# FURTHER RESULTS ON THE STRUCTURE OF (CO)ENDS IN FINITE TENSOR CATEGORIES 

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#### Abstract

Let $\mathcal{C}$ be a finite tensor category, and let $\mathcal{M}$ be an exact left $\mathcal{C}$ module category. The action of $\mathcal{C}$ on $\mathcal{M}$ induces a functor $\rho: \mathcal{C} \rightarrow \operatorname{Rex}(\mathcal{M})$, where $\operatorname{Rex}(\mathcal{M})$ is the category of $k$-linear right exact endofunctors on $\mathcal{M}$. Our key observation is that $\rho$ has a right adjoint $\rho^{\text {ra }}$ given by the end $$
\rho^{\mathrm{ra}}(F)=\int_{M \in \mathcal{M}} \underline{\operatorname{Hom}}(M, M) \quad(F \in \operatorname{Rex}(\mathcal{M})) .
$$

As an application, we establish the following results: (1) We give a description of the composition of the induction functor $\mathcal{C}_{\mathcal{M}}^{*} \rightarrow \mathcal{Z}\left(\mathcal{C}_{\mathcal{M}}^{*}\right)$ and Schauenburg's equivalence $\mathcal{Z}\left(\mathcal{C}_{\mathcal{M}}^{*}\right) \approx \mathcal{Z}(\mathcal{C})$. (2) We introduce the space $\mathrm{CF}(\mathcal{M})$ of 'class functions' of $\mathcal{M}$ and initiate the character theory for pivotal module categories. (3) We introduce a filtration for $\mathrm{CF}(\mathcal{M})$ and discuss its relation with some ringtheoretic notions, such as the Reynolds ideal and its generalizations. (4) We show that $\operatorname{Ext}_{\mathcal{C}}^{\bullet}\left(1, \rho^{\mathrm{ra}}\left(\mathrm{id}_{\mathcal{M}}\right)\right)$ is isomorphic to the Hochschild cohomology of $\mathcal{M}$. As an application, we show that the modular group acts projectively on the Hochschild cohomology of a modular tensor category.


## 1. Introduction

Let $\mathcal{C}$ be a finite tensor category. In recent study of finite tensor categories and their applications, it is important to consider the end $A=\int_{X \in \mathcal{C}} X \otimes X^{*}$ and the coend $L=\int^{X \in \mathcal{C}} X^{*} \otimes X$. The end $A$ is a categorical counterpart of the adjoint representation of a Hopf algebra. By using the end $A$, we have established the character theory and the integral theory for finite tensor categories in Shil7b and Shil7e, respectively. The coend $L$, which is isomorphic to $A^{*}$, plays a central role in Lyubashenko's work on 'non-semisimple' modular tensor categories Lyu95a, Lyu95b, Lyu95c, KL01. These results are used in recent progress of topological quantum field theory and conformal field theories GR16, GR17, FGR17.

Since these objects are defined by the universal property, it is difficult to analyze its structure. The aim of this paper is to provide a general framework to deal with such (co)ends. Let $\mathcal{M}$ be an indecomposable exact left $\mathcal{C}$-module category in the sense of EO04]. We denote by $\operatorname{Rex}(\mathcal{M})$ the category of $k$-linear right exact endofunctors on $\mathcal{M}$. The action of $\mathcal{C}$ on $\mathcal{M}$ induces a functor $\rho: \mathcal{C} \rightarrow \operatorname{Rex}(\mathcal{M})$ given by $\rho(X)(M)=X \otimes M$. Our key observation is that a right adjoint of $\rho$, say $\rho^{\text {ra }}$, is a $k$-linear faithful exact functor such that

$$
\begin{equation*}
\rho^{\mathrm{ra}}(F)=\int_{M \in \mathcal{M}} \underline{\operatorname{Hom}}(M, F(M)) \quad(F \in \operatorname{Rex}(\mathcal{M})), \tag{1.1}
\end{equation*}
$$

where Hom is the internal Hom functor (Theorem (3.4). The end $A$ considered at the beginning of this paper is just the case where $\mathcal{M}=\mathcal{C}$ and $F=\mathrm{id}_{\mathcal{C}}$. This result allows us to discuss interaction between several ends through $\rho^{\text {ra }}$. As applications,
we obtain several results on finite tensor categories and their module categories as summarized below:
(1) Let $\mathcal{C}_{\mathcal{M}}^{*}$ be the dual of $\mathcal{C}$ with respect to $\mathcal{M}$. We give an explicit description of the composition of the induction functor $\mathcal{C}_{\mathcal{M}}^{*} \rightarrow \mathcal{Z}\left(\mathcal{C}_{\mathcal{M}}^{*}\right)$ and Schauenburg's equivalence $\mathcal{Z}\left(\mathcal{C}_{\mathcal{M}}^{*}\right) \approx \mathcal{Z}(\mathcal{C})$. We note that this kind of method has been utilized to compute higher Frobenius-Schur indicators [Sch16].
(2) Generalizing Shi17b, we introduce the space $\mathrm{CF}(\mathcal{M})$ of class functions of $\mathcal{M}$. We also introduce the notion of pivotal module category and develop the character theory for such a module category. Especially, we show that the characters of simple objects are linearly independent.
(3) We introduce a filtration $\mathrm{CF}_{1}(\mathcal{M}) \subset \mathrm{CF}_{2}(\mathcal{M}) \subset \cdots \subset \mathrm{CF}(\mathcal{M})$ for the space of class functions. If $\mathcal{M}$ is pivotal, then the first term $\mathrm{CF}_{1}(\mathcal{M})$ is spanned by the characters of simple objects of $\mathcal{M}$ and the second term has the following expression:

$$
\mathrm{CF}_{2}(\mathcal{M}) \cong \mathrm{CF}_{1}(\mathcal{M}) \oplus \bigoplus_{L \in \operatorname{Irr}(\mathcal{M})} \operatorname{Ext}_{\mathcal{M}}^{1}(L, L)
$$

(4) We show that $\operatorname{Ext}_{\mathcal{C}}^{*}\left(\mathbb{1}, A_{\mathcal{M}}\right)$ is isomorphic to the Hochschild cohomology of $\mathcal{M}$, where $\mathbb{1}$ is the unit object of $\mathcal{C}$ and $A_{\mathcal{M}}=\rho^{\text {ra }}\left(\mathrm{id}_{\mathcal{M}}\right)$. As an application, we show that the modular group $\mathrm{SL}_{2}(\mathbb{Z})$ acts projectively on the Hochschild cohomology of a modular tensor category, generalizing [LMSS17.

Organization of this paper. This paper is organized as follows: Section 2 collects several basic notions and facts on finite abelian categories, finite tensor categories and their module categories from ML98, EGNO15, DSS13, DSS14, FSS16.

In Section 3, we study adjoints of the action functor $\rho: \mathcal{C} \rightarrow \operatorname{Rex}(\mathcal{M})$ for a finite tensor category $\mathcal{C}$ and a finite left $\mathcal{C}$-module category $\mathcal{M}$. We show that $\rho$ is an exact functor, and thus has a left adjoint and a right adjoint. It turns out that a right adjoint $\rho^{\mathrm{ra}}$ of $\rho$ is expressed as in (1.1). Moreover, $\rho^{\mathrm{ra}}$ is $k$-linear faithful exact functor if $\mathcal{M}$ is indecomposable and exact (Theorem 3.4).

The functor $\rho^{\text {ra }}$ has a natural structure of a monoidal functor and a $\mathcal{C}$-bimodule functor as a right adjoint of $\rho$. The structure morphisms of $\rho^{\text {ra }}$ are expressed in terms of the universal dinatural transformation of $\rho^{\text {ra }}$ as an end (Lemmas 3.7 and (3.8). By using the structure morphisms of $\rho^{\text {ra }}$, we can 'lift' the adjoint pair $\left(\rho, \rho^{\text {ra }}\right)$ to an adjoint pair between the Drinfeld center $\mathcal{Z}(\mathcal{C})$ and the category $\operatorname{Rex}_{\mathcal{C}}(\mathcal{M})$ of $k$-linear right exact $\mathcal{C}$-module endofunctors on $\mathcal{M}$ (Theorem 3.11). As an application, we give an explicit description of the composition

$$
\mathcal{C}_{\mathcal{M}}^{*}:=\operatorname{Rex}_{\mathcal{C}}(\mathcal{M})^{\text {rev }} \xrightarrow{\text { induction }} \mathcal{Z}\left(\mathcal{C}_{\mathcal{M}}^{*}\right) \xrightarrow{\text { Schauenburg's equivalence }} \mathcal{Z}(\mathcal{C})
$$

in terms of the structure morphisms of $\rho^{\text {ra }}$ (Theorem 3.14).
In Section 4 we consider an end of the form $A_{\mathcal{S}}:=\int_{X \in \mathcal{S}} \underline{\operatorname{Hom}}(X, X)$ for some topologizing full subcategory $\mathcal{S}$ of $\mathcal{M}$ in the sense of Rosenberg [Ros95]. The end $A_{\mathcal{S}}$ has a natural structure of an algebra in $\mathcal{C}$. The main result of this section states that, if $\mathcal{M}$ is an indecomposable exact left $\mathcal{C}$-module category, then $A_{\mathcal{S}}$ is a quotient algebra of $A_{\mathcal{M}}$ and the map

$$
\left\{\begin{array}{c}
\text { topologizing full } \\
\text { subcategories of } \mathcal{M}
\end{array}\right\} \rightarrow\left\{\begin{array}{c}
\text { quotient algebras } \\
\text { of } A_{\mathcal{M}} \text { in } \mathcal{C}
\end{array}\right\}, \quad \mathcal{S} \mapsto A_{\mathcal{S}}
$$

preserves and reflects the order in a certain sense (Theorem4.6). Another important result in Section 4 is that, if $\mathcal{S}$ is closed under the action of $\mathcal{C}$, then $A_{\mathcal{S}}$ lifts to a commutative algebra $\mathbf{A}_{\mathcal{S}}$ in $\mathcal{Z}(\mathcal{C})$ (Theorem4.9).

In Section 5, we consider the space $\operatorname{CF}(\mathcal{M}):=\operatorname{Hom}_{\mathcal{C}}\left(A_{\mathcal{M}}, \mathbb{1}\right)$ of 'class functions' of $\mathcal{M}$. As we have seen in Shi17b, $\operatorname{CF}(\mathcal{M})$ is an algebra if $\mathcal{M}=\mathcal{C}$. We extend this result by constructing a map $\star: \mathrm{CF}(\mathcal{C}) \times \mathrm{CF}(\mathcal{M}) \rightarrow \mathrm{CF}(\mathcal{M})$ making $\mathrm{CF}(\mathcal{M})$ a left $\mathrm{CF}(\mathcal{C})$-module (Lemma 5.3). We also introduce the notion of pivotal structure of an exact module category over a pivotal finite tensor category (Definition 5.6) in terms of the relative Serre functor introduced in FSS16. Let $\mathcal{C}$ be a pivotal finite tensor category, and let $\mathcal{M}$ be a pivotal exact left $\mathcal{C}$-module category. Then, for each object $M \in \mathcal{M}$, the internal character $\operatorname{ch}_{\mathcal{M}}(M)$ is defined in an analogous way as Shi17b (Definition 5.8). Our main result in this section is the following generalization of Shi17b: The linear map

$$
\operatorname{ch}_{\mathcal{M}}: \operatorname{Gr}_{k}(\mathcal{M}) \rightarrow \mathrm{CF}(\mathcal{M}), \quad[M] \mapsto \operatorname{ch}_{\mathcal{M}}(M)
$$

is a well-defined injective map, where $\operatorname{Gr}_{k}(-)=k \otimes_{\mathbb{Z}} \operatorname{Gr}(-)$ is the coefficient extension of the Grothendieck group. Moreover, we have

$$
\operatorname{ch}_{\mathcal{M}}(X \otimes M)=\operatorname{ch}_{\mathcal{C}}(X) \star \operatorname{ch}_{\mathcal{M}}(M)
$$

for all objects $X \in \mathcal{C}$ and $M \in \mathcal{M}$.
In Section 6, we introduce a filtration to the space of class functions. Let $\mathcal{C}$ be a finite tensor category, and let $\mathcal{M}$ be an exact left $\mathcal{C}$-module category. There is the socle filtration $\mathcal{M}_{1} \subset \mathcal{M}_{2} \subset \cdots$ of $\mathcal{M}$. By the result of Section [4, we have a series $A_{\mathcal{M}} \rightarrow \cdots \rightarrow A_{\mathcal{M}_{2}} \rightarrow A_{\mathcal{M}_{1}}$ of epimorphisms in $\mathcal{C}$. Thus, by applying the functor $\operatorname{Hom}_{\mathcal{C}}(-, \mathbb{1})$ to this series, we have a filtration

$$
\mathrm{CF}_{1}(\mathcal{M}) \subset \mathrm{CF}_{2}(\mathcal{M}) \subset \mathrm{CF}_{3}(\mathcal{M}) \subset \cdots \subset \mathrm{CF}(\mathcal{M}),
$$

where $\operatorname{CF}_{n}(\mathcal{M})=\operatorname{Hom}_{\mathcal{C}}\left(A_{\mathcal{M}_{n}}, \mathbb{1}\right)$. We investigate relations between this filtration and some ring-theoretic notions, such as the Jacobson radical, the Reynolds ideal and the space of symmetric linear forms. We see that $\mathrm{CF}_{1}(\mathcal{M})$ is spanned by the characters of simple objects of $\mathcal{M}$. The second term $\mathrm{CF}_{2}(\mathcal{M})$ is expressed in terms of $\operatorname{Ext}_{\mathcal{M}}^{1}(L, L)$ for simple objects $L \in \mathcal{M}$. For $\operatorname{CF}_{n}(\mathcal{M})$ with $n \geq 3$, we have no general results but study some examples.

In Section 7 we discuss the Hochschild (co)homology of finite tensor categories and their module categories. One can define the Hochschild homology HH. ( $\mathcal{M}$ ) and the Hochschild cohomology $\mathrm{HH}^{\bullet}(\mathcal{M})$ of a finite abelian category $\mathcal{M}$ in terms of the Ext functor in $\operatorname{Rex}(\mathcal{M})$. We then show that, if $\mathcal{M}$ is an exact $\mathcal{C}$-module category, then there is an isomorphism

$$
\begin{equation*}
\operatorname{HH}^{\bullet}(\mathcal{M}) \cong \operatorname{Ext}_{\mathcal{C}}^{\bullet}\left(\mathbb{1}, A_{\mathcal{M}}\right) \tag{1.2}
\end{equation*}
$$

If, in addition, $\mathcal{M}$ is pivotal, then there is also an isomorphism

$$
\operatorname{HH}_{\bullet}(\mathcal{M}) \cong \operatorname{Ext}_{\mathcal{C}}^{\bullet}\left(A_{\mathcal{M}}, \mathbb{1}\right)^{*}
$$

The isomorphism (1.2) is a generalization of the known fact that the Hochschild cohomology of a Hopf algebra can be computed by the cohomology of the adjoint representation. We use (1.2) to extend recent results of [LMSS17].

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## 2. Preliminaries

2.1. Ends and coends. For basic theory on categories, we refer the reader to the book of Mac Lane ML98. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $S$ and $T$ be functors from $\mathcal{C}^{\text {op }} \times \mathcal{C}$ to $\mathcal{D}$. A dinatural transformation [ML98, IX] from $S$ to $T$ is a family $\xi=\left\{\xi_{X}: S(X, X) \rightarrow T(X, X)\right\}_{X \in \mathcal{C}}$ of morphisms in $\mathcal{D}$ satisfying

$$
T\left(\operatorname{id}_{X}, f\right) \circ \xi_{X} \circ S\left(f, \operatorname{id}_{X}\right)=T\left(f, \mathrm{id}_{Y}\right) \circ \xi_{Y} \circ S\left(\operatorname{id}_{Y}, f\right)
$$

for all morphisms $f: X \rightarrow Y$ in $\mathcal{C}$. An end of $S$ is an object $E \in \mathcal{D}$ equipped with a dinatural transformation $\pi: E \rightarrow S$ that is universal in a certain sense (here the object $E$ is regarded as a constant functor from $\mathcal{C}^{\text {op }} \times \mathcal{C}$ to $\left.\mathcal{D}\right)$. Dually, a coend of $T$ is an object $C \in \mathcal{D}$ equipped with a 'universal' dinatural transformation from $T$ to $C$. An end of $S$ and a coend of $T$ are denoted by $\int_{X \in \mathcal{C}} S(X, X)$ and $\int^{X \in \mathcal{C}} T(X, X)$, respectively.

A (co)end does not exist in general. We note the following useful criteria for the existence of (co)ends. Suppose that $\mathcal{C}$ is essentially small. Let $\mathcal{C}, \mathcal{D}$ and $S$ be as above. Since the category Set of all sets is complete, the end

$$
S^{\natural}(W):=\int_{X \in \mathcal{C}} \operatorname{Hom}_{\mathcal{D}}(W, S(X, X))
$$

exists for each object $W \in \mathcal{D}$. By the parameter theorem for ends ML98, XI.7], we extend the assignment $W \mapsto S^{\natural}(W)$ to the contravariant functor $S^{\natural}: \mathcal{D} \rightarrow$ Set. The following lemma is the dual of [Shi17c, Lemma 3.1].

Lemma 2.1. An end of $S$ exists if and only if $S^{\natural}$ is representable.
We also note the following lemma:
Lemma 2.2. Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{V}$ be categories, and let $L: \mathcal{A} \rightarrow \mathcal{B}, R: \mathcal{B} \rightarrow \mathcal{A}$ and $H: \mathcal{B}^{\mathrm{op}} \times \mathcal{A} \rightarrow \mathcal{V}$ be functors. Suppose that $L$ is left adjoint to $R$. Then we have an isomorphism

$$
\begin{equation*}
\int_{V \in \mathcal{A}} H(V, L(V)) \cong \int_{W \in \mathcal{B}} H(R(W), W) \tag{2.1}
\end{equation*}
$$

meaning that if either one of these ends exists, then both exist and they are canonically isomorphic.

This lemma is the dual of BV12, Lemma 3.9]. For later use, we recall the construction of the canonical isomorphism (2.1). Let $\mathcal{E}$ and $\mathcal{E}^{\prime}$ be the left and the right hand side of (2.1), respectively, and let

$$
\pi(V): \mathcal{E} \rightarrow H(V, L(V)) \quad \text { and } \quad \pi^{\prime}(W): \mathcal{E}^{\prime} \rightarrow H(R(W), W)
$$

be the respective universal dinatural transformations. We assume that $(L, R)$ is an adjoint pair with unit $\eta: \operatorname{id}_{\mathcal{D}} \rightarrow R L$ and counit $\varepsilon: L R \rightarrow \mathrm{id}_{\mathcal{C}}$. By the universal property of $\mathcal{E}$, there is a unique morphism $\alpha: \mathcal{E}^{\prime} \rightarrow \mathcal{E}$ in $\mathcal{V}$ satisfying

$$
\pi(V) \circ \alpha=H\left(\eta_{V}, \operatorname{id}_{L(V)}\right) \circ \pi^{\prime}(L(V))
$$

for all objects $V \in \mathcal{A}$. Similarly, by the universal property of $\mathcal{E}^{\prime}$, there is a unique morphism $\beta: \mathcal{E} \rightarrow \mathcal{E}^{\prime}$ satisfying

$$
\pi^{\prime}(W) \circ \beta=\pi(R(W)) \circ H\left(\operatorname{id}_{R(W)}, \varepsilon_{W}\right)
$$

for all objects $W \in \mathcal{B}$. By the zigzag identities and the dinaturality of $\pi$ and $\pi^{\prime}$, one can verify that $\alpha$ and $\beta$ are mutually inverse.
2.2. Monoidal categories. A monoidal category ML98, VII] is a category $\mathcal{C}$ equipped with a functor $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, an object $\mathbb{1} \in \mathcal{C}$ and natural isomorphisms $(X \otimes Y) \otimes Z \cong X \otimes(Y \otimes Z)$ and $\mathbb{1} \otimes X \cong X \cong X \otimes \mathbb{1}(X, Y, Z \in \mathcal{C})$ satisfying the pentagon and the triangle axiom. If these natural isomorphisms are identities, then $\mathcal{C}$ is said to be strict. By the Mac Lane coherence theorem, we may assume that every monoidal category is strict.

We fix several conventions on monoidal categories and related notions: Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories. A monoidal functor [ML98, XI.2] from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ equipped with a natural transformation

$$
f_{X, Y}^{(2)}: F(X) \otimes F(Y) \rightarrow F(X \otimes Y) \quad(X, Y \in \mathcal{C})
$$

and a morphism $f^{(0)}: \mathbb{1} \rightarrow F(\mathbb{1})$ in $\mathcal{D}$ satisfying certain axioms. A monoidal functor $F=\left(F, f^{(2)}, f^{(0)}\right)$ is said to be strong if the structure morphisms $f^{(2)}$ and $f^{(0)}$ are invertible.

Let $L$ and $R$ be objects of a monoidal category $\mathcal{C}$, and let $\varepsilon: L \otimes R \rightarrow \mathbb{1}$ and $\eta: \mathbb{1} \rightarrow R \otimes L$ be morphisms in $\mathcal{C}$. We say that $(L, \varepsilon, \eta)$ is a left dual object of $R$ and $(R, \varepsilon, \eta)$ is a right dual object of $L$ if the equations

$$
\left(\varepsilon \otimes \mathrm{id}_{L}\right) \circ\left(\mathrm{id}_{L} \otimes \eta\right)=\mathrm{id}_{L} \quad \text { and } \quad\left(\mathrm{id}_{R} \otimes \varepsilon\right) \circ\left(\eta \otimes \mathrm{id}_{R}\right)=\mathrm{id}_{R}
$$

hold. If this is the case, then the morphisms $\varepsilon$ and $\eta$ are called the evaluation and the coevaluation, respectively.

A monoidal category $\mathcal{C}$ is said to be rigid if every object of $\mathcal{C}$ has a left dual object and a right dual object. If $\mathcal{C}$ is rigid, then we denote by $\left(X^{*}, \mathrm{ev}_{X}, \operatorname{coev}_{X}\right)$ the left dual object of $X \in \mathcal{C}$. Let $\mathcal{C}^{\text {rev }}$ denote the category $\mathcal{C}$ equipped with the reversed tensor product $X \otimes^{\text {rev }} Y=Y \otimes X$. The assignment $X \mapsto X^{*}$ gives rise to a strong monoidal functor $(-)^{*}: \mathcal{C}^{\mathrm{op}} \rightarrow \mathcal{C}^{\mathrm{rev}}$ called the left duality functor of $\mathcal{C}$. The right duality functor ${ }^{*}(-)$ of $\mathcal{C}$ is also defined by taking the right dual object. The left and the right duality functor are mutually quasi-inverse to each other.
2.3. Module categories. Let $\mathcal{C}$ be a monoidal category. A left $\mathcal{C}$-module category EGNO15] is a category $\mathcal{M}$ equipped with a functor $\otimes: \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$, called the action of $\mathcal{C}$, and natural isomorphisms

$$
\begin{equation*}
(X \otimes Y) \otimes M \cong X \otimes(Y \otimes M) \quad \text { and } \quad \mathbb{1} \otimes M \cong M \quad(X, Y \in \mathcal{C}, M \in \mathcal{M}) \tag{2.2}
\end{equation*}
$$

satisfying certain axioms similar to those for monoidal categories. There is an analogue of the Mac Lane coherence theorem for $\mathcal{C}$-module categories. Thus, without loss of generality, we may assume that the natural isomorphisms (2.2) are the identity; see EGNO15, Remark 7.2.4].

Let $\mathcal{M}$ and $\mathcal{N}$ be left $\mathcal{C}$-module categories. A lax left $\mathcal{C}$-module functor from $\mathcal{M}$ to $\mathcal{N}$ is a functor $F: \mathcal{M} \rightarrow \mathcal{N}$ equipped with a natural transformation

$$
s_{X, M}: X \otimes F(M) \rightarrow F(X \otimes M) \quad(X \in \mathcal{C}, M \in \mathcal{M})
$$

such that the equations

$$
s_{\mathbb{1}, M}=\operatorname{id}_{M} \quad \text { and } \quad s_{X \otimes Y, M}=s_{X, Y \otimes M} \circ\left(\mathrm{id}_{X} \otimes s_{Y, M}\right)
$$

hold for all objects $X, Y \in \mathcal{C}$ and $M \in \mathcal{M}$. We omit the definition of morphisms of lax $\mathcal{C}$-module functors; see DSS13, DSS14.

An oplax left $\mathcal{C}$-module functor from $\mathcal{M}$ to $\mathcal{N}$ is, in a word, a lax left $\mathcal{C}^{\text {op }}$-module functor from $\mathcal{M}^{\text {op }}$ to $\mathcal{N}^{\text {op }}$; see DSS14. Now let $L: \mathcal{M} \rightarrow \mathcal{N}$ be a functor with right adjoint $R: \mathcal{N} \rightarrow \mathcal{M}$, and let $\eta: \operatorname{id}_{\mathcal{M}} \rightarrow R L$ and $\varepsilon: L R \rightarrow \mathrm{id}_{\mathcal{N}}$ be the unit
and the counit of the adjunction $L \dashv R$. If $(L, v)$ is an oplax left $\mathcal{C}$-module functor, then $R$ is a lax left $\mathcal{C}$-module functor by the structure morphism defined by

$$
\begin{align*}
X \otimes R(N) & \xrightarrow{\eta_{X \otimes R(N)}} R L(X \otimes R(N)) \\
& \xrightarrow{R\left(v_{X, R(N)}\right)}  \tag{2.3}\\
& R(X \otimes L R(N)) \xrightarrow{R\left(\operatorname{id}_{X} \otimes \varepsilon_{N}\right)} R(X \otimes N)
\end{align*}
$$

for $X \in \mathcal{C}$ and $N \in \mathcal{N}$. Conversely, if $(R, w)$ is a lax $\mathcal{C}$-module functor, then $L$ is an oplax $\mathcal{C}$-module functor by

$$
\begin{align*}
L(X \otimes M) & \xrightarrow{L\left(\operatorname{id}_{X} \otimes \eta_{M}\right)} L(X \otimes R L(M))  \tag{2.4}\\
& \xrightarrow{L\left(w_{X, L(M)}\right)} L R(X \otimes L(M)) \xrightarrow{\varepsilon_{X \otimes L(M)}} R(X \otimes N)
\end{align*}
$$

for $X \in \mathcal{C}$ and $M \in \mathcal{M}$ DSS14, Lemma 2.11].
We say that an (op)lax $\mathcal{C}$-module functor $(F, s)$ is strong if the natural transformation $s$ is invertible. If $\mathcal{C}$ is rigid, then every (op)lax $\mathcal{C}$-module functor is strong [DSS14, Lemma 2.10] and thus we refer to an (op)lax $\mathcal{C}$-module functor simply as a $\mathcal{C}$-module functor.
2.4. Closed module categories. Let $\mathcal{C}$ be a monoidal category. A left $\mathcal{C}$-module category $\mathcal{M}$ is said to be closed if, for all objects $M \in \mathcal{M}$, the functor

$$
\begin{equation*}
\mathcal{C} \rightarrow \mathcal{M}, \quad X \mapsto X \otimes M \tag{2.5}
\end{equation*}
$$

has a right adjoint (cf. the definition of a closed monoidal category). Suppose that $\mathcal{M}$ is closed. For each object $M \in \mathcal{M}$, we fix a right adjoint $\underline{\operatorname{Hom}}(M,-)$ of the functor (2.5). Thus, by definition, there is a natural isomorphism

$$
\begin{equation*}
\phi: \operatorname{Hom}_{\mathcal{M}}(X \otimes M, N) \rightarrow \operatorname{Hom}_{\mathcal{C}}(X, \underline{\operatorname{Hom}}(M, N)) \tag{2.6}
\end{equation*}
$$

for $N \in \mathcal{M}$ and $X \in \mathcal{C}$. If we denote by

$$
\underline{\operatorname{coev}}_{X, M}: X \rightarrow \underline{\operatorname{Hom}}(M, X \otimes M) \quad \text { and } \quad \underline{\mathrm{ev}}_{M, N}: \underline{\operatorname{Hom}}(M, N) \otimes M \rightarrow N
$$

the unit and the counit of the adjunction $(-) \otimes M \dashv \underline{\operatorname{Hom}}(M,-)$, respectively, then the isomorphism (2.5) is given by

$$
\begin{equation*}
\phi(f)=\underline{\operatorname{Hom}}(M, f) \circ \underline{\operatorname{coev}}_{M, X} \quad \text { and } \quad \phi^{-1}(g)=\underline{\mathrm{ev}}_{M, N} \circ\left(g \otimes \operatorname{id}_{M}\right) \tag{2.7}
\end{equation*}
$$

for morphisms $f: X \otimes M \rightarrow N$ in $\mathcal{M}$ and $g: X \rightarrow \underline{\operatorname{Hom}}(M, N)$ in $\mathcal{C}$.
By [ML98, IV.7], one can extend the assignment $(M, N) \mapsto \underline{\operatorname{Hom}}(M, N)$ to a functor from $\mathcal{M}^{\mathrm{op}} \times \mathcal{M}$ to $\mathcal{C}$ in such a way that the isomorphism (2.6) is natural also in $M$. We call the functor Hom the internal Hom functor of $\mathcal{M}$. This makes $\mathcal{M}$ a $\mathcal{C}$-enriched category: The composition

$$
\begin{equation*}
\underline{\operatorname{comp}}_{M_{1}, M_{2}, M_{3}}: \underline{\operatorname{Hom}}\left(M_{2}, M_{3}\right) \otimes \underline{\operatorname{Hom}}\left(M_{1}, M_{2}\right) \rightarrow \underline{\operatorname{Hom}}\left(M_{1}, M_{3}\right) \tag{2.8}
\end{equation*}
$$

is defined to be the morphism corresponding to

$$
\begin{equation*}
\underline{\mathrm{ev}}_{M_{1}, M_{2}, M_{3}}^{(3)}:=\underline{\mathrm{ev}}_{M_{2}, M_{3}} \circ\left(\mathrm{id}_{\underline{\mathrm{Hom}}\left(M_{2}, M_{3}\right)} \otimes \underline{\mathrm{ev}}_{M_{1}, M_{2}}\right) \tag{2.9}
\end{equation*}
$$

via the isomorphism (2.6) with $X=\underline{\operatorname{Hom}}\left(M_{2}, M_{3}\right) \otimes \underline{\operatorname{Hom}}\left(M_{1}, M_{2}\right), M=M_{1}$ and $N=M_{3}$. The identity on $M \in \mathcal{M}$ is $\underline{\operatorname{coev}}_{\mathbb{1}, M}$.

We suppose that $\mathcal{C}$ is rigid. Let $M \in \mathcal{M}$ be an object. Since the functor (2.5) is a $\mathcal{C}$-module functor, so is its right adjoint $\underline{\operatorname{Hom}(M,-) \text {. We denote by }}$

$$
\begin{equation*}
\mathfrak{a}_{X, M, N}: X \otimes \underline{\operatorname{Hom}}(M, N) \rightarrow \underline{\operatorname{Hom}}(M, X \otimes N) \quad(X \in \mathcal{C}, N \in \mathcal{M}) \tag{2.10}
\end{equation*}
$$

the left $\mathcal{C}$-module structure of $\underline{\operatorname{Hom}}(M,-)$. There is also an isomorphism

$$
\begin{equation*}
\mathfrak{b}_{X, M, N}: \underline{\operatorname{Hom}}(X \otimes M, N) \rightarrow \underline{\operatorname{Hom}}(M, N) \otimes X^{*} \tag{2.11}
\end{equation*}
$$

induced by the natural isomorphisms

$$
\begin{gathered}
\operatorname{Hom}_{\mathcal{C}}(W, \underline{\operatorname{Hom}}(Y \otimes M, N)) \cong \operatorname{Hom}_{\mathcal{M}}(W \otimes Y \otimes M, N) \\
\cong \operatorname{Hom}_{\mathcal{C}}(W \otimes Y, \underline{\operatorname{Hom}}(M, N)) \cong \operatorname{Hom}_{\mathcal{C}}\left(W, \underline{\operatorname{Hom}}(M, N) \otimes Y^{*}\right)
\end{gathered}
$$

for $W, Y \in \mathcal{C}$ and $N, M \in \mathcal{M}$. It is convenient to introduce the morphism

$$
\begin{equation*}
\mathfrak{b}_{X, M, N}^{\natural}: \operatorname{Hom}(X \otimes M, N) \otimes X \rightarrow \underline{\operatorname{Hom}}(M, N) \tag{2.12}
\end{equation*}
$$

defined by $\mathfrak{b}_{X, M, N}^{\natural}=\left(\operatorname{id}_{\underline{H o m}(M, N)} \otimes \operatorname{ev}_{X}\right) \circ\left(\mathfrak{b}_{X, M, N} \otimes \operatorname{id}_{X}\right)$. We note that $\mathfrak{b}_{X, M, N}^{\natural}$ is natural in the variables $M$ and $N$ and dinatural in $X$.

Lemma 2.3. For all objects $X, Y \in \mathcal{C}$ and $M, N \in \mathcal{M}$, we have the equation

$$
\begin{equation*}
\mathfrak{b}_{\mathbb{1}, M, N}=\operatorname{id}_{\underline{\operatorname{Hom}(M, N)}} \tag{2.13}
\end{equation*}
$$

and the following commutative diagrams:

$$
\begin{align*}
& \underline{\operatorname{Hom}}(X \otimes Y \otimes M, N) \xrightarrow{\mathfrak{b}_{X \otimes Y, M, N}} \underline{\operatorname{Hom}(M, N) \otimes(X \otimes Y)^{*}} \tag{2.14}
\end{align*}
$$

$$
\begin{align*}
& X \otimes \underline{\operatorname{Hom}}(Y \otimes M, N) \xrightarrow{\mathrm{id}_{X} \otimes \mathfrak{b}_{Y, M, N}} X \otimes \underline{\operatorname{Hom}}(M, N) \otimes Y^{*} \tag{2.15}
\end{align*}
$$

The category $\mathcal{M}^{\mathrm{op}} \times \mathcal{M}$ is a $\mathcal{C}$-bimodule category by the actions given by

$$
\begin{equation*}
X \otimes(M, N)=\left(M^{\mathrm{op}}, X \otimes N\right) \quad \text { and } \quad\left(M^{\mathrm{op}}, N\right) \otimes X=\left({ }^{*} X \otimes M, N\right) \tag{2.16}
\end{equation*}
$$

for $X \in \mathcal{C}$ and $M, N \in \mathcal{M}$. The above lemma means that the internal Hom functor of $\mathcal{M}$ is a $\mathcal{C}$-bimodule functor from $\mathcal{M}^{\mathrm{op}} \times \mathcal{M}$ to $\mathcal{C}$ with left $\mathcal{C}$-module structure $\mathfrak{a}$ and the right $\mathcal{C}$-module structure given by

$$
\left(\mathfrak{b}^{*} Y, M, N\right)^{-1}: \underline{\operatorname{Hom}}(M, N) \otimes Y \rightarrow \underline{\operatorname{Hom}}\left({ }^{*} Y \otimes M, N\right) \quad(Y \in \mathcal{C}, M, N \in \mathcal{M}) .
$$

Although the above lemma seems to be well-known, we give its proof in Appen$\operatorname{dix} \AA$ for the sake of completeness. We will also give some equations involving the natural isomorphisms $\mathfrak{a}$ and $\mathfrak{b}$ in Appendix A,

In view of this lemma, we define the isomorphism

$$
\begin{equation*}
\mathfrak{c}_{X, M, N, Y}: X \otimes \underline{\operatorname{Hom}}(M, N) \otimes Y^{*} \rightarrow \underline{\operatorname{Hom}}(Y \otimes M, X \otimes N) \tag{2.17}
\end{equation*}
$$

for $X, Y \in \mathcal{C}$ and $M, N \in \mathcal{M}$ by

$$
\begin{equation*}
\mathfrak{c}_{X, M, N, Y}=\mathfrak{b}_{Y, M, X \otimes N}^{-1} \circ\left(\mathfrak{a}_{X, M, N} \otimes \operatorname{id}_{Y^{*}}\right)=\mathfrak{a}_{X, Y \otimes M, N} \circ\left(\operatorname{id}_{X} \otimes \mathfrak{b}_{Y, M, N}^{-1}\right) . \tag{2.18}
\end{equation*}
$$

2.5. Finite abelian categories. Throughout this paper, we work over an algebraically closed field $k$ of arbitrary characteristic. Given algebras $A$ and $B$ over $k$, we denote by $A$-mod, mod- $B$ and $A$-mod- $B$ the category of finite-dimensional left $A$-modules, the category of finite-dimensional right $B$-modules, and the category of finite-dimensional $A$ - $B$-bimodules, respectively.

A finite abelian category EGNO15, Definition 1.8.5] is a $k$-linear category that is equivalent to $A$-mod for some finite-dimensional algebra $A$ over $k$. For finite abelian categories $\mathcal{M}$ and $\mathcal{N}$, we denote by $\operatorname{Rex}(\mathcal{M}, \mathcal{N})$ the category of $k$-linear right exact functors from $\mathcal{M}$ to $\mathcal{N}$. If $A$ and $B$ are finite-dimensional algebras over $k$, then the Eilenberg-Watts theorem gives an equivalence

$$
\begin{equation*}
B-\bmod -A \xrightarrow{\approx} \operatorname{Rex}(A-\bmod , B-\bmod ), \quad M \mapsto M \otimes_{A}(-) \tag{2.19}
\end{equation*}
$$

of $k$-linear categories. Thus $\operatorname{Rex}(\mathcal{M}, \mathcal{N})$ is a finite abelian category. The above equivalence also implies that a $k$-linear functor $F: \mathcal{M} \rightarrow \mathcal{N}$ is right exact if and only if $F$ has a right adjoint.

A $k$-linear category $\mathcal{M}$ is finite abelian if and only if $\mathcal{M}^{\text {op }}$ is. Thus, by the dual argument, we see that a $k$-linear functor $F: \mathcal{M} \rightarrow \mathcal{N}$ is left exact if and only if $F$ has a left adjoint. We denote by $\operatorname{Lex}(\mathcal{M}, \mathcal{N})$ the category of $k$-linear left exact functors from $\mathcal{M}$ to $\mathcal{N}$. For a $k$-linear functor $F$, we denote by $F^{\text {la }}$ and $F^{\text {ra }}$ a left and a right adjoint of $F$, respectively, if it exists.

Let $M$ be an object of $\mathcal{M}$. Then the functor $\operatorname{Hom}_{\mathcal{M}}(M,-): \mathcal{M} \rightarrow k$-mod is left exact, and thus has a left adjoint. We denote it by $(-) \otimes_{k} M$. By definition, there is a natural isomorphism

$$
\operatorname{Hom}_{\mathcal{M}}\left(X \otimes_{k} M, N\right) \cong \operatorname{Hom}_{k}\left(X, \operatorname{Hom}_{\mathcal{N}}(M, N)\right) \quad(X \in k-\bmod , N \in \mathcal{M})
$$

For two finite abelian categories $\mathcal{M}$ and $\mathcal{N}$, we denote by $\mathcal{M} \boxtimes \mathcal{N}$ their Deligne tensor product [EGNO15, $\S 1.11]$. If $\mathcal{M}=A$-mod and $\mathcal{N}=B$-mod for some finitedimensional algebras $A$ and $B$, then $\mathcal{M} \boxtimes \mathcal{N}$ is identified with $\left(A \otimes_{k} B\right)$-mod. In view of the equivalence (2.19), one has:

Lemma 2.4 ([Shi17d, Lemma 3.3]). The $k$-linear functor

$$
\begin{equation*}
\Phi_{\mathcal{M}, \mathcal{N}}: \mathcal{M}^{\mathrm{op}} \boxtimes \mathcal{N} \rightarrow \operatorname{Rex}(\mathcal{M}, \mathcal{N}), \quad M^{\mathrm{op}} \boxtimes N \mapsto \operatorname{Hom}_{\mathcal{M}}(-, M)^{*} \otimes_{k} N \tag{2.20}
\end{equation*}
$$

is an equivalence. Moreover, the functor

$$
\begin{equation*}
\operatorname{Rex}(\mathcal{M}, \mathcal{N}) \rightarrow \mathcal{M}^{\mathrm{op}} \boxtimes \mathcal{N}, \quad F \mapsto \int_{M \in \mathcal{M}} M^{\mathrm{op}} \boxtimes F(M) \tag{2.21}
\end{equation*}
$$

is a quasi-inverse of (2.20)
The following lemma is proved by utilizing the equivalences (2.20) and (2.21).
Lemma 2.5 (Shi17a, Lemma 2.5]). Let $\mathcal{M}$ and $\mathcal{N}$ be finite abelian categories. For a $k$-linear right exact functor $F: \mathcal{M} \rightarrow \mathcal{N}$, the following are equivalent:
(1) $F$ is a projective object of the abelian category $\operatorname{Rex}(\mathcal{M}, \mathcal{N})$.
(2) $F(M)$ is a projective object of $\mathcal{N}$ for all objects $M \in \mathcal{M}$ and $F^{\mathrm{ra}}(N)$ is an injective object of $\mathcal{M}$ for all objects $N \in \mathcal{N}$.

For finite abelian categories $\mathcal{M}$ and $\mathcal{N}$, there is also an equivalence

$$
\Psi_{\mathcal{M}, \mathcal{N}}: \operatorname{Lex}(\mathcal{M}, \mathcal{N}) \rightarrow \mathcal{M}^{\mathrm{op}} \boxtimes \mathcal{N}, \quad F \mapsto \int^{M \in \mathcal{M}} M^{\mathrm{op}} \boxtimes F(M)
$$

Shi17c, Lemmas 3.2 and 3.3]. Fuchs, Schaumann and Schweigert [FSS16] defined the Nakayama functor of $\mathcal{M}$ by

$$
\mathbb{N}_{\mathcal{M}}:=\Phi_{\mathcal{M}, \mathcal{M}} \Psi_{\mathcal{M}, \mathcal{M}}\left(\mathrm{id}_{\mathcal{M}}\right) \in \operatorname{Rex}(\mathcal{M}, \mathcal{M})
$$

For later use, we recall from FSS16 the following results:
(1) We say that $\mathcal{M}$ is Frobenius if the class of injective objects of $\mathcal{M}$ coincides with the class of projective objects of $\mathcal{M}$ (or, equivalently, $\mathcal{M} \approx A$-mod for some Frobenius algebra $A$ ). The Nakayama functor $\mathbb{N}_{\mathcal{M}}$ is an equivalence if and only if $\mathcal{M}$ is Frobenius.
(2) We say that $\mathcal{M}$ is symmetric Frobenius if $\mathcal{M} \approx A$-mod for some symmetric Frobenius algebra $A$. The Nakayama functor $\mathbb{N}_{\mathcal{M}}$ is isomorphic to id ${ }_{\mathcal{M}}$ if and only if $\mathcal{M}$ is symmetric Frobenius.
(3) If $F: \mathcal{M} \rightarrow \mathcal{N}$ is a $k$-linear exact functor between finite abelian categories $\mathcal{M}$ and $\mathcal{N}$, then there is an isomorphism $\mathbb{N}_{\mathcal{M}} \circ F^{\text {la }} \cong F^{\text {ra }} \circ \mathbb{N}_{\mathcal{N}}$.
(4) The Nakayama functor of $\operatorname{Rex}(\mathcal{M}, \mathcal{N})$ is given by

$$
\mathbb{N}_{\operatorname{Rex}(\mathcal{M}, \mathcal{N})}(F)=\mathbb{N}_{\mathcal{N}} \circ F \circ \mathbb{N}_{\mathcal{M}} \quad(F \in \operatorname{Rex}(\mathcal{M}, \mathcal{N}))
$$

2.6. Finite tensor categories and their modules. A finite tensor category [EO04] is a rigid monoidal category $\mathcal{C}$ such that $\mathcal{C}$ is a finite abelian category, the tensor product of $\mathcal{C}$ is $k$-linear in each variable, and the unit object $\mathbb{1}$ of $\mathcal{C}$ is a simple object. A finite tensor category is Frobenius. The tensor product of a finite tensor category is exact in each variable.

Let $\mathcal{C}$ be a finite tensor category. A finite left $\mathcal{C}$-module category is a left $\mathcal{C}$-module category $\mathcal{M}$ such that $\mathcal{M}$ is a finite abelian category and the action of $\mathcal{C}$ on $\mathcal{M}$ is $k$-linear and right exact in each variable. One can define a finite right $\mathcal{C}$-module category and a finite $\mathcal{C}$-bimodule category in a similar manner.

Given an algebra $A \in \mathcal{C}\left(=\right.$ a monoid in $\mathcal{C}$ (ML98]), we denote by $\mathcal{C}_{A}$ the category of right $A$-modules in $\mathcal{C}$. The category $\mathcal{C}_{A}$ is a finite left $\mathcal{C}$-module category in a natural way. Moreover, every finite left $\mathcal{C}$-module category is equivalent to $\mathcal{C}_{A}$ for some algebra $A \in \mathcal{C}$ as a $\mathcal{C}$-module category. This implies that the action of $\mathcal{C}$ on a finite $\mathcal{C}$-module category is exact in each variable [DSS14, Corollary 2.26], although only the right exactness is assumed in our definition.

An exact left $\mathcal{C}$-module category EO04 is a finite left $\mathcal{C}$-module category $\mathcal{M}$ such that $P \otimes M$ is a projective object of $\mathcal{M}$ for all projective objects $P \in \mathcal{C}$ and for all objects $M \in \mathcal{M}$. It is known that exact module categories are Frobenius.

## 3. Adjoint of the action functor

3.1. The action functor. Let $\mathcal{C}$ be a finite tensor category, and let $\mathcal{M}$ and $\mathcal{N}$ be two finite left $\mathcal{C}$-module categories. Then $\operatorname{Rex}(\mathcal{M}, \mathcal{N})$ is a $\mathcal{C}$-bimodule category by the left action and the right action given by

$$
\begin{equation*}
(X \otimes F)(M)=X \otimes F(M) \quad \text { and } \quad(F \otimes X)(M)=F(X \otimes M) \tag{3.1}
\end{equation*}
$$

respectively, for $F \in \operatorname{Rex}(\mathcal{M}, \mathcal{N}), X \in \mathcal{C}$ and $M \in \mathcal{M}$. The category $\mathcal{M}^{\text {op }} \boxtimes \mathcal{N}$ is also a $\mathcal{C}$-bimodule category by the left and the right action determined by

$$
X \otimes\left(M^{\mathrm{op}} \boxtimes N\right)=M^{\mathrm{op}} \boxtimes(X \otimes N) \quad \text { and } \quad\left(M^{\mathrm{op}} \boxtimes N\right) \otimes X=\left({ }^{*} X \otimes M\right)^{\mathrm{op}} \boxtimes N
$$

respectively, for $M \in \mathcal{M}, N \in \mathcal{N}$ and $X \in \mathcal{C}$. It is easy to see that the equivalence (2.20) is in fact an equivalence of $\mathcal{C}$-bimodule categories. Since $\mathcal{M}^{\mathrm{op}} \boxtimes \mathcal{N}$ is a finite $\mathcal{C}$-bimodule category, so is $\operatorname{Rex}(\mathcal{M}, \mathcal{N})$.

Now we define the functor $\rho_{\mathcal{M}}: \mathcal{C} \rightarrow \operatorname{Rex}(\mathcal{M}):=\operatorname{Rex}(\mathcal{M}, \mathcal{M})$ by $X \mapsto X \otimes \mathrm{id}_{\mathcal{M}}$ and call $\rho_{\mathcal{M}}$ the action functor of $\mathcal{M}$. Since the action of $\mathcal{C}$ on a finite $\mathcal{C}$-module category is $k$-linear and exact in each variable, we have:

Lemma 3.1. The action functor $\rho_{\mathcal{M}}$ is $k$-linear and exact.
Thus the action functor $\rho_{\mathcal{M}}$ has a left adjoint and a right adjoint. The aim of this section is to study properties of adjoints of $\rho_{\mathcal{M}}$. Before doing so, we characterize some properties of $\mathcal{M}$ in terms of $\rho_{\mathcal{M}}$.

Lemma 3.2. $\mathcal{M}$ is exact if and only if $\rho_{\mathcal{M}}$ preserves projective objects.
Proof. Suppose that $\mathcal{M}$ is an exact $\mathcal{C}$-module category. We fix a projective object $P \in \mathcal{C}$ and set $F=\rho_{\mathcal{M}}(P)$. By the definition of an exact module category, the object $F(M)=P \otimes M$ is projective for all $M \in \mathcal{M}$. Since $\mathcal{C}$ and $\mathcal{M}$ are Frobenius, the object $F^{\mathrm{ra}}(M) \cong{ }^{*} P \otimes M$ is injective for all $M \in \mathcal{M}$. Thus, by Lemma 2.5, $F$ is a projective object of $\operatorname{Rex}(\mathcal{M})$. Hence $\rho_{\mathcal{M}}$ preserves projective objects. The converse is easily proved by Lemma 2.5 ,

Let $\mathcal{A}$ and $\mathcal{B}$ be finite abelian categories. A $k$-linear functor $F: \mathcal{A} \rightarrow \mathcal{B}$ is said to be dominant if every object of $\mathcal{B}$ is a subobject of an object of the form $F(X)$, $X \in \mathcal{A}$. Suppose that $F$ is exact and $\mathcal{B}$ is Frobenius. Then, as remarked in EG17, Lemma 2.3], the functor $F$ is dominant if and only if every object of $\mathcal{B}$ is a quotient of $F(X)$ for some $X \in \mathcal{A}$.

Lemma 3.3. An exact left $\mathcal{C}$-module category $\mathcal{M}$ is indecomposable if and only if the action functor $\rho_{\mathcal{M}}$ is dominant.

Proof. Suppose that there are non-zero $\mathcal{C}$-module full subcategories $\mathcal{M}_{1}$ and $\mathcal{M}_{2}$ of $\mathcal{M}$ such that $\mathcal{M}=\mathcal{M}_{1} \oplus \mathcal{M}_{2}$. Then we have the decomposition

$$
\operatorname{Rex}(\mathcal{M})=\mathcal{E}_{11} \oplus \mathcal{E}_{12} \oplus \mathcal{E}_{21} \oplus \mathcal{E}_{22}, \quad \mathcal{E}_{i j}=\operatorname{Rex}\left(\mathcal{M}_{i}, \mathcal{M}_{j}\right)
$$

into four non-zero full subcategories. Since the image of $\rho_{\mathcal{M}}$ is contained in the diagonal part $\mathcal{E}_{11} \oplus \mathcal{E}_{22}$, the action functor $\rho_{\mathcal{M}}$ cannot be dominant. Thus the 'if' part has been proved. The 'only if' part is [EG17, Proposition 2.6 (ii)].
3.2. Description of adjoints. For a while, we fix a finite tensor category $\mathcal{C}$ and a finite left $\mathcal{C}$-module category $\mathcal{M}$. We write $\rho=\rho_{\mathcal{M}}$ for simplicity. By Lemma 3.1, the functor $\rho$ has a right adjoint.

Theorem 3.4. For all $k$-linear right exact functor $F: \mathcal{M} \rightarrow \mathcal{M}$, the end of

$$
\mathcal{M}^{\mathrm{op}} \times \mathcal{M} \rightarrow \mathcal{C}, \quad\left(M, M^{\prime}\right) \mapsto \underline{\operatorname{Hom}}\left(M, F\left(M^{\prime}\right)\right)
$$

exists and a right adjoint of $\rho$ is given by

$$
\rho^{\mathrm{ra}}: \operatorname{Rex}(\mathcal{M}) \rightarrow \mathcal{C}, \quad F \mapsto \int_{M \in \mathcal{M}} \underline{\operatorname{Hom}}(M, F(M))
$$

We also have:
(a) If $\mathcal{M}$ is exact, then $\rho^{\mathrm{ra}}$ is exact.
(b) If $\mathcal{M}$ is exact and indecomposable, then $\rho^{\text {ra }}$ is faithful.

Proof. Let $\rho^{\text {ra }}$ be a right adjoint of $\rho$. Then we have

$$
\begin{aligned}
\operatorname{Hom}_{\mathcal{C}}\left(X, \rho^{\mathrm{ra}}(F)\right) & \cong \operatorname{Nat}(\rho(X), F) \\
& \cong \int_{M \in \mathcal{M}} \operatorname{Hom}_{\mathcal{M}}(X \otimes M, F(M)) \\
& \cong \int_{M \in \mathcal{M}} \operatorname{Hom}_{\mathcal{M}}(X, \underline{\operatorname{Hom}}(M, F(M)))
\end{aligned}
$$

for all $X \in \mathcal{C}$ and $F \in \operatorname{Rex}(\mathcal{M})$. Thus, by Lemma 2.1, we see that the end in question exists and $\rho^{\text {ra }}$ is given as stated.
(a) We suppose that $\mathcal{M}$ is exact. Let $P$ be a projective generator of $\mathcal{C}$. Then, by Lemma 3.2, the object $\rho(P) \in \operatorname{Rex}(\mathcal{M})$ is projective. Thus the functor

$$
\operatorname{Hom}_{\mathcal{C}}\left(P, \rho^{\mathrm{ra}}(-)\right) \cong \operatorname{Hom}_{\mathcal{C}}(\rho(P),-): \operatorname{Rex}(\mathcal{M}) \rightarrow k-\bmod
$$

is exact. Since $P$ is a projective generator, we conclude that $\rho^{\text {ra }}$ is exact.
(b) We suppose that $\mathcal{M}$ is exact and indecomposable. Since the functor $\rho^{\text {ra }}$ is exact by Part (a), it is enough to show that $\rho^{\text {ra }}$ reflects zero objects. Let $F$ be an object of $\operatorname{Rex}(\mathcal{M})$ such that $\rho^{\text {ra }}(F)=0$. By Lemma 3.3, there is an object $X \in \mathcal{C}$ such that $F$ is an epimorphic image of $\rho(X)$. If $F \neq 0$, then we have

$$
0=\operatorname{Hom}_{\mathcal{C}}\left(X, \rho^{\mathrm{ra}}(F)\right) \cong \operatorname{Nat}(\rho(X), F) \neq 0
$$

a contradiction. Thus $F=0$. The proof is done.
For $M, M^{\prime} \in \mathcal{M}$, we set $\underline{\operatorname{coHom}}\left(M, M^{\prime}\right)={ }^{*} \underline{\operatorname{Hom}}\left(M^{\prime}, M\right)$. It is easy to see that there is a natural isomorphism

$$
\operatorname{Hom}_{\mathcal{M}}\left(M, X \otimes M^{\prime}\right) \cong \operatorname{Hom}_{\mathcal{M}}\left(\underline{\operatorname{coHom}}\left(M, M^{\prime}\right), X\right)
$$

for $X \in \mathcal{C}$ and $M, M^{\prime} \in \mathcal{M}$. A left adjoint of $\rho$ is expressed as follows:
Theorem 3.5. For all $k$-linear right exact functor $F: \mathcal{M} \rightarrow \mathcal{M}$, the coend of

$$
\mathcal{M}^{\mathrm{op}} \times \mathcal{M} \rightarrow \mathcal{C}, \quad\left(M^{\mathrm{op}}, M^{\prime}\right) \mapsto \underline{\operatorname{coHom}}\left(M, F\left(M^{\prime}\right)\right)
$$

exists. Moreover, a left adjoint of $\rho$ is given by

$$
\rho^{\mathrm{la}}: \operatorname{Rex}(\mathcal{M}) \rightarrow \mathcal{C}, \quad F \mapsto \int^{M \in \mathcal{M}} \underline{\operatorname{coHom}}(M, F(M))
$$

We also have:
(a) If $\mathcal{M}$ is exact, then $\rho^{\text {la }}$ is exact.
(b) If $\mathcal{M}$ is exact and indecomposable, then $\rho^{\text {la }}$ is faithful.

Proof. Let $\rho^{\text {la }}$ be a left adjoint of $\rho$. Then we have

$$
\begin{aligned}
\operatorname{Hom}_{\mathcal{C}}\left(\rho^{\text {la }}(F), X\right) & \cong \operatorname{Nat}(F, \rho(X)) \\
& \cong \int_{M \in \mathcal{M}} \operatorname{Hom}_{\mathcal{M}}(F(M), X \otimes M) \\
& \left.\cong \int_{M \in \mathcal{M}} \operatorname{Hom}_{\mathcal{C}} \underline{(\operatorname{coHom}}(M, F(M)), X\right)
\end{aligned}
$$

for all $X \in \mathcal{C}$ and $F \in \operatorname{Rex}(\mathcal{M})$. Thus, by the dual of Lemma 2.1 we see that the coend in question exists and $\rho^{1 a}$ is given as stated.

Suppose that $\mathcal{M}$ is exact. Then, since $\mathcal{M}$ is Frobenius, the Nakayama functor of $\operatorname{Rex}(\mathcal{M}) \approx \mathcal{M}^{\mathrm{op}} \boxtimes \mathcal{M}$ is an equivalence. Parts (a) and (b) of this theorem follow from Theorem 3.4 and $\rho^{\text {la }} \cong \mathbb{N}_{\mathcal{C}}^{-1} \circ \rho^{\mathrm{ra}} \circ \mathbb{N}_{\operatorname{Rex}(\mathcal{M})}$.

Remark 3.6. In summary, for $F \in \operatorname{Rex}(\mathcal{M})$, we have

$$
\rho^{\mathrm{ra}}(F)=\int_{M \in \mathcal{M}} \underline{\operatorname{Hom}}(M, F(M)) \quad \text { and } \quad \rho^{\mathrm{la}}(F)=\int^{M \in \mathcal{M}} \underline{\operatorname{coHom}}(M, F(M)) .
$$

There is a left exact version of the action functor

$$
\lambda_{\mathcal{M}}: \mathcal{C} \rightarrow \operatorname{Lex}(\mathcal{M}):=\operatorname{Lex}(\mathcal{M}, \mathcal{M}), \quad \lambda_{\mathcal{M}}(X)(M)=X \otimes M
$$

By the same way as above, one can prove that $\lambda=\lambda_{\mathcal{M}}$ is a $k$-linear exact functor and its adjoints are given by

$$
\lambda^{\mathrm{ra}}(F)=\int_{M \in \mathcal{C}} \underline{\operatorname{Hom}}(M, F(M)) \quad \text { and } \quad \lambda^{\mathrm{la}}(F)=\int^{M \in \mathcal{C}} \underline{\operatorname{coHom}}(M, F(M))
$$

for $F \in \operatorname{Lex}(\mathcal{M})$. Moreover, there are natural isomorphisms

$$
\begin{equation*}
\lambda^{\mathrm{ra}}(F) \cong{ }^{*}\left(\rho^{\mathrm{la}}\left(F^{\mathrm{la}}\right)\right) \quad \text { and } \quad \lambda^{\mathrm{la}}(F) \cong{ }^{*}\left(\rho^{\mathrm{ra}}\left(F^{\mathrm{la}}\right)\right) \tag{3.2}
\end{equation*}
$$

for $F \in \operatorname{Lex}(\mathcal{M})$. Indeed, we have natural isomorphisms

$$
\begin{gathered}
\operatorname{Hom}_{\mathcal{C}}\left(X,{ }^{*}\left(\rho^{\mathrm{la}}\left(F^{\mathrm{la}}\right)\right)\right) \cong \operatorname{Hom}_{\mathcal{C}}\left(\rho^{\mathrm{la}}\left(F^{\mathrm{la}}\right), X^{*}\right) \cong \operatorname{Nat}\left(F^{\mathrm{la}}, \rho\left(X^{*}\right)\right) \\
\cong \operatorname{Nat}\left(F^{\mathrm{la}}, \rho(X)^{\mathrm{la}}\right) \cong \operatorname{Nat}(\rho(X), F)=\operatorname{Nat}(\lambda(X), F)
\end{gathered}
$$

for $F \in \operatorname{Lex}(\mathcal{M})$ and $X \in \mathcal{C}$. The second isomorphism of (3.2) is established in a similar way. Theorems 3.4 and 3.5 imply the following results:
(a) If $\mathcal{M}$ is exact, then $\lambda^{\text {la }}$ and $\lambda^{\text {ra }}$ are exact.
(b) If $\mathcal{M}$ is exact and indecomposable, then $\lambda^{\text {la }}$ and $\lambda^{\text {ra }}$ are faithful.
3.3. The unit and the counit of $\left(\rho, \rho^{\mathrm{ra}}\right)$. In what follows, we concentrate to study the structures of the right adjoint of $\rho=\rho_{\mathcal{M}}$. For this purpose, it is useful to describe the unit and the counit of the adjunction $\rho \dashv \rho^{\text {ra }}$. For $F \in \operatorname{Rex}(\mathcal{M})$ and $M \in \mathcal{M}$, we denote by

$$
\begin{equation*}
\pi_{F}(M): \rho^{\mathrm{ra}}(F) \rightarrow \underline{\operatorname{Hom}}(M, F(M)) \tag{3.3}
\end{equation*}
$$

the universal dinatural transformation and define

$$
\begin{equation*}
\varepsilon_{F, M}=\underline{\mathrm{ev}}_{M, F(M)} \circ\left(\pi_{F}(M) \otimes \operatorname{id}_{M}\right) \tag{3.4}
\end{equation*}
$$

By the proof of Theorem [3.4] the adjunction isomorphism

$$
\begin{equation*}
\operatorname{Hom}_{\mathcal{C}}\left(X, \rho^{\mathrm{ra}}(F)\right) \xrightarrow{\cong} \operatorname{Hom}_{\operatorname{Rex}(\mathcal{M})}(\rho(X), F)=\operatorname{Nat}(\rho(X), F) \tag{3.5}
\end{equation*}
$$

sends $a \in \operatorname{Hom}_{\mathcal{C}}\left(X, \rho^{\text {ra }}(F)\right)$ to the natural transformation $\widetilde{a}$ given by

$$
\begin{equation*}
\widetilde{a}_{M}=\varepsilon_{F, M} \circ\left(a \otimes \mathrm{id}_{M}\right) \quad(M \in \mathcal{M}) \tag{3.6}
\end{equation*}
$$

This implies that $\varepsilon=\left\{\varepsilon_{F, M}\right\}_{F, M}$ is the counit of (3.5). We also observe that the morphism $a$ is characterized by the property that the equation

$$
\begin{equation*}
\underline{\operatorname{Hom}}\left(M, \widetilde{a}_{M}\right) \circ \underline{\operatorname{coev}}_{X, M}=\pi_{F}(M) \circ a \tag{3.7}
\end{equation*}
$$

holds for all objects $M \in \mathcal{M}$. Let $\eta: \operatorname{id}_{\mathcal{C}} \rightarrow \rho^{\text {ra }} \circ \rho$ be the unit of the adjunction isomorphism (3.5). By substituting $a=\eta_{X}$ and $F=\rho(X)$ into (3.7), we see that $\eta$ is characterized by the property that the equation

$$
\begin{equation*}
\pi_{\rho(X)}(M) \circ \eta_{X}=\underline{\operatorname{coev}}_{X, M} \tag{3.8}
\end{equation*}
$$

holds for all objects $X \in \mathcal{C}$ and $M \in \mathcal{M}$. We also have

$$
\begin{equation*}
\pi_{F}(M)=\underline{\operatorname{Hom}}\left(M, \varepsilon_{F, M}\right) \circ{\underline{\operatorname{coev}^{\mathrm{ra}}}}_{\rho^{\mathrm{ra}}}(F), M \tag{3.9}
\end{equation*}
$$

by substituting $X=\rho^{\mathrm{ra}}(F)$ and $a=\mathrm{id}$ into (3.7).
3.4. Bimodule structure of $\rho^{\text {ra }}$. Since $\rho: \mathcal{C} \rightarrow \operatorname{Rex}(\mathcal{M})$ is a $\mathcal{C}$-bimodule functor, its right adjoint $\rho^{\text {ra }}$ is also a $\mathcal{C}$-bimodule functor such that the unit and the counit are $\mathcal{C}$-bimodule transformations. We denote by

$$
\xi_{X, F}^{(\ell)}: X \otimes \rho^{\mathrm{ra}}(F) \rightarrow \rho^{\mathrm{ra}}(X \otimes F) \quad \text { and } \quad \xi_{F, X}^{(r)}: \rho^{\mathrm{ra}}(F) \otimes X \rightarrow \rho^{\mathrm{ra}}(F \otimes X)
$$

the left and the right $\mathcal{C}$-module structure of $\rho^{\text {ra }}$. These morphisms are expressed in terms of the universal dinatural transformation $\pi$ as follows:
Lemma 3.7. For all objects $F \in \operatorname{Rex}(\mathcal{M}), X \in \mathcal{C}$ and $M \in \mathcal{M}$, we have

$$
\begin{align*}
& \pi_{X \otimes F}(M) \circ \xi_{X, F}^{(\ell)}=\mathfrak{a}_{X, M, F(M)} \circ\left(\operatorname{id}_{X} \otimes \pi_{F}(M)\right)  \tag{3.10}\\
& \pi_{F \otimes X}(M) \circ \xi_{X, F}^{(r)}=\mathfrak{b}_{X, M, F(X \otimes M)}^{\natural} \circ\left(\pi_{F}(X \otimes M) \otimes \operatorname{id}_{X}\right) \tag{3.11}
\end{align*}
$$

See Subsection 2.4 for definitions of $\mathfrak{a}$ and $\mathfrak{b}^{\natural}$. By the universal property of $\rho^{\text {ra }}(F)$ as an end, the isomorphisms $\xi_{X, F}^{(\ell)}$ and $\xi_{F, X}^{(r)}$ are characterized by equations (3.10) and (3.11), respecively. We postpone the proof of this lemma to Appendix $A$ since it is straightforward but lengthy.
3.5. Monoidal structure of $\rho^{\text {ra }}$. Since the action functor $\rho: \mathcal{C} \rightarrow \operatorname{Rex}(\mathcal{M})$ is a strong monoidal functor, its right adjoint $\rho^{\text {ra }}$ has a canonical structure of a (lax) monoidal functor. We denote the structure morphisms of $\rho^{\text {ra }}$ by

$$
\mu_{F, G}^{(2)}: \rho^{\mathrm{ra}}(F) \otimes \rho^{\mathrm{ra}}(G) \rightarrow \rho^{\mathrm{ra}}(F \circ G) \quad \text { and } \quad \mu^{(0)}: \mathbb{1} \rightarrow \rho^{\mathrm{ra}}\left(\mathrm{id}_{\mathcal{M}}\right)
$$

for $F, G \in \operatorname{Rex}(\mathcal{M})$. They are expressed in terms of the universal dinatural transformation $\pi$ as follows:
Lemma 3.8. For all objects $F, G \in \operatorname{Rex}(\mathcal{M})$ and $M \in \mathcal{M}$, we have

$$
\begin{gather*}
\pi_{F G}(M) \circ \mu_{F, G}^{(2)}={\underline{\operatorname{comp}_{M, G(M), F G(M)}}} \circ\left(\pi_{F}(G(M)) \otimes \pi_{G}(M)\right),  \tag{3.12}\\
\pi_{\mathrm{id}_{\mathcal{M}}}(M) \circ \mu^{(0)}=\underline{\operatorname{coev}}_{\mathbb{1}, M} \tag{3.13}
\end{gather*}
$$

By the universal property, $\mu^{(2)}$ and $\mu^{(0)}$ are characterized by equations (3.12) and (3.13), respectively. The proof is postponed to Appendix A.
3.6. Lifting the adjunction to the Drinfeld center. Given two finite left $\mathcal{C}$ module categories $\mathcal{M}$ and $\mathcal{N}$, we denote by $\operatorname{Rex}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ the category of $k$-linear right exact $\mathcal{C}$-module functors from $\mathcal{M}$ to $\mathcal{N}$. The aim of this subsection is to show that the adjoint pair ( $\rho, \rho^{\text {ra }}$ ) can be 'lifted' to an adjoint pair between the Drinfeld center of $\mathcal{C}$ and $\operatorname{Rex}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$.

We first introduce the following generalization of the Drinfeld center construction: For a $\mathcal{C}$-bimodule category $\mathcal{M}$, we define the category $\mathcal{Z}(\mathcal{M})$ as follows: An object of this category is a pair $(M, \sigma)$ consisting of an object $M \in \mathcal{M}$ and a natural isomorphism $\sigma_{X}: M \otimes X \rightarrow X \otimes M(X \in \mathcal{C})$ satisfying the equations

$$
\sigma_{\mathbb{1}}=\operatorname{id}_{M} \quad \text { and } \quad \sigma_{X \otimes Y}=\left(\operatorname{id}_{X} \otimes \sigma_{Y}\right) \circ\left(\sigma_{X} \otimes \operatorname{id}_{Y}\right)
$$

for all objects $X, Y \in \mathcal{C}$. If $\mathbf{M}=\left(M, \sigma_{M}\right)$ and $\mathbf{N}=\left(N, \sigma_{N}\right)$ are objects of $\mathcal{Z}(\mathcal{M})$, then a morphism $f: \mathbf{M} \rightarrow \mathbf{N}$ is a morphism $f: M \rightarrow N$ satisfying

$$
\left(\mathrm{id}_{X} \otimes f\right) \circ \sigma_{M ; X}=\sigma_{N ; X} \circ\left(f \otimes \operatorname{id}_{X}\right)
$$

for all objects $X \in \mathcal{C}$. The composition of morphisms in $\mathcal{Z}(\mathcal{M})$ is defined by the composition of morphisms in $\mathcal{M}$.

Example 3.9. The category $\mathcal{C}$ is a finite $\mathcal{C}$-bimodule category by the tensor product of $\mathcal{C}$. The category $\mathcal{Z}(\mathcal{C})$ is the Drinfeld center of $\mathcal{C}$. If this is the case, then $\mathcal{Z}(\mathcal{C})$ is not only a category but a braided finite tensor category [E04].
Example 3.10. If $\mathcal{M}$ and $\mathcal{N}$ are finite left $\mathcal{C}$-module categories, then $\mathcal{F}:=\operatorname{Rex}(\mathcal{M}, \mathcal{N})$ is a finite $\mathcal{C}$-bimodule category by the actions given by (3.1). The category $\mathcal{Z}(\mathcal{F})$ can be identified with $\operatorname{Rex}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$.

Now let $\mathcal{C}$-bimod be the 2 -category of finite $\mathcal{C}$-bimodule categories, $k$-linear right exact $\mathcal{C}$-bimodule functors and $\mathcal{C}$-bimodule natural transformations. Given a 1 -cell $F: \mathcal{M} \rightarrow \mathcal{N}$ in $\mathcal{C}$-bimod with structure morphisms

$$
\ell_{X, M}: X \otimes F(M) \rightarrow F(X \otimes M) \quad \text { and } \quad r_{M, X}: F(M) \otimes X \rightarrow F(M \otimes X),
$$

we define the $k$-linear functor $\mathcal{Z}(F): \mathcal{Z}(\mathcal{M}) \rightarrow \mathcal{Z}(\mathcal{N})$ by

$$
\mathcal{Z}(F)(\mathbf{M})=\left(F(M), \ell^{-1} \circ F(\sigma) \circ r\right)
$$

for $\mathbf{M}=(M, \sigma)$ in $\mathcal{Z}(\mathcal{M})$. It is routine to check that these constructions extends to a 2 -functor $\mathcal{Z}: \mathcal{C}$-bimod $\rightarrow k$ - Cat, where $k$ - Cat is the 2 -category of essentially small $k$-linear categories, $k$-linear functors and natural transformations.

We apply the 2 -functor $\mathcal{Z}$ to the action functor and its adjoint. Let $\mathcal{M}$ be a finite left $\mathcal{C}$-module category. Since $\rho=\rho_{\mathcal{M}}$ is a $\mathcal{C}$-bimodule functor, its right adjoint $\rho^{\text {ra }}$ is a $\mathcal{C}$-bimodule functor in such a way that the unit and the counit of the adjunction are bimodule natural transformations. Namely, there is an adjoint pair ( $\rho, \rho^{\mathrm{ra}}$ ) in the 2 -category $\mathcal{C}$-bimod. By applying the 2 -functor $\mathcal{Z}$, we obtain:

Theorem 3.11. There is an adjoint pair

$$
\begin{equation*}
\left(\mathcal{Z}(\rho): \mathcal{Z}(\mathcal{C}) \rightarrow \operatorname{Rex}_{\mathcal{C}}(\mathcal{M}), \mathcal{Z}\left(\rho^{\mathrm{ra}}\right): \operatorname{Rex}_{\mathcal{C}}(\mathcal{M}) \rightarrow \mathcal{Z}(\mathcal{C})\right) \tag{3.14}
\end{equation*}
$$

where we have identified $\mathcal{Z}(\operatorname{Rex}(\mathcal{M}))$ with $\operatorname{Rex}_{\mathcal{C}}(\mathcal{M})$.
It is instructive to describe the functors $\mathcal{Z}(\rho)$ and $\mathcal{Z}\left(\rho^{\text {ra }}\right)$ explicitly. Given an object $\mathbf{X}=(X, \sigma) \in \mathcal{Z}(\mathcal{C})$, we have $\mathcal{Z}(\rho)(\mathbf{X})=\rho(X)$. The left $\mathcal{C}$-module structure of $\mathcal{X}:=\mathcal{Z}(\rho)(\mathbf{X})$ is given by

$$
\left(\sigma_{W}\right)^{-1} \otimes \operatorname{id}_{M}: W \otimes \mathcal{X}(M)=W \otimes X \otimes M \rightarrow X \otimes W \otimes M=\mathcal{X}(W \otimes M)
$$

for $W \in \mathcal{C}$ and $M \in \mathcal{M}$. For an object $\mathbf{F}=(F, s) \in \operatorname{Rex}_{\mathcal{C}}(\mathcal{M})$, we have

$$
\mathcal{Z}\left(\rho^{\mathrm{ra}}\right)(\mathbf{F})=\left(\rho^{\mathrm{ra}}(F), \sigma^{\mathbf{F}}\right), \quad \text { where } \quad \sigma_{X}^{\mathbf{F}}=\left(\xi_{X, F}^{(\ell)}\right)^{-1} \circ \rho^{\mathrm{ra}}\left(s^{-1}\right) \circ \xi_{F, X}^{(r)}
$$

for $X \in \mathcal{C}$. More explicitly:
Lemma 3.12. The half-braiding $\sigma^{\mathbf{F}}$ is a unique natural transformation such that the following diagram commutes for all objects $X \in \mathcal{C}$ and $M \in \mathcal{M}$.


Proof. The commutativity of this diagram follows from Lemma 3.7. By the Fubini theorem for ends, we see that $X \otimes \rho^{\mathrm{ra}}(F)$ is an end of the functor

$$
\mathcal{M}^{\mathrm{op}} \times \mathcal{M} \rightarrow \mathcal{C}, \quad\left(M^{\mathrm{op}}, M^{\prime}\right) \mapsto X \otimes \underline{\operatorname{Hom}}\left(M, F\left(M^{\prime}\right)\right) .
$$

The universal property proves the 'uniqueness' part of this lemma.
3.7. Induction to the Drinfeld center. The $k$-linear monoidal category

$$
\mathcal{C}_{\mathcal{M}}^{*}:=\operatorname{Rex}_{\mathcal{C}}(\mathcal{M})^{\mathrm{rev}}
$$

is called the dual of $\mathcal{C}$ with respect to $\mathcal{M}$. By Schauenburg's result Sch01] (which we recall later), there is an equivalence $\mathcal{Z}(\mathcal{C}) \approx \mathcal{Z}\left(\mathcal{C}_{\mathcal{M}}^{*}\right)$ of $k$-linear braided monoidal categories. In this subsection, we show that $\mathcal{Z}\left(\rho_{\mathcal{M}}^{\mathrm{ra}}\right)$ is right adjoint to the composition

$$
\mathcal{Z}(\mathcal{C}) \xrightarrow{\text { Schauenburg's equivalence }} \mathcal{Z}\left(\mathcal{C}_{\mathcal{M}}^{*}\right) \xrightarrow{\text { the forgetful functor }} \mathcal{C}_{\mathcal{M}}^{*}
$$

We first recall Schauenburg's result Sch01 on the Drinfeld center of the category of bimodules. Let $A$ be an algebra in $\mathcal{C}$ with multiplication $m: A \otimes A \rightarrow A$ and unit $u: \mathbb{1} \rightarrow A$. Then the category ${ }_{A} \mathcal{C}_{A}$ of $A$-bimodules in $\mathcal{C}$ is a $k$-linear abelian monoidal category with respect to the tensor product over $A$. There is a $k$-linear braided strong monoidal functor $\theta_{A}: \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{Z}\left({ }_{A} \mathcal{C}_{A}\right)$ defined as follows: For an object $\mathbf{V}=(V, \sigma) \in \mathcal{Z}(\mathcal{C})$, we set $\theta_{A}(\mathbf{V})=(A \otimes V, \widetilde{\sigma})$, where the left action of $A$ on $A \otimes V$ is given by $m \otimes \mathrm{id}_{V}$, the right action is given by the composition

$$
(A \otimes V) \otimes A \xrightarrow{\operatorname{id}_{A} \otimes \sigma_{A}} A \otimes A \otimes V \xrightarrow{m \otimes \operatorname{id}_{V}} A \otimes V,
$$

and the half-braiding $\widetilde{\sigma}$ is determined by the commutative diagram

for an $A$-bimodule $M$ in $\mathcal{C}$ with left action $\triangleright_{M}: A \otimes M \rightarrow M$. For a morphism $f$ in $\mathcal{Z}(\mathcal{C})$, we set $\theta_{A}(f)=\operatorname{id}_{A} \otimes f$. The monoidal structure of $\theta_{A}$ is given by the canonical isomorphism

$$
\theta_{A}(\mathbf{V}) \otimes_{A} \theta_{A}(\mathbf{W})=(A \otimes V) \otimes_{A}(A \otimes W) \cong A \otimes(V \otimes W)=\theta_{A}(\mathbf{V} \otimes \mathbf{W})
$$

for $\mathbf{V}=(V, \sigma), \mathbf{W}=(W, \tau) \in \mathcal{Z}(\mathcal{C})$. Schauenburg Sch01] showed that the functor $\theta_{A}$ is in fact an equivalence of $k$-linear braided monoidal categories.

Now let $\mathcal{M}$ be a finite left $\mathcal{C}$-module category. Then there is an algebra $A$ in $\mathcal{C}$ such that $\mathcal{M} \approx \mathcal{C}_{A}$ as a left $\mathcal{C}$-module categories. Moreover, the functor

$$
\begin{equation*}
{ }_{A} \mathcal{C}_{A} \rightarrow \operatorname{Rex}_{\mathcal{C}}\left(\mathcal{C}_{A}, \mathcal{C}_{A}\right)^{\mathrm{rev}}, \quad M \mapsto(-) \otimes_{A} M \tag{3.15}
\end{equation*}
$$

is an equivalence of $k$-linear monoidal categories. Thus $\mathcal{Z}\left(\mathcal{C}_{\mathcal{M}}^{*}\right)$ and $\mathcal{Z}(\mathcal{C})$ are equivalent as $k$-linear braided monoidal categories.

Since we are interested in the general theory of finite tensor categories and their module categories, it is preferable to describe the equivalence $\mathcal{Z}(\mathcal{C}) \approx \mathcal{Z}\left(\mathcal{C}_{\mathcal{M}}^{*}\right)$ without referencing the algebra $A$ such that $\mathcal{M} \approx \mathcal{C}_{A}$. Thus, for a finite left $\mathcal{C}$-module category $\mathcal{M}$, we define the functor $\theta_{\mathcal{M}}: \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{Z}\left(\mathcal{C}_{\mathcal{M}}^{*}\right)$ as follows: For an object $\mathbf{V}=(V, \sigma) \in \mathcal{Z}(\mathcal{C})$, we set $\theta_{\mathcal{M}}(\mathbf{V})=\rho(V)$ as an object of $\operatorname{Rex}(\mathcal{M})$. We make $\rho(V)$ into a left $\mathcal{C}$-module functor by the structure morphism given by

$$
\left(\sigma_{X}\right)^{-1} \otimes \operatorname{id}_{M}: X \otimes \rho(V)(M)=X \otimes V \otimes M \rightarrow V \otimes X \otimes M=\rho(V)(X \otimes M)
$$

for $X \in \mathcal{C}$ and $M \in \mathcal{M}$. The half-braiding of $\theta_{\mathcal{M}}(\mathbf{V})$ is given by

$$
s_{V, M}:\left(\theta_{\mathcal{M}}(\mathbf{V}) \circ \mathbf{F}\right)(M)=V \otimes F(M) \rightarrow F(V \otimes M)=\left(\mathbf{F} \circ \theta_{\mathcal{M}}(\mathbf{V})\right)(M)
$$

for $\mathbf{F}=(F, s) \in \mathcal{C}_{\mathcal{M}}^{*}$ and $M \in \mathcal{M}$. The following theorem is obtained by rephrasing Schauenburg's result.

Theorem 3.13. The functor $\theta_{\mathcal{M}}: \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{Z}\left(\mathcal{C}_{\mathcal{M}}^{*}\right)$ is an equivalence of $k$-linear braided monoidal categories.

Proof. If finite left $\mathcal{C}$-module categories $\mathcal{M}$ and $\mathcal{N}$ are equivalent, then there is an equivalence $F: \mathcal{C}_{\mathcal{M}}^{*} \rightarrow \mathcal{C}_{\mathcal{N}}^{*}$ of $k$-linear monoidal categories. It is easy to check

$$
\widetilde{F} \circ \theta_{\mathcal{M}}=\theta_{\mathcal{N}}
$$

where $\widetilde{F}: \mathcal{Z}\left(\mathcal{C}_{\mathcal{M}}^{*}\right) \rightarrow \mathcal{Z}\left(\mathcal{C}_{\mathcal{N}}^{*}\right)$ is the braided monoidal equivalence induced by the monoidal equivalence $F$. Thus, to show that $\theta_{\mathcal{M}}$ is an equivalence, we may assume that $\mathcal{M}=\mathcal{C}_{A}$ for some algebra $A$ in $\mathcal{C}$. We consider the equivalence

$$
\theta_{\mathcal{M}}^{\prime}:=\left(\mathcal{Z}(\mathcal{C}) \xrightarrow{\theta_{A}} \mathcal{Z}\left({ }_{A} \mathcal{C}_{A}\right) \xrightarrow{\text { by (3.15) }} \mathcal{Z}\left(\mathcal{C}_{\mathcal{M}}^{*}\right)\right)
$$

of $k$-linear braided monoidal categories. One can check that $\theta_{\mathcal{M}} \cong \theta_{\mathcal{M}}^{\prime}$ as monoidal functors via the isomorphism given by

$$
\theta_{\mathcal{M}}(\mathbf{V})(M)=V \otimes M \xrightarrow{\sigma_{M}} M \otimes V \xrightarrow{\cong} M \otimes_{A}(A \otimes V)=\theta_{\mathcal{M}}^{\prime}(\mathbf{V})(M)
$$

for $\mathbf{V}=(V, \sigma) \in \mathcal{Z}(\mathcal{C})$ and $M \in \mathcal{M}$. Thus $\theta_{\mathcal{M}}$ is also an equivalence of $k$-linear braided monoidal categories.

Now we prove the result mentioned at the beginning of this subsection:
Theorem 3.14. Let $\mathrm{U}: \mathcal{Z}\left(\mathcal{C}_{\mathcal{M}}^{*}\right) \rightarrow \mathcal{C}_{\mathcal{M}}^{*}$ be the forgetful functor. Then

$$
\left(U \circ \theta_{\mathcal{M}}: \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}_{\mathcal{M}}^{*}, \quad \mathcal{Z}\left(\rho_{\mathcal{M}}^{\mathrm{ra}}\right): \mathcal{C}_{\mathcal{M}}^{*} \rightarrow \mathcal{Z}(\mathcal{C})\right)
$$

is an adjoint pair.
Proof. By Theorem 3.13 the functor $\mathrm{U} \circ \theta_{\mathcal{M}}$ is identical to $\mathcal{Z}\left(\rho_{\mathcal{M}}\right)$ and therefore it is left adjoint to $\mathcal{Z}\left(\rho_{\mathcal{M}}^{\mathrm{ra}}\right)$.

Corollary 3.15. Let $\mathrm{U}_{\mathcal{C}}: \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$ and $\mathrm{U}_{\mathcal{D}}: \mathcal{Z}(\mathcal{D}) \rightarrow \mathcal{D}$ be the forgetful functors, where $\mathcal{D}=\mathcal{C}_{\mathcal{M}}^{*}$. Then $\mathrm{U}_{\mathcal{D}}$ has a right adjoint. The composition

$$
\mathcal{D} \xrightarrow{\mathrm{U}_{\mathcal{D}}^{\mathrm{ra}}} \mathcal{Z}(\mathcal{D}) \xrightarrow{\theta_{\mathcal{M}}^{-1}} \mathcal{Z}(\mathcal{C}) \xrightarrow{\mathrm{U}_{\mathcal{C}}} \mathcal{C}
$$

sends an object $\mathbf{F}=(F, s) \in \mathcal{D}$ to the end $\int_{M \in \mathcal{M}} \underline{\operatorname{Hom}}(M, F(M))$.
Proof. Theorem 3.14 implies that $\theta_{\mathcal{M}} \circ \mathcal{Z}\left(\rho_{\mathcal{M}}^{\mathrm{ra}}\right)$ is right adjoint to $\mathrm{U}_{\mathcal{D}}$. Thus $\mathrm{U}_{\mathcal{D}}^{\text {ra }}$ exists and is isomorphic to $\theta_{\mathcal{M}} \circ \mathcal{Z}\left(\rho_{\mathcal{M}}^{\mathrm{ra}}\right)$. Hence the composition in question is isomorphic to $U_{\mathcal{C}} \circ \mathcal{Z}\left(\rho_{\mathcal{M}}^{\mathrm{ra}}\right)$. Now the result follows from the explicit description of $\mathcal{Z}\left(\rho_{\mathcal{M}}^{\mathrm{ra}}\right)$ given in the previous subsection.

## 4. Integral over a topologizing full Subcategory

4.1. Topologizing full subcategory. We first introduce the following terminology and notation: A full subcategory of an abelian category is said to be topologizing Ros95] if it is closed under finite direct sums and subquotients. We denote by $\mathfrak{T o p}(\mathcal{A})$ the class of topologizing full subcategories of an abelian category $\mathcal{A}$.

Let $\mathcal{M}$ be a finite module category over a finite tensor category. In Section 3, we have considered several 'integrals' over the category $\mathcal{M}$. In this section, based on our results on adjoints of the action functor, we extend techniques used in Shil7b and provide a framework to deal with 'integrals' of the form $\int_{X \in \mathcal{S}} \underline{\operatorname{Hom}}(X, X)$ for some $\mathcal{S} \in \mathfrak{T o p}(\mathcal{M})$.

We first summarize basic results on topologizing full subcategories of a finite abelian category. Let $\mathcal{M}$ be a finite abelian category, and let $\mathcal{S}$ be a topologizing full subcategory of $\mathcal{M}$ with inclusion functor $i: \mathcal{S} \rightarrow \mathcal{M}$. For $M \in \mathcal{M}$, we set

$$
\begin{equation*}
i^{\sharp}(M)=(\text { the largest subobject of } M \text { belonging to } \mathcal{S}) . \tag{4.1}
\end{equation*}
$$

By the assumption that $\mathcal{S}$ is a topologizing full subcategory, one can extend the assignment $M \mapsto i^{\sharp}(M)$ to a $k$-linear functor from $\mathcal{M}$ to $\mathcal{S}$. Dually, we set

$$
\begin{equation*}
\mathrm{\kappa}_{\mathcal{S}}(M)=\bigcap\{X \subset M \mid M / X \in \mathcal{S}\} \quad \text { and } \quad i^{b}(M)=M / \mathrm{k}_{\mathcal{S}}(M) \tag{4.2}
\end{equation*}
$$

for $M \in \mathcal{M}$. One can also extend the assignment $M \mapsto i^{b}(M)$ to a $k$-linear functor from $\mathcal{M}$ to $\mathcal{S}$. It is easy to see that $i^{\sharp}$ and $i^{b}$ are a right and a left adjoint of $i$, respectively. We now define

$$
\begin{equation*}
\tau_{\mathcal{S}}:=i \circ i^{b} \quad \text { and } \quad \tau_{\mathcal{S}}^{\prime}:=i \circ i^{\sharp} . \tag{4.3}
\end{equation*}
$$

Since $i^{b} \dashv i \dashv i^{\sharp}$, we have natural isomorphisms

$$
\begin{equation*}
\operatorname{Hom}_{\mathcal{M}}\left(\tau_{\mathcal{S}}(M), N\right) \cong \operatorname{Hom}_{\mathcal{S}}\left(i^{b}(M), i^{\sharp}(N)\right) \cong \operatorname{Hom}_{\mathcal{M}}\left(M, \tau_{\mathcal{S}}^{\prime}(N)\right) \tag{4.4}
\end{equation*}
$$

for $M, N \in \mathcal{M}$. Thus $\tau_{\mathcal{S}} \in \operatorname{Rex}(\mathcal{M})$ and $\tau_{\mathcal{S}}^{\prime}=\tau_{\mathcal{S}}^{\mathrm{ra}}$. Moreover, since $i^{\mathrm{b}} \circ i=\operatorname{id}$, the endofunctor $\tau_{\mathcal{S}}$ is an idempotent monad on $\mathcal{M}$ whose category of modules coincides with $\mathcal{S}$. By this observation, we have the following consequence:

Lemma 4.1. A topologizing full subcategory of a finite abelian category is a finite abelian category such that the inclusion functor preserves and reflects exact sequences.

Now we choose a finite-dimensional algebra $A$ such that $\mathcal{M} \approx A$-mod. If we identify $\operatorname{Rex}(\mathcal{M})$ with $A$ - $\bmod -A$, then $\operatorname{id}_{\mathcal{M}} \in \operatorname{Rex}(\mathcal{M})$ corresponds to the $A$-bimodule $A$. Thus a subobject of $\operatorname{id}_{\mathcal{M}}$ in $\operatorname{Rex}(\mathcal{M})$ corresponds to an ideal of $A$. By abuse of terminology, we call a subobject of $\operatorname{id}_{\mathcal{M}}$ in $\operatorname{Rex}(\mathcal{M})$ an ideal of $\mathcal{M}$. Then we have the following correspondence ( $c f$. Rosenberg Ros95, Chapter III]):

Lemma 4.2. For a finite abelian category $\mathcal{M}$, there is a one-to-one correspondence between the class $\mathfrak{T o p}(\mathcal{M})$ and the set of ideals of $\mathcal{M}$.

For $\mathcal{S} \in \mathfrak{T}_{\mathfrak{o p}}(\mathcal{M})$, we define $\kappa_{\mathcal{S}}$ by (4.2). The correspondence of the above lemma assigns $\kappa_{\mathcal{S}} \subset \operatorname{id}_{\mathcal{S}}[$ to $\mathcal{S}$. Conversely, given an ideal $I$ of $\mathcal{M}$, we consider the quotient $\tau:=\operatorname{id}_{\mathcal{M}} / I$. If we identify $\mathcal{M}$ with $A$-mod as above, then $I$ can be regarded as an ideal of the algebra $A$ and the functor $\tau$ is identified with $(A / I) \otimes_{A}(-)$. Thus $\tau$ is a $k$-linear right exact idempotent monad on $\mathcal{M}$. The correspondence of Lemma 4.2 assigns the category of $\tau$-modules to the ideal $I$.
4.2. Integral over a full subcategory. Let $\mathcal{C}$ be a finite tensor category, and let $\mathcal{M}$ be a finite left $\mathcal{C}$-module category. Given $\mathcal{S} \in \mathfrak{T o p}(\mathcal{M})$, we consider the end

$$
A_{\mathcal{S}}^{\prime}:=\rho_{\mathcal{M}}^{\mathrm{ra}}\left(\tau_{\mathcal{S}}\right)=\int_{M \in \mathcal{M}} \underline{\operatorname{Hom}}\left(M, \tau_{\mathcal{S}}(M)\right)
$$

where $\tau_{\mathcal{S}}$ is defined by (4.3). Let $i: \mathcal{S} \rightarrow \mathcal{M}$ be the inclusion functor. By applying Lemma 2.2 to the adjunction $i^{b} \dashv i$, we see that the end of the functor

$$
\begin{equation*}
\mathcal{S}^{\mathrm{op}} \times \mathcal{S} \rightarrow \mathcal{C}, \quad\left(X, X^{\prime}\right) \mapsto \underline{\operatorname{Hom}}\left(i(X), i\left(X^{\prime}\right)\right) \tag{4.5}
\end{equation*}
$$

exists and is canonically isomorphic to $A_{\mathcal{S}}^{\prime}$. We denote the end of (4.5) by

$$
A_{\mathcal{S}}=\int_{X \in \mathcal{S}} \underline{\operatorname{Hom}}(X, X)
$$

with omitting the inclusion functor. Let $\beta_{\mathcal{S}}: A_{\mathcal{S}}^{\prime} \rightarrow A_{\mathcal{S}}$ be the canonical isomorphism given by Lemma 2.2. If we denote by

$$
\pi_{\mathcal{S}}(X): A_{\mathcal{S}} \rightarrow \underline{\operatorname{Hom}}(X, X) \quad \text { and } \quad \pi_{\mathcal{S}}^{\prime}(M): A_{\mathcal{S}}^{\prime} \rightarrow \underline{\operatorname{Hom}}\left(M, \tau_{\mathcal{S}}(M)\right)
$$

the respective universal dinatural transformations, then the isomorphism $\beta_{\mathcal{S}}$ is characterized as a unique morphism in $\mathcal{C}$ such that the equation

$$
\begin{equation*}
\pi_{\mathcal{S}}(X) \circ \beta_{\mathcal{S}}=\pi_{\mathcal{S}}^{\prime}(X) \tag{4.6}
\end{equation*}
$$

holds for all $X \in \mathcal{S}$.
We recall that $\tau_{\mathcal{S}}$ is an idempotent monad on $\mathcal{M}$. Thus $A_{\mathcal{S}}^{\prime}$ is an algebra in $\mathcal{C}$ as the image of an algebra under the monoidal functor $\rho_{\mathcal{M}}^{\mathrm{ra}}$. On the other hand, by the universal property of the end $A_{\mathcal{S}}$, we can define

$$
\begin{equation*}
m_{\mathcal{S}}: A_{\mathcal{S}} \otimes A_{\mathcal{S}} \rightarrow A_{\mathcal{S}} \quad \text { and } \quad u_{\mathcal{S}}: \mathbb{1} \rightarrow A_{\mathcal{S}} \tag{4.7}
\end{equation*}
$$

to be unique morphisms such that the equations

$$
\pi_{\mathcal{S}}(X) \circ m_{\mathcal{S}}=\underline{\operatorname{comp}}_{X, X, X}^{\mathcal{M}} \circ\left(\pi_{\mathcal{S}}(X) \otimes \pi_{\mathcal{S}}(X)\right) \quad \text { and } \quad \pi_{\mathcal{S}}(X) \circ u_{\mathcal{S}}=\underline{\operatorname{coev}}_{\mathbb{1}, X}
$$

hold for all objects $X \in \mathcal{S}$. It is easy to see that $A_{\mathcal{S}}$ is an algebra in $\mathcal{C}$ with multiplication $m_{\mathcal{S}}$ and unit $u_{\mathcal{S}}$.

Lemma 4.3. The morphism $\beta_{\mathcal{S}}$ is an isomorphism of algebras in $\mathcal{C}$.
Proof. Noting $\tau_{\mathcal{S}}(X)=X$ for all $X \in \mathcal{S}$, we easily verify that the equations

$$
\begin{gathered}
\pi_{\mathcal{S}}(X) \circ \beta_{\mathcal{S}} \circ \mu_{\tau_{\mathcal{S}}, \tau_{\mathcal{S}}}^{(2)}=\underline{\operatorname{comp}}_{X, X, X}=\pi_{\mathcal{S}}(X) \circ m_{\mathcal{S}} \circ\left(\beta_{\mathcal{S}} \otimes \beta_{\mathcal{S}}\right), \\
\pi_{\mathcal{S}}(X) \circ \beta_{\mathcal{S}} \circ \mu^{(0)}=\underline{\operatorname{coev}}_{\mathbb{1}, X}=\pi_{\mathcal{S}}(X) \circ u_{\mathcal{S}}
\end{gathered}
$$

hold for all objects $X \in \mathcal{S}$. By the universal property of the end $A_{\mathcal{S}}$, we conclude that $\beta_{\mathcal{S}}$ is a morphism of algebras.

For $\mathcal{S} \in \mathfrak{T o p}(\mathcal{M})$, we denote by $q_{\mathcal{S}}: \operatorname{id}_{\mathcal{M}} \rightarrow \tau_{\mathcal{S}}$ the quotient morphism. We recall that the kernel of $q_{\mathcal{S}}$ is $\kappa_{\mathcal{S}}$. For $\mathcal{S}_{1}, \mathcal{S}_{2} \in \mathfrak{T} \mathfrak{o p}(\mathcal{M})$ with $\mathcal{S}_{1} \supset \mathcal{S}_{2}$, we have $\kappa_{\mathcal{S}_{1}} \subset \kappa_{\mathcal{S}_{2}}$ as subobjects of $\mathrm{id}_{\mathcal{M}}$. Thus there is a unique morphism $q_{\mathcal{S}_{1} \mid \mathcal{S}_{2}}: \tau_{\mathcal{S}_{1}} \rightarrow \tau_{\mathcal{S}_{2}}$ such that $q_{\mathcal{S}_{1} \mid \mathcal{S}_{2}} \circ q_{\mathcal{S}_{1}}=q_{\mathcal{S}_{2}}$.

For $\mathcal{S}_{1}, \mathcal{S}_{2}$ with $\mathcal{S}_{1} \supset \mathcal{S}_{2}$, we also define a morphism $\phi_{\mathcal{S}_{1} \mid \mathcal{S}_{2}}: A_{\mathcal{S}_{1}} \rightarrow A_{\mathcal{S}_{2}}$ to be a unique morphism such that the equation

$$
\begin{equation*}
\pi_{\mathcal{S}_{2}}(X) \circ \phi_{\mathcal{S}_{1} \mid \mathcal{S}_{2}}=\pi_{\mathcal{S}_{1}}(X) \tag{4.8}
\end{equation*}
$$

holds for all objects $X \in \mathcal{S}_{2}$.

Lemma 4.4. With the above notation, we have

$$
\phi_{\mathcal{S}_{1} \mid \mathcal{S}_{2}} \circ \beta_{\mathcal{S}_{1}}=\beta_{\mathcal{S}_{2}} \circ \rho_{\mathcal{M}}^{\mathrm{ra}}\left(q_{\mathcal{S}_{1} \mid \mathcal{S}_{2}}\right) .
$$

Proof. For all objects $X \in \mathcal{S}_{2}$, we have

$$
\pi_{\mathcal{S}_{2}}(X) \circ \phi_{\mathcal{S}_{1} \mid \mathcal{S}_{2}} \circ \beta_{\mathcal{S}_{1}}=\pi_{\mathcal{S}_{1}}(X) \circ \beta_{\mathcal{S}_{1}}=\pi_{\mathcal{S}_{1}}^{\prime}(X)
$$

by (4.6) and (4.8). Noting $\tau_{\mathcal{S}_{1}}(X)=X$ and $\left(q_{\mathcal{S}_{1} \mid \mathcal{S}_{2}}\right)_{X}=\mathrm{id}_{X}$, we also have

$$
\begin{aligned}
\pi_{\mathcal{S}_{2}}(X) \circ \beta_{\mathcal{S}_{2}} \circ \rho_{\mathcal{M}}^{\mathrm{ra}}\left(q_{\mathcal{S}_{1} \mid \mathcal{S}_{2}}\right) & =\pi_{\mathcal{S}_{2}}^{\prime}(X) \circ \rho_{\mathcal{M}}^{\mathrm{ra}}\left(q_{\mathcal{S}_{1} \mid \mathcal{S}_{2}}\right) \\
& =\underline{\operatorname{Hom}}\left(\operatorname{id}_{X},\left(q_{\mathcal{S}_{1} \mid \mathcal{S}_{2}}\right)_{X}\right) \circ \pi_{\mathcal{S}_{2}}^{\prime}(X)=\pi_{\mathcal{S}_{2}}^{\prime}(X)
\end{aligned}
$$

The claim follows from the universal property of $A_{\mathcal{S}_{2}}$.
For $\mathcal{S}_{1}, \mathcal{S}_{2}, \mathcal{S}_{3} \in \mathfrak{T o p}(\mathcal{M})$ with $\mathcal{S}_{1} \supset \mathcal{S}_{2} \supset \mathcal{S}_{3}$, we have

$$
\begin{equation*}
q_{\mathcal{S}_{2} \mid \mathcal{S}_{3}} \circ q_{\mathcal{S}_{1} \mid \mathcal{S}_{2}}=q_{\mathcal{S}_{1} \mid \mathcal{S}_{3}} \quad \text { and } \quad \phi_{\mathcal{S}_{2} \mid \mathcal{S}_{3}} \circ \phi_{\mathcal{S}_{1} \mid \mathcal{S}_{2}}=\phi_{\mathcal{S}_{1} \mid \mathcal{S}_{3}} \tag{4.9}
\end{equation*}
$$

Lemma 4.4 says that the inverse system $\left(\left\{A_{\mathcal{S}}\right\},\left\{\phi_{\mathcal{S}_{1} \mid \mathcal{S}_{2}}\right\}\right)$ in $\mathcal{C}$ is obtained from the inverse system $\left(\left\{\tau_{\mathcal{S}}\right\},\left\{q_{\mathcal{S}_{1} \mid \mathcal{S}_{2}}\right\}\right)$ in $\operatorname{Rex}(\mathcal{M})$ by applying $\rho_{\mathcal{M}}^{\mathrm{ra}}$. We note that an exact functor preserves epimorphisms. By Theorem 3.4 and Lemma 4.4 we have:

Lemma 4.5. If $\mathcal{M}$ is an exact $\mathcal{C}$-module category, then $\left(\left\{A_{\mathcal{S}}\right\},\left\{\phi_{\mathcal{S}_{1} \mid \mathcal{S}_{2}}\right\}\right)$ is an inverse system of epimorphisms in $\mathcal{C}$.

We use the above observation to state the main result of this section. For an object $X$ of an essentially small category $\mathcal{E}$, we denote by $\mathfrak{Q u o}(X)$ and $\mathfrak{S u b}(X)$ the set of quotient objects of $X$ and the set of subobjects of $X$, respectively. We introduce partial orders on these sets as follows: For $Q_{1}, Q_{2} \in \mathfrak{Q u o}(X)$, we write $Q_{1} \geq Q_{2}$ if there is a morphism $Q_{1} \rightarrow Q_{2}$ in $\mathcal{E}$ compatible with the quotient morphisms from $X$. Dually, for $S_{1}, S_{2} \in \mathfrak{S u b}(X)$, we write $S_{1} \geq S_{2}$ if there is a morphism $S_{2} \rightarrow S_{1}$ in $\mathcal{E}$ compatible with the inclusion morphisms to $X$.

Theorem 4.6. Let $\mathcal{M}$ be an exact $\mathcal{C}$-module category. Then the map

$$
\mathfrak{T o p}(\mathcal{M}) \rightarrow \mathfrak{Q u o}\left(A_{\mathcal{M}}\right), \quad \mathcal{S} \mapsto A_{\mathcal{S}}=\int_{X \in \mathcal{S}} \underline{\operatorname{Hom}}(X, X)
$$

preserves the order. If, moreover, $\mathcal{M}$ is indecomposable, then this map reflects the order.

Proof. Lemma 4.5means that the map in question preserves the order. To complete the proof, we suppose that $\mathcal{M}$ is indecomposable. Let $\mathcal{S}_{1}$ and $\mathcal{S}_{2}$ be topologizing full subcategory of $\mathcal{M}$ such that $A_{\mathcal{S}_{1}} \geq A_{\mathcal{S}_{2}}$ in $\mathfrak{Q u o}\left(A_{\mathcal{M}}\right)$. Then we have $\rho_{\mathcal{M}}^{\mathrm{ra}}\left(\kappa_{\mathcal{S}_{1}}\right) \leq$ $\rho_{\mathcal{M}}^{\mathrm{ra}}\left(\kappa_{\mathcal{S}_{2}}\right)$ in $\mathfrak{S u b}\left(A_{\mathcal{M}}\right)$. Since $\rho_{\mathcal{M}}^{\mathrm{ra}}$ is exact, we have

$$
\rho_{\mathcal{M}}^{\mathrm{ra}}\left(\frac{\kappa_{\mathcal{S}_{2}}}{\kappa_{\mathcal{S}_{2}} \cap \kappa_{\mathcal{S}_{1}}}\right)=\frac{\rho_{\mathcal{M}}^{\mathrm{ra}}\left(\kappa_{\mathcal{S}_{2}}\right)}{\rho_{\mathcal{M}}^{\mathrm{ra}}\left(\kappa_{\mathcal{S}_{2}}\right) \cap \rho_{\mathcal{M}}^{\mathrm{ra}}\left(\kappa_{\mathcal{S}_{1}}\right)}=\frac{\rho_{\mathcal{M}}^{\mathrm{ra}}\left(\kappa_{\mathcal{S}_{2}}\right)}{\rho_{\mathcal{M}}^{\mathrm{ra}}\left(\kappa_{\mathcal{S}_{2}}\right)}=0 .
$$

Since $\rho_{\mathcal{M}}^{\mathrm{ra}}$ is faithful by Theorem [3.4] we have $\kappa_{\mathcal{S}_{2}} /\left(\kappa_{\mathcal{S}_{2}} \cap \kappa_{\mathcal{S}_{1}}\right)=0$. This implies that $\kappa_{\mathcal{S}_{1}} \subset \kappa_{\mathcal{S}_{2}}$. Hence $\mathcal{S}_{1} \subset \mathcal{S}_{2}$. The proof is done.

The dual of Theorem 4.6 is also interesting. Let $\lambda_{\mathcal{M}}: \mathcal{C} \rightarrow \operatorname{Lex}(\mathcal{M})$ be the left exact version of the action functor. By Remark 3.6 and the dual of Lemma 2.2 (see [BV12, Lemma 3.9]), the coend of the functor

$$
\begin{equation*}
\mathcal{S}^{\mathrm{op}} \times \mathcal{S} \rightarrow \mathcal{C}, \quad\left(X, X^{\prime}\right) \mapsto \underline{\operatorname{coHom}}\left(X, X^{\prime}\right) \tag{4.10}
\end{equation*}
$$

exists for all $\mathcal{S} \in \mathfrak{T o p}(\mathcal{M})$ and is canonically isomorphic to the coend

$$
\lambda_{\mathcal{M}}^{\mathrm{la}}\left(\tau_{\mathcal{S}}^{\mathrm{ra}}\right)=\int^{M \in \mathcal{M}} \underline{\operatorname{coHom}}\left(M, \tau_{\mathcal{S}}^{\mathrm{ra}}(M)\right)
$$

We denote the coend of (4.10) by $L_{\mathcal{S}}=\int^{X \in \mathcal{S}}$ coHom $(X, X)$. Since the duality functor is an anti-equivalence, the object ${ }^{*} A_{\mathcal{S}}$ is also a coend of the functor (4.10) with universal dinatural transformation

$$
{ }^{*} \pi_{\mathcal{S}}:{ }^{*} A_{\mathcal{S}} \rightarrow{ }^{*} \underline{\operatorname{Hom}}(X, X)=\underline{\mathrm{coHom}}(X, X) \quad(X \in \mathcal{S}) .
$$

Thus there is an isomorphism ${ }^{*} A_{\mathcal{S}} \cong L_{\mathcal{S}}$ respecting the universal dinatural transformations. By the above observation, we now obtain the following theorem:

Theorem 4.7. Let $\mathcal{M}$ be an exact $\mathcal{C}$-module category. Then the map

$$
\mathfrak{T o p}(\mathcal{M}) \rightarrow \mathfrak{S u b}\left(L_{\mathcal{M}}\right), \quad \mathcal{S} \mapsto L_{\mathcal{S}}
$$

preserves the order. If, moreover, $\mathcal{M}$ is indecomposable, then this map reflects the order.
4.3. Integral over a module full subcategory. Let $\mathcal{C}$ be a finite tensor category, and let $\mathcal{M}$ be a finite left $\mathcal{C}$-module category. We introduce the following terminology:

Definition 4.8. A $\mathcal{C}$-module full subcategory of $\mathcal{M}$ is a topologizing full subcategory of $\mathcal{M}$ closed under the action of $\mathcal{C}$.

Let $\mathcal{S}$ be a $\mathcal{C}$-module full subcategory of $\mathcal{M}$, and let $\kappa_{\mathcal{S}}$ and $\tau_{\mathcal{S}}$ be the endofunctors on $\mathcal{M}$ defined by (4.2) and (4.3), respectively. Then we have

$$
(V \otimes M)\left(V \otimes \kappa_{\mathcal{S}}(M)\right) \cong V \otimes\left(M / \kappa_{\mathcal{S}}(M)\right)=V \otimes \tau_{\mathcal{S}}(M) \in \mathcal{S}
$$

for all $V \in \mathcal{C}$ and $M \in \mathcal{M}$. Thus we have a natural transformation

$$
\tau_{\mathcal{S}}(V \otimes M) \rightarrow V \otimes \tau_{\mathcal{S}}(M) \quad(V \in \mathcal{C}, M \in \mathcal{M})
$$

making $\tau_{\mathcal{S}} \in \operatorname{Rex}(\mathcal{M})$ an oplax $\mathcal{C}$-module endofunctor on $\mathcal{M}$. Since $\mathcal{C}$ is rigid, we may regard $\tau_{\mathcal{S}}$ as a strong $\mathcal{C}$-module functor.

By Theorem 3.11 we endow the algebra $A_{\mathcal{S}}^{\prime}=\rho^{\mathrm{ra}}\left(\tau_{\mathcal{S}}\right)$ with a half-braiding $\sigma_{\mathcal{S}}^{\prime}$ such that $\left(A_{\mathcal{S}}^{\prime}, \sigma_{\mathcal{S}}^{\prime}\right)$ is an algebra in $\mathcal{Z}(\mathcal{C})$. Since $A_{\mathcal{S}}$ is isomorphic to $A_{\mathcal{S}}^{\prime}$, the algebra $A_{\mathcal{S}}$ also give rise to an algebra in $\mathcal{Z}(\mathcal{C})$. By Lemma 3.12, the half-braiding

$$
\sigma_{\mathcal{S}}(V): A_{\mathcal{S}} \otimes V \rightarrow V \otimes A_{\mathcal{S}} \quad(V \in \mathcal{C})
$$

of $A_{\mathcal{S}}$ inherited from $\rho^{\mathrm{ra}}\left(\tau_{\mathcal{S}}\right)$ is the unique morphism such that the diagram

commutes for all objects $X \in \mathcal{S}$. We write $\mathbf{A}_{\mathcal{S}}:=\left(A_{\mathcal{S}}, \sigma_{\mathcal{S}}\right) \in \mathcal{Z}(\mathcal{C})$. The following result is well-known in the case where $\mathcal{M}=\mathcal{S}=\mathcal{C}$.

Theorem 4.9. The algebra $\mathbf{A}_{\mathcal{S}} \in \mathcal{Z}(\mathcal{C})$ is commutative.
Proof. We postpone the proof of this theorem to Appendix A since it requires some technical results on the natural isomorphisms $\mathfrak{a}$ and $\mathfrak{b}$.

Let $\mathcal{S}_{1}$ and $\mathcal{S}_{2}$ be $\mathcal{C}$-module full subcategories of $\mathcal{M}$ such that $\mathcal{S}_{1} \supset \mathcal{S}_{2}$. We have introduced the morphism $\phi_{\mathcal{S}_{1} \mid \mathcal{S}_{2}}: A_{\mathcal{S}_{1}} \rightarrow A_{\mathcal{S}_{2}}$ in $\mathcal{C}$ in the previous subsection. By the definition of $\phi_{\mathcal{S}_{1} \mid \mathcal{S}_{2}}$ and the above explicit description of the half-braiding, we have the following result:

Theorem 4.10. $\phi_{\mathcal{S}_{1} \mid \mathcal{S}_{2}}: \mathbf{A}_{\mathcal{S}_{1}} \rightarrow \mathbf{A}_{\mathcal{S}_{2}}$ is a morphism in $\mathcal{Z}(\mathcal{C})$.
Thus, if $\mathcal{M}$ is exact, then we have an order-preserving map
$\{\mathcal{C}$-module full subcategories of $\mathcal{M}\} \rightarrow\left\{\right.$ quotient algebras of $\mathbf{A}_{\mathcal{M}}$ in $\left.\mathcal{Z}(\mathcal{C})\right\}$
defined by $\mathcal{S} \mapsto \mathbf{A}_{\mathcal{S}}$. If, moreover, $\mathcal{M}$ is indecomposable, then this map reflects the order.

## 5. Class functions and characters

5.1. The space of class functions. Let $\mathcal{C}$ be a finite tensor category. By the result of the last section, we have the algebra $A_{\mathcal{M}} \in \mathcal{C}$ for each finite left $\mathcal{C}$-module category $\mathcal{M}$. The vector space $\operatorname{Hom}_{\mathcal{C}}\left(A_{\mathcal{M}}, \mathbb{1}\right)$ with $\mathcal{M}=\mathcal{C}$ is called the space of class functions in Shi17b as it generalizes the usual notion of class functions on a finite group. The aim of this section is to explore the structure of the space of class functions and its generalization to module categories. We first introduce the following notation:

Definition 5.1. For a finite left $\mathcal{C}$-module category $\mathcal{M}$, we define the space of class functions of $\mathcal{M}$ by $\operatorname{CF}(\mathcal{M})=\operatorname{Hom}_{\mathcal{C}}\left(A_{\mathcal{M}}, \mathbb{1}\right)$.

Let $\mathrm{U}: \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$ be the forgetful functor. To study class functions, we consider the functor $Z:=\mathrm{U}^{\mathrm{ra}} \circ \mathrm{U}$. There is an equivalence $\rho^{\prime}: \mathcal{C} \rightarrow \mathcal{C}_{\mathcal{C}}^{*}$ given by $\rho^{\prime}(V)=$ $\mathrm{id}_{\mathcal{C}} \otimes V$. By applying Theorem 3.14 to $\mathcal{M}=\mathcal{C}$, we have

$$
\mathrm{Z}(V)=\rho_{\mathcal{C}}^{\mathrm{ra}} \rho^{\prime}(V)=\int_{X \in \mathcal{C}} X \otimes V \otimes X^{*} \quad(V \in \mathcal{C})
$$

Now let $\pi_{V}^{\mathrm{Z}}(X): \mathrm{Z}(V) \rightarrow X \otimes V \otimes X^{*}(V, X \in \mathcal{C})$ be the universal dinatural transformation for the end $\mathbf{Z}(V)$. The assignment $V \mapsto \mathbf{Z}(V)$ extends to an endofunctor on $\mathcal{C}$ in such a way that $\pi_{V}^{Z}(X)$ is natural in $V$ and dinatural in $X$. By the universal property, we define natural transformations $\Delta^{\mathrm{Z}}: \mathrm{Z} \rightarrow \mathrm{Z}^{2}$ and $\varepsilon^{\mathrm{Z}}: \mathrm{Z} \rightarrow \mathrm{id}_{\mathcal{C}}$ by

$$
\left(\operatorname{id}_{X} \otimes \pi_{V}^{\mathrm{Z}}(Y) \otimes \operatorname{id}_{X^{*}}\right) \circ \pi_{\mathrm{Z}(V)}^{\mathrm{Z}}(X) \circ \Delta_{V}^{\mathrm{Z}}=\pi_{V}^{\mathrm{Z}}(X \otimes Y) \quad \text { and } \quad \varepsilon_{V}^{\mathrm{Z}}=\pi_{X}^{\mathrm{Z}}(\mathbb{1})
$$

for all objects $V, X, Y \in \mathcal{C}$. The functor Z is a comonad on $\mathcal{C}$ with comultiplication $\Delta^{\mathrm{Z}}$ and counit $\varepsilon^{\mathrm{Z}}$.

Given an object $\mathbf{V}=(V, \sigma) \in \mathcal{Z}(\mathcal{C})$, we define the morphism $\delta: V \rightarrow \mathbf{Z}(V)$ in $\mathcal{C}$ to be the unique morphism such that the equation

$$
\pi_{V}^{\mathrm{Z}}(X) \circ \delta=\left(\sigma_{X} \otimes \operatorname{id}_{X^{*}}\right) \circ\left(\operatorname{id}_{V} \otimes \operatorname{coev}_{X}\right)
$$

holds for all objects $X \in \mathcal{C}$. The assignment $(V, \sigma) \mapsto(V, \delta)$ allows us to identify $\mathcal{Z}(\mathcal{C})$ with the category of Z-comodules. If we identify them, then a right adjoint of U is given by the free Z -comodule functor

$$
\mathrm{U}^{\mathrm{ra}}: \mathcal{C} \rightarrow \text { (the category of Z-comodules), } \quad V \mapsto\left(\mathrm{Z}(V), \Delta_{V}^{\mathrm{Z}}\right)
$$

By Theorem 4.9, for each finite left $\mathcal{C}$-module category $\mathcal{M}$, there is a commutative algebra $\mathbf{A}_{\mathcal{M}}=\left(A_{\mathcal{M}}, \sigma_{\mathcal{M}}\right)$ in $\mathcal{Z}(\mathcal{C})$ such that $A_{\mathcal{M}}=\mathrm{U}\left(\mathbf{A}_{\mathcal{M}}\right)$.

Definition 5.2. Let $\mathcal{M}$ be as above, and let $\delta_{\mathcal{M}}: A_{\mathcal{M}} \rightarrow \mathrm{Z}\left(A_{\mathcal{M}}\right)$ be the coaction of $\mathbf{Z}$ associated to the half-braiding $\sigma_{\mathcal{M}}$. For $f \in \operatorname{CF}(\mathcal{C})$ and $g \in \operatorname{CF}(\mathcal{M})$, we define their product $f \star g \in \mathrm{CF}(\mathcal{M})$ by

$$
\begin{equation*}
f \star g=f \circ \mathbf{Z}(g) \circ \delta_{\mathcal{M}} \tag{5.1}
\end{equation*}
$$

In particular, we have a binary operation on $\operatorname{CF}(\mathcal{C})$ by considering the case where $\mathcal{M}=\mathcal{C}$ in the above definition. As we have observed in Shi17b] $\mathrm{CF}(\mathcal{C})$ is an associative unital algebra with respect $\star$. Moreover, we have:

Lemma 5.3. $\mathrm{CF}(\mathcal{M})$ is a left $\mathrm{CF}(\mathcal{C})$-module by $\star$.
Proof. We remark $\mathbf{A}_{\mathcal{C}}=\mathrm{U}^{\text {ra }}(\mathbb{1})$. Thus there is an isomorphism

$$
\Phi_{\mathcal{M}}: \mathrm{CF}(\mathcal{M}) \rightarrow \operatorname{Hom}_{\mathcal{Z}(\mathcal{C})}\left(\mathbf{A}_{\mathcal{M}}, \mathbf{A}_{\mathcal{C}}\right), \quad f \mapsto \mathrm{Z}(f) \circ \delta_{\mathcal{M}}
$$

Then, noting $\delta_{\mathcal{C}}=\Delta_{\mathbb{1}}^{\mathrm{Z}}$, we compute

$$
\begin{gathered}
\Phi_{\mathcal{C}}(f) \circ \Phi_{\mathcal{M}}(g)=\mathrm{Z}(f) \circ \Delta_{\mathbb{1}}^{\mathrm{Z}} \circ \mathrm{Z}(g) \circ \delta_{\mathcal{M}}=\mathrm{Z}(f) \circ \mathrm{Z}^{2}(g) \circ \Delta_{A_{\mathcal{M}}}^{\mathrm{Z}} \circ \delta_{\mathcal{M}} \\
=\mathrm{Z}(f) \circ \mathrm{Z}^{2}(g) \circ \mathrm{Z}\left(\delta_{\mathcal{M}}\right) \circ \delta_{\mathcal{M}}=\mathrm{Z}(f \star g) \circ \delta_{\mathcal{M}}=\Phi_{\mathcal{M}}(f \star g)
\end{gathered}
$$

for all elements $f \in \mathrm{CF}(\mathcal{C})$ and $g \in \mathrm{CF}(\mathcal{M})$. Since the composition of morphisms is unital and associative, the action $\star: \operatorname{CF}(\mathcal{C}) \times \operatorname{CF}(\mathcal{M}) \rightarrow \mathrm{CF}(\mathcal{M})$ is also unital and associative. The proof is done.

We set $F=\operatorname{CF}(\mathcal{C})$ and $E=\operatorname{End}_{\mathcal{Z}}\left(\mathbf{A}_{\mathcal{C}}\right)$ for simplicity. The proof of the above lemma implies the following interesting consequence:

Theorem 5.4. There is an isomorphism $E \cong F$ of algebras. Moreover, the left $F$-module $\operatorname{CF}(\mathcal{M})$ corresponds to the left E-module $\operatorname{Hom}_{\mathcal{Z}(\mathcal{C})}\left(\mathbf{A}_{\mathcal{M}}, \mathbf{A}_{\mathcal{C}}\right)$ under the isomorphism $E \cong F$.
5.2. Pivotal module category. We recall that a pivotal monoidal category is a rigid monoidal category $\mathcal{C}$ equipped with a pivotal structure, that is, an isomorphism $X \rightarrow X^{* *}(X \in \mathcal{C})$ of monoidal functors. Let $\mathcal{C}$ be a pivotal finite tensor category with pivotal structure $j$. For an object $X \in \mathcal{C}$, we set

$$
\operatorname{tr}_{\mathcal{C}}(X)=\operatorname{ev}_{X^{*}} \circ\left(j_{X} \otimes \mathrm{id}_{X^{*}}\right)
$$

and define the internal character Shi17b of $X$ by

$$
\operatorname{ch}(X)=\operatorname{tr}_{\mathcal{C}}(X) \circ \pi_{\mathcal{C}}(X) \in \mathrm{CF}(\mathcal{C})
$$

Some applications of this notion are given in Shil7b. It is interesting to extend results of Shi17b to module categories. We first introduce the notion of pivotal module category. To give its precise definition, we recall the following notion:

Definition 5.5 (FSS16]). For an exact $\mathcal{C}$-module category $\mathcal{M}$, there is a unique functor $\mathbb{\$}_{\mathcal{M}}: \mathcal{M} \rightarrow \mathcal{M}$ equipped with a natural isomorphism

$$
\begin{equation*}
\underline{\operatorname{Hom}}(M, N)^{*} \cong \underline{\operatorname{Hom}}\left(N, \mathbb{S}_{\mathcal{M}}(M)\right) \tag{5.2}
\end{equation*}
$$

for $M, N \in \mathcal{M}$. We call $\mathbb{S}_{\mathcal{M}}$ the relative Serre functor of $\mathcal{M}$.
Let $\mathcal{M}$ be an exact left $\mathcal{C}$-module category. We make $\mathcal{M}^{\mathrm{op}} \times \mathcal{M}$ a $\mathcal{C}$-bimodule category by (2.16). Then Hom is a $\mathcal{C}$-bimodule functor. Given a strong monoidal functor $T: \mathcal{C} \rightarrow \mathcal{C}$ and a left $\mathcal{C}$-module category $\mathcal{N}$, we denote by ${ }_{T} \mathcal{N}$ the left
$\mathcal{C}$-module category whose underlying category is $\mathcal{N}$ but the action of $\mathcal{C}$ on $\mathcal{N}$ is twisted by $T$. The functor

$$
\mathcal{M}^{\mathrm{op}} \times \mathcal{M} \rightarrow(-)^{* *} \mathcal{C}, \quad(M, N) \mapsto \underline{\operatorname{Hom}}(N, M)^{*}
$$

is a $\mathcal{C}$-bimodule functor by (2.10) and (A.2). By [FSS16, Lemma 4.22], there is a unique natural isomorphism

$$
\begin{equation*}
X^{* *} \otimes \mathbb{S}_{\mathcal{M}}(M) \rightarrow \mathbb{S}_{\mathcal{M}}(X \otimes M) \quad(X \in \mathcal{C}, M \in \mathcal{M}) \tag{5.3}
\end{equation*}
$$

making $\mathbb{S}_{\mathcal{M}}$ a left $\mathcal{C}$-module functor $\mathbb{S}_{\mathcal{M}}: \mathcal{M} \rightarrow{ }_{(-)^{* *}} \mathcal{M}$ such that (5.2) is an isomorphism of $\mathcal{C}$-bimodule functors.
Definition 5.6. Let $\mathcal{C}$ be a pivotal finite tensor category with pivotal structure $j$, and let $\mathcal{M}$ be an exact $\mathcal{C}$-module category. A pivotal structure of $\mathcal{M}$ is a natural isomorphism $j^{\prime}: \operatorname{id}_{\mathcal{M}} \rightarrow \$_{\mathcal{M}}$ such that the equation

$$
\begin{equation*}
j_{X \otimes M}=\left(X \otimes M \xrightarrow{j_{X} \otimes j_{M}} X^{* *} \otimes \mathbb{S}_{\mathcal{M}}(M) \xrightarrow[\cong]{\cong 5.3)} \mathbb{S}_{\mathcal{M}}(X \otimes M)\right) \tag{5.4}
\end{equation*}
$$

holds for all $X \in \mathcal{C}$ and $M \in \mathcal{M}$. A pivotal left $\mathcal{C}$-module category is an exact left $\mathcal{C}$-module category equipped with a pivotal structure. Let $\mathcal{M}$ be such a category, and let $j^{\prime}$ be the pivotal structure of $\mathcal{M}$. Then we define the trace

$$
\operatorname{tr}_{\mathcal{M}}(M): \underline{\operatorname{Hom}}(M, M) \rightarrow \mathbb{1} \quad(M \in \mathcal{M})
$$

to be the morphism corresponding to $j_{M}^{\prime}: M \rightarrow \mathbb{S}_{\mathcal{M}}(M)$ via

$$
\begin{aligned}
& \operatorname{Hom}_{\mathcal{M}}\left(M, \mathbb{S}_{\mathcal{M}}(M)\right) \cong \operatorname{Hom}_{\mathcal{M}}\left(\mathbb{1}, \underline{\operatorname{Hom}}\left(M, \mathbb{S}_{\mathcal{M}}(M)\right)\right. \\
\cong & \left.\operatorname{Hom}_{\mathcal{M}}\left(\mathbb{1}, \underline{\operatorname{Hom}}(M, M)^{*}\right) \cong \operatorname{Hom}_{\mathcal{M}} \underline{\operatorname{Hom}}(M, M), \mathbb{1}\right) .
\end{aligned}
$$

Remark 5.7. Let $\mathcal{C}$ and $\mathcal{M}$ be as above. Then, for a morphism $f: M \rightarrow M$ in $\mathcal{M}$, the pivotal trace $\operatorname{ptr}(f) \in k$ is defined by

$$
\begin{equation*}
\operatorname{ptr}(f) \cdot \operatorname{id}_{\mathbb{1}}=\operatorname{tr}_{\mathcal{M}}(M) \circ \underline{\operatorname{Hom}}\left(\operatorname{id}_{M}, f\right) \circ{\underline{\operatorname{coev}_{\mathbb{1}}}}_{\mathbb{1}, M} . \tag{5.5}
\end{equation*}
$$

As in the ordinary trace, the pivotal trace is cyclic, multiplicative with respect to $\otimes$ and additive with respect to exact sequences; see Propositions B. 2 and B.4 in Appendix B
5.3. Internal characters for module categories. Let $\mathcal{C}$ be a pivotal finite tensor category, and let $\mathcal{M}$ be an pivotal exact left $\mathcal{C}$-module category. We now define:

Definition 5.8. The internal character of $M \in \mathcal{M}$ is defined by

$$
\begin{equation*}
\operatorname{ch}_{\mathcal{M}}(M)=\operatorname{tr}_{\mathcal{M}}(M) \circ \pi_{\mathcal{M}}(M) \in \mathrm{CF}(\mathcal{M}) \tag{5.6}
\end{equation*}
$$

We give basic properties of internal characters:
Lemma 5.9. For all $X \in \mathcal{C}$ and $M \in \mathcal{M}$, we have

$$
\operatorname{ch}_{\mathcal{C}}(X) \star \operatorname{ch}_{\mathcal{M}}(M)=\operatorname{ch}_{\mathcal{M}}(X \otimes M)
$$

Proof. Straightforward. See Appendix B for the detail.
Lemma 5.10. The internal character is additive in exact sequences: For any exact sequence $0 \rightarrow M_{1} \rightarrow M_{2} \rightarrow M_{3} \rightarrow 0$ in $\mathcal{M}$, we have

$$
\operatorname{ch}_{\mathcal{M}}\left(M_{2}\right)=\operatorname{ch}_{\mathcal{M}}\left(M_{1}\right)+\operatorname{ch}_{\mathcal{M}}\left(M_{2}\right)
$$

Proof. It is well-known that the pivotal trace is additive in exact sequences. One can find a detailed proof of this fact in GKPM11, Lemma 2.5.1]. The proof of this lemma goes along the same line; see Appendix B for the detail.

For a finite abelian category $\mathcal{A}$, we denote by $\operatorname{Gr}(\mathcal{A})$ the Grothendieck group of $\mathcal{A}$, that is, the quotient of the additive group generated by the isomorphism classes of objects of $\mathcal{A}$ by the relation $\left[M_{2}\right]=\left[M_{1}\right]+\left[M_{3}\right]$ for all exact sequences $0 \rightarrow M_{1} \rightarrow M_{2} \rightarrow M_{3} \rightarrow 0$ in $\mathcal{A}$. We set $\operatorname{Gr}_{k}(\mathcal{A})=k \otimes_{\mathbb{Z}} \operatorname{Gr}(\mathcal{A})$.

Now let $\left\{L_{1}, \ldots, L_{n}\right\}$ be a complete set of representatives of isomorphism classes of simple objects of $\mathcal{M}$. As a generalization of the main result of Shi17b, we prove the following theorem:

Theorem 5.11. The set $\left\{\operatorname{ch}\left(L_{i}\right)\right\}_{i=1}^{n} \subset \operatorname{CF}(\mathcal{M})$ is linearly independent.
Proof. The proof goes along the same way as Shi17b]. Let $\mathcal{S}$ be the full subcategory of $\mathcal{M}$ consisting of all semisimple objects of $\mathcal{M}$. Then, since $\mathcal{S}$ is semisimple, we may assume $A_{\mathcal{S}}=\bigoplus_{i=1}^{m} \underline{\operatorname{Hom}}\left(L_{i}, L_{i}\right)$ and $\pi_{\mathcal{S}}\left(L_{i}\right)$ is the projection to Hom $\left(L_{i}, L_{i}\right)$ for $i=1, \ldots, n$. Let $\phi_{\mathcal{M} \mid \mathcal{S}}: A_{\mathcal{M}} \rightarrow A_{\mathcal{S}}$ be the morphism defined by (4.8). Since $\phi_{\mathcal{M} \mid \mathcal{S}}$ is an epimorphism, the map

$$
\bigoplus_{i=1}^{n} \operatorname{Hom}_{\mathcal{C}}\left(\underline{\operatorname{Hom}}\left(L_{i}, L_{i}\right), \mathbb{1}\right)=\operatorname{CF}(\mathcal{S}) \xrightarrow{\operatorname{Hom}_{\mathcal{C}}\left(\phi_{\mathcal{M} \mid \mathcal{S}}, \mathbb{1}\right)} \mathrm{CF}(\mathcal{M})
$$

is injective. Since the morphism $\operatorname{ch}_{\mathcal{M}}\left(L_{i}\right)$ is the image of the morphism $\operatorname{tr}_{\mathcal{M}}\left(L_{i}\right)$ under this map, the set $\left\{\operatorname{ch}\left(L_{i}\right)\right\}_{i=1}^{n}$ is linearly independent in $\operatorname{CF}(\mathcal{M})$.

Let $\mathcal{C}$ and $\mathcal{M}$ be as above. By Lemma 5.10 and Theorem 5.11, the linear map

$$
\operatorname{ch}_{\mathcal{M}}: \operatorname{Gr}_{k}(\mathcal{M}) \rightarrow \mathrm{CF}(\mathcal{M}), \quad[M] \mapsto \operatorname{ch}_{\mathcal{M}}(M) \quad(M \in \mathcal{M})
$$

is well-defined and injective. Lemma 5.9 implies that $\operatorname{ch}_{\mathcal{C}}: \operatorname{Gr}_{k}(\mathcal{C}) \rightarrow \operatorname{CF}(\mathcal{C})$ is an algebra map and $\operatorname{ch}_{\mathcal{M}}: \operatorname{Gr}_{k}(\mathcal{M}) \rightarrow \operatorname{CF}(\mathcal{M})$ is $\operatorname{Gr}_{k}(\mathcal{C})$-linear if we view $\operatorname{CF}(\mathcal{M})$ as a left $\operatorname{Gr}_{k}(\mathcal{C})$-module through the algebra map $\mathrm{ch}_{\mathcal{C}}$.

By the proof of the above lemma, we see that the linear map $\operatorname{ch}_{\mathcal{M}}$ is bijective if $\mathcal{M}$ is semisimple. We have proved that, under the assumption that $\mathcal{C}$ is unimodular in the sense of [EO04], the map $\operatorname{ch}_{\mathcal{C}}: \operatorname{Gr}_{k}(\mathcal{C}) \rightarrow \operatorname{CF}(\mathcal{C})$ is bijective if and only if $\mathcal{C}$ is semisimple Shi17b]. It would be interesting to establish an analogous result for module categories. The unimodularity of module categories, introduced in [FSS16], may be useful to formulate such a result.
5.4. Class functions of the dual tensor category. Let $\mathcal{C}$ be a finite tensor category, and let $\mathcal{M}$ be an indecomposable exact left $\mathcal{C}$-module category. As an application of our results, we give the following description of the algebra of class functions of the dual tensor category:

Theorem 5.12. $\operatorname{CF}\left(\mathcal{C}_{\mathcal{M}}^{*}\right) \cong \operatorname{End}_{\mathcal{Z}(\mathcal{C})}\left(\mathbf{A}_{\mathcal{M}}\right)$ as algebras.
Proof. Set $\mathcal{D}=\mathcal{C}_{\mathcal{M}}^{*}$. Let $\mathrm{U}: \mathcal{Z}(\mathcal{D}) \rightarrow \mathcal{D}$ be the forgetful functor. By Theorems 3.13 3.14 and 5.4, we have isomorphisms

$$
\mathrm{CF}(\mathcal{D}) \cong \operatorname{End}_{\mathcal{Z}(\mathcal{D})}\left(\mathrm{U}_{\mathcal{D}}^{\mathrm{ra}}\left(\mathbb{1}_{\mathcal{D}}\right)\right) \cong \operatorname{End}_{\mathcal{Z}(\mathcal{C})}\left(\theta_{\mathcal{M}}^{-1} \mathrm{U}_{\mathcal{D}}^{\mathrm{ra}}\left(\mathbb{1}_{\mathcal{D}}\right)\right) \cong \operatorname{End}_{\mathcal{Z}(\mathcal{C})}\left(\mathbf{A}_{\mathcal{M}}\right)
$$

of algebras. The proof is done.
A semisimple finite tensor category is called a fusion category [ENO05]. Our results give some new results on fusion categories. For example:

Corollary 5.13. Suppose that the base field $k$ is of characteristic zero. Let $\mathcal{C}$ be $a$ fusion category, and let $\mathcal{M}$ be an indecomposable exact left $\mathcal{C}$-module category such that $\mathcal{C}_{\mathcal{M}}^{*}$ admits a pivotal structure. Then there is an isomorphism

$$
\operatorname{Gr}_{k}\left(\mathcal{C}_{\mathcal{M}}^{*}\right) \cong \operatorname{End}_{\mathcal{Z}(\mathcal{C})}\left(\mathbf{A}_{\mathcal{M}}\right)
$$

of algebras.
Proof. $\mathcal{C}_{\mathcal{M}}^{*}$ is a pivotal fusion categories by the assumption ENO05. Thus the result follows from the results of the previous subsection.

The following result generalizes Ost13, Example 2.18]:
Corollary 5.14. Under the same assumption on the above corollary, the following two assertions are equivalent:
(1) The Grothendieck ring of $\mathcal{C}_{\mathcal{M}}^{*}$ is commutative.
(2) The object $\mathbf{A}_{\mathcal{M}} \in \mathcal{Z}(\mathcal{C})$ is multiplicity-free.

Proof. Since $k$ is of characteristic zero, $\mathcal{Z}(\mathcal{C})$ is a fusion category ENO05. Moreover, the $\operatorname{ring} \operatorname{Gr}(\mathcal{D})$ is commutative if and only if the $k$-algebra $\operatorname{Gr}_{k}(\mathcal{D})$ is. Now the claim follows from the above corollary.

## 6. A filtration on the space of class functions

6.1. A filtration on the space of class functions. Let $\mathcal{M}$ be a finite abelian category. For an object $M \in \mathcal{M}$, we denote by $\operatorname{soc}(M)$ the socle of $M$. Every object $M \in \mathcal{M}$ has a canonical filtration

$$
0=M_{0} \subset M_{1} \subset M_{2} \subset M_{3} \subset \cdots \subset M
$$

such that $M_{i+1} / M_{i}=\operatorname{soc}\left(M / M_{i}\right)$. We denote $M_{n}$ by $\operatorname{soc}_{n}(M)$. Then the assignment $M \mapsto \operatorname{soc}_{n}(M)$ extends to a $k$-linear left exact endofunctor on $\mathcal{M}$, which we call the $n$-th socle functor. The number

$$
\operatorname{Lw}(M)=\min \left\{n=0,1,2, \ldots \mid \operatorname{soc}_{n}(M)=M\right\}
$$

is called the Loewy length of $M$. We define $\mathcal{M}_{n}$ to be the full subcategory of $\mathcal{M}$ consisting of all objects $M$ with $\operatorname{Lw}(M) \leq n$. Since $\mathcal{M}$ is finite, the number

$$
\operatorname{Lw}(\mathcal{M}):=\min \left\{n=0,1,2, \ldots \mid \mathcal{M}_{n}=\mathcal{M}\right\}=\max \{\operatorname{Lw}(M) \mid M \in \mathcal{M}\}
$$

is finite. We call $\operatorname{Lw}(\mathcal{M})$ the Loewy length of $\mathcal{M}$ and the filtration

$$
\begin{equation*}
0=\mathcal{M}_{0} \subset \mathcal{M}_{1} \subset \mathcal{M}_{2} \subset \cdots \subset \mathcal{M}_{w}=\mathcal{M} \quad(w=\operatorname{Lw}(\mathcal{M})) \tag{6.1}
\end{equation*}
$$

the socle filtration of $\mathcal{M}$.
It is easy to see that each $\mathcal{M}_{n}$ is a topologizing full subcategory of $\mathcal{M}$. Thus, if $\mathcal{C}$ is a finite tensor category and $\mathcal{M}$ is an exact left $\mathcal{C}$-module category with Loewy length $w$, then we have a series

$$
\begin{equation*}
A_{\mathcal{M}}=A_{\mathcal{M}_{w}} \rightarrow A_{\mathcal{M}_{w-1}} \rightarrow \cdots \rightarrow A_{\mathcal{M}_{2}} \rightarrow A_{\mathcal{M}_{1}} \tag{6.2}
\end{equation*}
$$

of epimorphisms of of algebras in $\mathcal{C}$ by Theorem 4.6. Applying $\operatorname{Hom}_{\mathcal{C}}(-, \mathbb{1})$ to this series, we obtain the filtration of the space of class functions

$$
\begin{equation*}
\mathrm{CF}_{1}(\mathcal{M}) \subset \mathrm{CF}_{2}(\mathcal{M}) \subset \cdots \subset \mathrm{CF}_{w-1}(\mathcal{M}) \subset \mathrm{CF}_{w}(\mathcal{M})=\mathrm{CF}(\mathcal{M}) \tag{6.3}
\end{equation*}
$$

where $\operatorname{CF}_{n}(\mathcal{M})=\operatorname{Hom}_{\mathcal{C}}\left(A_{\mathcal{M}_{n}}, \mathbb{1}\right)$. In this section, we investigate how this filtration relates to representation-theoretic properties of $\mathcal{M}$.
6.2. Jacobson radical functor. For further study of the series (6.2) and the filtration (6.3), we introduce the following abstract definition of the Jacobson radical: Let $\mathcal{M}$ be a finite abelian category. For an object $M \in \mathcal{M}$, we define the subobject $\operatorname{rad}(M)$ of $M$ to be the intersection of all maximal subobjects of $M$. It is easy to see that $M \mapsto \operatorname{rad}(M)$ extends to a $k$-linear right exact endofunctor on $\mathcal{M}$. We call $\operatorname{rad} \in \operatorname{Rex}(\mathcal{M})$ the Jacobson radical functor of $\mathcal{M}$.

We rephrase several known results in the representation theory in terms of the Jacobson radical functor. Let $A$ be a finite-dimensional algebra such that $\mathcal{M} \approx$ $A$-mod, and let $J$ be the Jacobson radical of $A$. Then the Jacobson radical functor may be identified with $J \otimes_{A}(-)$. Thus we have the series

$$
\begin{equation*}
\operatorname{id}_{\mathcal{M}}=: \operatorname{rad}^{0} \supset \operatorname{rad} \supset \operatorname{rad}^{2} \supset \cdots \supset \operatorname{rad}^{w-1} \supset \operatorname{rad}^{w}=0 \quad(w=\operatorname{Lw}(\mathcal{M})) \tag{6.4}
\end{equation*}
$$

of subobjects in $\operatorname{Rex}(\mathcal{M})$. We have $\operatorname{rad}_{\mathcal{M}}^{i} \neq \operatorname{rad}_{\mathcal{M}}^{i+1}$ for all $i=0, \ldots, w-1$ by the Nakayama lemma.

For a positive integer $n$, we define the $n$-th capital functor $\operatorname{cap}_{n} \in \operatorname{Rex}(\mathcal{M})$ as the quotient object $\operatorname{id}_{\mathcal{M}} / \operatorname{rad}^{n}$. If we identify $\operatorname{Rex}(\mathcal{M})$ with $A-\bmod -A$, then this functor corresponds to the bimodule $A / J^{n}$ and therefore

$$
\begin{equation*}
\operatorname{cap}_{n}(M)=\left(A / J^{n}\right) \otimes_{A} M \cong M / J^{n} M \tag{6.5}
\end{equation*}
$$

for all $M \in \mathcal{M}$. By Sakurai Sak17b Lemma 2.3], there is an adjunction

$$
\begin{equation*}
\operatorname{Hom}_{\mathcal{M}}\left(\operatorname{cap}_{n}(M), M^{\prime}\right) \cong \operatorname{Hom}_{\mathcal{M}}\left(M, \operatorname{soc}_{n}\left(M^{\prime}\right)\right) \quad\left(M, M^{\prime} \in \mathcal{M}\right) \tag{6.6}
\end{equation*}
$$

The $n$-th term $\mathcal{M}_{n}$ of the socle filtration (6.1) coincides with the full subcategory of $\mathcal{M}$ consisting of all objects $M$ such that $\operatorname{soc}_{n}(M)=M$. Comparing (6.6) with (4.4), we have $\operatorname{cap}_{n}=\tau_{\mathcal{M}_{n}}$ with the notation in Subsection 4.1. In other words, $\mathcal{M}_{n}$ corresponds to $\mathrm{rad}^{n}$ via the correspondence of Lemma 4.2,

Now we consider the case where $\mathcal{C}$ is a finite tensor category and $\mathcal{M}$ is an exact left $\mathcal{C}$-module category with Loewy length $w$. There is a series

$$
\begin{equation*}
\mathrm{id}_{\mathcal{M}}=\operatorname{cap}_{w} \rightarrow \operatorname{cap}_{w-1} \rightarrow \cdots \rightarrow \operatorname{cap}_{2} \rightarrow \operatorname{cap}_{1} \tag{6.7}
\end{equation*}
$$

of epimorphisms in $\operatorname{Rex}(\mathcal{M})$. We have a canonical isomorphism

$$
\rho^{\mathrm{ra}}\left(\operatorname{cap}_{n}\right) \cong \int_{X \in \mathcal{M}_{n}} \underline{\operatorname{Hom}}(X, X)
$$

and the series (6.2) is obtained by applying $\rho^{\mathrm{ra}}$ to (6.7).
6.3. Reynolds ideal and its generalization. Let $A$ be a finite-dimensional algebra. For $n \in \mathbb{Z}_{+}$, we define the $n$-th Reynolds ideal Sak17a] of $A$ by

$$
\begin{equation*}
\operatorname{Rey}_{n}(A)=\operatorname{soc}_{n}(A) \cap Z(A), \tag{6.8}
\end{equation*}
$$

where $\operatorname{soc}_{n}(A)$ is the $n$-th socle of the left $A$-module $A$. As $\operatorname{Rey}_{n}(A)$ is a Morita invariant Sak17a, it is natural to expect that the $n$-th Reynolds ideal of a finite abelian category is defined in an intrinsic way. For $n=1$, this was achieved by Gainutdinov and Runkel in GR17. By using the Jacobson radical functor, we propose the following definition, which is different to GR17:

Definition 6.1. Let $\mathcal{M}$ be a finite abelian category. For a non-negative positive integer $n$, we define the $n$-th Reynolds ideal of $\mathcal{M}$ by

$$
\operatorname{Rey}_{n}(\mathcal{M})=\left\{\xi \in \operatorname{End}\left(\operatorname{id}_{\mathcal{M}}\right) \mid \xi \circ i_{n}=0\right\}
$$

where $i_{n}: \operatorname{rad}^{n} \rightarrow \operatorname{id}_{\mathcal{M}}$ is the inclusion morphism.

Let $A$ be a finite-dimensional algebra. We explain that $\operatorname{Rey}_{n}(\mathcal{M})$ can be identified with the $n$-th Reynolds ideal of $A$ when $\mathcal{M}=A$-mod. Let $J$ be the Jacobson radical of $A$. Then the $n$-th socle of $M \in A$-mod is given by

$$
\operatorname{soc}_{n}(M)=\left\{m \in M \mid r m=0 \text { for all } r \in J^{n}\right\}
$$

and hence the $n$-th Reynolds ideal of $A$ is expressed as follows:

$$
\begin{equation*}
\operatorname{Rey}_{n}(A)=\left\{z \in Z(A) \mid r z=0 \text { for all } r \in J^{n}\right\} \tag{6.9}
\end{equation*}
$$

If $\mathcal{M}=A$-mod, then $\operatorname{Rex}(\mathcal{M})$ can be identified with $A$-mod- $A$. Under this identification, $\mathrm{id}_{\mathcal{M}}$ and $\operatorname{rad}^{n}$ correspond to the $A$-bimodule $A$ and its subbimodule $J^{n}$, respectively. By (6.9), it is easy to check that the isomorphism $Z(A) \cong \operatorname{End}\left(\mathrm{id}_{\mathcal{M}}\right)$ restricts to an isomorphism $\operatorname{Rey}_{n}(A) \cong \operatorname{Rey}_{n}(A$-mod) for each $n$.

We consider the case where $\mathcal{C}$ is a finite tensor category and $\mathcal{M}$ is an indecomposable exact left $\mathcal{C}$-module category with action functor $\rho=\rho_{\mathcal{M}}$. Then we have an adjunction isomorphism

$$
\begin{equation*}
\operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, A_{\mathcal{M}}\right)=\operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, \rho^{\mathrm{ra}}\left(\operatorname{id}_{\mathcal{M}}\right)\right) \cong \operatorname{Nat}\left(\rho(\mathbb{1}), \operatorname{id}_{\mathcal{M}}\right)=\operatorname{End}\left(\mathrm{id}_{\mathcal{C}}\right) \tag{6.10}
\end{equation*}
$$

Moreover, since $\rho^{\text {ra }}$ is exact by Theorem 3.4 the object $J_{\mathcal{M}}^{n}:=\rho^{\text {ra }}\left(\operatorname{rad}^{n}\right)$ is a subobject of $A_{\mathcal{M}}$. The following description of $\operatorname{Rey}_{n}(\mathcal{M})$ may be regarded as a generalization of (6.9).
Lemma 6.2. For an indecomposable exact $\mathcal{C}$-module category $\mathcal{M}$, we define

$$
R_{n}(\mathcal{M})=\left\{a \in \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, A_{\mathcal{M}}\right) \mid m \circ(a \otimes i)=0\right\}
$$

where $m$ is the multiplication of $A_{\mathcal{M}}$ and $i: J_{\mathcal{M}}^{n} \rightarrow A_{\mathcal{M}}$ is the inclusion morphism. Then (6.10) restricts to an isomorphism between $R_{n}(\mathcal{M})$ and $\operatorname{Rey}_{n}(\mathcal{M})$.

Proof. We use the monoidal structure of $\rho^{\text {ra }}$ described in Lemma 3.8. Let $a: \mathbb{1} \rightarrow$ $A_{\mathcal{M}}$ be a morphism in $\mathcal{C}$, and let $\widetilde{a} \in \operatorname{End}\left(\mathrm{id}_{\mathcal{C}}\right)$ be the natural transformation corresponding to $a$ via (6.10). By the definition of $\mu^{(0)}$, we have $a=\rho^{\mathrm{ra}}(\widetilde{a}) \circ \mu^{(0)}$. Let $i_{n}: \operatorname{rad}^{n} \rightarrow \operatorname{id}_{\mathcal{M}}$ be the inclusion morphism. Since $i=\rho^{\mathrm{ra}}\left(i_{n}\right)$, we have

$$
\begin{aligned}
m \circ(a \otimes i) & =\mu_{\mathrm{id}_{\mathcal{M}}, \mathrm{id}_{\mathcal{M}}}^{(2)} \circ\left(\rho^{\mathrm{ra}}(\widetilde{a}) \otimes \rho^{\mathrm{ra}}\left(i_{n}\right)\right) \circ\left(\mu^{(0)} \otimes \operatorname{id}_{J_{\mathcal{M}}^{n}}\right) \\
& =\rho^{\mathrm{ra}}\left(\widetilde{a} \circ i_{n}\right) \circ \mu_{\mathrm{id}_{\mathcal{M}}, \mathrm{id}_{\mathcal{M}}}^{(2)} \circ\left(\mu^{(0)} \otimes \operatorname{id}_{J_{\mathcal{M}}^{n}}\right)=\rho^{\mathrm{ra}}\left(\widetilde{a} \circ i_{n}\right) .
\end{aligned}
$$

Thus, by the faithfulness of $\rho^{\text {ra }}$ (Theorem 3.4), the morphism $a$ belongs to $R_{n}(\mathcal{M})$ if and only if $\widetilde{a} \in \operatorname{Rey}_{n}(\mathcal{M})$. The proof is done.

We recall that an algebra $A$ in $\mathcal{C}$ with multiplication $m$ is said to be Frobenius if there is an isomorphism $\phi: A \rightarrow A^{*}$ of right $A$-modules in $\mathcal{C}$. Given such an isomorphism $\phi$, we define

$$
e_{\phi}=\operatorname{ev}_{A} \circ\left(\phi \otimes \mathrm{id}_{A}\right) \quad \text { and } \quad d_{\phi}=\left(\mathrm{id}_{A} \otimes \phi^{-1}\right) \circ \operatorname{coev}_{A}
$$

Then the triple $\left(A, e_{\phi}, d_{\phi}\right)$ is a left dual object of $A$. Thus the map

$$
\begin{equation*}
\operatorname{Hom}_{\mathcal{C}}(A, \mathbb{1}) \rightarrow \operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, A) \quad \xi \mapsto\left(\xi \otimes \operatorname{id}_{A}\right) \circ d_{\phi} \tag{6.11}
\end{equation*}
$$

is an isomorphism of vector spaces with inverse

$$
\begin{equation*}
\operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, A) \rightarrow \operatorname{Hom}_{\mathcal{C}}(A, \mathbb{1}), \quad a \mapsto e_{\phi} \circ\left(a \otimes \operatorname{id}_{A}\right) \tag{6.12}
\end{equation*}
$$

The $A$-linearity of $\phi$ imply

$$
\begin{align*}
e_{\phi} \circ\left(m \otimes \mathrm{id}_{A}\right) & =e_{\phi} \circ\left(\mathrm{id}_{A} \otimes m\right),  \tag{6.13}\\
\left(\mathrm{id}_{A} \otimes m\right) \circ\left(d_{\phi} \otimes \mathrm{id}_{A}\right) & =\left(m \otimes \mathrm{id}_{A}\right) \circ\left(\mathrm{id}_{A} \otimes d_{\phi}\right) . \tag{6.14}
\end{align*}
$$

The following lemma may be well-known:
Lemma 6.3. Let $J$ be an ideal of $A$ with inclusion morphism $i: J \rightarrow A$. Then the isomorphisms (6.11) and (6.12) restricts to an isomorphism

$$
\operatorname{Hom}_{\mathcal{C}}(A / J, \mathbb{1}) \cong\left\{a \in \operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, A) \mid m \circ(a \otimes i)=0\right\}
$$

Proof. Let $\xi: A \rightarrow \mathbb{1}$ be a morphism in $\mathcal{C}$, and let $a: \mathbb{1} \rightarrow A$ be the morphism corresponding to $\xi$ by (6.11) and (6.12). We first suppose that $\xi$ belongs to $\operatorname{Hom}_{\mathcal{C}}(A / J, \mathbb{1})$, that is, $\xi \circ i=0$. Then we compute

$$
\begin{aligned}
m \circ(a \otimes i) & =\left(\xi \otimes \mathrm{id}_{A}\right) \circ\left(\mathrm{id}_{A} \otimes m\right) \circ\left(d_{\phi} \otimes \mathrm{id}_{A}\right) \circ i \\
& =\left(\xi \otimes \mathrm{id}_{A}\right) \circ\left(m \otimes \mathrm{id}_{A}\right) \circ\left(\mathrm{id}_{A} \otimes d_{\phi}\right) \circ i \\
& =\left(\left(\xi \circ m \circ\left(i \otimes \operatorname{id}_{A}\right)\right) \otimes \operatorname{id}_{A}\right) \circ\left(\operatorname{id}_{J} \otimes d_{\phi}\right)
\end{aligned}
$$

by (6.14). Since $J$ is an ideal of $A$, the image of $m \circ\left(i \otimes \mathrm{id}_{A}\right)$ is contained in $J$. Thus we have $\xi \circ m \circ\left(i \otimes \operatorname{id}_{A}\right)=0$. Therefore $m \circ(a \otimes i)=0$. If, conversely, this equation holds, then we have

$$
\begin{aligned}
\xi \circ i=e_{\phi} \circ(i \otimes a) & =e_{\phi} \circ\left(m \otimes \operatorname{id}_{A}\right) \circ(u \otimes i \otimes a) \\
& =e_{\phi} \circ\left(\operatorname{id}_{A} \otimes m\right) \circ(u \otimes i \otimes a)=0
\end{aligned}
$$

by (6.13), where $u: \mathbb{1} \rightarrow A$ is the unit of $A$. Thus $\xi \in \operatorname{Hom}_{\mathcal{C}}(A / J, \mathbb{1})$. The proof is done.

Now we have the following representation-theoretic description of $\mathrm{CF}_{n}$.
Theorem 6.4. Let $\mathcal{M}$ be an indecomposable exact $\mathcal{C}$-module category. If $A_{\mathcal{M}}$ is $a$ Frobenius algebra, then the isomorphism

$$
\begin{equation*}
\mathrm{CF}(\mathcal{M})=\operatorname{Hom}_{\mathcal{C}}\left(A_{\mathcal{M}}, \mathbb{1}\right) \xrightarrow[\cong]{\cong 6.11} \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, A_{\mathcal{M}}\right) \xrightarrow[\cong]{\cong} \operatorname{End}\left(\mathrm{id}_{\mathcal{M}}\right) \tag{6.15}
\end{equation*}
$$

restricts to isomorphisms

$$
\operatorname{CF}_{n}(\mathcal{M}) \cong \operatorname{Rey}_{n}(\mathcal{M}) \quad(n=1,2,3, \ldots)
$$

Proof. The subobject $\rho^{\mathrm{ra}}\left(\operatorname{rad}^{n}\right)$ is an ideal of $A_{\mathcal{M}}=\rho^{\mathrm{ra}}\left(\mathrm{id} \mathcal{M}_{\mathcal{M}}\right)$. The proof is done by applying the above two lemmas to this ideal.

A finite tensor category $\mathcal{D}$ is said to be unimodular EO04 if the projective cover of the unit object $\mathbb{1} \in \mathcal{D}$ is also an injective hull of $\mathbb{1}$. Following Shi17c , a finite tensor category $\mathcal{D}$ is unimodular if and only if the algebra $R(\mathbb{1}) \in \mathcal{Z}(\mathcal{D})$ is Frobenius, where $\mathrm{R}: \mathcal{D} \rightarrow \mathcal{Z}(\mathcal{D})$ is a right adjoint of the forgetful functor.

Let $\mathcal{M}$ be an indecomposable exact left $\mathcal{C}$-module category. Then $\mathcal{D}:=\mathcal{C}_{\mathcal{M}}^{*}$ is a finite tensor category. By Theorem 3.14 and the above-mentioned fact, the algebra $\mathbf{A}_{\mathcal{M}} \in \mathcal{Z}(\mathcal{C})$ is Frobenius if and only if $\mathcal{D}$ is unimodular. Thus the algebra $A_{\mathcal{M}} \in \mathcal{C}$ is Frobenius if $\mathcal{D}$ is unimodular. By the above theorem, we have:

Corollary 6.5. Let $\mathcal{M}$ be an indecomposable exact $\mathcal{C}$-module category. If $\mathcal{C}_{\mathcal{M}}^{*}$ is unimodular, then we have $\operatorname{CF}_{n}(\mathcal{M}) \cong \operatorname{Rey}_{n}(\mathcal{M})$.

In particular, if $\mathcal{C}$ is unimodular, then $\operatorname{CF}_{n}(\mathcal{C}) \cong \operatorname{Rey}_{n}(\mathcal{C})$.
6.4. Symmetric linear forms on an algebra. For a finite-dimensional algebra $A$ with Jacobson radical $J$, we set

$$
\begin{aligned}
\operatorname{SLF}(A) & =\left\{f \in A^{*} \mid f(a b)=f(b a) \text { for all } a, b \in A\right\} \\
\operatorname{SLF}_{n}(A) & =\left\{f \in \operatorname{SLF}(A) \mid f\left(J^{n}\right)=0\right\} \quad\left(n \in \mathbb{Z}_{+}\right)
\end{aligned}
$$

If $G$ is a finite group, then $\operatorname{SLF}(k G)$ is the space of class functions on $G$. Thus, for a finite module category $\mathcal{M}$ such that $\mathcal{M} \approx A$-mod, it is natural to ask how $\mathrm{CF}(\mathcal{M})$ relates to $\operatorname{SLF}(A)$. To consider this problem, we first introduce the following categorical definition of the space of symmetric linear forms:

Definition 6.6. For a finite abelian category $\mathcal{M}$ and $n \in \mathbb{Z}_{+}$, we set

$$
\operatorname{SLF}(\mathcal{M}):=\operatorname{Nat}\left(\operatorname{id}_{\mathcal{M}}, \mathbb{N}_{\mathcal{M}}\right) \quad \text { and } \quad \operatorname{SLF}_{n}(\mathcal{M})=\left\{f \in \operatorname{SLF}(\mathcal{M}) \mid f \circ i_{n}=0\right\}
$$

where $i_{n}: \operatorname{rad}^{n} \rightarrow \mathrm{id}_{\mathcal{M}}$ is the inclusion morphism.
If $\mathcal{M}$ is a finite abelian category such that $\mathcal{M} \approx A$ - $\bmod$, then $\operatorname{Rex}(\mathcal{M})$ can be identified with $A$-mod- $A$. Under this identification, $\mathrm{id}_{\mathcal{M}}$ and $\mathbb{N}_{\mathcal{M}}$ correspond to the $A$-bimodules $A$ and $A^{*}$, respectively. Thus we have

$$
\begin{equation*}
\operatorname{SLF}(\mathcal{M}) \cong \operatorname{Hom}_{A-\bmod -A}\left(A, A^{*}\right) \cong \operatorname{SLF}(A) \tag{6.16}
\end{equation*}
$$

where the second isomorphism is given by $f \mapsto f(1)$. If we identify $\operatorname{SLF}(\mathcal{M})$ with $\operatorname{SLF}(A)$ by this isomorphism, then $\operatorname{SLF}_{n}(\mathcal{M})$ is identified with $\operatorname{SLF}_{n}(A)$.

Remark 6.7. Let $\mathcal{M}$ be a finite abelian category. We suppose that $\mathcal{M}$ is symmetric Frobenius and choose an isomorphism $\lambda: \mathrm{id}_{\mathcal{M}} \rightarrow \mathbb{N}_{\mathcal{M}}$. Then the map

$$
\operatorname{End}\left(\operatorname{id}_{\mathcal{M}}\right) \rightarrow \operatorname{SLF}(\mathcal{M}), \quad z \mapsto \lambda \circ z
$$

is an isomorphism. By Definitions 6.1 and 6.6. we also have isomorphisms

$$
\operatorname{Rey}_{n}(\mathcal{M}) \rightarrow \operatorname{SLF}_{n}(\mathcal{M}), \quad z \mapsto \lambda \circ z \quad\left(n \in \mathbb{Z}_{+}\right)
$$

In ring-theoretic terms, this means: Let $A$ be a symmetric Frobenius algebra, and let $\lambda: A \rightarrow A^{*}$ be an isomorphism of $A$-bimodules. For each $n \in \mathbb{Z}_{+}$, the isomorphism $\lambda$ restricts to an isomorphism between $\operatorname{Rey}_{n}(A)$ and $\operatorname{SLF}_{n}(A)$.

Now we consider the case where $\mathcal{M}$ is an exact module category over a finite tensor category $\mathcal{C}$. Although $\operatorname{CF}(\mathcal{M})$ is an analogue of the space of class functions, it does not seem to be isomorphic to $\operatorname{SLF}(\mathcal{M})$ in general. To see when they are isomorphic, we provide the following lemma:

Lemma 6.8. There is a natural isomorphism

$$
\operatorname{Hom}_{\mathcal{C}}\left(\rho^{\mathrm{ra}}\left(\mathbb{S}_{\mathcal{M}} \circ F\right), X^{* *}\right) \cong \operatorname{Nat}\left(F, X \otimes \mathbb{N}_{\mathcal{M}}\right) \quad(F \in \operatorname{Rex}(\mathcal{M}), V \in \mathcal{C})
$$

Proof. Let $D$ be the distinguished invertible object of $\mathcal{C}$ introduced in [ENO04]. Then there are natural isomorphisms

$$
\mathbb{N}_{\mathcal{C}}(X) \cong D^{*} \otimes X^{* *} \quad \text { and } \quad \mathbb{N}_{\mathcal{M}}(M) \cong D^{*} \otimes \mathbb{S}_{\mathcal{M}}(M)
$$

for $X \in \mathcal{C}$ and $M \in \mathcal{M}$ [FSS16]. Since $\mathbb{S}_{\mathcal{M}}: \mathcal{M} \rightarrow_{(-)^{* *}} \mathcal{M}$ is a $\mathcal{C}$-module functor, and since $D$ is an invertible object, we have natural isomorphisms

$$
\left(\mathbb{N}_{\mathcal{M}}^{-1} \circ \mathbb{S}_{\mathcal{M}}\right)(M) \cong \mathbb{S}_{\mathcal{M}}^{-1}\left(D \otimes \mathbb{S}_{\mathcal{M}}(M)\right) \cong D^{* *} \otimes \mathbb{S}_{\mathcal{M}}^{-1} \mathbb{S}_{\mathcal{M}}(M) \cong D \otimes M
$$

for $M \in \mathcal{M}$. By using these isomorphisms and basic results on the Nakayama functor recalled in Subsection 2.5, we have natural isomorphisms

$$
\begin{aligned}
& \operatorname{Hom}_{\mathcal{C}}\left(\rho^{\mathrm{ra}}\left(\mathbb{S}_{\mathcal{M}} \circ F\right), X^{* *}\right) \\
& \cong \operatorname{Hom}_{\mathcal{C}}\left(\rho^{\operatorname{la}}\left(\mathbb{N}_{\mathcal{M}}^{-1} \circ \mathbb{S}_{\mathcal{M}} \circ F \circ \mathbb{N}_{\mathcal{M}}^{-1}\right), \mathbb{N}_{\mathcal{C}}^{-1}\left(X^{* *}\right)\right) \\
& \cong \operatorname{Nat}\left(\mathbb{N}_{\mathcal{M}}^{-1} \circ \mathbb{S}_{\mathcal{M}} \circ F \circ \mathbb{N}_{\mathcal{M}}^{-1}, \mathbb{N}_{\mathcal{C}}^{-1}\left(X^{* *}\right) \otimes \operatorname{id}{ }_{\mathcal{M}}\right) \\
& \cong \operatorname{Nat}\left(\mathbb{N}_{\mathcal{M}}^{-1} \circ \mathbb{S}_{\mathcal{M}} \circ F, \mathbb{N}_{\mathcal{C}}^{-1}\left(X^{* *}\right) \otimes \mathbb{N}_{\mathcal{M}}\right) \\
& \cong \operatorname{Nat}\left(D \otimes F, D \otimes X \otimes \mathbb{N}_{\mathcal{M}}\right) \cong \operatorname{Nat}\left(F, X \otimes \mathbb{N}_{\mathcal{M}}\right)
\end{aligned}
$$

for $F \in \operatorname{Rex}(\mathcal{M})$ and $X \in \mathcal{C}$. The proof is done.
The following theorem is an immediate consequence of the above lemma.
Theorem 6.9. If $\mathcal{M}$ is an exact $\mathcal{C}$-module category whose relative Serre functor is isomorphic to the identity functor, then there is a natural isomorphism

$$
\operatorname{Hom}_{\mathcal{C}}\left(\rho^{\mathrm{ra}}(F), X^{* *}\right) \cong \operatorname{Nat}\left(F, X \otimes \mathbb{N}_{\mathcal{M}}\right)
$$

for $F \in \operatorname{Rex}(\mathcal{M})$ and $X \in \mathcal{C}$. In particular, we have an isomorphism

$$
\operatorname{CF}(\mathcal{M}) \cong \operatorname{SLF}(\mathcal{M})
$$

which restricts to isomorphisms

$$
\operatorname{CF}_{n}(\mathcal{M}) \cong \operatorname{SLF}_{n}(\mathcal{M}) \quad\left(n \in \mathbb{Z}_{+}\right)
$$

6.5. Dimension of $\mathrm{CF}_{1}$. For a finite abelian category $\mathcal{A}$, we denote by $\operatorname{Irr}(\mathcal{A})$ the set of isomorphism classes of simple objects of $\mathcal{M}$. Let $\mathcal{C}$ be a finite tensor category, and let $\mathcal{M}$ be an exact $\mathcal{C}$-module category. Then, by the proof of Theorem 5.11 we have isomorphisms

$$
\begin{equation*}
\mathrm{CF}_{1}(\mathcal{M}) \cong \bigoplus_{L \in \operatorname{Irr}(\mathcal{M})} \operatorname{Hom}_{\mathcal{C}}(\underline{\operatorname{Hom}}(L, L), \mathbb{1}) \cong \bigoplus_{L \in \operatorname{Irr}(\mathcal{M})} \operatorname{Hom}_{\mathcal{C}}\left(L, \mathbb{S}_{\mathcal{M}}(L)\right) \tag{6.17}
\end{equation*}
$$

Thus, by Schur's lemma, we have

$$
\operatorname{dim}_{k} \operatorname{CF}_{1}(\mathcal{M})=\#\left\{L \in \operatorname{Irr}(\mathcal{C}) \mid \mathbb{S}_{\mathcal{M}}(L) \cong L\right\}
$$

We suppose, moreover, that $\mathcal{C}$ is a pivotal finite tensor category and $\mathcal{M}$ is a pivotal $\mathcal{C}$-module category with pivotal structure $j^{\prime}$. Again by the proof of Theorem 5.11, the internal character of $L \in \operatorname{Irr}(\mathcal{M})$ corresponds to $j_{L}^{\prime}$ via (6.17). Thus the set $\{\operatorname{ch}(L) \mid L \in \operatorname{Irr}(\mathcal{M})\}$ of 'irreducible characters' is a basis of $\mathrm{CF}_{1}(\mathcal{M})$.
6.6. Dimension of $\mathrm{CF}_{2}$. As we have seen in the above, the dimension of $\mathrm{CF}_{1}$ is expressed in representation-theoretic terms. It is interesting to give such an expression for the dimension of $\mathrm{CF}_{n}$ for $n \geq 2$. Here we give the following result:

Theorem 6.10. Let $\mathcal{C}$ be a finite tensor category. For an exact $\mathcal{C}$-module category $\mathcal{M}$ such that $\mathbb{S}_{\mathcal{M}} \cong \mathrm{id}_{\mathcal{M}}$, there is an isomorphism

$$
\mathrm{CF}_{2}(\mathcal{M})=\mathrm{CF}_{1}(\mathcal{M}) \oplus \bigoplus_{L \in \operatorname{Irr}(\mathcal{M})} \operatorname{Ext}_{\mathcal{M}}^{1}(L, L)
$$

To prove Theorem 6.10, we recall the following expression of Ext ${ }^{1}$ : Let $A$ be a finite-dimensional algebra. Given $X \in A$-mod, we denote by $g_{X}: A \rightarrow \operatorname{End}_{k}(X)$ the algebra map induced by the action of $A$ on $X$. For $V, W \in A$-mod, the vector space $\operatorname{Ext}_{A}^{1}(V, W)$ is identified with the set of equivalence classes of short exact
sequences of the form $0 \rightarrow W \rightarrow X \rightarrow V \rightarrow 0$ in $A$-mod. If $X \in A$-mod fits into such an exact sequence, then we may assume that $X=V \oplus W$ as a vector space and the algebra map $g_{X}$ is given by

$$
g_{X}(a)=\left(\begin{array}{cc}
g_{V}(a) & 0 \\
\xi(a) & g_{W}(a)
\end{array}\right) \in \operatorname{End}_{k}(X) \quad(a \in A)
$$

for some $\xi \in \operatorname{Hom}_{k}\left(A, \operatorname{Hom}_{k}(V, W)\right)$. Since $g_{X}$ is an algebra map, we have

$$
\begin{equation*}
\xi(1)=0 \quad \text { and } \quad \xi(a b)=\xi(a) \circ g_{V}(b)+g_{W}(a) \circ \xi(b) \quad(a, b \in A) \tag{6.18}
\end{equation*}
$$

We define $\partial: \operatorname{Hom}_{k}(V, W) \rightarrow \operatorname{Hom}_{k}\left(A, \operatorname{Hom}_{k}(V, W)\right)$ by

$$
\partial(f)(a)=f \circ g_{V}(a)-g_{W}(a) \circ f \quad\left(f \in \operatorname{Hom}_{k}(V, W), a \in A\right)
$$

For two linear maps $\xi_{i}: A \rightarrow \operatorname{Hom}_{k}(V, W)(i=1,2)$ satisfying (6.18), the corresponding short exact sequences are equivalent if and only if $\xi_{1}-\xi_{2} \in \operatorname{Im}(\partial)$. Thus the vector space $\operatorname{Ext}_{A}^{1}(V, W)$ is identified with

$$
\begin{equation*}
\mathcal{E}_{A}^{1}(V, W):=\left\{\xi \in \operatorname{Hom}_{k}\left(A, \operatorname{Hom}_{k}(V, W)\right) \text { satisfying (6.18) }\right\} / \operatorname{Im}(\partial) \tag{6.19}
\end{equation*}
$$

Theorem 6.10 is in fact an immediate consequence of Theorem 6.9 and the following theorem:

Theorem 6.11. For $M \in A$-mod, we define

$$
\operatorname{Tr}_{A, M}^{*}: \operatorname{Ext}_{A}^{1}(M, M)=\mathcal{E}_{A}^{1}(M, M) \rightarrow \operatorname{SLF}(A), \quad \xi \mapsto \operatorname{Tr} \circ \xi
$$

The map $\operatorname{Tr}_{A, L}^{*}$ is injective for all $L \in \operatorname{Irr}(A)$. Moreover, we have

$$
\operatorname{SLF}_{2}(A)=\operatorname{SLF}_{1}(A) \oplus \bigoplus_{L \in \operatorname{Irr}(A)} \operatorname{Im}\left(\operatorname{Tr}_{A, L}^{*}\right)
$$

Remark 6.12. Our construction of the map $\operatorname{Tr}_{A, M}^{*}$ is inspired from the construction of pseudo-trace functions introduced by Miyamoto Miy04 and further studied in Ari10b, Ari10a, AN13] in relation with conformal field theory and vertex operator algebras.

Remark 6.13. Theorem 6.11 is inspired by the following Okuyama's result: For a symmetric Frobenius algebra $A$, Okuyama [Oku81] showed

$$
\operatorname{dim}_{k} \operatorname{Rey}_{2}(A)=|\operatorname{Irr}(A)|+\sum_{L \in \operatorname{Irr}(A} \operatorname{dim}_{k} \operatorname{Ext}_{A}^{1}(L, L)
$$

(see also Koshitani's review Koshi16, Section 2]). This formula follows from the above theorem and Remark 6.7. We note that Theorem 6.11 does not require $A$ to be a symmetric Frobenius algebra.

We give a proof of Theorem 6.11 Let $A$ be a finite-dimensional algebra, and write $\operatorname{Irr}(A)=\left\{S_{1}, \ldots, S_{m}\right\}$. For each $i=1, \ldots, m$, we fix a primitive idempotent $e_{i} \in A$ such that $A e_{i}$ is a projective cover of $S_{i}$. Set $e=e_{1}+\cdots+e_{m}$. Then $A^{b}:=e A e$ is a basic algebra and the functor

$$
\begin{equation*}
A-\bmod \rightarrow A^{b}-\bmod , \quad X \mapsto e X \tag{6.20}
\end{equation*}
$$

is an equivalence. The following lemma is well-known [NS43, but we give a proof from the viewpoint of Definition 6.6.

Lemma 6.14. The following map is bijective:

$$
\begin{equation*}
\operatorname{SLF}(A) \rightarrow \operatorname{SLF}\left(A^{b}\right),\left.\quad f \mapsto f\right|_{A^{b}} \tag{6.21}
\end{equation*}
$$

Proof. The equivalence (6.20) induces an equivalence $A$-mod- $A \approx A^{b}$-mod- $A^{b}$ send$\operatorname{ing} M \in A$-mod- $A$ to $e M e$. By (6.16) and this equivalence, we have

$$
\operatorname{SLF}(A) \cong \operatorname{Hom}_{A-\bmod -A}\left(A, A^{*}\right) \cong \operatorname{Hom}_{A^{b}-\bmod -A^{b}}\left(A^{b},\left(A^{b}\right)^{*}\right) \cong \operatorname{SLF}\left(A^{b}\right)
$$

The composition yields the map (6.21).
The equivalence (6.20) also induces an isomorphism

$$
\begin{equation*}
\mathcal{E}_{A}^{1}(V, W)=\operatorname{Ext}_{A}^{1}(V, W) \cong \operatorname{Ext}_{A^{b}}^{1}(e V, e W)=\mathcal{E}_{A^{b}}^{1}(e V, e W) \tag{6.22}
\end{equation*}
$$

for $V, W \in \mathcal{M}$, which sends $\xi \in \mathcal{E}_{A}^{1}(V, W)$ to

$$
\xi^{b}: A^{b} \rightarrow \operatorname{Hom}_{k}(e V, e W), \quad e a e \mapsto e \xi(e a e)(e v) \quad(a \in A, v \in V)
$$

Lemma 6.15. For $V \in A$-mod, the following diagram commutes:


Proof. Let $g: A \rightarrow \operatorname{End}_{k}(V)$ be the algebra map defined by the action of $A$. For $\xi \in \mathcal{E}_{A}^{1}(V, V)$ and $a \in A$, we have $\xi(e)=\xi\left(e^{2}\right)=\xi(e) g(e)+g(e) \xi(e)$ by (6.18). By multiplying $g(e)$ to both sides, we obtain $g(e) \xi(e) g(e)=0$. Again by (6.18),

$$
\begin{aligned}
\operatorname{Tr}(\xi(e a e)) & =\operatorname{Tr}(\xi(e \cdot e a e \cdot e)) \\
& =\operatorname{Tr}(\xi(e) g(e a e) g(e)+g(e) \xi(e a e) g(e)+g(e) g(e a e) \xi(e)) \\
& =\operatorname{Tr}(\xi(e) g(e a e) g(e))+\operatorname{Tr}(g(e) \xi(e a e) g(e))+\operatorname{Tr}(g(e) g(e a e) \xi(e))
\end{aligned}
$$

The first term is zero, since $\operatorname{Tr}(\xi(e) g(e a e) g(e))=\operatorname{Tr}(g(e) \xi(e) g(e) g(e a e))=\operatorname{Tr}(0)=$ 0 . By a similar computation, the third term is also zero. Thus we have

$$
\operatorname{Tr}(\xi(e a e))=\operatorname{Tr}(g(e) \xi(e a e) g(e))=\operatorname{Tr}\left(\xi^{b}(e a e)\right)
$$

This means that the diagram in question commutes.
Proof of Theorem 6.11. Let $A$ be a finite-dimensional algebra, and let $J$ be the Jacobson radical of $A$. Since the $n$-th power of the Jacobson radical of $A^{b}$ is $e J^{n} e$, we see that (6.21) restricts to isomorphisms

$$
\begin{equation*}
\operatorname{SLF}_{n}(A) \rightarrow \operatorname{SLF}_{n}\left(A^{b}\right),\left.\quad f \mapsto f\right|_{A^{b}} \quad\left(n \in \mathbb{Z}_{+}\right) \tag{6.23}
\end{equation*}
$$

Thus, by Lemma 6.15, it is sufficient to consider the case where $A$ is basic.
We assume that $A$ is basic. Then $C:=A^{*}$ is a pointed coalgebra. Let $\Delta$ and $\varepsilon$ denote the comultiplication and the counit of $C$, respectively. We note that the set $\operatorname{Irr}(A)$ is identified with the set

$$
G(C):=\{c \in C \mid \Delta(c)=c \otimes c \text { and } \varepsilon(c)=1\}
$$

of grouplike elements of $C$. Let $g, h \in G(C)$. By (6.19), the vector space $\operatorname{Ext}_{A}^{1}(g, h)$ is identified with the space of $(g, h)$-skew-primitive elements

$$
P_{g, h}:=\{x \in C \mid \Delta(x)=x \otimes g+h \otimes x\},
$$

and the map $\operatorname{Tr}_{A, g}^{*}$ is just the inclusion map $P_{g, g} \rightarrow C$. Thus, to prove this theorem, it is enough to show the following equation:

$$
\begin{equation*}
\operatorname{SLF}_{2}(A)=C_{0} \oplus \bigoplus_{g \in G(C)} P_{g, g} \tag{6.24}
\end{equation*}
$$

Let $J$ be the Jacobson radical of $A$. By Mon93, Proposition 5.2.9], the coradical filtration $\left\{C_{n}\right\}_{n \geq 0}$ of $C$ is given by $C_{n}=\left(A / J^{n+1}\right)^{*}$. Thus,

$$
\begin{equation*}
\operatorname{SLF}_{n}(A)=C_{n-1} \cap \operatorname{SLF}(A) \quad\left(n \in \mathbb{Z}_{+}\right) \tag{6.25}
\end{equation*}
$$

For each $g, h \in G(C)$, we choose a subspace $P_{g, h}^{\prime}$ of $P_{g, h}$ such that $P_{g, h}=P_{g, h}^{\prime} \oplus$ $k(g-h)$. The Taft-Wilson theorem [Mon93, Theorem 5.4.1] states

$$
\begin{equation*}
C_{1}=C_{0} \oplus \bigoplus_{g, h \in G(C)} P_{g, h}^{\prime} \tag{6.26}
\end{equation*}
$$

Thus $C_{1} \otimes C_{1}$ is decomposed as follows:

$$
\begin{align*}
C_{1} \otimes C_{1} & =\left(C_{0} \otimes C_{0}\right) \\
& \oplus \bigoplus_{f, g, h \in G(C)}\left(f \otimes P_{g, h}^{\prime}\right) \oplus\left(P_{g, h}^{\prime} \otimes f\right) \oplus \bigoplus_{e, f, g, h \in G(C)} P_{e, f}^{\prime} \otimes P_{g, h}^{\prime} \tag{6.27}
\end{align*}
$$

Let $x \in \mathrm{SLF}_{2}(A)$. By (6.25) and (6.26), we have $x=x_{0}+\sum_{g, h \in G(C)} x_{g, h}$ for some $x_{0} \in C_{0}$ and $x_{g, h} \in P_{g, h}^{\prime}$. Let $\pi_{f, g, h}$ be the projection to $f \otimes P_{g, h}^{\prime}$ along the direct sum decomposition (6.27). Since $x \in \operatorname{SLF}(A)$, we have

$$
\delta_{f, h} \cdot h \otimes x_{g, h}=\pi_{f, g, h} \Delta(x)=\pi_{f, g, h} \Delta^{\mathrm{cop}}(x)=\delta_{f, g} \cdot g \otimes x_{g, h}
$$

This implies that $x_{g, h}=0$ unless $g=h$. Hence,

$$
\operatorname{SLF}_{2}(A) \subset C_{0} \oplus \bigoplus_{g \in G} P_{g, g}^{\prime}=C_{0} \oplus \bigoplus_{g \in G} P_{g, g}
$$

Thus the left-hand side of (6.24) is contained in the right-hand side. It is easy to show the converse inclusion. The proof is done.
6.7. Examples. We have no general results on $\mathrm{CF}_{n}$ for $n \geq 3$. Here we give some computational results on $\mathrm{CF}_{n}(\mathcal{C})$ for the case where $\mathcal{C}$ is the category of modules over a finite-dimensional Hopf algebra.

Let $H$ be a finite-dimensional Hopf algebra with comultiplication $\Delta$, counit $\varepsilon$ and antipode $S$. We use the Sweedler notation $\Delta(h)=h_{(1)} \otimes h_{(2)}$ to express the comultiplication of $h \in H$. Set $\mathcal{C}=H$-mod. If we identify $\operatorname{Rex}(\mathcal{C})$ with $H$-mod- $H$, then the action functor $\rho: \mathcal{C} \rightarrow \operatorname{Rex}(\mathcal{C})$ is given by $\rho(X)=X \otimes_{k} H$, where the left and the right action of $H$ on $\rho(X)$ are given by

$$
h \cdot\left(x \otimes h^{\prime}\right)=h_{(1)} x \otimes h_{(2)} h^{\prime} \quad \text { and } \quad\left(x \otimes h^{\prime}\right) \cdot h=x \otimes h^{\prime} h
$$

respectively, for $x \in X$ and $h, h^{\prime} \in H$. A right adjoint of $\rho$ is given as follows: As a vector space, $\rho^{\mathrm{ra}}(M)=M$. The action of $H$ on $\rho^{\mathrm{ra}}(M)$ is given by

$$
h \cdot m=h_{(1)} m S\left(h_{(2)}\right) \quad(h \in H, m \in M) .
$$

Indeed, one can check that the map

$$
\operatorname{Hom}_{H}(\rho(X), M) \rightarrow \operatorname{Hom}_{H-\bmod -H}\left(X, \rho^{\mathrm{ra}}(M)\right), \quad f \mapsto f\left(1_{H} \otimes-\right)
$$

is a natural isomorphism for $X \in H-\bmod$ and $M \in H-\bmod -H$.
In particular, as remarked in Shil7b, the algebra $A_{\mathcal{C}}=\rho^{\mathrm{ra}}\left(\mathrm{id}_{\mathcal{C}}\right) \in \mathcal{C}$ is the adjoint representation of $H$. Thus the space of class functions is given by

$$
\begin{aligned}
\mathrm{CF}(H-\mathrm{mod}) & =\left\{f \in H^{*} \mid f\left(h_{(1)} x S\left(h_{(2)}\right)\right)=\varepsilon(h) f(x) \text { for all } h, x \in H\right\} \\
& =\left\{f \in H^{*} \mid f(a b)=f\left(b S^{2}(a)\right) \text { for all } a, b \in H\right\}
\end{aligned}
$$

By the above description of $\rho^{\text {ra }}$, we also have

$$
\mathrm{CF}_{n}(H-\mathrm{mod})=\left\{f \in \mathrm{CF}(H-\bmod ) \mid f\left(J^{n}\right)=0\right\}
$$

for all positive integer $n$, where $J$ is the Jacobson radical of $H$. As these expressions show, if $S^{2}$ is inner, then there are isomorphisms

$$
\begin{equation*}
\mathrm{CF}(H-\bmod ) \cong \operatorname{SLF}(H) \quad \text { and } \quad \mathrm{CF}_{n}(H-\bmod ) \cong \operatorname{SLF}_{n}(H) \tag{6.28}
\end{equation*}
$$

Example 6.16. Suppose that the base field $k$ is of characteristic $p>0$. We consider the cyclic group $G=\left\langle g \mid g^{p}=1\right\rangle$ of order $p$. It is easy to see that the Jacobson radical of $k G$ is generated by $x:=g-1$. We note that the set $\left\{1, x, \ldots, x^{p-1}\right\}$ is a basis of $k G$. Since the square of the antipode of $k G$ is the identity, we have

$$
\mathrm{CF}_{n}(k G \text {-mod })=\operatorname{SLF}_{n}(k G)=\left\{f \in(k G)^{*} \mid f\left(x^{r}\right)=0 \text { for all } r \geq n\right\}
$$

Hence the dimension of $\mathrm{CF}_{n}:=\mathrm{CF}_{n}(k G$-mod $)$ is given by

$$
\operatorname{dim}_{k} \mathrm{CF}_{n}=n \quad(n=1,2, \ldots, p) \quad \text { and } \quad \operatorname{dim}_{k} \mathrm{CF}_{n}=p \quad(n>p)
$$

A basis of $\mathrm{CF}:=\mathrm{CF}(k G$-mod) can be constructed in the following roundabout but interesting way: There is a matrix representation

$$
\rho: k G \rightarrow \operatorname{Mat}_{p}(k), \quad g \mapsto\left(\begin{array}{cccc}
1 & & & 0 \\
1 & 1 & & \\
& \ddots & \ddots & \\
0 & & 1 & 1
\end{array}\right)
$$

Let $\rho_{i j} \in(k G)^{*}$ be the $(i, j)$-entry of $\rho$. Then the set $\left\{\rho_{n 1}\right\}_{n=1, \ldots, p}$ is a basis of CF such that $\rho_{n 1} \in \mathrm{CF}_{n}$ and $\rho_{n 1} \notin \mathrm{CF}_{n-1}$ for all $n=1, \ldots, p$ (with convention $\mathrm{CF}_{0}=\{0\}$ ).

Example 6.17. Suppose that the base field $k$ is of characteristic zero. Let $p \geq 2$ be an integer, and let $q \in k$ be a primitive $2 p$-th root of unity. The algebra $\bar{U}_{q}:=\bar{U}_{q}\left(\mathfrak{s l}_{2}\right)$ is generated by $E, F$ and $K$ subject to the relations

$$
E^{p}=F^{p}=0, K^{2 p}=1, K E=q^{2} E K, K F=q^{-2} F K,[E, F]=\frac{K-K^{-1}}{q-q^{-1}}
$$

The algebra $\bar{U}_{q}$ has the Hopf algebra structure determined by

$$
\Delta(E)=E \otimes K+1 \otimes E, \quad \Delta(F)=F \otimes 1+K^{-1} \otimes F, \quad \Delta(K)=K \otimes K
$$

The antipode is given by $S(E)=-E K^{-1}, S(F)=-K F$ and $S(K)=K^{-1}$ on the generators. Thus $S^{2}$ is the inner automorphism implemented by $K$. In view of (6.28), we consider $\operatorname{SLF}_{n}\left(\bar{U}_{q}\right)$ instead of $\mathrm{CF}_{n}\left(\bar{U}_{q^{-}}\right.$mod $)$.

An explicit basis of $\operatorname{SLF}\left(\bar{U}_{q}\right)$ is given by Arike Ari10a. We recall his construction: For $\alpha \in\{+,-\}$ and $s \in\{1, \ldots, p\}$, there is an $s$-dimensional simple left $\bar{U}_{q}$-module $\mathcal{X}_{s}^{\alpha}$ (see Ari10a, Subsection 3.4] for notations). The module $\mathcal{X}_{p}^{\alpha}$ is projective. For $s<p$, the module $\mathcal{X}_{s}^{\alpha}$ is not projective. Let $\mathcal{P}_{s}^{\alpha}$ be the projective cover of $\mathcal{X}_{s}^{\alpha}$. Arike Ari10a, Subsection 5.1] showed that $\mathcal{P}_{s}^{\alpha}$ has a matrix presentation of the form

$$
\rho_{s}^{\alpha}: \bar{U}_{q} \rightarrow \operatorname{Mat}_{2 p}(k), \quad \rho_{s}^{\alpha}(x)=\left(\begin{array}{cccc}
g_{s}^{\alpha}(x) & 0 & 0 & 0 \\
a_{s}^{\alpha}(x) & g_{p-s}^{-\alpha}(x) & 0 & 0 \\
b_{s}^{\alpha}(x) & 0 & g_{p-s}^{-\alpha}(x) & 0 \\
h_{s}^{\alpha}(x) & a_{p-s}^{-\alpha}(x) & b_{p-s}^{-\alpha}(x) & g_{s}^{\alpha}(x)
\end{array}\right)
$$

where $g_{s}^{\alpha}: \bar{U}_{q} \rightarrow \operatorname{Mat}_{s}(k)$ is a matrix presentation of $\mathcal{X}_{s}^{\alpha}$ and $a_{s}^{\alpha}, b_{s}^{\alpha}$ and $h_{s}^{\alpha}$ are certain matrix-valued linear functions on $\bar{U}_{q}$ (given by $a_{s}^{+}=A_{p-s, s}, a_{s}^{-}=C_{s, p-s}$, $b_{s}^{+}=B_{p-s, s}, b_{s}^{-}=D_{s, p-s} h_{s}^{+}=H_{s}$ and $h_{s}^{-}=\tilde{H}_{s}$ with Arike's original notation). Now we define linear forms $\chi_{s}^{\alpha}(\alpha \in\{+,-\}, s=1, \ldots, p)$ and $\varphi_{s^{\prime}}\left(s^{\prime}=1, \ldots, p-1\right)$ on $\bar{U}_{q}$ by

$$
\chi_{s}^{\alpha}(x)=\operatorname{Tr}\left(g_{s}^{\alpha}(x)\right) \quad \text { and } \quad \varphi_{s^{\prime}}(x)=\operatorname{Tr}\left(h_{s^{\prime}}^{+}(x)\right)+\operatorname{Tr}\left(h_{p-s^{\prime}}^{-}(x)\right) \quad\left(x \in \bar{U}_{q}\right)
$$

Then the following set is a basis of $\operatorname{SLF}\left(\bar{U}_{q}\right)$ Ari10a, Theorem 5.5]:

$$
\left\{\chi_{s}^{+}, \chi_{s}^{-} \mid s=1, \ldots, p\right\} \cup\left\{\varphi_{s} \mid s=1, \ldots, p-1\right\}
$$

Arike's basis respects the filtration of $\operatorname{SLF}\left(\bar{U}_{q}\right)$. More precisely, we have:

$$
\begin{align*}
& \operatorname{SLF}_{1}\left(\bar{U}_{q}\right)=\operatorname{span}\left\{\chi_{s}^{+}, \chi_{s}^{-} \mid 1 \leq s \leq p\right\}  \tag{6.29}\\
& \operatorname{SLF}_{2}\left(\bar{U}_{q}\right)=\operatorname{SLF}_{1}\left(\bar{U}_{q}\right)  \tag{6.30}\\
& \operatorname{SLF}_{3}\left(\bar{U}_{q}\right)=\operatorname{SLF}_{2}\left(\bar{U}_{q}\right) \oplus \operatorname{span}\left\{\varphi_{s} \mid 1 \leq s \leq p-1\right\} \tag{6.31}
\end{align*}
$$

Indeed, (6.29) follows from the fact that $\operatorname{SLF}_{1}\left(\bar{U}_{q}\right)$ is spanned by the characters of simple modules. Equation (6.30) follows from Theorem 6.11 and the fact that the self-extension vanishes for every simple $\bar{U}_{q}$-module [Sut94, p.379]. To show (6.31), we note $\operatorname{Lw}\left(\bar{U}_{q}\right)=3$ [Sut94, p.367]. Thus we have $\operatorname{SLF}_{n}\left(\bar{U}_{q}\right)=\operatorname{SLF}\left(\bar{U}_{q}\right)$ for $n \geq 3$. This implies (6.31).

## 7. Hochschild (co) homology

7.1. Hochschild (co)homology of a finite abelian category. For an algebra $A$, the Hochschild homology and the Hochschild cohomology of $A$ are defined by

$$
\operatorname{HH}_{\bullet}(A)=\operatorname{Tor}_{\bullet}^{A^{e}}(A, A) \quad \text { and } \quad \operatorname{HH}^{\bullet}(A)=\operatorname{Ext}_{A^{e}}^{\bullet}(A, A),
$$

respectively, where $A^{e}=A \otimes_{k} A^{\mathrm{op}}$. We note that the 0-th Hochschild cohomology $\mathrm{HH}^{0}(A)=\operatorname{Hom}_{A^{e}}(A, A)$ is the center of $A$. It has been known that the modular group $\mathrm{SL}_{2}(\mathbb{Z})$ acts projectively on the center of a ribbon factorizable Hopf algebra [SZ12. Recently, Lentner, Mierach, Schweigert and Sommerhäuser LMSS17] showed that $\mathrm{SL}_{2}(\mathbb{Z})$ also acts projectively on the higher Hochschild cohomology of such a Hopf algebra.

A modular tensor category (in the sense of Kerler-Lyubashenko [KL01]) is a category-theoretical counterpart of a ribbon factorizable Hopf algebra. The aim of this section is to extend the construction of LMSS17] to modular tensor categories. To accomplish this, we first need to discuss what the Hochschild cohomology of a finite abelian category is. Our proposal is the following definition:

Definition 7.1. For a finite abelian category $\mathcal{M}$, we define the Hochschild coho$\operatorname{mology} \mathrm{HH}^{\bullet}(\mathcal{M})$ of $\mathcal{M}$ by $\mathrm{HH}^{\bullet}(\mathcal{M})=\operatorname{Ext}_{\operatorname{Rex}(\mathcal{M})}^{\bullet}\left(\mathrm{id}_{\mathcal{M}}, \mathrm{id}_{\mathcal{M}}\right)$.

If $\mathcal{M} \approx A$ - $\bmod$ for some finite-dimensional algebra $A$, then $\operatorname{Rex}(\mathcal{M})$ is equivalent to $A$-mod $-A$ and the identity functor $\operatorname{id}_{\mathcal{M}} \in \operatorname{Rex}(\mathcal{M})$ corresponds to $A$ via the equivalence. Since a category equivalence preserves Ext ${ }^{\bullet}$, we have

$$
\mathrm{HH}^{\bullet}(\mathcal{M})=\operatorname{Ext}_{\operatorname{Rex}(\mathcal{M})}^{\bullet}\left(\operatorname{id}_{\mathcal{M}}, \operatorname{id}_{\mathcal{M}}\right) \cong \operatorname{Ext}_{A-\text { mod }-A}^{\bullet}(A, A)=\mathrm{HH}^{\bullet}(A)
$$

which justifies the definition.

Although it is not directly related to our main purpose of this section, it is also interesting to give a definition of the Hochschild homology of a finite abelian category. Our proposal is:
Definition 7.2. For a finite abelian category $\mathcal{M}$, we define the Hochschild homology $\mathrm{HH}_{\bullet}(\mathcal{M})$ of $\mathcal{M}$ by $\mathrm{HH}_{\bullet}(\mathcal{M})=\operatorname{Ext}_{\operatorname{Rex}(\mathcal{M})}^{\bullet}\left(\operatorname{id}_{\mathcal{M}}, \mathbb{N}_{\mathcal{M}}\right)^{*}$, where $\mathbb{N}_{\mathcal{M}}$ is the Nakayama functor on $\mathcal{M}$.

This definition is justified as follows: If $M \leftarrow P_{0} \leftarrow P_{1} \leftarrow \cdots$ is a projective resolution of $M \in A$-mod- $A$, then $\operatorname{Tor}_{\bullet}^{A^{e}}(A, M)$ is the homology of

$$
\begin{equation*}
0 \leftarrow A \otimes_{A^{e}} P_{0} \leftarrow A \otimes_{A^{e}} P_{1} \leftarrow \cdots \tag{7.1}
\end{equation*}
$$

By the tensor-hom adjunction, the dual of this chain complex is:

$$
\left(A \otimes_{A^{e}} P_{\bullet}\right)^{*}=\operatorname{Hom}_{k}\left(A \otimes_{A^{e}} P_{\bullet}, k\right) \cong \operatorname{Hom}_{A^{e}}\left(P_{\bullet}, \operatorname{Hom}_{k}(A, k)\right)=\operatorname{Hom}_{A^{e}}\left(P_{\bullet}, A^{*}\right)
$$

Thus we have $\operatorname{Tor}_{\bullet}^{A^{e}}(A, M)^{*} \cong \operatorname{Ext}_{A^{e}}^{\bullet}\left(M, A^{*}\right)$ by taking the cohomology of the dual of (7.1). If $\mathcal{M}=A$-mod, then $A$ and $A^{*}$ corresponds to $\operatorname{id}_{\mathcal{M}}$ and $\mathbb{N}_{\mathcal{M}}$, respectively, via the equivalence $A$ - $\bmod -A \approx \operatorname{Rex}(\mathcal{M})$. Hence we have

$$
\operatorname{HH}_{\bullet}(\mathcal{M})=\operatorname{Ext}_{\operatorname{Rex}(\mathcal{M})}^{\bullet}\left(\operatorname{id}_{\mathcal{M}}, \mathbb{N}_{\mathcal{M}}\right)^{*} \cong \operatorname{Ext}_{A^{e}}^{\bullet}\left(A, A^{*}\right)^{*} \cong \operatorname{Tor}_{\bullet}^{A^{e}}(A, A)=\operatorname{HH} \cdot(A)
$$

7.2. Formulas of $\mathrm{HH}^{\bullet}$ and HH . by the adjoint algebra. Let $H$ be a finitedimensional Hopf algebra, and let $A$ be the adjoint representation of $H$. It is known that there is an isomorphism

$$
\begin{equation*}
\operatorname{Ext}_{H}^{\bullet}(k, A) \cong \mathrm{HH}^{\bullet}(H) \tag{7.2}
\end{equation*}
$$

where $k$ is the trivial $H$-module. We now generalize this result to exact module categories. Let $\mathcal{C}$ be a finite tensor category, and let $\mathcal{M}$ be a finite left $\mathcal{C}$-module category over $\mathcal{C}$ with action functor $\rho: \mathcal{C} \rightarrow \operatorname{Rex}(\mathcal{M})$.

Theorem 7.3. If $\mathcal{M}$ is exact, then there is a natural isomorphism

$$
\operatorname{Ext}_{\mathcal{C}}^{\bullet}\left(V, \rho^{\mathrm{ra}}(F)\right) \cong \operatorname{Ext}_{\operatorname{Rex}(\mathcal{M})}^{\bullet}(\rho(V), F)
$$

for $V \in \mathcal{C}, F \in \operatorname{Rex}(\mathcal{M})$.
Proof. We set $\mathcal{E}=\operatorname{Rex}(\mathcal{M})$ for simplicity. Let $V \leftarrow P^{0} \leftarrow P^{1} \leftarrow \cdots$ be a projective resolution of $V$ in $\mathcal{C}$. By applying $\operatorname{Hom}_{\mathcal{E}}\left(-, \rho^{\mathrm{ra}}(F)\right)$ to this resolution, we have the following commutative diagram:


By Lemmas 3.1 and 3.2, the sequence $0 \leftarrow \rho(V) \leftarrow \rho\left(P^{0}\right) \leftarrow \rho\left(P^{1}\right) \leftarrow \cdots$ is a projective resolution of $\rho(V)$. Now the claim is proved by taking the cohomology of the rows of the above commutative diagram.

Theorem 7.4. If $\mathcal{M}$ is an exact $\mathcal{C}$-module category and the relative Serre functor of $\mathcal{M}$ is isomorphic to $\mathrm{id}_{\mathcal{M}}$, then there is a natural isomorphism

$$
\operatorname{Ext}_{\mathcal{C}}^{\bullet}\left(\rho^{\mathrm{ra}}(F), V^{* *}\right) \cong \operatorname{Ext}_{\operatorname{Rex}(\mathcal{M})}^{\bullet}\left(F, V \otimes \mathbb{N}_{\mathcal{M}}\right)
$$

for $V \in \mathcal{C}$ and $F \in \operatorname{Rex}(\mathcal{M})$

Proof. We set $\mathcal{E}=\operatorname{Rex}(\mathcal{M})$ for simplicity. Let $F \leftarrow P^{0} \leftarrow P^{1} \leftarrow \cdots$ be a projective resolution of $F$ in $\mathcal{E}$. By applying $\operatorname{Hom}_{\mathcal{C}}\left(-, V \otimes \mathbb{N}_{\mathcal{M}}\right)$ to this resolution and using Lemma 6.8 we obtain the following commutative diagram:


The claim is proved by taking the cohomology of the rows of this commutative diagram.

Specializing the above theorems, we obtain:
Corollary 7.5. For an exact left $\mathcal{C}$-module category $\mathcal{M}$, we have

$$
\begin{equation*}
\operatorname{HH}^{\bullet}(\mathcal{M}) \cong \operatorname{Ext}_{\mathcal{C}}^{\bullet}\left(\mathbb{1}, A_{\mathcal{M}}\right) \tag{7.3}
\end{equation*}
$$

If $\mathbb{S}_{\mathcal{M}} \cong \mathrm{id}_{\mathcal{M}}$, then we also have an isomorphism

$$
\begin{equation*}
\operatorname{HH}_{\bullet}(\mathcal{M}) \cong \operatorname{Ext}_{\mathcal{C}}^{\bullet}\left(A_{\mathcal{M}}, \mathbb{1}\right)^{*} \tag{7.4}
\end{equation*}
$$

We consider the case where $\mathcal{C}=H$-mod for some finite-dimensional Hopf algebra $H$ and $\mathcal{M}=\mathcal{C}$. Let $A$ be the adjoint representation of $H$. Since $A_{\mathcal{M}} \cong A$ in this case, the isomorphism (7.3) specializes to (7.2) in this case. Since the relative Serre functor of $\mathcal{M}$ is the double dual functor, $\mathbb{S}_{\mathcal{M}} \cong \mathrm{id}_{\mathcal{M}}$ if and only if the square of the antipode of $H$ is inner. If this is the case, then we have an isomorphism HH. $(H) \cong \operatorname{Ext}_{H}^{\bullet}(A, k)^{*}$ by (7.4).
7.3. Modular group action on the Hochschild cohomology. Let $\mathcal{C}$ be a ribbon finite tensor category with braiding $\sigma$ and twist $\theta$. Then the coend $L:=$ $\int^{X \in \mathcal{C}} X^{*} \otimes X$ has a natural structure of a Hopf algebra in $\mathcal{C}$. We note that the algebra $A:=\left(A_{\mathcal{C}}, m_{\mathcal{C}}, u_{\mathcal{C}}\right)$ is dual to the coalgebra $L$, and thus $A$ is also a Hopf algebra (see 4.7) for the definition of $m_{\mathcal{C}}$ and $u_{\mathcal{C}}$ ). By using the universal property, we define $Q: \mathbb{1} \rightarrow A \otimes A$ to be the unique morphism such that the equation

$$
\left(\pi_{\mathcal{C}}(X) \otimes \pi_{\mathcal{C}}(Y)\right) \circ Q=\left(\operatorname{id}_{X} \otimes \sigma_{Y, X^{*}} \sigma_{X^{*}, Y} \otimes \operatorname{id}_{Y^{*}}\right) \circ\left(\operatorname{coev}_{X} \otimes \operatorname{coev}_{Y}\right)
$$

holds for all $X, Y \in \mathcal{C}$. The morphism $Q$ is dual to the Hopf pairing $\omega: L \otimes L \rightarrow \mathbb{1}$ used in Lyu95a, Lyu95b, Lyu95c to define the modularity of $\mathcal{C}$. Thus we say that $\mathcal{C}$ is a modular tensor category if there is a morphism $e: A \otimes A \rightarrow \mathbb{1}$ such that $(A, e, Q)$ is a left dual object of $A$ KL01, Definition 5.2.7].

Now we suppose that $\mathcal{C}$ is a modular tensor category. Then the Hopf algebra $A$ has a morphism $\lambda: A \rightarrow \mathbb{1}$, unique up to sign, such that

$$
\begin{equation*}
\left(\operatorname{id}_{A} \otimes \lambda\right) \circ \underline{\Delta}=u_{\mathcal{C}} \circ \lambda=\left(\lambda \otimes \operatorname{id}_{A}\right) \circ \underline{\Delta} \quad \text { and } \quad(\lambda \otimes \lambda) \circ Q=\mathrm{id}_{\mathbb{1}} \tag{7.5}
\end{equation*}
$$

where $\underline{\Delta}$ is the comultiplication of $A$. We fix such a morphism $\lambda$ and then define two morphisms $\mathfrak{S}, \mathfrak{T}: A \rightarrow A$ by

$$
\mathfrak{S}=\left(\lambda \otimes \operatorname{id}_{A}\right) \circ\left(m_{\mathcal{C}} \otimes \operatorname{id}_{A}\right) \circ\left(\operatorname{id}_{A} \otimes Q\right) \quad \text { and } \quad \mathfrak{T}=\int_{X \in \mathcal{C}} \theta_{X} \otimes \operatorname{id}_{X^{*}}
$$

The morphisms $\mathfrak{S}$ and $\mathfrak{T}$ are the dual of the morphisms $S$ and $T$, respectively, given in Lyu95a, Definition 6.3]. Thus $\mathfrak{S}$ and $\mathfrak{T}$ are invertible and there is an element $c \in k^{\times}$such that the following 'modular relation' hold:

$$
\begin{equation*}
(\mathfrak{S T})^{3}=c \cdot \mathfrak{S}^{2} \quad \text { and } \quad \mathfrak{S}^{4}=\theta_{A}^{-1} \tag{7.6}
\end{equation*}
$$

Theorem 7.6. With the above notation, we set

$$
\widetilde{\mathfrak{S}}=\operatorname{Ext}_{\mathcal{C}}^{\bullet}(\mathbb{1}, \mathfrak{S}) \quad \text { and } \quad \widetilde{\mathfrak{T}}=\operatorname{Ext}_{\mathcal{C}}^{\bullet}(\mathbb{1}, \mathfrak{T})
$$

Then we have a well-defined projective representation

$$
\mathrm{SL}_{2}(\mathbb{Z}) \rightarrow \operatorname{PGL}\left(\operatorname{HH}^{\bullet}(\mathcal{C})\right), \quad\left(\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right) \mapsto \widetilde{\mathfrak{S}}, \quad\left(\begin{array}{ll}
1 & 1 \\
0 & 1
\end{array}\right) \mapsto \widetilde{\mathfrak{T}}
$$

Proof. It is enough to show that $\widetilde{\mathfrak{S}}$ and $\widetilde{\mathfrak{T}}$ satisfy $(\widetilde{\mathfrak{S}} \widetilde{\mathfrak{T}})^{3}=\widetilde{\mathfrak{S}}^{2}$ and $\widetilde{\mathfrak{S}}^{4}=$ id up to scalar multiple. By the funtorial property of Ext and (7.6), we have

$$
(\widetilde{\mathfrak{S}} \widetilde{\mathfrak{T}})^{3}=c \cdot \widetilde{\mathfrak{S}}^{2} \quad \text { and } \quad \widetilde{\mathfrak{S}}^{4}=\operatorname{Ext}_{\mathcal{C}}^{\bullet}\left(\mathbb{1}, \theta_{A}^{-1}\right)=\operatorname{Ext}_{\mathcal{C}}^{\bullet}\left(\theta_{\mathbb{1}}^{-1}, A\right)=\mathrm{id}
$$

Let $H$ be a finite-dimensional ribbon Hopf algebra with universal R-matrix $R$ and ribbon element $v$. We consider the case where $\mathcal{C}=H$-mod. Then $A$ is the adjoint representation of $H$. For $X \in H$-mod, the composition

$$
A \otimes X \xrightarrow{\pi_{\mathcal{C}}(X) \otimes \mathrm{id}_{X}} X \otimes X^{*} \otimes X \xrightarrow{\mathrm{id}_{X} \otimes \mathrm{ev}_{X}} X
$$

sends $a \otimes x \in A \otimes X$ to $a x$ [Shi17b, Subsection 3.7]. From this, we see that $\pi_{\mathcal{C}}(X)$ is given as follows: Let $\left\{x_{i}\right\}$ be a basis of $X$, and let $\left\{x^{i}\right\}$ be the dual basis of $\left\{x_{i}\right\}$. With the Einstein notation, we have

$$
\begin{equation*}
\pi_{\mathcal{C}}(X)(a)=a x_{i} \otimes x^{i} \quad(a \in A) \tag{7.7}
\end{equation*}
$$

The morphism $\lambda$ is in fact a suitably normalized left integral on $H$. The morphism $Q$ can be regarded as an element of $A \otimes_{k} A$. For simplicity, we express $R$ and $Q$ as $R=R_{1} \otimes R_{2}$ and $Q=Q_{1} \otimes Q_{2}$, respectively. Then the braiding is given by

$$
\sigma_{X, Y}(x \otimes y)=R_{2} y \otimes R_{1} x \quad(x \in X, y \in Y)
$$

for $X, Y \in H$-mod. Let $\left\{h_{i}\right\}$ and $\left\{h^{i}\right\}$ be a basis of $H$ and the dual basis of $H^{*}$, respectively. Then, by (7.7) and the definition of $Q$, we have

$$
\begin{aligned}
Q_{1} h_{i} \otimes h^{i} \otimes Q_{2} h_{i} \otimes h^{i} & =\left(\pi_{\mathcal{C}}(H) \otimes \pi_{\mathcal{C}}(H)\right)(Q) \\
& =h_{i} \otimes \sigma_{H, H^{*}} \sigma_{H^{*}, H}\left(h^{i} \otimes h_{i}\right) \otimes h^{i} \\
& =h_{i} \otimes R_{2}^{\prime} R_{1} h^{i} \otimes R_{1}^{\prime} R_{2} h_{i} \otimes h^{i} \\
& =S\left(R_{2}^{\prime} R_{1}\right) h_{i} \otimes h^{i} \otimes R_{1}^{\prime} R_{2} h_{i} \otimes h^{i}
\end{aligned}
$$

where $R^{\prime}=R_{1}^{\prime} \otimes R_{2}^{\prime}$ is a copy of $R$. Thus we have $Q=S\left(R_{2}^{\prime} R_{1}\right) \otimes R_{1}^{\prime} R_{2}$. By using the element $Q$, the morphisms $\mathfrak{S}, \mathfrak{T}: A \rightarrow A$ are given by

$$
\mathfrak{S}(a)=\lambda\left(a Q_{1}\right) Q_{2} \quad \text { and } \quad \mathfrak{T}(a)=v a \quad(a \in A)
$$

respectively. Thus our $\mathfrak{S}$ and $\mathfrak{T}$ coincide with those in [LM94, Theorem 4.4]. If we replace $(H, R, v)$ with ( $H^{\text {op,cop }}, R, v$ ), then the morphisms $\mathfrak{S}$ and $\mathfrak{T}$ coincide with the morphisms considered in LMSS17.

In LMSS17, the action $\mathrm{SL}_{2}(\mathbb{Z}) \rightarrow \mathrm{PGL}\left(\mathrm{HH}^{n}(H)\right)$ is defined as follows: First, they extend $\mathfrak{S}$ and $\mathfrak{T}$ to cochain maps $\mathfrak{S}^{\bullet}$ and $\mathfrak{T}^{\bullet}$ of a cochain complex $C_{1}^{\bullet}$ computing the cohomology $\operatorname{Ext}_{H}^{\bullet}(k, A)$. They also established an explicit isomorphism between the complex $C_{1}^{\bullet}$ and the Hochschild complex $C_{2}^{\bullet}$ computing the Hochschild cohomology of $H$. The isomorphism $C_{1}^{\bullet} \cong C_{2}^{\bullet}$ induces (7.2). The projective action of $\mathrm{SL}_{2}(\mathbb{Z})$ on $\mathrm{HH}^{\bullet}(H)$ is then given by $\mathfrak{S}^{\bullet}$ and $\mathfrak{T}^{\bullet}$ through (7.2). By the definition of Ext functor, we see that their action is expressed as in Theorem 7.6. Thus, in conclusion, we have obtained a generalization of LMSS17].

## Appendix A. Computation of structure morphisms of $\rho^{\text {ra }}$

A.1. Bimodule structure of Hom. Let $\mathcal{C}$ be a rigid monoidal category, and let $\mathcal{M}$ be a closed left $\mathcal{C}$-module category in the sense of Subsection 2.4. We establish some results on the natural transformations $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{b}^{\mathfrak{\natural}}$ introduced in that subsection. For simplicity, we write $\underline{\operatorname{Hom}}(M, N)=[M, N]$. We recall that $[M,-]$ is defined to be a right adjoint of the functor $\mathcal{C} \rightarrow \mathcal{M}$ given by $X \mapsto X \otimes M$. As before, we denote by $\underline{\operatorname{coev}}_{(-), M}$ and $\underline{\mathrm{ev}}_{M,(-)}$ the unit and the counit of this adjunction, respectively. Then, by (2.3), we have

$$
\begin{equation*}
\mathfrak{a}_{X, M, N}=\underline{\operatorname{Hom}}\left(\operatorname{id}_{M}, \operatorname{id}_{X} \otimes \underline{\mathrm{ev}}_{M, N}\right) \circ \underline{\operatorname{coev}}_{X \otimes \underline{\operatorname{Hom}}(M, N), M} \tag{A.1}
\end{equation*}
$$

for $X \in \mathcal{C}$ and $M, N \in \mathcal{M}$. By (2.7), we have

$$
\begin{equation*}
\mathfrak{b}_{Y, M, N}=\left(\mathfrak{b}_{Y, M, N}^{\natural} \otimes \operatorname{id}_{Y^{*}}\right) \circ\left(\operatorname{id}_{\underline{\operatorname{Hom}}(M, N)} \otimes \operatorname{coev}_{Y}\right) \tag{A.2}
\end{equation*}
$$

for $Y \in \mathcal{C}$ and $M, N \in \mathcal{M}$, where

$$
\begin{equation*}
\mathfrak{b}_{Y, M, N}^{\natural}=\underline{\operatorname{Hom}}\left(M, \underline{\mathrm{ev}}_{Y \otimes M, N}\right) \circ \underline{\operatorname{coev}}_{\underline{\operatorname{Hom}}(Y \otimes M, N) \otimes Y, M} . \tag{A.3}
\end{equation*}
$$

By the zig-zag identities for the adjunction $(-) \otimes M \dashv \underline{\operatorname{Hom}}(M,-)$, we have

$$
\begin{align*}
\underline{\mathrm{ev}}_{M, X \otimes M} \circ(\underline{\operatorname{coev}} & X, M  \tag{A.4}\\
& \left.\operatorname{id}_{M}\right) \tag{A.5}
\end{align*}=\operatorname{id}_{X \otimes M},
$$

for $X \in \mathcal{C}$ and $N \in \mathcal{M}$. By (A.4), A.5) and the naturality of $\mathrm{ev}_{M,(-)}$, we have

$$
\begin{gather*}
\underline{\mathrm{ev}}_{M, N} \circ\left(\mathfrak{a}_{X, M, N} \otimes \operatorname{id}_{M}\right)=\mathrm{id}_{X} \otimes \underline{\mathrm{ev}}_{M, N},  \tag{A.6}\\
\underline{\mathrm{ev}}_{M, N} \circ\left(\mathfrak{b}_{Y, M, N}^{\natural} \otimes \operatorname{id}_{M}\right)=\underline{\mathrm{ev}}_{Y \otimes M, N} \tag{A.7}
\end{gather*}
$$

for all objects $X, Y \in \mathcal{C}$ and $M, N \in \mathcal{M}$. We now prove:
Lemma A. 1 (= Lemma 2.3). The equations
(2.13)

$$
\begin{gathered}
\mathfrak{b}_{\mathbb{1}, M, N}=\operatorname{id}_{[M, N]}, \\
\mathfrak{b}_{X \otimes Y, M, N}=\left(\mathfrak{b}_{Y, M, N} \otimes \operatorname{id}_{X^{*}}\right) \circ \mathfrak{b}_{X, Y \otimes M, N}, \\
\left(\mathfrak{a}_{X, M, N} \otimes \operatorname{id}_{Y}\right) \circ\left(\operatorname{id}_{X} \otimes \mathfrak{b}_{Y, M, N}\right)=\mathfrak{b}_{Y, M, X \otimes N} \circ \mathfrak{a}_{X, Y \otimes M, N}
\end{gathered}
$$

(2.14)
(2.15)
hold for all objects $X, Y \in \mathcal{C}$ and $M, N \in \mathcal{M}$.
Proof. Equation (2.13) is trivial. By the canonical isomorphisms

$$
\begin{aligned}
\operatorname{Hom}_{\mathcal{C}}([X & \left.\otimes Y \otimes M, N],[M, N] \otimes Y^{*} \otimes X^{*}\right) \\
& \cong \operatorname{Hom}_{\mathcal{C}}([X \otimes Y M, N] \otimes X \otimes Y,[M, N]) \\
& \cong \operatorname{Hom}_{\mathcal{C}}([X \otimes Y M, N] \otimes X \otimes Y \otimes M, N),
\end{aligned}
$$

we see that (2.14) is equivalent to the equation

$$
\underline{\mathrm{ev}}_{M, N} \circ\left(\mathfrak{b}_{X \otimes Y, M, N}^{\natural} \otimes \operatorname{id}_{M}\right)=\underline{\mathrm{ev}}_{M, N} \circ\left(\mathfrak{b}_{Y, M, N}^{\natural} \otimes \operatorname{id}_{M}\right) \circ\left(\mathfrak{b}_{X, Y \otimes M, N}^{\natural} \otimes \mathrm{id}_{Y} \otimes \operatorname{id}_{M}\right) .
$$

By (A.7), the both sides are equal to $\underline{\mathrm{ev}}_{X \otimes Y \otimes M, N}$. Thus (2.14) is verified. In a similar way, we see that (2.15) is equivalent to the equation

$$
\begin{aligned}
\underline{\mathrm{ev}}_{M, X \otimes N} \circ & \left(\mathfrak{a}_{X, M, N} \otimes \operatorname{id}_{M}\right) \circ\left(\operatorname{id}_{X} \otimes \mathfrak{b}_{Y, M, N}^{\natural} \otimes \operatorname{id}_{M}\right) \\
& =\underline{\mathrm{ev}}_{M, X \otimes N} \circ\left(\mathfrak{b}_{Y, M, X \otimes N}^{\natural} \otimes \operatorname{id}_{M}\right) \circ\left(\mathfrak{a}_{X, Y \otimes M, N} \otimes \operatorname{id}_{Y} \otimes \operatorname{id}_{M}\right)
\end{aligned}
$$

By (A.4)-(A.7), the both sides are equal to $\operatorname{id}_{X} \otimes_{\underline{\mathrm{ev}}}^{Y \otimes M, N}{ }^{\text {A. }}$. The proof is done.

In view of this lemma, we have defined the natural isomorphism

$$
\mathfrak{c}_{X, M, N, Y}: X \otimes[M, N] \otimes Y^{*} \rightarrow[Y \otimes M, X \otimes N] \quad(X, Y \in \mathcal{C}, M, N \in \mathcal{M})
$$

by (2.18). The following Lemmas A. 2 and A.3 will be used in later:
Lemma A.2. For all $X \in \mathcal{C}$ and $M \in \mathcal{M}$, the following equation holds:

$$
\begin{equation*}
\underline{\operatorname{coev}}_{\mathbb{1}, X \otimes M}=\mathfrak{c}_{X, M, M, X} \circ\left(\operatorname{id}_{X} \otimes \underline{\operatorname{coev}}_{\mathbb{1}, M} \otimes \operatorname{id}_{X^{*}}\right) \circ \operatorname{coev}_{X} . \tag{A.8}
\end{equation*}
$$

Proof. By the definition of $\mathfrak{c}$, the claim is equivalent to that the equation

$$
\mathfrak{b}_{X, M, X \otimes M}^{\mathfrak{\natural}} \circ\left({\underline{\operatorname{coev}_{1}}}_{\mathbb{1}, X \otimes M} \otimes \operatorname{id}_{X}\right)=\mathfrak{a}_{X, M, M} \circ\left(\mathrm{id}_{X} \otimes{\underline{\operatorname{coev}_{\mathbb{1}}}}_{\mathbb{1}, M}\right)
$$

holds for all $X \in \mathcal{C}$ and $M \in \mathcal{M}$. By (A.4)-(A.7), the both sides correspond to the identity morphism under the canonical isomorphism

$$
\operatorname{Hom}_{\mathcal{C}}(X, \underline{\operatorname{Hom}}(X \otimes M, M)) \cong \operatorname{Hom}_{\mathcal{M}}(X \otimes M, X \otimes M)
$$

Lemma A.3. For all $M_{1}, M_{2}, M_{3} \in \mathcal{M}$, the following diagram commutes:


Proof. By the naturality of $\mathrm{ev}_{M_{1},(-)}$ and (A.6), we compute

$$
\begin{aligned}
& \underline{\mathrm{ev}}_{M_{1}, M_{3}} \circ\left(\left[\mathrm{id}_{M_{1}}, \underline{\mathrm{ev}}_{M_{2}, M_{3}}\right] \mathfrak{a}_{\left[M_{2}, M_{3}\right], M_{1}, M_{3}} \otimes \operatorname{id}_{M_{1}}\right) \\
& \quad=\underline{\mathrm{ev}}_{M_{2}, M_{3}} \circ \underline{\mathrm{ev}}_{M_{1},\left[M_{2}, M_{3}\right] \otimes M_{2}} \circ\left(\mathfrak{a}_{\left[M_{2}, M_{3}\right], M_{1}, M_{3}} \otimes \operatorname{id}_{M_{1}}\right) \\
& \quad=\underline{\mathrm{ev}}_{M_{2}, M_{3}} \circ\left(\operatorname{id}_{\left[M_{2}, M_{3}\right]} \otimes \underline{\mathrm{ev}}_{M_{1}, M_{2}}\right)=\underline{\mathrm{ev}}_{M_{1}, M_{2}, M_{3}}^{(3)} .
\end{aligned}
$$

This shows the commutativity of the left triangle of the diagram. Similarly, by (A.7) and the dinaturality of $\mathrm{ev}_{(-), M_{3}}$, we compute

$$
\begin{aligned}
\underline{\mathrm{ev}}_{M_{1}, M_{3}} \circ & \left(\left(\mathfrak{b}_{\left[M_{1}, M_{2}\right], M_{1}, M_{3}} \circ\left(\left[\underline{\mathrm{ev}}_{M_{2}, M_{1}}, \operatorname{id}_{M_{3}}\right] \otimes \operatorname{id}_{\left[M_{1}, M_{2}\right]}\right)\right) \otimes \operatorname{id}_{M_{1}}\right) \\
& =\underline{\mathrm{ev}}_{\left[M_{1}, M_{2}\right] \otimes M_{1}, M_{3}} \circ\left(\left[\underline{\mathrm{ev}}_{M_{2}, M_{1}}, \operatorname{id}_{M_{3}}\right] \otimes \operatorname{id}_{\left[M_{1}, M_{2}\right]} \otimes \operatorname{id}_{M_{1}}\right) \\
& =\underline{\mathrm{ev}}_{M_{2}, M_{3}} \circ\left(\operatorname{id}_{\left[M_{2}, M_{3}\right]} \otimes \underline{\mathrm{ev}}_{M_{1}, M_{2}}\right)=\underline{\mathrm{ev}}_{M_{1}, M_{2}, M_{3}}^{(3)} .
\end{aligned}
$$

Thus the left triangle of the diagram also commutes.
A.2. Structure morphisms of $\rho^{\text {ra }}$. Now we consider the case where $\mathcal{C}$ is a finite tensor category and $\mathcal{M}$ is a finite left $\mathcal{C}$-module category. To save space, we set $\bar{\rho}=\rho_{\mathcal{M}}^{\mathrm{ra}}$. Let $\xi^{(\ell)}$ and $\xi^{(r)}$ be the left and the right $\mathcal{C}$-module structure of $\bar{\rho}$, respectively. By (2.3) and its right module version, we have

$$
\begin{equation*}
\xi_{X, F}^{(\ell)}=\bar{\rho}\left(\operatorname{id}_{X} \otimes \varepsilon_{F}\right) \circ \eta_{X \otimes \bar{\rho}(F)} \quad \text { and } \quad \xi_{F, X}^{(r)}=\bar{\rho}\left(\varepsilon_{F} \otimes \operatorname{id}_{X}\right) \circ \eta_{\bar{\rho}(F) \otimes X} \tag{A.9}
\end{equation*}
$$

for $F \in \operatorname{Rex}(\mathcal{M})$ and $X \in \mathcal{C}$. By using the universal dinatural transformation $\pi_{F}$ of the end $\bar{\rho}(F)$, these structure morphisms are given as follows:
Lemma A. 4 (= Lemma 3.7). The equations
(3.10)
$\pi_{X \otimes F}(M) \circ \xi_{X, F}^{(\ell)}=\mathfrak{a}_{X, M, F(M)} \circ\left(\operatorname{id}_{X} \otimes \pi_{F}(M)\right)$,
(3.11)

$$
\pi_{F \otimes X}(M) \circ \xi_{X, F}^{(r)}=\mathfrak{b}_{X, M, F(X \otimes M)}^{\natural} \circ\left(\pi_{F}(X \otimes M) \otimes \operatorname{id}_{X}\right)
$$

hold for all $F \in \operatorname{Rex}(\mathcal{M})$ and $X \in \mathcal{C}$.
Proof. Equation (3.10) is proved as follows:

$$
\begin{aligned}
& \pi_{X \otimes F}(M) \circ \xi_{X, F}^{(\ell)} \\
& =\left[\operatorname{id}_{M}, \operatorname{id}_{X} \otimes \varepsilon_{F, M}\right] \circ \pi_{\rho(X \otimes \overline{\mathbf{\rho}}(F))}(M) \circ \eta_{X \otimes \overline{\mathbf{\rho}}(F)} \\
& =\left[\operatorname{id}_{M},\left(\operatorname{id}_{X} \otimes \underline{\mathrm{ev}}_{M, F(M)}\right) \circ\left(\operatorname{id}_{X} \otimes \pi_{F}(M) \otimes \operatorname{id}_{M}\right)\right] \circ \underline{\operatorname{coev}}_{X \otimes \overline{\mathbf{\rho}}(F), M} \\
& =\left[\operatorname{id}_{M}, \operatorname{id}_{X} \otimes \underline{\operatorname{ev}}_{M, F(M)}\right] \circ \underline{\operatorname{coev}} X \otimes[M, F(M)], M \circ\left(\operatorname{id}_{X} \otimes \pi_{F}(M)\right) \\
& =\mathfrak{a}_{X, M, F(M)} \circ\left(\operatorname{id}_{X} \otimes \pi_{F}(M)\right) .
\end{aligned}
$$

Here, the first equality follows from (A.9) and the naturality of $\pi_{(-)}(M)$, the second from (3.4) and (3.8), and the third from the naturality of $\underline{\mathrm{coev}}_{(-), M}$.

To prove equation (3.11), we note that the symbol $\varepsilon_{F} \otimes \operatorname{id}_{X}$ in (A.9) means the natural transformation whose component is given by $\left(\varepsilon_{F} \otimes \operatorname{id}_{X}\right)_{M}=\varepsilon_{F, X \otimes M}$ for $M \in \mathcal{M}$. Thus we compute:

$$
\begin{aligned}
& \pi_{F \otimes X}(M) \circ \xi_{X, F}^{(r)} \\
& =\left[\operatorname{id}_{M}, \varepsilon_{F, X \otimes M}\right] \circ \pi_{\rho(\overline{\boldsymbol{\rho}}(F) \otimes X)}(M) \circ \eta_{\bar{\rho}(F) \otimes X} \\
& =\left[\operatorname{id}_{M}, \underline{\mathrm{ev}}_{X \otimes M, F(X \otimes M)}\right] \circ\left[\operatorname{id}_{M}, \pi_{F}(X \otimes M) \otimes \operatorname{id}_{X \otimes M}\right] \circ \operatorname{coev}_{\overline{\boldsymbol{\rho}}(F) \otimes X, M} \\
& =\left[\operatorname{id}_{M}, \underline{\mathrm{ev}}_{X \otimes M, F(X \otimes M)}\right] \circ \operatorname{coev}_{[X \otimes M, F(X \otimes M)] \otimes X, M} \circ\left(\pi_{F}(X \otimes M) \otimes \operatorname{id}_{X}\right) \\
& =\mathfrak{b}_{X, M, F(X \otimes M)}^{\natural} \circ\left(\pi_{F}(X \otimes M) \otimes \operatorname{id}_{X}\right)
\end{aligned}
$$

in a similar way as above. The proof is done.
We recall that $\rho=\rho_{\mathcal{M}}$ is a strict monoidal functor. Hence its right adjoint $\bar{\rho}$ has a structure of a monoidal functor. We denote the structure morphisms by

$$
\mu_{F, G}^{(2)}: \bar{\rho}(F) \otimes \bar{\rho}(G) \rightarrow \bar{\rho}(F G) \quad \text { and } \quad \mu^{(0)}: \mathbb{1} \rightarrow \bar{\rho}\left(\mathrm{id}_{\mathcal{M}}\right)
$$

for $F, G \in \operatorname{Rex}(\mathcal{M})$. With the use of $\eta$ and $\varepsilon$, they are expressed by

$$
\begin{equation*}
\mu_{F, G}^{(2)}=\bar{\rho}\left(\varepsilon_{F} \circ \varepsilon_{G}\right) \circ \eta_{\bar{\rho}(F) \otimes \bar{\rho}(G)} \quad \text { and } \quad \mu^{(0)}=\eta_{\mathbb{1}}, \tag{A.10}
\end{equation*}
$$

where $\varepsilon_{F} \circ \varepsilon_{G}$ means the tensor product of $\varepsilon_{F}$ and $\varepsilon_{G}$ in $\operatorname{Rex}(\mathcal{M})$, or, equivalently, the horizontal composition of $\varepsilon_{F}$ and $\varepsilon_{G}$.
Lemma A. 5 (= Lemma (3.8). The equations
(3.12)

$$
\pi_{F G}(M) \circ \mu_{F, G}^{(2)}=\underline{\operatorname{comp}}_{M, G(M), F G(M)} \circ\left(\pi_{F}(G(M)) \otimes \pi_{G}(M)\right),
$$

(3.13)

$$
\pi_{\mathrm{id} \mathcal{M}}(M) \circ \mu^{(0)}=\underline{\operatorname{coev}}_{\mathbb{1}, M}
$$

hold for all $F, G \in \operatorname{Rex}(\mathcal{M})$ and $M \in \mathcal{M}$.
Proof. Equation (3.13) follows from (3.8) and (A.10). To prove (3.12), we set

$$
w=\operatorname{ev}_{M, G(M), F G(M)}^{(3)} \circ\left(\pi_{F G}(M) \otimes \pi_{G}(M) \otimes \operatorname{id}_{M}\right) .
$$

We note that there is an isomorphism
(A.11) $\operatorname{Hom}_{\mathcal{C}}(\bar{\rho}(F) \otimes \bar{\rho}(G),[M, F G(M)]) \cong \operatorname{Hom}_{\mathcal{C}}(\bar{\rho}(F) \otimes \bar{\rho}(G) \otimes M, F G(M))$.

The right-hand side of (3.12) corresponds to $w$ via (A.11). On the other hand, the left-hand side of (3.12) corresponds to $\left(\varepsilon_{F} \circ \varepsilon_{G}\right)_{M}$ via (A.11). By (3.4) and the definition of the horizontal composition, we have

$$
\left(\varepsilon_{F} \circ \varepsilon_{G}\right)_{M}=\varepsilon_{F, G(M)} \circ\left(\operatorname{id}_{\bar{\rho}(F)} \otimes \varepsilon_{G, M}\right)=w .
$$

Thus (3.12) is verified. The proof is done.
A.3. Commutativity of $A_{\mathcal{S}}$. For a $\mathcal{C}$-module full subcategory $\mathcal{S} \subset \mathcal{M}$, we have proved that the end $A_{\mathcal{S}}:=\int_{S \in \mathcal{S}} \underline{\operatorname{Hom}}(S, S)$ has the half-braiding $\sigma_{\mathcal{S}}$ given by the commutative diagram (4.11). Namely, the equation

$$
\begin{equation*}
\mathfrak{a}_{X, W, W} \circ\left(\operatorname{id}_{X} \otimes \pi_{\mathcal{S}}(W)\right) \circ \sigma_{\mathcal{S}}(X)=\mathfrak{b}_{X, W, X \otimes W}^{\natural} \circ\left(\pi_{\mathcal{S}}(X \otimes W) \otimes \operatorname{id}_{X}\right) \tag{A.12}
\end{equation*}
$$

holds for all $X \in \mathcal{C}$ and $W \in \mathcal{S}$, where $\pi_{\mathcal{S}}(X): A_{\mathcal{S}} \rightarrow \underline{\operatorname{Hom}}(X, X)$ is the universal dinatural transformation. Let $m_{\mathcal{S}}: A_{\mathcal{S}} \otimes A_{\mathcal{S}} \rightarrow A_{\mathcal{S}}$ be the multiplication of $A_{\mathcal{S}}$, and set $m_{\mathcal{S}}^{\mathrm{op}}=m_{\mathcal{S}} \circ \sigma_{\mathcal{S}}\left(A_{\mathcal{S}}\right)$.
Proof of Theorem 4.9. The claim of this theorem is that $\mathbf{A}_{\mathcal{S}}=\left(A_{\mathcal{S}}, \sigma_{\mathcal{S}}\right)$ is a commutative algebra in $\mathcal{Z}(\mathcal{C})$. Thus it is sufficient to show $m_{\mathcal{S}}^{\mathrm{op}}=m_{\mathcal{S}}$. We fix $W \in \mathcal{S}$ and set $E=[W, W]$ for simplicity. Then we compute

$$
\begin{aligned}
\pi_{\mathcal{S}}(W) & \circ m_{\mathcal{S}}^{\mathrm{op}}=\underline{\operatorname{comp}}_{W, W, W} \circ\left(\pi_{\mathcal{S}}(W) \otimes \pi_{\mathcal{S}}(W)\right) \circ \sigma_{\mathcal{S}}\left(A_{\mathcal{S}}\right) \\
& =\left[\operatorname{id}_{W}, \underline{\mathrm{ev}}_{W, W}\right] \circ \mathfrak{a}_{E, W, W} \circ\left(\pi_{\mathcal{S}}(W) \otimes \operatorname{id}_{X}\right) \circ \sigma_{\mathcal{S}}(E) \circ\left(\operatorname{id}_{A_{\mathcal{S}}} \otimes \pi_{\mathcal{S}}(W)\right) \\
& =\left[\operatorname{id}_{W}, \underline{\mathrm{ev}}_{W, W}\right] \circ \mathfrak{b}_{E, W, E \otimes W}^{\mathrm{\natural}} \circ\left(\pi_{\mathcal{S}}(E \otimes W) \otimes \pi_{\mathcal{S}}(W)\right) \\
& =\mathfrak{b}_{E, W, W}^{\natural} \circ\left(\left[\mathrm{id}_{E \otimes W}, \underline{\mathrm{ev}}_{W, W}\right] \otimes \operatorname{id}_{E}\right) \circ\left(\pi_{\mathcal{S}}(E \otimes W) \otimes \pi_{\mathcal{S}}(W)\right) \\
& =\mathfrak{b}_{E, W, W}^{\natural} \circ\left(\left[\underline{\mathrm{ev}}_{W, W}, \mathrm{id}_{W}\right] \otimes \operatorname{id}_{E}\right) \circ\left(\pi_{\mathcal{S}}(W) \otimes \pi_{\mathcal{S}}(W)\right) \\
& =\underline{\operatorname{comp}}_{W, W, W} \circ\left(\pi_{\mathcal{S}}(W) \otimes \pi_{\mathcal{S}}(W)\right)=\left(\pi_{\mathcal{S}}(W) \otimes \pi_{\mathcal{S}}(W)\right) \circ m_{\mathcal{S}} .
\end{aligned}
$$

Here, the first and the last equalities follow from the definition of $m_{\mathcal{S}}$, the second and the sixth from Lemma A.3, the third from (A.12), the fourth from the naturality of $\mathfrak{b}_{E, W,(-)}^{\natural}$, and the fifth from the dinaturality of $\pi_{\mathcal{S}}$. We now obtain $m_{\mathcal{S}}^{\mathrm{op}}=m_{\mathcal{S}}$ by the universal property of $A_{\mathcal{S}}$. The proof is done.

## Appendix B. On the properties of the pivotal trace

We complement properties of the trace in a pivotal module category. Let $\mathcal{C}$ be a finite tensor category, and let $\mathcal{M}$ be an exact left $\mathcal{C}$-module category. For simplicity, we write $[M, N]=\underline{\operatorname{Hom}}(M, N)$. By the definition of the relative Serre functor, there is a natural isomorphism
(5.2) $\mathfrak{d}_{M, N}:[M, N]^{*} \rightarrow[N, \mathbb{S}(M)] \quad(M, N \in \mathcal{M})$,
where $\mathbb{S}=\mathbb{S}_{\mathcal{M}}$. There is also a natural isomorphism

$$
\text { (5.3) } \quad \zeta_{X, M}: X^{* *} \otimes \mathbb{S}(M) \rightarrow \mathbb{S}(X \otimes M) \quad(X \in \mathcal{C}, M \in \mathcal{M})
$$

such that $\mathfrak{d}$ is an isomorphism of $\mathcal{C}$-bimodule functors from $\mathcal{M}^{\mathrm{op}} \times \mathcal{M}$ to ${ }_{(-)^{* *} \mathcal{C} \text {, }}$ that is, the equations $\zeta_{\mathbb{1}, M}=\mathrm{id}_{\mathbb{S}(M)}$ and

$$
\begin{align*}
{\left[\operatorname{id}_{X \otimes N}, \zeta_{Y, M}\right] \circ } & \mathfrak{d}_{Y \otimes M, X \otimes N} \circ\left(\mathfrak{c}_{X, M, N, Y}^{*}\right)^{-1}  \tag{B.1}\\
& =\mathfrak{c}_{Y^{* *}, N, \mathbb{S}(M), X} \circ\left(\mathrm{id}_{Y^{* *}} \otimes \mathfrak{d}_{M, N} \otimes \operatorname{id}_{X^{*}}\right)
\end{align*}
$$

hold for all objects $X, Y \in \mathcal{C}$ and $M, N \in \mathcal{M}$.
Now we suppose that $\mathcal{C}$ and $\mathcal{M}$ are pivotal with pivotal structures $j$ and $j^{\prime}$, respectively. By Definition 5.6, we have

$$
\begin{equation*}
\zeta_{X, M} \circ j_{X \otimes M}^{\prime}=j_{X} \otimes j_{M}^{\prime} \quad(X \in \mathcal{C}, M \in \mathcal{M}) \tag{B.2}
\end{equation*}
$$

The trace $\operatorname{tr}_{\mathcal{M}}$, defined in Subsection 5.2, is characterized by

$$
\begin{equation*}
\mathfrak{d}_{M, M} \circ \operatorname{tr}_{\mathcal{M}}(M)^{*}=\left[\operatorname{id}_{M}, j_{M}^{\prime}\right] \circ \underline{\operatorname{coev}}_{\mathbb{1}, M} \quad(M \in \mathcal{M}) \tag{B.3}
\end{equation*}
$$

We recall that $\operatorname{tr}_{\mathcal{C}}$ is defined by $\operatorname{tr}_{\mathcal{C}}(X)=\mathrm{ev}_{X^{*}} \circ\left(j_{X} \otimes \mathrm{id}_{X}\right)$ for $X \in \mathcal{C}$. Thus,

$$
\begin{equation*}
\operatorname{tr}_{\mathcal{C}}(X)^{*}=\left(\mathrm{id}_{X^{* *}} \otimes j_{X}^{*}\right) \circ \operatorname{coev}_{X^{* *}}=\left(j_{X} \otimes \operatorname{id}_{X^{*}}\right) \circ \operatorname{coev}_{X} \tag{B.4}
\end{equation*}
$$

Lemma B.1. The trace $\operatorname{tr}_{\mathcal{M}}$ is dinatural, and the equation

$$
\begin{equation*}
\operatorname{tr}_{\mathcal{M}}(X \otimes M) \circ \mathfrak{c}_{X, M, M, X}=\operatorname{tr}_{\mathcal{C}}(X) \circ\left(\operatorname{id}_{X} \otimes \operatorname{tr}_{\mathcal{M}}(M) \otimes \operatorname{id}_{X^{*}}\right) \tag{B.5}
\end{equation*}
$$

holds for all objects $M \in \mathcal{M}$ and $X \in \mathcal{C}$.
Proof. The dinaturality of $\operatorname{tr}_{\mathcal{M}}$ follows from the naturality of $j^{\prime}$ and the dinaturality of $\underline{\operatorname{coev}}_{\mathbb{1},(-)}$. Equation (B.5) is proved as follows:

$$
\begin{aligned}
& \left(\operatorname{id}_{X^{* *}} \otimes \mathfrak{d}_{M, M} \otimes \operatorname{id}_{X^{*}}\right) \circ\left(\operatorname{tr}_{\mathcal{C}}(X) \circ\left(\operatorname{id}_{X} \otimes \operatorname{tr}_{\mathcal{M}}(M) \otimes \operatorname{id}_{X^{*}}\right)\right)^{*} \\
& =\left(\mathrm{id}_{X^{* *}} \otimes\left[\mathrm{id}_{M}, j_{M}^{\prime}\right] \underline{\operatorname{coev}}_{\mathbb{1}, M} \otimes \operatorname{id}_{X^{*}}\right) \circ\left(j_{X} \otimes \mathrm{id}_{X^{*}}\right) \circ \operatorname{coev}_{X} \quad(\mathrm{by}(\mathrm{~B} .3), \text { (B.4) }) \\
& =\left(j_{X} \otimes\left[\operatorname{id}_{M}, j_{M}^{\prime}\right] \otimes \operatorname{id}_{X^{*}}\right) \circ\left(\mathrm{id}_{X} \otimes \underline{\operatorname{coev}}_{\mathbb{1}, M} \otimes \mathrm{id}_{X^{*}}\right) \circ \operatorname{coev}_{X} \\
& =\left(j_{X} \otimes\left[\operatorname{id}_{M}, j_{M}^{\prime}\right] \otimes \operatorname{id}_{X^{*}}\right) \circ \mathfrak{c}_{X, M, M, X}^{-1} \circ \underline{\operatorname{coev}_{X \otimes M} \quad(\text { by (A.8) })} \\
& =\mathfrak{c}_{X^{* *}, M, \mathscr{S}(M), X}^{-1} \circ\left[\operatorname{id}_{X \otimes M}, j_{X} \otimes j_{M}^{\prime}\right] \circ \underline{\operatorname{coev}_{X}}{ }_{\otimes M} \quad \text { (by the naturality of } \mathfrak{c} \text { ) } \\
& =\mathfrak{c}_{X^{* *}, M, \Phi(M), X}^{-1} \circ\left[\operatorname{id}_{X \otimes M}, \zeta_{X, M}\right] \circ\left[\operatorname{id}_{X \otimes M}, j_{X \otimes M}^{\prime}\right] \circ{\underline{\operatorname{coev}_{\mathbb{1}}, X \otimes M}} \quad \text { (by (B.2)) } \\
& =\mathfrak{c}_{X^{* *}, M, \Phi}^{-1}(M), X \circ\left[\mathrm{id}_{X \otimes M}, \zeta_{X, M}\right] \circ \mathfrak{d}_{X \otimes M, X \otimes M} \circ \operatorname{tr}_{\mathcal{M}}(X \otimes M)^{*} \quad(\text { by (B.3) }) \\
& =\left(\operatorname{id}_{X^{* *}} \otimes \mathfrak{d}_{M, M} \otimes \operatorname{id}_{X^{*}}\right) \circ \mathfrak{c}_{X, M, M, X}^{*} \circ \operatorname{tr}_{\mathcal{M}}(X \otimes M)^{*} \quad(\text { by }(\overline{\text { B. } 1)}) \text {. }
\end{aligned}
$$

For a morphism $f: M \rightarrow M$ in $\mathcal{M}$, we have defined $\operatorname{ptr}_{\mathcal{M}}(f) \in k$ of $f$ by

$$
\text { (5.5) } \quad \operatorname{tr}(M) \circ\left[\mathrm{id}_{M}, f\right] \circ{\underline{\operatorname{coev}_{\mathbb{1}}, M}}=\operatorname{ptr}(f) \cdot \mathrm{id}_{\mathbb{1}} .
$$

Proposition B.2. For morphisms $f: M \rightarrow N$ and $g: N \rightarrow M$ in $\mathcal{M}$, we have

$$
\begin{equation*}
\operatorname{ptr}(f g)=\operatorname{ptr}(g f) \tag{B.6}
\end{equation*}
$$

For morphisms $f: M \rightarrow M$ in $\mathcal{M}$ and $a: X \rightarrow X$ in $\mathcal{C}$, we have

$$
\begin{equation*}
\operatorname{ptr}(a \otimes f)=\operatorname{ptr}(a) \cdot \operatorname{ptr}(f) \tag{B.7}
\end{equation*}
$$

Proof. Equation ( $\overline{\mathrm{B} .6}$ ) follows from the dinaturality of $\operatorname{tr}_{\mathcal{M}}$ and $\underline{\operatorname{coev}}_{\mathbb{1},(-)}$. Equation (B.7) follows from (B.5).

For $M \in \mathcal{M}$, we have defined the internal character $\operatorname{ch}_{\mathcal{M}}(M) \in \operatorname{CF}(\mathcal{M})$ by

$$
\begin{equation*}
\operatorname{ch}_{\mathcal{M}}(M)=\operatorname{tr}_{\mathcal{M}}(M) \circ \pi_{\mathcal{M}}(M) \tag{5.6}
\end{equation*}
$$

where $\pi_{\mathcal{M}}: A_{\mathcal{M}} \rightarrow[M, M]$ is the universal dinatural transformation.
Proposition B. 3 (=Lemma 5.9). For all $X \in \mathcal{C}$ and $M \in \mathcal{M}$, we have

$$
\operatorname{ch}_{\mathcal{M}}(X \otimes M)=\operatorname{ch}_{\mathcal{C}}(X) \star \operatorname{ch}_{\mathcal{M}}(M)
$$

where $\star$ is the action (5.1) of $\mathrm{CF}(\mathcal{C})$ on $\operatorname{CF}(\mathcal{M})$.
Proof. We recall that $A_{\mathcal{M}}$ has the Z-coaction $\delta_{\mathcal{M}}: A_{\mathcal{M}} \rightarrow \mathrm{Z}\left(A_{\mathcal{M}}\right)$ induced from the half-braiding of $A_{\mathcal{M}}$. By definition,

$$
\left(\operatorname{id}_{X} \otimes \pi_{\mathcal{M}}(M) \otimes \operatorname{id}_{X^{*}}\right) \circ \pi^{\mathrm{Z}}\left(A_{\mathcal{M}}\right) \circ \delta_{\mathcal{M}}=\mathfrak{c}_{X, M, M, X}^{-1} \circ \pi_{\mathcal{M}}(X \otimes M)
$$

for all objects $X \in \mathcal{C}$ and $M \in \mathcal{M}$. Thus, by Lemma B. 1 we have

$$
\begin{aligned}
& \operatorname{ch}_{\mathcal{C}}(X) \star \operatorname{ch}_{\mathcal{M}}(M)=\operatorname{ch}_{\mathcal{C}}(X) \circ \mathrm{Z}\left(\operatorname{ch}_{\mathcal{M}}(M)\right) \circ \delta_{\mathcal{M}} \\
& \left.=\operatorname{tr}_{\mathcal{C}}(X) \circ\left(\operatorname{id}_{X} \otimes \operatorname{tr}_{\mathcal{M}}(M) \circ \pi_{\mathcal{M}}(M)\right) \otimes \operatorname{id}_{X^{*}}\right) \circ \pi^{\mathrm{Z}}\left(A_{\mathcal{M}}\right) \circ \delta_{\mathcal{M}} \\
& =\operatorname{tr}_{\mathcal{C}}(X) \circ\left(\operatorname{id}_{X} \otimes \operatorname{tr}_{\mathcal{M}}(M) \otimes \operatorname{id}_{X^{*}}\right) \circ \mathfrak{c}_{X, M, M, X}^{-1} \circ \pi_{\mathcal{M}}(X \otimes M) \\
& =\operatorname{tr}_{\mathcal{C}}(X \otimes M) \circ \pi_{\mathcal{M}}(X \otimes M)=\operatorname{ch}_{\mathcal{M}}(X \otimes M)
\end{aligned}
$$

Finally, we give a proof of Lemma 5.10 in the following general form: Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and let $\langle-,-\rangle: \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \rightarrow \mathcal{B}$ be a functor that is additive and exact in each variable. Let $X$ and $Y$ be objects of $\mathcal{B}$. Suppose that there are two dinatural transformations $d(M): X \rightarrow\langle M, M\rangle$ and $e(M):\langle M, M\rangle \rightarrow Y$ $(M \in \mathcal{A})$. For a morphism $f: M \rightarrow M$ in $\mathcal{A}$, we define

$$
t(f)=e(M) \circ\left\langle\operatorname{id}_{M}, f\right\rangle \circ d(M) \in \operatorname{Hom}_{\mathcal{B}}(X, Y)
$$

Proposition B.4. Suppose that

is a commutative diagram in $\mathcal{A}$ with exact rows. Then we have

$$
t\left(f_{2}\right)=t\left(f_{1}\right)+t\left(f_{3}\right)
$$

We note that the internal Hom functor of an exact module category is exact in each variable. Lemma 5.10 is the case where $\mathcal{A}=\mathcal{M}, \mathcal{B}=\mathcal{C},\langle-,-\rangle=[-,-]$, $d=\pi_{\mathcal{M}}, e=\operatorname{tr}_{\mathcal{M}}$ and $f_{i}=\operatorname{id}_{M_{i}}$ for $i=1,2,3$. If we consider $d=\underline{\operatorname{coev}}_{\mathbb{1},(-)}$ instead of $d=\pi_{\mathcal{M}}$, then we obtain the additivity of the pivotal trace with respect to exact sequences.
Proof. By the assumption on $\langle-,-\rangle$, we obtain the following commutative diagram with exact rows and exact columns:


We set $K=\operatorname{Ker}\left(\langle r, s\rangle:\left\langle M_{2}, M_{2}\right\rangle \rightarrow\left\langle M_{1}, M_{3}\right\rangle\right)$. Then we have $K=I_{1}+I_{2}$, where

$$
\begin{aligned}
I_{1} & =\operatorname{Im}\left(\left\langle\mathrm{id}_{M_{2}}, r\right\rangle:\left\langle M_{2}, M_{1}\right\rangle \rightarrow\left\langle M_{2}, M_{2}\right\rangle\right) \\
I_{2} & =\operatorname{Im}\left(\left\langle s, \mathrm{id}_{M_{2}}\right\rangle:\left\langle M_{3}, M_{2}\right\rangle \rightarrow\left\langle M_{2}, M_{2}\right\rangle\right)
\end{aligned}
$$

Moreover, there are morphisms $p_{i}: K \rightarrow\left\langle M_{i}, M_{i}\right\rangle(i=1,3)$ such that

$$
\begin{equation*}
\left\langle\mathrm{id}_{M_{1}}, r\right\rangle \circ p_{1}=\left\langle r, \operatorname{id}_{M_{2}}\right\rangle \quad \text { and } \quad\left\langle s, \mathrm{id}_{M_{3}}\right\rangle \circ p_{3}=\left\langle\operatorname{id}_{M_{2}}, s\right\rangle . \tag{B.8}
\end{equation*}
$$

These claims are checked by chasing the diagram. See GKPM11, Lemma 2.5.1] for the detail of the verification, since the proof up to here is completely same.

For simplicity of notation, we set $d_{i}=\left\langle\operatorname{id}_{M_{i}}, f_{i}\right\rangle \circ d\left(M_{i}\right)$. By $f_{2} s=s f_{3}$ and the dinaturality of $d$, we have

$$
\begin{align*}
& \left\langle\operatorname{id}_{M_{2}}, s\right\rangle \circ d_{2}=\left\langle s, \operatorname{id}_{M_{3}}\right\rangle \circ d_{3},  \tag{B.9}\\
& \left\langle r, \operatorname{id}_{M_{2}}\right\rangle \circ d_{2}=\left\langle\operatorname{id}_{M_{1}}, r\right\rangle \circ d_{3} . \tag{B.10}
\end{align*}
$$

Thus $\langle r, s\rangle \circ d_{2}=\langle r, \mathrm{id}\rangle \circ\langle\mathrm{id}, s\rangle \circ d_{2}=\langle r, \mathrm{id}\rangle \circ\langle s, \mathrm{id}\rangle \circ d_{3}=0$, that is, $\operatorname{Im}\left(d_{2}\right) \subset K$. Hence the following morphism is defined:

$$
\begin{equation*}
\Gamma:=e\left(M_{1}\right) \circ p_{1} \circ d_{2}+e\left(M_{3}\right) \circ p_{3} \circ d_{2} . \tag{B.11}
\end{equation*}
$$

We first show that $\Gamma=t\left(f_{1}\right)+t\left(f_{3}\right)$. By (B.8) and (B.10), we have

$$
\left\langle\operatorname{id}_{M_{1}}, r\right\rangle \circ p_{1} \circ d_{2}=\left\langle r, \operatorname{id}_{M_{2}}\right\rangle \circ d_{2}=\left\langle\operatorname{id}_{M_{1}}, r\right\rangle \circ d_{1} .
$$

Since $r$ is monic, so is $\left\langle\operatorname{id}_{M_{1}}, r\right\rangle$. Thus we have $p_{1} \circ d_{2}=d_{1}$. Therefore the first term of (B.11) is $t\left(f_{1}\right)$. Similarly, we have

$$
\left\langle s, \operatorname{id}_{M_{3}}\right\rangle \circ p_{3} \circ d_{2}=\left\langle\operatorname{id}_{M_{2}}, s\right\rangle \circ d_{2}=\left\langle s, \operatorname{id}_{M_{1}}\right\rangle \circ d_{3}
$$

by (B.8) and (B.9), and thus $p_{3} \circ d_{2}=d_{3}$. From this, we see that the second term of (B.11) is $t\left(f_{3}\right)$. Thus the claim follows.

To complete the proof, we show $\Gamma=t\left(f_{2}\right)$. To see this, we remark

$$
\begin{aligned}
& \left\langle\operatorname{id}_{M_{1}}, r\right\rangle \circ p_{1} \circ\left\langle s, \operatorname{id}_{M_{2}}\right\rangle=\left\langle r, \operatorname{id}_{M_{2}}\right\rangle \circ\left\langle s, \operatorname{id}_{M_{2}}\right\rangle=0, \\
& \left\langle s, \operatorname{id}_{M_{3}}\right\rangle \circ p_{3} \circ\left\langle\operatorname{id}_{M_{2}}, r\right\rangle=\left\langle\operatorname{id}_{M_{2}}, s\right\rangle \circ\left\langle\operatorname{id}_{M_{2}}, r\right\rangle=0
\end{aligned}
$$

by (B.8). Since both $\left\langle\operatorname{id}_{M_{1}}, r\right\rangle$ and $\left\langle s, \operatorname{id}_{M_{3}}\right\rangle$ are monic, $p_{1} \circ\left\langle s, \operatorname{id}_{M_{2}}\right\rangle$ and $p_{3} \circ\left\langle\operatorname{id}_{M_{2}}, r\right\rangle$ are zero morphisms. Set $\Gamma^{\prime}=e\left(M_{1}\right) \circ p_{1}+e\left(M_{3}\right) \circ p_{3}$. We have

$$
\Gamma^{\prime} \circ\left\langle s, \mathrm{id}_{M_{2}}\right\rangle=e\left(M_{2}\right) \circ\left\langle s, \operatorname{id}_{M_{2}}\right\rangle \quad \text { and } \quad \Gamma^{\prime} \circ\left\langle\operatorname{id}_{M_{2}}, r\right\rangle=e\left(M_{2}\right) \circ\left\langle\operatorname{id}_{M_{2}}, r\right\rangle
$$

by the dinaturality of $e$. These equation imply that $\Gamma^{\prime}=e\left(M_{2}\right)$ on $K=I_{1}+I_{2}$. Since $\operatorname{Im}\left(d_{3}\right) \subset K$, we conclude that $\Gamma=t\left(f_{2}\right)$. The proof is done.

## References

[AM07] Nicolás Andruskiewitsch and Juan Martín Mombelli. On module categories over finitedimensional Hopf algebras. J. Algebra, 314(1):383-418, 2007.
[AN13] Yusuke Arike and Kiyokazu Nagatomo. Some remarks on pseudo-trace functions for orbifold models associated with symplectic fermions. Internat. J. Math., 24(2):1350008, 29, 2013.
[Ari10a] Yusuke Arike. A construction of symmetric linear functions on the restricted quantum group $\bar{U}_{q}\left(\mathrm{sl}_{2}\right)$. Osaka J. Math., 47(2):535-557, 2010.
[Ari10b] Yusuke Arike. Some remarks on symmetric linear functions and pseudotrace maps. Proc. Japan Acad. Ser. A Math. Sci., 86(7):119-124, 2010.
[BK01] Bojko Bakalov and Alexander Kirillov, Jr. Lectures on tensor categories and modular functors, volume 21 of University Lecture Series. American Mathematical Society, Providence, RI, 2001.
[BKLT00] Yuri Bespalov, Thomas Kerler, Volodymyr Lyubashenko, and Vladimir Turaev. Integrals for braided Hopf algebras. J. Pure Appl. Algebra, 148(2):113-164, 2000.
[BLV11] Alain Bruguières, Steve Lack, and Alexis Virelizier. Hopf monads on monoidal categories. Adv. Math., 227(2):745-800, 2011.
[BV07] Alain Bruguières and Alexis Virelizier. Hopf monads. Adv. Math., 215(2):679-733, 2007.
[BV12] Alain Bruguières and Alexis Virelizier. Quantum double of Hopf monads and categorical centers. Trans. Amer. Math. Soc., 364(3):1225-1279, 2012.
[CW08] Miriam Cohen and Sara Westreich. Characters and a Verlinde-type formula for symmetric Hopf algebras. J. Algebra, 320(12):4300-4316, 2008.
[DSS13] C. L. Douglas, C. Schommer-Pries, and N. Snyder. Dualizable tensor categories. arXiv:1312.7188 2013.
[DSS14] C. L. Douglas, C. Schommer-Pries, and N. Snyder. The balanced tensor product of module categories. arXiv:1406.4204, 2014.
[EG17] Pavel Etingof and Shlomo Gelaki. Exact sequences of tensor categories with respect to a module category. Adv. Math., 308:1187-1208, 2017.
[EGNO15] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. Tensor categories, volume 205 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2015.
[ENO04] Pavel Etingof, Dmitri Nikshych, and Viktor Ostrik. An analogue of Radford's $S^{4}$ formula for finite tensor categories. Int. Math. Res. Not., (54):2915-2933, 2004.
[ENO05] Pavel Etingof, Dmitri Nikshych, and Viktor Ostrik. On fusion categories. Ann. of Math. (2), 162(2):581-642, 2005.
[EO04] Pavel Etingof and Viktor Ostrik. Finite tensor categories. Mosc. Math. J., 4(3):627654, 782-783, 2004.
[FGR17] V. Farsad, A. M. Gainutdinov, and I. Runkel. SL(2,Z)-action for ribbon quasi-Hopf algebras. arXiv:1702.01086, 2017.
[FSS16] J. Fuchs, G. Schaumann, and C. Schweigert. Eilenberg-Watts calculus for finite categories and a bimodule Radford $S^{4}$ theorem. arXiv:1612.04561, 2016.
[GKPM11] Nathan Geer, Jonathan Kujawa, and Bertrand Patureau-Mirand. Generalized trace and modified dimension functions on ribbon categories. Selecta Math. (N.S.), 17(2):453-504, 2011.
[GR16] A. M. Gainutdinov and I. Runkel. The non-semisimple Verlinde formula and pseudotrace functions. arXiv:1605.04448 2016.
[GR17] A. M. Gainutdinov and I. Runkel. Projective objects and the modified trace in factorisable finite tensor categories. arXiv:1703.00150 2017.
[KL01] Thomas Kerler and Volodymyr V. Lyubashenko. Non-semisimple topological quantum field theories for 3-manifolds with corners, volume 1765 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2001.
[Koshi16] S. Koshitani, Endo-trivial modules for finite groups with dihedral Sylow 2-groups, RIMS Kôkyûroku 2003 (2016) 128-132.
[LM94] Volodimir Lyubashenko and Shahn Majid. Braided groups and quantum Fourier transform. J. Algebra, 166(3):506-528, 1994.
[LMSS17] S. Lentner, S. N. Mierach, C. Schweigert, and Y. Sommerhaeuser. Hochschild Cohomology and the Modular Group. arXiv:1707.04032 2017.
[Lyu95a] V. Lyubashenko. Modular transformations for tensor categories. J. Pure Appl. Algebra, 98(3):279-327, 1995.
[Lyu95b] Volodimir Lyubashenko. Modular properties of ribbon abelian categories. In Proceedings of the 2nd Gauss Symposium. Conference A: Mathematics and Theoretical Physics (Munich, 1993), Sympos. Gaussiana, pages 529-579, Berlin, 1995. de Gruyter.
[Lyu95c] Volodymyr V. Lyubashenko. Invariants of 3-manifolds and projective representations of mapping class groups via quantum groups at roots of unity. Comm. Math. Phys., 172(3):467-516, 1995.
[Miy04] Masahiko Miyamoto. Modular invariance of vertex operator algebras satisfying $C_{2}$ cofiniteness. Duke Math. J., 122(1):51-91, 2004.
[ML98] Saunders Mac Lane. Categories for the working mathematician, volume 5 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1998.
[Mon93] Susan Montgomery. Hopf algebras and their actions on rings, volume 82 of CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1993.
[NS43] C. Nesbitt and W. M. Scott. Some remarks on algebras over an algebraically closed field. Ann. of Math. (2), 44:534-553, 1943.
[Oku81] T. Okuyama, $\operatorname{Ext}^{1}(S, S)$ for a simple $k G$-module $S$, in: Proceedings of the Symposium "Representations of Groups and Rings and Its applications" (in Japanese), December 16-19 1981, Edited by S. Endo. pp.238-249.
[Ost13] V. Ostrik. Pivotal fusion categories of rank 3 (with an Appendix written jointly with Dmitri Nikshych). arXiv:1309.4822, 2013.
[Rad94] David E. Radford. The trace function and Hopf algebras. J. Algebra, 163(3):583-622, 1994.
[Ros95] Alexander L. Rosenberg. Noncommutative algebraic geometry and representations of quantized algebras, volume 330 of Mathematics and its Applications. Kluwer Academic Publishers Group, Dordrecht, 1995.
[Sak17a] T. Sakurai. Central elements of the Jennings basis and certain Morita invariants. arXiv:1701.03799 2017.
[Sak17b] Taro Sakurai. A generalization of dual symmetry and reciprocity for symmetric algebras. J. Algebra, 484:265-274, 2017.
[Sch01] Peter Schauenburg. The monoidal center construction and bimodules. J. Pure Appl. Algebra, 158(2-3):325-346, 2001.
[Sch16] Peter Schauenburg. Computing higher Frobenius-Schur indicators in fusion categories constructed from inclusions of finite groups. Pacific J. Math., 280(1):177-201, 2016.
[Shi16] K. Shimizu. Non-degeneracy conditions for braided finite tensor categories. arXiv:1602.06534 2016.
[Shi17a] K. Shimizu. Ribbon structures of the Drinfeld center. arXiv:1707.09691, 2017.
[Shi17b] Kenichi Shimizu. The monoidal center and the character algebra. J. Pure Appl. Algebra, 221(9):2338-2371, 2017.
[Shi17c] Kenichi Shimizu. On unimodular finite tensor categories. Int. Math. Res. Not. IMRN, (1):277-322, 2017.
[Shi17d] Kenichi Shimizu. The relative modular object and Frobenius extensions of finite Hopf algebras. J. Algebra, 471:75-112, 2017.
[Shi17e] K. Shimizu. Integrals for finite tensor categories arXiv:1702.02425 2017.
[Sut94] Ruedi Suter. Modules over $\mathfrak{U}_{q}\left(\mathfrak{s l}_{2}\right)$. Comm. Math. Phys., 163(2):359-393, 1994.
[SZ12] Yorck Sommerhäuser and Yongchang Zhu. Hopf algebras and congruence subgroups. Mem. Amer. Math. Soc., 219(1028):vi+134, 2012.
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