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CORRECTION

# Correction to: The Karoubi envelope and weak idempotent completion of an extriangulated category

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## 1 Introduction

In the original article [2], we showed that given an extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, its idempotent completion $\tilde{\mathcal{C}}$ is also an extriangulated category $(\tilde{\mathcal{C}}, \mathbb{F}, \mathfrak{r})$. An important technical result is [2, Proposition 3.10], which states that the correspondence $\mathfrak{r}$ is well-defined. The proof of the proposition given in the original article was incorrect. We will give a correct proof of this statement in this corrigendum. The statement of the proposition is as follows.

**Proposition 3.10** *Let $\delta$ be an extension in $\mathbb{F}((C, p), (A, q))$ realised under $\mathfrak{s}$ by the following sequences,*

$$A \xrightarrow{a} B \xrightarrow{b} C, \tag{1}$$
$$A \xrightarrow{x} Y \xrightarrow{y} C. \tag{2}$$

*Then given idempotents $r: B \to B$ and $w: Y \to Y$ such that*

$$aq = ra, \ \ pb = br \text{ and } xq = wx, \ \ py = yw \tag{3}$$

*the sequences*

$$(A, q) \xrightarrow{aq} (B, r) \xrightarrow{pb} (C, p), \tag{4}$$
$$(A, q) \xrightarrow{xq} (Y, w) \xrightarrow{py} (C, p). \tag{5}$$

*are equivalent. That is to say, $\mathfrak{r}$ is well-defined.*

To prove the equivalence of the sequences (4) and (5) in $\tilde{\mathcal{C}}$, the strategy used in [2] was to prove that the morphism $wfr: (B, r) \to (Y, w)$ is an isomorphism, where $f: B \to Y$ is an isomorphism in $\mathcal{C}$ which gives the equivalence of the sequences (1) and (2) in $\mathcal{C}$. We claimed

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this could be done by first showing that $wfr = wf$, using the fact that $wfraq = wfaq$ and hence $(wfr - wf)aq = 0$, then employing the fact that $py$ is a weak cokernel of $aq$ in $\tilde{\mathcal{C}}$ to further deduce that $wfr = wf$. However, it is not clear if $wf$ is a morphism in $\tilde{\mathcal{C}}$, so we cannot take advantage of the fact that $py$ is a weak cokernel of $aq$ in $\tilde{\mathcal{C}}$ in this way.

## 2 Corrigendum

Recall for an extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, we defined the biadditive functor $\mathbb{F}\colon \tilde{\mathcal{C}}^{\mathrm{op}} \times \tilde{\mathcal{C}} \to Ab$ on the idempotent completion as follows. Given a pair of objects $(X, p)$ and $(Y, q)$ in $\tilde{\mathcal{C}}$, we define $\mathbb{F}$ on objects by setting,

$$\mathbb{F}((X, p), (Y, q)) := p^* q_* \mathbb{E}(X, Y) = \{p^* q_* \delta \mid \delta \in \mathbb{E}(X, Y)\} \subseteq \mathbb{E}(X, Y).$$

This is just the image of $\mathbb{E}(X, Y)$ under the group homomorphism $\mathbb{E}(p, q)$. For morphisms $f\colon (A, a) \to (A', a')$ and $g\colon (C, c) \to (C', c') \in \tilde{\mathcal{C}}$, we then defined $\mathbb{F}(f, g) := \mathbb{E}(f, g)_{|\mathbb{F}((C,c),(A,a))}$, the restriction of the group homomorphism $\mathbb{E}(f, g)$ to $\mathbb{F}((C, c), (A, a))$. Before we can give the proof of Proposition 3.10, we first need to collect some lemmas which will be needed.

**Lemma 2.1** [1, Lemma 4.1] *Let $\mathcal{C}$ be an additive category with biadditive functor $\mathbb{E}\colon \mathcal{C}^{op} \times \mathcal{C} \to Ab$. Let $X_\bullet = A \xrightarrow{x_1} X \xrightarrow{x_2} C$ and $Y_\bullet = A \xrightarrow{y_1} Y \xrightarrow{y_2} C$ be a pair of complexes in $\mathcal{C}$. Suppose that the following sequences of functors are exact,*

$$\mathcal{C}(C, -) \overset{\mathcal{C}(x_2,-)}{\Longrightarrow} \mathcal{C}(X, -) \overset{\mathcal{C}(x_1,-)}{\Longrightarrow} \mathcal{C}(A, -)$$

$$\mathcal{C}(-, A) \overset{\mathcal{C}(-,x_1)}{\Longrightarrow} \mathcal{C}(-, X) \overset{\mathcal{C}(-,x_2)}{\Longrightarrow} \mathcal{C}(-, C)$$

*and likewise for $Y_\bullet$. Then for any commutative diagram $f_\bullet = (1_A, f, 1_C)\colon X_\bullet \to Y_\bullet$*

$$\begin{array}{ccccc} A & \xrightarrow{x_1} & X & \xrightarrow{x_2} & C \\ \| & & \downarrow{f} & & \| \\ A & \xrightarrow{y_1} & Y & \xrightarrow{y_2} & C \end{array}$$

*the following statements are equivalent.*

1. *$f_\bullet$ is a homotopy equivalence.*
2. *$f_\bullet$ is an equivalence of the sequences $X_\bullet$ and $Y_\bullet$, i.e. $f$ is an isomorphism and the squares in the above diagram commute.*
3. *$f\colon X \to Y$ is an isomorphism.*

**Lemma 2.2** [1, Proposition 2.21] *Let $\delta \in \mathbb{E}(C, A)$ be an extension, and let $X_\bullet = A \xrightarrow{x_1} X \xrightarrow{x_2} C$ and $Y_\bullet = A \xrightarrow{y_1} Y \xrightarrow{y_2} C$ be a pair of complexes in $\mathcal{C}$. Suppose that the following sequences of functors are exact,*

$$\mathcal{C}(C, -) \overset{\mathcal{C}(x_2,-)}{\Longrightarrow} \mathcal{C}(X, -) \overset{\mathcal{C}(x_1,-)}{\Longrightarrow} \mathcal{C}(A, -) \overset{\delta^\#}{\Longrightarrow} \mathbb{E}(C, -)$$

$$\mathcal{C}(-, A) \overset{\mathcal{C}(-,x_1)}{\Longrightarrow} \mathcal{C}(-, X) \overset{\mathcal{C}(-,x_2)}{\Longrightarrow} \mathcal{C}(-, C) \overset{\delta_\#}{\Longrightarrow} \mathbb{E}(-, A)$$

*and likewise for $Y_\bullet$. Let $f_\bullet = (1_A, f, 1_C)\colon X_\bullet \to Y_\bullet$ be a commutative diagram:*

$$
\begin{array}{ccccc}
A & \xrightarrow{x_1} & X & \xrightarrow{x_2} & C \\
\| & & \downarrow f & & \| \\
A & \xrightarrow{y_1} & Y & \xrightarrow{y_2} & C
\end{array}
$$

*If there exists a commutative diagram* $g_\bullet = (1_A, g, 1_C)\colon Y_\bullet \to X_\bullet$*:*

$$
\begin{array}{ccccc}
A & \xrightarrow{y_1} & Y & \xrightarrow{y_2} & C \\
\| & & \downarrow g & & \| \\
A & \xrightarrow{x_1} & X & \xrightarrow{x_2} & C
\end{array}
$$

*Then* $f_\bullet$ *is a homotopic equivalence.*

We also need to strengthen [2, Lemma 3.9] as follows.

**Lemma 2.3** *Let* $\delta = p^* q_* \varepsilon$ *be an extension in* $\mathbb{F}((Z, p), (X, q))$ *such that*

$$\mathfrak{s}(p^* q_* \varepsilon) = [X \xrightarrow{x} Y \xrightarrow{y} Z].$$

*Then the following sequences of functors are exact;*

$$\tilde{\mathcal{C}}((Z, p), -) \xRightarrow{\tilde{\mathcal{C}}(py,-)} \tilde{\mathcal{C}}((Y, r), -) \xRightarrow{\tilde{\mathcal{C}}(xq,-)} \tilde{\mathcal{C}}((X, q), -) \xRightarrow{\delta^\#_-} \mathbb{F}((Z, p), -)$$

$$\tilde{\mathcal{C}}(-, (X, q)) \xRightarrow{\tilde{\mathcal{C}}(-,xq)} \tilde{\mathcal{C}}(-, (Y, r)) \xRightarrow{\tilde{\mathcal{C}}(-,py)} \tilde{\mathcal{C}}(-, (Z, p)) \xRightarrow{\delta^-_\#} \mathbb{F}(-, (X, q))$$

*where* $r\colon Y \to Y$ *is an idempotent morphism such that* $rx = xq$ *and* $yr = py$*, obtained by an application of [2, Lemma 3.5].*

***Proof*** We will only show the exactness of the first sequence. The proof of the exactness of the second sequence is dual. Exactness at $\tilde{\mathcal{C}}((Y, r), -)$ is as in [2, Lemma 3.9]. So what is left is to prove exactness at $\tilde{\mathcal{C}}((X, q), -)$. Since $(\mathcal{C}, \mathbb{E}, \mathfrak{r})$ is an extriangulated category, the following sequence

$$\mathcal{C}(Z, -) \xRightarrow{\mathcal{C}(y,-)} \mathcal{C}(Y, -) \xRightarrow{\mathcal{C}(x,-)} \mathcal{C}(X, -) \xRightarrow{\delta^\#-} \mathbb{E}(Z, -) \tag{6}$$

is exact.

Let $(A, e)$ be an arbitrary object in $\tilde{\mathcal{C}}$. Take any morphism $f\colon (Y, r) \to (A, e) \in \tilde{\mathcal{C}}((Y, r), (A, e))$. Then

$$(\delta^\#_{(A,e)} \circ \tilde{\mathcal{C}}(xq, (A, e)))(f) = (fxq)_*\delta = (f(xq))_*\delta = (f(rx))_*\delta = ((fr)x)_*\delta) = 0$$

by the exactness of (6). We conclude that $\mathrm{im}(\tilde{\mathcal{C}}(xq, (A, e)) \subseteq \ker(\delta^\#_{(A,e)})$.

Now take any morphism $g\colon (X, q) \to (A, e) \in \tilde{\mathcal{C}}((X, q), (A, e))$. Recall that this means $g$ is a morphism $g\colon X \to A$ in $\mathcal{C}$ such that $gq = eg = g$. Suppose $\delta^\#_{(A,e)}(g) = g_*\delta = 0$. Since $g$ is also a morphism in $\mathcal{C}$ and $\delta$ is an $\mathbb{E}$-extension, we have by the exactness of (6) that there exists $h\colon Y \to A$ such that $g = hx$. Now consider the morphism $h' = ehr\colon (Y, r) \to (A, e)$. We have that

$$h'xq = (ehr)xq = eh(rx)q = eh(xq)q = e(hx)q = e(g)q = g.$$

We conclude that $\ker(\delta^\#_{(A,e)}) \subseteq \mathrm{im}(\tilde{\mathcal{C}}(xq, (A, e))$. Therefore we have exactness at $\tilde{\mathcal{C}}((X, q), -)$ as required. □

We are now able to give a proof of Proposition 3.10.

### 2.1 Proof of Proposition 3.10

***Proof*** Since the sequences $A \xrightarrow{a} B \xrightarrow{b} C$, and $A \xrightarrow{x} Y \xrightarrow{y} C$ both realise $\delta$, they are by definition equivalent in $\mathcal{C}$. That is to say we have the following commutative diagram,

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, equal] & B \arrow[r, "b"] \arrow[d, "f"] & C \arrow[d, equal] \\ A \arrow[r, "x"] & Y \arrow[r, "y"] & C \end{tikzcd} \tag{7}$$

where the morphism $f : B \to Y$ is an isomorphism. Now consider the following diagram.

$$\begin{tikzcd} (A, q) \arrow[r, "aq"] \arrow[d, equal] & (B, r) \arrow[r, "pb"] \arrow[d, "wfr"] & (C, p) \arrow[d, equal] \\ (A, q) \arrow[r, "xq"] & (Y, w) \arrow[r, "py"] & (C, p) \end{tikzcd} \tag{8}$$

From the relations in (3) and those arising from the commutative diagram (7), we can observe the following:

$$wf(ra)q = wf(aq)q = wf(aq) = w(fa)q = w(x)q = (wx)q = (xq)q = xq, \tag{9}$$
$$p(yw)fr = p(py)fr = (py)fr = p(yf)r = p(b)r = p(br) = p(pb) = pb. \tag{10}$$

That is to say, diagram (8) is a commutative diagram. Now consider the following diagram.

$$\begin{tikzcd} (A, q) \arrow[r, "xq"] \arrow[d, equal] & (Y, w) \arrow[r, "py"] \arrow[d, "rf^{-1}w"] & (C, p) \arrow[d, equal] \\ (A, q) \arrow[r, "aq"] & (B, r) \arrow[r, "pb"] & (C, p) \end{tikzcd} \tag{11}$$

From the relations in (3) and those arising from the commutative diagram (7), we can observe the following:

$$rf^{-1}(wx)q = rf^{-1}(xq)q = r(f^{-1}x)q = r(a)q = (ra)q = (aq)q = aq, \tag{12}$$
$$p(br)f^{-1}w = p(pb)f^{-1}w = p(bf^{-1})w = p(y)w = p(yw) = p(py) = py. \tag{13}$$

That is to say, diagram (11) is a commutative diagram. By [2, Lemma 3.9] both (4) and (5) are complexes, that is to say $pb \circ aq = 0$ and $py \circ xq = 0$. We apply Lemma 2.2 to conclude that $wfr_\bullet = (1_{(A,q)}, wfr, 1_{(C,p)})$ is a homotopy equivalence, and hence by Lemma 2.1, the morphism $wfr$ is an isomorphism. We conclude that (8) is an equivalence, that is to say $\mathfrak{r}$ is well-defined. □

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