R-LINEAR TRIANGULATED CATEGORIES AND STABILITY CONDITIONS

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ABSTRACT. Let R be a commutative ring. We introduce the notion of support of a object in an R-linear triangulated category. As an application, we study the nonexistence of Bridgeland stability condition on R-linear triangulated categories.

1. INTRODUCTION

Recall that a stability condition introduced by Bridgeland [Bri07] on a triangulated category **D** defines a stability of objects in the category **D**. If the category is the bounded derived category $\mathbf{D}^{b}(X)$ of coherent sheaves on a projective variety X (over a field), Analogously to the slope/Gieseker-Maruyama stability for coherent sheaves, one can define the stability of complexes of coherent sheaves via stability conditions on $\mathbf{D}^{b}(X)$ if it does exist. Recalling the stability of sheaves are defined by an ample line bundle on the variety, a stability condition on $\mathbf{D}^{b}(X)$ can be regarded as a huge generalization of ample line bundles on the projecitve variety. Thus the stability condition gives a powerful tool to study birational geometry for moduli spaces not only of sheaves, but also of complexes and much work has been done in this direction. For instance, moduli spaces on K3 surfaces or abelian suffaces are studied by many authors (Arcara and Bertram in [AB11] or [AB13], Bayer and Macri in [BM14a] and [BM14b] or Minaminde, Yanagida and Yoshioka in [MYY18]. the case of other surfaces are studied by [ABCH13], [LZ19], [Nue16], or [NY20]

However the existence of stability conditions a triangulated category is sensitive, even if the category is the derived category $\mathbf{D}^{b}(X)$ of a smooth projective variety X. For instance, if dim $X \leq 2$, then $\operatorname{Stab} \mathbf{D}^{b}(X)$ does exists. When dim X = 3, the existence follows from generalized Bogomolov inequality proposed by Bayer-Macri-Toda [BMT14].

Let us discuss the case where stability conditions do not exist. Since a stability condition on a triangulated category naturally induces a bounded *t*-structure, if the category has no bounded *t*-structure, then there are no stability condition. Such a category can be found by using schemes with singularities or varieties which are non-proper. For instance, as mentioned by Antieau, Gepner, and Heller [AGH19], if the scheme X is a nodal cubic curve, then the triangulated category $\mathbf{D}^{\text{perf}}(X)$ of perfect complexes on X

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has no bounded t-structure and hence no stability condition. They also conjecture that the category of perfect complexes on a finite dimensional Noetherian scheme X has a bounded t-structure if and only if X is regular. If the scheme X is affine, the conjecture holds by Smith [Smi22]. Recently Neeman studies the conjecture for separated case (cf. [Nee22]).

Let us introduce another example for the non-existence. Suppose that X is an affine scheme Spec R of a Noetherian ring R. Then $\operatorname{Stab} \mathbf{D}^b(X)$ is non-empty if and only if dim X = 0 by the author [Kaw20]. Unlike the case of perfect complexes, the category $\mathbf{D}^b(X)$ has a natural bounded t-structure. In stead of the non-existence of t-structures, our proof is based on the supports of complexes in $\mathbf{D}^b(X)$.

The aim of this paper is to extend the results in [Kaw20] to more general *R*-linear triangulated categories. The main theorem is the following:

Theorem 1.1. Let R be a Noetherian ring and let $\mathcal{X} \to \operatorname{Spec} R$ be a proper morphism of schemes. If the dimension of the image is positive, then $\mathbf{D}^{b}(\mathcal{X})$ and $\mathbf{D}^{\operatorname{perf}}(\mathcal{X})$ have no stability condition.

Recall that the support Supp E of a complex E of (quasi-)coherent sheaves is the union of the support of each cohomology $H^i(E)$. To develop the argument in [Kaw20], we give a generalization of the support for objects in an R-linear triangulated category \mathbf{D} . Precisely if E is an object in the R-linear category \mathbf{D} , we define $\operatorname{Supp}_R E$ as the support of the R-module $\operatorname{Hom}_{\mathbf{D}}(E, E)$ of endomorphisms. The generalized support $\operatorname{Supp}_R E$ coincides with $\operatorname{Supp} E$ if E is a bounded complex of (not necessarily finite) R-modules.

Using the generalized support $\operatorname{Supp}_R E$, we give a sufficient condition for the nonexistence of stability conditions under a certain finiteness assumption on **D**. The properness in Theorem 1.1 guarantee the finiteness.

For a familiy $\mathcal{X} \to S$ of schemes, our theorem says that there is no "absolute" stability conditions unless the dimension of the base is zero. On the other hand, a stability condition is one of generalizations of slope stability of locally free sheaves on projective varieties. Recall that the slope stability derived from an ample line bundle and that we have relatively ample line bundles for a family $\mathcal{X} \to S$ of projective varieties. Thus it might be natural to introduce a notion of "relative" stability conditions so that a stability condition over a base does exist. Interestingly the notion of stability conditions over a base scheme for a flat family of projective varieties is introduced by Bayer et al. in [BLM+21] which gives a kind of "relative" stability conditions, to study Kuznetsov's non-commutative K3 surfaces. Our theorem gives an algebraic evidence of the necessity of relative stability conditions.

2. *R*-linear triangulated categories

From now on, R is a commutative ring.

Definition 2.1. Let \mathbf{D} be an *R*-linear triangulated category and *M* an object in \mathbf{D} .

(1) We denote by $\mu: R \to \operatorname{Hom}_{\mathbf{D}}(M, M)$ the morphism defined by the following composition:

(2.1)
$$R \times {\mathrm{id}}_M \subset R \times \mathrm{Hom}_{\mathbf{D}}(M, M) \to \mathrm{Hom}_{\mathbf{D}}(M, M).$$

(2) The support of the object M is defined as the support of $\operatorname{Hom}_{\mathbf{D}}(M, M)$ as R-modules and is denoted by $\operatorname{Supp}_{R} M$:

 $\operatorname{Supp}_{B} M := \operatorname{Supp} \operatorname{Hom}_{\mathbf{D}}(M, M).$

Lemma 2.2. Let **D** be an *R*-linear triangulated category. Suppose that a distinguished triangle $A \xrightarrow{i} B \xrightarrow{p} C$ in **D** satisfies $\operatorname{Hom}_{\mathbf{D}}^{0}(A, C) = \operatorname{Hom}_{\mathbf{D}}^{-1}(A, C) = 0$.

- (1) Any morphism $\varphi \colon B \to B$ uniquely indues morphisms $\varphi_A \colon A \to A$ and $\varphi_C \colon C \to C$ such that $i \cdot \varphi_A = \varphi_B \cdot i$ and $p \cdot \varphi_B = \varphi_C \cdot p$.
- (2) If $\varphi = 0$ then the morphisms φ_C and $\varphi_A = 0$ are zero.

Proof. We obtain the diagram of exact sequences of R-modules:

The assumptions imply both α and γ are isomorphisms. Then φ_A and φ_C are given by

$$\varphi_A = \alpha^{-1}(i_*\varphi)$$
 and $\varphi_C = \gamma^{-1}(p_*\varphi)$.

The second assertion is obvious from the above diagram.

Proposition 2.3. Let **D** be an *R*-linear triangulated category. Suppose that a distinguished triangle $A \xrightarrow{i} B \xrightarrow{p} C$ in **D** satisfies $\operatorname{Hom}_{\mathbf{D}}^{0}(A, C) = \operatorname{Hom}_{\mathbf{D}}^{-1}(A, C) = 0$.

Then the following holds:

$$(2.2) \qquad \qquad \operatorname{Supp}_R A \cup \operatorname{Supp}_R C \subset \operatorname{Supp}_R B$$

Proof. Suppose that $\mathfrak{p} \notin \operatorname{Supp}_R B$. It is enough to show that $\mathfrak{p} \notin \operatorname{Supp}_R A$ and $\mathfrak{p} \notin \operatorname{Supp}_R C$. By the assumption $\mathfrak{p} \notin \operatorname{Supp}_R B$, there exists $r \in R - \mathfrak{p}$ such that $r \cdot \operatorname{id} = \mu_r = 0$ in $\operatorname{End}(B)$. Since **D** is *R*-linear, the endomorphism $\mu_r \in \operatorname{End}(B)$ also induces the endomorphisms of A and C which make the following diagram commutative:

$$\begin{array}{ccc} A \xrightarrow{i} & B \xrightarrow{p} & C \\ \mu_r & & \mu_r & & \mu_r \\ A \xrightarrow{i} & B \xrightarrow{p} & C \end{array}$$

By Lemma 2.2, both μ_r and μ_r are zero. Thus the localizations $\operatorname{End}(A) \otimes_R R_{\mathfrak{p}}$ and $\operatorname{End}(C) \otimes_R R_{\mathfrak{p}}$ are zero.

Corollary 2.4. Let R be an R-linear triangulated category. The *i*th cohomology of $E \in \mathbf{D}$ with respect to a bounded t-structure $(\mathbf{D}^{\leq 0}, \mathbf{D}^{\geq 1})$ on \mathbf{D} is denoted by $H^{i}(E)$. Then the following holds:

$$\operatorname{Supp}_R H^i(E) \subset \operatorname{Supp}_R E.$$

Proof. Set p and q by

$$p = \max\{i \in \mathbb{Z} \mid H^{i}(E) \neq 0\}, \text{ and}$$
$$q = \min\{i \in \mathbb{Z} \mid H^{i}(E) \neq 0\}.$$

The proof is by induction on p-q. If p-q=0, then the assertion is clear.

Taking the filtration with respect to the *t*-structure, we obtain the following triangle

$$E^{p-1} \longrightarrow E \longrightarrow H^p(E)[-p].$$

where $E^{p-1} \in \mathbf{D}^{\leq 0}[1-p]$. Lemma 2.2 implies $\operatorname{Supp}_R E^{p-1} \subset \operatorname{Supp}_R E$ and $\operatorname{Supp}_R H^p(E) \subset \operatorname{Supp}_R E$. Then the assumption of induction implies the desired assertion.

Lemma 2.5. Let **D** be an *R*-linear triangulated category. Consider a diagram of distinguished triangle in **D**:

$$\begin{array}{ccc} A \xrightarrow{i} & B \xrightarrow{p} & C \\ \psi_A & \downarrow & \psi_B & \downarrow & \psi_C \\ A \xrightarrow{i} & B \xrightarrow{p} & C. \end{array}$$

If $\psi_A = 0$ and $\psi_C = 0$ then the composite ψ_B^2 is zero.

Proof. Since $p \cdot \psi_B = \psi_C \cdot p = 0$, there exists a $\varphi \colon B \to A$ such that $i \cdot \varphi = \psi_B$. Thus we see

$$\psi_B^2 = \psi_B \cdot i \cdot \varphi = i \cdot \psi_A \cdot \varphi = 0.$$

Lemma 2.6. Let $A \xrightarrow{i} B \xrightarrow{p} C$ be a distinguished triangle in an *R*-linear triangulated category **D**. Then the following holds:

$$\operatorname{Supp}_R B \subset \operatorname{Supp}_R A \cup \operatorname{Supp}_R C.$$

Proof. Take a prime ideal \mathfrak{p} such that $\mathfrak{p} \notin \operatorname{Supp}_R A$ and $\mathfrak{p} \notin \operatorname{Supp}_R C$. Then there exists r_A (resp. r_C) in $R - \mathfrak{p}$ such that $r_A \cdot \operatorname{id}_A = 0$ (resp. $r_C \cdot \operatorname{id}_C = 0$). Then $r_0 = r_A r_C$ satisfies $r_0 \operatorname{id}_A = 0$ and $r_0 \operatorname{id}_C = 0$. Since the category **D** is *R*-linear, the following diagram commutes:

$$\begin{array}{ccc} A \xrightarrow{i} & B \xrightarrow{p} & C \\ \mu_{r_0} & & \mu_{r_0} & \mu_{r_0} \\ A \xrightarrow{i} & B \xrightarrow{p} & C. \end{array}$$

By Lemma 2.5, we see $\mu_{r_0}^2 = \mu_{r_0}^2 \colon B \to B$ is the zero morphism. Hence $\mathrm{id}_B \in \mathrm{End}(B)$ is zero via localization to $\mathfrak{p} \in \mathrm{Spec} R$. Thus we see $\mathfrak{p} \notin \mathrm{Supp}_R B = \mathrm{Supp} \mathrm{End}(B)$. \Box

Corollary 2.7. Let **D** be an *R*-linear triangulated category. The *i*th cohomology of $E \in \mathbf{D}$ with respect to a bounded t-structure on **D** is denoted by $H^i(E)$. The the following holds:

$$\operatorname{Supp}_R E = \bigcup_{i \in \mathbb{Z}} \operatorname{Supp}_R H^i(E).$$

Proof. By Corollary 2.4, it is enough to show $\operatorname{Supp}_R E \subset \bigcup_{i \in \mathbb{Z}} \operatorname{Supp}_R H^i(E)$. Set p and q by

$$p = \max\{i \in \mathbb{Z} \mid H^i(E) \neq 0\}, \text{ and}$$
$$q = \min\{i \in \mathbb{Z} \mid H^i(E) \neq 0\}.$$

The proof is by the induction on p-q. Similarly to the proof of Corollary 2.4, we have the distinguished triangle

$$E^{p-1} \longrightarrow E \longrightarrow H^p(E)[-p]$$

Lemma 2.6 implies

$$\operatorname{Supp}_R E \subset \operatorname{Supp}_R E^{p-1} \cup \operatorname{Supp}_R H^p(E).$$

Then the assumption of the induction implies

$$\operatorname{Supp}_{R} E^{p-1} = \bigcup_{i \in \mathbb{Z}} H^{i}(E^{p-1})$$

which completes the proof.

In the last of this section, we show that the generalized support coincides with "usual supports" of complexes of R-modules. So let us suppose a triangulated category is the unbounded derived category $\mathbf{D}(\operatorname{Mod} R)$ of (not necessarily finite) R-modules. Recall that the support of an object E in $\mathbf{D}(\operatorname{Mod} R)$ is the union of the supports of the *i*th cohomology $H^i(E)$:

$$\operatorname{Supp} E := \bigcup_{i \in \mathbb{Z}} \operatorname{Supp} H^i(E).$$

Lemma 2.8. Let R be commutative ring and let E be a complex of R-modules.

- (1) We have $\operatorname{Supp} E \subset \operatorname{Supp}_R E$.
- (2) If the complex E is bounded, then $\operatorname{Supp} E = \operatorname{Supp}_{R} E$.

Proof. Suppose $\mathfrak{p} \notin \operatorname{Supp}_R E$. Then there exists $r \in R - \mathfrak{p}$ such that $r \operatorname{id} = \mu_r$ is zero in $\operatorname{Hom}(E, E)$. Taking cohomology with respect to the standard *t*-structure, μ_r does imply the multiplication μ_r on $H^i(E)$. Since $\mu_r \in \operatorname{End}(E)$ is zero, so does $\mu_r \in \operatorname{End}(H^i(E))$ for any $i \in \mathbb{Z}$. Hence \mathfrak{p} is not in $\operatorname{Supp} H^i(E)$ for any $i \in \mathbb{Z}$. Thus we have

$$\operatorname{Supp} E = \bigcup_{i \in \mathbb{Z}} \operatorname{Supp} H^i(E) \subset \operatorname{Supp}_R E.$$

Suppose that E is bounded and take a $\mathfrak{p} \in \operatorname{Spec} R - \bigcup_{i \in \mathbb{Z}} \operatorname{Supp} H^i(E)$. Then there exists $r_i \in R - \mathfrak{p}$ such that $\mu_{r_i} \colon H^i(E) \to H^i(E)$ is zero for each i with $H^i(E) \neq 0$. Since E is bounded, let s be the product of such r_i .

Then the morphism $\mu_s \in \text{End}(H^i(E))$ is zero for any $i \in \mathbb{Z}$. By the lemma below, there exists $N \in \mathbb{N}$ such that the Nth composite $\mu_s^N = \mu_{s^N}$ is the zero morphism of E. Hence we see $\mathfrak{p} \notin \text{Supp}_R E$.

Lemma 2.9. Let *E* be a bounded complex of *R*-modules. For an $r \in R$, if $\mu_r \in \text{End}(H^i(E))$ is zero for any $i \in \mathbb{Z}$, then there exists $N \in \mathbb{N}$ such that the composite $\mu_s^N = \mu_{s^N}$ is zero in End(E).

Proof. Set $p := \max\{i \in \mathbb{Z} \mid H^i(E) \neq 0\}$ and $q := \{i \in \mathbb{Z} \mid H^i(E) \neq 0\}$. The proof is by induction on p - q.

If p - q = 0, then N is taken to be 1. Suppose the assertion holds for $p - q = \ell - 1$. Taking truncation of E, we have the following distinguished triangle:

$$E^{p-1} \longrightarrow E \longrightarrow H^p(E)[-p].$$

By the assumption on induction, there exists $N \in \mathbb{N}$ such that $\mu_s^N \colon E^{p-1} \to E^{p-1}$ is zero. Since $\mu_s^N \colon H^p(E) \to H^p(E)$ is zero, Lemma 2.5 implies that the endomorphism $\mu_s^{2N} \colon E \to E$ is zero.

3. Stability conditions and R-linear categories

The aim in this section is to study a property of $\operatorname{Supp}_R E$ for σ -stable objects with respect to a stability condition σ . The following definition reflects properties of stable objects.

Definition 3.1. Let **D** be an *R*-linear triangulated category. An object $M \in \mathbf{D}$ satisfies the isomorphic property if the following holds:

(Ism) $\forall r \in R, \mu_r \colon M \to M$ is an isomorphism if μ_r is non-zero.

For well behavior of $\operatorname{Supp}_R E$, we need a finiteness assumption on the triangulated category **D**.

Definition 3.2. Let **D** be an *R*-linear triangulated category. **D** is said to be *finite* over *R* if the *R*-module Hom_{**D**}(*E*, *E*) is finite for any object $E \in$ **D**.

Lemma 3.3. Assume that the commutative ring R is an integral domain with dim R > 0and an R-linear triangulated category **D** is finite. If an object M in **D** satisfies

• the morphism $\mu_r \colon M \to M$ is an isomorphism for any $r \in R \setminus \{0\}$,

then M is zero.

Proof. It is enough to show $\text{Hom}_{\mathbf{D}}(M, M) = 0$, since the category **D** is additive.

If $\operatorname{Hom}_{\mathbf{D}}(M, M)$ is non-zero, then we have $\operatorname{Supp}_R M = \operatorname{Spec} R$ since $\operatorname{Hom}_{\mathbf{D}}(M, M)$ contains R. On the other hand, there exists a non-unit r in R. Then the assumption implies that μ_r gives an isomorphism of M. Hence we have the following isomorphism

$$\mu_r^* \colon \operatorname{Hom}_{\mathbf{D}}(M, M) \to \operatorname{Hom}_{\mathbf{D}}(M, M).$$

Hence the *R*-module $\operatorname{Hom}_{\mathbf{D}}(M, M)$ satisfies $\operatorname{Hom}_{\mathbf{D}}(M, M) \otimes R/(r) = 0$ which implies $\operatorname{Supp}_R M \cap \operatorname{Spec} R/(r) = \emptyset$ since $\operatorname{Hom}_{\mathbf{D}}(M, M)$ is finitely generated. This contradicts the fact $\operatorname{Supp}_R M = \operatorname{Spec} R$.

Lemma 3.4. Suppose an object M in a finite R-linear triangulated category D satisfies the condition (Ism). Then the following holds:

- (1) The ideal $\operatorname{ann}(\operatorname{End}(M))$ of R is prime.
- (2) If M is non-zero, then $\operatorname{ann}(\operatorname{End}(M)) = \operatorname{ann}(f)$ for any $f \in \operatorname{End}(M) \setminus \{0\}$.
- (3) If M is non-zero, then dim $\operatorname{Supp}_R(M) = 0$.

Proof. Let a and b be in R. Suppose the product ab is in $\operatorname{ann}(\operatorname{End}(M))$ and $a \notin \operatorname{ann}(\operatorname{End}(M))$. Then the endomorphism μ_a is non-zero and hence is invertible. Thus μ_b is zero by $\mu_{ab} = \mu_a \mu_b$. Hence $b \in \operatorname{ann}(\operatorname{End}(M))$.

Take $a \in \operatorname{ann}(f)$. If μ_a is non-zero, then μ_a is an isomorphism by the assumption. This contradicts $a \in \operatorname{ann}(f)$. Thus μ_a is the zero-morphism and we have $\operatorname{ann}(f) \subset \operatorname{ann}(\operatorname{End}(M))$. The opposite inclusion is clear.

Put $\mathfrak{p} = \operatorname{ann}(\operatorname{End}(M))$. End(M) is an R/\mathfrak{p} -module and M satisfies the condition (Ism). If the prime ideal \mathfrak{p} is not maximal, then dim $R/\mathfrak{p} > 0$ and Lemma 3.3 implies M = 0. Hence \mathfrak{p} has to be maximal and dim $\operatorname{Supp}_R M = 0$.

Lemma 3.5. Let **D** be a finite *R*-linear triangulated category and let σ be a locally finite stability condition on **D**. If an object $E \in \mathbf{D}$ is σ -stable then dim Supp_R E = 0.

Proof. If E is σ -stable, then the condition (Ism) holds. Lemma 3.4 implies dim $\operatorname{Supp}_R E = 0$.

Proposition 3.6. Let **D** be a finite *R*-linear triangulated category and σ be a locally finite stability condition on **D**. If an object $E \in \mathbf{D}$ is σ -semistable, then dim Supp_R E = 0.

Proof. Let $\mathcal{P}(\phi)$ be the slicing of σ with phase ϕ . Recall that $\mathcal{P}(\phi)$ is an abelian category. Since σ is locally finite, any object $E \in \mathcal{P}(\phi)$ is given by a successive extension of finite σ -stable objects.

The proof is induction on the number of stable factors of $E \in \mathcal{P}(\phi)$. If the number is 1, the assertion follows from Lemma 3.5 since E is σ -stable.

Now E is not σ -stable but σ -semistable. Take a σ -stable subobject A of E. Then the quotient E/A satisfies the assumption on the induction. Hence Lemma 2.6 implies

 $\operatorname{Supp}_{R} E \subset \operatorname{Supp}_{R} A \cup \operatorname{Supp}_{R} E/A.$

and the assumption on induction implies dim $\operatorname{Supp}_R E/A = 0$. Then Corollary 2.7 implies the desired assertion.

Theorem 3.7. Let \mathbf{D} be a finite *R*-linear triangulated category. Suppose that there exists a locally finite stability condition σ on \mathbf{D} . For any object $E \in \mathbf{D}$, the dimension of the support $\operatorname{Supp}_R E$ is zero. Moreover any ith cohomology $H^i(E)$ of E has the zero dimensional support. *Proof.* The proof is by induction on the length $\ell(E)$ of the Harder-Narasimhan filtration of E with respect to σ . If $\ell(E) = 1$, the assertion follows from Proposition 3.6.

Suppose that the assertion holds for $\ell(E) - 1$. Taking the last term of the Harder-Narasimhan filtration of E, we obtain the distinguished triangle

 $(3.1) E_{n-1} \longrightarrow E \longrightarrow A_n,$

where A_n is σ -semistable and $\ell(E_{n-1}) = \ell(E) - 1$. Thus we see dim Supp_R $E_{n-1} = 0$ and dim Supp_R $A_n = 0$. Then Lemma 2.6 implies the desired assertion. The last assertion follows from Corollary 2.7.

Corollary 3.8. Let **D** be a finite *R*-linear triangulated category. If there exists an object $M \in \mathbf{D}$ such that $\dim \operatorname{Supp}_R M > 0$, then there exists no locally finite stability condition on **D**.

Proof. If there exists a locally finite stability condition on \mathbf{D} , the dimension the support of any object in \mathbf{D} is zero. This contradicts Theorem 3.7.

Corollary 3.9. Let $f: \mathcal{X} \to \operatorname{Spec} R$ be a proper morphism to the affine Noetherian scheme $\operatorname{Spec} R$. If the dimension of the image of f is positive, then both $\operatorname{Stab} \mathbf{D}^{b}(\mathcal{X})$ and $\operatorname{Stab} \mathbf{D}^{\operatorname{perf}}(\mathcal{X})$ are empty.

Proof. Recall that $\mathbf{D}^{\text{perf}}(\mathcal{X})$ is a full subcategory of $\mathbf{D}^{b}(\mathcal{X})$ and the structure sheaf $\mathcal{O}_{\mathcal{X}}$ is in $\mathbf{D}^{\text{perf}}(\mathcal{X})$.

Let Z be the image of f. Note that Z is a closed subscheme of Spec R. Then the morphism $\tilde{f}: \mathcal{X} \to Z$ is proper and hence $\mathbf{D}^b(X)$ is linear and finite over the ring $H^0(Z, \mathcal{O}_Z)$. Since \tilde{f} is surjective, the ring $H^0(X, \mathcal{O}_X)$ containes $H^0(Z, \mathcal{O}_Z)$. Then the assumption implies

$$\dim \operatorname{Supp}_{H^0(Z,\mathcal{O}_Z)} \mathcal{O}_X \geq \dim \operatorname{Supp}_{H^0(Z,\mathcal{O}_Z)} H^0(Z,\mathcal{O}_Z) > 0.$$

Then Corollary 3.8 implies the desired assertion.

Remark 3.10. In [Kaw20], we show that the bounded derived category of a affine Noetherian scheme Spec R has a locally finite stability condition. Corollary 3.9 gives a generalization.

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Appendix A. by Hiroyuki Minamoto

Inspired by main body this paper, we prove the following theorem.

Theorem A.1. Let X be a Noetherian scheme. Assume that the Krull dimension of the ring $\Gamma(X, \mathcal{O}_X)$ is positive. Then both Stab $\mathbf{D}^b(X)$ and Stab $\mathbf{D}^{\text{perf}}(X)$ are empty.

We need preparations.

Lemma A.2. Let X be a Noetherian scheme, $s \in \Gamma(X, \mathcal{O}_X)$ a global section, and $Z := Spec \mathcal{O}_X/(s)$ the vanishing locus of the section s. For an object $M \in \mathbf{D}^b(X)$, we denote by μ_s the multiplication map $\mu_s^M \colon M \to M$. Then for $M \in \mathbf{D}^b(X)$, the following assertion hold:

- (1) If μ_s is zero, then Supp $M \subset Z$.
- (2) If μ_s is an isomorphism, then Supp $M \subset X \setminus Z$.

Proof. (1) By the assumption, the support of *i*th cohomology $H^i(M)$ of M with respect to standard *t*-structure is contained in Z since the cohomology of the morphism μ_s is zero for all $i \in \mathbb{Z}$. This gives the proof.

(2) By the assumption, the multiplication map $\mu_s^i \colon H^i(M) \to H^i(M)$ is an isomorphism for each $i \in \mathbb{Z}$. Hence the localization $(\mu_s^i)_x \colon H^i(M)_x \to H^i(M)_x$ of the map μ_s^i on each $x \in X$ is an isomorphism of $\mathcal{O}_{X,x}$ modules. If x in Z, then clearly the germ s_x is in the maximal ideal $\mathfrak{m}_{X,x}$ of $\mathcal{O}_{X,x}$. Hence we see $\mathfrak{m}_{X,x}H^i(M)_x = H^i(M)_x$. By Nakayama's lemma, we have $H^i(M)_x$ is zero and hence $\operatorname{Supp} M \subset X \setminus Z$.

Proposition A.3. Let X be a connected Noetherian scheme, and Z the vanishing locus $Spec \mathcal{O}_X/(s)$ of a global section $s \in \Gamma(X, \mathcal{O}_X)$. Assume that s is neither nilpotent nor invertible. Then both Stab $\mathbf{D}^b(X)$ and Stab $\mathbf{D}^{perf}(X)$ are empty.

Proof. We deal with $\mathbf{D}^{b}(X)$. The same proof works for $\mathbf{D}^{\text{perf}}(X)$.

Suppose to the contrary that $\mathbf{D}^{b}(X)$ has a locally finite stability condition. Then by the argument of the proofs of Proposition 3.6 and Theorem 3.7, there exists finite objects $M_1, \dots, M_n \in \mathbf{D}^{b}(X)$ having the property (Ism) such that \mathcal{O}_X is in the thick hull of M_1, \dots, M_n .

We set

 $I = \{i \mid 1 \le i \le n, \mu_s^{M_i} \text{ is an isomorphism}\}, J = \{j \mid 1 \le j \le n, \mu_s^{M_j} \text{ is zero}\}.$

Note that $I \sqcup J = \{1, 2, \dots, n\}$. We set $Y_I := \bigcup_{i \in I} \operatorname{Supp} M_i$ and $Y_J := \bigcup_{j \in J} \operatorname{Supp} M_j$. Then we see $Y_I \subset X \setminus Z$ and $Y_J \subset Z$ by Lemma A.2. On the other hand, we have $X = \operatorname{Supp} \mathcal{O}_X \subset \bigcup_{i=1}^n \operatorname{Supp} M_i = Y_I \sqcup Y_J \subset Y_I \sqcup Z$ and hence $X = Y_I \sqcup Z$. Since X is connected by the assumption, we have either Z = X or $Z = \emptyset$. However the condition Z = X contradicts to the assumption s is non-nilpotent and the condition $Z = \emptyset$ contradicts to the assumption that s is non-invertible. \Box

We proceed a proof of the main theorem of appendix.

Proof of Theorem A.1. Take a connected component X' of X such that dim $\Gamma(X', \mathcal{O}_{X'}) \geq 1$. Then it has a global section which is neither nilpotent nor invertible. Then the assertion follows from Proposition A.3.

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