

PSEUDO-DUALIZING COMPLEXES OF BICOMODULES AND PAIRS OF T-STRUCTURES

LEONID POSITSELSKI

ABSTRACT. This paper is a coalgebra version of [23] and a sequel to [21]. We present the definition of a pseudo-dualizing complex of bicomodules over a pair of coassociative coalgebras $\mathcal{C}$ and $\mathcal{D}$. For any such complex $\mathcal{L}^\bullet$, we construct a triangulated category endowed with a pair of (possibly degenerate) t-structures of the derived type, whose hearts are the abelian categories of left $\mathcal{C}$ -comodules and left $\mathcal{D}$ -contramodules. A weak version of pseudo-derived categories arising out of (co)resolving subcategories in abelian/exact categories with enough homotopy adjusted complexes is also considered. Quasi-finiteness conditions for coalgebras, comodules, and contramodules are discussed as a preliminary material.

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1. INTRODUCTION

1.1. The philosophy of pseudo-derived equivalences was invented for the purposes of ∞ -tilting theory in [25] and applied to pseudo-dualizing complexes of bimodules in [23]. The latter are otherwise known as “semi-dualizing complexes” in the literature [2, 12].

In its full form, a pseudo-derived equivalence between two abelian categories means an equivalence between their exotic derived categories standing in some sense “in between” the conventional unbounded derived and co/contraderived categories. To be more precise, a “pseudo-derived category” means either a pseudo-coderived or a pseudo-contraderived category. A construction of such pseudo-derived categories

associated with (co)resolving subcategories closed under (co)products in abelian categories with exact (co)products was suggested in [25] and used in [23].

Let $\mathbf{A}$ be an abelian category with exact coproduct functors. The *coderived category* $D^{\text{co}}(\mathbf{A})$ is defined as the triangulated Verdier quotient category of the homotopy category of unbounded complexes $\text{Hot}(\mathbf{A})$ by its minimal full triangulated subcategory $\text{Acycl}^{\text{co}}(\mathbf{A})$ containing the totalizations of short exact sequences of complexes in $\mathbf{A}$ and closed under coproducts. The condition of exactness of coproducts in $\mathbf{A}$ guarantees that all the complexes from $\text{Acycl}^{\text{co}}(\mathbf{A})$ are acyclic in the conventional sense, $\text{Acycl}^{\text{co}}(\mathbf{A}) \subset \text{Acycl}(\mathbf{A})$; hence the conventional derived category $D(\mathbf{A})$ is a Verdier quotient category of the coderived category $D^{\text{co}}(\mathbf{A})$.

Let $\mathbf{E} \subset \mathbf{A}$ be a *coresolving subcategory*, i. e., a full subcategory closed under extensions and cokernels of monomorphisms, and such that every object of $\mathbf{A}$ is a subobject of an object from $\mathbf{E}$. Assume further that the full subcategory $\mathbf{E}$ is closed under coproducts in $\mathbf{A}$. As explained in [25, Section 4] and [23, Section 1], the derived category $D(\mathbf{E})$ of the exact category $\mathbf{E}$ is an intermediate Verdier quotient category between $D^{\text{co}}(\mathbf{A})$ and $D(\mathbf{A})$; so the Verdier quotient functor $D^{\text{co}}(\mathbf{A}) \twoheadrightarrow D(\mathbf{A})$ is the composition of two triangulated Verdier quotient functors $D^{\text{co}}(\mathbf{A}) \twoheadrightarrow D(\mathbf{E}) \twoheadrightarrow D(\mathbf{A})$. The category $D(\mathbf{E})$ is called the *pseudo-coderived category* of $\mathbf{A}$ associated with the coresolving subcategory $\mathbf{E} \subset \mathbf{A}$ closed under coproducts.

Dually, let $\mathbf{B}$ be an abelian category with exact products. The *contraderived category* $D^{\text{ctr}}(\mathbf{B})$ is the triangulated Verdier quotient category of the homotopy category $\text{Hot}(\mathbf{B})$ by its minimal full triangulated subcategory containing the totalizations of short exact sequences in $\mathbf{B}$ and closed under products. The conventional derived category $D(\mathbf{B})$ is naturally a Verdier quotient category of the contraderived category.

Let $\mathbf{F} \subset \mathbf{B}$ be a *resolving subcategory*, i. e., a full subcategory closed under extensions and kernels of monomorphisms such that every object of $\mathbf{B}$ is a quotient object of an object from $\mathbf{F}$. Assume that $\mathbf{F}$ is closed under products in $\mathbf{B}$. Then the derived category $D(\mathbf{F})$ of the exact category $\mathbf{F}$ is an intermediate Verdier quotient category between the contraderived category $D^{\text{ctr}}(\mathbf{B})$ and the derived category $D(\mathbf{B})$; so there are triangulated Verdier quotient functors $D^{\text{ctr}}(\mathbf{B}) \twoheadrightarrow D(\mathbf{F}) \twoheadrightarrow D(\mathbf{B})$. The triangulated category $D(\mathbf{F})$ is called the *pseudo-contraderived category* of $\mathbf{B}$ associated with the resolving subcategory $\mathbf{F} \subset \mathbf{B}$ closed under products.

A *pseudo-derived equivalence* between abelian categories $\mathbf{A}$ and $\mathbf{B}$ is a triangulated equivalence $D(\mathbf{E}) \simeq D(\mathbf{F})$ between their pseudo-derived categories associated with some full subcategories $\mathbf{E} \subset \mathbf{A}$ and $\mathbf{F} \subset \mathbf{B}$. Following the papers [25, 23], pseudo-derived equivalences occur in the ∞ -tilting theory and in connection with pseudo-dualizing complexes of bimodules over a pair of associative rings.

1.2. It appears, though, that coresolving subcategories closed under coproducts and resolving subcategories closed under products are not as common as one would like them to be. For example, if $\mathbf{A}$ is an abelian category with enough injective objects, then the full subcategory of injective objects $\mathbf{A}_{\text{inj}}$ is coresolving in $\mathbf{A}$, but it is rarely

closed under coproducts. Similarly, if $\mathcal{B}$ is an abelian category with enough projectives, then the full subcategory of projective objects $\mathcal{B}_{\text{proj}} \subset \mathcal{B}$ is resolving in $\mathcal{B}$, but it is rarely closed under products.

What can one say about the derived category $D(\mathcal{E})$ or $D(\mathcal{F})$ of a (co)resolving subcategory $\mathcal{E} \subset \mathcal{A}$ or $\mathcal{F} \subset \mathcal{B}$ that is not closed under (co)products? How does one interpret a triangulated equivalence $D(\mathcal{E}) \simeq D(\mathcal{F})$ in terms of the original abelian categories $\mathcal{A}$ and $\mathcal{B}$? For this purpose, a weak version of pseudo-derived equivalences with an equivalence of pseudo-derived categories replaced by a pair of t-structures on a single triangulated category was discussed in [25, Section 5].

Specifically, let $\mathcal{A}$ be an abelian category and $\mathcal{E} \subset \mathcal{A}$ be a coresolving subcategory. Denote by $D^{\geq 0}(\mathcal{E})$ the full subcategory in $D(\mathcal{E})$ consisting of all the objects that can be represented by nonnegatively cohomologically graded complexes in $\mathcal{E}$. Furthermore, denote by $D_{\mathcal{A}}^{\leq 0}(\mathcal{E}) \subset D(\mathcal{E})$ the full subcategory of all objects $E^\bullet$ whose images in $D(\mathcal{A})$ have their cohomology objects $H_A^n(E^\bullet) \in \mathcal{A}$ concentrated in nonpositive cohomological degrees n . Then the pair of full subcategories $D_{\mathcal{A}}^{\leq 0}(\mathcal{E})$ and $D^{\geq 0}(\mathcal{E}) \subset D(\mathcal{E})$ is a t-structure on $D(\mathcal{E})$ with the heart naturally equivalent to $\mathcal{A}$. Moreover, it is a t-structure *of the derived type*, i. e., the Ext groups between objects of the heart computed in the triangulated category $D(\mathcal{E})$ agree with the Ext groups in the abelian category $\mathcal{A}$ [25, Proposition 5.5].

Dually, if $\mathcal{B}$ is an abelian category and $\mathcal{F} \subset \mathcal{B}$ is a resolving subcategory, then the pair of full subcategories $D^{\leq 0}(\mathcal{F})$ and $D_{\mathcal{B}}^{\geq 0}(\mathcal{F})$ is a t-structure of the derived type on $D(\mathcal{F})$ with the heart naturally equivalent to $\mathcal{B}$. Such t-structures can well be *degenerate*, though: the intersections $\bigcap_{n \geq 0} D^{\geq n}(\mathcal{E})$ and $\bigcap_{n \leq 0} D^{\leq n}(\mathcal{F})$ always vanish, but the intersections $\bigcap_{n \leq 0} D_{\mathcal{A}}^{\leq n}(\mathcal{E})$ and $\bigcap_{n \geq 0} D_{\mathcal{B}}^{\geq n}(\mathcal{F})$ are often nontrivial [25, Remark 5.6].

Thus, if the triangulated categories $D(\mathcal{E})$ and $D(\mathcal{F})$ happen to be equivalent, we obtain a pair of t-structures of the derived type on one and the same triangulated category $D(\mathcal{E}) = D = D(\mathcal{F})$. The hearts of these two t-structures are the abelian categories $\mathcal{A}$ and $\mathcal{B}$, respectively. Under a natural additional assumption of nontriviality, we call such a situation a *t-derived pseudo-equivalence* between the abelian categories $\mathcal{A}$ and $\mathcal{B}$. The aim of this paper is to show that such a situation does occur in connection with what we call a *pseudo-dualizing complex of bicomodules* over a pair of coalgebras $\mathcal{C}$ and $\mathcal{D}$.

1.3. We also suggest an alternative point of view on pseudo-derived categories, which is in some way intermediate between the approaches of Section 1.1 (i. e., [25, Section 4]) and Section 1.2 (i. e., [25, Section 5]).

Namely, let $\mathcal{A}$ be an exact category. We say that $\mathcal{A}$ *has enough homotopy injective complexes of injectives* if every object of the derived category $D(\mathcal{A})$ can be represented by a homotopy injective complex of injective objects in $\mathcal{A}$, i. e., a complex of injectives that is right orthogonal to all acyclic complexes in the homotopy category $\text{Hot}(\mathcal{A})$. For example, the abelian category of $\mathcal{C}$ -comodules has enough homotopy injective complexes of injectives for any coalgebra $\mathcal{C}$ [16, Theorem 2.4(a)], [21, Theorem 1.1(c)]; moreover, any Grothendieck abelian category has enough homotopy injective complexes of injectives [1, 27, 9].

Let $E \subset A$ be a coresolving subcategory containing the injective objects. Then the functor $D(E) \rightarrow D(A)$ induced by the inclusion $E \rightarrow A$ is a Verdier quotient functor, and it also has a (fully faithful) right adjoint. Moreover, there is the following diagram of triangulated functors:

$$(1) \quad \begin{array}{c} \text{Hot}(A_{\text{inj}}) \\ \downarrow \\ D(E) \\ \downarrow \\ D(A) \end{array} \quad \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array}$$

Here arrows with a tail denote fully faithful functors and arrows with two heads denote Verdier quotient functors. The downwards directed curvilinear arrow is the composition of the two straight downwards directed arrows, while the upwards directed curvilinear arrows denote right adjoint functors to the downwards directed arrows. The functor $D(A) \rightarrow D(E)$ is the composition $D(A) \rightarrow \text{Hot}(A_{\text{inj}}) \rightarrow D(E)$.

Dually, let B be an exact category with enough homotopy projective complexes of projectives. For example, the abelian category of $\mathcal{C}$ -contramodules has enough homotopy projective complexes of projectives for any coalgebra $\mathcal{C}$ [16, Theorem 2.4(b)], [21, Theorem 1.1(a)]; moreover, any locally presentable abelian category with enough projective objects has enough homotopy projective complexes of projectives [26, Corollary 6.7]. Let $F \subset B$ be a resolving subcategory containing the projective objects. Then the functor $D(F) \rightarrow D(B)$ induced by the inclusion $F \rightarrow B$ is a Verdier quotient functor, and it also has a (fully faithful) left adjoint. Moreover, there is the following diagram of triangulated functors:

$$(2) \quad \begin{array}{c} \text{Hot}(B_{\text{proj}}) \\ \downarrow \\ D(F) \\ \downarrow \\ D(B) \end{array} \quad \begin{array}{c} \curvearrowleft \\ \curvearrowright \end{array}$$

Here the downwards directed curvilinear arrow is the composition of the two straight downwards directed arrows, while the upwards directed curvilinear arrows denote left adjoint functors to the downwards directed arrows. The functor $D(B) \rightarrow D(F)$ is the composition $D(B) \rightarrow \text{Hot}(B_{\text{proj}}) \rightarrow D(F)$.

1.4. The philosophy of *dualizing* and *dedualizing complexes* was discussed in the paper [19] and, in the context of coalgebras, in the paper [21]. Briefly put, dualizing complexes induce equivalences between the coderived and contraderived categories, while dedualizing complexes provide equivalences between conventional derived categories. In particular, given an associative ring A , the one-term complex

of A - A -bimodules A is the simplest example of a dedualizing complex of bimodules, related to the identity derived equivalence $D(A\text{-mod}) = D(A\text{-mod})$; while the datum of a dualizing complex $D^\bullet$ for a pair of rings A and B leads to a triangulated equivalence $D^{\text{co}}(A\text{-mod}) \simeq D^{\text{ctr}}(B\text{-mod})$ [20, 23].

Let k be a fixed ground field and $\mathcal{C}$ be a (coassociative, counital) coalgebra over k . Then there are two kinds of abelian categories that one can assign to $\mathcal{C}$: in addition to the more familiar categories of left and right $\mathcal{C}$ -comodules $\mathcal{C}\text{-comod}$ and $\text{comod-}\mathcal{C}$, there are also less familiar, but no less natural abelian categories of left and right $\mathcal{C}$ -contramodules $\mathcal{C}\text{-contra}$ and $\text{contra-}\mathcal{C}$ [18].

The abelian category of $\mathcal{C}$ -comodules has exact functors of infinite coproducts and enough injective objects, while the abelian category of $\mathcal{C}$ -contramodules has exact functors of infinite products and enough projective objects. There is a fundamental homological phenomenon of *comodule-contramodule correspondence*, meaning a natural triangulated equivalence between the coderived category of comodules and the contraderived category of contramodules [15, Sections 0.2.6–0.2.7], and [16, Sections 4.4 and 5.2]

$$(3) \quad D^{\text{co}}(\mathcal{C}\text{-comod}) \simeq \text{Hot}(\mathcal{C}\text{-comod}_{\text{inj}}) \simeq \text{Hot}(\mathcal{C}\text{-contra}_{\text{proj}}) \simeq D^{\text{ctr}}(\mathcal{C}\text{-contra}).$$

The triangulated equivalence (3) is induced by an equivalence between the additive categories of injective left $\mathcal{C}$ -comodules and projective left $\mathcal{C}$ -contramodules, $\mathcal{C}\text{-comod}_{\text{inj}} \simeq \mathcal{C}\text{-contra}_{\text{proj}}$. The latter equivalence is provided by the adjoint functors of left $\mathcal{C}$ -comodule homomorphisms from and the *contratensor product* with the $\mathcal{C}$ - $\mathcal{C}$ -bicomodule $\mathcal{C}$,

$$(4) \quad \text{Hom}_{\mathcal{C}}(\mathcal{C}, -) : \mathcal{C}\text{-comod}_{\text{inj}} \simeq \mathcal{C}\text{-contra}_{\text{proj}} : \mathcal{C} \odot_{\mathcal{C}} -.$$

Thus the one-term complex of $\mathcal{C}$ - $\mathcal{C}$ -bicomodules $\mathcal{C}$ is the simplest example of a dualizing complex of bicomodules.

Let $\mathcal{C}$ and $\mathcal{D}$ be two coalgebras over the same ground field k . The definition of a *dedualizing complex* of $\mathcal{C}$ - $\mathcal{D}$ -bicomodules was given in the paper [21, Section 3]. According to [21, Theorem 3.6], the datum of a dedualizing complex of bicomodules $\mathcal{B}^\bullet$ for a left cocoherent coalgebra $\mathcal{C}$ and a right cocoherent coalgebra $\mathcal{D}$ over a field k induces an equivalence of derived categories

$$(5) \quad D^\star(\mathcal{C}\text{-comod}) \simeq D^\star(\mathcal{D}\text{-contra})$$

for any conventional derived category symbol $\star = \mathbf{b}, +, -, \text{ or } \emptyset$. Moreover, similar triangulated equivalences can be constructed for any *absolute* derived category symbol $\star = \text{abs}+, \text{abs}-, \text{ or } \text{abs}$. The triangulated equivalences (5) are provided by the mutually inverse right derived functor of comodule homomorphisms $\mathbb{R}\text{Hom}_{\mathcal{C}}(\mathcal{B}^\bullet, -)$ and left derived functor of contratensor product $\mathcal{B}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} -$.

1.5. All the definitions of a dualizing complex of (bi)modules [11, 32, 33, 14, 3, 20] involve three kinds of conditions: (i) finite injective dimension, (ii) finite generatedness, and (iii) homothety isomorphisms. Analogously, the definitions of a dedualizing complex in [19, 21] involve (roughly) three kinds of conditions: (i) finite projective dimension, (ii) finite (co)generatedness, and (iii) homothety isomorphisms.

The definition of a pseudo-dualizing complex [23] (or, in the more traditional terminology, a “semi-dualizing complex” [2, 12]) is obtained from that of a (de)dualizing complex by dropping the finite injective/projective dimension condition (i), while retaining the finite generatedness and homothety isomorphism conditions (ii–iii).

In all these situations, the (derived) homothety isomorphism conditions are rather straightforward to formulate, but the finite (co)generatedness conditions are complicated, with many alternative versions of them considered in various papers. In particular, the paper [21] starts with a discussion of finite (co)generatedness and (co)presentability conditions in comodule and contramodule categories in [21, Section 2]. As we mentioned in Section 1.4, the construction of the derived equivalence in [21, Theorem 3.6] (see (5) above) was given in the assumption of cocoherence conditions on the coalgebras $\mathcal{C}$ and $\mathcal{D}$.

Finite cogeneratedness and finite copresentability conditions are not very natural for coalgebras, though, as they are not invariant under the Morita–Takeuchi equivalences of coalgebras. Quasi-finite cogeneratedness and quasi-finite copresentability conditions, going back to Takeuchi’s classical paper [30], are generally preferable. So this paper starts with a discussion of quasi-finitely cogenerated and copresented comodules and quasi-finitely generated and presented contramodules in Section 3.

We strived to relax the finite generatedness/presentability conditions as much as possible in the paper [23], and we do likewise in the present paper. The result is that no coherence assumptions about the rings A and B are used in the main results of [23], and no cocoherence or quasi-cocoherence assumptions about the coalgebras $\mathcal{C}$ and $\mathcal{D}$ are made in the main results of the present paper. Instead, we imposed appropriate finite presentability conditions on the pseudo-dualizing complex of bimodules $L^\bullet$ in the paper [23], and we impose appropriate quasi-finite copresentability conditions on the pseudo-dualizing complex of bicomodules $\mathcal{L}^\bullet$ in this paper.

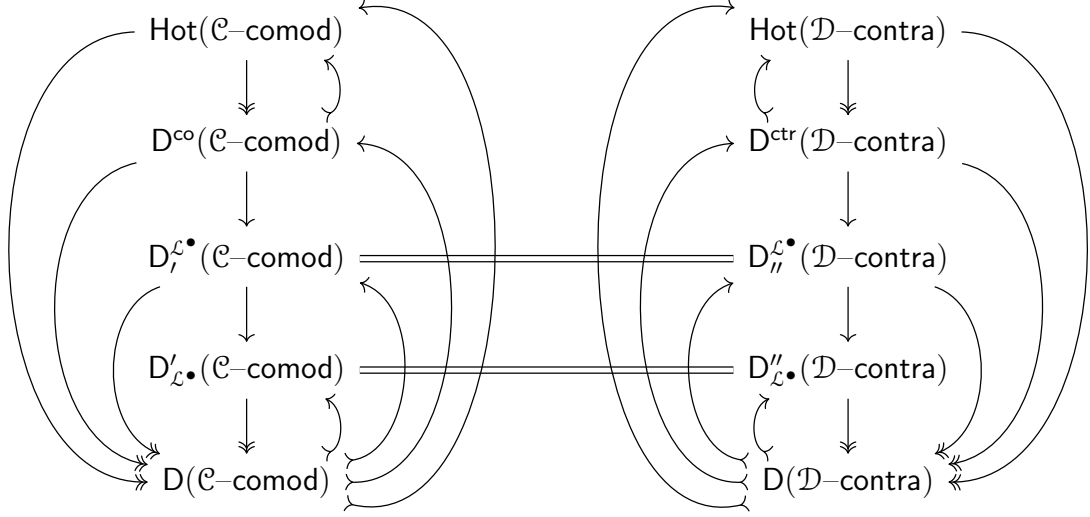
1.6. Let $\mathcal{C}$ and $\mathcal{D}$ be coassociative coalgebras over a fixed field k . A *pseudo-dualizing complex* $\mathcal{L}^\bullet$ for the coalgebras $\mathcal{C}$ and $\mathcal{D}$ (cf. the discussion of “semidualizing bicomodules” in [8]) is a finite complex of $\mathcal{C}$ - $\mathcal{D}$ -bicomodules satisfying the following two conditions:

- (ii) as a complex of left $\mathcal{C}$ -comodules, $\mathcal{L}^\bullet$ is quasi-isomorphic to a bounded below complex of quasi-finitely cogenerated injective $\mathcal{C}$ -comodules, and similarly, as a complex of right $\mathcal{D}$ -comodules, $\mathcal{L}^\bullet$ is quasi-isomorphic to a bounded below complex of quasi-finitely cogenerated injective $\mathcal{D}$ -comodules;
- (iii) the homothety maps $\mathcal{C}^* \longrightarrow \mathrm{Hom}_{\mathrm{D}^b(\mathrm{comod}\text{-}\mathcal{D})}(\mathcal{L}^\bullet, \mathcal{L}^\bullet[*])$ and $\mathcal{D}^{*\mathrm{op}} \longrightarrow \mathrm{Hom}_{\mathrm{D}^b(\mathcal{C}\text{-}\mathrm{comod})}(\mathcal{L}^\bullet, \mathcal{L}^\bullet[*])$ are isomorphisms of graded rings.

This definition is obtained by dropping the finite projective and contraflat dimension condition (i) from the definition of a *dedualizing* complex of $\mathcal{C}$ - $\mathcal{D}$ -bicomodules $\mathcal{B}^\bullet$ in [21, Section 3], removing the cocoherence conditions on the coalgebras, and rewriting the finite copresentability condition (ii) accordingly. Here the quasi-finite cogeneratedness is a natural weakening of the finite cogeneratedness condition on

comodules, having the advantage of being Morita-invariant [30], as discussed above in Section 1.5.

The main result of this paper provides the following diagram of triangulated functors associated with a pseudo-dualizing complex of $\mathcal{C}$ - $\mathcal{D}$ -bicomodules $\mathcal{L}^\bullet$ (cf. [23]):
(6)



Here the functors shown by arrows with two heads are Verdier quotient functors, the functors shown by arrows with a tail are fully faithful, and double lines show triangulated equivalences. The outer, downwards directed curvilinear arrows (with two heads) are the compositions of the vertical straight arrows. The inner, upwards directed curvilinear arrows (with a tail) are adjoint on the right (in the comodule part of the diagram) and on the left (in the contra-module part of the diagram) to the compositions of the vertical straight arrows.

In particular, when $\mathcal{L}^\bullet = \mathcal{B}^\bullet$ is a dedualizing complex for a pair of coalgebras $\mathcal{C}$ and $\mathcal{D}$, i. e., the finite projective/contraflat dimension condition (i) of [21, Section 3] is satisfied, one has $D'_{\mathcal{L}^\bullet}(\mathcal{C}\text{-comod}) = D(\mathcal{C}\text{-comod})$ and $D'_{\mathcal{L}^\bullet}(\mathcal{D}\text{-contra}) = D(\mathcal{D}\text{-contra})$. In other words, the lower two vertical arrows in the diagram (6) are isomorphisms of triangulated categories. The lower triangulated equivalence in the diagram (6) coincides with the one provided by [21, Theorem 3.6] in this case.

When $\mathcal{L}^\bullet = \mathcal{C} = \mathcal{D}$, one has $D_{\mathcal{L}^\bullet}^{\mathcal{L}^\bullet}(\mathcal{C}\text{-comod}) = D^{\text{co}}(\mathcal{C}\text{-comod})$ and $D_{\mathcal{L}^\bullet}^{\mathcal{L}^\bullet}(\mathcal{D}\text{-contra}) = D^{\text{ctr}}(\mathcal{D}\text{-contra})$, that is the next-to-upper two vertical arrows in the diagram (6) are isomorphisms of triangulated categories. The upper triangulated equivalence in the diagram (6) is the derived comodule-contra-module correspondence (3) in this case. More generally, the upper triangulated equivalence in the diagram (6) corresponding to a *dualizing complex* $\mathcal{L}^\bullet = \mathcal{K}^\bullet$ for a pair of coalgebras $\mathcal{C}$ and $\mathcal{D}$ can be thought of as a part of derived Morita–Takeuchi equivalence (cf. [5]).

1.7. Among the five pairs of triangulated categories on the diagram (6), there are two pairs which depend on the pseudo-dualizing complex $\mathcal{L}^\bullet$. These four triangulated

categories $D_{\mathcal{L}^\bullet}'(\mathcal{C}\text{-comod})$, $D_{\mathcal{L}^\bullet}''(\mathcal{D}\text{-contra})$, $D_{\mathcal{L}^\bullet}'(\mathcal{C}\text{-comod})$, and $D_{\mathcal{L}^\bullet}''(\mathcal{D}\text{-contra})$ are constructed in the following way.

Suppose that the finite complex $\mathcal{L}^\bullet$ is situated in the cohomological degrees $-d_1 \leq m \leq d_2$. Then there are two pairs of sequences of full subcategories

$$\dots \subset E^{d_2+2} \subset E^{d_2+1} \subset E^{d_2} \subset E_{d_1} \subset E_{d_1+1} \subset E_{d_1+2} \subset \dots \subset A$$

and

$$\dots \subset F^{d_2+2} \subset F^{d_2+1} \subset F^{d_2} \subset F_{d_1} \subset F_{d_1+1} \subset F_{d_1+2} \subset \dots \subset B$$

in the abelian categories $A = \mathcal{C}\text{-comod}$ and $B = \mathcal{D}\text{-contra}$. The subcategories with lower indices form increasing sequences, while the subcategories with upper indices form decreasing sequences. The full subcategories E_{l_1} and $E^{l_2} \subset A$, where $l_1 \geq d_1$ and $l_2 \geq d_2$, are coresolving, while the full subcategories F_{l_1} and $F^{l_2} \subset B$ are resolving.

For any two integers $l_1'' \geq l_1' \geq d_1$, the category $E_{l_1''}$ has finite $E_{l_1'}$ -coresolution dimension, while the category $F_{l_1''}$ has finite $F_{l_1'}$ -resolution dimension. Therefore, the derived category $D(E_{l_1})$ of the exact category E_{l_1} does not depend on the choice of an integer $l_1 \geq d_1$, and similarly, the derived category $D(F_{l_1})$ of the exact category F_{l_1} does not depend on the choice of l_1 . We set

$$D_{\mathcal{L}^\bullet}'(\mathcal{C}\text{-comod}) = D(E_{l_1}) \quad \text{and} \quad D_{\mathcal{L}^\bullet}''(\mathcal{D}\text{-contra}) = D(F_{l_1}).$$

For any two integers $l_2'' \geq l_2' \geq d_2$, the category $E^{l_2''}$ has finite $E^{l_2'}$ -coresolution dimension, while the category $F^{l_2''}$ has finite $F^{l_2'}$ -resolution dimension. Therefore, the derived category $D(E^{l_2})$ does not depend on the choice of an integer $l_2 \geq d_2$, and similarly, the derived category $D(F^{l_2})$ does not depend on the choice of l_2 . We set

$$D_{\mathcal{L}^\bullet}'(\mathcal{C}\text{-comod}) = D(E^{l_2}) \quad \text{and} \quad D_{\mathcal{L}^\bullet}''(\mathcal{D}\text{-contra}) = D(F^{l_2}).$$

Let us now briefly explain where the full subcategories E_{l_1} , E^{l_2} , F_{l_1} , and F^{l_2} come from. The full subcategories $E_{l_1} \subset \mathcal{C}\text{-comod}$ and $F_{l_1} \subset \mathcal{D}\text{-contra}$ are our analogues of what are known as the *Auslander* and *Bass classes* in the literature [2, 6, 3, 12, 8]. So they are defined as the classes of all left $\mathcal{C}$ -comodules and left $\mathcal{D}$ -contra modules satisfying certain conditions with respect to the derived functors $\mathbb{R}\mathrm{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, -)$ and $\mathcal{L}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} -$, with the parameter l_1 meaning a certain (co)homological degree. The full subcategory $F_{l_1} \subset \mathcal{D}\text{-contra}$ is an analogue of the Auslander class and the full subcategory $E_{l_1} \subset \mathcal{C}\text{-comod}$ is a version of the Bass class. These are the *maximal corresponding classes* of objects in the categories $A = \mathcal{C}\text{-comod}$ and $B = \mathcal{D}\text{-contra}$ with respect to the covariant duality defined by the pseudo-dualizing complex $\mathcal{L}^\bullet$ (cf. the discussions of the Auslander and Bass classes in [23, Sections 0.7 and 3] and the maximal ∞ -tilting and ∞ -cotilting classes in [25, Sections 2–3]).

The full subcategories $E^{l_2} \subset \mathcal{C}\text{-comod}$ and $F^{l_2} \subset \mathcal{D}\text{-contra}$ are the *minimal corresponding classes* in the categories $A = \mathcal{C}\text{-comod}$ and $B = \mathcal{D}\text{-contra}$. They are defined by a certain iterative generation procedure, starting from the full subcategory of injectives in A and the full subcategory of projectives in B , and proceeding using the derived functors $\mathbb{R}\mathrm{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, -)$ and $\mathcal{L}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} -$, with the parameter l_2 , once again, meaning a certain (co)homological degree. These are the analogues of the minimal

corresponding classes from [23, Sections 0.7 and 5] and of the minimal ∞ -tilting and ∞ -cotilting classes from [25, Lemma 3.6 and Example 3.7].

The derived equivalences

$$D^*(E_{l_1}) \simeq D^*(F_{l_1}) \quad \text{and} \quad D^*(E^{l_2}) \simeq D^*(F^{l_2})$$

(including, in particular, the triangulated equivalences in the diagram (6) arising as the particular cases for $\star = \emptyset$) are provided by *appropriately constructed* derived functors of comodule homomorphisms and contratensor product $\mathbb{R} \operatorname{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, -)$ and $\mathcal{L}^\bullet \odot_{\mathcal{D}}^\mathbb{L} -$. The rather complicated constructions of these derived functors are based on the technique developed in [23, Appendix A].

Following the discussion in Section 1.2, each of the triangulated categories

$$D'_{\mathcal{L}^\bullet}(\mathcal{C}\text{-comod}) \simeq D''_{\mathcal{L}^\bullet}(\mathcal{D}\text{-contra}) \quad \text{and} \quad D'^{\mathcal{L}^\bullet}_I(\mathcal{C}\text{-comod}) \simeq D''^{\mathcal{L}^\bullet}_{II}(\mathcal{D}\text{-contra})$$

carries two (very possibly degenerate) t-structures of the derived type, whose abelian hearts are the categories of left $\mathcal{C}$ -comodules and left $\mathcal{D}$ -contramodules $\mathbf{A} = \mathcal{C}\text{-comod}$ and $\mathbf{B} = \mathcal{D}\text{-contra}$.

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2. HOMOTOPY ADJUSTED COMPLEXES AND (CO)RESOLVING SUBCATEGORIES

In this section we prove the results promised in Section 1.3 of the introduction.

Following the terminology in [23] (originating from the assumptions of [17, Sections A.3 and A.5]), which differs slightly from the standard terminology, we do *not* include closedness under direct summands into the definition of a (co)resolving subcategory (see Section 1.1). This allows us to avoid the unnecessary closure under direct summands in the conditions (I–IV) of Section 5 below. That is why we sometimes need to assume separately that a coresolving subcategory contains the injective objects, or that a resolving subcategory contains the projective objects.

Let $\mathbf{A}$ be an exact category (the reader will lose little by assuming that $\mathbf{A}$ is abelian). An (unbounded) complex $J^\bullet$ in $\mathbf{A}$ is said to be *homotopy injective* if for any acyclic complex $X^\bullet$ in $\mathbf{A}$ the complex of abelian groups $\operatorname{Hom}_{\mathbf{A}}(X^\bullet, J^\bullet)$ is acyclic. We say that an exact category $\mathbf{A}$ has *enough homotopy injective complexes* if for any complex $M^\bullet$ in $\mathbf{A}$ there exists a homotopy injective complex $J^\bullet$ together with a quasi-isomorphism $M^\bullet \rightarrow J^\bullet$ of complexes in $\mathbf{A}$.

Consider the canonical Verdier quotient functor from the homotopy category to the derived category of unbounded complexes,

$$(7) \quad \operatorname{Hot}(\mathbf{A}) \longrightarrow D(\mathbf{A}).$$

An exact category $\mathbf{A}$ has enough homotopy injective complexes if and only if the functor (7) has a right adjoint. If this is the case, such a right adjoint functor assigns to any complex $M^\bullet$ in $\mathbf{A}$ its homotopy injective resolution, i. e., a homotopy injective complex $J^\bullet$ endowed with a quasi-isomorphism $M^\bullet \rightarrow J^\bullet$.

Moreover, we will say that an exact category $\mathbf{A}$ has *enough homotopy injective complexes of injectives* if for any complex $M^\bullet$ in $\mathbf{A}$ there exists a homotopy injective complex of injective objects $J^\bullet$ together with a quasi-isomorphism $M^\bullet \rightarrow J^\bullet$ of complexes in $\mathbf{A}$. If this is the case, then the right adjoint functor $D(\mathbf{A}) \rightarrow \text{Hot}(\mathbf{A})$ to the functor (7) factorizes through the homotopy category of complexes of injective objects, $D(\mathbf{A}) \rightarrow \text{Hot}(\mathbf{A}_{\text{inj}}) \rightarrow \text{Hot}(\mathbf{A})$. We will denote the resulting functor by $\theta: D(\mathbf{A}) \rightarrow \text{Hot}(\mathbf{A}_{\text{inj}})$.

For example, any Grothendieck abelian category $\mathbf{A}$ has enough homotopy injective complexes [1, Theorem 5.4]; moreover, it has enough homotopy injective complexes of injective objects [27, Theorem 3.13 and Lemma 3.7(ii)], [9, Corollary 7.1]. In particular, for any coassociative coalgebra $\mathcal{C}$ over a field k , the abelian category of left $\mathcal{C}$ -comodules $\mathbf{A} = \mathcal{C}\text{-comod}$ has enough homotopy injective complexes of injectives [16, Sections 2.4 and 5.5] (see also [21, Theorem 1.1(c)]).

Dually, let $\mathbf{B}$ be an exact category (which is also going to be abelian in most applications). An (unbounded) complex $P^\bullet$ in $\mathbf{B}$ is said to be *homotopy projective* if for any acyclic complex $Y^\bullet$ in $\mathbf{B}$ the complex of abelian groups $\text{Hom}_{\mathbf{B}}(P^\bullet, Y^\bullet)$ is acyclic. We say that an exact category $\mathbf{B}$ has *enough homotopy projective complexes* if for any complex $T^\bullet$ in $\mathbf{B}$ there exists a homotopy projective complex $P^\bullet$ together with a quasi-isomorphism $P^\bullet \rightarrow T^\bullet$ of complexes in $\mathbf{B}$.

An exact category $\mathbf{B}$ has enough homotopy projective complexes if and only if the canonical Verdier quotient functor $\text{Hot}(\mathbf{B}) \rightarrow D(\mathbf{B})$ has a left adjoint (which then assigns to a complex $T^\bullet$ in $\mathbf{B}$ its homotopy projective resolution).

Moreover, we will say that an exact category $\mathbf{B}$ has *enough homotopy projective complexes of projectives* if for any complex $T^\bullet$ in $\mathbf{B}$ there exists a homotopy projective complex of projective objects $P^\bullet$ together with a quasi-isomorphism $P^\bullet \rightarrow T^\bullet$. If this is the case, then the above-mentioned left adjoint functor $D(\mathbf{B}) \rightarrow \text{Hot}(\mathbf{B})$ to the canonical Verdier quotient functor factorizes through the homotopy category of complexes of projective objects, $D(\mathbf{B}) \rightarrow \text{Hot}(\mathbf{B}_{\text{proj}}) \rightarrow \text{Hot}(\mathbf{B})$. We will denote the resulting functor by $\varkappa: D(\mathbf{B}) \rightarrow \text{Hot}(\mathbf{B}_{\text{proj}})$.

For example, any locally presentable abelian category $\mathbf{B}$ with enough projective objects has enough homotopy projective complexes of projectives [26, Lemma 6.1 and Corollary 6.7]. In particular, for any coassociative coalgebra $\mathcal{D}$ over a field k , the abelian category of left $\mathcal{D}$ -contra modules $\mathbf{B} = \mathcal{D}\text{-contra}$ has enough homotopy projective complexes of projective objects [16, Sections 2.4 and 5.5] (see also [21, Theorem 1.1(a)]).

Theorem 2.1. (a) *Let $\mathbf{A}$ be an exact category with enough homotopy injective complexes of injectives, and let $\mathbf{E} \subset \mathbf{A}$ be a coresolving subcategory containing the injective objects. Then the triangulated functor $D(\mathbf{E}) \rightarrow D(\mathbf{A})$ induced by the inclusion of exact categories $\mathbf{E} \rightarrow \mathbf{A}$ is a Verdier quotient functor, and it has a right adjoint functor*

$\rho: D(A) \longrightarrow D(E)$. Moreover, there is a diagram of triangulated functors

$$\begin{array}{c}
 \text{Hot}(A_{\text{inj}}) \\
 \downarrow \\
 D(E) \\
 \downarrow \\
 D(A)
 \end{array}
 \begin{array}{c}
 \nearrow \theta \\
 \nearrow \rho
 \end{array}$$

where the functor $\text{Hot}(A_{\text{inj}}) \longrightarrow D(E)$ is induced by the inclusion $A_{\text{inj}} \longrightarrow E$, the functors shown by two-headed arrows are Verdier quotient functors, the long downwards directed curvilinear arrow is the composition of two straight downwards directed arrows, and the upwards directed curvilinear arrows with a tail show fully faithful right adjoint functors to the downwards directed arrows. The functor $\rho: D(A) \longrightarrow D(E)$ is the composition $D(A) \xrightarrow{\theta} D(A_{\text{inj}}) \longrightarrow D(E)$.

(b) Let B be an exact category with enough homotopy projective complexes of projectives, and let $F \subset B$ be a resolving subcategory containing the projective objects. Then the triangulated functor $D(F) \longrightarrow D(B)$ induced by the inclusion of exact categories $F \longrightarrow B$ is a Verdier quotient functor, and it has a left adjoint functor $\lambda: D(B) \longrightarrow D(F)$. Moreover, there is a diagram of triangulated functors

$$\begin{array}{c}
 \text{Hot}(B_{\text{proj}}) \\
 \downarrow \\
 D(F) \\
 \downarrow \\
 D(B)
 \end{array}
 \begin{array}{c}
 \nearrow \lambda \\
 \nearrow \zeta
 \end{array}$$

where the functor $\text{Hot}(B_{\text{proj}}) \longrightarrow D(F)$ is induced by the inclusion $B_{\text{proj}} \longrightarrow F$, the functors shown by two-headed arrows are Verdier quotient functors, the long downwards directed curvilinear arrow is the composition of two straight downwards directed arrows, and the upwards directed curvilinear arrows with a tail show fully faithful left adjoint functors to the downwards directed arrows. The functor $\lambda: D(B) \longrightarrow D(F)$ is the composition $D(B) \xrightarrow{\zeta} D(B_{\text{proj}}) \longrightarrow D(F)$.

Proof. We will prove part (a), as part (b) is dual. We have already explained how the functors are constructed. Any triangulated functor with a fully faithful adjoint is a Verdier quotient functor, and conversely any triangulated functor adjoint to a Verdier quotient functor is fully faithful [7, Proposition I.1.3].

The adjunction between the two functors $\text{Hot}(A_{\text{inj}}) \longrightarrow D(A)$ and $D(A) \xrightarrow{\theta} \text{Hot}(A_{\text{inj}})$ is obtained by restricting the adjunction between the two functors $\text{Hot}(A) \longrightarrow D(A)$ and $D(A) \longrightarrow \text{Hot}(A)$ to the full subcategory $\text{Hot}(A_{\text{inj}}) \subset \text{Hot}(A)$. Since the image of the functor $D(A) \longrightarrow \text{Hot}(A)$ is contained in $\text{Hot}(A_{\text{inj}}) \subset \text{Hot}(A)$,

such restriction makes sense. The functor $\text{Hot}(\mathbf{A}) \rightarrow \mathbf{D}(\mathbf{A})$ is a Verdier quotient functor, hence the functor $\mathbf{D}(\mathbf{A}) \rightarrow \text{Hot}(\mathbf{A})$ is fully faithful, so the functor $\mathbf{D}(\mathbf{A}) \xrightarrow{\theta} \text{Hot}(\mathbf{A}_{\text{inj}})$ is fully faithful as well, and it follows that the functor $\text{Hot}(\mathbf{A}_{\text{inj}}) \rightarrow \mathbf{D}(\mathbf{A})$ is a Verdier quotient functor.

It remains to show that the functor $\rho: \mathbf{D}(\mathbf{A}) \rightarrow \mathbf{D}(\mathbf{E})$ is fully faithful and right adjoint to the triangulated functor $\mathbf{D}(\mathbf{E}) \rightarrow \mathbf{D}(\mathbf{A})$ induced by the inclusion $\mathbf{E} \rightarrow \mathbf{A}$; then it will follow that the latter functor is a Verdier quotient functor. For this purpose, we decompose the functor $\text{Hot}(\mathbf{A}_{\text{inj}}) \rightarrow \mathbf{D}(\mathbf{E})$ as $\text{Hot}(\mathbf{A}_{\text{inj}}) \rightarrow \text{Hot}(\mathbf{E}) \rightarrow \mathbf{D}(\mathbf{E})$, where $\text{Hot}(\mathbf{A}_{\text{inj}}) \rightarrow \text{Hot}(\mathbf{E})$ is the functor induced by the inclusion of additive categories $\mathbf{A}_{\text{inj}} \rightarrow \mathbf{E}$ and $\text{Hot}(\mathbf{E}) \rightarrow \mathbf{D}(\mathbf{E})$ is the canonical Verdier quotient functor. Then the functor ρ decomposes as

$$\mathbf{D}(\mathbf{A}) \xrightarrow{\theta} \text{Hot}(\mathbf{A}_{\text{inj}}) \longrightarrow \text{Hot}(\mathbf{E}) \longrightarrow \mathbf{D}(\mathbf{E}).$$

For simplicity, in the following lemma we denote the compositions $\mathbf{D}(\mathbf{A}) \xrightarrow{\theta} \text{Hot}(\mathbf{A}_{\text{inj}}) \rightarrow \text{Hot}(\mathbf{E})$ and $\mathbf{D}(\mathbf{A}) \xrightarrow{\theta} \text{Hot}(\mathbf{A}_{\text{inj}}) \rightarrow \text{Hot}(\mathbf{A})$ also by θ .

Lemma 2.2. *Let $\mathbf{H}$ be a triangulated category and $\mathbf{Y} \subset \mathbf{G} \subset \mathbf{H}$, $\mathbf{Y} \subset \mathbf{X} \subset \mathbf{H}$ be its full triangulated subcategories. Suppose that the Verdier quotient functor $\mathbf{H} \rightarrow \mathbf{H}/\mathbf{X}$ has a right adjoint $\theta: \mathbf{H}/\mathbf{X} \rightarrow \mathbf{H}$, whose image is contained in $\mathbf{G}$. Then the triangulated functor $\mathbf{G}/\mathbf{Y} \rightarrow \mathbf{H}/\mathbf{X}$ induced by the inclusion $(\mathbf{G}, \mathbf{Y}) \hookrightarrow (\mathbf{H}, \mathbf{X})$ is a Verdier quotient functor with a right adjoint functor ρ , which can be computed as the composition $\mathbf{H}/\mathbf{X} \xrightarrow{\theta} \mathbf{G} \rightarrow \mathbf{G}/\mathbf{Y}$.*

Proof. The functor θ is fully faithful as an adjoint to a Verdier quotient functor. Denote by ρ the composition $\mathbf{H}/\mathbf{X} \rightarrow \mathbf{G} \rightarrow \mathbf{G}/\mathbf{Y}$; so $\rho(C) = \theta(C)/\mathbf{Y} \in \mathbf{G}/\mathbf{Y}$ for all $C \in \mathbf{H}/\mathbf{X}$. To prove that the functor ρ is fully faithful, it suffices to check that all the objects in the image of θ are right orthogonal to $\mathbf{Y}$, that is $\text{Hom}_{\mathbf{G}}(\mathbf{Y}, \theta(C)) = 0$ for all $C \in \mathbf{H}/\mathbf{X}$ and $\mathbf{Y} \in \mathbf{Y}$. Indeed, we have $\text{Hom}_{\mathbf{H}}(\mathbf{X}, \theta(C)) = 0$ for all $\mathbf{X} \in \mathbf{X}$, as the image of $\mathbf{X}$ in $\mathbf{H}/\mathbf{X}$ vanishes.

It remains to show that ρ is right adjoint to the functor $\mathbf{G}/\mathbf{Y} \rightarrow \mathbf{H}/\mathbf{X}$; it will then follow that the latter is a Verdier quotient functor. Let $E \in \mathbf{G}$ and $C \in \mathbf{H}/\mathbf{X}$ be two objects. Then we have

$$\begin{aligned} \text{Hom}_{\mathbf{G}/\mathbf{Y}}(E/\mathbf{Y}, \rho(C)) &= \text{Hom}_{\mathbf{G}/\mathbf{Y}}(E/\mathbf{Y}, \theta(C)/\mathbf{Y}) \simeq \text{Hom}_{\mathbf{G}}(E, \theta(C)) \\ &= \text{Hom}_{\mathbf{H}}(E, \theta(C)) \simeq \text{Hom}_{\mathbf{H}/\mathbf{X}}(E/\mathbf{X}, C), \end{aligned}$$

since $\theta(C)$ is right orthogonal to $\mathbf{Y}$. □

To finish the proof of part (a), it remains to set

$$\mathbf{H} = \text{Hot}(\mathbf{A}) \supset \mathbf{G} = \text{Hot}(\mathbf{E})$$

and denote further by $\mathbf{X} \subset \mathbf{H}$ the full subcategory of acyclic complexes in the exact category $\mathbf{A}$ and by $\mathbf{Y} \subset \mathbf{X} \cap \mathbf{G}$ the full subcategory of acyclic complexes in the exact category $\mathbf{E}$. So $\mathbf{H}/\mathbf{X} = \mathbf{D}(\mathbf{A})$ and $\mathbf{G}/\mathbf{Y} = \mathbf{D}(\mathbf{E})$. The right adjoint functor to the

Verdier quotient functor $\text{Hot}(\mathbf{A}) \longrightarrow \mathbf{D}(\mathbf{A})$ lands inside $\text{Hot}(\mathbf{A}_{\text{inj}}) \subset \text{Hot}(\mathbf{E}) \subset \text{Hot}(\mathbf{A})$, as required in the lemma. $\square$

3. QUASI-FINITENESS CONDITIONS FOR COALGEBRAS

We refer to the classical book [29], the introductory section and appendix [15, Section 0.2 and Appendix A], the memoir [16], the overview [18], the paper [21], and the references therein for a general discussion of coassociative coalgebras over fields and module objects (comodules and contramodules) over them.

Let k be a fixed ground field and $\mathcal{C}$ be a coassociative coalgebra (with counit) over k . We denote by $\mathcal{C}\text{-comod}$ and $\text{comod-}\mathcal{C}$ the abelian categories of left and right $\mathcal{C}$ -comodules. The abelian category of left $\mathcal{C}$ -contramodules is denoted by $\mathcal{C}\text{-contra}$.

For any two left $\mathcal{C}$ -comodules $\mathcal{M}$ and $\mathcal{N}$, we denote by $\text{Hom}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ the k -vector space of left $\mathcal{C}$ -comodule morphisms $\mathcal{M} \longrightarrow \mathcal{N}$. For any two left $\mathcal{C}$ -contramodules $\mathfrak{S}$ and $\mathfrak{T}$, we denote by $\text{Hom}^{\mathcal{C}}(\mathfrak{S}, \mathfrak{T})$ the k -vector space of left $\mathcal{C}$ -contramodule morphisms $\mathfrak{S} \longrightarrow \mathfrak{T}$. The coalgebra opposite to $\mathcal{C}$ is denoted by $\mathcal{C}^{\text{op}}$; so a right $\mathcal{C}$ -comodule is the same thing as a left $\mathcal{C}^{\text{op}}$ -comodule.

We recall that for any right $\mathcal{C}$ -comodule $\mathcal{N}$ and k -vector space V the vector space $\text{Hom}_k(\mathcal{N}, V)$ has a natural left $\mathcal{C}$ -contramodule structure [18, Sections 1.1–2]. We refer to [21, Section 2], [18, Section 3.1], [16, Section 2.2], or [15, Sections 0.2.6 and 5.1.1–2] for the definition and discussion of the functor of *contratensor product* $\mathcal{N} \odot_{\mathcal{C}} \mathfrak{T}$ of a right $\mathcal{C}$ -comodule $\mathcal{N}$ and a left $\mathcal{C}$ -contramodule $\mathfrak{T}$.

The construction of the *cotensor product* $\mathcal{N} \square_{\mathcal{C}} \mathcal{M}$ of a right $\mathcal{C}$ -comodule $\mathcal{N}$ and a left $\mathcal{C}$ -comodule $\mathcal{M}$ goes back at least to the paper [13, Section 2]. The dual-analogous construction involving contramodules is the vector space of *cohomomorphisms* $\text{Cohom}_{\mathcal{C}}(\mathcal{M}, \mathfrak{T})$ from a left $\mathcal{C}$ -comodule $\mathcal{M}$ to a left $\mathcal{C}$ -contramodule $\mathfrak{T}$. We refer to [21, Section 2], [18, Sections 2.5–6], or [15, Sections 0.2.1, 0.2.4, 1.2.1, and 3.2.1] for the definitions and discussion of these constructions.

Finiteness and quasi-finiteness conditions for coalgebras and comodules were studied in [30, 31, 10, 21] and many other papers. The next two lemmas are well-known and included here for the reader's convenience.

Given a subcoalgebra $\mathcal{B} \subset \mathcal{C}$ and a left $\mathcal{C}$ -comodule $\mathcal{M}$, let ${}_{\mathcal{B}}\mathcal{M} \subset \mathcal{M}$ denote the maximal $\mathcal{B}$ -subcomodule in $\mathcal{M}$ whose $\mathcal{C}$ -comodule structure comes from a $\mathcal{B}$ -comodule structure. The $\mathcal{B}$ -comodule ${}_{\mathcal{B}}\mathcal{M}$ can be computed as the full preimage of the subcomodule $\mathcal{B} \otimes_k \mathcal{M} \subset \mathcal{C} \otimes_k \mathcal{M}$ under the coaction map $\mathcal{M} \longrightarrow \mathcal{C} \otimes_k \mathcal{M}$, or as the cotensor product $\mathcal{B} \square_{\mathcal{C}} \mathcal{M}$ [21, Section 2].

Lemma 3.1. *Let $\mathcal{C}$ be a coassociative coalgebra over k and $\mathcal{M}$ be a left $\mathcal{C}$ -comodule. Then the following four conditions are equivalent:*

- *for any finite-dimensional subcoalgebra $\mathcal{B} \subset \mathcal{C}$, the k -vector space ${}_{\mathcal{B}}\mathcal{M}$ is finite-dimensional;*
- *for any cosimple subcoalgebra $\mathcal{A} \subset \mathcal{C}$, the k -vector space ${}_{\mathcal{A}}\mathcal{M}$ is finite-dimensional;*

- for any finite-dimensional left $\mathcal{C}$ -comodule $\mathcal{K}$, the k -vector space $\mathrm{Hom}_{\mathcal{C}}(\mathcal{K}, \mathcal{M})$ is finite-dimensional.
- for any irreducible left $\mathcal{C}$ -comodule $\mathcal{J}$, the k -vector space $\mathrm{Hom}_{\mathcal{C}}(\mathcal{J}, \mathcal{M})$ is finite-dimensional.

Proof. This is essentially a statement about comodules over finite-dimensional coalgebras, or which is the same thing, modules over finite-dimensional algebras. Basically, the assertion is that a module over a finite-dimensional algebra is finite-dimensional if and only if its socle is finite-dimensional.

We refer to [29, Section 2] for the background material. In particular, one should keep in mind that $\mathcal{C}$ is the union of its finite-dimensional subcoalgebras, all $\mathcal{C}$ -comodules are the unions of their finite-dimensional subcomodules, and all finite-dimensional $\mathcal{C}$ -comodules are comodules over finite-dimensional subcoalgebras of $\mathcal{C}$; so irreducible left $\mathcal{C}$ -comodules correspond bijectively to cosimple subcoalgebras in $\mathcal{C}$. $\square$

We will say that a left $\mathcal{C}$ -comodule $\mathcal{M}$ is *quasi-finitely cogenerated* if it satisfies the equivalent conditions of Lemma 3.1. (Such comodules were called “quasi-finite” in [30, 10].) Recall that a left $\mathcal{C}$ -comodule is said to be *finitely cogenerated* [30, 31, 21] if it is a subcomodule of a cofree left $\mathcal{C}$ -comodule $\mathcal{C} \otimes_k V$ with a finite-dimensional vector space of cogenerators V . Any finitely cogenerated $\mathcal{C}$ -comodule is quasi-finitely cogenerated, while the cofree left $\mathcal{C}$ -comodule $\mathcal{C} \otimes_k V$ cogenerated by an infinite-dimensional k -vector space V is not quasi-finitely cogenerated when $\mathcal{C} \neq 0$.

One can see from [21, Lemma 2.2(e)] that the classes of finitely cogenerated and quasi-finitely cogenerated left $\mathcal{C}$ -comodules coincide if and only if the maximal cosemisimple subcoalgebra $\mathcal{C}^{\mathrm{ss}}$ of the coalgebra $\mathcal{C}$ is finite-dimensional (cf. [10, Proposition 1.6 in journal version or Proposition 2.5 in arXiv version]). Unlike the finite cogeneratedness condition, the quasi-finite cogeneratedness condition on comodules is *Morita invariant*, i. e., it is preserved by equivalences of the categories of comodules $\mathcal{C}\text{-comod} \simeq \mathcal{D}\text{-comod}$ over different coalgebras $\mathcal{C}$ and $\mathcal{D}$ [30].

Lemma 3.2. (a) *The class of all quasi-finitely cogenerated left $\mathcal{C}$ -comodules is closed under extensions and the passages to arbitrary subcomodules.*

(b) *Any quasi-finitely cogenerated $\mathcal{C}$ -comodule is a subcomodule of a quasi-finitely cogenerated injective $\mathcal{C}$ -comodule.* $\square$

Proof. To prove part (a), notice that the functor $\mathcal{M} \mapsto {}_{\mathcal{B}}\mathcal{M}$ is left exact for any subcoalgebra $\mathcal{B} \subset \mathcal{C}$. In part (b), it suffices to say that the injective envelope of a quasi-finitely cogenerated comodule is quasi-finitely cogenerated. Indeed, if $\mathcal{M}$ is a left $\mathcal{C}$ -comodule, $\mathcal{J}$ is an injective envelope of $\mathcal{M}$, and $\mathcal{A}$ is a co(semi)simple subcoalgebra in $\mathcal{C}$, then any $\mathcal{A}$ -subcomodule in $\mathcal{J}$ is contained in $\mathcal{M}$. $\square$

The following result can be found in [10, Theorem 2.1 in journal version or Theorem 3.1 in arXiv version].

Proposition 3.3. *Let $\mathcal{C}$ be a coassociative coalgebra over a field k . Then the following four conditions are equivalent:*

- any quotient comodule of a quasi-finitely cogenerated left $\mathcal{C}$ -comodule is quasi-finitely cogenerated;
- any quotient comodule of a quasi-finitely cogenerated injective left $\mathcal{C}$ -comodule is quasi-finitely cogenerated;
- any quotient comodule of a finitely cogenerated left $\mathcal{C}$ -comodule is quasi-finitely cogenerated;
- any quotient comodule of the left $\mathcal{C}$ -comodule $\mathcal{C}$ is quasi-finitely cogenerated.

Proof. We will prove that the fourth condition implies the second one (the other implications being obvious in view of Lemma 3.2(b)).

As in any locally Noetherian Grothendieck abelian category, every injective $\mathcal{C}$ -comodule is a direct sum of indecomposable injectives. In fact, the category $\mathcal{C}\text{-comod}$ is even locally finite (any object is the union of its subobjects of finite length); hence an injective $\mathcal{C}$ -comodule is indecomposable if and only if its socle is irreducible. The correspondence assigning to an injective left $\mathcal{C}$ -comodule $\mathcal{J}$ its socle $\mathcal{K}$ restricts to a bijection between the isomorphism classes of indecomposable injective left $\mathcal{C}$ -comodules $\mathcal{J}_i$ and the isomorphism classes of irreducible left $\mathcal{C}$ -comodules $\mathcal{J}_i$. The latter corespond bijectively to the cosimple subcoalgebras $\mathcal{A}_i \subset \mathcal{C}$. It follows that an injective left $\mathcal{C}$ -comodule $\mathcal{J}$ is quasi-finitely cogenerated if and only if its socle $\mathcal{K}$ contains any irreducible left $\mathcal{C}$ -comodule $\mathcal{J}_i$ with at most finite multiplicity.

Let $\mathcal{J}$ be a quasi-finitely cogenerated injective left $\mathcal{C}$ -comodule. Consider a decomposition of $\mathcal{J}$ into a direct sum of indecomposable injective left $\mathcal{C}$ -comodules $\mathcal{J}_a$, choose a well-ordering of the set of indices a , and consider the ordinal-indexed increasing filtration of $\mathcal{J}$ associated with this direct sum decomposition and this ordering of the summands. Then any subcomodule $\mathcal{L}$ and any quotient comodule $\mathcal{M}$ of the $\mathcal{C}$ -comodule $\mathcal{J}$ acquires the induced increasing filtration indexed by the same ordinal. The successive quotient comodules $\mathcal{M}_a$ of the induced filtration on $\mathcal{M}$ are certain quotients of the indecomposable injectives $\mathcal{J}_a$.

This argument shows that it suffices to check quasi-finite cogeneratedness of the quotient comodules $\mathcal{M} = \mathcal{J}/\mathcal{L}$ of the form $\mathcal{M} = \bigoplus_a \mathcal{M}_a$, where $\mathcal{M}_a$ are certain quotient comodules of $\mathcal{J}_a$. We arrive to the following criterion. All quotient comodules of quasi-finitely cogenerated injective left $\mathcal{C}$ -comodules are quasi-finitely cogenerated if and only if both of the next two conditions hold:

- (*) for any cosimple subcoalgebra $\mathcal{B} \subset \mathcal{C}$ and any quotient comodule $\mathcal{M}$ of an indecomposable injective left $\mathcal{C}$ -comodule $\mathcal{J}_i$, the subcomodule ${}_{\mathcal{B}}\mathcal{M} \subset \mathcal{M}$ is finite-dimensional;
- (**) for any cosimple subcoalgebra $\mathcal{B} \subset \mathcal{C}$, there exist at most finite number of isomorphism classes of indecomposable injective left $\mathcal{C}$ -comodules $\mathcal{J}_i$ for which $\mathcal{J}_i$ has a quotient comodule $\mathcal{M}$ with ${}_{\mathcal{B}}\mathcal{M} \neq 0$.

Now let us consider the direct sum $\mathcal{J} = \bigoplus_i \mathcal{J}_i$ of all the indecomposable injective left $\mathcal{C}$ -comodules, exactly one copy of each. Then $\mathcal{J}$ is a direct summand of the left $\mathcal{C}$ -comodule $\mathcal{C}$. If at least one of the conditions (*) and (**) is *not* satisfied, then $\mathcal{J}$ has a quotient comodule $\mathcal{M} = \bigoplus_i \mathcal{M}_i$ which is not quasi-finitely cogenerated. This observation finishes the proof. $\square$

A coalgebra $\mathcal{C}$ is said to be *left quasi-co-Noetherian* if it satisfies the equivalent conditions of Proposition 3.3, i. e., if any quotient comodule of a quasi-finitely cogenerated left $\mathcal{C}$ -comodule is quasi-finitely cogenerated. (Such coalgebras were called “left strictly quasi-finite” in [10].) Over a left quasi-co-Noetherian coalgebra $\mathcal{C}$, quasi-finitely cogenerated left $\mathcal{C}$ -comodules form an abelian category. Recall that a coalgebra $\mathcal{C}$ is said to be *left co-Noetherian* [31, 10, 21] if any quotient comodule of a finitely cogenerated left $\mathcal{C}$ -comodule is finitely cogenerated. It is clear from Proposition 3.3 that any left co-Noetherian coalgebra is left quasi-co-Noetherian.

An example of a quasi-co-Noetherian coalgebra that is not co-Noetherian is given in [10, Example 1.5 in journal version or Example 2.3 in [arXiv](#) version]. An example of a right co-Noetherian coalgebra that is not left quasi-co-Noetherian can be found in [10, Example 2.7 in journal version or Example 3.6 in [arXiv](#) version].

A $\mathcal{C}$ -comodule $\mathcal{M}$ is said to be *quasi-finitely copresented* if it is isomorphic to the kernel of a morphism of quasi-finitely cogenerated injective $\mathcal{C}$ -comodules. Any finitely copresented $\mathcal{C}$ -comodule in the sense of [31] and [21, Section 2] is quasi-finitely copresented. Any quasi-finitely copresented $\mathcal{C}$ -comodule is quasi-finitely cogenerated.

An example of a quasi-finitely cogenerated comodule that is not quasi-finitely copresented can be easily extracted from [10, Example 2.1 in [arXiv](#) version] using part (c) of the next lemma.

Lemma 3.4. (a) *The kernel of a morphism from a quasi-finitely copresented $\mathcal{C}$ -comodule to a quasi-finitely cogenerated one is quasi-finitely copresented.*

(b) *The class of quasi-finitely copresented $\mathcal{C}$ -comodules is closed under extensions.*

(c) *The cokernel of an injective morphism from a quasi-finitely copresented $\mathcal{C}$ -comodule to a quasi-finitely cogenerated one is quasi-finitely cogenerated.*

Proof. Follows from Lemma 3.2 (cf. the proof of [21, Lemma 2.8(a)]). $\square$

Given a subcoalgebra $\mathcal{B} \subset \mathcal{C}$ and a left $\mathcal{C}$ -contramodule $\mathfrak{T}$, we denote by ${}^{\mathcal{B}}\mathfrak{T}$ the maximal quotient contramodule of $\mathfrak{T}$ whose $\mathcal{C}$ -contramodule structure comes from a $\mathcal{B}$ -contramodule structure. The $\mathcal{B}$ -contramodule ${}^{\mathcal{B}}\mathfrak{T}$ can be computed as the cokernel of the composition $\mathrm{Hom}_k(\mathcal{C}/\mathcal{B}, \mathfrak{T}) \longrightarrow \mathfrak{T}$ of the natural embedding $\mathrm{Hom}_k(\mathcal{C}/\mathcal{B}, \mathfrak{T}) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, \mathfrak{T})$ with the contraaction map $\mathrm{Hom}_k(\mathcal{C}, \mathfrak{T}) \longrightarrow \mathfrak{T}$, or as the space of cohomomorphisms ${}^{\mathcal{B}}\mathfrak{T} = \mathrm{Cohom}_{\mathcal{C}}(\mathcal{B}, \mathfrak{T})$ [21, Section 2].

Lemma 3.5. *Let $\mathcal{C}$ be a coassociative coalgebra over k and $\mathfrak{T}$ be a left $\mathcal{C}$ -contramodule. Then the following four conditions are equivalent:*

- *for any finite-dimensional subcoalgebra $\mathcal{B} \subset \mathcal{C}$, the k -vector space ${}^{\mathcal{B}}\mathfrak{T}$ is finite-dimensional;*
- *for any cosimple subcoalgebra $\mathcal{A} \subset \mathcal{C}$, the k -vector space ${}^{\mathcal{A}}\mathfrak{T}$ is finite-dimensional;*
- *for any finite-dimensional left $\mathcal{C}$ -contramodule $\mathfrak{K}$, the k -vector space $\mathrm{Hom}^{\mathcal{C}}(\mathfrak{T}, \mathfrak{K})$ is finite-dimensional;*
- *for any irreducible left $\mathcal{C}$ -contramodule $\mathfrak{I}$, the k -vector space $\mathrm{Hom}^{\mathcal{C}}(\mathfrak{T}, \mathfrak{I})$ is finite-dimensional.*

Proof. This is also essentially a statement about contramodules over finite-dimensional coalgebras, or which is the same thing, modules over finite-dimensional algebras. Basically, the assertion is that a module over a finite-dimensional algebra is finite-dimensional if and only if its quotient module by its cosocle is finite-dimensional.

We refer to [15, Appendix A] and [18, Section 1] for the background material. In particular, one has to be careful in that *not* every $\mathcal{C}$ -contramodule embeds into the projective limit of its finite-dimensional quotient contramodules; nevertheless, any nonzero $\mathcal{C}$ -contramodule has a nonzero finite-dimensional quotient contramodule, and therefore all irreducible $\mathcal{C}$ -contramodules are finite-dimensional. Furthermore, *not* every finite-dimensional $\mathcal{C}$ -contramodule is a contramodule over a finite-dimensional subcoalgebra of $\mathcal{C}$, generally speaking; but every irreducible $\mathcal{C}$ -contramodule is, so irreducible left $\mathcal{C}$ -contramodules still correspond bijectively to cosimple subcoalgebras in $\mathcal{C}$. $\square$

We will say that a left $\mathcal{C}$ -contramodule $\mathfrak{T}$ is *quasi-finitely generated* if it satisfies the equivalent conditions of Lemma 3.5. Recall that a left $\mathcal{C}$ -contramodule is said to be *finitely generated* [21, Section 2] if it is a quotient contramodule of a free left $\mathcal{C}$ -contramodule $\mathrm{Hom}_k(\mathcal{C}, V)$ with a finite-dimensional space of generators V . According to [21, Lemma 2.5(a) and the proof of Lemma 2.5(b)], any finitely generated $\mathcal{C}$ -contramodule is quasi-finitely generated, while the free left $\mathcal{C}$ -contramodule $\mathrm{Hom}_k(\mathcal{C}, V)$ generated by an infinite-dimensional vector space V is not quasi-finitely generated when $\mathcal{C} \neq 0$.

One can see from [21, Lemma 2.5(e)] that the classes of finitely generated and quasi-finitely generated left $\mathcal{C}$ -contramodules coincide if and only if the maximal cosemisimple subcoalgebra $\mathcal{C}^{\mathrm{ss}}$ of the coalgebra $\mathcal{C}$ is finite-dimensional. Unlike the finite generatedness condition, the quasi-finite generatedness condition on contramodules is Morita invariant, i. e., it is preserved by equivalences of the categories of contramodules $\mathcal{C}\text{-contra} \simeq \mathcal{D}\text{-contra}$ over different coalgebras $\mathcal{C}$ and $\mathcal{D}$ (see [15, Section 7.5.3] for a discussion).

Lemma 3.6. (a) *The class of quasi-finitely generated left $\mathcal{C}$ -contramodules is closed under extensions and the passages to arbitrary quotient contramodules.*

(b) *Any quasi-finitely generated left $\mathcal{C}$ -contramodule is a quotient contramodule of a quasi-finitely generated projective $\mathcal{C}$ -contramodule.*

Proof. To prove part (a), notice that the functor $\mathfrak{T} \mapsto {}^{\mathcal{B}}\mathfrak{T}$ is right exact for any subcoalgebra $\mathcal{B} \subset \mathcal{C}$.

The proof of part (b) is based on the arguments in the first half of the proof of [15, Lemma A.3]. Given a left $\mathcal{C}$ -contramodule $\mathfrak{T}$, one considers the left $\mathcal{C}^{\mathrm{ss}}$ -contramodule $\mathfrak{K} = {}^{\mathcal{C}^{\mathrm{ss}}}\mathfrak{T}$. The key step is to construct for any left $\mathcal{C}^{\mathrm{ss}}$ -contramodule $\mathfrak{K}$ a projective left $\mathcal{C}$ -contramodule $\mathfrak{P}$ such that the left $\mathcal{C}^{\mathrm{ss}}$ -contramodule ${}^{\mathcal{C}^{\mathrm{ss}}}\mathfrak{P}$ is isomorphic to $\mathfrak{K}$. Then one applies the contramodule Nakayama lemma [15, Lemma A.2.1] in order to show that $\mathfrak{T}$ is a quotient $\mathcal{C}$ -contramodule of $\mathfrak{P}$.

The above proof of part (b) looks different from the proof of Lemma 3.2(b), but in fact they are very similar (or dual-analogous). Any $\mathcal{C}$ -contramodule has a projective cover [24, Example 11.3], and the $\mathcal{C}$ -contramodule $\mathfrak{P}$ in the above argument is a projective cover of the $\mathcal{C}$ -contramodule $\mathfrak{T}$ (cf. the construction in [24, Section 9]). $\square$

A $\mathcal{C}$ -contramodule $\mathfrak{T}$ is said to be *quasi-finitely presented* if it is isomorphic to the cokernel of a morphism of quasi-finitely generated projective $\mathcal{C}$ -contramodules. Any finitely presented contramodule in the sense of [21, Section 2] is quasi-finitely presented. Any quasi-finitely presented $\mathcal{C}$ -contramodule is quasi-finitely generated.

Lemma 3.7. (a) *The cokernel of a morphism from a quasi-finitely generated $\mathcal{C}$ -contramodule to a quasi-finitely presented one is quasi-finitely presented.*

(b) *The class of quasi-finitely presented $\mathcal{C}$ -contramodules is closed under extensions.*

(c) *The kernel of a surjective morphism from a quasi-finitely generated $\mathcal{C}$ -contramodule to a quasi-finitely presented one is quasi-finitely generated.*

Proof. Follows from Lemma 3.6 (cf. [20, Lemma 1.1] and [21, Lemma 2.8(b)]). $\square$

The following proposition is our version of [21, Proposition 2.9].

Proposition 3.8. (a) *The functor $\mathcal{N} \mapsto \mathcal{N}^* = \text{Hom}_k(\mathcal{N}, k)$ restricts to an anti-equivalence between the additive category of quasi-finitely copresented right $\mathcal{C}$ -comodules and the additive category of quasi-finitely presented left $\mathcal{C}$ -contramodules.*

(b) *For any right $\mathcal{C}$ -comodule $\mathcal{M}$, any quasi-finitely cogenerated right $\mathcal{C}$ -comodule $\mathcal{N}$, and any k -vector space V , the natural k -linear map $\text{Hom}_k(\mathcal{M}, \mathcal{N} \otimes_k V) \rightarrow \text{Hom}_k(\mathcal{N}^*, \text{Hom}_k(\mathcal{M}, V))$ restricts to an isomorphism of the Hom spaces in the categories of right $\mathcal{C}$ -comodules and left $\mathcal{C}$ -contramodules*

$$\text{Hom}_{\mathcal{C}^{\text{op}}}(\mathcal{M}, \mathcal{N} \otimes_k V) \simeq \text{Hom}^{\mathcal{C}}(\mathcal{N}^*, \text{Hom}_k(\mathcal{M}, V)).$$

(c) *For any right $\mathcal{C}$ -comodule $\mathcal{M}$, any quasi-finitely cogenerated right $\mathcal{C}$ -comodule $\mathcal{N}$, and any k -vector space V , the natural k -linear map $(\mathcal{M} \otimes_k \mathcal{N}^*) \otimes_k V \rightarrow \mathcal{M} \otimes_k \text{Hom}_k(\mathcal{N}, V)$ induces an isomorphism of the (contra)tensor product spaces*

$$(\mathcal{M} \odot_{\mathcal{C}} \mathcal{N}^*) \otimes_k V \simeq \mathcal{M} \odot_{\mathcal{C}} \text{Hom}_k(\mathcal{N}, V).$$

Proof. Part (b): for a right $\mathcal{C}$ -comodule $\mathcal{M}$ and a subcoalgebra $\mathcal{B} \subset \mathcal{C}$, we denote the maximal subcomodule of $\mathcal{M}$ whose $\mathcal{C}$ -comodule structure comes from a $\mathcal{B}$ -comodule structure by $\mathcal{M}_{\mathcal{B}}$. Then for any k -vector space V we have ${}^{\mathcal{B}}\text{Hom}_k(\mathcal{M}, V) = \text{Hom}_k(\mathcal{M}_{\mathcal{B}}, V)$. Since any right $\mathcal{C}$ -comodule $\mathcal{M}$ is the union of its subcomodules $\mathcal{M}_{\mathcal{A}}$ over the finite-dimensional subcoalgebras $\mathcal{A} \subset \mathcal{C}$, it follows that

$$\text{Hom}_k(\mathcal{M}, V) = \varinjlim_{\mathcal{A}} {}^{\mathcal{A}}\text{Hom}_k(\mathcal{M}, V).$$

Therefore,

$$\begin{aligned} \text{Hom}^{\mathcal{C}}(\mathcal{N}^*, \text{Hom}_k(\mathcal{M}, V)) &= \varinjlim_{\mathcal{A}} \text{Hom}^{\mathcal{C}}(\mathcal{N}^*, {}^{\mathcal{A}}\text{Hom}_k(\mathcal{M}, V)) \\ &= \varinjlim_{\mathcal{A}} \text{Hom}^{\mathcal{C}}({}^{\mathcal{A}}(\mathcal{N}^*), {}^{\mathcal{A}}\text{Hom}_k(\mathcal{M}, V)) = \varinjlim_{\mathcal{A}} \text{Hom}^{\mathcal{A}}((\mathcal{N}_{\mathcal{A}})^*, \text{Hom}_k(\mathcal{M}_{\mathcal{A}}, V)) \\ &\simeq \varinjlim_{\mathcal{A}} \text{Hom}_{\mathcal{A}^{\text{op}}}(\mathcal{M}_{\mathcal{A}}, \mathcal{N}_{\mathcal{A}} \otimes_k V) = \text{Hom}_{\mathcal{C}^{\text{op}}}(\mathcal{M}, \mathcal{N} \otimes_k V) \end{aligned}$$

because the right $\mathcal{A}$ -comodule $\mathcal{N}_{\mathcal{A}}$ is finite-dimensional.

To prove part (a), we notice from the computations above that the left $\mathcal{C}$ -contra-module $\mathcal{N}^*$ is quasi-finitely generated if and only if a right $\mathcal{C}$ -comodule $\mathcal{N}$ is quasi-finitely cogenerated. Furthermore, substituting $V = k$ into the assertion (b), we see that the dualization functor $\mathcal{N} \mapsto \mathcal{N}^*: \mathbf{comod}\text{-}\mathcal{C} \rightarrow \mathcal{C}\text{-}\mathbf{contra}$ is fully faithful on the full subcategory of quasi-finitely cogenerated comodules in $\mathbf{comod}\text{-}\mathcal{C}$. It remains to prove the essential surjectivity.

As the left $\mathcal{C}$ -contramodule $\mathcal{J}^*$ is projective for any injective right $\mathcal{C}$ -comodule $\mathcal{J}$, the dualization functor takes quasi-finitely cogenerated injective right $\mathcal{C}$ -comodules to quasi-finitely generated projective left $\mathcal{C}$ -contramodules. A projective left $\mathcal{C}$ -contramodule $\mathfrak{P}$ is uniquely determined, up to isomorphism, by the left $\mathcal{C}^{\text{ss}}$ -contramodule $\mathcal{C}^{\text{ss}}\mathfrak{P}$, and all the quasi-finitely generated $\mathcal{C}^{\text{ss}}$ -contramodules belong to the image of the dualization functor; therefore so do all the quasi-finitely generated projective $\mathcal{C}$ -contramodules (cf. the proofs of Lemmas 3.2(b) and 3.6(b)).

Finally, any quasi-finitely presented $\mathcal{C}$ -contramodule is the cokernel of a morphism of quasi-finitely generated projective $\mathcal{C}$ -contramodules, this morphism comes from a morphism of quasi-finitely cogenerated injective $\mathcal{C}$ -comodules, the kernel of the latter morphism is a quasi-finitely copresented $\mathcal{C}$ -comodule, and the dualization functor takes the kernels to the cokernels.

Part (c): For any right $\mathcal{C}$ -comodule $\mathcal{M}$ and left $\mathcal{C}$ -contramodule $\mathfrak{T}$, one has $\mathcal{M} \odot_{\mathcal{C}} \mathfrak{T} = (\varinjlim_{\mathcal{A}} \mathcal{M}_{\mathcal{A}}) \odot_{\mathcal{C}} \mathfrak{T} = \varinjlim_{\mathcal{A}} (\mathcal{M}_{\mathcal{A}} \odot_{\mathcal{C}} \mathfrak{T}) = \varinjlim_{\mathcal{A}} (\mathcal{M}_{\mathcal{A}} \odot_{\mathcal{A}} {}^{\mathcal{A}}\mathfrak{T})$, where the inductive limit is taken over all the finite-dimensional subcoalgebras $\mathcal{A} \subset \mathcal{C}$. In particular,

$$\begin{aligned} \mathcal{M} \odot_{\mathcal{C}} \text{Hom}_k(\mathcal{N}, V) &= \varinjlim_{\mathcal{A}} (\mathcal{M}_{\mathcal{A}} \odot_{\mathcal{C}} \text{Hom}_k(\mathcal{N}_{\mathcal{A}}, V)) \\ &\simeq \varinjlim_{\mathcal{A}} ((\mathcal{M}_{\mathcal{A}} \odot_{\mathcal{C}} \mathcal{N}_{\mathcal{A}}^*) \otimes_k V) = (\mathcal{M} \odot_{\mathcal{C}} \mathcal{N}^*) \otimes_k V, \end{aligned}$$

because $\mathcal{N}_{\mathcal{A}}$ is finite-dimensional. $\square$

Remark 3.9. The following construction of a (partially defined) functor of two comodule arguments plays a major role in the classical paper [30]. Let $\mathcal{N}$ be a quasi-finitely cogenerated right $\mathcal{C}$ -comodule. Then, for any right $\mathcal{C}$ -comodule $\mathcal{M}$, the vector space of comodule homomorphisms $\text{Hom}_{\mathcal{C}^{\text{op}}}(\mathcal{M}, \mathcal{N})$ is the projective limit of finite-dimensional vector spaces. Hence it is naturally the dual vector space to a certain vector space, which is called “co-hom” and denoted by $h_{\mathcal{C}}(\mathcal{N}, \mathcal{M})$ in [30]. The vector space $h_{\mathcal{C}}(\mathcal{N}, \mathcal{M})$ is characterized by the property that, for any k -vector space V , there is a natural adjunction isomorphism of k -vector spaces $\text{Hom}_{\mathcal{C}^{\text{op}}}(\mathcal{M}, \mathcal{N} \otimes_k V) \simeq \text{Hom}_k(h_{\mathcal{C}}(\mathcal{N}, \mathcal{M}), V)$.

Proposition 3.8 explains that the functor $h_{\mathcal{C}}$ can be expressed in our terms as $h_{\mathcal{C}}(\mathcal{N}, \mathcal{M}) = \mathcal{M} \odot_{\mathcal{C}} \mathcal{N}^*$. Indeed, for any k -vector space V we have

$$\text{Hom}_k(\mathcal{M} \odot_{\mathcal{C}} \mathcal{N}^*, V) \simeq \text{Hom}_{\mathcal{C}}(\mathcal{N}^*, \text{Hom}_k(\mathcal{M}, V)) \simeq \text{Hom}_{\mathcal{C}^{\text{op}}}(\mathcal{M}, \mathcal{N} \otimes_k V)$$

by part (b) of the proposition.

A coalgebra $\mathcal{C}$ is called *right quasi-cocoherent* if any quasi-finitely cogenerated quotient comodule of a quasi-finitely copresented right $\mathcal{C}$ -comodule is quasi-finitely

copresented, or equivalently, if any quasi-finitely generated subcontramodule of a quasi-finitely presented left $\mathcal{C}$ -contramodule is quasi-finitely presented. Over a right co-coherent coalgebra $\mathcal{C}$, the categories of quasi-finitely copresented right comodules and quasi-finitely presented left contramodules are abelian. Any left quasi-co-Noetherian coalgebra is left quasi-cocoherent, and any quasi-finitely cogenerated left comodule over such a coalgebra is quasi-finitely copresented.

Recall that a coalgebra $\mathcal{C}$ is said to be *right cocoherent* [21, Section 2] if any finitely cogenerated quotient comodule of a finitely copresented right $\mathcal{C}$ -comodule is finitely copresented, or equivalently, if any finitely generated subcontramodule of a finitely presented left $\mathcal{C}$ -contramodule is finitely presented. We do *not* know whether any right cocoherent coalgebra needs to be right quasi-cocoherent.

A further discussion of quasi-finiteness conditions for coalgebras, comodules, and contramodules can be found in [22, Sections 5.1–5.4].

A $\mathcal{C}$ -comodule is said to be *strongly quasi-finitely copresented* if it has an injective coresolution consisting of quasi-finitely cogenerated injective $\mathcal{C}$ -comodules. Similarly one could define “strongly quasi-finitely presented contramodules”; and the following two lemmas have their obvious dual-analogous contramodule versions.

Lemma 3.10. *Let $0 \rightarrow \mathcal{K} \rightarrow \mathcal{L} \rightarrow \mathcal{M} \rightarrow 0$ be a short exact sequence of $\mathcal{C}$ -comodules. Then whenever two of the three comodules $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$ are strongly quasi-finitely copresented, so is the third one.*

Proof. Dual to the proof of [23, Lemma 2.2] (see *loc. cit.* for relevant references). $\square$

Abusing terminology, we will say that a bounded below complex of $\mathcal{C}$ -comodules is *strongly quasi-finitely copresented* if it is quasi-isomorphic to a bounded below complex of quasi-finitely cogenerated injective $\mathcal{C}$ -comodules. Clearly, the class of all strongly quasi-finitely copresented complexes is closed under shifts and cones in $D^+(\mathcal{C}\text{-comod})$.

Lemma 3.11. (a) *Any bounded below complex of strongly quasi-finitely copresented $\mathcal{C}$ -comodules is strongly quasi-finitely copresented.*

(b) *Let $\mathcal{M}^\bullet$ be a complex of $\mathcal{C}$ -comodules concentrated in the cohomological degrees $\geq n$, where n is a fixed integer. Then $\mathcal{M}^\bullet$ is strongly quasi-finitely copresented if and only if it is quasi-isomorphic to a complex of strongly quasi-finitely copresented $\mathcal{C}$ -comodules concentrated in the cohomological degrees $\geq n$.*

(c) *Let $\mathcal{M}^\bullet$ be a finite complex of $\mathcal{C}$ -comodules concentrated in the cohomological degrees $n_1 \leq m \leq n_2$. Then $\mathcal{M}^\bullet$ is strongly quasi-finitely copresented if and only if it is quasi-isomorphic to a complex of $\mathcal{C}$ -comodules $\mathcal{K}^\bullet$ concentrated in the cohomological degrees $n_1 \leq m \leq n_2$ such that the $\mathcal{C}$ -comodules $\mathcal{K}^m$ are quasi-finitely cogenerated and injective for all $n_1 \leq m \leq n_2 - 1$, while the $\mathcal{C}$ -comodule $\mathcal{K}^{n_2}$ is strongly quasi-finitely copresented.*

Proof. Dual to [23, Lemma 2.3]. $\square$

Lemmas 3.10–3.11 are very similar to the respective results from [23, Section 2]. The following examples of pseudo-derived categories of comodules and contramodules,

intended to illustrate the discussion in Section 1.1 of the introduction, are different from [23, Examples 2.5–2.6], though, in that no finiteness or quasi-finiteness conditions (of the kind discussed above in this section) play any role in Examples 3.12–3.13.

Given a left $\mathcal{C}$ -comodule $\mathcal{M}$ and a right $\mathcal{C}$ -comodule $\mathcal{N}$, the k -vector spaces $\text{Cotor}_i^{\mathcal{C}}(\mathcal{N}, \mathcal{M})$, $i = 0, -1, -2, \dots$ are defined as the right derived functors of the left exact functor of cotensor product $\mathcal{N} \square_{\mathcal{C}} \mathcal{M}$, constructed by replacing any one or both of the comodules $\mathcal{N}$ and $\mathcal{M}$ by its injective coresolution, taking the cotensor product and computing the cohomology [15, Sections 0.2.2 and 1.2.2], [16, Section 4.7].

Examples 3.12. (1) Let $\mathcal{C}$ be a coassociative coalgebra and $\mathcal{S}$ be a class of right $\mathcal{C}$ -comodules. Denote by $\mathbf{E} \subset \mathbf{A} = \mathcal{C}\text{-comod}$ the full subcategory formed by all the left $\mathcal{C}$ -comodules $\mathcal{E}$ such that $\text{Cotor}_i^{\mathcal{C}}(\mathcal{S}, \mathcal{E}) = 0$ for all $\mathcal{S} \in \mathcal{S}$ and all $i < 0$. Then the full subcategory $\mathbf{E} \subset \mathcal{C}\text{-comod}$ is a coresolving subcategory closed under infinite direct sums. Thus the derived category $\mathbf{D}(\mathbf{E})$ of the exact category $\mathbf{E}$ is a pseudo-coderived category of the abelian category $\mathcal{C}\text{-comod}$, that is an intermediate quotient category between the coderived category $\mathbf{D}^{\text{co}}(\mathcal{C}\text{-comod})$ and the derived category $\mathbf{D}(\mathcal{C}\text{-comod})$, as explained in [25, Section 4] or [23, Section 1].

(2) In particular, if $\mathcal{S} = \emptyset$, then one has $\mathbf{E} = \mathcal{C}\text{-comod}$. On the other hand, if $\mathcal{S}$ is the class of all right $\mathcal{C}$ -comodules, then $\mathbf{E} = \mathcal{C}\text{-comod}_{\text{inj}}$ is the full subcategory of all injective left $\mathcal{C}$ -comodules. In fact, it suffices to take $\mathcal{S}$ to be the set of all finite-dimensional right $\mathcal{C}$ -comodules, or just irreducible right $\mathcal{C}$ -comodules, to force $\mathbf{E} = \mathcal{C}\text{-comod}_{\text{inj}}$ [18, Lemma 3.1(a)]. In this case, the derived category $\mathbf{D}(\mathbf{E}) = \text{Hot}(\mathbf{E})$ of the (split) exact category $\mathbf{E}$ is equivalent to the coderived category of left $\mathcal{C}$ -comodules $\mathbf{D}^{\text{co}}(\mathcal{C}\text{-comod})$ [15, Theorem 5.4(a) or 5.5(a)], [16, Theorem 4.4(c)].

Given a left $\mathcal{C}$ -comodule $\mathcal{M}$ and a left $\mathcal{C}$ -contramodule $\mathfrak{T}$, the k -vector spaces $\text{Coext}_i^{\mathcal{C}}(\mathcal{M}, \mathfrak{T})$, $i = 0, -1, -2, \dots$ are defined as the left derived functors of the right exact functor of cohomomorphisms $\text{Cohom}_{\mathcal{C}}(\mathcal{M}, \mathfrak{T})$, constructed by replacing either the comodule argument $\mathcal{M}$ by its injective coresolution, or the contramodule argument $\mathfrak{T}$ by its projective resolution, or both, taking the $\text{Cohom}_{\mathcal{C}}$ and computing the homology [15, Sections 0.2.5 and 3.2.2], [16, Section 4.7].

Examples 3.13. (1) Let $\mathcal{D}$ be a coassociative coalgebra over k and $\mathcal{S}$ be a class of left $\mathcal{D}$ -comodules. Denote by $\mathbf{F} \subset \mathbf{B} = \mathcal{D}\text{-contra}$ the full subcategory formed by all the left $\mathcal{D}$ -contramodules $\mathfrak{F}$ such that $\text{Coext}_i^{\mathcal{D}}(\mathcal{S}, \mathfrak{F}) = 0$ for all $\mathcal{S} \in \mathcal{S}$ and all $i < 0$. Then the full subcategory $\mathbf{F} \subset \mathcal{D}\text{-contra}$ is a resolving subcategory closed under infinite products. Thus the derived category $\mathbf{D}(\mathbf{F})$ of the exact category $\mathbf{F}$ is a pseudo-contraderived category of the abelian category $\mathcal{D}\text{-contra}$, that is an intermediate quotient category between the contraderived category $\mathbf{D}^{\text{ctr}}(\mathcal{D}\text{-contra})$ and the derived category $\mathbf{D}(\mathcal{D}\text{-contra})$, as explained in [25, Section 4] or [23, Section 1].

(2) In particular, if $\mathcal{S} = \emptyset$, then one has $\mathbf{F} = \mathcal{D}\text{-contra}$. On the other hand, if $\mathcal{S}$ is the class of all left $\mathcal{D}$ -comodules, then $\mathbf{F} = \mathcal{D}\text{-contra}_{\text{proj}}$ is the full subcategory of all projective left $\mathcal{D}$ -contramodules [18, Lemma 3.1(b)]. In fact, it suffices to take $\mathcal{S}$ to be the set of all finite-dimensional left $\mathcal{D}$ -comodules, or just irreducible left $\mathcal{D}$ -comodules, to force $\mathbf{F} = \mathcal{D}\text{-contra}_{\text{proj}}$ [15, Lemma A.3]. In this case, the

derived category $D(F) = \text{Hot}(F)$ of the (split) exact category F is equivalent to the contraderived category of left $\mathcal{D}$ -contramodules $D^{\text{ctr}}(\mathcal{D}\text{-contra})$ [15, Theorem 5.4(b) or 5.5(b)], [16, Theorem 4.4(d)], [17, Corollary A.6.2].

4. AUSLANDER AND BASS CLASSES

We recall the definition of a pseudo-dualizing complex of bicomodules from Section 1.6. Let $\mathcal{C}$ and $\mathcal{D}$ be coassociative coalgebras over a field k .

A *pseudo-dualizing complex* $\mathcal{L}^\bullet$ for the coalgebras $\mathcal{C}$ and $\mathcal{D}$ is a finite complex of $\mathcal{C}\text{-}\mathcal{D}$ -bicomodules satisfying the following two conditions:

- (ii) the complex $\mathcal{L}^\bullet$ is strongly quasi-finitely copresented as a complex of left $\mathcal{C}$ -comodules and as a complex of right $\mathcal{D}$ -comodules;
- (iii) the homothety maps $\mathcal{C}^* \rightarrow \text{Hom}_{D^b(\text{comod-}\mathcal{D})}(\mathcal{L}^\bullet, \mathcal{L}^\bullet[*])$ and $\mathcal{D}^{*\text{op}} \rightarrow \text{Hom}_{D^b(\mathcal{C}\text{-comod})}(\mathcal{L}^\bullet, \mathcal{L}^\bullet[*])$ are isomorphisms of graded rings.

Here the condition (ii) refers to the definition of a strongly quasi-finitely copresented complex of comodules in Section 3. The complex $\mathcal{L}^\bullet$ is viewed as an object of the bounded derived category of $\mathcal{C}\text{-}\mathcal{D}$ -bicomodules $D^b(\mathcal{C}\text{-comod-}\mathcal{D})$.

Given a $\mathcal{C}\text{-}\mathcal{D}$ -bicomodule $\mathcal{K}$, the functor of contratensor product $\mathcal{K} \odot_{\mathcal{D}} - : \mathcal{D}\text{-contra} \rightarrow \mathcal{C}\text{-comod}$ is left adjoint to the functor of comodule homomorphisms $\text{Hom}_{\mathcal{C}}(\mathcal{K}, -) : \mathcal{C}\text{-comod} \rightarrow \mathcal{D}\text{-contra}$. Hence, in particular, the functor of contratensor product of complexes $\mathcal{L}^\bullet \odot_{\mathcal{D}} - : \text{Hot}(\mathcal{D}\text{-contra}) \rightarrow \text{Hot}(\mathcal{C}\text{-comod})$ is left adjoint to the functor $\text{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, -) : \text{Hot}(\mathcal{C}\text{-comod}) \rightarrow \text{Hot}(\mathcal{D}\text{-contra})$.

We will use the existence theorem of homotopy injective resolutions of complexes of comodules and homotopy projective resolutions of complexes of contramodules [16, Theorem 2.4] in order to work with the conventional unbounded derived categories of comodules and contramodules $D(\mathcal{C}\text{-comod})$ and $D(\mathcal{D}\text{-contra})$. Using the homotopy projective and homotopy injective resolutions of the second arguments, one constructs the derived functors $\mathcal{L}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} - : D(\mathcal{D}\text{-contra}) \rightarrow D(\mathcal{C}\text{-comod})$ and $\mathbb{R} \text{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, -) : D(\mathcal{C}\text{-comod}) \rightarrow D(\mathcal{D}\text{-contra})$. As a particular case of the general property of the left and right derived functors (e. g., in the sense of Deligne [4, 1.2.1–2]) of left and right adjoint functors, the functor $\mathcal{L}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} -$ is left adjoint to the functor $\mathbb{R} \text{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, -)$ [15, Lemma 8.3].

We will use the following simplified notation. Given two complexes of left $\mathcal{C}$ -comodules $\mathcal{M}^\bullet$ and $\mathcal{N}^\bullet$, we denote by $\text{Ext}_{\mathcal{C}}^n(\mathcal{M}^\bullet, \mathcal{N}^\bullet)$ the vector spaces $H^n \mathbb{R} \text{Hom}_{\mathcal{C}}(\mathcal{M}^\bullet, \mathcal{N}^\bullet) \simeq \text{Hom}_{D(\mathcal{C}\text{-comod})}(\mathcal{M}^\bullet, \mathcal{N}^\bullet[n])$ of cohomology of the complex $\mathbb{R} \text{Hom}_{\mathcal{C}}(\mathcal{M}^\bullet, \mathcal{N}^\bullet) = \text{Hom}_{\mathcal{C}}(\mathcal{M}^\bullet, \mathcal{J}^\bullet)$, where $\mathcal{N}^\bullet \rightarrow \mathcal{J}^\bullet$ is a quasi-isomorphism of complexes of left $\mathcal{C}$ -comodules and $\mathcal{J}^\bullet$ is a homotopy injective complex of left $\mathcal{C}$ -comodules. Given a complex of right $\mathcal{D}$ -comodules $\mathcal{N}^\bullet$ and a complex of left $\mathcal{D}$ -contramodules $\mathfrak{T}^\bullet$, we denote by $\text{Ctrtor}_n^{\mathcal{D}}(\mathcal{N}^\bullet, \mathfrak{T}^\bullet)$ the vector spaces $H^{-n}(\mathcal{N}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} \mathfrak{T}^\bullet)$ of cohomology of the complex $\mathcal{N}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} \mathfrak{T}^\bullet = \mathcal{N}^\bullet \odot_{\mathcal{D}} \mathfrak{P}^\bullet$, where $\mathfrak{P}^\bullet \rightarrow \mathfrak{T}^\bullet$ is a quasi-isomorphism of complexes of left $\mathcal{D}$ -contramodules and $\mathfrak{P}^\bullet$ is a homotopy projective complex of left $\mathcal{D}$ -contramodules.

Suppose that the finite complex $\mathcal{L}^\bullet$ is situated in the cohomological degrees $-d_1 \leq m \leq d_2$. Then one has $\text{Ext}_{\mathcal{C}}^n(\mathcal{L}^\bullet, \mathcal{J}) = 0$ for all $n > d_1$ and all injective left $\mathcal{C}$ -comodules $\mathcal{J}$. Similarly, one has $\text{Ctrtor}_{\mathcal{D}}^n(\mathcal{L}^\bullet, \mathfrak{P}) = 0$ for all $n > d_1$ and all projective left $\mathcal{D}$ -contramodules $\mathfrak{P}$. Choose an integer $l_1 \geq d_1$ and consider the following full subcategories in the abelian categories of left $\mathcal{C}$ -comodules and $\mathcal{D}$ -contramodules:

- $E_{l_1} = E_{l_1}(\mathcal{L}^\bullet) \subset \mathcal{C}\text{-comod}$ is the full subcategory consisting of all the $\mathcal{C}$ -comodules $\mathcal{E}$ such that $\text{Ext}_{\mathcal{C}}^n(\mathcal{L}^\bullet, \mathcal{E}) = 0$ for all $n > l_1$ and the adjunction morphism $\mathcal{L}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} \mathbb{R} \text{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, \mathcal{E}) \rightarrow \mathcal{E}$ is an isomorphism in $\mathcal{D}^-(\mathcal{C}\text{-comod})$;
- $F_{l_1} = F_{l_1}(\mathcal{L}^\bullet) \subset \mathcal{D}\text{-contra}$ is the full subcategory consisting of all the $\mathcal{D}$ -contramodules $\mathfrak{F}$ such that $\text{Ctrtor}_{\mathcal{D}}^n(\mathcal{L}^\bullet, \mathfrak{F}) = 0$ for all $n > l_1$ and the adjunction morphism $\mathfrak{F} \rightarrow \mathbb{R} \text{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, \mathcal{L}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} \mathfrak{F})$ is an isomorphism in $\mathcal{D}^+(\mathcal{D}\text{-contra})$.

Clearly, for any $l_1'' \geq l_1' \geq d_1$, one has $E_{l_1'} \subset E_{l_1''} \subset \mathcal{C}\text{-comod}$ and $F_{l_1'} \subset F_{l_1''} \subset \mathcal{D}\text{-contra}$. The category F_{l_1} can be called the *Auslander class of contramodules* corresponding to a pseudo-dualizing complex $\mathcal{L}^\bullet$, while the category E_{l_1} is the *Bass class of comodules*.

Lemma 4.1. (a) *The full subcategory $E_{l_1} \subset \mathcal{C}\text{-comod}$ is closed under the cokernels of injective morphisms, extensions, and direct summands.*

(b) *The full subcategory $F_{l_1} \subset \mathcal{D}\text{-contra}$ is closed under the kernels of surjective morphisms, extensions, and direct summands.* $\square$

The formulation of the next lemma is similar to that of [23, Lemma 3.2], but the proof is quite different. Rather, it resembles the related arguments in the proofs of [21, Theorem 3.6] and [19, Theorem 4.9].

This lemma shows, in particular, that in the case of a pseudo-dualizing bicomodule (= one-term complex) $\mathcal{L}^\bullet = \mathcal{L}$, the pair of adjoint functors $\text{Hom}_{\mathcal{C}}(\mathcal{L}, -): \mathcal{C}\text{-comod} \rightarrow \mathcal{D}\text{-contra}$ and $\mathcal{L} \odot_{\mathcal{D}} -: \mathcal{D}\text{-contra} \rightarrow \mathcal{C}\text{-comod}$ is a “left and right semidualizing adjoint pair” in the sense of [8, Definition 2.1].

Lemma 4.2. (a) *The full subcategory $E_{l_1} \subset \mathcal{C}\text{-comod}$ contains all the injective left $\mathcal{C}$ -comodules.*

(b) *The full subcategory $F_{l_1} \subset \mathcal{D}\text{-contra}$ contains all the projective left $\mathcal{D}$ -contramodules.*

Proof. Part (a): we have to check that for any injective left $\mathcal{C}$ -comodule $\mathcal{E}$ the adjunction morphism $\mathcal{L}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} \mathbb{R} \text{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, \mathcal{E}) = \mathcal{L}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} \text{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, \mathcal{E}) \rightarrow \mathcal{E}$ is a quasi-isomorphism. It suffices to consider the case of a cofree left $\mathcal{C}$ -comodule $\mathcal{E} = \mathcal{C} \otimes_k V$, where V is a k -vector space. Then one has $\text{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, \mathcal{E}) \simeq \text{Hom}_k(\mathcal{L}^\bullet, V)$.

According to the condition (ii), there exists a bounded below complex of quasi-finitely cogenerated injective right $\mathcal{D}$ -comodules $\mathcal{J}^\bullet$ endowed with a quasi-isomorphism of complexes of right $\mathcal{D}$ -comodules $\mathcal{L}^\bullet \rightarrow \mathcal{J}^\bullet$. Then we have a quasi-isomorphism of complexes of left $\mathcal{D}$ -contramodules $\text{Hom}_k(\mathcal{J}^\bullet, V) \rightarrow \text{Hom}_k(\mathcal{L}^\bullet, V)$, and $\text{Hom}_k(\mathcal{J}^\bullet, V)$ is a bounded above complex of projective left $\mathcal{D}$ -contramodules. Hence $\mathcal{L}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} \text{Hom}_k(\mathcal{L}^\bullet, V) = \mathcal{L}^\bullet \odot_{\mathcal{D}} \text{Hom}_k(\mathcal{J}^\bullet, V)$, and it remains to show that the morphism of

complexes of left $\mathcal{C}$ -comodules

$$(8) \quad \mathcal{L}^\bullet \odot_{\mathcal{D}} \operatorname{Hom}_k(\mathcal{J}^\bullet, V) \longrightarrow \mathcal{C} \otimes_k V$$

is a quasi-isomorphism. The morphism (8) is constructed in terms of the morphism of complexes of right $\mathcal{D}$ -comodules $\mathcal{L}^\bullet \rightarrow \mathcal{J}^\bullet$ and the left $\mathcal{C}$ -coaction in the complex $\mathcal{L}^\bullet$.

In particular, substituting $V = k$ into (8), we have a morphism of complexes of left $\mathcal{C}$ -comodules

$$(9) \quad \mathcal{L}^\bullet \odot_{\mathcal{D}} \mathcal{J}^{\bullet*} \longrightarrow \mathcal{C}.$$

Passing to the dual vector spaces in (9), we obtain a map $\mathcal{C}^* \rightarrow \operatorname{Hom}^{\mathcal{D}}(\mathcal{J}^{\bullet*}, \mathcal{L}^{\bullet*})$, which is equal to the composition of the homothety map $\mathcal{C}^* \rightarrow \operatorname{Hom}_{\mathcal{D}}(\mathcal{L}^\bullet, \mathcal{J}^\bullet) = \mathbb{R} \operatorname{Hom}_{\mathcal{D}}(\mathcal{L}^\bullet, \mathcal{L}^\bullet)$ with the dualization map $\operatorname{Hom}_{\mathcal{D}}(\mathcal{L}^\bullet, \mathcal{J}^\bullet) \rightarrow \operatorname{Hom}^{\mathcal{D}}(\mathcal{J}^{\bullet*}, \mathcal{L}^{\bullet*})$.

As the homothety map is a quasi-isomorphism by the condition (iii) and the dualization map is an isomorphism of complexes by Proposition 3.8(b) (because $\mathcal{J}^\bullet$ is a complex of quasi-finitely cogenerated right $\mathcal{D}$ -comodules, while $\mathcal{L}^\bullet$ is a finite complex), it follows that passing to the dual vector spaces in (9) produces a quasi-isomorphism. Hence the map (9) is a quasi-isomorphism, too. Finally, by Proposition 3.8(c) the natural map $(\mathcal{L}^\bullet \odot_{\mathcal{D}} \mathcal{J}^{\bullet*}) \otimes_k V \rightarrow \mathcal{L}^\bullet \odot_{\mathcal{D}} \operatorname{Hom}_k(\mathcal{J}^\bullet, V)$ is an isomorphism of complexes. Therefore, the map (8) is also a quasi-isomorphism.

Part (b): we have to check that for any projective left $\mathcal{D}$ -contramodule $\mathfrak{F}$ the adjunction morphism $\mathfrak{F} \rightarrow \mathbb{R} \operatorname{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, \mathcal{L}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} \mathfrak{F}) = \mathbb{R} \operatorname{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, \mathcal{L}^\bullet \odot_{\mathcal{D}} \mathfrak{F})$ is a quasi-isomorphism. It suffices to consider the case of a free left $\mathcal{D}$ -contramodule $\mathfrak{F} = \operatorname{Hom}_k(\mathcal{D}, V)$, where V is a k -vector space. Then one has $\mathcal{L}^\bullet \odot_{\mathcal{D}} \operatorname{Hom}_k(\mathcal{D}, V) \simeq \mathcal{L}^\bullet \otimes_k V$.

According to the condition (ii), there exists a bounded below complex of quasi-finitely cogenerated injective left $\mathcal{C}$ -comodules $\mathcal{J}^\bullet$ endowed with a quasi-isomorphism of complexes of left $\mathcal{C}$ -comodules $\mathcal{L}^\bullet \rightarrow \mathcal{J}^\bullet$. Then $\mathbb{R} \operatorname{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, \mathcal{L}^\bullet \otimes_k V) = \operatorname{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, \mathcal{J}^\bullet \otimes_k V)$, and it remains to show that the morphism of complexes of left $\mathcal{D}$ -contramodules

$$(10) \quad \operatorname{Hom}_k(\mathcal{D}, V) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, \mathcal{J}^\bullet \otimes_k V)$$

is a quasi-isomorphism. The morphism (10) is constructed in terms of the morphism of complexes of left $\mathcal{C}$ -comodules $\mathcal{L}^\bullet \rightarrow \mathcal{J}^\bullet$ and the right $\mathcal{D}$ -coaction in the complex $\mathcal{L}^\bullet$.

In the same way as in the proof of part (a), one deduces from the condition (iii) using Proposition 3.8(b) that the natural morphism of complexes of right $\mathcal{D}$ -comodules

$$(11) \quad \mathcal{L}^\bullet \odot_{\mathcal{C}^{\operatorname{op}}} \mathcal{J}^{\bullet*} \longrightarrow \mathcal{D}$$

is a quasi-isomorphism. Applying the functor $\operatorname{Hom}_k(-, V)$ to (11), we see that the natural map

$$(12) \quad \operatorname{Hom}_k(\mathcal{D}, V) \longrightarrow \operatorname{Hom}_k(\mathcal{L}^\bullet \odot_{\mathcal{C}^{\operatorname{op}}} \mathcal{J}^{\bullet*}, V) \simeq \operatorname{Hom}^{\mathcal{C}^{\operatorname{op}}}(\mathcal{J}^{\bullet*}, \operatorname{Hom}_k(\mathcal{L}^\bullet, V))$$

is a quasi-isomorphism, too. It remains to use Proposition 3.8(b) again in order to identify the right-hand sides of (10) and (12). $\square$

Remark 4.3. It would be interesting to know whether the analogue of [23, Lemma 3.3] holds in our present context, i. e., whether the Bass class of comodules $E_{l_1} \subset \mathcal{C}\text{-comod}$ is always closed under infinite direct sums and whether the Auslander class of contra modules $F_{l_1} \subset \mathcal{D}\text{-contra}$ is closed under infinite products. A positive answer to these questions would allow to strengthen our main results in the context of the diagram (6) in Section 1.6, as discussed in Sections 1.1–1.2 (cf. [23, Sections 0.5 and 0.8]). Notice that the class of all injective objects in $\mathcal{C}\text{-comod}$ is always closed under direct sums, and the class of all projective objects in $\mathcal{D}\text{-contra}$ is closed under products [18, Section 1.2]. By the previous lemma, all such injective objects belong to E_{l_1} , and all such projective objects belong to F_{l_1} .

In the rest of this section, as well as in the next Sections 5–6, we largely follow the exposition in [23, Sections 3–5] with obvious minimal variations. Most proofs are omitted and replaced with references to [23].

Lemma 4.4. (a) *Let $\mathcal{M}^\bullet$ be a complex of left $\mathcal{C}$ -comodules concentrated in the cohomological degrees $-n_1 \leq m \leq n_2$. Then $\mathcal{M}^\bullet$ is quasi-isomorphic to a complex of left $\mathcal{C}$ -comodules concentrated in the cohomological degrees $-n_1 \leq m \leq n_2$ with the terms belonging to the full subcategory $E_{l_1} \subset \mathcal{C}\text{-comod}$ if and only if $\text{Ext}_{\mathcal{C}}^n(\mathcal{L}^\bullet, \mathcal{M}^\bullet) = 0$ for $n > n_2 + l$ and the adjunction morphism $\mathcal{L}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} \mathbb{R} \text{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, \mathcal{M}^\bullet) \rightarrow \mathcal{M}^\bullet$ is an isomorphism in $D^-(\mathcal{C}\text{-comod})$.*

(b) *Let $\mathcal{T}^\bullet$ be a complex of left $\mathcal{D}$ -contra modules concentrated in the cohomological degrees $-n_1 \leq m \leq n_2$. Then $\mathcal{T}^\bullet$ is quasi-isomorphic to a complex of left $\mathcal{D}$ -contra modules concentrated in the cohomological degrees $-n_1 \leq m \leq n_2$ with the terms belonging to the full subcategory $F_{l_1} \subset \mathcal{D}\text{-contra}$ if and only if $\text{Ctrtor}_n^{\mathcal{D}}(\mathcal{L}^\bullet, \mathcal{T}^\bullet) = 0$ for $n > n_1 + l_1$ and the adjunction morphism $\mathcal{T}^\bullet \rightarrow \mathbb{R} \text{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, \mathcal{L}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} \mathcal{T}^\bullet)$ is an isomorphism in $D^+(\mathcal{D}\text{-contra})$.*

Proof. Similar to [23, Lemma 3.4]. □

It follows from Lemma 4.4 that the full subcategory $D^b(E_{l_1}) \subset D(\mathcal{C}\text{-comod})$ consists of all the complexes of left $\mathcal{C}$ -comodules $\mathcal{M}^\bullet$ with bounded cohomology such that the complex $\mathbb{R} \text{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, \mathcal{M}^\bullet)$ also has bounded cohomology and the adjunction morphism $\mathcal{L}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} \mathbb{R} \text{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, \mathcal{M}^\bullet) \rightarrow \mathcal{M}^\bullet$ is an isomorphism. Similarly, the full subcategory $D^b(F_{l_1}) \subset D(\mathcal{D}\text{-contra})$ consists of all the complexes of left $\mathcal{D}$ -contra modules $\mathcal{T}^\bullet$ with bounded cohomology such that the complex $\mathcal{L}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} \mathcal{T}^\bullet$ also has bounded cohomology and the adjunction morphism $\mathcal{T}^\bullet \rightarrow \mathbb{R} \text{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, \mathcal{L}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} \mathcal{T}^\bullet)$ is an isomorphism.

These two full subcategories can be called the *derived Bass class of comodules* and the *derived Auslander class of contra modules*. General category-theoretic consideration with adjoint functors [6, Theorem 1.1] (cf. [8, Proposition 2.1]) show that the functors $\mathbb{R} \text{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, -)$ and $\mathcal{L}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} -$ restrict to a triangulated equivalence between the derived Auslander and Bass classes,

$$(13) \quad D^b(E_{l_1}) \simeq D^b(F_{l_1}).$$

Lemma 4.5. (a) For any $\mathcal{C}$ -comodule $\mathcal{E} \in \mathbf{E}_{l_1}$, the object $\mathbb{R}\mathrm{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, \mathcal{E}) \in \mathbf{D}^b(\mathcal{D}\text{-contra})$ can be represented by a complex of $\mathcal{D}$ -contramodules concentrated in the cohomological degrees $-d_2 \leq m \leq l_1$ with the terms belonging to $\mathbf{F}_{l_1}$.

(b) For any $\mathcal{D}$ -contramodule $\mathfrak{F} \in \mathbf{F}_{l_1}$, the object $\mathcal{L}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} \mathfrak{F} \in \mathbf{D}^b(\mathcal{C}\text{-comod})$ can be represented by a complex of $\mathcal{C}$ -comodules concentrated in the cohomological degrees $-l_1 \leq m \leq d_2$ with the terms belonging to $\mathbf{E}_{l_1}$.

Proof. Similar to [23, Lemma 3.5]. $\square$

The definitions and discussions of the *coresolution dimension* of objects of an exact category $\mathbf{A}$ with respect to its coresolving subcategory $\mathbf{E}$ and the *resolution dimension* of objects of an exact category $\mathbf{B}$ with respect to its resolving subcategory $\mathbf{F}$ can be found in [28, Section 2] or [17, Section A.5] (the terminology in the latter reference is the *right E-homological dimension* and *left F-homological dimension*).

Lemma 4.6. (a) For any integers $l''_1 \geq l'_1 \geq d_1$, the full subcategory $\mathbf{E}_{l''_1} \subset \mathcal{C}\text{-comod}$ consists precisely of all the left $\mathcal{C}$ -comodules whose $\mathbf{E}_{l'_1}$ -coresolution dimension does not exceed $l''_1 - l'_1$.

(b) For any integers $l''_1 \geq l'_1 \geq d_1$, the full subcategory $\mathbf{F}_{l''_1} \subset \mathcal{D}\text{-contra}$ consists precisely of all the left $\mathcal{D}$ -contramodules whose $\mathbf{F}_{l'_1}$ -resolution dimension does not exceed $l''_1 - l'_1$.

Proof. Similar to [23, Lemma 3.6]. $\square$

Remark 4.7. It follows from Lemmas 4.2 and 4.6 that, for any finite $n \geq 0$, all the left $\mathcal{C}$ -comodules of injective dimension $\leq n$ belong to $\mathbf{E}_{d_1+n}$ and all the left $\mathcal{D}$ -contramodules of projective dimension $\leq n$ belong to $\mathbf{F}_{d_1+n}$.

We refer to [17, Section A.1] or [19, Appendix A] for the definitions of the absolute derived categories appearing in the next proposition.

Proposition 4.8. For any $l''_1 \geq l'_1 \geq d_1$ and any conventional or absolute derived category symbol $\star = \mathbf{b}, +, -, \emptyset, \mathbf{abs}+, \mathbf{abs}-$, or $\mathbf{abs}$, the exact inclusion functors of the Bass/Auslander classes of co/contramodules with varying parameter, $\mathbf{E}_{l'_1} \longrightarrow \mathbf{E}_{l''_1}$ and $\mathbf{F}_{l'_1} \longrightarrow \mathbf{F}_{l''_1}$, induce triangulated equivalences

$$\mathbf{D}^\star(\mathbf{E}_{l'_1}) \simeq \mathbf{D}^\star(\mathbf{E}_{l''_1}) \quad \text{and} \quad \mathbf{D}^\star(\mathbf{F}_{l'_1}) \simeq \mathbf{D}^\star(\mathbf{F}_{l''_1}).$$

Proof. Follows from Lemma 4.6 and [17, Proposition A.5.6]. $\square$

In particular, the unbounded derived category of the Bass class of left $\mathcal{C}$ -comodules $\mathbf{D}(\mathbf{E}_{l_1})$ is the same for all $l_1 \geq d_1$ and the unbounded derived category of the Auslander class of left $\mathcal{D}$ -contramodules $\mathbf{D}(\mathbf{F}_{l_1})$ is the same for all $l_1 \geq d_1$. We put

$$\mathbf{D}'_{\mathcal{L}^\bullet}(\mathcal{C}\text{-comod}) = \mathbf{D}(\mathbf{E}_{l_1}) \quad \text{and} \quad \mathbf{D}''_{\mathcal{L}^\bullet}(\mathcal{D}\text{-contra}) = \mathbf{D}(\mathbf{F}_{l_1}).$$

According to the discussion in Section 1.2, it follows from Lemmas 4.1 and 4.2 by virtue of [25, Proposition 5.5] that the pair of full subcategories $\mathbf{D}_A^{\leq 0}(\mathbf{E}_{l_1})$ and $\mathbf{D}^{\geq 0}(\mathbf{E}_{l_1})$ forms a t-structure of the derived type with the heart $\mathbf{A} = \mathcal{C}\text{-comod}$ on the triangulated category $\mathbf{D}'_{\mathcal{L}^\bullet}(\mathcal{C}\text{-comod})$, while the pair of full subcategories $\mathbf{D}^{\leq 0}(\mathbf{F}_{l_1})$

and $D_B^{\geq 0}(F_{l_1})$ forms a t-structure of the derived type with the heart $B = \mathcal{D}\text{-contra}$ on the triangulated category $D'_{\mathcal{L}^\bullet}(\mathcal{D}\text{-contra})$. It is easy to see that these t-structures do not depend on the choice of a parameter $l_1 \geq d_1$.

Moreover, following the discussion in Section 2, the canonical triangulated functor $D'_{\mathcal{L}^\bullet}(\mathcal{C}\text{-comod}) \rightarrow D(\mathcal{C}\text{-comod})$ induced by the inclusion of exact/abelian categories $E_{l_1} \rightarrow A$ is a Verdier quotient functor that has a right adjoint, and similarly, the canonical triangulated functor $D''_{\mathcal{L}^\bullet}(\mathcal{D}\text{-contra}) \rightarrow D(\mathcal{D}\text{-contra})$ induced by the inclusion of exact/abelian categories $F_{l_1} \rightarrow B$ is a Verdier quotient functor that has a left adjoint.

The next theorem, generalizing the equivalence (13), provides, in particular, a triangulated equivalence

$$D'_{\mathcal{L}^\bullet}(\mathcal{C}\text{-comod}) = D(E_{l_1}) \simeq D(F_{l_1}) = D''_{\mathcal{L}^\bullet}(\mathcal{D}\text{-contra}).$$

Thus we obtain a pair of t-structures of the derived type with the hearts A and B on one and the same triangulated category.

Theorem 4.9. *For any symbol $\star = b, +, -, \emptyset, \text{abs}+, \text{abs}-, \text{or } \text{abs}$, there is a triangulated equivalence $D^\star(E_{l_1}) \simeq D^\star(F_{l_1})$ provided by (appropriately defined) mutually inverse derived functors $\mathbb{R}\text{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, -)$ and $\mathcal{L}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} -$.*

Proof. This is a particular case of Theorem 5.2 below. $\square$

5. ABSTRACT CORRESPONDING CLASSES

More generally, suppose that $E \subset \mathcal{C}\text{-comod}$ and $F \subset \mathcal{D}\text{-contra}$ are two full subcategories satisfying the following conditions (for some fixed integers l_1 and l_2):

- (I) the class of objects E is closed under extensions and the cokernels of injective morphisms in $\mathcal{C}\text{-comod}$, and contains all the injective left $\mathcal{C}$ -comodules;
- (II) the class of objects F is closed under extensions and the kernels of surjective morphisms in $\mathcal{D}\text{-contra}$, and contains all the projective left $\mathcal{D}$ -contramodules;
- (III) for any $\mathcal{C}$ -comodule $\mathcal{E} \in E$, the derived category object $\mathbb{R}\text{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, \mathcal{E}) \in D^+(\mathcal{D}\text{-contra})$ can be represented by a complex of $\mathcal{D}$ -contramodules concentrated in the cohomological degrees $-l_2 \leq m \leq l_1$ with the terms belonging to F ;
- (IV) for any $\mathcal{D}$ -contramodule $\mathfrak{F} \in F$, the derived category object $\mathcal{L}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} \mathfrak{F} \in D^-(\mathcal{C}\text{-comod})$ can be represented by a complex of $\mathcal{C}$ -comodules concentrated in the cohomological degrees $-l_1 \leq m \leq l_2$ with the terms belonging to E .

Just as in [23, Section 4], one can see from the conditions (I) and (III), or (II) and (IV), that $l_1 \geq d_1$ and $l_2 \geq d_2$ if $H^{-d_1}(\mathcal{L}^\bullet) \neq 0 \neq H^{d_2}(\mathcal{L}^\bullet)$.

According to Lemmas 4.1, 4.2, and 4.5, the Bass and Auslander classes $E = E_{l_1}$ and $F = F_{l_1}$ satisfy the conditions (I–IV) with $l_2 = d_2$. The following lemma provides a kind of converse implication.

Lemma 5.1. (a) For any $\mathcal{C}$ -comodule $\mathcal{E} \in \mathbf{E}$, the adjunction morphism $\mathcal{L}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} \mathbb{R} \operatorname{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, \mathcal{E}) \longrightarrow \mathcal{E}$ is an isomorphism in $\mathbf{D}^b(\mathcal{C}\text{-comod})$.

(b) For any $\mathcal{D}$ -contramodule $\mathfrak{F} \in \mathbf{F}$, the adjunction morphism $\mathfrak{F} \longrightarrow \mathbb{R} \operatorname{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, \mathcal{L}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} \mathfrak{F})$ is an isomorphism in $\mathbf{D}^b(\mathcal{D}\text{-contra})$.

Proof. This can be proved directly in the way similar to the proof of [23, Lemma 4.1], or obtained as a byproduct of the proof of Theorem 5.2 below. (In any case, the argument is based on Lemma 4.2.) $\square$

Assuming that $l_1 \geq d_1$ and $l_2 \geq d_2$, it is clear from the conditions (III–IV) and Lemma 5.1 that the inclusions $\mathbf{E} \subset \mathbf{E}_{l_1}$ and $\mathbf{F} \subset \mathbf{F}_{l_1}$ hold for any two classes of objects $\mathbf{E} \subset \mathcal{C}\text{-comod}$ and $\mathbf{F} \subset \mathcal{D}\text{-contra}$ satisfying (I–IV). Furthermore, it follows from the conditions (I–II) that the triangulated functors $\mathbf{D}^b(\mathbf{E}) \longrightarrow \mathbf{D}^b(\mathcal{C}\text{-comod})$ and $\mathbf{D}^b(\mathbf{F}) \longrightarrow \mathbf{D}^b(\mathcal{D}\text{-contra})$ induced by the exact inclusions $\mathbf{E} \longrightarrow \mathcal{C}\text{-comod}$ and $\mathbf{F} \longrightarrow \mathcal{D}\text{-contra}$ are fully faithful. Therefore, the triangulated functors $\mathbf{D}^b(\mathbf{E}) \longrightarrow \mathbf{D}^b(\mathbf{E}_{l_1})$ and $\mathbf{D}^b(\mathbf{F}) \longrightarrow \mathbf{D}^b(\mathbf{F}_{l_1})$ are fully faithful, too. Using again the conditions (III–IV), we can conclude that the equivalence (13) restricts to a triangulated equivalence

$$(14) \quad \mathbf{D}^b(\mathbf{E}) \simeq \mathbf{D}^b(\mathbf{F}).$$

The following theorem is our main result.

Theorem 5.2. Let $\mathbf{E} \subset \mathcal{C}\text{-comod}$ and $\mathbf{F} \subset \mathcal{D}\text{-contra}$ be a pair of full subcategories of comodules and contramodules satisfying the conditions (I–IV) for a pseudo-dualizing complex of $\mathcal{C}$ - $\mathcal{D}$ -bicomodules $\mathcal{L}^\bullet$. Then for any conventional or absolute derived category symbol $\star = \mathbf{b}, +, -, \emptyset, \text{abs}+, \text{abs}-, \text{or } \text{abs}$, there is a triangulated equivalence $\mathbf{D}^\star(\mathbf{E}) \simeq \mathbf{D}^\star(\mathbf{F})$ provided by (appropriately defined) mutually inverse derived functors $\mathbb{R} \operatorname{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, -)$ and $\mathcal{L}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} -$.

Proof. The words “appropriately defined” here mean “defined or constructed using the technology of [23, Appendix A]”. In the context of the latter, we set

$$\begin{aligned} \mathbf{A} &= \mathcal{C}\text{-comod} \supset \mathbf{E} \supset \mathbf{J} = \mathcal{C}\text{-comod}_{\text{inj}} \\ \mathbf{B} &= \mathcal{D}\text{-contra} \supset \mathbf{F} \supset \mathbf{P} = \mathcal{D}\text{-contra}_{\text{proj}}. \end{aligned}$$

Consider the adjoint pair of DG-functors

$$\begin{aligned} \Psi &= \operatorname{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, -): \mathbf{C}^+(\mathbf{J}) \longrightarrow \mathbf{C}^+(\mathbf{B}) \\ \Phi &= \mathcal{L}^\bullet \odot_{\mathcal{D}} -: \mathbf{C}^-(\mathbf{P}) \longrightarrow \mathbf{C}^-(\mathbf{A}). \end{aligned}$$

Then the constructions of [23, Sections A.2–A.3] provide the derived functors $\mathbb{R}\Psi: \mathbf{D}^\star(\mathbf{E}) \longrightarrow \mathbf{D}^\star(\mathbf{F})$ and $\mathbb{L}\Phi: \mathbf{D}^\star(\mathbf{F}) \longrightarrow \mathbf{D}^\star(\mathbf{E})$. The arguments in [23, Section A.4] show that the functor $\mathbb{L}\Phi$ is left adjoint to the functor $\mathbb{R}\Psi$, and the first assertion of [23, Theorem A.7] allows to deduce the claim that they are mutually inverse triangulated equivalences from the particular case of $\star = \mathbf{b}$, which is the triangulated equivalence (14).

Alternatively, applying the second assertion of [23, Theorem A.7] together with Lemma 4.2 allows to reprove the triangulated equivalence (14) instead of using it,

thus obtaining a proof of Lemma 5.1. We refer to [23, proof of Theorem 4.2] for the more informal discussion and further details. $\square$

According to the discussion in Section 1.2, it follows from the conditions (I–II) by virtue of [25, Proposition 5.5] that the pair of full subcategories $D_A^{\leq 0}(E)$ and $D_A^{\geq 0}(E)$ forms a t-structure of the derived type with the heart $A = \mathcal{C}\text{-comod}$ on the triangulated category $D(E)$, while the pair of full subcategories $D_B^{\leq 0}(F)$ and $D_B^{\geq 0}(F)$ forms a t-structure of the derived type with the heart $B = \mathcal{D}\text{-contra}$ on the triangulated category $D(F)$. By Theorem 5.2, there is a triangulated equivalence

$$(15) \quad D(E) \simeq D(F).$$

Thus, as in Section 4, we have a pair of t-structures of the derived type with the hearts $\mathcal{C}\text{-comod}$ and $\mathcal{D}\text{-contra}$ on one and the same triangulated category (15).

Moreover, following the discussion in Section 2, the triangulated functor $D(E) \rightarrow D(\mathcal{C}\text{-comod})$ induced by the inclusion of exact/abelian categories $E \rightarrow \mathcal{C}\text{-comod}$ is a Verdier quotient functor with a right adjoint, while the triangulated functor $D(F) \rightarrow D(\mathcal{D}\text{-contra})$ induced by the inclusion of exact/abelian categories $F \rightarrow \mathcal{D}\text{-contra}$ is a Verdier quotient functor with a left adjoint.

Now suppose that we have two pairs of full subcategories $E, \subset E' \subset \mathcal{C}\text{-comod}$ and $F, \subset F' \subset \mathcal{D}\text{-contra}$ such that both the pairs (E, F) and (E', F') satisfy the conditions (I–IV). Then for any symbol $\star = b, +, -, \emptyset, \text{abs}+, \text{abs}-$, or abs there is a commutative diagram of triangulated functors and triangulated equivalences

$$(16) \quad \begin{array}{ccc} D^\star(E) & \xlongequal{\quad} & D^\star(F) \\ \downarrow & & \downarrow \\ D^\star(E') & \xlongequal{\quad} & D^\star(F') \end{array}$$

The vertical functors are induced by the exact inclusions of exact categories $E \rightarrow E'$ and $F \rightarrow F'$. When $\star = \emptyset$, the vertical functors on the diagram (16) are t-exact with respect to both the respective t-structures and induce the identity equivalences $A \xrightarrow{\sim} A$ and $B \xrightarrow{\sim} B$ on the hearts.

Remark 5.3. Can one make a *concept* out of the above construction of a pair of t-structures (and the similar construction of the pair of ∞ -(co)tilting t-structures in [25, Section 5])? Let A and B be abelian categories. One can define a *t-derived pseudo-equivalence* between A and B as a triangulated category D endowed with a pair of (possibly degenerate) t-structures of the derived type $({}^{\prime}D^{\leq 0}, {}^{\prime}D^{\geq 0})$ and $({}^{\prime\prime}D^{\leq 0}, {}^{\prime\prime}D^{\geq 0})$ whose hearts are ${}^{\prime}D^{\leq 0} \cap {}^{\prime}D^{\geq 0} \simeq A$ and ${}^{\prime\prime}D^{\leq 0} \cap {}^{\prime\prime}D^{\geq 0} \simeq B$.

With such a definition, though, any pair of abelian categories is connected by a trivial t-derived pseudo-equivalence. Set D to be the Cartesian product $D(A) \times D(B)$. Then the pair of full subcategories ${}^{\prime}D^{\leq 0} = D^{\leq 0}(A) \times D(B)$ and ${}^{\prime}D^{\geq 0} = D^{\geq 0}(A) \times 0 \subset D$ is a degenerate t-structure of the derived type on D with the heart A , while the pair of

full subcategories ${}''\mathcal{D}^{\leq 0} = 0 \times \mathcal{D}^{\leq 0}(\mathcal{B})$ and ${}''\mathcal{D}^{\geq 0} = \mathcal{D}(\mathcal{A}) \times \mathcal{D}^{\geq 0}(\mathcal{B}) \subset \mathcal{D}$ is a degenerate t-structure of the derived type on $\mathcal{D}$ with the heart $\mathcal{B}$.

Perhaps imposing an additional nontriviality condition would make the notion of a t-derived pseudo-equivalence more meaningful. An obvious idea is to demand existence of a pair of integers l_1 and l_2 such that for any object $X \in \mathcal{D}$ there exist distinguished triangles $Y' \rightarrow X \rightarrow Z'' \rightarrow Y'[1]$ and $Y'' \rightarrow X \rightarrow Z' \rightarrow Y''[1]$ with $Y' \in {}'\mathcal{D}^{\geq 0}$, $Z'' \in {}''\mathcal{D}^{\leq l_1-1}$, $Y'' \in {}''\mathcal{D}^{\geq 0}$, and $Z' \in {}'\mathcal{D}^{\leq l_2-1}$. This condition holds for the pair of t-structures $({}'\mathcal{D}^{\leq 0}, {}'\mathcal{D}^{\geq 0}) = (\mathcal{D}_\mathcal{A}^{\leq 0}(\mathcal{E}), \mathcal{D}^{\geq 0}(\mathcal{E}))$ and $({}''\mathcal{D}^{\leq 0}, {}''\mathcal{D}^{\geq 0}) = (\mathcal{D}^{\leq 0}(\mathcal{F}), \mathcal{D}_\mathcal{B}^{\geq 0}(\mathcal{F}))$ constructed in this section, and does *not* hold for the trivial pair of t-structures from the previous paragraph.

6. MINIMAL CORRESPONDING CLASSES

Let $\mathcal{C}$ and $\mathcal{D}$ be coassociative coalgebras over a field k , and let $\mathcal{L}^\bullet$ be a pseudo-dualizing complex of $\mathcal{C}$ - $\mathcal{D}$ -bicomodules concentrated in the cohomological degrees $-d_1 \leq m \leq d_2$.

Proposition 6.1. *Fix $l_1 = d_1$ and $l_2 \geq d_2$. Then there exists a unique minimal pair of full subcategories $\mathcal{E}^{l_2} = \mathcal{E}^{l_2}(\mathcal{L}^\bullet) \subset \mathcal{C}\text{-comod}$ and $\mathcal{F}^{l_2} = \mathcal{F}^{l_2}(\mathcal{L}^\bullet) \subset \mathcal{D}\text{-contra}$ satisfying the conditions (I–IV). For any pair of full subcategories $\mathcal{E} \subset \mathcal{C}\text{-comod}$ and $\mathcal{F} \subset \mathcal{D}\text{-contra}$ satisfying the conditions (I–IV) one has $\mathcal{E}^{l_2} \subset \mathcal{E}$ and $\mathcal{F}^{l_2} \subset \mathcal{F}$.*

Proof. The full subcategories $\mathcal{E}^{l_2} \subset \mathcal{C}\text{-comod}$ and $\mathcal{F}^{l_2} \subset \mathcal{D}\text{-contra}$ are constructed simultaneously by a generation process similar to the one in [23, proof of Proposition 5.1]. The difference is that, unlike in [23], we do not require the classes $\mathcal{E}^{l_2}$ and $\mathcal{F}^{l_2}$ to be closed under infinite direct sums and products, and accordingly do not do the direct sum/product closure in their construction. (Indeed, the direct sum/product closure would be problematic, as we do not know whether the Bass and Auslander classes $\mathcal{E}_{l_1}$ and $\mathcal{F}_{l_1}$ are closed under direct sums/products; see Remark 4.3.) Accordingly, it suffices to repeat the iterative process over the poset of nonnegative integers and no transfinite iterations are needed. $\square$

Remark 6.2. Moreover, for any two integers $l_1 \geq d_1$ and $l_2 \geq d_2$ and any two full subcategories $\mathcal{E} \subset \mathcal{C}\text{-comod}$ and $\mathcal{F} \subset \mathcal{D}\text{-contra}$ satisfying the conditions (I–IV) with the parameters l_1 and l_2 , one has $\mathcal{E}^{l_2} \subset \mathcal{E}$ and $\mathcal{F}^{l_2} \subset \mathcal{F}$. This is easily provable by induction with respect to the iterative construction of the categories $\mathcal{E}^{l_2}$ and $\mathcal{F}^{l_2}$ in Proposition 6.1 (cf. [23, Remark 5.2]).

One observes that the conditions (III–IV) become weaker as the parameter l_2 increases. Therefore, $\mathcal{E}^{l_2} \supset \mathcal{E}^{l_2+1}$ and $\mathcal{F}^{l_2} \supset \mathcal{F}^{l_2+1}$ for all $l_2 \geq d_2$.

Lemma 6.3. *Let $n \geq 0$ and $l_1 \geq d_1$, $l_2 \geq d_2 + n$ be some integers, and let $\mathcal{E} \subset \mathcal{C}\text{-comod}$ and $\mathcal{F} \subset \mathcal{D}\text{-contra}$ be a pair of full subcategories satisfying the conditions (I–IV) with the parameters l_1 and l_2 . Denote by $\mathcal{E}(n) \subset \mathcal{C}\text{-comod}$ the full subcategory of all left $\mathcal{C}$ -comodules of $\mathcal{E}$ -coresolution dimension $\leq n$ and by*

$F(n) \subset \mathcal{D}\text{-contra}$ the full subcategory of all left $\mathcal{D}$ -contramodules of F -resolution dimension $\leq n$. Then the pair of classes of comodules and contramodules $E(n)$ and $F(n)$ satisfies the conditions (I–IV) with the parameters $l_1 + n$ and $l_2 - n$.

Proof. Similar to [23, Lemma 5.3]. $\square$

Proposition 6.4. *For any $l_2'' \geq l_2' \geq d_2$ and any conventional or absolute derived category symbol $\star = \mathbf{b}, +, -, \emptyset, \mathbf{abs}+, \mathbf{abs}-$, or $\mathbf{abs}$, the exact inclusions $E^{l_2''} \rightarrow E^{l_2'}$ and $F^{l_2''} \rightarrow F^{l_2'}$ induce triangulated equivalences*

$$D^\star(E^{l_2''}) \simeq D^\star(E^{l_2'}) \quad \text{and} \quad D^\star(F^{l_2''}) \simeq D^\star(F^{l_2'}).$$

Proof. Similar to [23, Proposition 5.4]. Using Lemma 6.3, one checks that the $E^{l_2''}$ -coresolution dimension of any object of $E^{l_2'}$ does not exceed $l_2'' - l_2'$ and the $F^{l_2''}$ -resolution dimension of any object of $F^{l_2'}$ does not exceed $l_2'' - l_2'$. Then one applies [17, Proposition A.5.6], as in the proof of Proposition 4.8. $\square$

In particular, the unbounded derived category $D(E^{l_2})$ of the minimal corresponding class of left $\mathcal{C}$ -comodules E^{l_2} is the same for all $l_2 \geq d_2$ and the unbounded derived category $D(F^{l_2})$ of the minimal corresponding class of left $\mathcal{D}$ -contramodules F^{l_2} is the same for all $l_2 \geq d_2$. We put

$$D_{\mathcal{C}}^{\mathcal{L}^\bullet}(\mathcal{C}\text{-comod}) = D(E^{l_2}) \quad \text{and} \quad D_{\mathcal{D}}^{\mathcal{L}^\bullet}(\mathcal{D}\text{-contra}) = D(F^{l_2}).$$

As a particular case of the discussion in Section 5, the pair of full subcategories $D_{\mathcal{A}}^{\leq 0}(E^{l_2})$ and $D_{\mathcal{A}}^{\geq 0}(E^{l_2})$ forms a t-structure of the derived type with the heart $\mathcal{A} = \mathcal{C}\text{-comod}$ on the triangulated category $D_{\mathcal{C}}^{\mathcal{L}^\bullet}(\mathcal{C}\text{-comod})$, while the pair of full subcategories $D_{\mathcal{B}}^{\leq 0}(F^{l_2})$ and $D_{\mathcal{B}}^{\geq 0}(F^{l_2})$ forms a t-structure of the derived type with the heart $\mathcal{B} = \mathcal{D}\text{-contra}$ on the triangulated category $D_{\mathcal{D}}^{\mathcal{L}^\bullet}(\mathcal{D}\text{-contra})$. It is easy to see that these t-structures do not depend on the choice of a parameter $l_2 \geq d_2$.

Moreover, there is a canonical Verdier quotient functor $D_{\mathcal{C}}^{\mathcal{L}^\bullet}(\mathcal{C}\text{-comod}) \rightarrow D(\mathcal{C}\text{-comod})$ that has a right adjoint, and similarly, there is a canonical Verdier quotient functor $D_{\mathcal{D}}^{\mathcal{L}^\bullet}(\mathcal{D}\text{-contra}) \rightarrow D(\mathcal{D}\text{-contra})$ that has a left adjoint.

The next theorem provides, in particular, a triangulated equivalence

$$D_{\mathcal{C}}^{\mathcal{L}^\bullet}(\mathcal{C}\text{-comod}) = D(E^{l_2}) \simeq D(F^{l_2}) = D_{\mathcal{D}}^{\mathcal{L}^\bullet}(\mathcal{D}\text{-contra}).$$

Thus, once again, we obtain a pair of t-structures of the derived type with the hearts $\mathcal{A}$ and $\mathcal{B}$ on one and the same triangulated category.

Theorem 6.5. *For any symbol $\star = \mathbf{b}, +, -, \emptyset, \mathbf{abs}+, \mathbf{abs}-$, or $\mathbf{abs}$, there is a triangulated equivalence $D^\star(E^{l_2}) \simeq D^\star(F^{l_2})$ provided by (appropriately defined) mutually inverse derived functors $\mathbb{R}\mathrm{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, -)$ and $\mathcal{L}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} -$.*

Proof. This is another particular case of Theorem 5.2. $\square$

7. DUALIZING COMPLEXES

Let $\mathcal{C}$ and $\mathcal{D}$ be coassociative coalgebras over k . A *dualizing complex* of $\mathcal{C}$ - $\mathcal{D}$ -bicomodules $\mathcal{L}^\bullet = \mathcal{K}^\bullet$ is a pseudo-dualizing complex (according to the definition in Section 4) satisfying the following additional condition:

- (i) As a complex of left $\mathcal{C}$ -comodules, $\mathcal{K}^\bullet$ is quasi-isomorphic to a finite complex of injective $\mathcal{C}$ -comodules, and as a complex of right $\mathcal{D}$ -comodules, $\mathcal{K}^\bullet$ is quasi-isomorphic to a finite complex of injective $\mathcal{D}$ -comodules.

In view of Lemma 3.11, the conditions (i) and (ii) taken together can be equivalently restated by saying that, as a complex of left $\mathcal{C}$ -comodules, $\mathcal{K}^\bullet$ is quasi-isomorphic to a finite complex of quasi-finitely cogenerated injective $\mathcal{C}$ -comodules, and as a complex of right $\mathcal{D}$ -comodules, $\mathcal{K}^\bullet$ is quasi-isomorphic to a finite complex of quasi-finitely cogenerated injective $\mathcal{D}$ -comodules.

Let us choose the parameter l_2 in such a way that $\mathcal{K}^\bullet$ is quasi-isomorphic to a complex of (quasi-finitely cogenerated) injective left $\mathcal{C}$ -comodules concentrated in the cohomological degrees $-d_1 \leq m \leq l_2$ and to a complex of (quasi-finitely cogenerated) injective right $\mathcal{D}$ -comodules concentrated in the cohomological degrees $-d_1 \leq m \leq l_2$.

Proposition 7.1. *Let $\mathcal{L}^\bullet = \mathcal{K}^\bullet$ be a dualizing complex of $\mathcal{C}$ - $\mathcal{D}$ -bicomodules, and let the parameter l_2 be chosen as stated above. Then the related minimal corresponding classes $\mathbf{E}^{l_2} = \mathbf{E}^{l_2}(\mathcal{K}^\bullet)$ and $\mathbf{F}^{l_2} = \mathbf{F}^{l_2}(\mathcal{K}^\bullet)$ coincide with the classes of injective left $\mathcal{C}$ -comodules and projective left $\mathcal{D}$ -contra modules, $\mathbf{E}^{l_2} = \mathcal{C}\text{-comod}_{\text{inj}}$ and $\mathbf{F}^{l_2} = \mathcal{D}\text{-contra}_{\text{proj}}$.*

Proof. It suffices to check that the conditions (I–IV) hold for $\mathbf{E} = \mathcal{C}\text{-comod}_{\text{inj}}$ and $\mathbf{F} = \mathcal{D}\text{-contra}_{\text{proj}}$. Indeed, the conditions (I–II) are obvious in this case. To check (III), one can assume that $\mathcal{E}$ is a cofree left $\mathcal{C}$ -comodule, $\mathcal{E} = \mathcal{C} \otimes_k V$, where V is a k -vector space. Then $\mathbb{R}\text{Hom}_{\mathcal{C}}(\mathcal{K}^\bullet, \mathcal{E}) = \text{Hom}_{\mathcal{C}}(\mathcal{K}^\bullet, \mathcal{E}) \simeq \text{Hom}_k(\mathcal{K}^\bullet, V)$ as a complex of left $\mathcal{D}$ -contra modules. Choose a complex of injective right $\mathcal{D}$ -comodules ${}'\mathcal{J}^\bullet$ quasi-isomorphic to $\mathcal{K}^\bullet$ and concentrated in the cohomological degrees $-d_1 \leq m \leq l_2$. Then the derived category object $\mathbb{R}\text{Hom}_{\mathcal{C}}(\mathcal{K}^\bullet, \mathcal{E}) \in \mathbf{D}^+(\mathcal{D}\text{-contra})$ is represented by the complex of projective left $\mathcal{D}$ -contra modules $\text{Hom}_k({}'\mathcal{J}^\bullet, V)$ concentrated in the cohomological degrees $-l_2 \leq m \leq d_1$. Similarly, to check (IV), one can assume that $\mathfrak{F}$ is a free left $\mathcal{D}$ -contra module, $\mathfrak{F} = \text{Hom}_k(\mathcal{D}, V)$. Then $\mathcal{K}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} \mathfrak{F} = \mathcal{K}^\bullet \odot_{\mathcal{D}} \mathfrak{F} \simeq \mathcal{K}^\bullet \otimes_k V$ as a complex of left $\mathcal{C}$ -comodules. Choose a complex of injective left $\mathcal{C}$ -comodules ${}''\mathcal{J}^\bullet$ quasi-isomorphic to $\mathcal{K}^\bullet$ and concentrated in the cohomological degrees $-d_1 \leq m \leq l_2$. Then the derived category object $\mathcal{K}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} \mathfrak{F} \in \mathbf{D}^-(\mathcal{C}\text{-comod})$ is represented by the complex of injective left $\mathcal{C}$ -comodules ${}''\mathcal{J}^\bullet \otimes_k V$ concentrated in the cohomological degrees $-d_1 \leq m \leq l_2$. $\square$

It is clear from Proposition 7.1, in view of the triangulated equivalences $\text{Hot}(\mathcal{C}\text{-comod}_{\text{inj}}) \simeq \mathbf{D}^{\text{co}}(\mathcal{C}\text{-comod})$ and $\text{Hot}(\mathcal{D}\text{-contra}_{\text{proj}}) \simeq \mathbf{D}^{\text{ctr}}(\mathcal{D}\text{-contra})$ [16, Section 4.4] mentioned in the formula (3) in Section 1.4, that for a dualizing complex of $\mathcal{C}$ - $\mathcal{D}$ -bicomodules $\mathcal{K}^\bullet$ one has

$$\mathbf{D}'^{\mathcal{K}^\bullet}(\mathcal{C}\text{-comod}) = \mathbf{D}^{\text{co}}(\mathcal{C}\text{-comod}) \quad \text{and} \quad \mathbf{D}''^{\mathcal{K}^\bullet}(\mathcal{D}\text{-contra}) = \mathbf{D}^{\text{ctr}}(\mathcal{D}\text{-contra}).$$

Corollary 7.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be coassociative coalgebras over k , and $\mathcal{K}^\bullet$ be a dualizing complex of $\mathcal{C}$ - $\mathcal{D}$ -bicomodules. Then there is a triangulated equivalence $D^\infty(\mathcal{C}\text{-comod}) \simeq D^{\text{ctr}}(\mathcal{D}\text{-contra})$ provided by the mutually inverse derived functors $\mathbb{R} \text{Hom}_{\mathcal{C}}(\mathcal{K}^\bullet, -)$ and $\mathcal{K}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} -$.*

Furthermore, there is a commutative diagram of triangulated equivalences

$$\begin{array}{ccc}
 D^\infty(\mathcal{C}\text{-comod}) & \xleftarrow{\mathcal{K}^\bullet \square_{\mathcal{D}}^{\mathbb{R}} -} & D^\infty(\mathcal{D}\text{-comod}) \\
 \parallel & \swarrow \mathcal{K}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} - & \parallel \\
 D^{\text{ctr}}(\mathcal{C}\text{-contra}) & \xrightarrow{\mathbb{L} \text{Cohom}_{\mathcal{C}}(\mathcal{K}^\bullet, -)} & D^{\text{ctr}}(\mathcal{D}\text{-contra})
 \end{array}$$

$\mathbb{R} \text{Hom}_{\mathcal{C}}(\mathcal{K}^\bullet, -)$ (diagonal arrow from top-left to bottom-right)

where the vertical double lines denote the derived comodule-contramodule correspondence equivalences (3), and the horizontal arrows are the derived functors of cotensor product $\square_{\mathcal{D}}$ and cohomomorphisms $\text{Cohom}_{\mathcal{C}}$.

Proof. The first assertion is a particular case of Theorem 6.5. In fact, since $\mathbf{E}^{l_2} = \mathcal{C}\text{-comod}_{\text{inj}}$ and $\mathbf{F}^{l_2} = \mathcal{D}\text{-contra}_{\text{proj}}$ are split exact categories, the assertion of Theorem 6.5 for a dualizing complex $\mathcal{L}^\bullet = \mathcal{K}^\bullet$ reduces to triangulated equivalences between the (bounded or unbounded) homotopy categories

$$(17) \quad \text{Hom}_{\mathcal{C}}(\mathcal{K}^\bullet, -) : \text{Hot}^\star(\mathcal{C}\text{-comod}_{\text{inj}}) \simeq \text{Hot}^\star(\mathcal{D}\text{-contra}_{\text{proj}}) : \mathcal{K}^\bullet \odot_{\mathcal{D}} -$$

for all symbols $\star = \mathbf{b}, +, -, \text{ or } \emptyset$. Here the functors $\text{Hom}_{\mathcal{C}}(\mathcal{K}^\bullet, -)$ and $\mathcal{K}^\bullet \odot_{\mathcal{D}} -$ do not even need to be derived as, following the construction of [23, Appendix A], they are simply applied to any complex of injective $\mathcal{C}$ -comodules or (respectively) any complex of projective $\mathcal{D}$ -contramodules. Then the resulting complex is replaced by a complex of projective/injective objects isomorphic to it in the absolute derived category of contra/comodules to obtain the triangulated functors providing the equivalence (17). Identifying $D^\infty(\mathcal{C}\text{-comod})$ with $\text{Hot}(\mathcal{C}\text{-comod}_{\text{inj}})$ and $D^{\text{ctr}}(\mathcal{D}\text{-contra})$ with $\text{Hot}(\mathcal{D}\text{-contra}_{\text{proj}})$, as mentioned above, we obtain the desired triangulated equivalence between the coderived and the contraderived category,

$$(18) \quad \mathbb{R} \text{Hom}_{\mathcal{C}}(\mathcal{K}^\bullet, -) : D^\infty(\mathcal{C}\text{-comod}) \simeq D^{\text{ctr}}(\mathcal{D}\text{-contra}) : \mathcal{K}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} -.$$

Similarly, the right derived functor

$$\mathcal{K}^\bullet \square_{\mathcal{D}}^{\mathbb{R}} - : D^\infty(\mathcal{D}\text{-comod}) \longrightarrow D^\infty(\mathcal{C}\text{-comod})$$

is constructed as the composition

$$D^\infty(\mathcal{D}\text{-comod}) \simeq \text{Hot}(\mathcal{D}\text{-comod}_{\text{inj}}) \xrightarrow{\mathcal{K}^\bullet \square_{\mathcal{D}} -} \text{Hot}(\mathcal{C}\text{-comod}) \longrightarrow D^\infty(\mathcal{C}\text{-comod})$$

of the underived cotensor product functor $\mathcal{K}^\bullet \square_{\mathcal{D}} - : \text{Hot}(\mathcal{D}\text{-comod}_{\text{inj}}) \longrightarrow \text{Hot}(\mathcal{C}\text{-comod})$ with the Verdier quotient functor $\text{Hot}(\mathcal{C}\text{-comod}) \longrightarrow D^\infty(\mathcal{C}\text{-comod})$.

The left derived functor

$$\mathbb{L} \operatorname{Cohom}_{\mathcal{C}}(\mathcal{K}^\bullet, -) : \mathcal{D}^{\operatorname{ctr}}(\mathcal{C}\text{-}\operatorname{contra}) \longrightarrow \mathcal{D}^{\operatorname{ctr}}(\mathcal{D}\text{-}\operatorname{contra})$$

is constructed as the composition

$$\mathcal{D}^{\operatorname{ctr}}(\mathcal{C}\text{-}\operatorname{contra}) \simeq \operatorname{Hot}(\mathcal{C}\text{-}\operatorname{contra}_{\operatorname{proj}}) \xrightarrow{\operatorname{Cohom}_{\mathcal{C}}(\mathcal{K}^\bullet, -)} \operatorname{Hot}(\mathcal{D}\text{-}\operatorname{contra}) \longrightarrow \mathcal{D}^{\operatorname{ctr}}(\mathcal{D}\text{-}\operatorname{contra})$$

of the underived cohomomorphism functor $\operatorname{Cohom}_{\mathcal{C}}(\mathcal{K}^\bullet, -) : \operatorname{Hot}(\mathcal{C}\text{-}\operatorname{contra}_{\operatorname{proj}}) \longrightarrow \operatorname{Hot}(\mathcal{D}\text{-}\operatorname{contra})$ with the Verdier quotient functor $\operatorname{Hot}(\mathcal{D}\text{-}\operatorname{contra}) \longrightarrow \mathcal{D}^{\operatorname{ctr}}(\mathcal{D}\text{-}\operatorname{contra})$.

We recall that the downwards directed leftmost vertical equivalence is constructed by applying the functor $\operatorname{Hom}_{\mathcal{C}}(\mathcal{C}, -)$ to complexes of injective left $\mathcal{C}$ -comodules, while the upwards directed rightmost vertical equivalence is constructed by applying the functor $\mathcal{D} \odot_{\mathcal{D}} -$ to complexes of projective left $\mathcal{D}$ -contramodules (see formula (4) in Section 1.4). Now the upper triangle commutes due to the natural isomorphism of left $\mathcal{C}$ -comodules $\mathcal{K} \square_{\mathcal{D}} (\mathcal{D} \odot_{\mathcal{D}} \mathfrak{P}) \simeq (\mathcal{K} \square_{\mathcal{D}} \mathcal{D}) \odot_{\mathcal{D}} \mathfrak{P} = \mathcal{K} \odot_{\mathcal{D}} \mathfrak{P}$, which holds for any projective left $\mathcal{D}$ -contramodule $\mathfrak{P}$ and any $\mathcal{C}$ - $\mathcal{D}$ -bicomodule $\mathcal{K}$ [18, Proposition 3.1.1], [15, Proposition 5.2.1(a)]. Similarly, the lower triangle commutes due to the natural isomorphism of left $\mathcal{D}$ -contramodules $\operatorname{Cohom}_{\mathcal{C}}(\mathcal{K}, \operatorname{Hom}_{\mathcal{C}}(\mathcal{C}, \mathcal{J})) \simeq \operatorname{Hom}_{\mathcal{C}}(\mathcal{C} \square_{\mathcal{C}} \mathcal{K}, \mathcal{J}) = \operatorname{Hom}_{\mathcal{C}}(\mathcal{K}, \mathcal{J})$, which holds for any injective left $\mathcal{C}$ -comodule $\mathcal{J}$ and any $\mathcal{C}$ - $\mathcal{D}$ -bicomodule $\mathcal{K}$ [18, Proposition 3.1.2], [15, Proposition 5.2.2(a)].

Finally, the horizontal functors are triangulated equivalences, since so are the vertical and diagonal functors. $\square$

Lemma 7.3. *Let $\mathcal{K}^\bullet$ be a dualizing complex of $\mathcal{C}$ - $\mathcal{D}$ -bicomodules. Then*

- (a) *there exists a finite complex of injective left $\mathcal{D}$ -comodules $'\mathcal{K}^{\bullet\vee}$ such that the finite complex of left $\mathcal{C}$ -comodules $\mathcal{K}^\bullet \square_{\mathcal{D}} '\mathcal{K}^{\bullet\vee}$ is quasi-isomorphic to $\mathcal{C}$;*
- (b) *there exists a finite complex of injective right $\mathcal{C}$ -comodules $''\mathcal{K}^{\bullet\vee}$ such that the finite complex of right $\mathcal{D}$ -comodules $''\mathcal{K}^{\bullet\vee} \square_{\mathcal{C}} \mathcal{K}^\bullet$ is quasi-isomorphic to $\mathcal{D}$.*

Proof. Part (a): by Proposition 7.1, the derived category object $\mathbb{R} \operatorname{Hom}_{\mathcal{C}}(\mathcal{K}^\bullet, \mathcal{C}) \in \mathcal{D}^+(\mathcal{D}\text{-}\operatorname{contra})$ can be represented by a complex of projective left $\mathcal{D}$ -contramodules $\mathfrak{P}^\bullet$ concentrated in the cohomological degrees $-l_2 \leq m \leq d_1$. Denote by $'\mathcal{K}^{\bullet\vee}$ the complex of injective left $\mathcal{D}$ -comodules $\mathcal{D} \odot_{\mathcal{D}} \mathfrak{P}^\bullet$. Then we have a natural isomorphism of complexes of left $\mathcal{C}$ -comodules $\mathcal{K}^\bullet \square_{\mathcal{D}} '\mathcal{K}^{\bullet\vee} \simeq \mathcal{K}^\bullet \odot_{\mathcal{D}} \mathfrak{P}^\bullet$ by [18, Proposition 3.1.1] or [15, Proposition 5.2.1(a)], and a natural quasi-isomorphism of complexes of left $\mathcal{C}$ -comodules $\mathcal{K}^\bullet \odot_{\mathcal{D}} \mathfrak{P}^\bullet \longrightarrow \mathcal{C}$ by Lemma 4.2(a) (see formula (9)).

Part (b): since the definition of a dualizing complex of bicomodules is symmetric with respect to switching the left and right sides and the roles of the coalgebras $\mathcal{C}$ and $\mathcal{D}$, one can simply apply part (a) to the dualizing complex of $\mathcal{D}^{\operatorname{op}}\text{-}\mathcal{C}^{\operatorname{op}}$ -bicomodules $\mathcal{K}^{\operatorname{op}}$. $\square$

Theorem 7.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be coassociative coalgebras over k , and $\mathcal{K}^\bullet$ be a dualizing complex of $\mathcal{C}$ - $\mathcal{D}$ -bicomodules. Then*

- (a) *for any conventional derived category symbol $\star = \mathbf{b}, +, -, \text{ or } \varnothing$, there is a triangulated equivalence $\mathcal{D}^\star(\mathcal{C}\text{-}\operatorname{comod}) \simeq \mathcal{D}^\star(\mathcal{D}\text{-}\operatorname{comod})$ provided by (appropriately defined) right derived functor $\mathcal{K}^\bullet \square_{\mathcal{D}}^{\mathbb{R}} - : \mathcal{D}^\star(\mathcal{D}\text{-}\operatorname{comod}) \longrightarrow \mathcal{D}^\star(\mathcal{C}\text{-}\operatorname{comod})$;*

(b) for any conventional derived category symbol $\star = \mathbf{b}, +, -, \text{ or } \emptyset$, there is a triangulated equivalence $D^\star(\mathcal{C}\text{-contra}) \simeq D^\star(\mathcal{D}\text{-contra})$ provided by (appropriately defined) left derived functor $\mathbb{L}\text{Cohom}_{\mathcal{C}}(\mathcal{K}^\bullet, -): D^\star(\mathcal{C}\text{-contra}) \longrightarrow D^\star(\mathcal{D}\text{-contra})$.

Proof. Part (a): as in Theorem 5.2, the words “appropriately defined” here mean “constructed as in [23, Appendix A]”. In fact, a less powerful technology of [19, Appendix B] is already sufficient for our purposes here. In the context of either of these references, we put

$$\begin{aligned} A &= \mathcal{D}\text{-comod} = E \supset J = \mathcal{C}\text{-comod}_{\text{inj}} \\ B &= \mathcal{C}\text{-comod} = F. \end{aligned}$$

Consider the DG-functor

$$\Psi = \mathcal{K}^\bullet \square_{\mathcal{D}} -: C^+(J) \longrightarrow C^+(B).$$

Then the construction of [23, Section A.2] provides the derived functors

$$\mathbb{R}\Psi: D^\star(\mathcal{D}\text{-comod}) \longrightarrow D^\star(\mathcal{C}\text{-comod})$$

for all derived category symbols $\star = \mathbf{b}, +, -, \emptyset, \mathbf{abs}+, \mathbf{abs}-, \mathbf{co}$, or $\mathbf{abs}$.

The key observation here is that, for any left $\mathcal{D}$ -comodule $\mathcal{N}$ and its injective coresolution $\mathcal{J}^\bullet$, the complex of left $\mathcal{C}$ -comodules $\mathcal{K}^\bullet \square_{\mathcal{D}} \mathcal{J}^\bullet$ has cohomology comodules concentrated in the cohomological degrees $-d_1 \leq m \leq l_2$. Indeed, let $'\mathcal{K}^\bullet$ be a complex of injective right $\mathcal{D}$ -comodules quasi-isomorphic to $\mathcal{K}^\bullet$ whose terms are concentrated in the cohomological degrees $-d_1 \leq m \leq l_2$. Then the complex $\mathcal{K}^\bullet \square_{\mathcal{D}} \mathcal{J}^\bullet$ is quasi-isomorphic, as a complex of k -vector spaces, to the complex $'\mathcal{K}^\bullet \square_{\mathcal{D}} \mathcal{J}^\bullet$, since for any injective left $\mathcal{D}$ -comodule $\mathcal{J}$ the functor $-\square_{\mathcal{D}} \mathcal{J}$ is exact. Now for any injective right $\mathcal{D}$ -comodule $'\mathcal{K}$ the complex of k -vector spaces $'\mathcal{K} \square_{\mathcal{D}} \mathcal{J}^\bullet$ is a coresolution of the vector space $'\mathcal{K} \square_{\mathcal{D}} \mathcal{N}$, since the functor $'\mathcal{K} \square_{\mathcal{D}} -$ is exact. So the cohomology spaces of the complex $'\mathcal{K}^\bullet \square_{\mathcal{D}} \mathcal{J}^\bullet$ are concentrated in the desired cohomological degrees. This makes the construction of a derived functor of finite homological dimension from [23, Appendix A] or [19, Appendix B] applicable.

The derived functors $\mathbb{R}\Psi$ agree with each other as the derived category symbol $\star$ varies. In particular, for $\star = \mathbf{co}$ and $\star = \emptyset$ we have a commutative diagram of triangulated functors

$$\begin{array}{ccc} D^{\mathbf{co}}(\mathcal{D}\text{-comod}) & \xrightarrow{\mathcal{K}^\bullet \square_{\mathcal{D}}^{\mathbb{R}} -} & D^{\mathbf{co}}(\mathcal{C}\text{-comod}) \\ \downarrow \Downarrow & & \downarrow \Downarrow \\ D(\mathcal{D}\text{-comod}) & \xrightarrow{\mathcal{K}^\bullet \square_{\mathcal{D}}^{\mathbb{R}} -} & D(\mathcal{C}\text{-comod}) \end{array}$$

where the vertical arrows are the canonical Verdier quotient functors. By Corollary 7.2, the upper horizontal arrow is a triangulated equivalence, hence it follows that the lower horizontal arrow is, at worst, a Verdier quotient functor.

Let us check that the kernel of the functor $\mathcal{K}^\bullet \square_{\mathcal{D}}^{\mathbb{R}} -: D(\mathcal{D}\text{-comod}) \longrightarrow D(\mathcal{C}\text{-comod})$ vanishes. By Lemma 7.3(b), there exists a finite complex of injective right

$\mathcal{C}$ -comodules ${}''\mathcal{K}^{\bullet\vee}$ such that the cotensor product ${}''\mathcal{K}^{\bullet\vee} \square_{\mathcal{C}} \mathcal{K}^{\bullet}$ is quasi-isomorphic to $\mathcal{D}$ as a complex of right $\mathcal{D}$ -comodules. The cotensor product with the complex ${}''\mathcal{K}^{\bullet\vee}$ defines a triangulated functor

$$(19) \quad {}''\mathcal{K}^{\bullet\vee} \square_{\mathcal{C}} -: \mathcal{D}(\mathcal{C}\text{-comod}) \longrightarrow \mathcal{D}(k\text{-vect})$$

taking values in the derived category of k -vector spaces. The composition

$$(20) \quad \mathcal{D}(\mathcal{D}\text{-comod}) \xrightarrow{\mathcal{K}^{\bullet} \square_{\mathcal{D}}^{\mathbb{R}} -} \mathcal{D}(\mathcal{C}\text{-comod}) \xrightarrow{{}''\mathcal{K}^{\bullet\vee} \square_{\mathcal{C}} -} \mathcal{D}(k\text{-vect})$$

is the derived functor of cotensor product with ${}''\mathcal{K}^{\bullet\vee} \square_{\mathcal{C}} \mathcal{K}^{\bullet}$, which is isomorphic to the forgetful functor $\mathcal{D}(\mathcal{D}\text{-comod}) \longrightarrow \mathcal{D}(k\text{-vect})$. It follows that any complex of left $\mathcal{D}$ -comodules annihilated by the functor $\mathcal{K}^{\bullet} \square_{\mathcal{D}}^{\mathbb{R}} -$ is acyclic. We have proved that the functor

$$\mathcal{K}^{\bullet} \square_{\mathcal{D}}^{\mathbb{R}} -: \mathcal{D}(\mathcal{D}\text{-comod}) \longrightarrow \mathcal{D}(\mathcal{C}\text{-comod})$$

is a triangulated equivalence.

Finally, for $\star = \mathbf{b}, +$, or $-$ we have a commutative diagram of triangulated functors

$$\begin{array}{ccc} \mathcal{D}^{\star}(\mathcal{D}\text{-comod}) & \xrightarrow{\mathcal{K}^{\bullet} \square_{\mathcal{D}}^{\mathbb{R}} -} & \mathcal{D}^{\star}(\mathcal{C}\text{-comod}) \\ \downarrow & & \downarrow \\ \mathcal{D}(\mathcal{D}\text{-comod}) & \xrightarrow{\mathcal{K}^{\bullet} \square_{\mathcal{D}}^{\mathbb{R}} -} & \mathcal{D}(\mathcal{C}\text{-comod}) \end{array}$$

where the vertical arrows are the canonical fully faithful inclusions. We already know that the lower horizontal arrow is an equivalence; hence the upper horizontal arrow is, at worst, fully faithful.

Let us check that a complex of left $\mathcal{D}$ -comodules has $\star$ -bounded cohomology comodules whenever its image under the functor $\mathcal{K}^{\bullet} \square_{\mathcal{D}}^{\mathbb{R}} -$ has $\star$ -bounded cohomology. Indeed, the triangulated functor (19) takes $\star$ -bounded complexes to $\star$ -bounded complexes,

$${}''\mathcal{K}^{\bullet\vee} \square_{\mathcal{C}} -: \mathcal{D}^{\star}(\mathcal{C}\text{-comod}) \longrightarrow \mathcal{D}^{\star}(k\text{-vect}),$$

so the desired assertion follows from the fact that the composition of functors (20) is isomorphic to the forgetful functor $\mathcal{D}(\mathcal{D}\text{-comod}) \longrightarrow \mathcal{D}(k\text{-vect})$. The proof of part (a) is finished.

Part (b) is similar. In the context of [23, Appendix A] or [19, Appendix B], we put

$$\begin{aligned} \mathbf{B} &= \mathcal{C}\text{-contra} = \mathbf{F} \supset \mathbf{P} = \mathcal{C}\text{-contra}_{\text{proj}} \\ \mathbf{A} &= \mathcal{D}\text{-contra} = \mathbf{E}. \end{aligned}$$

Consider the DG-functor

$$\Phi = \text{Cohom}_{\mathcal{C}}(\mathcal{K}^{\bullet}, -): \mathcal{C}^{-}(\mathbf{P}) \longrightarrow \mathcal{C}^{-}(\mathbf{A}).$$

Then the construction of [23, Section A.3] provides the derived functors

$$\mathbb{L}\Phi: \mathcal{D}^{\star}(\mathcal{C}\text{-contra}) \longrightarrow \mathcal{D}^{\star}(\mathcal{D}\text{-contra})$$

for all derived category symbols $\star = \mathbf{b}, +, -, \emptyset, \mathbf{abs+}, \mathbf{abs-}, \mathbf{ctr},$ or $\mathbf{abs}$. The key observation here is that, for any left $\mathcal{C}$ -contramodule $\mathfrak{T}$ and its projective resolution $\mathfrak{P}^\bullet$, the complex of left $\mathcal{D}$ -contramodules $\mathrm{Cohom}_{\mathcal{C}}(\mathcal{K}^\bullet, \mathfrak{P}^\bullet)$ has cohomology contramodules concentrated in the cohomological degrees $-l_2 \leq m \leq d_1$.

The rest of the argument uses Lemma 7.3(a) in the way similar to the above. $\square$

Remark 7.5. The exposition above simplifies considerably when a dualizing complex $\mathcal{K}^\bullet$ is quasi-isomorphic to a (finite) complex of $\mathcal{C}$ - $\mathcal{D}$ -bicomodules that are *simultaneously* quasi-finitely cogenerated injective left $\mathcal{C}$ -comodules and quasi-finitely cogenerated injective right $\mathcal{D}$ -comodules. Then, following [30, Sections 1.8–1.9] (cf. Remark 3.9 above), both the complexes $'\mathcal{K}^{\bullet\vee}$ and $''\mathcal{K}^{\bullet\vee}$ in Lemma 7.3 can be constructed as complexes of $\mathcal{D}$ - $\mathcal{C}$ -bicomodules together with a natural quasi-isomorphism of complexes of $\mathcal{C}$ - $\mathcal{C}$ -bicomodules $\mathcal{C} \rightarrow \mathcal{K}^\bullet \square_{\mathcal{D}} ''\mathcal{K}^{\bullet\vee}$ and a natural quasi-isomorphism of complexes of $\mathcal{D}$ - $\mathcal{D}$ -bicomodules $\mathcal{D} \rightarrow '\mathcal{K}^{\bullet\vee} \square_{\mathcal{C}} \mathcal{K}^\bullet$ (implying that $'\mathcal{K}^{\bullet\vee}$ and $''\mathcal{K}^{\bullet\vee}$ are quasi-isomorphic to each other as complexes of $\mathcal{D}$ - $\mathcal{C}$ -bicomodules). Given such “inverse complex(es) of $\mathcal{D}$ - $\mathcal{C}$ -bicomodules” to $\mathcal{K}^\bullet$, one easily constructs the derived equivalences in Theorem 7.4 (including even the case of $\star = \mathbf{abs+}, \mathbf{abs-},$ or $\mathbf{abs}$) similarly to the underived theory of [30] (cf. [5, Sections 2–3]; see also the discussion of Morita morphisms and Morita equivalences in [15, Section 7.5]).

8. DEDUALIZING COMPLEXES

Recall the following definitions from [21, Section 3]. A finite complex of left $\mathcal{C}$ -comodules $\mathcal{M}^\bullet$ is said to have *projective dimension* $\leq l$ if $\mathrm{Ext}_{\mathcal{C}}^n(\mathcal{M}^\bullet, \mathcal{N}) = 0$ for all left $\mathcal{C}$ -comodules $\mathcal{N}$ and all the integers $n > l$. A finite complex of right $\mathcal{D}$ -comodules $\mathcal{N}^\bullet$ is said to have *contraflat dimension* $\leq l$ if $\mathrm{Ctrtor}_n^{\mathcal{D}}(\mathcal{N}^\bullet, \mathfrak{T}) = 0$ for all left $\mathcal{D}$ -contramodules $\mathfrak{T}$ and all the integers $n > l$. Here we use the notation $\mathrm{Ext}_{\mathcal{C}}^*$ and $\mathrm{Ctrtor}_*^{\mathcal{D}}$ introduced in the beginning of Section 4.

A *dedualizing complex* of $\mathcal{C}$ - $\mathcal{D}$ -bicomodules $\mathcal{L}^\bullet = \mathcal{B}^\bullet$ is a pseudo-dualizing complex (according to the definition in Section 4) satisfying the following additional condition:

- (i) the complex $\mathcal{B}^\bullet$ has finite projective dimension as a complex of left $\mathcal{C}$ -comodules and finite contraflat dimension as a complex of right $\mathcal{D}$ -comodules.

This is a version of the definition of a dedualizing complex in [21, Section 3], extended from the case of cocommutative coalgebras to arbitrary ones and from the case of finitely copresented comodules to quasi-finitely copresented ones.

Let us choose the parameter l_1 in such a way that the projective dimension of the complex of left $\mathcal{C}$ -comodules $\mathcal{B}^\bullet$ does not exceed l_1 and the contraflat dimension of the complex of right $\mathcal{D}$ -comodules $\mathcal{B}^\bullet$ does not exceed l_1 . One can easily see that any one of these two conditions implies $l_1 \geq d_1$ (take $\mathcal{N} = \mathcal{C}$ or $\mathfrak{T} = \mathcal{D}^*$ in the above definitions of the projective and contraflat dimensions).

Lemma 8.1. *Let $\mathcal{B}^\bullet$ be a dedualizing complex of $\mathcal{C}$ - $\mathcal{D}$ -bicomodules, and let the parameter l_1 be chosen as stated above. Then the related Bass and Auslander classes*

$E_{l_1} = E_{l_1}(\mathcal{B}^\bullet)$ and $F_{l_1} = F_{l_1}(\mathcal{B}^\bullet)$ coincide with the whole categories of left $\mathcal{C}$ -comodules and left $\mathcal{D}$ -contramodules, $E_{l_1} = \mathcal{C}\text{-comod}$ and $F_{l_1} = \mathcal{D}\text{-contra}$.

Proof. In view of Lemma 5.1 and the subsequent discussion, it suffices to check that the conditions (I–IV) of Section 5 hold for the classes $E = \mathcal{C}\text{-comod}$ and $F = \mathcal{D}\text{-contra}$ with the given parameter l_1 and some $l_2 \geq d_2$. Indeed, let us take $l_2 = d_2$. Then the conditions (I–II) are obvious, and the conditions (III–IV) follow from (i). $\square$

It is clear from Lemma 8.1 that for a dedualizing complex of $\mathcal{C}$ - $\mathcal{D}$ -bicomodules $\mathcal{B}^\bullet$ one has

$$D'_{\mathcal{B}^\bullet}(\mathcal{C}\text{-comod}) = D(\mathcal{C}\text{-comod}) \quad \text{and} \quad D''_{\mathcal{B}^\bullet}(\mathcal{D}\text{-contra}) = D(\mathcal{D}\text{-contra}).$$

The next corollary is a generalization of [21, Theorem 3.6].

Corollary 8.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be coassociative coalgebras over k , and $\mathcal{B}^\bullet$ be a dedualizing complex of $\mathcal{C}$ - $\mathcal{D}$ -bicomodules. Then for any conventional or absolute derived category symbol $\star = \mathbf{b}, +, -, \emptyset, \mathbf{abs}+, \mathbf{abs}-$, or $\mathbf{abs}$, there is a triangulated equivalence $D^\star(\mathcal{C}\text{-comod}) \simeq D^\star(\mathcal{D}\text{-contra})$ provided by (appropriately defined) mutually inverse derived functors $\mathbb{R}\mathrm{Hom}_{\mathcal{C}}(\mathcal{B}^\bullet, -)$ and $\mathcal{B}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} -$.*

Proof. This is a particular case of Theorem 4.9. $\square$

9. MAIN DIAGRAM

The aim of this section is to discuss the results formulated in Section 1.6, and in particular, the ones encoded in the diagram (6). In fact, all of them have been proved already in the preceding sections; see, in particular, the diagram (16) in Section 5, generalizing the middle square in (6). So in this section we present a kind of overview and conclusion.

Let $\mathcal{L}^\bullet$ be a pseudo-dualizing complex of bicomodules for a pair of coalgebras $\mathcal{C}$ and $\mathcal{D}$, and let $E \subset \mathcal{C}\text{-comod}$ and $F \subset \mathcal{D}\text{-contra}$ be a pair of classes of comodules and contramodules satisfying the conditions (I–IV) with some parameters $l_1 \geq d_1$ and $l_2 \geq d_2$. Then there is the following diagram of triangulated functors:

$$(21) \quad \begin{array}{ccc} \text{Hot}(\mathcal{C}\text{-comod}) & & \text{Hot}(\mathcal{D}\text{-contra}) \\ \downarrow & & \downarrow \\ D^{\mathrm{co}}(\mathcal{C}\text{-comod}) & & D^{\mathrm{ctr}}(\mathcal{D}\text{-contra}) \\ \downarrow & & \downarrow \\ D(E) & \xlongequal{\quad} & D(F) \\ \downarrow & & \downarrow \\ D(\mathcal{C}\text{-comod}) & & D(\mathcal{D}\text{-contra}) \end{array}$$

The diagram consists of two vertical sequences of functors connected by a horizontal double arrow. The left sequence is: $\text{Hot}(\mathcal{C}\text{-comod}) \rightarrow D^{\mathrm{co}}(\mathcal{C}\text{-comod}) \rightarrow D(E) \rightarrow D(\mathcal{C}\text{-comod})$. The right sequence is: $\text{Hot}(\mathcal{D}\text{-contra}) \rightarrow D^{\mathrm{ctr}}(\mathcal{D}\text{-contra}) \rightarrow D(F) \rightarrow D(\mathcal{D}\text{-contra})$. Curved arrows indicate isomorphisms: $\text{Hot}(\mathcal{C}\text{-comod}) \xrightarrow{\sim} D(\mathcal{C}\text{-comod})$, $D^{\mathrm{co}}(\mathcal{C}\text{-comod}) \xrightarrow{\sim} D(E)$, $D(E) \xrightarrow{\sim} D(\mathcal{C}\text{-comod})$, and similarly for the right side. A horizontal double arrow connects $D(E)$ and $D(F)$.

Here the uppermost straight vertical arrows $\text{Hot}(\mathcal{C}\text{-comod}) \longrightarrow D^\infty(\mathcal{C}\text{-comod})$ and $\text{Hot}(\mathcal{D}\text{-contra}) \longrightarrow D^{\text{ctr}}(\mathcal{D}\text{-contra})$ are the canonical Verdier quotient functors. The functor $\text{Hot}(\mathcal{C}\text{-comod}) \longrightarrow D^\infty(\mathcal{C}\text{-comod})$ has a fully faithful right adjoint, identifying the coderived category $D^\infty(\mathcal{C}\text{-comod})$ with the homotopy category of complexes of injective comodules $\text{Hot}(\mathcal{C}\text{-comod}_{\text{inj}})$ [16, Theorem 4.4(a,c)]. Similarly, the functor $\text{Hot}(\mathcal{D}\text{-contra}) \longrightarrow D^{\text{ctr}}(\mathcal{D}\text{-contra})$ has a fully faithful left adjoint, identifying the contraderived category $D^{\text{ctr}}(\mathcal{D}\text{-contra})$ with the homotopy category of complexes of projective contramodules $\text{Hot}(\mathcal{D}\text{-contra}_{\text{proj}})$ [16, Theorem 4.4(b,d)]. These fully faithful adjoints are shown on the diagram as the uppermost short curvilinear arrows.

The functor $D^\infty(\mathcal{C}\text{-comod}) \longrightarrow D(E)$ shown by the leftmost middle straight vertical arrow is constructed as the composition

$$D^\infty(\mathcal{C}\text{-comod}) \simeq \text{Hot}(\mathcal{C}\text{-comod}_{\text{inj}}) \longrightarrow D(E)$$

of the natural equivalence $D^\infty(\mathcal{C}\text{-comod}) \simeq \text{Hot}(\mathcal{C}\text{-comod}_{\text{inj}})$ with the triangulated functor $\text{Hot}(\mathcal{C}\text{-comod}_{\text{inj}}) \longrightarrow D(E)$ induced by the inclusion of additive/exact categories $\mathcal{C}\text{-comod}_{\text{inj}} \longrightarrow E$.

Similarly, the functor $D^{\text{ctr}}(\mathcal{D}\text{-contra}) \longrightarrow D(F)$ shown by the rightmost middle straight vertical arrow is constructed as the composition

$$D^{\text{ctr}}(\mathcal{D}\text{-contra}) \simeq \text{Hot}(\mathcal{D}\text{-contra}_{\text{proj}}) \longrightarrow D(F)$$

of the natural equivalence $D^{\text{ctr}}(\mathcal{D}\text{-contra}) \simeq \text{Hot}(\mathcal{D}\text{-contra}_{\text{proj}})$ with the triangulated functor $\text{Hot}(\mathcal{D}\text{-contra}_{\text{proj}}) \longrightarrow D(F)$ induced by the inclusion of additive/exact categories $\mathcal{D}\text{-contra}_{\text{proj}} \longrightarrow F$.

The horizontal double line is the triangulated equivalence of Theorem 5.2 for $\star = \emptyset$.

The left lower straight vertical arrow $D(E) \longrightarrow D(\mathcal{C}\text{-comod})$ is induced by the inclusion of exact/abelian categories $E \longrightarrow \mathcal{C}\text{-comod}$. Similarly, the right lower straight vertical arrow $D(F) \longrightarrow D(\mathcal{D}\text{-contra})$ is induced by the inclusion of exact/abelian categories $F \longrightarrow \mathcal{D}\text{-contra}$. It was shown in Section 2 that these two functors are Verdier quotient functors.

The composition $D^\infty(\mathcal{C}\text{-comod}) \longrightarrow D(E) \longrightarrow D(\mathcal{C}\text{-comod})$ is the canonical Verdier quotient functor. So is the long composition $\text{Hot}(\mathcal{C}\text{-comod}) \longrightarrow D^\infty(\mathcal{C}\text{-comod}) \longrightarrow D(\mathcal{C}\text{-comod})$. These Verdier quotient functors (shown on the diagram by the two-headed long curvilinear arrows in the left-hand side) have fully faithful right adjoints (shown by the long curvilinear arrows with tails), which were constructed in [16, Sections 2.4 and 5.5] (see also [21, Theorem 1.1(c)]).

Similarly, the composition $D^{\text{ctr}}(\mathcal{D}\text{-contra}) \longrightarrow D(F) \longrightarrow D(\mathcal{D}\text{-contra})$ is the canonical Verdier quotient functor. So is the long composition $\text{Hot}(\mathcal{D}\text{-contra}) \longrightarrow D^{\text{ctr}}(\mathcal{D}\text{-contra}) \longrightarrow D(\mathcal{D}\text{-contra})$. These Verdier quotient functors (shown on the diagram by the two-headed long curvilinear arrows in the right-hand side) have fully faithful left adjoints (shown by the long curvilinear arrows with tails), which were constructed in [16, Sections 2.4 and 5.5] (see also [21, Theorem 1.1(a)]).

The lower curvilinear arrows with tails show triangulated functors $D(\mathcal{C}\text{-comod}) \rightarrow D(E)$ and $D(\mathcal{D}\text{-contra}) \rightarrow D(F)$, which are fully faithful and adjoint to the above-mentioned functors $D(E) \rightarrow D(\mathcal{C}\text{-comod})$ and $D(F) \rightarrow D(\mathcal{D}\text{-contra})$ on the respective sides. These functors are described in the following proposition.

Proposition 9.1. (a) *The triangulated functor $D(E) \rightarrow D(\mathcal{C}\text{-comod})$ induced by the inclusion of exact/abelian categories $E \rightarrow \mathcal{C}\text{-comod}$ is a Verdier quotient functor, and has a fully faithful right adjoint.*

(b) *The triangulated functor $D(F) \rightarrow D(\mathcal{D}\text{-contra})$ induced by the inclusion of exact/abelian categories $F \rightarrow \mathcal{D}\text{-contra}$ is a Verdier quotient functor, and has a fully faithful left adjoint.*

Proof. This is a particular case of Theorem 2.1, which was mentioned already in Section 5. In part (a), the desired functor $\rho: D(\mathcal{C}\text{-comod}) \rightarrow D(E)$ is constructed as the composition

$$D(\mathcal{C}\text{-comod}) \xrightarrow{\theta} D^{\text{co}}(\mathcal{C}\text{-comod}) \longrightarrow D(E),$$

or which is the same, the composition

$$D(\mathcal{C}\text{-comod}) \longrightarrow \text{Hot}(\mathcal{C}\text{-comod}) \longrightarrow D(E)$$

of the functors on the diagram. In part (b), the desired functor $\lambda: D(\mathcal{D}\text{-contra}) \rightarrow D(F)$ is constructed as the composition

$$D(\mathcal{D}\text{-contra}) \xrightarrow{\simeq} D^{\text{ctr}}(\mathcal{D}\text{-contra}) \longrightarrow D(F),$$

or which is the same, the composition

$$D(\mathcal{D}\text{-contra}) \longrightarrow \text{Hot}(\mathcal{D}\text{-contra}) \longrightarrow D(F)$$

of the functors on the diagram. □

Finally, according to Sections 1.2 and 5, there are two (possibly degenerate) t-structures of the derived type on the triangulated category $D(E) \simeq D(F)$. The abelian hearts are $\mathbf{A} = \mathcal{C}\text{-comod}$ and $\mathbf{B} = \mathcal{D}\text{-contra}$. This pair of t-structures satisfies the nontriviality condition of Remark 5.3.

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INSTITUTE OF MATHEMATICS OF THE CZECH ACADEMY OF SCIENCES, ŽITNÁ 25, 115 67
PRAGUE 1, CZECH REPUBLIC; AND

LABORATORY OF ALGEBRA AND NUMBER THEORY, INSTITUTE FOR INFORMATION TRANSMIS-
SION PROBLEMS, MOSCOW 127051, RUSSIA

Email address: positselski@yandex.ru