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Neighbourhood Operators: Additivity, Idempotency and Convergence

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Abstract

We define and discuss the notions of additivity and idempotency for neighbourhood and interior operators. We then propose an order-theoretic description of the notion of convergence that was introduced by D. Holgate and J. Šlapal with the help of these two properties. This will provide a rather convenient setting in which compactness and completeness can be studied via neighbourhood operators. We prove, among other things, a Frolík-type theorem with the introduction of *reflecting morphisms*.

Keywords Neighbourhood operators · Interior operators · Idempotency · Additivity · Kleisli composition · Kan extension · Compactness · Convergence · Filters

Mathematics Subject Classification 18B30 · 54B30 · 54C10 · 18B35

1 Introduction

The theory of categorical neighbourhood operators traces its roots back to the introduction of closure operators on categories equipped with a factorisation system by Dikranjan and Giuli [12]. Several authors went on to develop the theory of closure operators in the following decades.[1] Among the most interesting problems that have arisen is the depiction of epimorphisms in topological categories [11,13,14]. The effort to capture the notion of convergence in the presence of a categorical closure operator led to the introduction of the notion of a neighbourhood in a category. This can be seen in two steps: first neighbourhoods were defined from closure operators as seen in [19], and then as a primitive notion as is done in [24]. Some earlier development on the subject includes [25,32,33], where neighbourhood spaces are studied. However, we would like to mention two independent studies that could arguably be considered as forerunners of the study of neighbourhoods on categories: the work of Á.

[1] Dikranjan and Tholen [15] and Castellini [4] give a detailed account of the topic.

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Császár on syntopogenous structures [10] and that of D. Doitchinov on supertopological spaces [16,17].[2]

Independently, S. Vorster introduced interior operators in [37] with the motivation that, in the absence of complement in the subobject lattices, they will provide an (order-theoretic) dual notion to that of closure operators. There follows a few studies on the structure of interior operators [5–7,29]. It was shown in [22] that interior operations and neighbourhood operations on a given poset relate each other via a Galois connection that reduces to an equivalence between interior operations and so-called left-adjoint neighbourhood operations. This correspondence interacts well with the adjunction that is induced by taking images and pre-images of subobjects and consequently it is easily preserved when we extend the operations to the ambient category, bringing the two theories together.

The present paper resumes the work in [23] and discusses the notions of additivity and idempotency of neighbourhood operators and their eventual applications to convergence. As the presence (or absence) of these two properties more or less affects the preservation of the convergence of filters or/and rasters through various constructions, they will determine two distinct subcategories, one reflective and the other coreflective. Our notion of a neighbourhood operator departs from the previous ones in that it does not rely on the presence of a factorisation system. Thus for each object X, we assign a poset PX on which a neighbourhood operation is defined and then concentrate on Galois correspondences between such posets. When $P = Sub(-)$ is the subobject functor, then we are in the presence of categorical neighbourhood operators in the sense of [24]. As is shown in [23], this formalism presents various advantages. In particular, it allows one to incorporate various examples that are not captured in a framework that is constrained by the presence of a factorisation system. Though we mainly follow the concepts from the available literature on interior and neighbourhood operators, our approach has been largely influenced by the lax method [28] that is being used to describe convergence.

The structure of the paper is as follows. The categorical setting will be discussed in the preliminaries. The notions of additivity and idempotency are presented in Sect. 3 along with their interaction with Galois connections. The coreflective and reflective subcategories that arise from this interaction will be summed up in a diagram and illustrated with an example in Proposition 6. Before we discuss the notion of convergence, we wish to give in Sect. 4 a description of the initial structures with respect to a neighbourhood operator. It is in this section that the close interaction between interior and neighbourhood operations through the introduction of right Kan extension becomes essential. We shall also give a brief equivalence between closed maps with respect to a closure operator and those closed maps with respect to a neighbourhood operator when the subobject lattices are Boolean. As a consequence, the stably closed maps that one obtains in each case are exactly the same. Proper maps, which are precisely the stably closed maps for topological spaces, were initially defined by Bourbaki via ultrafilters [3] and hence, under certain conditions, the result in this section allows one to study convergence in parallel with the topology that is induced by stably closed maps in the sense of [35]. In other words, this allows a discussion of convergence with respect to a closure operator but on a morphism level, or more formally on a slice category. Though this section is independent, its rather modest role is important in providing this link.

The last section shall be devoted to convergence. Continuity can be construed as preservation of convergence. However, for the convergence (of filters or rasters) to be preserved under certain constructions (such as the formation of products), one needs to *reflect* them.

[2] See also [2,26,27,36,38].

We thus define the notion of *reflecting morphisms* and present a few properties about the stability of these morphisms under the formation of pullback and product.

2 Preliminaries

[22] If Q is a partially ordered set (poset), then an *interior operation* on Q is a monotone map $i : Q \to Q$ such that $i \leq 1_Q$. A *neighbourhood operation* on Q is a monotone map $\nu : Q \to \mathcal{U}(Q)$ with $\uparrow \preccurlyeq \nu$, where:

- $\mathcal{U}(Q) = \{A \subseteq Q \mid A$ is upward closed$\}$ endowed with the reverse set inclusion $\preccurlyeq$;
- $\uparrow : Q \to \mathcal{U}(Q)$ is given by $\uparrow(x) = \{y \mid x \leq y\}$ for each $x \in Q$.

Given two monotone maps $f : P \to Q$ and $g : P \to R$, the *right Kan extension* $Ran_g(f) : R \to Q$ of f along g is defined as follows: for any other monotone map $h : R \to Q$, the inequalities $h \leq Ran_g(f)$ and $hg \leq f$ are equivalent. In particular $Ran_g(f).g \leq f$. Such a right Kan extension exists when Q is complete, and is explicitly given by $Ran_g(f)(x) = \inf\{f(a) \mid c \leq g(a)\}$ for each $x \in R$. We mention the following useful properties:

- For any monotone map h, $Ran_h(Ran_g(f)) = Ran_{hg}(f)$ when the composition hg makes sense;
- If r is a right adjoint map, i.e. commutes with infima, then $rRan_g(f) = Ran_g(rf)$.[3]

Now, since $\mathcal{U}(Q)$ is complete, with the meet denoted by $\sqcap$ (set-union), any interior operation i on Q gives rise to a neighbourhood operation given by the right Kan extension $Ran_i(\uparrow)$ where:

$$Ran_i(\uparrow)(x) = \sqcap\{\uparrow(a) \mid x \leq i(a)\} = \{b \mid x \leq i(b)\} \quad \text{for all } x \in Q.$$

Trivially we have $\uparrow = Ran_{1_Q}(\uparrow)$. When Q is complete and and ν admits a left adjoint $j : \mathcal{U}(Q) \to Q$, then the composition $j\uparrow$ becomes an interior operation on Q [22]. If $Int(Q)$ denotes the set of interior operations on Q and $Nbh(Q)$ that of neighbourhood operations on Q, both ordered pointwise, then the above processes provide an equivalence that are part of a Galois connection [22]:

$$\begin{tikzcd} Int(Q)^{op} \arrow[r, shift left, "Ran_{-}(\uparrow)"] \arrow[r, phantom, "\perp"] & Nbh(Q). \arrow[l, shift left, "\mathfrak{I}"] \end{tikzcd}$$

The neighbourhood operations that are equivalent to interior operations are called *left-adjoint neighbourhood operations*, and in this case $x \leq i(y)$ if and only if $y \in Ran_i(\uparrow)(x)$.

For the remainder of the paper, $\mathbf{C}$ will denote a finitely complete (and possibly small complete) category and $P : \mathbf{C^{op}} \to \mathbf{Pos}$ a pseudo-functor to the 2-category of complete posets that are small. P assigns to each morphism $f : X \to Y$ a map $f^\circ : PY \to PX$ that we refer to as a pre-image. Depending on whether f° commutes with joins and/or meets, it will admit a left-adjoint $f_\circ$—that we shall refer to as image—and/or a right-adjoint f_*. Now, we consider the "upset functor" $\mathcal{U} : \mathbf{Pos} \to \mathbf{Pos}$ that assigns to each map $h : Q \to R$ the map given by $\mathcal{U}(h)(U) = \{b \in R \mid h(a) \leq b$ for some $a \in U\}$. Each morphism $f : X \to Y$ will then give the following Galois connections:[4]

$$\mathcal{U}(f_\circ) \dashv \mathcal{U}(f^\circ) \dashv \mathcal{U}(f_*),$$

[3] For further properties, see [30].

[4] We have adopted this notation, unlike in [22, Section 2], as we shall deal with more than one endofunctor.

so that $\uparrow$ becomes a natural transformation from $1_{\mathbf{Pos}}$ to $\mathcal{U}$.

Definition 1 [23] A neighbourhood operator ν (on $\mathbf{C}$, with respect to P) is lax natural transformation $\nu : P \to \mathcal{U}P$ such that each ν_X is a neighbourhood operation on PX, for each $X \in \mathbf{C}$.

Thus for each morphism $f : X \to Y$, we have $\mathcal{U}(f_\circ)\nu_X \preccurlyeq \nu_Y f_\circ$ or equivalently $\nu_X f^\circ \preccurlyeq \mathcal{U}(f^\circ)\nu_Y$. If each ν_X is a left-adjoint neighbourhood operation, then it determines an *interior operator* i on $\mathbf{C}$; in this case $f^\circ i_Y \leq i_X f^\circ$ [22]. The category of objects of the form (X, ν_X) together with the morphisms from $\mathbf{C}$ shall be denoted by $\mathbf{C}[\nu]$. If ν is left-adjoint and i is the interior associated to ν, then we also write $\mathbf{C}[i]$.

A Word on Atoms The notion of convergence contemplated in [24] eventually requires the use of atoms which, for a given poset Q, are determined by particular maps in $\mathbf{Pos}(\{\perp, \top\}, Q)$. For this reason and because most of the examples used to illustrate this notion are concrete categories over sets (with the notable exceptions of locales/frames [23] and sieves [29]) it is reasonable to directly deal with points $x : 1 \to X$ and their images $x_\circ : P1 \to PX$, for a given object $X \in \mathbf{C}$. Also we shall assume that $|P1| = 2$.

Example 1 To avoid repetition, we shall only cite examples that are recent and refer the reader to [5–7,21,23,24] for further illustrations.

1. Among the neighbourhood operators that are used in [18, Section 4], we point out the following ones:
   - The neighbourhood operator ν_1 on **Top**: where $B \in \nu(A)$ if and only if $cl(A) \subseteq int(B)$.
   - The *coarse neighbourhood operator* ν_2 on **Set** (with the presence of large scale structures): where $B \in \nu(A)$ if and only if $A \subseteq B$ and for any uniformly bounded cover $\mathcal{U}$ of X, $st(A, \mathcal{U}) \subseteq B \bigcup K$ for some weakly bounded set K. "ν-continuous maps" are precisely the *slowly oscillating maps*.
   - The *hybrid neighbourhood operator* ν_3 on **Top** (with the presence of large scale structures): where $B \in \nu(A)$ if and only if $cl(A) \subseteq int(B)$ and B is a coarse neighbourhood of A. "ν-continuous maps" are precisely the continuous and slowly oscillating maps.
2. [23, Example 4] Consider the functor $\mathcal{O} : \mathbf{Loc}^{\mathbf{op}} \to \mathbf{Pos}$ that assigns to each locale X its frame of formal open sets $\mathcal{O}X$ and to each locale map $f : X \to Y$ the corresponding frame homomorphism $f^\circ : \mathcal{O}Y \to \mathcal{O}X$. We have two particular neighbourhood (or interior) operators with respect to $\mathcal{O}$: $a \prec b$ (rather below) and $a \prec\!\!\prec b$ (way below) for all $a, b \in \mathcal{O}X$.
3. [10,21] The topogenous orders are neighbourhood operators on **Set**.
4. [2,16,17,36] A supertopology on a set X is a pair $(\mathcal{M}, \mathcal{V})$, where $\mathcal{M} \subseteq \mathcal{P}(X)$ is a collection of subsets and $\mathcal{V} : \mathcal{M} \to \mathcal{P}(\mathcal{P}(X))$ a function such that:
   (a) $\{\{x\} \mid x \in X\} \subseteq \mathcal{M}$;
   (b) if $A \in \mathcal{M}$ and $U \in \mathcal{V}(A)$, then $A \subseteq U$;
   (c) if $A \in \mathcal{M}$ and $U \in \mathcal{V}(A)$, then there is $V \in \mathcal{V}(A)$ such that $U \in \mathcal{V}(B)$ for all $B \in \mathcal{M}$ with $B \subseteq V$.

   Each such a supertopology gives a left-adjoint neighbourhood operation ν^s with respect to the powerset functor $\mathcal{P}$ defined by

$$\nu^s(A) = \bigcap\{\mathcal{V}(\{x\}) \mid x \in A\} = \bigcap\{\mathcal{V}(M) \mid M \subseteq A \text{ and } M \in \mathcal{M}\}.$$

Conversely, each left-adjoint neighbourhood operation on a set X induces a supertopology by restricting it to $\mathcal{M} \subseteq \mathcal{P}(X)$. We note that these two processes are inverse to each other and can be also understood via *left Kan extensions*. This however goes beyond the scope of this paper.

3 Additivity and Idempotency

Hereafter we consider a complete lattice Q, that is $Q \in \mathbf{Pos}$, and let $\mathcal{F}(Q)$ be the set of filters (including the degenerated filter) on Q, equipped with the reverse inclusion denoted by $\preccurlyeq$. The inclusion $e : \mathcal{F}(Q) \to \mathcal{U}(Q)$ admits a right adjoint $\rho : \mathcal{U}(Q) \to \mathcal{F}(Q)$, so that $e\rho \preccurlyeq 1$ and $\rho e = 1$. Here, ρ takes the upset generated by the finite meets from the members of an upset. Clearly $\uparrow = e\rho \uparrow \preccurlyeq e\rho\nu$ so that $e\rho\nu$ is a neighbourhood operation on Q.

Lemma 1 *Let i be an interior operation on Q and let ν be the neighbourhood operation associated to i. The following are equivalent* [24, Theorem 2 (a)]:

1. *i is additive: $i(x \wedge y) = i(x) \wedge i(y)$ for all $x, y \in Q$.*
2. *$e\rho\nu = \nu$; in which case we shall say that ν is additive.*

Consider the underlying set of Q (that we shall denote with the same letter for convenience) and the powerset monad $(\mathcal{P}, \eta, \mu)$ on **Set**. Let u_Q be the inclusion map from $\mathcal{U}(Q)$ to $\mathcal{P}(Q)$. Since the set-union of upsets is an upset, the operation μ restricts to $\bar{\mu}_Q : \mathcal{P}(\mathcal{U}(Q)) \to \mathcal{U}(Q)$ so that the following diagram commutes (seen in **Set**):

$$\begin{tikzcd}
\mathcal{P}(\mathcal{U}(Q)) \arrow[r, "\bar{\mu}_Q"] \arrow[d, "\mathcal{P}(u_Q)"'] & \mathcal{U}(Q) \arrow[d, "u_Q"] \\
\mathcal{P}(\mathcal{P}(Q)) \arrow[r, "\mu_Q"'] & \mathcal{P}(Q)
\end{tikzcd}$$

Now, consider the Kleisli composition $(u_Q\nu) \circ (u_Q\nu) = \mu_Q\mathcal{P}(u_Q\nu)(u_Q\nu)$. We define

$$\nu * \nu = \bar{\mu}_Q\mathcal{P}(\nu)(u_Q\nu),$$

so that $\nu*\nu$ is the (necessarily) unique map such that $(u_Q\nu)\circ(u_Q\nu) = u_Q(\nu*\nu)$. A pointwise computation of $\nu * \nu$ shows that for each $x \in Q$, we have

$$(\nu * \nu)(x) = \bigcup\{\nu(y) \mid y \in \nu(x)\}.$$

Composition of two different neighbourhood operations are computed in a similar fashion. It is straightforward to see that:

Lemma 2 *For any complete lattice Q, $\nu*\nu$ is a neighbourhood operation on Q with $\nu \preccurlyeq \nu*\nu$. Furthermore, the map $(-) * (-) : Nbh(Q) \times Nbh(Q) \to Nbh(Q)$ is associative and is monotone in each variable.*

Lemma 3 *If $e\rho\nu = \nu$, then $e\rho(\nu * \nu) = \nu * \nu$.*

Definition 2 We say that ν is idempotent if $\nu * \nu = \nu$.

Example 2 1. Idempotency is equivalent to the condition (N4) in [18]:

(a) A topological space is normal if and only if ν_1 is idempotent.

(b) An hybrid large scale space is hybrid large scale-normal if and only if ν_3 is idempotent.

2. In a locale, $\prec$ interpolates if and only if it is idempotent as a neighbourhood operator. In this case, it coincides with $\prec\prec$.
3. A neighbourhood space $(X, \mathcal{N})$ [24] is a supratopological space if and only if $\mathcal{N}$ is idempotent.
4. A pretopological space is a topological space if and only if the neighbourhood operator obtained from the "pre-open" sets is idempotent.
5. A topogenous order $\sqsubseteq$ on a set is interpolative if and only if the neighbourhood operator associated to it is idempotent [21].
6. The neighbourhood operation ν^s induced by a supertopology is always idempotent, thanks to the third axiom in the definition.
7. Completeness of the poset Q may dramatically affect idempotency: consider the set of natural $\mathbb{N}$ with the neighbourhood operation given by $\nu(n) = \{n^2, n^2+1, n^2+2, \dots\}$ for each $n \in \mathbb{N}$. For any $n, k \in \mathbb{N}$, we have $\nu^k(n) \neq \nu(n)$.

If ν is a left-adjoint neighbourhood operation and i its associated interior operation, then for all $x \in Q$:

$$\begin{aligned}(\nu * \nu)(x) &= \bigcup\{\nu(y) \mid x \leq i(y)\} \\ &= \{z \in Q \mid x \leq (i \circ i)(z)\} \\ &= Ran_{i\circ i}(\uparrow)(x) \\ &= Ran_i(Ran_i(\uparrow))(x) \\ &= Ran_i(\nu)(x).\end{aligned}$$

Thus $\nu * \nu = Ran_{i\circ i}(\uparrow) = Ran_i(\nu)$. This shows the following observation:

Lemma 4 *The interior operation i is idempotent, i.e. $i \circ i = i$, if and only if ν is idempotent* [24, Theorem 2(b)].

Proposition 1 *A neighbourhood operation on Q is idempotent if and only if for any $\mathcal{A} \supseteq \nu(p)$ and any $\mathcal{B}_x \supseteq \nu(x)$, where $p, x \in Q$, one has $\nu(p) \subseteq \bigcup_{a\in\mathcal{A}} \bigcap_{x\leq a} \mathcal{B}_x$.*

Proof Assume that the necessary condition stated in the proposition is true and let $q \in \nu(p)$. Let $\mathcal{A} = \nu(p)$ and $\mathcal{B}_x = \nu(x)$ for any $x \in Q$. By assumption, there is $a \in \nu(p)$ such that for all $x \leq a$, $q \in \nu(x)$. But then

$$q \in \bigcap_{x\leq a} \nu(x) = \nu(a).$$

Thus $q \in \bigcup\{\nu(a) \mid a \in \nu(p)\} = (\nu * \nu)(p)$. The reverse inclusion is always true.

Conversely, let $\mathcal{A} \supseteq \nu(p)$ and $\mathcal{B}_x \supseteq \nu(x)$, for each $x \in Q$. Let $q \in \nu(p)$. By idempotency, there is $r \in \nu(p)$ such that $q \in \nu(r)$. On the other hand, there is $a \in \mathcal{A}$ such that $a \leq r$. Thus, for all $x \leq a$:

$$\mathcal{B}_x \supseteq \nu(x) \supseteq \nu(a) \supseteq \nu(r).$$

Therefore, there is $a \in \mathcal{A}$ such that for all $x \leq a$, $q \in \mathcal{B}_x$. □

The above proposition is in fact the expression of condition (Top) in [31, Proposition 2.1] and condition (F4) in [1, Proposition 17] in terms of neighbourhood structures. Now, consider the following iteration:

$$\begin{aligned}\nu^0 &= \uparrow\\ \nu^{\alpha+1} &= \nu^\alpha * \nu \text{ (where } \alpha+1 \text{ is a successor ordinal)}\\ \nu^\beta &= \sqcup_{\alpha<\beta}\nu^\alpha \text{ (where } \beta \text{ is a limit ordinal)}\end{aligned}$$

Taking into account the fact that Q is a set, that is small, there is a smallest ordinal ∞ such that $\nu^{\infty+1} = \nu^\infty$.

Lemma 5 *For a neighbourhood operation ν on Q:*

1. *$e\rho\nu$ is additive and ν^∞ is idempotent.*
2. *$e\rho\nu$ and ν^∞ are left-adjoint whenever ν is.*

Proof The first statement is trivial. Suppose that ν is a left-adjoint neighbourhood operation, that is $\nu = Ran_i(\uparrow)$ for some interior operator i. Since $Ran_{-}(\uparrow)$ preserves joins and compositions, we have $\nu^\infty = Ran_{i^\infty}(\uparrow)$, where i^∞ is the interior operation obtained after iteration of i. On the other hand, let $\hat{i} = \bigwedge\{k \geq i \mid k \text{ is additive}\}$. It is clear that $\hat{i}$ is additive as well. Let $\hat{\nu} = Ran_{\hat{i}}(\uparrow)$. $\hat{\nu}$ is additive by Lemma 1. We have $\rho\nu = \rho\hat{\nu}$ and so $\hat{\nu} = e\rho\hat{\nu} = e\rho\nu$. □

Lemma 6 *$e\rho\nu * e\rho\nu$ is additive.*

Proof From Lemma 3 we have $e\rho(e\rho\nu * e\rho\nu) = e\rho\nu * e\rho\nu$. □

Proposition 2 *Let ν be a neighbourhood operation.*

1. *If ν is idempotent, then so is $e\rho\nu$.*
2. *If ν is additive, then so is ν^∞.*

Proof 1. Since $e\rho\nu * e\rho\nu \preccurlyeq \nu * \nu = \nu$, we have $e\rho(e\rho\nu * e\rho\nu) \preccurlyeq e\rho\nu$. On the other hand, since $e\rho\nu \preccurlyeq e\rho\nu * e\rho\nu$, we have $e\rho\nu = e\rho e\rho\nu \preccurlyeq e\rho(e\rho\nu * e\rho\nu)$. By Lemma 6, $e\rho\nu = e\rho(e\rho\nu * e\rho\nu) = e\rho\nu * e\rho\nu$.

2. Since ν^∞ is idempotent, so is $e\rho\nu^\infty$. And since $\nu = e\rho\nu \preccurlyeq e\rho\nu^\infty$, we have $\nu^\infty \preccurlyeq e\rho\nu^\infty$. □

We shall now extend ν^∞ and $e\rho\nu$ with respect to $P : \mathbf{C^{op}} \to \mathbf{Pos}$.

Lemma 7 *Let $f : X \to Y$ be a morphism in $\mathbf{C}$. We have:*

1. *$u_{PX}\mathcal{U}(f^\circ) \preccurlyeq \mathcal{P}(f^\circ)u_{PY}$;*
2. *If $h, g : Q \to \mathcal{U}(Q)$ such that $h \preccurlyeq g$, then $\bar{\mu}_Q\mathcal{P}(h) \preccurlyeq \bar{\mu}_Q\mathcal{P}(g)$;*
3. *$\bar{\mu}_{PX}\mathcal{P}(\mathcal{U}(f^\circ)) = \mathcal{U}(f^\circ)\bar{\mu}_{PY}$.*

Proof The first statement is trivial. 2. For any $A \subseteq Q$, we have

$$\begin{aligned}\bar{\mu}_Q\mathcal{P}(h)(A) &= \{z \mid (\exists a \in A),\ z \in h(a)\}\\ &\preccurlyeq \{z \mid (\exists a \in A),\ z \in g(a)\} = \bar{\mu}_Q\mathcal{P}(g)(A).\end{aligned}$$

3. The algebras of the powerset monad are precisely the sup-lattices. The poset $\mathcal{U}(PX)$ with set-inclusion and equipped with the operation set-union $\bar{\mu}_{PX} : \mathcal{P}(\mathcal{U}(PX)) \to \mathcal{U}(PX)$, is a sup-lattice and $\mathcal{U}(f^\circ) : \mathcal{U}(PY) \to \mathcal{U}(PY)$ is a sup-lattice homomorphism. □

Proposition 3 *Given any morphism $f : X \to Y$ in $\mathbf{C}$, if ν_Y is idempotent, then so is the neighbourhood operation $\mathcal{U}(f^\circ)\nu_Y f_\circ : PX \to \mathcal{U}(PX)$.*

Proof First we note that $\mathcal{U}(f^\circ)\nu_Y f_\circ \preccurlyeq (\mathcal{U}(f^\circ)\nu_Y f_\circ) * (\mathcal{U}(f^\circ)\nu_Y f_\circ)$. Now

$$\begin{aligned}
(\mathcal{U}(f^\circ)\nu_Y f_\circ) * (\mathcal{U}(f^\circ)\nu_Y f_\circ) &= \bar{\mu}_{PX}\mathcal{P}(\mathcal{U}(f^\circ)\nu_Y f_\circ)u_{PX}\mathcal{U}(f^\circ)\nu_Y f_\circ \\
&\preccurlyeq \bar{\mu}_{PX}\mathcal{P}(\mathcal{U}(f^\circ)\nu_Y f_\circ)\mathcal{P}(f^\circ)u_{PY}\nu_Y f_\circ \text{ (Lemma 7.1)} \\
&= \bar{\mu}_{PX}\mathcal{P}(\mathcal{U}(f^\circ)\nu_Y f_\circ f^\circ)u_{PY}\nu_Y f_\circ \\
&\preccurlyeq \bar{\mu}_{PX}\mathcal{P}(\mathcal{U}(f^\circ)\nu_Y)u_{PY}\nu_Y f_\circ \text{ (Lemma 7.2)} \\
&= \mathcal{U}(f^\circ)\bar{\mu}_{PY}\mathcal{P}(\nu_Y)u_{PY}\nu_Y f_\circ \text{ (Lemma 7.3)} \\
&= \mathcal{U}(f^\circ)(\nu_Y * \nu_Y) f_\circ \\
&= \mathcal{U}(f^\circ)\nu_Y f_\circ.
\end{aligned}$$

Thus we have equality. □

Let ν^∞ be the family of maps defined by $(\nu^\infty)_X = (\nu_X)^\infty$ for all $X \in \mathbf{C}$.

Lemma 8 *ν^∞ is a neighbourhood operator on* $\mathbf{C}$.

Proof For any morphism $f : X \to Y$, we have $\nu_X f^\circ \preccurlyeq \mathcal{U}(f^\circ)\nu_Y \preccurlyeq \mathcal{U}(f^\circ)\nu_Y^\infty$, or equivalently $\nu_X \preccurlyeq \mathcal{U}(f^\circ)\nu_Y^\infty f_\circ$. Now, since $\mathcal{U}(f^\circ)\nu_Y^\infty f_\circ$ is idempotent (Proposition 3), we have $\nu_X^\infty \preccurlyeq \mathcal{U}(f^\circ)\nu_Y^\infty f_\circ$. □

Proposition 4 $\mathbf{C}[\nu^\infty]$ *is a full reflective subcategory of* $\mathbf{C}[\nu]$.

Proof The unit of the reflector from $\mathbf{C}[\nu]$ to $\mathbf{C}[\nu^\infty]$ is given by $(X, \nu_X) \mapsto (X, \nu_X^\infty)$ for each object X in $\mathbf{C}$. □

Next, for each pseudofunctor $P : \mathbf{C}^{\mathbf{op}} \to \mathbf{Pos}$, let $\mathcal{F}(PX)$ be the collection of filters on PX. For each morphism $f : X \to Y$ in $\mathbf{C}$, $F \in \mathcal{F}(PX)$ and $G \in \mathcal{F}(PY)$, we have $\mathcal{U}(f_\circ)(F) \in \mathcal{F}(PX)$ and $\mathcal{U}(f^\circ)(G) \in \mathcal{F}(PY)$. Thus we have two monotone maps $\mathcal{F}(f_\circ) : \mathcal{F}(PX) \to \mathcal{F}(PY)$ and $\mathcal{F}(f^\circ) : \mathcal{F}(PY) \to \mathcal{F}(PX)$. Extending the inclusion e and its right adjoint ρ for all objects $X \in \mathbf{C}$ in an obvious way, we have a pseudofunctor $\mathcal{F} : \mathbf{Pos} \to \mathbf{Pos}$ such that $e_Y\mathcal{F}(f_\circ) = \mathcal{U}(f_\circ)e_X$ and $\mathcal{U}(f^\circ)e_Y = e_X\mathcal{F}(f^\circ)$. It follows that

Lemma 9 *For any morphism $f : X \to Y$ in* $\mathbf{C}$*:*

1. $\mathcal{F}(f^\circ)\rho_Y = \rho_X\mathcal{U}(f^\circ)$.
2. $\mathcal{F}(f_\circ) = \rho_Y\mathcal{U}(f_\circ)e_X$ *and* $\mathcal{F}(f^\circ) = \rho_X\mathcal{U}(f^\circ)e_Y$.
3. $\mathcal{F}(f_\circ) \dashv \mathcal{F}(f^\circ)$.

Proof 1. Since $e_Y\mathcal{F}(f_\circ) = \mathcal{U}(f_\circ)e_X$, both $\mathcal{F}(f^\circ)\rho_Y$ and $\rho_X\mathcal{U}(f^\circ)$ are right adjoint to $\mathcal{U}(f_\circ)e_X$.
2. Follows from the identities $e_Y\mathcal{F}(f_\circ) = \mathcal{U}(f_\circ)e_X$ and $\mathcal{U}(f^\circ)e_Y = e_X\mathcal{F}(f^\circ)$ by composing with ρ_Y and ρ_X respectively on the left.
3. $\mathcal{F}(f_\circ)\mathcal{F}(f^\circ) = \rho_Y\mathcal{U}(f_\circ)e_X\rho_X\mathcal{U}(f^\circ)e_Y \preccurlyeq \rho_Y\mathcal{U}(f_\circ)\mathcal{U}(f^\circ)e_Y \preccurlyeq \rho_Y e_Y = 1$. On the other hand, since $e_Y\rho_Y\mathcal{U}(f_\circ)e_X = e_Y\rho_Y e_Y\mathcal{F}(f_\circ) = e_Y\mathcal{F}(f_\circ) = \mathcal{U}(f_\circ)e_X$, it follows that $\mathcal{F}(f^\circ)\mathcal{F}(f_\circ) = \rho_X\mathcal{U}(f^\circ)\mathcal{U}(f_\circ)e_X \succcurlyeq \rho_X e_X = 1$.

□

Let $\hat{\nu}$ be the family of maps defined by $\hat{\nu}_X = e_X\rho_X\nu_X$ for all $X \in \mathbf{C}$.

Lemma 10 *$\hat{\nu}$ is a neighbourhood operator on* $\mathbf{C}$.

Proof For any $f : X \to Y$ in $\mathbf{C}$, we have:

$$\hat{\nu}_X \mathcal{U}(f^\circ) = e_X \rho_X \nu_X \mathcal{U}(f^\circ) \preccurlyeq e_X \rho_X \mathcal{U}(f^\circ) \nu_Y = e_X \mathcal{F}(f^\circ) \rho_Y \nu_Y = \mathcal{U}(f^\circ) \hat{\nu}_Y.$$

□

Proposition 5 $\mathbf{C}[\hat{\nu}]$ *is a full coreflective subcategory of* $\mathbf{C}[\nu]$.

Proof The co-unit of the coreflector is clearly given by $(X, \hat{\nu}_X) \mapsto (X, \nu_X)$ for each object X in $\mathbf{C}$. □

Corollary 1 *Let* ν *be a left-adjoint neighbourhood operator and* i *its associated interior operator. Then* i^∞ *and* $\hat{i}$, *where* $i^\infty_X = (i_X)^\infty$ *and* $\hat{i}_X = \widehat{(i_X)}$ *for each object* X, *are interior operators on* $\mathbf{C}$. *Furthermore,* $\mathbf{C}[i^\infty]$ *is a full reflective subcategory of* $\mathbf{C}[i]$ *and* $\mathbf{C}[\hat{i}]$ *is a full coreflective subcategory of* $\mathbf{C}[i]$.

Proof Follows from Lemma 5.2. □

The following embeddings illustrate the above:

$$\begin{tikzcd}
 & \mathbf{C}[\nu] & & \\
\mathbf{C}[\nu^\infty] \arrow[ur] & & & \mathbf{C}[i] \arrow[ull] \\
 & \mathbf{C}[\hat{\nu}] \arrow[uu] & \mathbf{C}[i^\infty] \arrow[ull] \arrow[ur] & \\
\mathbf{C}[\hat{\nu}^\infty] \arrow[uu] \arrow[ur] & & & \mathbf{C}[\hat{i}] \arrow[ull] \arrow[uu] \\
 & & \mathbf{C}[\hat{i}^\infty] \arrow[uu] \arrow[ull] \arrow[ur] &
\end{tikzcd}$$

Proposition 6 *Let* $\mathbf{C} = \mathbf{Set}$ *and* $P = Sub(-)$. *Let* ν *be a left-adjoint neighbourhood operator on* $\mathbf{Set}$. *Then*

1. $\mathbf{Set}[\nu]$ *is equivalent to the category* **Neigh** *of neighbourhood spaces* [24,25].
2. $\mathbf{Set}[\nu]$ *is equivalent to the category* **PrTop** *of pretopological spaces* [1,31] *if and only if* ν *is additive.*
3. $\mathbf{Set}[\nu]$ *is equivalent to the category* **SuTop** *of supratopological spaces* [24,25] *if and only if* ν *is idempotent* [24,25].
4. $\mathbf{Set}[\nu]$ *is equivalent to the category* **Top** *of topological spaces* [1,24,31] *if and only if* ν *is additive and idempotent.*

We then have the following known embeddings, where ‘r’ stands for full reflection and ‘c’ for full coreflection:

$$
\begin{tikzcd}
 & \mathbf{Neigh} & \\
\mathbf{PreTop} \arrow[ur, "c"] & & \mathbf{SuTop} \arrow[ul, "r"'] \\
 & \mathbf{Top} \arrow[ul, "r"] \arrow[ur, "c"'] &
\end{tikzcd}
$$

Example 3 The neighbourhood operation ν^s is additive (by definition) and idempotent. Thus, this neighbourhood operation actually is a topology. The fact that a topology always lies beside a supertopology has already been mentioned in Doitchinov’s paper [17].

4 Initial Structures

As we are mostly concerned with finite limits and products in general, an optimal way to look at these is to consider inverse limits [3,8,9,15]. Thus the limit that we are discussing here should be construed as an inverse limit of finite limits. Given a small diagram $D : I \to \mathbf{C}$ and a limit cone $\langle f \rangle : \Delta X \to D$, the neighbourhood operation:

$$\nu_{\langle f \rangle} = \sqcap_I \mathcal{U}(f_i^\circ)\nu_{D_i} f_{i\circ},$$

where $\sqcap$ is the meet on $\mathbf{Pos}(PX, \mathcal{U}(PX))$, provides $\mathbf{C}[\nu]$ with an initial structure and makes $\mathbf{C}[\nu]$ topological over $\mathbf{C}$ [23]. The initial structures on $\mathbf{C}[\hat{\nu}]$ and $\mathbf{C}[\nu^\infty]$ are then given by $e_X \rho_X \nu_{\langle f \rangle}$ and $\nu_{\langle f \rangle}$ respectively.

We now assume that pre-images commute with joins and let $i = \mathfrak{I}(\nu)$. The embedding $\mathbf{C}[i] \to \mathbf{C}[\nu]$ admits a coreflector provided by

$$(X, Ran_{\mathfrak{I}(\nu_X)}(\uparrow_X)) \to (X, \nu_X).$$

The corresponding initial structure with respect to i is given by [23, Section 4]:

$$
\begin{aligned}
\mathfrak{I}(\nu_{\langle f \rangle}) &= \mathfrak{I}(\sqcap_I \mathcal{U}(f_i^\circ)\nu_{D_i} f_{i\circ}) \\
&= \bigvee_I \mathfrak{I}(\mathcal{U}(f_i^\circ)\nu_{D_i} f_{i\circ}) \\
&= \bigvee_I f_i^\circ i_{D_i} f_{i*}.
\end{aligned}
$$

We shall denote $i_{\langle f \rangle} = \bigvee_I f_i^\circ i_{D_i} f_{i*}$.

Lemma 11 *let $f : X \to Y$ be a morphism in $\mathbf{C}$.*

1. *If i is additive, then $f^\circ i_Y f_*$, as an interior operation on PX, is also additive.*
2. *If i is idempotent, then $f^\circ i_Y f_*$ is idempotent.*

Proof The second statement is given by Proposition 3. Now,

$$\begin{aligned} e_X \rho_X \mathcal{U}(f^\circ) Ran_{i_Y}(\uparrow_Y) f_\circ &= e_X \rho_X \mathcal{U}(f^\circ) e_Y \rho_Y Ran_{i_Y}(\uparrow_Y) f_\circ \\ &= e_X \mathcal{F}(f^\circ) \rho_Y Ran_{i_Y}(\uparrow_Y) f_\circ \\ &= \mathcal{U}(f) e_Y \rho_Y Ran_{i_Y}(\uparrow_Y) f_\circ \\ &= \mathcal{U}(f^\circ) Ran_{i_Y}(\uparrow_Y) f_\circ . \end{aligned}$$

From Lemma 1, $f^\circ i_Y f_* = \mathfrak{I}(\mathcal{U}(f^\circ) Ran_{i_Y}(\uparrow_Y) f_\circ)$ is additive. □

Remark 1 In the proof above, one can just point out that $f^\circ i_Y f_*$ preserves binary meets once i_Y does. The proof above applies to neighbourhood operation in general and circumvents the existence of the right adjoint f_*.

Proposition 7 *Consider a cone $\langle f \rangle : \Delta X \to D$ on a diagram $D : I \to \mathbf{C}$.*

1. *If i is additive and PX is a frame, then $i_{\langle f \rangle}$ is additive.*
2. *If i is idempotent, then so is $i_{\langle f \rangle}$.*

Proof Idempotency of $i_{\langle f \rangle}$ is clear. Now, since PX is a frame, $i_{\langle f \rangle}$ preserves finite meets as well. □

Proposition 8 *Let $\langle f \rangle : \Delta X \to D$ be a limit on a diagram $D : I \to \mathbf{C}$ and let ν be a left-adjoint neighbourhood operator on $\mathbf{C}$.*

1. *If ν is additive, then (X, ν_X) is the limit in $\mathbf{C}[\hat{\nu}]$ if and only if $\nu_X = e_X \rho_X Ran_{\mathfrak{I}(\nu_{\langle f \rangle})}(\uparrow_X)$;*
2. *If ν is idempotent, then (X, ν_X) is the limit in $\mathbf{C}[\nu^\infty]$ if and only if $\nu_X = Ran_{\mathfrak{I}(\nu_{\langle f \rangle})}(\uparrow_X)$.*

Corollary 2 *If ν is additive and idempotent, then (X, ν_X) is the limit in $\mathbf{C}[\nu]$ if and only if $\nu_X = e_X \rho_X Ran_{\mathfrak{I}(\nu_{\langle f \rangle})}(\uparrow_X)$.*

Corollary 3 *Let $x : 1 \to X$ be a point. If ν is idempotent, then $\nu_X x_\circ = \nu_{\langle f \rangle} x_\circ$. If ν is additive, then $\nu_X x_\circ = e_X \rho_X \nu_{\langle f \rangle} x_\circ$.*

Proof By adjunction $Ran_{\mathfrak{I}(\nu_{\langle f \rangle})}(\uparrow_X) \preccurlyeq \nu_{\langle f \rangle}$ holds. Let $U : P1 \to \mathcal{U}(PX)$ be a monotone map such that $Ran_{\mathfrak{I}(\nu_{\langle f \rangle})}(\uparrow_X) x_\circ \preccurlyeq U$, or $x_\circ \leq \left(\bigvee_I f_i^\circ j_{D_i} f_{i*}\right) U$, where $\nu_{D_i} \dashv j_{D_i}$. There is $i \in I$ such that $x_\circ \leq f_i^\circ j_{D_i} f_{i*} U$, or equivalently $f_i^\circ \nu_{D_i} f_{i\circ} x_\circ \preccurlyeq U$. Therefore $\nu_{\langle f \rangle} x_\circ \preccurlyeq U$ and we have equality. If ν is additive, then $\nu_X x_\circ = e_X \rho_X Ran_{\mathfrak{I}(\nu_{\langle f \rangle})}(\uparrow_X) x_\circ = e_X \rho_X \nu_{\langle f \rangle} x_\circ$. □

5 Remark on Closed Maps

When the subobject lattices are Boolean algebras, then closure and interior operators uniquely determine each other. However, it is not trivial to see that closed maps with respect to each of these operators are identical in such a setting. We wish to briefly show that this is indeed the case and that furthermore this identity does not assume additivity or idempotency. In this particular section, we assume that $\mathbf{C}$ is endowed with an $(\mathcal{E}, \mathcal{M})$-factorisation system and that each subobject lattice $Sub(X)$ is Boolean for each object X. We shall also assume the Frobenius reciprocity law in this section, that is for any subobjects p and n, and any morphism f we have $f_\circ(p \wedge f^\circ(n)) = f_\circ(p) \wedge n$.

Definition 3 [24] We say that an interior operator i is compatible with a closure operator c if for any $m \in Sub(X)$ and $X \in \mathbf{C}$, $i_X(m) = \overline{c_X(\overline{m})}$, where $\overline{()}$ is the complement map.

In what follows, c is a fixed closure operator, i the interior operator compatible with c and ν the left-adjoint neighbourhood operator associated to i. Let us recall that a morphism $f : X \to Y$ is c-closed [15] if for any $m \in Sub(X)$, $f_\circ(c_X(m)) = c_Y(f_\circ(m))$ and that it is ν-closed [23] if $(\mathcal{U}(f^\circ))\nu_Y(p) = (\nu_X f^\circ)(p)$ for each $p \in Sub(Y)$.

Proposition 9 *If $f : X \to Y$ is c-closed, then it is ν-closed.*

Proof We must show that for all $n \in Sub(Y)$, $\nu_X(f^\circ(n)) \subseteq \mathcal{U}(f^\circ)(\nu_Y(n))$. Let $p \in \nu_X(f^\circ(n))$. Then $f^\circ(n) \le i_X(p) = \overline{c_X(\overline{p})}$. Therefore $c_X(\overline{p}) \wedge f^\circ(n) = 0_X$ and so $f_\circ(c_X(\overline{p})) \wedge n = 0_Y$. Since f is c-closed, this amounts to $n \wedge c_Y(f_\circ(\overline{p})) = 0_Y$. It follows that $n \le \overline{c_Y(f_\circ(\overline{p}))} \le i_Y(\overline{f_\circ(\overline{p})})$. But then $\overline{f_\circ(\overline{p})} \in \nu_Y(n)$ and $f^\circ(\overline{f_\circ(\overline{p})}) \in \mathcal{U}(f^\circ)(\nu_Y(n))$. Since $f^\circ(\overline{f_\circ(\overline{p})}) \le p$, we have $p \in \mathcal{U}(f^\circ)(\nu_Y(n))$. Thus f is ν-closed. $\square$

Proposition 10 *If $f : X \to Y$ is ν-closed, then it is c-closed.*

Proof Let $m \in Sub(X)$ and $p \in Sub(X)$ such that $p \wedge f_\circ(c_X(m)) = 0_Y$. Then $f^\circ(p) \wedge \overline{i_X(\overline{m})} = 0_X$ or equivalently $f^\circ(p) \le i_X(\overline{m})$. Thus $\overline{m} \in \nu_X(f^\circ(p)) = \mathcal{U}(f^\circ)(\nu_Y(p))$ and so there is $q \in \nu_Y(p)$ such that $f^\circ(q) \le \overline{m}$. We then have $p \wedge c_Y(\overline{q}) = p \wedge \overline{i_Y(q)} = 0_Y$. Now, since $f^\circ(q) \le \overline{m}$, we have $m \le \overline{f^\circ(q)} = f^\circ(\overline{q})$ or equivalently $f_\circ(m) \le \overline{q}$. Hence $p \wedge c_Y(f_\circ(m)) = 0_Y$. Since p is arbitrary, we have $c_Y(f_\circ(m)) \le f_\circ(c_X(m))$, as desired. $\square$

Note that the existence of the right a adjoint for each pre-image was not necessary here. The collection of maps which are ν-closed is denoted by $\mathcal{K}(\nu)$ [23].

Proposition 11 *Let $f : X \to Y$ be a morphism such that f° commutes with joins. Suppose that ν is a left-adjoint neighbourhood operator and that $Sub(-)$ has enough points. Then f is ν-closed if and only if for any point $y : 1 \to Y$ we have $\mathcal{U}(f^\circ)\nu_Y y_\circ = \nu_X f^\circ y_\circ$.*

Proof The necessary condition is clear. Now, let $m : M \to Y$ be a subobject with $m = \bigvee\{y : 1 \to Y \mid y \le m\}$. Then

$$\begin{aligned}
(\mathcal{U}(f^\circ)\nu_Y)(m) &= (\mathcal{U}(f^\circ)\nu_Y)\left(\bigvee y\right)\\
&= \mathcal{U}(f^\circ)(\sqcup\, \nu_Y(y))\\
&= \sqcup\, \mathcal{U}(f^\circ)\nu_Y(y)\\
&= \sqcup\, \nu_X f^\circ(y)\\
&= \nu_X\left(\bigvee f^\circ(y)\right)\\
&= \nu_X f^\circ(m),
\end{aligned}$$

where $\sqcup$ is the join (pointwise set intersection) on $\mathbf{Pos}(PY, \mathcal{U}(PX))$. $\square$

6 Convergence

The idea of convergence developed here mainly follows [24,34]. Among the most important classes of maps that are relevant to the study of convergence are arguably those that preserve and reflect convergence. For the remainder of the section, we shall assume that ν is left-adjoint and pre-images commute with joins.

We now add another parameter and consider a subfunctor $s : \mathcal{S} \Rightarrow \mathcal{U}$ in such a way that for any morphism $f : X \to Y$, $s_Y\mathcal{S}(f_\circ) = \mathcal{U}(f_\circ)s_X$. Thus we look at $\mathbf{C}$ together with a neighbourhood operator ν and an appropriate choice of $\mathcal{S}$:

Example 4 1. If $\mathbf{C} = \mathbf{Top}$, then we consider $\mathcal{S} = \mathscr{U}$, where $\mathscr{U}(PX) = \{\text{ultrafilters on } PX\}$ or $\mathcal{S} = \mathcal{F}$.

2. If $\mathbf{C} = \mathbf{Unif}$ with the uniform neighbourhood operator [18,21], then we consider $\mathcal{S} = \mathscr{C}$, where $\mathscr{C}(PX) = \{\text{Cauchy filters on } PX\}$ and/or $\mathcal{S} = \mathscr{U}$.
3. For the categories that are similar to **Neigh**, **PrTop** and **SuTop**, we consider $\mathcal{S} = \mathscr{U}, \mathcal{F}$, but also $\mathcal{S} = \mathscr{R}$, where $\mathscr{R}(PX) = \{\text{rasters on } PX\}$ [24].

The following proposition indicates that, in the current setting, one can use neighbourhood operators to deal with limits in general.

Proposition 12 *Any convergence structure $\pi \subseteq T(-) \times (-)$ in the sense of [33] on $\mathbf{C}$ gives rise to a neighbourhood operator ν as follows*

$$\nu_X^\pi x_\circ = \sqcup\{e_X\phi \mid (\phi, x) \in \pi_X\} \quad \textit{for any } X \in \mathbf{C} \textit{ and any point } x.$$

On the other hand, any neighbourhood operator on $\mathbf{C}$ gives rise to a convergence structure π^ν by declaring $(\phi, x) \in \pi_X^\nu$ whenever $e_X\phi \preccurlyeq \nu_X x_\circ$. We always have $\nu^{\pi^\nu} = \nu$ and if π is a limit structure, then $\pi^{\nu^\pi} = \pi$.

Next, we look at the description of closed maps with respect to neighbourhood operators.

Proposition 13 *Let ν be a left-adjoint neighbourhood operator. For a morphism $f : X \to Y$ in $\mathbf{C}$, the following are equivalent:*

(i) *f is ν-closed;*
(ii) *$\nu_X f^\circ = Ran_{i_Y}(\mathcal{U}(f^\circ) \uparrow_Y)$;*
(iii) *For any monotone maps $h : P1 \to PX$ and $\phi : P1 \to \mathcal{U}(PY)$, the relations $\mathcal{U}(f_\circ)\phi \preccurlyeq \nu_Y h$ and $\phi \preccurlyeq \nu_X f^\circ h$ are equivalent.*

Proof (ii) implies (i): Since $\mathcal{U}(f^\circ)$ is a right adjoint, $\mathcal{U}(f^\circ)\nu_Y$ is a right Kan extension of $\mathcal{U}(f^\circ) \uparrow_Y$ along i_Y. The result follows from universality. (i) implies (iii): One of the other implication is equivalent to ν-continuity. For the other implication, let ϕ and h be monotone maps such that $\mathcal{U}(f_\circ)\phi \preccurlyeq \nu_Y h$. We have:

$$\phi \preccurlyeq \mathcal{U}(f^\circ f_\circ)\phi \preccurlyeq \mathcal{U}(f^\circ)\nu_Y h = \nu_X f^\circ h.$$

(iii) implies (ii): Suppose that $\phi i_Y \preccurlyeq \mathcal{U}(f^\circ) \uparrow_Y$, then $\mathcal{U}(f_\circ)\phi i_Y \preccurlyeq \uparrow_Y$. Since ν_Y is the right Kan extension of $\uparrow_Y$ along i_Y, we have $\mathcal{U}(f_\circ)\phi \preccurlyeq \nu_Y$. By hypothesis the latter is equivalent to $\phi \preccurlyeq \nu_X f^\circ$. □

Corollary 4 *Suppose that ν is additive. f is ν-closed if and only if for any maps $h : P1 \to PX$ and $\phi : P1 \to \mathcal{F}(PY)$, the relations $\mathcal{U}(f_\circ)e_X\phi \preccurlyeq \nu_Y h$ and $e_X\phi \preccurlyeq \nu_X f^\circ h$ are equivalent.*

Condition (iii) in Proposition 13 indicates that, in some sense closed maps are weaker versions of maps that reflect convergence. This will naturally lead us to the notion of $\mathcal{S}$-reflecting maps.

Definition 4 A monotone map $\phi : P1 \to \mathcal{S}PX$ converges to a point $x : 1 \to X$ if $s_X\phi \preccurlyeq \nu_X x_\circ$.

Definition 5 [3] A morphism $f : X \to Y$ in $\mathbf{C}$ is said to be $\mathcal{S}$-reflecting if for any monotone map $\phi : P1 \to \mathcal{S}X$ there is given a point $y : 1 \to Y$ such that $\mathcal{U}(f_\circ)s_X\phi \preccurlyeq \nu_Y y_\circ$, then there is a point $x : 1 \to X$ such that $f_\circ x_\circ = y_\circ$ and $s_X\phi \preccurlyeq \nu_X x_\circ$.

Remark 2 Note that $f_\circ x_\circ = y_\circ$ and $fx = y$ are equivalent.

Example 5 1. A topological space is compact if and only if $X \to 1$ is $\mathscr{U}$-reflecting; and a continuous map f is $\mathscr{U}$-reflecting if and only if it is proper in the usual sense [3].
2. A uniform space X is complete if and only if $X \to 1$ is $\mathscr{C}$-reflecting.
3. If a subspace A of a topological space is $\mathscr{F}$-reflecting, then it is *extension closed* [20], but the converse fails.

The class of all $\mathcal{S}$-reflecting morphisms in $\mathbf{C}[\nu]$ shall be denoted by $\mathcal{R}(\mathcal{S})$. The definition could be described in the following lax diagram:

$$\begin{tikzcd}
P1 \arrow[dr, "x_\circ"] \arrow[drr, bend left, "y_\circ"] \arrow[ddr, bend right, "s_X\phi"'] & & \\
& \cdot \arrow[r, "f_\circ"] \arrow[d, "\nu_X"] & \cdot \arrow[d, "\nu_Y"] \\
& \cdot \arrow[r, "\mathcal{U}(f_\circ)"'] \arrow[ur, phantom, "\preccurlyeq"] & \cdot
\end{tikzcd}$$

Though the diagram is not a 2-pullback, many essential features of pullbacks that are familiar to us allow to obtain some interesting results (compare with [9,35]).

Proposition 14 1. *$\mathcal{R}(\mathcal{S})$ contains isomorphisms and is stable under composition. Also, if $\mathcal{S} \subseteq \mathcal{S}'$, then $\mathcal{R}(\mathcal{S}') \subseteq \mathcal{R}(\mathcal{S})$.*
2. *If $gf \in \mathcal{R}(\mathcal{S})$ and $g^\circ g_\circ = 1$, then $f \in \mathcal{R}(\mathcal{S})$.*
3. *If $gf \in \mathcal{R}(\mathcal{U})$ (resp. $\mathcal{R}(\mathscr{F})$, $\mathcal{R}(\mathscr{U})$) and $f_\circ f^\circ = 1$, then $g \in \mathcal{R}(\mathcal{U})$ (resp. $\mathcal{R}(\mathscr{F})$, $\mathcal{R}(\mathscr{U})$). On the other hand if $gf \in \mathcal{R}(\mathscr{C})$ and f is a retraction, then $g \in \mathcal{R}(\mathscr{C})$.*

Remark 3 We note that $\mathcal{U}(f^\circ)$ restricts to $\mathscr{U}(f^\circ)$ when $f_\circ f^\circ = 1$.

Proposition 15 *Suppose that $P = Sub(-)$. If $m^\circ m_\circ = 1$ and m is ν-closed, then $m \in \mathcal{R}(\mathcal{S})$. In particular we have the inclusion $\mathcal{K}(\nu) \bigcap Mono(\mathbf{C}) \subseteq \mathcal{R}(\mathcal{S})$.*

Proof Let $\phi : P1 \to \mathcal{S}X$ be monotone map and $y : 1 \to X$ be a point such that $\mathcal{U}(m_\circ)s_X\phi \preccurlyeq \nu_X y_\circ$. Since m is closed, $s_X\phi \preccurlyeq \nu_M m^\circ y_\circ$. Let $m' : M' \to 1$ be the pullback of m along y. We have $m' \in Sub(1)$ and therefore $m' \cong 0_1 = 0_M$ or $M' \cong 1$. (Since $|Sub(1)| = |P1| = 2$.) The first case implies $\nu_M m^\circ y_\circ = \nu_M(0_M) = Sub(M)$ since ν is left-adjoint. Because ϕ cannot be trivial, we have $M' \cong 1$ and so there is $x : 1 \to M$ such that $s_X\phi \preccurlyeq \nu_M x_\circ$ and $m_\circ x_\circ = y_\circ$. □

We single out the following fact which is a consequence of Propositions 14.1 and 15:

Corollary 5 *Suppose that $P = Sub(-)$. If m is ν-closed and f is $\mathcal{S}$-reflecting, then fm is $\mathcal{S}$-reflecting whenever the composition makes sense.*

Instances of the above result are the following well-known facts: a closed subspace of a compact topological space is compact and a closed subspace of a complete uniform space is complete.

Proposition 16 *Suppose that ν is additive and that P and $\mathcal{F}$ have enough points. If f is $\mathscr{U}$-reflecting (resp. $\mathscr{F}$-reflecting), then f is ν-closed. The converse fails in general.*

Proof Let $\phi : P1 \to \mathcal{F}(PX)$ be monotone map and $y : 1 \to Y$ a point such that $e_X\phi \preccurlyeq f^{\circ}\nu_Y y_{\circ}$. For each ultrafilter $\theta : P1 \to \mathscr{U}(PX)$ such that $t_X\theta \preccurlyeq \phi$ (where t_X is the natural inclusion), we have $\mathcal{U}(f_{\circ})e_X t_X\theta \preccurlyeq \nu_Y y_{\circ}$. There are x^{θ} such that $e_X t_X \theta \preccurlyeq \nu_X x_{\circ}^{\theta}$ and $f_{\circ}x_{\circ}^{\theta} \leq y_{\circ}$, so that $e_X t_X\theta \preccurlyeq \nu_X f^{\circ} y_{\circ}$. Since $\mathcal{F}$ has enough points, we have

$$\begin{aligned} e_X\phi &= e_X(\sqcup\{t_X\theta \mid \theta : P1 \to \mathscr{U}(PX)\}) \\ &= \sqcup\{e_X t_X\theta \mid \theta : P1 \to \mathscr{U}(PX)\} \\ &\preccurlyeq \nu_X f^{\circ} y_{\circ}. \end{aligned}$$

The result follows from Proposition 11. □

Corollary 6 *Suppose that $P = Sub(-)$. With the assumptions of Proposition 16, we have the identity $\mathcal{K}(\nu) \bigcap Mono(\mathbf{C}) = \mathcal{R}(\mathscr{U}) \bigcap Mono(\mathbf{C})$.*

Proposition 17 *Let $s : \mathcal{S} \to \mathcal{U}$ be a subfunctor. Let $\langle f\rangle : \Delta X \to D$ be a limit cone and $x : 1 \to X$ a point. For any monotone map $\phi : P1 \to \mathcal{S}X$ we have $s_X\phi \preccurlyeq \nu_X x_{\circ}$ if and only if $\mathcal{U}(f_{i\circ})s_X\phi \preccurlyeq \nu_{D_i} f_{i\circ} x_{\circ}$ for each $i \in I$.*

Proof The necessary condition follows from continuity. Suppose that for each $i \in I$ we have $\mathcal{U}(f_{i\circ})s_X\phi \preccurlyeq \nu_{D_i} f_{i\circ} x_{\circ}$. Then $s_X\phi \preccurlyeq \nu_{\langle f\rangle} x_{\circ}$, or equivalently $s_X\phi \preccurlyeq \nu_X x_{\circ}$. □

Corollary 7 *With the assumptions of Proposition 17:*

1. *If ν is additive and s factors through e, say $s = e \circ t$, then $s_X\phi \preccurlyeq \nu_X x_{\circ}$ if and only if $\mathcal{U}(f_{i\circ})s_X\phi \preccurlyeq \nu_{D_i} f_{i\circ} x_{\circ}$ for each $i \in I$;*
2. *If ν is idempotent, then $s_X\phi \preccurlyeq \nu_X x_{\circ}$ if and only if $\mathcal{U}(f_{i\circ})s_X\phi \preccurlyeq \nu_{D_i} f_{i\circ} x_{\circ}$ for each $i \in I$.*

Proof Results follow from Corollary 3. If ν is additive, then:

$$s_X\phi = e_X t_X\phi = e_X(\rho_X s_X)\phi \preccurlyeq e_X\rho_X \nu_{\langle f\rangle} x_{\circ} = \nu_X x_{\circ}.$$

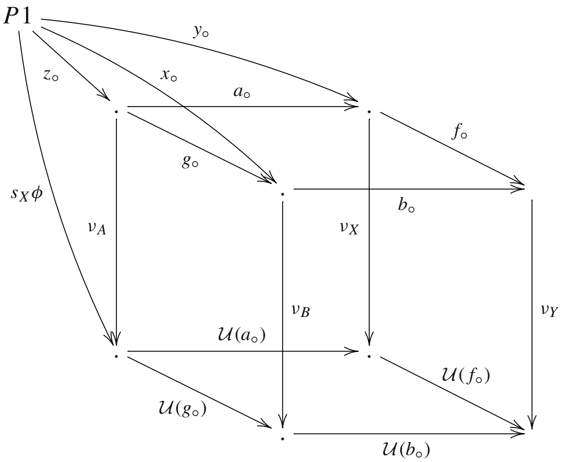
The case where ν is idempotent is clear. □

Theorem 1 *$\mathcal{R}(S)$ is pullback stable in $\mathbf{C}[\mu]$, in each case where $\mu = \hat{\nu}, \nu^{\infty}$ or $\hat{\nu}^{\infty}$ and assuming that $s = e \circ t$ where additivity is involved.*

Proof Let us have a pullback diagram:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Assume that f is $\mathcal{S}$-reflecting. We shall prove that g is $\mathcal{S}$-reflecting. Consider the following diagram (the inequalities are not written for convenience)



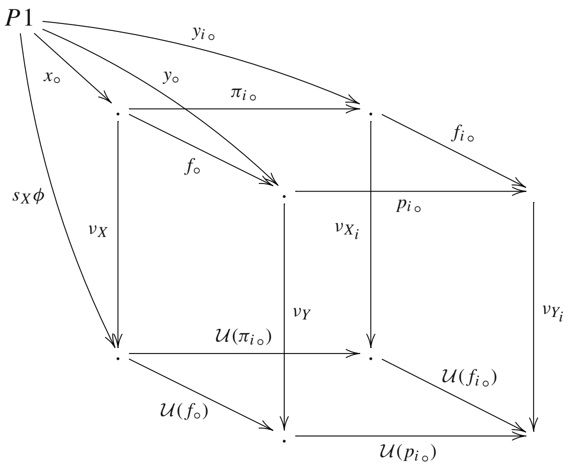
Let $x : 1 \to B$ be a point and ϕ a(n appropriate) monotone map such that $\mathcal{U}(g_\circ)s_X\phi \preccurlyeq \nu_B x_\circ$. Then $\mathcal{U}(f_\circ a_\circ)s_X\phi \preccurlyeq \nu_Y b_\circ x_\circ$. Since f is $\mathcal{S}$-reflecting, there is a point y such that $f_\circ y_\circ = b_\circ x_\circ$ and $\mathcal{U}(a_\circ)s_X\phi \preccurlyeq \nu_X y_\circ$. By the property of pullback there is a point z such that $g_\circ z_\circ = x_\circ$ and $a_\circ z_\circ = y_\circ$. We have the following inequalities:

$$\begin{aligned} \mathcal{U}(f_\circ)\mathcal{U}(a_\circ)s_X\phi &\preccurlyeq \nu_Y f_\circ a_\circ z_\circ \\ \mathcal{U}(a_\circ)s_X\phi &\preccurlyeq \nu_X a_\circ z_\circ \\ \mathcal{U}(g_\circ)s_X\phi &\preccurlyeq \nu_B g_\circ z_\circ, \end{aligned}$$

which imply $s_X\phi \preccurlyeq \nu_A z_\circ$ by Proposition 17 and Corollary 7. Thus g is $\mathcal{S}$-reflecting. □

Theorem 2 (Frolik's theorem) *$\mathcal{R}(S)$ is closed under the formation of direct products in* $\mathbf{C}[\mu]$, *in each case where $\mu = \hat{\nu}$, ν^∞ or $\hat{\nu}^\infty$ and assuming that $s = e \circ t$ where additivity is involved.*

Proof Let $f_i : X_i \to Y_i$, $i \in I$ be a family of morphisms and let $f : X \to Y$ be its product with natural projections $\pi_i : X \to X_i$ and $p_i : Y \to Y_i$, $i \in I$. Assume that each f_i is $\mathcal{S}$-reflecting and consider the following diagram:



Let $y : 1 \to Y$ be a point and ϕ a(n appropriate) monotone map such that $\mathcal{U}(f_\circ)s_X\phi \preccurlyeq \nu_Y y_\circ$, then for each $i \in I$, we have $\mathcal{U}(f_{i\circ}\pi_{i\circ})s_X\phi \preccurlyeq \nu_{Y_i} p_{i\circ} y_\circ$. Each f_i is $\mathcal{S}$-reflecting, therefore there are points y_i such that $f_{i\circ} y_{i\circ} = p_{i\circ} y_\circ$ and $\mathcal{U}(\pi_{i\circ})s_X\phi \preccurlyeq \nu_{X_i} y_{i\circ}$ for each $i \in I$. There is a point x such that $\pi_{i\circ} x_\circ = y_{i\circ}$ for all $i \in I$ so that $p_{i\circ} y_\circ = p_{i\circ} f_\circ x_\circ$. By the fact that we have a product, $f_\circ x_\circ = y_\circ$.

On the other hand the inequalities $\mathcal{U}(\pi_{i\circ})s_X\phi \preccurlyeq \nu_{X_i}\pi_{i\circ}x_\circ$ imply $s_X\phi \preccurlyeq \nu_X x_\circ$ (Proposition 17 and Corollary 7). Thus f is $\mathcal{S}$-reflecting as desired. □

We end this section by discussing under which condition the "fill-in" arrow in Definition 5 is unique. The answer to this is obviously a separation condition and we use our notion of $\mathcal{S}$-reflecting morphisms to define this notion. Though ν is considered to be a left-adjoint neighbourhood operator, the notion of separation discussed here is not to be confused with the notion of separation with respect to interior operators introduced in [7]. It is rather related to the notion of *convergence separation* in [33].

Definition 6 An object X is said to be ν-separated if the diagonal $\delta_X = \langle 1_X, 1_X \rangle$ is ν-closed.

Lemma 12 *Suppose that $P = Sub(-)$. Suppose in addition that P and $\mathcal{F}$ have enough points. Then X is ν-separated if and only if δ_X is $\mathcal{U}$-reflecting. If ν is additive, then X is ν-separated if and only if δ_X is $\mathscr{U}$-reflecting.*

Proof Corollary 6. □

If δ_X is $\mathcal{S}$-reflecting, then looking at the following diagram

$$\begin{tikzcd}
P1 \arrow[dr, "x_\circ"] \arrow[drr, bend left, "y_\circ"] \arrow[ddr, bend right, "s_X\phi"'] & & \\
& \cdot \arrow[r, "\delta_{X\circ}"] \arrow[d, "\nu_X"] \arrow[dr, phantom, "\preccurlyeq"] & \cdot \arrow[d, "\nu_{X\times X}"] \\
& \cdot \arrow[r, "\mathcal{U}(\delta_{X\circ})"'] & \cdot
\end{tikzcd}$$

one sees that x is necessarily unique such that $\delta_{X\circ}x_\circ \leq y_\circ$ and $s_X\phi \preccurlyeq \nu_X$ since $\delta_X^\circ \delta_{X\circ} = 1$. This motivates the following definition:

Definition 7 We say that X is $\mathcal{S}$-separated if δ_X is $\mathcal{S}$-reflecting.

Lemma 13 *Let $f : X \to Y$ be a morphism and suppose that X is $\mathcal{S}$-separated. Then f is $\mathcal{S}$-reflecting if and only if the continuity diagram $f_\circ \nu_X \preccurlyeq \nu_Y f_\circ$ is a* pointed lax pullback*, i.e. the fill-in arrow in Definition 5 is unique.*

The above observations motivate a notion of *separated morphisms* that are suitable for neighbourhood operators. Properties of separated morphisms depend merely on the behaviour of the class $\mathcal{R}(\mathcal{U}) \bigcap Mono(\mathbf{C})$ [9, Section 4.2]. Proposition 14 ensures that we have a well-behaved class that meets the criteria. We shall mention the following facts which are of interest (compare with [9, Paragraph 5.5.1]).

Proposition 18 [9] *Any morphism $f : X \to Y$ is $\mathcal{S}$-reflecting provided that $X \to 1$ is $\mathcal{S}$-reflecting and Y is $\mathcal{S}$-separated.*

Proof [9] f is a composition of the morphism $\langle 1_X, f\rangle : X \to X \times Y$ and the projection $X \times Y \to Y$ which are $\mathcal{S}$-reflecting as consequences of Theorem 1. By Proposition 14.1, f is $\mathcal{S}$-reflecting. $\square$

Instances of the above result are the well-known facts that compact (resp. complete) subspaces of a Hausdorff (resp. uniform) spaces are closed.

Corollary 8 [9] *Suppose that $Y \to 1$ is $\mathcal{S}$-reflecting and $\mathcal{S}$-separated, then any morphism $f : X \to Y$ is $\mathcal{S}$-reflecting if and only if $X \to 1$ is $\mathcal{S}$-reflecting.*

Remark 4 By merging the three parameters $\mathcal{F}$, $\mathscr{C}$ and $\mathscr{U}$ in the categories such as **Unif**, one can obtain satisfactory notions more or less related to convergence such as clustering and pre-compactness: one says that a point x is an adherence point of $\mathcal{F}$ if there is an ultrafilter $\mathscr{U} \supseteq \mathcal{F}$ converging to x, and an object X is pre-compact if $\mathscr{U}(X) \subseteq \mathscr{C}(X)$.

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