsf{f}(\mathcal{V}) \times \varrho F \xrightarrow{\mathrm{id}\times r} \mathfrak{S}_\mathsf{f}(\mathcal{V}) \times F(\varrho F) \xrightarrow{\lambda_{\varrho F}} F(\mathfrak{S}_\mathsf{f}(\mathcal{V}) \times \varrho F).$$

We first prove that this coalgebra is lfp. Let $(\pi, x) \in \mathfrak{S}_\mathsf{f}(\mathcal{V}) \times \varrho F$. Since ϱF is an lfp coalgebra, we obtain an orbit-finite subcoalgebra (S, s) of ϱF containing x. For the finite subgroup $G = \mathfrak{S}_\mathsf{f}(\mathsf{supp}(\pi))$ of $\mathfrak{S}_\mathsf{f}(\mathcal{V})$, we have a restriction $\lambda_S^G : G \times FS \to F(G \times S)$ by localizability. Now consider the diagram below (where, as usual, we abuse objects to denote their identity, here: G in place of id_G):

$$\begin{tikzcd}
 & \mathfrak{S}_\mathsf{f}(\mathcal{V}) \times \varrho F \arrow[r, "\mathfrak{S}_\mathsf{f}(\mathcal{V}) \times r^F"] & \mathfrak{S}_\mathsf{f}(\mathcal{V}) \times F\varrho F \arrow[r, "\lambda_{\varrho F}"] & F(\mathfrak{S}_\mathsf{f}(\mathcal{V}) \times \varrho F) \\
1 \arrow[ur, bend left, "{(\pi, x)}"] \arrow[r, "{(\pi, x)}"] \arrow[dr, bend right, "{(\pi, x)}"'] & \mathfrak{S}_\mathsf{f}(\mathcal{V}) \times S \arrow[u, "\mathfrak{S}_\mathsf{f}(\mathcal{V}) \times \mathsf{in}_S"'] \arrow[r, "\mathfrak{S}_\mathsf{f}(\mathcal{V}) \times s"] \arrow[ur, phantom, "(1)"] & \mathfrak{S}_\mathsf{f}(\mathcal{V}) \times FS \arrow[u, "\mathfrak{S}_\mathsf{f}(\mathcal{V}) \times F\mathsf{in}_S"'] \arrow[r, "\lambda_S"] \arrow[ur, phantom, "(2)"] & F(\mathfrak{S}_\mathsf{f}(\mathcal{V}) \times S) \arrow[u, "F(\mathfrak{S}_\mathsf{f}(\mathcal{V}) \times \mathsf{in}_S)"'] \\
 & G \times S \arrow[u, "\mathsf{in}_G \times S"'] \arrow[r, "G \times s"] \arrow[ur, phantom, "(3)"] & G \times FS \arrow[u, "\mathsf{in}_G \times FS"'] \arrow[r, "\lambda_S^G"] \arrow[ur, phantom, "(4)"] & F(G \times S) \arrow[u, "F(\mathsf{in}_G \times S)"']
\end{tikzcd}$$

It commutes because all its inner parts do:

(1) (S, s) is a subcoalgebra of $(\varrho F, r^F)$.
(2) Naturality of λ .
(3) Properties of products.
(4) λ_S^G restricts λ_S.

Hence $(G \times S, \lambda_S^G \cdot (G \times s))$ is a finite subcoalgebra of $\mathfrak{S}_\mathsf{f}(\mathcal{V}) \times \varrho F$ containing (π, x) proving $\mathfrak{S}_\mathsf{f}(\mathcal{V}) \times \varrho F$ to be an lfp coalgebra.

Now we obtain a unique coalgebra homomorphism $u : \mathfrak{S}_\mathsf{f}(\mathcal{V}) \times \varrho F \to \varrho F$. It remains to show that u is a group action. To this end, we show that u is the restriction of the group action on the final F-coalgebra to ϱF.

From [7, Theorem 3.2.3] and [31] we know that the final F-coalgebra in the $\mathfrak{S}_\mathsf{f}(\mathcal{V})$-sets is just the final F-coalgebra $(\nu F, t)$, with the group action on the carrier defined by coinduction, i.e., the group action is the unique map a such that the diagram below commutes:

$$\begin{tikzcd}
\mathfrak{S}_\mathsf{f}(\mathcal{V}) \times \nu F \arrow[r, "\mathfrak{S}_\mathsf{f}(\mathcal{V}) \times t"] \arrow[d, dashed, "a"'] & \mathfrak{S}_\mathsf{f}(\mathcal{V}) \times F\nu F \arrow[r, "\lambda_{\nu F}"] & F(\mathfrak{S}_\mathsf{f}(\mathcal{V}) \times \nu F), \arrow[d, "Fa"] \\
\nu F \arrow[rr, "t"] & & F\nu F.
\end{tikzcd}$$

Since F preserves monos, the rational fixpoint is a subcoalgebra of $(\nu F, t)$, i.e. the unique coalgebra homomorphism $j : (\varrho F, r) \rightarrowtail (\nu F, t)$ is monic. Then $\mathfrak{S}_\mathsf{f}(\mathcal{V}) \times j$

also is a coalgebra homomorphism:

$$\begin{tikzcd}
\mathfrak{S}_{\mathsf{f}}(\mathcal{V}) \times \varrho F \arrow[r, "\mathfrak{S}_{\mathsf{f}}(\mathcal{V}) \times r"] \arrow[d, "\mathfrak{S}_{\mathsf{f}}(\mathcal{V}) \times j"'] \arrow[dr, phantom, "(1)"] & \mathfrak{S}_{\mathsf{f}}(\mathcal{V}) \times F\varrho F \arrow[r, "\lambda_{\varrho F}"] \arrow[d, "\mathfrak{S}_{\mathsf{f}}(\mathcal{V}) \times Fj"'] \arrow[dr, phantom, "(2)"] & F(\mathfrak{S}_{\mathsf{f}}(\mathcal{V}) \times \varrho F) \arrow[d, "F(\mathfrak{S}_{\mathsf{f}}(\mathcal{V}) \times j)"] \\
\mathfrak{S}_{\mathsf{f}}(\mathcal{V}) \times \nu F \arrow[r, "\mathfrak{S}_{\mathsf{f}}(\mathcal{V}) \times t"] & \mathfrak{S}_{\mathsf{f}}(\mathcal{V}) \times F\nu F \arrow[r, "\lambda_{\nu F}"] & F(\mathfrak{S}_{\mathsf{f}}(\mathcal{V}) \times \nu F)
\end{tikzcd}$$

This diagram commutes because (1) j is a coalgebra homomorphism and (2) λ is natural. By finality of νF, $j \cdot u = a \cdot (\mathrm{id}_{\mathfrak{S}_{\mathsf{f}}(\mathcal{V})} \times j)$. As j is monic, this means that u is the restriction of the group action a to ϱF and hence a group action. ◻

Definition 4.7 For a coalgebra $c : C \to HC$ of a functor $H : \mathcal{C} \to \mathcal{C}$, we denote the *iterated* coalgebra structure by $c^{(n)} : C \to H^n C$, $n \geq 0$, which is inductively defined by $c^{(0)} = \mathrm{id}_C$ and

$$c^{(n+1)} \equiv \big(\; C \xrightarrow{c^{(n)}} H^n C \xrightarrow{H^n c} H^{n+1} C \; \big).$$

Moreover, given another functor M on $\mathcal{C}$ and a natural transformation $\varphi : MH \to HM$, we define the *iterated* transformation $\varphi^{(n)} : MH^n \to H^n M$ by $\varphi^{(0)} = \mathrm{id}$ and $\varphi^{(n+1)} = H^n \varphi \cdot \varphi^{(n)} H$.

It is easy to verify that in the case where M is a monad and φ a distributive law of M over H, the iterated transformation $\lambda^{(n)}$ is a distributive law of M over H^n. These two notions of iteration interact nicely:

Lemma 4.8 *For φ and c as in Definition 4.7, $(\varphi_C \cdot Mc)^{(n)} = \varphi_C^{(n)} \cdot Mc^{(n)}$.*

Proof For $n = 0$, the equality reduces to $\mathrm{id}_{MC} = \mathrm{id}_{MC}$. For the induction step, we have that

$$\begin{tikzcd}
MC \arrow[r, "Mc^{(n)}"] \arrow[rr, bend left, "c^{(n+1)}"] \arrow[dr, "(\varphi_C \cdot Mc)^{(n)}"'] \arrow[drrr, bend right, "(\varphi_C \cdot Mc)^{(n+1)}"'] & MH^n C \arrow[r, "MH^n c"] \arrow[d, "\varphi_C^{(n)}"] \arrow[dr, phantom, "(2)"] & MH^{n+1} C \arrow[d, "\varphi_{HC}^{(n)}"'] \arrow[dr, "\varphi_C^{(n+1)}"] & \\
& H^n MC \arrow[r, "H^n Mc"'] & H^n MHC \arrow[r, "H^n \varphi_C"'] & H^{n+1} MC
\end{tikzcd}$$

commutes, using (1) the induction hypothesis, (2) naturality of $\varphi^{(n)}$, and (3) the definition of $\varphi^{(n+1)}$. ◻

In Lemma 4.10 below we will establish a coinduction principle using iterated coalgebra structures. For its soundness proof, we use that for a finitary set endofunctor the terminal coalgebra can be obtained by an iterative construction that we now recall.

Remark 4.9 (1) Let $H : \mathsf{Set} \to \mathsf{Set}$ be a finitary endofunctor. The *terminal sequence* of H is the op-chain $(H^n 1)_{n<\omega}$ with the connecting maps

$$H^n! : H^{n+1} 1 \to H^n 1 \qquad \text{for every } n < \omega.$$

Its limit $H^\omega 1$ does not in general yield the terminal coalgebra. However, Worrell [35] shows that by continuing the terminal sequence for ω more steps, one does obtain the terminal coalgebra. Indeed, denote by $\ell_{\omega,n} : H^\omega 1 \to H^n 1$ the limit projections and let $H^{\omega+n}1 = H^n(H^\omega 1)$. We define the connecting map $\ell_{\omega+1,\omega} : H^{\omega+1}1 \to H^\omega 1$ as the unique morphism such that $\ell_{\omega,n} \cdot \ell_{\omega+1,\omega} = H\ell_{\omega,n}$, and by applying H iteratively we obtain $\ell_{\omega+n+1,\omega+n} = H^n \ell\omega + 1, \omega : H^{\omega+n+1}1 \to H^{\omega+n}1$. Worrell proves that the limit of the ensuing op-chain formed by the $H^{\omega+n}1$ is the terminal coalgebra νH. Moreover, he shows that all connecting morphisms $\ell_{\omega+n+1,\omega+n}$ are injective maps; it follows that νH is actually the intersection of all $H^{\omega+n}1$.

(2) Recall that every coalgebra $c : C \to HC$ induces a *canonical cone* $c_n : C \to H^n 1$, $n < \omega + \omega$ on the above op-chain defined by (transfinite) induction as follows: $c_0 : C \to 1$ is uniquely determined, for isolated steps one has $c_{n+1} = Hc_n \cdot c$ and for the limit step we define c_ω to be the unique map such that $\ell_{\omega,n} \cdot c_\omega = c_n$. Note that the unique H-coalgebra morphism $c^\dagger : C \to \nu H$ can be obtained as the unique map such that $\ell_{\omega+\omega,n} \cdot c^\dagger = c_n$ for every $n < \omega + \omega$, where the maps $\ell_{\omega+\omega,n} : \nu H \to H^n 1$ are the limit projections.

Lemma 4.10 *Let $H : \mathsf{Set} \to \mathsf{Set}$ be a finitary endofunctor. If for H-coalgebras (C, c) and (D, d) there is an object X with maps $p_1 : X \to C$ and $p_2 : X \to D$ such that*

$$\begin{tikzcd} X \arrow[r, "p_1"] \arrow[dr, "p_2"'] & C \arrow[r, "c^{(n)}"] & H^n C \arrow[dr, "H^n !"] & \\ & D \arrow[r, "d^{(n)}"] & H^n D \arrow[r, "H^n !"] & H^n 1 \end{tikzcd} \tag{4.1}$$

commutes for all $n < \omega$, then $c^\dagger \cdot p_1 = d^\dagger \cdot p_2$.

Proof First, an easy induction shows that for the canonical cone $c_n : C \to H^n 1$ we have $c_n = H^n ! \cdot c^{(n)}$ for every $n < \omega$. Now it follows from Remark 4.9 that elements $x \in C$ and $y \in D$ are behaviourally equivalent, i.e. $c^\dagger(x) = d^\dagger(y)$, if and only if $c_n(x) = d_n(y)$ for all $n < \omega$. Indeed, necessity is obvious since $c_n = \ell_{\omega+\omega,n} \cdot c^\dagger$ (and similarly for d) and sufficiency follows from the fact that all $\ell_{\omega+n,\omega}$ are injective.

By hypothesis we have for every $x \in X$ that

$$c_n(p_1(x)) = H^n ! \cdot c^{(n)} \cdot p_1(x) = H^n ! \cdot d^{(n)} \cdot p_1 = d_n(p_2(x)),$$

and equivalently, $c^\dagger(p_1(x)) = d^\dagger(p_2(x))$, which completes the proof. □

Remark 4.11 Consider the nominal sets $\bar{F}^n D(\varrho F)$ for $n < \omega$ (recalling that DX is X equipped with the trivial nominal structure). We denote by

$$\beta_n : \mathfrak{S}_{\mathsf{f}}(\mathcal{V}) \times F^n(\varrho F) \to F^n(\varrho F)$$

the group action on $\bar{F}^n D(\varrho F)$. Note that for $n = 0$, the action β_0 is trivial, i.e. it is the projection

$$\beta_0 = \mathsf{outr} : \mathfrak{S}_{\mathsf{f}}(\mathcal{V}) \times \varrho F \to \varrho F.$$

By Assumption 4.1, the lifting $\bar{F}$ is specified by a distributive law. Hence, we have $\beta_{n+1} = F\beta_n \cdot \lambda_{F^n \varrho}$. An easy induction thus shows that β_n has the form

$$\beta_n = \big(\mathfrak{S}_{\mathsf{f}}(\mathcal{V}) \times F^n(\varrho F) \xrightarrow{\lambda^{(n)}_{\varrho F}} F^n(\mathfrak{S}_{\mathsf{f}}(\mathcal{V}) \times \varrho F) \xrightarrow{F^n \,\mathsf{outr}} F^n(\varrho F)\big). \tag{4.2}$$

Lemma 4.12 *For the $\mathfrak{S}_\mathsf{f}(\mathcal{V})$-set structure from Lemma 4.6, $t \in \varrho F$ is supported by*

$$s(t) = \bigcup_{n\geq 0} \mathsf{supp}(r^{(n)}(t)) \quad \textit{where } r^{(n)} : \varrho F \to F^n(\varrho F)$$

and where the support of $r^{(n)}(t)$ is taken in $\bar{F}^n D(\varrho F)$.

Proof Let $\pi \in \mathsf{Fix}(s(t))$. Abbreviate the coalgebra structure on $\mathfrak{S}_\mathsf{f}(\mathcal{V}) \times \varrho F$ by $p := \lambda_{\varrho F} \cdot (\mathrm{id}_{\mathfrak{S}_\mathsf{f}(\mathcal{V})} \times r)$, and recall from Lemma 4.6 that the induced algebra structure on ϱF is $p^\dagger$. Now consider the following diagram:

$$\begin{tikzcd}
1 \arrow[r, "{(\pi,t)}"] \arrow[dd, "t"'] & \mathfrak{S}_\mathsf{f}(\mathcal{V}) \times \varrho F \arrow[d, "\mathfrak{S}_\mathsf{f}(\mathcal{V}) \times r^{(n)}"'] \arrow[dr, bend left, "p^{(n)}"] \arrow[dr, phantom, "\text{Lemma 4.8}"] & & \\
\arrow[r, phantom, "(1)"] & \mathfrak{S}_\mathsf{f}(\mathcal{V}) \times F^n(\varrho F) \arrow[r, "\lambda^{(n)}_{\varrho F}"] \arrow[d, "\mathsf{outr}"'] \arrow[dr, "\beta_n"] \arrow[d, phantom, shift left=6, "(*)"] & F^n(\mathfrak{S}_\mathsf{f}(\mathcal{V}) \times \varrho F) \arrow[d, "F^n\,\mathsf{outr}"] \arrow[dr, bend left, "F^n!"] \arrow[dl, phantom, "(2)"] \arrow[dr, phantom, "(3)"] & \\
\varrho F \arrow[r, "r^{(n)}"] & F^n(\varrho F) \arrow[r, "F^n\mathrm{id}"] & F^n(\varrho F) \arrow[r, "F^n!"] & F^n 1
\end{tikzcd}$$

Part(1) commutes trivially, Part (2) by the definition of β_n, and Part (3) by finality. For

$$(\pi, r^{(n)}(t)) \in \mathfrak{S}_\mathsf{f}(\mathcal{V}) \times F^n(\varrho F),$$

Part $(*)$ commutes as well, because π fixes $\mathsf{supp}(r^{(n)}(t)) \subseteq s(t)$, and thus $\pi \cdot r^{(n)}(t) = r^{(n)}(t)$ holds in $\bar{F}D(\varrho F)$, i.e. we have $\beta_n(\pi, r^{(n)}(t)) = r^{(n)}(t)$. It follows that the outside of the diagram commutes, and we obtain by Lemma 4.10 that (π, t) and t are identified in νF and thus also in its subcoalgebra ϱF. In other words, $\pi \cdot t = t$ with respect to the algebra structure $p^\dagger : \mathfrak{S}_\mathsf{f}(\mathcal{V}) \times \varrho F \to \varrho F$, and therefore $s(t)$ supports t. ◻

Lemma 4.13 *For $t \in \varrho F$, $s(t)$ is finite.*

Proof Since ϱF is lfp in Set, we have a finite subcoalgebra $j : (C, c) \to (\varrho F, r)$ containing t. Then define

$$S = \bigcup_{x \in C} \mathsf{supp}(r \cdot j(x)) \subseteq \mathcal{V},$$

where the support is taken in $\bar{F}D(\varrho F)$. Clearly finiteness of C implies that S is finite. We will now show that $s(t) \subseteq S$ by proving that S supports $r^{(n)}(t) \in \bar{F}^n D(\varrho F)$ for every $n < \omega$. This follows once we show that the diagram

$$G \times C \xrightarrow{G \times c^{(n)}} G \times F^n C \underset{\mathsf{outr}'_n}{\overset{\beta'_n}{\longrightarrow}} F^n \varrho F \tag{4.3}$$

commutes, with $G = \mathfrak{S}_\mathsf{f}(\mathcal{V} \setminus S)$, $m : G \hookrightarrow \mathfrak{S}_\mathsf{f}(\mathcal{V})$, β'_n the restriction of group action β_n on $\bar{F}^n D(\varrho F)$ to G and C, i.e. $\beta'_n = \beta_n \cdot (m \times F^n j)$, and $\mathsf{outr}'_n = \mathsf{outr} \cdot (m \times F^n j) = F^n j \cdot \mathsf{outr}$.

To show commutation of (4.3), we proceed by induction in n. For $n = 0$, (4.3) is clear, because $\beta_0 = \mathsf{outr}$. For the induction step consider the diagram

$$
\begin{tikzcd}
 & & G \times F^{n+1}C \arrow[r, "\lambda^G_{F^nC}"] \arrow[rr, bend left, "\beta'_{n+1}"] & F(G \times F^nC) \arrow[dr, "F\beta'_n"] \arrow[d, phantom, "F(\mathrm{IH})"] & (1) \\
G \times C \arrow[r, "G \times c"] \arrow[urr, bend left, "G \times c^{(n+1)}"] \arrow[dr, "G \times c^{(n+1)}"'] & G \times FC \arrow[ur, "G \times Fc^{(n)}"] \arrow[r, "\lambda^G_C"] \arrow[d, "G \times Fc^{(n)}"] \arrow[drr, "\beta'_1"] \arrow[drr, bend right, "\mathsf{outr}'_{n+1}"'] \arrow[u, phantom, "\text{Def.}"] & F(G \times C) \arrow[ur, "F(G \times c^{(n)})"] \arrow[r, "F(G \times c^{(n)})"] \arrow[dr, "F\,\mathsf{outr}'_0"] \arrow[u, phantom, "\text{Naturality}"] \arrow[d, phantom, "(2)\ (3)"] & F(G \times F^nC) \arrow[r, "F\,\mathsf{outr}'_n"] \arrow[d, phantom, "(4)"] & F^{n+1}\varrho F \\
 & |[yshift=-5mm]| G \times F^{n+1}C \arrow[rr, "\mathsf{outr}'_1"] \arrow[u, phantom, "\text{Def.}"] \arrow[r, phantom, "(5)"] & & F\varrho F \arrow[ur, "Fr^{(n)}"] \arrow[ur, bend right] &
\end{tikzcd}
$$

Most parts commute by the definitions of β_n, β'_n, and $c^{(n+1)}$, respectively. For the remaining parts:

(1) is the commutative diagram below:

$$
\begin{tikzcd}
G \times F^{n+1}C \arrow[r, "m \times F^{n+1}C"] \arrow[d, "\lambda^G_{F^nC}"'] \arrow[rrrd, bend left, "\beta'_{n+1}"] \arrow[dr, phantom, "\text{Assumption 4.1}"] & \mathfrak{S}_{\mathsf{f}}(\mathcal{V}) \times F^{n+1}C \arrow[r, "\mathfrak{S}_{\mathsf{f}}(\mathcal{V}) \times F^{n+1}j"] \arrow[d, "\lambda_{F^nC}"] \arrow[dr, phantom, "\text{Naturality of } \lambda"] & \mathfrak{S}_{\mathsf{f}}(\mathcal{V}) \times F^{n+1}\varrho F \arrow[d, "\lambda_{\varrho F}"'] \arrow[dr, "\beta_{n+1}"] \arrow[dr, phantom, "\text{Remark 4.11}"'] & \\
F(G \times F^nC) \arrow[r, "F(m \times F^nC)"'] \arrow[rrr, bend right, "F\beta'_n"'] & F(\mathfrak{S}_{\mathsf{f}}(\mathcal{V}) \times F^nC) \arrow[r, "F(\mathfrak{S}_{\mathsf{f}}(\mathcal{V}) \times F^nj)"'] & F(\mathfrak{S}_{\mathsf{f}}(\mathcal{V}) \times F^n\varrho F) \arrow[r, "F^n\beta_n"'] & F^{n+1}\varrho F
\end{tikzcd}
$$

(2) is just the previous item for $n = 0$ using that $\mathsf{outr}'_0 = \beta'_0$.

(3) commutes, i.e. $\beta'_1 = \mathsf{outr}'_1$, because G is defined to consist of those permutations that fix every element in S and therefore fix every element in the image of $r \cdot j$.

(4) commutes because we can remove F and then prove $\mathsf{outr}'_n \cdot (\mathrm{id}_G \times c^{(n)}) = r^{(n)} \cdot \mathsf{outr}'_0$ by induction. For $n = 0$, the desired equation obviously holds. For the induction step consider the commutative diagram below:

$$
\begin{tikzcd}
G \times C \arrow[r, "G \times c^{(n)}"] \arrow[d, "\mathsf{outr}"] \arrow[rr, bend left, "G \times c^{(n+1)}"] \arrow[dd, bend right=60, "\mathsf{outr}'_0"'] & G \times F^nC \arrow[r, "G \times F^nc"] \arrow[d, "\mathsf{outr}"] \arrow[dr, phantom, "\text{Naturality}"] & G \times F^{n+1}C \arrow[d, "\mathsf{outr}"] \arrow[dd, bend left=60, "\mathsf{outr}'_{n+1}"] \\
C \arrow[d, "j"] \arrow[r, phantom, "(\mathrm{IH})"] & F^nC \arrow[r, "F^nc"] \arrow[d, "F^nj"] \arrow[dr, phantom, "\text{Coalgebra Hom.}"] & F^{n+1}C \arrow[d, "F^{n+1}j"] \\
\varrho F \arrow[r, "r^{(n)}"] \arrow[rr, bend right, "r^{(n+1)}"'] & F^n\varrho F \arrow[r, "F^nr"] & F^{n+1}\varrho F
\end{tikzcd}
$$

(5) commutes by a similar induction proof as the previous item, but starting with $n = 1$. ◻

The two previous lemmas combined imply that ϱF is a nominal set; by Lemma 4.6, $(\varrho F, r)$ is a $\bar{F}$-coalgebra, and by Lemma 4.2 this coalgebra is lfp.

Theorem 4.14 *The lifted coalgebra $(\varrho F, r)$ is the rational fixpoint of $\bar{F}$.*

Proof It remains only to show that for every $\bar{F}$-coalgebra (C, c) in Nom with orbit-finite carrier there exists a unique coalgebra homomorphism from C to ϱF. So let (C, c) be an orbit-finite $\bar{F}$ coalgebra. By Lemma 4.3, (C, c) is an lfp F-coalgebra, and thus induces a unique F-coalgebra homomorphism $h : (C, c) \to (\varrho F, r)$ (in Set). This homomorphism $h : C \to \varrho F$ is equivariant; to see this, recall first that the final F-coalgebra $(\nu F, t)$ lifts to the final $\hat{F}$-coalgebra, where $\hat{F}$ is the lifting of F to $\mathfrak{S}_{\mathsf{f}}(\mathcal{V})$-sets induced by the distributive law. Let $h' : (C, c) \to (\nu F, t)$ be the unique homomorphism into the final $\hat{F}$-coalgebra, which is an equivariant map. Recall further that $(\varrho F, r)$ is a subcoalgebra of $(\nu F, t)$ via $j : \varrho F \to \nu F$, say. Note that the group action on ϱF has been defined as the restriction of that on νF, and so j is equivariant. Then clearly $j \cdot h = h'$ by finality of νF; and since j is equivariant and monic it follows that h is equivariant. $\square$

Example 4.15 (1) For all canonical liftings $\bar{F}$ (e.g. the ones we mentioned in Example 3.10), the rational fixpoint $\varrho\bar{F}$ is the rational fixpoint ϱF equipped with the discrete action (cf. Example 2.13). Note that for the cyclic shift functor $\mathcal{Z}$ on Set, the final coalgebra consists of all finitely branching trees where the order of the children of any vertex is taken modulo cyclic shifting, and the rational fixpoint is given by all rational such trees; this follows from the results of [3].

(2) Recall that the final coalgebra for $LX = \mathcal{V} + X \times X + \mathcal{V} \times X$ on Set is carried by the set of all λ-trees and the rational fixpoint by the set of all rational λ-trees. It follows that the rational fixpoint of the (non-canonical but localizable) lifting $\bar{L}$ where $\mathcal{V}$ is equipped with the standard action is carried by the same set with the nominal structure given according to Lemma 4.6; this action applies the standard action of $\mathcal{V}$ to the labels of the leaves of λ-trees.

(3) For the functor $\mathcal{B}(-) + \mathcal{V}$ on Set, the final coalgebra is carried by all unordered trees some of whose leaves are labelled in $\mathcal{V}$. The rational fixpoint is then carried by the set of all rational such trees. To obtain the rational fixpoint of the non-canonical lifting $\bar{\mathcal{B}}(-) + \mathcal{V}$, one equips this set of trees with the action that applies the standard action of $\mathcal{V}$ on the labels of leaves. This follows once again from Lemma 4.6 and Theorem 4.14. A similar description can be given for the functor $\bar{\mathcal{Z}} + \mathcal{V}$; we obtain rational trees some of whose leaves are labelled in $\mathcal{V}$ and where the order of the children of a vertex is only determined up to cyclic shift.

5 Quotients of Nom-Functors

We next consider quotient functors on Nom. For the rest of this section we assume a finitary functor $H : \mathsf{Nom} \to \mathsf{Nom}$ that is a *quotient* of a finitary functor $F : \mathsf{Nom} \to \mathsf{Nom}$, i.e. we have a natural transformation $q : F \twoheadrightarrow H$ with surjective components. We present a sufficient condition on coalgebras for F and H that ensures that the rational fixpoint ϱH is a quotient of the rational fixpoint ϱF. We then introduce a simple, if ad-hoc, condition on F that ensures that the mentioned sufficient condition is satisfied for all quotients H of F. Combining this result with the ones from the previous section, we obtain a description of the rational fixpoint of endofunctors H on Nom arising from binding signatures and exponentiation.

Definition 5.1 An H-coalgebra (C, c) is a *quotient* of an F-coalgebra (A, a) if there is a surjective H-coalgebra homomorphism $h : (A, q_A \cdot a) \to (C, c)$.

Theorem 5.2 *Suppose that every orbit-finite H-coalgebra is a quotient of an orbit-finite F-coalgebra. Then the rational fixpoint of H is a quotient of the rational fixpoint of F.*

Proof Let $(\varrho H, r^H)$ and $(\varrho F, r^F)$ be the rational fixpoints of H and F, respectively. First we argue that the H-coalgebra

$$\varrho F \xrightarrow{r^F} F(\varrho F) \xrightarrow{q_{\varrho F}} H(\varrho F) \tag{5.1}$$

is lfp. To this end note that the object assignment that maps an F-coalgebra (A, a) to the H-coalgebra $(A, q_A \cdot a)$ extends to a finitary functor $\mathsf{Coalg}\, F \to \mathsf{Coalg}\, H$ that preserves orbit-finite coalgebras. So since $(\varrho F, r^F)$ is the filtered colimit of all orbit-finite F-coalgebras (A, a), the above H-coalgebra (5.1) on ϱF is the filtered colimit of all orbit-finite H-coalgebras of the form $(A, q_A \cdot a)$, whence (5.1) is an lfp coalgebra for H.

Now we obtain a unique H-coalgebra homomorphism $p : (\varrho F, q_{\varrho F} \cdot r^F) \to (\varrho H, r^H)$ by the finality of the latter coalgebra. It remains to show that p is surjective. To this end, let (C, c) be an orbit-finite H-coalgebra. By assumption, (C, c) is a quotient of some orbit-finite F-coalgebra (A, a), i.e. we have a diagram

$$\begin{tikzcd}
(A, q_A \cdot a) \arrow[r, "a^\dagger"] \arrow[d, two heads, "h"'] & (\varrho F, q_A \cdot r^F) \arrow[d, "p"] \\
(C, c) \arrow[r, "c^\dagger"] & (\varrho H, r^H)
\end{tikzcd} \tag{5.2}$$

Now the sink consisting of all these $c^\dagger \cdot h$, where (C, c) ranges over all orbit-finite H-coalgebras, is jointly surjective since the $c^\dagger$ are jointly surjective. Thus, it follows that p is surjective by commutativity of the diagrams (5.2). □

Remark 5.3 It follows from the previous theorem that ϱH is the image of $(\varrho F, q_{\varrho F} \cdot r^F)$ in the final H-coalgebra. In fact, ϱH is a subcoalgebra of νH via the injective H-coalgebra homomorphism $m : \varrho H \to \nu H$, say. Then $m \cdot p$ is the image-factorization of the unique H-coalgebra homormorphism from (5.1) to νH.

We next introduce the announced condition on F that ensures satisfaction of the assumptions of Theorem 5.2.

Definition 5.4 For nominal sets X and Y, we define the nominal subset

$$X < Y = \{(x, y) \in X \times Y \mid \mathsf{supp}(x) \subseteq \mathsf{supp}(y)\}$$

of $X \times Y$ (this subset is clearly equivariant, so $X < Y$ is indeed a nominal set). A *sub-strength* of F is a family of a equivariant maps

$$s_{X,Y} : FX < Y \to F(X < Y),$$

indexed by nominal sets X, Y (but not necessarily natural in X, Y) such that

$$\begin{tikzcd} FX < Y \arrow[r, "s_{X,Y}"] \arrow[dr, "\mathsf{outl}"'] & F(X < Y) \arrow[d, "F\,\mathsf{outl}"] \\ & FX \end{tikzcd} \tag{5.3}$$

where $\mathsf{outl} : X < Y \to X$ denotes the obvious projection map.

Example 5.5 Not every functor has a sub-strength. The finitary functor DU is a lifting of $\mathrm{Id}_{\mathsf{Set}}$ (in fact, it is a mono-preserving and localizable lifting) but has no sub-strength, because for $X = \mathcal{V}$, $Y = 1$, the nominal set $DU\mathcal{V} < 1$ is just $DU\mathcal{V}$ whereas $DU(\mathcal{V} < 1) = DU\emptyset = \emptyset$. Hence, there is no map $DU\mathcal{V} < 1 \to DU(\mathcal{V} < 1)$ at all.

For the rest of this section, we assume that F has a sub-strength s_X. Moreover, we fix an orbit-finite coalgebra $c : C \to HC$. We will show that c is a quotient of an F-coalgebra in the sense of Definition 5.1, thus showing that the rational fixpoint of H is a quotient of that of F, by Theorem 5.2.

We put

$$B = \max_{x \in C} |\mathsf{supp}(x)| \; + \; \max_{x \in C} \min_{\substack{y \in FC \\ q_C(y) = c(x)}} |\mathsf{supp}(y)|. \tag{5.4}$$

Intuitively, B is a bound on the total number of free and bound variables in any element $c(x)$. First observe that B exists because the numbers numbers $|\mathsf{supp}(x)|$ and $\min_{y \in FC, q_C(y) = c(x)} |\mathsf{supp}(y)|$ are constant on every orbit of C; for the former simply apply Lemma 2.10, and for the latter suppose that $x = \pi \cdot x'$ and let y and y', respectively, assume the above minimum. Then

$$q(y) = c(x) = c(\pi \cdot x') = \pi \cdot c(x) = \pi \cdot q(y') = q(\pi \cdot y')$$

using equivariance of c and q and therefore $|\mathsf{supp}(y)| \leq |\mathsf{supp}(\pi \cdot y')| = |\mathsf{supp}(y')|$ by minimality and Lemma 2.10. Similarly $|\mathsf{supp}(y')| \leq |\mathsf{supp}(y)|$ by starting from $x' = \pi^{-1} \cdot x$.

Next we define $W \subseteq \mathcal{V}^B$ to be the nominal set of tuples of B distinct atoms. Thus, for every $w \in W$, $|\mathsf{supp}(w)| = B$.

Note that W has only one orbit, in particular is orbit-finite. Hence $C \times W$ and thus also its subobject $C < W$ are orbit-finite. We will use $C < W$ as the carrier of the orbit-finite F-coalgebra we aim to construct.

Lemma 5.6 *The projection* $\mathsf{outl} : C < W \to C$ *is an epimorphism.*

Proof For $x \in C$, there is $w \in W$ with $\mathsf{supp}(x) \subseteq \mathsf{supp}(w)$, because $|\mathsf{supp}(x)| \leq B$. So $(x, w) \in C < W$ and $\mathsf{outl}(x, w) = x$. □

We recall the notion of strong nominal set:

Definition 5.7 [34] An element x of a nominal set X is *strongly supported* if $\mathsf{fix}(x) \subseteq \mathsf{Fix}(\mathsf{supp}(x))$ (so $\mathsf{fix}(x) = \mathsf{Fix}(\mathsf{supp}(x))$). A nominal set is *strong* if all its elements are strongly supported.

Example 5.8 (1) The nominal set W is strong: for $a = (a_1, \ldots, a_B) \in W$, the equality $\pi \cdot a = a$ implies that $\pi(a_i) = a_i$ for all i, i.e. $\pi \in \mathsf{Fix}(\mathsf{supp}(a))$.

(2) The nominal set of unordered pairs of atoms fails to be strong, because $(a\ b) \cdot \{a, b\} = \{a, b\}$.

From the first example, the following is immediate.

Lemma 5.9 *The nominal set $C < W$ is strong.*

Strong nominal sets are of interest due to the following extension property (mentioned already in [23]):

Proposition 5.10 *Let X be a strong nominal set, and let O be a subset of X containing precisely one element per orbit of X. Let Y be a nominal set, and let $f_0 : O \to Y$ be a map such that $\mathsf{supp}(f_0(x)) \subseteq \mathsf{supp}(x)$ for all $x \in O$. Then f_0 extends uniquely to an equivariant map $X \to Y$.*

Proof Uniqueness is clear. To show existence, define $f : X \to Y$ by $f(\pi \cdot x) = \pi \cdot f_0(x)$ for $x \in O$. We have to show well-definedness, so let $\pi' \cdot x = \pi \cdot x$. Since X is strong and $\mathsf{supp}(f_0(x)) \subseteq \mathsf{supp}(x)$, we then have $\pi^{-1}\pi' \in \mathsf{fix}(x) \subseteq \mathsf{Fix}(\mathsf{supp}(x)) \subseteq \mathsf{Fix}(\mathsf{supp}(f_0(x))) \subseteq \mathsf{fix}(f_0(x))$, so $\pi' \cdot f_0(x) = \pi \cdot f_0(x)$. Equivariance of f is immediate from the definition. □

This property is used in the construction of a part of our target coalgebra. In the construction, it is an essential observation that if an equivariant map drops certain atoms, then we can rename the atoms without changing the value:

Lemma 5.11 *Consider an equivariant map $e : X \to Y$ and $x \in X$. Then for any $S \in \mathcal{P}_{\mathsf{f}}(\mathcal{V})$ with $\mathsf{supp}(e(x)) \subseteq S$ and $|\mathsf{supp}(x)| \leq |S|$, there is some $\pi \in \mathfrak{S}_{\mathsf{f}}(\mathcal{V})$ with $\mathsf{supp}(\pi \cdot x) \subseteq S$ and $e(\pi \cdot x) = e(x)$.*

Proof Put $Y = \mathsf{supp}(x) \setminus \mathsf{supp}(e(x))$ and $N = S \setminus \mathsf{supp}(e(x))$. Then $|Y| \leq |N|$. Pick some injection $\pi' : Y \setminus N \rightarrowtail N \setminus Y$ and extend it to a finite permutation on $\mathcal{V}$ by

$$\pi(a) = \begin{cases} \pi'(a) & \text{if } a \in Y \setminus N \\ \pi'^{-1}(a) & \text{if } a \in \mathsf{Im}(\pi') \\ a & \text{otherwise} \end{cases}$$

where $\mathsf{Im}(\pi') \subseteq N \setminus Y$ denotes the image of π'. This definition implies $\pi[Y] \subseteq N$ and $\pi \cdot e(x) = e(x)$ since π fixes every $a \notin Y$ and therefore $\pi \in \mathsf{Fix}(\mathsf{supp}(e(x)))$. Hence,

$$\begin{aligned} \mathsf{supp}(\pi \cdot x) = \pi \cdot \mathsf{supp}(x) &\subseteq \pi \cdot (Y \cup \mathsf{supp}(e(x))) = (\pi \cdot Y) \cup \mathsf{supp}(\pi \cdot e(x)) \\ &= \pi[Y] \cup \mathsf{supp}(e(x)) \subseteq N \cup \mathsf{supp}(e(x)) = S. \end{aligned}$$ □

Lemma 5.12 *There is an equivariant map $f : C < W \to FC$ such that*

$$\begin{tikzcd} C < W \arrow[r, "f"] \arrow[d, "\mathsf{outl}"'] & FC \arrow[d, "q_C"] \\ C \arrow[r, "c"] & HC \end{tikzcd}$$

commutes.

Proof Pick a subset $O = \{(x_1, w_1), \dots, (x_n, w_n)\} \subseteq C < W$ containing precisely one element from each of the n orbits of $C < W$. We have for each i some $y_i \in FC$ such that $q_C(y_i) = c(x_i)$, and by (5.4), $|\,\mathsf{supp}(y_i)| \le B = |\,\mathsf{supp}(w_i)|$; in addition, $\mathsf{supp}(q_C(y_i)) \subseteq \mathsf{supp}(x_i) \subseteq \mathsf{supp}(w_i)$. By Lemma 5.11 applied to q_C, y_i and $S = \mathsf{supp}(w_i)$, there is some σ_i such that $q_C(\sigma_i \cdot y_i) = q_C(y_i) = c(x_i)$ and $\mathsf{supp}(\sigma_i \cdot y_i) \subseteq \mathsf{supp}(w_i) = \mathsf{supp}(x_i, w_i)$.

Now define $f_0 : O \to FC$, $f_0(x_i, w_i) = \sigma_i \cdot y_i$. Then $\mathsf{supp}(f_0(x_i, w_i)) \subseteq \mathsf{supp}(x_i, w_i)$. By Proposition 5.10, f_0 extends uniquely to an equivariant map $f : C < W \to FC$, and we have

$$q_C(f(x_j, w_j)) = q_C(f_0(x_j, w_j)) = q_C(\sigma_j \cdot y_j) = c(x_j) = c \cdot \mathsf{outl}(x_j, w_j) \qquad (5.5)$$

for all $1 \le j \le n$. This equality extends to all elements of $C < W$ by equivariance: any $p \in C < W$ has the form $p = \pi \cdot (x_i, w_i)$, and thus multiplying (5.5) by π yields $q_C(f(p)) = c \cdot \mathsf{outl}(p)$. ◻

In combination with the sub-strength, the map f now induces the required F-coalgebra:

Lemma 5.13 *The H-coalgebra (C, c) is, via outl, a quotient of the orbit-finite F-coalgebra*

$$C < W \xrightarrow{\bar f} FC < W \xrightarrow{s_{C,W}} F(C < W) \qquad \textit{where } \bar f(x, w) = (f(x), w).$$

Proof The map $\bar f$ is equivariant, and $\bar f(x, w) \in FC < W$ because f is equivariant. Moreover, the diagram below commutes:

$$\begin{tikzcd}
C < W \arrow[r, "\bar f"] \arrow[dd, "\mathsf{outl}"'] \arrow[drr, bend right, "f"'] \arrow[dr, phantom, "\text{Def. } \bar f"] & FC < W \arrow[r, "s_{C,W}"] \arrow[dr, "\mathsf{outl}"] \arrow[r, phantom, shift right=3ex, "(5.3)"] & F(C < W) \arrow[r, "q_{C<W}"] \arrow[d, "F\,\mathsf{outl}"] \arrow[dr, phantom, "\text{Naturality}"] & H(C < W) \arrow[dd, "H\,\mathsf{outl}"] \\
\arrow[r, phantom, "\text{Lemma 5.12}"] & & FC \arrow[dr, "q_C"] & \\
C \arrow[rrr, "c"] & & & HC
\end{tikzcd}$$

Thus, $\mathsf{outl} : C < W \to C$ is an H-coalgebra homomorphism, and surjective by Lemma 5.6. ◻

From Theorem 5.2 we now obtain:

Corollary 5.14 *If $F : \mathsf{Nom} \to \mathsf{Nom}$ is finitary and has a sub-strength, and H is a quotient of F, then the rational fixpoint ϱH is a quotient of the rational fixpoint ϱF.*

Example 5.15 Having a sub-strength is not a necessary condition for a quotient $F \to H$ to satisfy the requirements of Theorem 5.2. Recall from Example 5.5 that the functor $F = DU$ has no sub-strength. Take $q : DU \twoheadrightarrow H$ to be any quotient (e.g. $HX = 1$). Since $q_X : DUX \to HX$ is a surjective equivariant map and equivariant maps do no increase the support, HX is a discrete nominal set for all X. It follows that every splitting $s_X : HX \rightarrowtail FX$ of q_X is an equivariant map. Thus, every H-coalgebra $x : X \to HX$ is trivially a quotient of the DU-coalgebra $s_X \cdot x : X \to DUX$, i.e. the quotient $DU \to H$ satisfies the assumptions of Theorem 5.2.

However, a sub-strength does exist in many relevant examples:

Lemma 5.16 *(1) Every constant functor has a sub-strength.*
(2) The identity functor has a sub-strength.
(3) The class of functors having a sub-strength is closed under finite products, arbitrary coproducts, and functor composition.

Proof We give definitions of the sub-strength in all cases; commutation of (5.3) is obvious throughout.

(1) If K is constant, then we have $s_{X,Y} = \mathsf{outl} : KX < Y \to K(X < Y)$.
(2) Trivial.
(3) (a) For G and H having sub-strengths $s^G_{X,Y}$ and $s^H_{X,Y}$, respectively, we define

$$f : (GX \times HX) < Y \to (GX < Y) \times (HX < Y)$$

by $f(x, y, w) = \big((x, w), (y, w)\big)$, which is well-typed because $\mathsf{supp}(x) \subseteq \mathsf{supp}(x, y) \subseteq w$ (and analogously for y). We then obtain a sub-strength $s_{X,Y}$ for $G \times H$ as $s_{X,Y} = s^G_{X,Y} \times s^H_{X,Y} \circ f$.

(b) For each G_i having a sub-strength $s^i_{X,Y}$, we define

$$f : \big(\coprod_{i\in I} G_i X\big) < Y \to \coprod_{i\in I}(G_i X < Y)$$

by $f(\mathsf{in}_i x, w) = \mathsf{in}_i(x, w)$, again noting that $\mathsf{supp}(x) = \mathsf{supp}(\mathsf{in}_i x) \subseteq \mathsf{supp}(w)$. We then obtain a sub-strength $s_{X,Y}$ for $\coprod G_i$ as $s_{X,Y} = (\coprod s^i_{X,Y}) \circ f$.

(c) Given sub-strengths $s^F_{X,Y} : FX < Y \to F(X < Y)$ and $s^G_{X,Y} : GX < Y \to G(X < Y)$, the desired sub-strength for the composite GF is

$$GFX < Y \xrightarrow{s^G_{FX,Y}} G(FX < Y) \xrightarrow{Gs^F_{X,Y}} GF(X < Y). \qquad \square$$

Notation 5.17 We denote by $X^{n\neq}$ the subset of X^n consisting of all n-tuples with pairwise distinct components.

Proposition 5.18 *If a functor $F : \mathsf{Nom} \to \mathsf{Nom}$ has a sub-strength, then it preserves epimorphisms with orbit-finite codomain.*

Proof Take $e : X \twoheadrightarrow Y$ and suppose that Y is orbit-finite. Define

$$m = \max_{y\in Y} \min_{\substack{x\in X\\ e(x)=y}} |\mathsf{supp}(x)| \quad \text{and} \quad Z = \coprod_{k\geq m} \mathcal{V}^{k\neq}.$$

The maximum m exists, because Y is orbit-finite and because for any two elements y, y' of the same orbit, $|\mathsf{supp}(y)| = |\mathsf{supp}(y')|$ by Lemma 2.10. Pick a subset $O \subseteq Y < Z$ containing precisely one representative (x_i, z_i) of each orbit of $Y < Z$. We have for every i some x_i with $e(x_i) = y_i$ and $|\mathsf{supp}(x_i)| \leq m \leq |\mathsf{supp}(z_i)|$. By Lemma 5.11 applied to e, y_i and $S = \mathsf{supp}(z_i)$ we can assume w.l.o.g. that $\mathsf{supp}(x_i) \subseteq \mathsf{supp}(z_i)$.

Now define $c_0 : O \to X < Z$ by $c_0(y_i, z_i) = (x_i, z_i)$. By Proposition 5.10 we obtain a unique equivariant extension $c : Y < Z \to X < Z$, and for every $\pi \in \mathfrak{S}_\mathsf{f}(\mathcal{V})$ we have

$$(e < \mathrm{id}_Z)(c(\pi \cdot y_i, \pi \cdot z_i)) = \pi \cdot (e < \mathrm{id}_Z)(x_i, z_i) = \pi \cdot (y_i, z_i).$$

This implies that $(e < \mathrm{id}_Z) : X < Z \to Y < Z$ is a split epimorphism and thus preserved by F.

Next consider the commuting diagram

$$\begin{tikzcd}
F(X < Z) \arrow[r, two heads, "F(e<Z)"] \arrow[d, "F\,\mathsf{outl}"'] \arrow[dr, phantom, "\text{Naturality of } \mathsf{outl}"] & F(Y < Z) \arrow[d, "F\,\mathsf{outl}"'] \arrow[dr, phantom, "(5.3)"] & FY < Z \arrow[l, "s_{Y,Z}"'] \arrow[dl, bend left, "\mathsf{outl}"] \\
FX \arrow[r, "Fe"'] & FY &
\end{tikzcd}$$

For any $t \in FY$, there is some $z \in Z$ with $\mathsf{supp}(t) \subseteq \mathsf{supp}(z)$ since for every subset S of $\mathcal{V}$ of cardinality of least m there exist elements in Z whose support is S. Since $\mathsf{outl} : FY < Z \to FY$ is epimorphic, so is $F\,\mathsf{outl} : F(Y < Z) \to FY$. Hence, $F\,\mathsf{outl} \cdot F(e < Z)$ is epimorphic, thus so is Fe. ◻

For epimorphisms with non-orbit-finite codomain, preservation by functors having a sub-strength may fail:

Example 5.19 The functor $FX = X^\omega$ of finitely supported sequences has a sub-strength

$$s_{X,Y}\big((a_k)_{k\in\mathbb{N}}, y\big) = \big((a_k, y)\big)_{k\in\mathbb{N}}$$

because $\mathsf{supp}(a_k) \subseteq \mathsf{supp}\,\big((a_k)_{k\in\mathbb{N}}\big) \subseteq \mathsf{supp}(y)$.

However, F does not preserve all epimorphisms. To see this consider the discrete nominal set $\mathbb{N}$ of natural numbers and the equivariant surjection

$$e : \mathcal{P}_{\mathsf{f}}\mathcal{V} \twoheadrightarrow \mathbb{N}, \quad e(W) = |W|.$$

The image of Fe in $\mathbb{N}^\omega$ contains only bounded sequences: for any finitely supported sequence $s \in \mathcal{P}_{\mathsf{f}}\mathcal{V}^\omega$, the sequence $Fe(s)$ is bounded by $|\,\mathsf{supp}(s)|$. Since $\mathbb{N}$ is discrete, every sequence in $\mathbb{N}^\omega$ has finite (namely, empty) support; this shows that Fe is not surjective.

Note that $FX = X^\omega$ is not finitary (see Proposition 6.4). In fact, for finitary functors we have the following

Proposition 5.20 *Finitary* Nom*-functors with a sub-strength preserve epimorphisms.*

This easily follows from Proposition 5.18 since every epimorphism in Nom is the filtered colimit of epimorphisms with orbit-finite domain and codomain (see Proposition A.3 in the appendix).

Preservation of epimorphisms is convenient because quotients of epimorphism-preserving functors are closed under composition:

Lemma 5.21 *Let $q : F \twoheadrightarrow H$ and $q' : F' \twoheadrightarrow H'$ be quotients of functors on* Nom*. If F preserves epis then HH' is a quotient of FF' via*

$$FF' \xrightarrow{Fq'} FH' \xrightarrow{qH'} HH',$$

Proof Recall that we have defined quotients as natural transformations that are pointwise epi. ◻

This extends the closure properties of the class of functors with a sub-strength (Lemma 5.16) to the class of quotients of finitary functors having a sub-strength:

Corollary 5.22 *The class of quotients of finitary* Nom*-functors that have a sub-strength is closed under coproducts, finite products, composition, and quotients.*

Proof Closedness under coproducts, finite products, and quotients is trivial. For composition, recall that finitary functors that have a sub-strength preserve epimorphisms, and apply Lemma 5.21. □

6 Applications

6.1 Binding Signatures

One can describe various flavours of (possibly) infinite terms with variable binding operators (such as infinite λ-terms or process terms of the π-calculus) as the inhabitants of final coalgebras of so called *binding signatures*, see [22, Definition 5.8]. We refrain from defining the corresponding *binding signatures* explicitly here, focusing instead on the functor representation. The latter is given by a generalization of the class of polynomial functors:

Definition 6.1 The class of *binding functors* is the smallest class of functors on Nom that contains the identity functor and all constant functors and is closed under all coproducts, binary products, and left composition with the abstraction functor $[\mathcal{V}](-)$. The *raw functor* of a binding functor is the polynomial functor obtained by replacing all occurrences of $[\mathcal{V}](-)$ with $\mathcal{V} \times (-)$. (Strictly speaking this requires an explicit distinction between a syntax and a semantics for binding functors; we refrain from elaborating this distinction to avoid overformalization.)

Lemma 6.2 *Every binding functor is a quotient of its raw functor.*

By Lemma 5.16 and Corollary 5.14, we have in particular that for every quotient H of a polynomial functor F, the rational fixpoint ϱH is a quotient of ϱF. By the previous lemma, this applies in particular in the situation where H is a binding functor and F is its raw functor. One concrete instance is the main result of [27]:

Example 6.3 For $FX = \mathcal{V} + \mathcal{V} \times X + X \times X$ and $HX = \mathcal{V} + [\mathcal{V}]X + X \times X$ we already saw that the rational fixpoint ϱF is formed by all rational λ-trees (see Example 4.15(2)). Furthermore, we know that ϱH is a subcoalgebra of the final H-coalgebra, and the latter consists of the α-equivalence classes of λ-trees with finitely many free variables [22]. But now we also know that ϱH is a quotient of ϱF, therefore ϱH consists of those α-equivalence classes of λ-trees that contain a rational λ-tree.

Similarly, for a binding functor H arising from a binding signature one takes its raw functor F. Then the rational fixpoint of F consists of all rational trees for the given binding signature, and it follows that the rational fixpoint of H consists of all rational trees modulo α-equivalence, i.e. it contains precisely those α-equivalence classes of trees for the binding signature that have finitely many free variables and contain a rational tree.

6.2 Exponentiation by Orbit-Finite Strong Nominal Sets

As in Set, the core ingredient of functors that model various flavours of nominal automata as coalgebras is exponentiation by the input alphabet. Denote by X^P the internal hom-object witnessing the cartesian closedness of Nom (i.e. $(-)^P$ is right-adjoint to $(-) \times P$); this nominal set contains those maps $f : P \to X$ that are finitely supported w.r.t. the group action given by

$$(\pi \star f)(y) = \pi \cdot f(\pi^{-1} \cdot y).$$

We will see that for P orbit-finite and strong, the functor $(-)^P$ is a quotient of a polynomial functor, so that Corollary 5.14 applies to $(-)^P$. In fact, we are going to prove that

$$\begin{array}{c} P \text{ is strong} \\ \text{and orbit-finite} \end{array} \iff \begin{array}{c} (-)^P \text{ is a quotient of a} \\ \text{polynomial } \mathsf{Nom}\text{-functor.} \end{array} \tag{6.1}$$

It is not difficult to see that orbit-finiteness of P is necessary:

Proposition 6.4 *If $(-)^P$ is finitary, then P is orbit-finite.*

Proof Take the projection $\mathsf{outr} : 1 \times P \to P$ and consider its curried version $\overline{\mathsf{outr}} : 1 \to P^P$. Since $(-)^P$ is finitary and 1 is orbit-finite (i.e. finitely presentable), $\overline{\mathsf{outr}}$ factors through an orbit-finite subobject $j : A \hookrightarrow P$:

$$\begin{tikzcd} 1 \arrow[r, "\overline{\mathsf{outr}}"] \arrow[dr, "\bar{f}"'] & P^P \\ & A^P \arrow[u, "j^P"'] \end{tikzcd}$$

In other words, $\mathsf{outr} = j \cdot f$, where f is the uncurrying of $\bar{f}$. Since outr is surjective, j is surjective; hence, P is orbit-finite because orbit-finite sets are closed under epimorphisms. ◻

Secondly, we show that it is necessary that P is strong. For the sake of readability, we show the contraposition of (6.1) for a concrete example and then indicate how the construction generalizes.

Proposition 6.5 *Let $B = \{\{a, b\} \mid a, b \in \mathcal{V}, a \neq b\}$ be the (non-strong) nominal set of unordered pairs of distinct elements of X. Then the functor $(-)^B$ is not a natural quotient of any Nom-functor with a sub-strength.*

Proof Assume that we have a natural quotient $q_X : FX \twoheadrightarrow X^B$ and a sub-strength $s_{X,Y} : FX < Y \to F(X < Y)$. Then for any Y, the following diagram commutes:

$$\begin{tikzcd} FB < Y \arrow[r, "s_{B,Y}"] \arrow[dr, "\mathsf{outl}"'] & F(B < Y) \arrow[r, two heads, "q_{B<Y}"] \arrow[d, "F\,\mathsf{outl}"] \arrow[dr, phantom, "\text{Naturality}"] & (B < Y)^B \arrow[d, "\mathsf{outl}^B"] \\ & FB \arrow[r, two heads, "q_B"] \arrow[u, phantom, "(5.3)", near end, xshift=-3ex] & B^B \end{tikzcd}$$

The identity id_B is equivariant, hence finitely supported, i.e. $\mathrm{id}_B \in B^B$. Since q_B is surjective, we have $x \in FB$ such that $q_B(x) = \mathrm{id}_B$. Let $n = |\mathsf{supp}(x)|$ and

$Y = \mathcal{V}^n$. Then there exist $v_1, \dots, v_n \in \mathcal{V}$ such that $(x, (v_1, \dots, v_n)) \in FB < Y$. Put $g = q_{B<Y}(s_{B,Y}(x, v_1, \dots, v_n)) : B \to (B < Y)$. This is a finitely supported map, so we can pick distinct $a, b \in \mathcal{V}$ that are fresh for g, so that $(a\,b) \star g = g$. By commutativity of the above diagram,

$$\begin{aligned}\mathsf{outl} \circ g &= \mathsf{outl}^B(g) = \mathsf{outl}^B(q_{B<Y}(s_{B,Y}(x, v_1, \dots, v_n))) \\ &= q_B(\mathsf{outl}(x, v_1, \dots, v_n)) = q_B(x) = \mathrm{id}_B.\end{aligned}$$

In particular, $g(\{a,b\})$ has the form $g(\{a,b\}) = (\{a,b\}, u_1, \dots, u_n)$ with $u_1, \dots, u_n \in \mathcal{V}$. Since $\mathsf{supp}(\{a,b\}) \subseteq \mathsf{supp}(u_1, \dots, u_n)$, we have $1 \le i \le n$ such that $u_i = a$. Therefore,

$$\begin{aligned}g(\{a,b\}) &= \big((a\,b) \star g\big)(\{a,b\}) = (a\,b) \cdot g\big((a\,b)^{-1} \cdot \{a,b\}\big) = (a\,b) \cdot g\big(\{a,b\}\big) \\ &= (\{a,b\}, (a\,b) \cdot u_1, \dots, (a\,b) \cdot u_n) \ne g(\{a,b\}),\end{aligned}$$

in contradiction to $(a\,b) \cdot u_i = b \ne a = u_i$. □

The counterexample in Proposition 6.5 can be generalized to an arbitrary non-strong nominal set B by using, in lieu of $\{a,b\}$ and $(a\,b)$ in the above proof, an element $z \in B$ that fails to be strongly supported but is fresh for g (i.e. $\mathsf{supp}(z) \cap \mathsf{supp}(g) = \emptyset$) and $\pi \in (\mathsf{fix}(g) \cap \mathsf{fix}(z)) \setminus \mathsf{Fix}(\mathsf{supp}(z))$, respectively.

Remark 6.6 Proposition 6.5 has two consequences for an orbit-finite non-strong nominal set B:

- The exponentiation functor $E = (-)^B$ is not the quotient of any polynomial Nom-functor (in the sense of Definition 3.16), i.e. "$\Leftarrow$" in (6.1) holds.
- The exponentiation functor E has no sub-strength.

A basic example of a strong nominal set is the set $P = \mathcal{V}$ of all atoms. We will now show that $(-)^{\mathcal{V}}$ is a quotient of a polynomial functor. Later, we extend this to $P = \mathcal{V}^n$, and then conclude the desired result for arbitrary orbit-finite strong nominal sets P.

Notation 6.7 In the following we shall write $\mathbf{x}$ for a tuple $(x_1, \dots, x_n) \in X^n$ for any set X, and for a map $f : X \to Y$ we write $f(\mathbf{x})$ for the tuple $(f(x_1), \dots, f(x_n))$.

Consider the functor

$$FX = \mathcal{V} \times X \times \coprod_{n \in \mathbb{N}} \mathcal{V}^n \times X^n. \tag{6.2}$$

In order to identify $(-)^{\mathcal{V}}$ as a quotient of this functor, we define a map $\bar{q}_X : FX \times \mathcal{V} \to X$:

$$\bar{q}_X(a, d, \mathbf{v}, \mathbf{x}, b) = \begin{cases} x_i & \text{where } i \text{ is minimal s.t. } v_i = b \\ (a\ b) \cdot d & \text{if no such } i \text{ exists.} \end{cases}$$

This definition of $\bar{q}_X$ exploits the fact that a finitely supported map $f : \mathcal{V} \to X$ is equivariant w.r.t. permutations that fix elements in $\mathsf{supp}(f)$, i.e. whenever $\pi \in \mathsf{Fix}(\mathsf{supp}(f))$ then $f(\pi \cdot x) = \pi \cdot f(x)$ for every x. (In particular, the finitely supported maps with empty support are precisely the equivariant maps.) Therefore, in order to represent f, we fix a name $a \in \mathcal{V} \setminus \mathsf{supp}(f)$ and its image $d = f(a)$; these data then determine the action of f on all names of the form $\pi(a)$ for $\pi \in \mathsf{Fix}(\mathsf{supp}(f))$.

These are all names except those in $\mathsf{supp}(f)$; we therefore enumerate the names in $\mathsf{supp}(f)$ as a tuple $\mathbf{v}$, and their images as a tuple $\mathbf{x}$, arriving at a representation of f as a quadruple $(a, d, \mathbf{v}, \mathbf{x}) \in FX$.

Lemma 6.8 *The map $\bar{q}_X : FX \times \mathcal{V} \to X$ is equivariant and natural in X.*

Proof *Equivariance:* All operations used in the definition of $\bar{q}$ are equivariant, in particular the operation of picking the first occurrence of given name, if any, from a list of names, as well as the map $(a, b, d) \mapsto (a\,b) \cdot d$.

Naturality: Let $f : X \to Y$ be equivariant. Then

$$\begin{aligned} f\big(\bar{q}_X(a, d, \mathbf{v}, \mathbf{x}, b)\big) &= \begin{cases} f(x_i) & \text{where } i \text{ is minimal s.t. } v_i = b \\ f\big((a\ b) \cdot d\big) & \text{if no such } i \text{ exists.} \end{cases} \\ &= \begin{cases} f(x_i) & \text{where } i \text{ is minimal s.t. } v_i = b \\ (a\ b) \cdot f(d) & \text{if no such } i \text{ exists.} \end{cases} \\ &= \bar{q}_X(a, f(d), \mathbf{v}, f(\mathbf{x}), b). \end{aligned}$$

□

By currying, $\bar{q}$ induces a natural transformation

$$q : F \to (-)^{\mathcal{V}}.$$

Lemma 6.9 *The natural transformation $q : F \to (-)^{\mathcal{V}}$ is component-wise surjective. More specifically, given $f \in X^{\mathcal{V}}$, let $\{v_1, \ldots, v_n\} = \mathsf{supp}(f)$ and $a \in \mathcal{V} \setminus \mathsf{supp}(f)$; then we have*

$$q_X(a, f(a), \mathbf{v}, f(\mathbf{v})) = f.$$

Proof We just have to formalize the argument given in the informal explanation of the definition of $\bar{q}$: Let $b \in \mathcal{V}$, and put $\mathbf{v} = (v_1, \ldots, v_n)$, $g = q_X\big(a, f(a), \mathbf{v}, f(\mathbf{v})\big) : \mathcal{V} \to X$. We have to show $g(b) = f(b)$.

– If $b \in \mathsf{supp}(f)$, then $b = v_i$ for some i, so that $g(v_i) = f(v_i)$ by definition.
– If $b \in \mathcal{V} \setminus \mathsf{supp}(f)$, then $v_i \neq b$ for all $1 \leq i \leq n$, so $g(b) = (a\,b) \cdot f(a)$. Moreover, $a, b \notin \mathsf{supp}(f)$ implies $(a\ b) \star f = f$. Therefore,

$$g(b) = (a\,b) \cdot f(a) = (a\,b) \cdot f((a\,b)^{-1} \cdot (a\,b) \cdot a) = (a\,b) \star f((a\,b) \cdot a) = f((a\,b) \cdot a) = f(b).$$

□

Up to now, we have seen that exponentiation by $\mathcal{V}$ is a quotient of a polynomial functor, F (6.2). To extend this to exponentiation by $\mathcal{V}^n$, $n \geq 0$, recall from Lemma 5.21 that quotients of polynomial Set-functors compose. Now observe that by the usual exponentiation laws, $(-)^{\mathcal{V}^n}$ is just the n-fold composite of $(-)^{\mathcal{V}}$ with itself. Being a polynomial functor on a cartesian closed category, F preserves epis; so $(-)^{\mathcal{V}^n}$ is a quotient of F^n (i.e. of the n-fold composite $F \circ \cdots \circ F$) by Lemma 5.21, applied inductively with trivial base case $n = 0$.

Definition 6.10 Recall from Notation 5.17 that $X^{n\neq} \subseteq X^n$ denotes the set of tuples of n distinct elements, and let $m : X^{n\neq} \rightarrowtail X^n$ be the inclusion map. Define

$$\mathsf{uniq} : X^n \twoheadrightarrow \coprod_{1 \leq k \leq n} X^{k\neq}$$

to be the map that removes all duplicates:

$$\mathsf{uniq}(\mathbf{x}) = \big(v_i \mid 1 \leq i \leq n, \forall j < i : v_j \neq v_i\big).$$

Note that uniq is equivariant (although not natural), since $v_j \neq v_i$ iff $\pi \cdot v_j \neq \pi \cdot v_i$; moreover, we have

$$\begin{tikzcd} X^{n\neq} \arrow[r, tail, "m"] \arrow[rr, bend left, "\mathsf{in}_n"] & X^n \arrow[r, two heads, "\mathsf{uniq}"] & \coprod_{1\leq k\leq n} X^{k\neq} \end{tikzcd} \tag{6.3}$$

Definition 6.11 For $n \geq 1$, define a map

$$\mathsf{fill} : X^n \times X^{2n\neq} \to X^{n\neq}$$

where $\mathsf{fill}(\mathbf{v}, \mathbf{w})$ removes duplicates from $\mathbf{v}$ and fills the gap with components of $\mathbf{w}$ to obtain n distinct elements. Formally, we define $\mathsf{fill}(\mathbf{v}, \mathbf{w})$ as the length-n prefix of $\mathsf{uniq}(\mathbf{v})\mathbf{w}'$ where $\mathbf{w}' = (w_i \mid 1 \leq i \leq 2n, w_i \notin \mathbf{v})$, noting that $\mathbf{w}'$ has at least n elements. The map fill is equivariant because $w_i \notin \mathbf{v}$ iff $\pi \cdot w_i \notin \pi \cdot \mathbf{v}$. The diagram

$$\begin{tikzcd} X^{n\neq} \times X^{2n\neq} \arrow[r, tail, "m \times X^{2n\neq}"] \arrow[rr, bend left, "\mathsf{outl}"] & X^n \times X^{2n\neq} \arrow[r, "\mathsf{fill}"] & X^{n\neq}. \end{tikzcd} \tag{6.4}$$

commutes.

Lemma 6.12 *The restriction map $r_X : X^{\mathcal{V}^n} \to X^{\mathcal{V}^{n\neq}}$ (i.e. $r_X(g) = g \cdot m$) is equivariant, surjective, and natural in X.*

Proof Equivariance and naturality are by standard properties of cartesian closed categories. We show surjectivity. We write eval_X for the evaluation map $X^{\mathcal{V}^{n\neq}} \times \mathcal{V}^{n\neq} \to X$. We then have an equivariant map

$$\bar{g} : X^{\mathcal{V}^{n\neq}} \times \mathcal{V}^{2n\neq} \times \mathcal{V}^n \to X, \quad \bar{g}(f, \mathbf{w}, \mathbf{v}) = \mathsf{eval}_X(f, \mathsf{fill}(\mathbf{v}, \mathbf{w}))$$

whose curried version $g : X^{\mathcal{V}^{n\neq}} \times \mathcal{V}^{2n\neq} \to X^{\mathcal{V}^n}$ provides us with the desired preimage of a given $f \in X^{\mathcal{V}^{n\neq}}$. Indeed, pick any $\mathbf{w} \in \mathcal{V}^{2n\neq}$. Then $r_X(g(f, \mathbf{w})) = f$: for $\mathbf{u} \in \mathcal{V}^{n\neq}$, we have

$$\begin{aligned} r_X(g(f, \mathbf{w}))(\mathbf{u}) &= g(f, \mathbf{w})(m(\mathbf{u})) = \bar{g}(f, \mathbf{w}, m(\mathbf{u})) = \mathsf{eval}_X(f, \mathsf{fill}(m(\mathbf{u}), \mathbf{w})) \\ &\overset{(6.4)}{=} \mathsf{eval}_X(f, \mathsf{outl}(\mathbf{u}, \mathbf{w})) = \mathsf{eval}_X(f, \mathbf{u}) = f(\mathbf{u}). \end{aligned}$$ □

This result allows us to describe the exponentiation by a nominal set from a slightly larger class of nominal sets.

Lemma 6.13 *(1) Every single-orbit strong nominal set P is isomorphic to $\mathcal{V}^{n\neq}$ where $n = |\,\mathsf{supp}(p)|$, $p \in P$.*
(2) Every strong nominal set is isomorphic to a coproduct of nominal sets of the form $\mathcal{V}^{n\neq}$.

Proof

(1) Pick some $p \in P$ and choose some order $\{v_1, \ldots, v_n\} = \mathsf{supp}(p)$. Since $\mathsf{supp}(p) = \mathsf{supp}(v_1, \ldots, v_n)$, the isomorphism $\{p\} \cong \{(v_1, \ldots, v_n)\}$ induces an isomorphism $P \cong \mathcal{V}^{n\neq}$ by Proposition 5.10.
(2) Immediate from 1, noting that every nominal set is the coproduct of its orbits and orbits of strong nominal sets are strong. □

This combines nicely with the usual power law for coproducts:

Lemma 6.14 *Given quotients $FX \twoheadrightarrow (-)^P$, $GX \twoheadrightarrow (-)^Q$, exponentiation by $P + Q$ is a quotient of $F \times G$.*

Proof Epimorphisms are stable under products in Nom, and $(-)^P \times (-)^Q \cong (-)^{P+Q}$. □

In combination, these observations prove '$\Longrightarrow$' in (6.1):

Corollary 6.15 *For any orbit-finite strong nominal set P, the functor $(-)^P$ is the quotient of a polynomial functor.*

Proof We have observed that $(-)^{\mathcal{V}^n}$ is a quotient of a polynomial functor for every n. By Lemma 6.12, it follows that $(-)^{\mathcal{V}^{n\neq}}$ is a quotient of a polynomial functor. By Lemma 6.13.1, this property extends to $(-)^P$ for every single-orbit strong nominal set P, and by Lemma 6.13E.2 and Lemma 6.14 to every orbit-finite strong nominal set P. □

Putting all the previous examples together, we can sum up:

Corollary 6.16 *The class of quotients of Nom-liftings contains the constant functors, the identity functor, $\mathcal{P}_{\mathsf{f}}$, the abstraction functor $[\mathcal{V}]$, and the functor $(-)^P$ for any orbit-finite strong nominal set P, and is closed under coproducts, finite products, composition, and quotients.*

7 Conclusions and Future Work

We have identified a sufficient criterion for the rational fixpoint $\varrho\bar{F}$ of a functor $\bar{F}$ on Nom that lifts a functor F on Set to arise as a lifting of the rational fixpoint ϱF of F. Moreover, we have given a sufficient condition that guarantees that rational fixpoints survive quotienting of functors on Nom, that is, for the rational fixpoint ϱH of a quotient H of a Nom-functor G to be a quotient of the rational fixpoint ϱG of G. In combination, these results yield a description of the rational fixpoint for quotients of liftings of Set-functors to Nom. This applies in particular to functors arising from combinations of binding signatures and exponentiation by orbit-finite strong nominal sets. This includes type functors arising in the study of nominal automata, which typically contain exponentiation as in the functor $2 \times X^{\mathcal{V}} \times [\mathcal{V}]X$ defining deterministic nominal automata [21].

It remains to explore the scope of these results, and possibly extend them. Specifically, it is not currently clear how restrictive our sufficient condition on rational fixpoints of liftings actually is; we do give an example of a lifting that violates the condition, and for which indeed the fixpoint of the underlying functor does not lift, but that example is somewhat contrived and moreover can be dealt with by moving to an isomorphic functor. Our condition on quotients of functors in Theorem 5.2 makes explicit reference to coalgebras of the quotient; the presence of a sub-strength then is a condition that refers only to the structure of the quotiented functor as such, without mentioning its coalgebras. We leave a closer analysis of these conditions to future work, e.g. the question whether there are weaker conditions implying the condition on quotients in Theorem 5.2.

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Appendix: Finitary Functors and Preservation of Strong Epimorphisms

We prove that for finitary functors between locally finitely presentable categories, preservations of strong epimorphisms may be tested on strong epimorphisms with finitely generated domain and codomain.

Let $\mathcal{C}$ be a locally finitely presentable category. Recall that an object C of $\mathcal{C}$ is *finitely generated* (fg) if its covariant hom-functor $\mathcal{C}(X, -)$ preserves directed unions. Further recall that every object of $\mathcal{C}$ is the directed union of all its fg subobjects and that $\mathcal{C}$ has (strong epi, mono) factorizations (see [5, Proposition 1.61 and Theorem 1.70]).

Note that in general the classes of finitely presentable and finitely generated objects do not coincide. However, in the category Nom of nominal sets, the finitely generated objects are precisely the orbit-finite nominal sets and the strong epimorphisms are the surjective equivariant maps (i.e. all epis are strong).

Lemma A.1 *For any directed diagram $D : (I, \leq) \to \mathcal{C}$ of subobjects $m_i : C_i \rightarrowtail C$ of C, the colimit $(d_i : C_i \to \operatorname{colim} D)_{i \in I}$ is obtained by taking the (strong epi,mono)-factorization of $\coprod C_i \xrightarrow{[m_i]} C$.*

Proof First note that the $(m_i)_{i \in \mathcal{D}}$ form a cocone, so we have a unique $m : \operatorname{colim} D \to C$ with $m \cdot d_i = m_i$, and d_i is monic. As $\mathcal{C}$ is lfp and both d_i and m_i are monic, [5, Proposition 1.62(ii)] implies that m is monic, too. Recall that, in general, the copairing of colimit injections yields a strong epimorphism $[d_i] : \coprod C_i \to \operatorname{colim} D$. Therefore we have the factorization:

$$\begin{tikzcd}
\coprod C_i \arrow[rr, "{[m_i]}"] \arrow[dr, two heads, "{[d_i]}"'] & & C \\
& \operatorname{colim} D \arrow[ur, tail, "m"'] &
\end{tikzcd}$$

□

Lemma A.2 *Strong quotients of directed colimits are directed colimits of images. More precisely, for a diagram $D : \mathcal{D} \to \mathcal{C}$, given a colimit cocone $(c_i : Di \to C)_{i \in \mathcal{D}}$ and a strong epimorphism $e : C \twoheadrightarrow B$, define A_i by factorizing $e \cdot c_i$ into a strong epi and a mono. Then B is the directed colimit of the A_i together with the induced monomorphisms.*

Proof For each $i \in \mathcal{D}$, take the (strong epi,mono)-factorization

$$\begin{tikzcd} Di \arrow[r, two heads, "e_i"] \arrow[rr, bend left, "e \cdot c_i"] & A_i \arrow[r, tail, "m_i"] & B. \end{tikzcd}$$

For any morphism $g : Di \to Dj$ we get a morphism $\bar{g} : A_i \rightarrowtail A_j$ by diagonalization:

$$\begin{tikzcd} Di \arrow[r, two heads, "e_i"] \arrow[d, "g"'] & A_i \arrow[d, tail, dashed, "\bar{g}"'] \arrow[dr, tail, "m_i"] & \\ Dj \arrow[r, two heads, "e_j"] & A_j \arrow[r, tail, "m_j"'] & B \end{tikzcd}$$

Since $d_j \cdot \bar{g} = d_i$, we see that $\bar{g}$ is monic. It is easy to see that the A_i form a directed diagram of monos in $\mathcal{C}$. To see that B is indeed its colimit, consider the square

$$\begin{tikzcd} \coprod_i Di \arrow[r, two heads, "{[c_i]}"] \arrow[d, two heads, "\coprod e_i"'] & C \arrow[d, two heads, "e"] \\ \coprod A_i \arrow[r, "{[m_i]}"'] & B \end{tikzcd}$$

which commutes by the definition of e_i and m_i. The copairing of the colimit injections $[c_i]$ is a strong epi, hence so is $e \cdot [c_i]$. Since $\coprod e_i$ is a strong epi as well, we see that $[m_i]$ is a strong epi. By Lemma A.1, it follows that B is the colimit of the A_i as desired. $\square$

Proposition A.3 *Let $\mathcal{C}$ and $\mathcal{D}$ be locally finitely presentable categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a finitary functor preserving strong epimorphisms with finitely generated domain and codomain. Then F preserves all epimorphisms.*

Proof Let $e : X \to Y$ be a strong epimorphism. Write X as the colimit of the directed diagram of all its finitely generated subobjects $c_i : X_i \to X$. Take the (strong epi, mono)-factorizations of all $e_i\dot{c}$:

$$\begin{tikzcd} X_i \arrow[r, two heads, "e_i"] \arrow[rr, bend left, "e \cdot c_i"] & A_i \arrow[r, tail, "m_i"] & Y. \end{tikzcd}$$

Note that each A_i is finitely generated, being a strong quotient of the finitely generated object X_i. By Lemma A.2, Y is the directed colimit of the A_i with colimit injections m_i. This directed colimit is preserved by the finitary functor F, resulting in a colimit cocone $(Fm_i : FA_i \to FY)$. The family of colimit injections Fm_i is jointly strongly epic. By assumption, each of the strong epimorphisms e_i

is preserved by F. Hence the $Fm_i \cdot Fe_i$ form a jointly strongly epic family. Since the colimit injections Fc_i form jointly strongly epic family and

$$Fe \cdot Fc_i = Fm_i \cdot Fe_i,$$

we conclude that Fe is a strong epimorphism. □