Diagrammatic presentations of enriched monads and varieties for a subcategory of arities

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Abstract

The theory of presentations of enriched monads was developed by Kelly, Power, and Lack, following classic work of Lawvere, and has been generalized to apply to subcategories of arities in recent work of Bourke-Garner and the authors. We argue that, while theoretically elegant and structurally fundamental, such presentations of enriched monads can be inconvenient to construct directly in practice, as they do not directly match the definitional procedures used in constructing many categories of enriched algebraic structures via operations and equations.

Retaining the above approach to presentations as a key technical underpinning, we establish a flexible formalism for directly describing enriched algebraic structure borne by an object of a \mathcal{V} -category \mathcal{C} in terms of parametrized \mathcal{J} -ary operations and diagrammatic equations for a suitable subcategory of arities $\mathcal{J} \hookrightarrow \mathcal{C}$. On this basis we introduce the notions of diagrammatic \mathcal{J} -presentation and \mathcal{J} -ary variety, and we show that the category of \mathcal{J} -ary varieties is dually equivalent to the category of \mathcal{J} -ary \mathcal{V} -monads. We establish several examples of diagrammatic \mathcal{J} -presentations and \mathcal{J} -ary varieties relevant in both mathematics and theoretical computer science, and we define the sum and tensor product of diagrammatic \mathcal{J} -presentations. We show that both \mathcal{J} -relative monads and \mathcal{J} -pretheories give rise to diagrammatic \mathcal{J} -presentations that directly describe their algebras. Using diagrammatic \mathcal{J} -presentations as a method of proof, we generalize the pretheories-monads adjunction of Bourke and Garner beyond the locally presentable setting. Lastly, we generalize Birkhoff's Galois connection between classes of algebras and sets of equations to the above setting.

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1 Introduction

Universal algebra begins with the idea of a set A equipped with a family of finitary operations satisfying specified equations. This is usually formalized by beginning with a signature (or similarity type), i.e. a set Σ (whose elements we call operation symbols) equipped with an assignment to each $\sigma \in \Sigma$ a finite cardinal n_{σ} called the arity of σ . A Σ -algebra is then a set A equipped with a family of operations $\sigma^A:A^{n_\sigma}\to A$ indexed by the operation symbols $\sigma \in \Sigma$. Syntactic expressions or terms over the signature Σ allow the specification of equations over Σ , and by collecting together only those Σ -algebras that satisfy a specified set of equations E we arrive at the notion of variety of algebras that is fundamental to algebra in general. The Σ -algebras in the given variety, and their homomorphisms, form a category, and one may then call the pair $P = (\Sigma, E)$ a presentation of this category of algebras. One of the complications inherent in this approach to algebra is that several different presentations may present isomorphic categories of algebras. This issue was addressed by Lawvere [27]. whose algebraic theories (or Lawvere theories) classify varieties of algebras up to a suitable notion of isomorphism, and with the insights of Linton [28] it was soon realized that Lawvere theories are equivalently given by finitary monads on the category of sets, Set, so that each presentation $P = (\Sigma, E)$ presents a finitary monad.

It is known that universal algebraic concepts admit a generalization to a setting in which sets are replaced by the objects of a symmetric monoidal closed category \mathscr{V} or, more generally, a \mathscr{V} -enriched category \mathscr{C} . Early contributions to this line of development in enriched category theory include works of Kock [24], Dubuc [14, 15], Borceux and Day [9], Kelly and Power [23], Kelly and Lack [22]. Generalizing finitary monads, a recent theme in this area has been the study of \mathscr{V} -enriched monads for a subcategory of arities $\mathscr{J} \hookrightarrow \mathscr{C}$, i.e. a full subcategory that is dense in the enriched sense, and whose objects $J \in \mathsf{ob} \mathscr{J}$ play the role of arities. In the setting of ordinary Set-enriched category theory, this line of development originates with Linton [29] and includes the paper of Berger, Melliès, and Weber [4], while enriched monads for subcategories of arities have been studied by the first author [31], by Bourke and Garner [10], and by the authors [35]; also see [26] for related work in terms of classes of weights.

In the present paper, we develop a versatile framework and methodology for directly describing enriched algebraic structure in terms of operations and equations, relative to a suitable small subcategory of arities $\mathscr{J} \hookrightarrow \mathscr{C}$. A technical underpinning for this paper is provided by the authors' recent paper [35], which generalizes certain prior contributions of Kelly, Power, and Lack [23, 22, 25] in the study of presentations of finitary enriched monads via signatures.

But here we depart substantially from these prior works, in that we employ a more flexible formalism for describing enriched algebraic structure that more directly matches the definitional procedures that are typically used in constructing specific examples of enriched algebraic categories, thus more directly reflecting mathematical practice.

Indeed, here we formulate enriched algebraic structure borne by an object A of a suitable \mathcal{V} -category \mathcal{C} as consisting of a family of parametrized \mathcal{J} -ary operations, each of which is

given by a morphism of the form

$$\mathscr{C}(J,A)\otimes C\to A$$

in \mathscr{C} , where the *arity J* is an object of a given subcategory of arities $\mathscr{J} \hookrightarrow \mathscr{C}$ and C is an object of \mathscr{C} that we call the *parameter* (while \otimes here is the action of \mathscr{V} on \mathscr{C} by tensor). Such parametrized operations can be described equivalently as morphisms

$$\mathscr{C}(J,A) \to \mathscr{C}(C,A)$$

in \mathscr{V} . For example, if $\mathscr{C} = \mathscr{V} = \mathsf{Set}$ then a parametrized operation may be written in the form $A^J \times C \to A$ or $A^J \to A^C$.

We introduce the notion of free-form \mathcal{J} -signature, which is a span ob $\mathcal{J} \leftarrow \mathcal{S} \rightarrow \text{ob} \mathcal{C}$ (i.e. a span from ob \mathcal{J} to ob \mathcal{C}) in which \mathcal{S} is a small set. Equivalently, a free-form \mathcal{J} -signature is a small set \mathcal{S} , whose elements we call operation symbols, equipped with an assignment to each operation symbol $\sigma \in \mathcal{S}$ an object J_{σ} of \mathcal{J} called the arity and an object C_{σ} of \mathcal{C} called the parameter. An \mathcal{S} -algebra is then an object A of \mathcal{C} equipped with a family of parametrized \mathcal{J} -ary operations

$$\sigma^A : \mathscr{C}(J_{\sigma}, A) \longrightarrow \mathscr{C}(C_{\sigma}, A) \quad (\sigma \in \mathcal{S}),$$

and we obtain a \mathcal{V} -category of \mathcal{S} -algebras, \mathcal{S} -Alg, which we show is *strictly* \mathcal{J} -ary monadic over \mathcal{C} , i.e., isomorphic to the \mathcal{V} -category of Eilenberg-Moore algebras for an associated \mathcal{J} -ary \mathcal{V} -monad $\mathbb{T}_{\mathcal{S}}$.

In order to impose equations on S-algebras in an intuitive and practically convenient way, we systematically employ Lawvere's notion of algebraic structure [27] and its enriched analogue developed by Dubuc [14]. Given a free-form \mathscr{J} -signature S, we define a diagrammatic S-equation $\omega \doteq \nu$ to be a family of parallel pairs of the form

$$\omega_A, \nu_A : \mathscr{C}(J, A) \rightrightarrows \mathscr{C}(C, A),$$

 \mathscr{V} -natural in $A \in \mathcal{S}$ -Alg, where $J \in \mathsf{ob} \mathscr{J}$ and $C \in \mathsf{ob} \mathscr{C}$. Such diagrammatic equations correspond bijectively to parallel pairs $t, u : C \rightrightarrows T_{\mathcal{S}}J$ (i.e., pairs of Kleisli morphisms for $\mathbb{T}_{\mathcal{S}}$ from an object of \mathscr{C} to an object of \mathscr{J}), but the formulation in terms of diagrammatic \mathcal{S} -equations is by far the more convenient formalism for constructing specific examples in the enriched context, because, as the nomenclature suggests, we simply construct whatever diagrammatic laws we want an \mathcal{S} -algebra A to satisfy—using whatever constructions are available in the host \mathscr{V} -category \mathscr{C} —and we then observe that the resulting parallel pairs of composite morphisms are \mathscr{V} -natural in $A \in \mathcal{S}$ -Alg by construction.

We define a diagrammatic \mathcal{J} -presentation $\mathcal{P} = (\mathcal{S}, \mathsf{E})$ to consist of a free-form \mathcal{J} -signature \mathcal{S} together with a small family of diagrammatic \mathcal{S} -equations E , and we say that an \mathcal{S} -algebra is a \mathcal{P} -algebra if it satisfies the diagrammatic equations in E . We show that the \mathcal{V} -category of \mathcal{P} -algebras, \mathcal{P} -Alg, is strictly \mathcal{J} -ary monadic over \mathcal{C} , so that \mathcal{P} determines an associated \mathcal{J} -ary \mathcal{V} -monad $\mathbb{T}_{\mathcal{P}}$ on \mathcal{C} . We define a \mathcal{J} -ary variety to be a \mathcal{V} -category of the form \mathcal{P} -Alg, regarded as an object of the slice \mathcal{V} -CAT/ \mathcal{C} , and we show that the category

of \mathcal{J} -ary varieties is dually equivalent to the category of \mathcal{J} -ary \mathcal{V} -monads on \mathcal{C} , thus generalizing several results in the literature. We establish several examples of diagrammatic \mathcal{J} -presentations and \mathcal{J} -ary varieties, as well as certain constructions on diagrammatic \mathcal{J} -presentations including the sum and tensor product. We further illustrate the versatility of diagrammatic \mathcal{J} -presentations by showing that the algebras of \mathcal{J} -relative \mathcal{V} -monads [3] and of \mathcal{J} -pretheories [10] may be described as algebras for diagrammatic \mathcal{J} -presentations; thus we establish establish new and more general results on the monadicity of these categories of structures. We pursue further applications and ramifications of diagrammatic \mathcal{J} -presentations and \mathcal{J} -ary varieties, including a generalization of the pretheories-monads pretheories-monads adjunction of Bourke and Garner [10] beyond the locally presentable setting, as well as a generalization of Birkhoff's Galois connection [6] between classes of algebras and sets of equations to the present enriched setting of free-form \mathcal{J} -signatures \mathcal{S} and (parametrized) diagrammatic \mathcal{S} -equations.

We now contrast the above concepts with the more traditional formalism for presentations of \mathcal{J} -ary \mathcal{V} -monads used in the authors' paper [35], which provides a fundamental technical basis for this work and generalizes methods developed by Kelly, Power, and Lack [23, 22]. With this more traditional formalism, one defines a \mathcal{J} -signature Σ to be a family of objects ΣJ of \mathscr{C} , indexed by the objects $J \in \mathsf{ob} \mathscr{J}$, thinking of ΣJ as the object of internal operation symbols of arity J. The theoretical importance of this approach lies in the result that the category of \mathcal{J} -ary \mathcal{V} -monads on \mathcal{C} is monadic over the category of these 'traditional' I-signatures [35], generalizing a result for finitary monads proved by Lack [25]. Thus every \mathcal{J} -ary \mathcal{V} -monad has a canonical presentation as a coequalizer, leading to a more traditional concept of presentation, namely a parallel pair of morphisms $\Gamma \rightrightarrows \mathbb{T}_{\Sigma}$ from a given (traditional) \mathcal{J} -signature Γ into the \mathcal{J} -signature underlying the free \mathcal{J} -ary \mathcal{V} monad \mathbb{T}_{Σ} on Σ ; see [35]. This notion of \mathscr{J} -presentation is theoretically fundamental and yet practically inconvenient for capturing many specific examples of categories of enriched algebraic structures, because it forces the user of the formalism to (1) explicitly construct each object of internal operation symbols ΣJ ($J \in \mathsf{ob} \mathscr{J}$), often as a coproduct, and (2) similarly construct the objects ΓJ $(J \in \mathsf{ob} \mathscr{J})$ and the morphisms $\Gamma J \rightrightarrows T_{\Sigma} J$, a process that is often tedious and just as often does not resemble the manner in which the given enriched algebraic structures are defined in practice. Thus, we argue that diagrammatic I-presentations provide an essential practical tool for enriched algebra, enabling familiar definitional procedures to be directly employed in describing enriched algebraic structures, and thus providing an indispensable 'user-friendly interface' to the fundamental theory of presentations via monadicity of \mathcal{J} -ary monads.

The diagrammatic \mathscr{J} -presentations that we introduce here should also be compared and contrasted with the term equational systems of Fiore and Hur [18] and the monadic equational systems of Fiore [16]. Indeed, these works involve arbitrary sets of parallel pairs of morphisms $C \rightrightarrows TB$ for a \mathscr{V} -monad \mathbb{T} on a \mathscr{V} -category \mathscr{C} , and the interpretation of such morphisms as parametrized operations on \mathbb{T} -algebras. But these works do not employ a given subcategory of arities $\mathscr{J} \hookrightarrow \mathscr{C}$ and, moreover, they are not founded on the monadicity of any particular class of \mathscr{V} -monads over any class of signatures, whereas the present paper

begins with a subcategory of arities $\mathscr{J} \hookrightarrow \mathscr{C}$ and is founded upon the monadicity of \mathscr{J} -ary \mathscr{V} -monads over \mathscr{J} -signatures.

We now outline the structure of the paper. In §2, we review elements of the authors' paper [35], which provides a theoretical underpinning for this work, and we discuss our assumptions on the given closed category \mathcal{V} , the \mathcal{V} -category \mathcal{C} , and the subcategory of arities $\mathcal{J} \hookrightarrow \mathcal{C}$; these relatively mild assumptions allow the theory in this paper to be applied to wide classes of closed categories \mathcal{V} and \mathcal{V} -categories \mathcal{C} that need not be locally presentable, including the locally bounded closed categories of [21] and the locally bounded \mathcal{V} -categories of [34]. In §3, we delineate those aspects of parametrized operations and diagrammatic equations that can be formulated for an arbitrary \mathcal{V} -monad, without reference to the subcategory of arities.

In §4 we define systems of diagrammatic \mathcal{J} -ary equations on \mathcal{J} -ary monadic \mathcal{V} -categories \mathscr{A} over \mathscr{C} , and we show that the full sub- \mathcal{V} -category of \mathscr{A} described by such a system of equations is \mathcal{J} -ary monadic over \mathscr{C} .

In §5, we define the fundamental concepts of free-form \mathcal{J} -signature, diagrammatic \mathcal{J} -presentation, and \mathcal{J} -ary variety, and we show not only that the \mathcal{V} -category of algebras for a diagrammatic \mathcal{J} -presentation is \mathcal{J} -ary monadic over \mathcal{C} , but also that there is a dual equivalence between \mathcal{J} -ary varieties and \mathcal{J} -ary \mathcal{V} -monads on \mathcal{C} .

In §6 we construct several examples of diagrammatic \mathscr{J} -presentations and diagrammatic systems of \mathscr{J} -ary equations that present various \mathscr{V} -categories of enriched mathematical structures, including the following: internal R-modules and R-affine spaces for an internal rig (or semiring) R in a complete and cocomplete cartesian closed category \mathscr{V} (6.1, 8.15); monoidal categories, regarded as Cat-enriched algebraic structures in Cat (6.2); internal categories in a locally presentable closed category \mathscr{V} (6.3), regarded as \mathscr{V} -enriched algebraic structures in the \mathscr{V} -category of internal graphs in \mathscr{V} ; the global state algebras of Plotkin and Power [39] (6.5); and some of the parametrized algebraic theories of Staton [42] (6.6). The latter two examples illustrate the applicability of diagrammatic \mathscr{J} -presentations for algebraic computational effects.

In §7 we further illustrate the versatility of diagrammatic \mathscr{J} -presentations by constructing the sum and tensor product (when $\mathscr{C} = \mathscr{V}$) of a given pair of diagrammatic \mathscr{J} -presentations; the tensor product generalizes Freyd's tensor product of finitary algebraic presentations [19], and it can be expressed very conveniently in this formalism, since we simply write out the diagrammatic laws that express that the operations for the first presentation must commute with those for the second.

In §8, we show that every \mathscr{J} -relative \mathscr{V} -monad H (cf. [3]) gives rise to a diagrammatic \mathscr{J} -presentation whose algebras are precisely those of H. We thus establish general results on the monadicity of such algebras, deducing as a corollary that for any small full subcategory \mathscr{K} of a locally presentable \mathscr{V} -category \mathscr{C} over a locally presentable closed category \mathscr{V} , the \mathscr{V} -category of algebras for any \mathscr{K} -relative \mathscr{V} -monad is monadic by way of an accessible \mathscr{V} -monad.

In §9, as one of the further main contributions of the paper, we use diagrammatic \mathscr{J} presentations as a method of proof in order to generalize the *pretheories-monads adjunction*of Bourke and Garner [10] beyond the locally presentable setting. We achieve this by showing

that the algebras for a pretheory are precisely the algebras for an associated diagrammatic \mathcal{J} -presentation and so are the objects of a strictly \mathcal{J} -ary monadic \mathcal{V} -category over \mathcal{C} .

In §10, we generalize Birkhoff's well-known Galois connection between classes of algebras and sets of equations [6] to the present setting of free-form \mathcal{J} -signatures \mathcal{S} and (parametrized) diagrammatic \mathcal{S} -equations, thus enabling the consideration of the \mathcal{J} -ary variety over \mathcal{S} generated by a class of \mathcal{S} -algebras, and we prove characterization theorems for \mathcal{J} -ary varieties with reference to the resulting Galois connection.

2 Background and given data

We assume that the reader has familiarity with enriched category theory, which is exposited (for example) in [21, 14] and [8, Chapter 6]. We use the term "set" throughout to refer to a not-necessarily-small set (which we sometimes emphasize by prepending the phrase "possibly large" in parentheses); when we wish to specify that a set is *small*, we always use the terminology "small set". We have the category Set of small sets and the category SET of sets. We now recall the needed background, definitions, and results from our previous work [35], and in 2.4 we recall several examples.

2.1. Throughout, we let \mathscr{V} be a locally small, symmetric monoidal closed category that is complete and cocomplete. We also assume that \mathscr{V} is a **closed factegory** (see [35, 6.1.2]), meaning that \mathscr{V} is equipped with a \mathscr{V} -enriched factorization system $(\mathscr{E}, \mathscr{M})$ (see [30]) that need not be proper.

Throughout, we let \mathscr{C} be a \mathscr{V} -factegory [35, 6.1.5], i.e., a \mathscr{V} -category \mathscr{C} equipped with an enriched factorization system $(\mathscr{E}_{\mathscr{C}}, \mathscr{M}_{\mathscr{C}})$ such that each $\mathscr{C}(C, -) : \mathscr{C} \to \mathscr{V} \ (C \in \mathsf{ob}\,\mathscr{C})$ sends $\mathscr{M}_{\mathscr{C}}$ -morphisms to \mathscr{M} -morphisms. We assume that the \mathscr{V} -category \mathscr{C} is cocomplete and cotensored and has arbitrary conical cointersections (i.e. wide pushouts) of $\mathscr{E}_{\mathscr{C}}$ -morphisms. Also, in order to invoke certain theorems in [35], we suppose either that $(\mathscr{E}_{\mathscr{C}}, \mathscr{M}_{\mathscr{C}})$ is proper or that \mathscr{C} is $\mathscr{E}_{\mathscr{C}}$ -cowellpowered.

2.2. We assume that the \mathscr{V} -category \mathscr{C} is equipped with a small **subcategory of arities** $j: \mathscr{J} \hookrightarrow \mathscr{C}$, i.e. a small, full, and dense sub- \mathscr{V} -category. We suppose that $j: \mathscr{J} \hookrightarrow \mathscr{C}$ is **eleutheric**, which means that every \mathscr{V} -functor $H: \mathscr{J} \to \mathscr{C}$ has a left Kan extension along j that is preserved by each $\mathscr{C}(J,-): \mathscr{C} \to \mathscr{V}$ ($J \in \mathsf{ob} \mathscr{J}$), or equivalently that $j: \mathscr{J} \hookrightarrow \mathscr{C}$ presents \mathscr{C} as a free Φ -cocompletion of \mathscr{J} for a class of small weights Φ (see [35, 3.8]). In particular, j then presents \mathscr{C} as a free $\Phi_{\mathscr{J}}$ -cocompletion of \mathscr{J} for the class $\Phi_{\mathscr{J}}$ consisting of the weights $\mathscr{C}(j-,C): \mathscr{J}^{\mathsf{op}} \to \mathscr{V}$ with $C \in \mathsf{ob} \mathscr{C}$ [35, 3.6].

We also assume that $j: \mathscr{J} \hookrightarrow \mathscr{C}$ is **bounded**, meaning that (\mathscr{J} is small and) there is a regular cardinal α for which each $\mathscr{C}(J,-):\mathscr{C} \to \mathscr{V}$ ($J \in \mathsf{ob} \mathscr{J}$) preserves the \mathscr{E} -tightness of small α -filtered \mathscr{M} -cocones (see [35, 6.1.6, 6.1.10]). In the special case where both ($\mathscr{E}_{\mathscr{C}}, \mathscr{M}_{\mathscr{C}}$) and (\mathscr{E}, \mathscr{M}) are the trivial factorization system (Iso, All), this reduces to the requirement that each $\mathscr{C}(J,-):\mathscr{C} \to \mathscr{V}$ ($J \in \mathsf{ob} \mathscr{J}$) preserve small conical α -filtered colimits. See 2.4 for examples of bounded and eleutheric subcategories of arities.

2.3. A \mathscr{V} -endofunctor $H:\mathscr{C}\to\mathscr{C}$ is \mathscr{J} -ary (or j-ary) if it preserves left Kan extensions along $j:\mathscr{J}\to\mathscr{C}$, equivalently, if H is $\Phi_{\mathscr{J}}$ -cocontinuous for the class of weights $\Phi_{\mathscr{J}}$ of 2.2. Moreover, if Φ is any class of small weights such that j presents \mathscr{C} as a free Φ -cocompletion of \mathscr{J} (e.g. $\Phi_{\mathscr{J}}$), then a \mathscr{V} -functor $H:\mathscr{C}\to\mathscr{C}$ is \mathscr{J} -ary iff H is Φ -cocontinuous [35, 4.2]. We let $\operatorname{End}_{\mathscr{J}}(\mathscr{C})$ be the (ordinary) category of \mathscr{J} -ary \mathscr{V} -endofunctors on \mathscr{C} , which is a (non-symmetric) strict monoidal category under composition. A \mathscr{J} -ary \mathscr{V} -monad is a \mathscr{V} -monad \mathbb{T} on \mathscr{C} whose underlying \mathscr{V} -endofunctor T is \mathscr{J} -ary. We write $\operatorname{Mnd}_{\mathscr{J}}(\mathscr{C})$ to denote the category of \mathscr{J} -ary \mathscr{V} -monads, i.e., the category of monoids in the monoidal category $\operatorname{End}_{\mathscr{J}}(\mathscr{C})$. Since $j:\mathscr{J}\to\mathscr{C}$ is eleutheric, we have an equivalence of categories \mathscr{V} -CAT(\mathscr{J},\mathscr{C}) $\simeq \operatorname{End}_{\mathscr{J}}(\mathscr{C})$ given by precomposition with j and left Kan extension along j (see [35, 4.5]).

A \mathscr{V} -category over \mathscr{C} is an object of the (strict) slice category \mathscr{V} -CAT/ \mathscr{C} , i.e., a \mathscr{V} -category \mathscr{A} equipped with a \mathscr{V} -functor $G: \mathscr{A} \to \mathscr{C}$. We denote such an object of \mathscr{V} -CAT/ \mathscr{C} either by (\mathscr{A}, G) or simply by \mathscr{A} . Given a \mathscr{V} -monad \mathbb{T} on \mathscr{C} , we write \mathbb{T} -Alg to denote the \mathscr{V} -category of \mathbb{T} -algebras, and we regard \mathbb{T} -Alg as a \mathscr{V} -category over \mathscr{C} by way of the forgetful \mathscr{V} -functor $U^{\mathbb{T}}: \mathbb{T}$ -Alg $\to \mathscr{C}$.

We say that a \mathscr{V} -functor $G: \mathscr{A} \to \mathscr{C}$ is **strictly** \mathscr{J} -monadic (or **strictly** \mathscr{J} -ary monadic), or that \mathscr{A} is a **strictly** \mathscr{J} -monadic \mathscr{V} -category over \mathscr{C} , if $\mathscr{A} \cong \mathbb{T}$ -Alg in \mathscr{V} -CAT/ \mathscr{C} for some \mathscr{J} -ary \mathscr{V} -monad \mathbb{T} . It then follows that G has a left adjoint F such that the \mathscr{V} -monad induced by the \mathscr{V} -adjunction $F \dashv G$ is precisely \mathbb{T} . Writing \mathscr{J} -Monadic for the full subcategory of \mathscr{V} -CAT/ \mathscr{C} consisting of the strictly \mathscr{J} -monadic \mathscr{V} -categories over \mathscr{C} , we have an equivalence \mathscr{J} -Monadic \mathscr{V} -category by [35, §9.3]

Example 2.4. The following examples satisfy the blanket assumptions of 2.1 and 2.2; see [35, 4.7, 12.3]:

- 1. the subcategory of arities $\mathscr{C}_{\alpha} \hookrightarrow \mathscr{C}$ given by a skeleton of the full sub- \mathscr{V} -category of enriched α -presentable objects in a locally α -presentable \mathscr{V} -category \mathscr{C} over a locally α -presentable closed category \mathscr{V} [20]; in this class of examples, \mathscr{C}_{α} -ary \mathscr{V} -monads are precisely α -ary or α -accessible \mathscr{V} -monads, i.e. \mathscr{V} -monads that preserve conical α -filtered colimits, which correspond to the α -ary variant of the enriched Lawvere theories of [38];
- 2. the subcategory of arities $\mathsf{SF}(\mathscr{V}) \hookrightarrow \mathscr{V}$ consisting of the finite copowers of the unit object in a symmetric monoidal closed π -category [9] (e.g. a complete and cocomplete cartesian closed category); in this class of examples, $\mathsf{SF}(\mathscr{V})$ -ary \mathscr{V} -monads correspond to the enriched algebraic theories of Borceux and Day [9], and, in the case where \mathscr{V} is cartesian closed, $\mathsf{SF}(\mathscr{V})$ -ary \mathscr{V} -monads are precisely the strongly finitary \mathscr{V} -monads of Lack and Kelly [22], since in this case $\mathsf{SF}(\mathscr{V})$ is the free \mathscr{V} -category on the category of finite cardinals (with all maps between them);
- 3. the subcategory of arities $\{I\} \hookrightarrow \mathscr{V}$ consisting of the unit object in a complete and cocomplete symmetric monoidal closed category \mathscr{V} ; in this class of examples, $\{I\}$ -ary \mathscr{V} -monads correspond to monoids in $(\mathscr{V}, \otimes, I)$;

- 4. the subcategory of arities $y_{\mathscr{X}}: \mathscr{X}^{op} \hookrightarrow [\mathscr{X}, \mathscr{V}]$ consisting of the representables, for a small \mathscr{V} -category \mathscr{X} and a complete and cocomplete closed category \mathscr{V} ; in the special case where \mathscr{X} is the discrete \mathscr{V} -category on a set X, $y_{\mathscr{X}}$ -ary \mathscr{V} -monads on $[\mathscr{X}, \mathscr{V}] = \mathscr{V}^X$ correspond to \mathscr{V} -categories with object set X;
- 5. the subcategory of arities $y_{\Phi}: \mathscr{T}^{op} \hookrightarrow \Phi\text{-Mod}(\mathscr{T})$ consisting of the representables in the \mathscr{V} -category of models of a Φ -theory \mathscr{T} , where Φ is a locally small class of small weights satisfying Axiom A of [26] and \mathscr{V} is a locally bounded and \mathscr{E} -cowellpowered closed category; in this class of examples, y_{Φ} -ary \mathscr{V} -monads are precisely the Φ -accessible \mathscr{V} -monads on Φ -Mod(\mathscr{T}) in the sense defined by Lack and Rosický in [26];
- 6. any small and eleutheric subcategory of arities in a locally bounded \mathscr{V} -category \mathscr{C} over a locally bounded closed category \mathscr{V} [34].

In examples 1 through 4, both $\mathscr V$ and $\mathscr C$ carry the (Iso, AII) factorization system, while in examples 5 and 6 both $\mathscr V$ and $\mathscr C$ carry associated proper factorization systems as discussed in [34].

- **2.5.** In the case where $\mathscr{C} = \mathscr{V}$, the subcategory of arities $\mathscr{J} \hookrightarrow \mathscr{V}$ is said to be a **system** of arities if \mathscr{J} is closed under the monoidal product \otimes and contains the unit object I of \mathscr{V} [31, 3.8]. For an eleutheric system of arities $\mathscr{J} \hookrightarrow \mathscr{V}$, it was shown in [31, 11.8] that there is an equivalence $\mathsf{Mnd}_{\mathscr{J}}(\mathscr{V}) \simeq \mathsf{Th}_{\mathscr{J}}$ between \mathscr{J} -ary \mathscr{V} -monads on \mathscr{V} and \mathscr{J} -theories (see [31, 4.1]) that respects semantics, in the sense that if the \mathscr{J} -ary \mathscr{V} -monad \mathbb{T} and the \mathscr{J} -theory \mathscr{T} correspond under this equivalence, then the \mathscr{V} -category \mathscr{T} -Alg of \mathscr{T} -algebras [31, 5.1] is equivalent to \mathbb{T} -Alg, while the \mathscr{V} -category \mathscr{T} -Alg! of normal \mathscr{T} -algebras [31, 5.10] is isomorphic to \mathbb{T} -Alg in \mathscr{V} -CAT/ \mathscr{V} (see [31, 11.14]). The subcategories of arities $\mathsf{SF}(\mathscr{V}) \hookrightarrow \mathscr{V}$ and $\{I\} \hookrightarrow \mathscr{V}$ of 2.4 are systems of arities [31, 3.6, 3.7], as is the subcategory of arities $\mathscr{V}_{\alpha} \hookrightarrow \mathscr{V}$ in a locally α -presentable closed category \mathscr{V} [31, 3.4].
- **2.6.** A \mathscr{J} -signature (in \mathscr{C}) is a \mathscr{V} -functor Σ : ob $\mathscr{J} \to \mathscr{C}$, where we write ob \mathscr{J} to denote the discrete \mathscr{V} -category on ob \mathscr{J} ; hence Σ is just an ob \mathscr{J} -indexed family of objects of \mathscr{C} . We then have the ordinary category $\mathsf{Sig}_{\mathscr{J}}(\mathscr{C}) = \mathscr{V}$ -CAT(ob \mathscr{J},\mathscr{C}) of \mathscr{J} -signatures [35, 7.1]. If Σ is a \mathscr{J} -signature, then a Σ -algebra is an object A of \mathscr{C} equipped with structural morphisms $\alpha_J : \mathscr{C}(J,A) \otimes \Sigma J \to A$ ($J \in \mathsf{ob} \mathscr{J}$), or equivalently $\alpha_J : \mathscr{C}(J,A) \to \mathscr{C}(\Sigma J,A)$, or equivalently $\alpha_J : \Sigma J \to [\mathscr{C}(J,A),A]$, where we denote the tensor and cotensor of each object C of \mathscr{C} by each object X of \mathscr{V} by $X \otimes C$ and [X,C], respectively. We write such a Σ -algebra either as (A,α) or simply as A. Given Σ -algebras $A = (A,\alpha)$ and $B = (B,\beta)$, a Σ -homomorphism $f:A \to B$ is a morphism in \mathscr{C} that makes the following diagram commute for each $J \in \mathsf{ob} \mathscr{J}$:

$$\mathscr{C}(J,A) \otimes \Sigma J \xrightarrow{\mathscr{C}(J,f) \otimes \Sigma J} \mathscr{C}(J,B) \otimes \Sigma J$$

$$\uparrow^{\alpha_J} \qquad \qquad \downarrow^{\beta_J} \qquad \qquad$$

We then have a \mathscr{V} -category Σ -Alg of Σ -algebras with a faithful \mathscr{V} -functor $U^{\Sigma}: \Sigma$ -Alg $\to \mathscr{C}$ (see [35, 7.4]).

The forgetful functor $\mathcal{U}: \mathsf{Mnd}_{\mathscr{I}}(\mathscr{C}) \to \mathsf{Sig}_{\mathscr{I}}(\mathscr{C})$ given by $\mathbb{T} \mapsto (TJ)_{J \in \mathsf{ob} \mathscr{J}}$ has a left adjoint by [35, 7.9], and the free \mathscr{J} -ary \mathscr{V} -monad \mathbb{T}_{Σ} on a \mathscr{J} -signature Σ moreover satisfies \mathbb{T}_{Σ} -Alg $\cong \Sigma$ -Alg in \mathscr{V} -CAT/ \mathscr{C} by [35, 7.9]. The forgetful functor $\mathscr{U}: \mathsf{Mnd}_{\mathscr{J}}(\mathscr{C}) \to \mathsf{Sig}_{\mathscr{J}}(\mathscr{C})$ is also monadic by [35, 8.2].

- **2.7.** If $\mathbb{M}: \mathscr{K} \to \mathsf{Mnd}_{\mathscr{J}}(\mathscr{C})$ is a small diagram of \mathscr{J} -ary \mathscr{V} -monads on \mathscr{C} , then \mathbb{M} has a colimit $\mathbb{T}_{\mathbb{M}}$ that is **algebraic** (see [35, 9.3.8]), meaning that it is sent to a limit by the semantics functor $\mathsf{Alg}: \mathsf{Mnd}_{\mathscr{J}}(\mathscr{C})^{\mathsf{op}} \to \mathscr{V}\text{-}\mathsf{CAT}/\mathscr{C}$ (given on objects by $\mathbb{T} \to \mathbb{T}\text{-}\mathsf{Alg}$). In particular, $\mathsf{Mnd}_{\mathscr{J}}(\mathscr{C})$ has **algebraic coequalizers**, meaning that each parallel pair $\mathbb{T} \rightrightarrows \mathbb{T}'$ of \mathscr{J} -ary \mathscr{V} -monads has a coequalizer in $\mathsf{Mnd}_{\mathscr{J}}(\mathscr{C})$ that is sent to an equalizer by the semantics functor Alg .
- 2.8. Given a \mathscr{J} -ary \mathscr{V} -monad \mathbb{T} on \mathscr{C} , we write simply \mathbb{T} to denote the \mathscr{J} -signature underlying \mathbb{T} (2.6). A system of \mathscr{J} -ary equations over \mathbb{T} (see [35, 10.1.2]) is a parallel pair $E = (t, u : \Gamma \rightrightarrows \mathbb{T})$ of \mathscr{J} -signature morphisms, where Γ is a \mathscr{J} -signature. A system of \mathscr{J} -ary equations is then a pair (\mathbb{T}, E) consisting of a \mathscr{J} -ary \mathscr{V} -monad \mathbb{T} and a system of \mathscr{J} -ary equations E over \mathbb{T} . A \mathscr{J} -presentation is a pair $P = (\Sigma, E)$ consisting of a \mathscr{J} -signature Σ and a system of \mathscr{J} -ary equations $E = (t, u : \Gamma \rightrightarrows \mathbb{T}_{\Sigma})$ over the free \mathscr{J} -ary \mathscr{V} -monad \mathbb{T}_{Σ} on Σ . As we also consider other notions of signature and presentation relative to \mathscr{J} in this paper, we refer to \mathscr{J} -signatures, \mathscr{J} -presentations, and systems of \mathscr{J} -ary equations also as traditional \mathscr{J} -signatures, traditional \mathscr{J} -presentations, and traditional systems of \mathscr{J} -ary equations for clarity.
- If (\mathbb{T}, E) is a system of \mathscr{J} -ary equations, where $E = (t, u : \Gamma \rightrightarrows \mathbb{T})$, then we write \mathbb{T}/E to denote the algebraic coequalizer of the parallel pair t^{\sharp} , $u^{\sharp} : \mathbb{T}_{\Gamma} \rightrightarrows \mathbb{T}$ in $\mathsf{Mnd}_{\mathscr{I}}(\mathscr{C})$ induced by t, u via the adjunction between $\mathsf{Mnd}_{\mathscr{I}}(\mathscr{C})$ and $\mathsf{Sig}_{\mathscr{I}}(\mathscr{C})$, and we call the \mathscr{J} -ary \mathscr{V} -monad \mathbb{T}/E the **quotient of** \mathbb{T} by E [35, 10.1.6]. In particular, if $P = (\Sigma, E)$ is a \mathscr{J} -presentation, then (\mathbb{T}_{Σ}, E) is a system of \mathscr{J} -ary equations, and we call $\mathbb{T}_P := \mathbb{T}_{\Sigma}/E$ the \mathscr{J} -ary \mathscr{V} -monad presented by P. We then say that a given \mathscr{J} -ary \mathscr{V} -monad is presented by P if it is isomorphic to \mathbb{T}_P . Every \mathscr{J} -ary \mathscr{V} -monad \mathbb{T} has a canonical \mathscr{J} -presentation $P_{\mathbb{T}}$ (see [35, 10.1.9]), so that \mathbb{T} -Alg $\cong P_{\mathbb{T}}$ -Alg in \mathscr{V} -CAT/ \mathscr{C} .
- **2.9.** Let $\mathbb{T} = (T, \eta, \mu)$ be a \mathscr{J} -ary \mathscr{V} -monad on \mathscr{C} . Given any object C of \mathscr{C} , there is a \mathscr{J} -ary \mathscr{V} -monad $\langle C, C \rangle$ on \mathscr{C} with $\langle C, C \rangle J = [\mathscr{C}(J, C), C]$ for each $J \in \mathsf{ob} \mathscr{J}$ (see [35, 10.2.9]). Given a \mathbb{T} -algebra (A, a), which we also write simply as A, there is a canonical \mathscr{V} -monad morphism $\mathbf{i}^A : \mathbb{T} \to \langle A, A \rangle$, which we call the \mathscr{J} -ary interpretation morphism for A [35, 10.2.10]. For each $J \in \mathsf{ob} \mathscr{J}$ the component $\mathbf{i}_J^A : TJ \to \langle A, A \rangle J = [\mathscr{C}(J, A), A]$ is the transpose of the composite $\mathscr{C}(J, A) \xrightarrow{T_{JA}} \mathscr{C}(TJ, TA) \xrightarrow{\mathscr{C}(TJ, a)} \mathscr{C}(TJ, A)$ [35, 10.2.1].

Given a (traditional) system of \mathscr{J} -ary equations (\mathbb{T}, E) , where $E = (t, u : \Gamma \Rightarrow \mathbb{T})$, a (\mathbb{T}, E) -algebra is, by definition, a \mathbb{T} -algebra A such that $\mathfrak{i}_J^A \circ t_J = \mathfrak{i}_J^A \circ u_J : \Gamma J \to [\mathscr{C}(J, A), A]$ for each $J \in \mathsf{ob} \mathscr{J}$ (equivalently, such that $\mathfrak{i}^A \circ t^\sharp = \mathfrak{i}^A \circ u^\sharp : \mathbb{T}_\Gamma \to \langle A, A \rangle$ with the notation of 2.8). We let (\mathbb{T}, E) -Alg be the full sub- \mathscr{V} -category of \mathbb{T} -Alg consisting of the (\mathbb{T}, E) -algebras, and we regard (\mathbb{T}, E) -Alg as a \mathscr{V} -category over \mathscr{C} by means of the \mathscr{V} -functor (\mathbb{T}, E) -Alg $\to \mathscr{C}$

obtained as a restriction of $U^{\mathbb{T}}$. Writing \mathbb{T}/E for the quotient of \mathbb{T} by E (2.8), we have an isomorphism (\mathbb{T}/E) -Alg $\cong (\mathbb{T}, E)$ -Alg in \mathscr{V} -CAT/ \mathscr{C} by [35, 10.2.13], under which each (\mathbb{T}/E) -algebra (A, a) corresponds to the (\mathbb{T}, E) -algebra $(A, a \circ q_A^E)$ where $q^E : \mathbb{T} \to \mathbb{T}/E$ is the regular epimorphism that presents \mathbb{T}/E as a quotient.

Given a \mathscr{J} -presentation $P=(\Sigma,E)$, where $E=(t,u:\Gamma\rightrightarrows\mathbb{T}_{\Sigma})$, a P-algebra is a Σ -algebra A such that $\mathfrak{i}_J^A \circ t_J = \mathfrak{i}_J^A \circ u_J$ for all $J \in \mathsf{ob} \mathscr{J}$, where we write $\mathfrak{i}^A : \mathbb{T}_{\Sigma} \to \langle A, A \rangle$ for the interpretation morphism obtained by regarding A equivalently as a \mathbb{T}_{Σ} -algebra via the isomorphism Σ -Alg $\cong \mathbb{T}_{\Sigma}$ -Alg in \mathscr{V} -CAT/ \mathscr{C} (2.6). Hence, a P-algebra is precisely a Σ -algebra whose corresponding \mathbb{T}_{Σ} -algebra is a (\mathbb{T}_{Σ}, E) -algebra for the system of \mathscr{J} -ary equations (\mathbb{T}_{Σ}, E) . We write P-Alg for the full sub- \mathscr{V} -category of Σ -Alg consisting of P-algebras, and we regard P-Alg as a \mathscr{V} -category over \mathscr{C} by means of the \mathscr{V} -functor P-Alg $\to \mathscr{C}$ obtained as a restriction of U^{Σ} . Hence P-Alg $\cong (\mathbb{T}_{\Sigma}, E)$ -Alg $\cong \mathbb{T}_P$ -Alg in \mathscr{V} -CAT/ \mathscr{C} [35, 10.2.14] where $\mathbb{T}_P = \mathbb{T}_{\Sigma}/E$ is the \mathscr{J} -ary \mathscr{V} -monad presented by P (2.8).

3 Parametrized operations and interpretation relative to a \mathcal{V} -adjunction

Everything in the present section applies when \mathscr{C} is an arbitrary tensored and cotensored \mathscr{V} -category, so that the full force of the assumptions in §2 is not required.

Definition 3.1. Given objects $A, J, C \in \mathsf{ob}\,\mathscr{C}$, a **parametrized operation** on A with **arity** J and **parameter object** (or **parameter**) C is a morphism $\omega : \mathscr{C}(J,A) \to \mathscr{C}(C,A)$ in \mathscr{V} . Such a parametrized operation is equivalently given by a morphism $\omega : \mathscr{C}(J,A) \otimes C \to A$ in \mathscr{C} , or by a morphism $\omega : C \to [\mathscr{C}(J,A),A]$ in \mathscr{C} . When $\mathscr{C} = \mathscr{V}$, we may also write $\omega : C \otimes \mathscr{V}(J,A) \to A$.

Later we shall restrict attention to the case where the arity J is an object of the specified subcategory of arities $\mathscr{J} \hookrightarrow \mathscr{C}$, while crucially we shall *not* require the parameter C to be an object of \mathscr{J} . It is for this reason that we deliberately choose to call the object C a parameter rather than an (output) arity. The following generalizes 3.1 (which is recovered in the case where \mathscr{A} is the unit \mathscr{V} -category):

Definition 3.2. Let $G: \mathscr{A} \to \mathscr{C}$ be a \mathscr{V} -functor, so that we may regard \mathscr{A} as a \mathscr{V} -category over \mathscr{C} (2.3). Given objects $J, C \in \mathsf{ob}\,\mathscr{C}$, a **parametrized operation on** G (or **on** \mathscr{A}) with **arity** J and **parameter** C is a \mathscr{V} -natural transformation $\omega: \mathscr{C}(J, G-) \to \mathscr{C}(C, G-)$. A **diagrammatic equation on** G (or **on** \mathscr{A}), written as $\omega \doteq \nu$, is a parallel pair of \mathscr{V} -natural transformations $\omega, \nu: \mathscr{C}(J, G-) \rightrightarrows \mathscr{C}(C, G-)$ for a specified pair of objects J, C of \mathscr{C} ; an object $A \in \mathsf{ob}\,\mathscr{A}$ satisfies the equation $\omega \doteq \nu$ if $\omega_A = \nu_A$.

As a variation on [14, II.3], if $G: \mathscr{A} \to \mathscr{C}$ is a \mathscr{V} -functor then we write $\mathsf{Str}_0(G)$ to denote the (ordinary) category whose objects are those of \mathscr{C} and whose morphisms $\omega: X \to Y$ are the parametrized operations $\omega: \mathscr{C}(X, G-) \to \mathscr{C}(Y, G-)$ with arity X and parameter Y. The following is a consequence of [14, Proposition II.1.5 and p. 81]:

Proposition 3.3. Let $F \dashv G : \mathscr{A} \to \mathscr{C}$ be a \mathscr{V} -adjunction with unit η and counit ε , and let \mathbb{T} be the induced \mathscr{V} -monad on \mathscr{C} . Then the category $\mathsf{Str}_0(G)$ underlies a \mathscr{V} -category $\mathsf{Str}(G)$

(called the \mathcal{V} -structure of G) that is isomorphic to the opposite of the Kleisli \mathcal{V} -category $\mathscr{C}_{\mathbb{T}}$ for \mathbb{T} . Under the latter isomorphism $\mathsf{Str}(G) \cong \mathscr{C}_{\mathbb{T}}^{\mathsf{op}}$, if $t: C \to J$ is a morphism in $\mathscr{C}_{\mathbb{T}}$, given by a morphism $t: C \to TJ$ in \mathscr{C} with $C, J \in \mathsf{ob}\,\mathscr{C}$, then the corresponding morphism in $\mathsf{Str}(G)$ is the \mathscr{V} -natural transformation $\mathscr{C}(J, G-) \to \mathscr{C}(C, G-)$ consisting of the composite morphisms

$$\mathscr{C}(J,GA) \xrightarrow{T_{J,GA}} \mathscr{C}(TJ,TGA) \xrightarrow{\mathscr{C}(t,a_A)} \mathscr{C}(C,GA)$$
 (3.3.i)

associated to objects A of \mathscr{A} , where we write $a_A: TGA \to GA$ for the \mathbb{T} -algebra structure on GA defined as $a_A = G\varepsilon_A$. In the opposite direction, if $\omega: \mathscr{C}(J,G-) \to \mathscr{C}(C,G-)$ is a parametrized operation on G, then the corresponding morphism $C \to J$ in $\mathscr{C}_{\mathbb{T}}$ is the morphism $C \to TJ$ in \mathscr{C}_0 obtained as the composite $I \xrightarrow{\eta_J} \mathscr{C}(J,TJ) \xrightarrow{\omega_{FJ}} \mathscr{C}(C,TJ)$ in \mathscr{V} . \square

Definition 3.4. Given any \mathscr{V} -monad \mathbb{T} on \mathscr{C} , a **parametrized** \mathbb{T} -**term (with arity** J and **parameter** C) is a morphism $t:C\to TJ$ for a pair of objects $J,C\in \mathsf{ob}\mathscr{C}$. A **parametrized** \mathbb{T} -**equation**, denoted by $t\doteq u:C\rightrightarrows TJ$, is a parallel pair $t,u:C\rightrightarrows TJ$ of parametrized \mathbb{T} -terms for a specified pair of objects $J,C\in \mathsf{ob}\mathscr{C}$. \square

Definition 3.5. Let $G: \mathscr{A} \to \mathscr{C}$ be a right adjoint \mathscr{V} -functor, and let \mathbb{T} be the induced \mathscr{V} -monad on \mathscr{C} . Given a parametrized \mathbb{T} -term $t: C \to TJ$ and an object A of \mathscr{A} , we denote the composite morphism in (3.3.i) by

$$[\![t]\!]_A : \mathscr{C}(J, GA) \longrightarrow \mathscr{C}(C, GA)$$
 (3.5.i)

and we call $[\![t]\!]_A$ the **interpretation of** t **in** A. Hence $[\![t]\!]_A$ is a parametrized operation on GA with arity J and parameter C. In view of 3.3, the resulting family of morphisms (3.5.i) is \mathscr{V} -natural in $A \in \mathscr{A}$ and so is a parametrized operation on G that we write as

$$[\![t]\!] \ : \ \mathscr{C}(J,G-) \longrightarrow \mathscr{C}(C,G-)$$

and call simply the **interpretation of** t (in \mathscr{A}). Note that $\llbracket t \rrbracket$ is the morphism in $\mathsf{Str}(G)$ that corresponds to t under the isomorphism $\mathsf{Str}(G) \cong \mathscr{C}^{\mathsf{op}}_{\mathbb{T}}$ of 3.3. Given a parametrized \mathbb{T} -equation $t \doteq u : C \rightrightarrows TJ$, we say that an object A of \mathscr{A} satisfies $t \doteq u$, and we write $A \models t \doteq u$, if $\llbracket t \rrbracket_A = \llbracket u \rrbracket_A : \mathscr{C}(J, GA) \to \mathscr{C}(C, GA)$. In other words, A satisfies $t \doteq u$ iff A satisfies the diagrammatic equation $\llbracket t \rrbracket \doteq \llbracket u \rrbracket : \mathscr{C}(J, G-) \rightrightarrows \mathscr{C}(C, G-)$ consisting of the interpretations of t and u in \mathscr{A} .

Example 3.6. Given a \mathscr{V} -monad \mathbb{T} on \mathscr{C} , we may apply 3.3 and 3.5 to the right adjoint \mathscr{V} -functor $U^{\mathbb{T}}: \mathbb{T}$ -Alg $\to \mathscr{C}$. In particular, given a \mathbb{T} -algebra A=(A,a), we may consider the interpretation $[\![t]\!]_A=\mathscr{C}(t,a)\circ T_{JA}:\mathscr{C}(J,A)\to\mathscr{C}(C,A)$ of a parametrized \mathbb{T} -term $t:C\to TJ$ in A, and we may ask whether A satisfies a given parametrized \mathbb{T} -equation $t\doteq u$, i.e. whether $[\![t]\!]_A=[\![u]\!]_A$.

Proposition 3.7. Let $t \doteq u : C \rightrightarrows TJ$ be a parametrized \mathbb{T} -equation, where $\mathbb{T} = (T, \eta, \mu)$ is a \mathscr{V} -monad on \mathscr{C} . The following are equivalent: (1) every \mathbb{T} -algebra A satisfies $t \doteq u$; (2) the free \mathbb{T} -algebra (TJ, μ_J) satisfies $t \doteq u$; (3) t = u.

¹Fiore and Hur [18] use the terms generalised term and generalised equation to refer to these concepts.

3.8. Let $\lambda : \mathbb{T} \to \mathbb{S}$ be a morphism of \mathscr{V} -monads on \mathscr{C} , and let A = (A, a) be an \mathbb{S} -algebra. Then $(A, a \circ \lambda_A)$ is a \mathbb{T} -algebra, which we also denote simply by A, by abuse of notation. In particular, if $t : C \to TJ$ is a parametrized \mathbb{T} -term then we write $[\![t]\!]_A$ to denote the interpretation of t in the \mathbb{T} -algebra $(A, a \circ \lambda_A)$, and if $t \doteq u$ is a parametrized \mathbb{T} -equation

Proof. (1) holds iff [t] = [u], and the result now readily follows from 3.3.

then we say that that A satisfies $t \doteq u$ if $(A, a \circ \lambda_A)$ satisfies $t \doteq u$.

Proposition 3.9. Let $\lambda : \mathbb{T} \to \mathbb{S}$ be a morphism of \mathscr{V} -monads on \mathscr{C} , and let A be an \mathbb{S} -algebra. With the notation of 3.8, $[\![t]\!]_A = [\![\lambda_J \circ t]\!]_A$ for each parametrized \mathbb{T} -term $t : C \to TJ$. Hence, with the terminology of 3.8, A satisfies a parametrized \mathbb{T} -equation $t \doteq u : C \rightrightarrows TJ$ if and only if A satisfies the parametrized \mathbb{S} -equation $\lambda_J \circ t \doteq \lambda_J \circ u : C \rightrightarrows SJ$.

Proof. For the first claim, we may use 3.6 and the \mathscr{V} -naturality of λ to compute that $\llbracket t \rrbracket_A = \mathscr{C}(t, a \circ \lambda_A) \circ T_{JA} = \mathscr{C}(t, a) \circ \mathscr{C}(TJ, \lambda_A) \circ T_{JA} = \mathscr{C}(t, a) \circ \mathscr{C}(\lambda_J, SA) \circ S_{JA} = \mathscr{C}(\lambda_J \circ t, a) \circ S_{JA} = \llbracket \lambda_J \circ t \rrbracket_A$. The second claim follows from the first.

Corollary 3.10. Let $\lambda : \mathbb{T} \to \mathbb{S}$ be a morphism of \mathcal{V} -monads on \mathcal{C} , and let $t \doteq u : C \Rightarrow TJ$ be a parametrized \mathbb{T} -equation. Then the following are equivalent, where we employ the terminology of 3.8: (1) every \mathbb{S} -algebra A satisfies $t \doteq u$; (2) the free \mathbb{S} -algebra on J satisfies $t \doteq u$; (3) $\lambda_J \circ t = \lambda_J \circ u$.

Proof. This follows immediately from 3.9 and 3.7.

4 Systems of diagrammatic equations on \mathscr{J} -monadic \mathscr{V} -categories over \mathscr{C}

4.1. A parametrized \mathcal{J} -ary operation is a parametrized operation whose arity J is an object of the given subcategory of arities \mathcal{J} . Moreover, we add the word \mathcal{J} -ary to the various terms in 3.1, 3.2, 3.4 to signify that the arity J is an object of \mathcal{J} , thus arriving at the notions of diagrammatic \mathcal{J} -ary equation, parametrized \mathcal{J} -ary \mathbb{T} -term, and parametrized \mathcal{J} -ary \mathbb{T} -equation.

Definition 4.2. Let $G: \mathscr{A} \to \mathscr{C}$ be a strictly \mathscr{J} -monadic \mathscr{V} -functor, so that \mathscr{A} may be regarded as a strictly \mathscr{J} -monadic \mathscr{V} -category over \mathscr{C} . A **system of diagrammatic** \mathscr{J} -ary **equations on** G (or **on** \mathscr{A}) is a small family $\mathsf{E} = (\omega_{\delta} \doteq \nu_{\delta})_{\delta \in \mathcal{D}}$ of diagrammatic \mathscr{J} -ary equations $\omega_{\delta}, \nu_{\delta} : \mathscr{C}(J_{\delta}, G^{-}) \rightrightarrows \mathscr{C}(C_{\delta}, G^{-})$ on G, indexed by a small set \mathcal{D} . An **E-model** (or a **model of E**) is then an object A of \mathscr{A} that satisfies every equation $\omega_{\delta} \doteq \nu_{\delta}$ ($\delta \in \mathcal{D}$). We write \mathscr{A}_{E} for the full sub- \mathscr{V} -category of \mathscr{A} consisting of the E-models, and we regard \mathscr{A}_{E} as a \mathscr{V} -category over \mathscr{C} by restricting G to \mathscr{A}_{E} .

We shall show in 4.9 that \mathscr{A}_{E} is strictly \mathscr{J} -monadic over \mathscr{C} in the situation of 4.2. But in order to do this, we first consider the monad-based counterpart of the concept in 4.2:

Definition 4.3. Given a \mathscr{J} -ary \mathscr{V} -monad \mathbb{T} , a system of parametrized \mathscr{J} -ary equations over \mathbb{T} is a small family $\mathsf{E} = (t_\delta \doteq u_\delta)_{\delta \in \mathcal{D}}$ of parametrized \mathscr{J} -ary \mathbb{T} -equations

- $t_{\delta} \doteq u_{\delta} : C_{\delta} \rightrightarrows TJ_{\delta}$ (indexed by a small set \mathcal{D}). A system of parametrized \mathscr{J} -ary equations is a pair (\mathbb{T}, E) consisting of a \mathscr{J} -ary \mathscr{V} -monad \mathbb{T} and a system of parametrized \mathscr{J} -ary equations E over \mathbb{T} . A (\mathbb{T}, E) -algebra is a \mathbb{T} -algebra A that satisfies every equation in E . We write (\mathbb{T}, E) -Alg to denote the full sub- \mathscr{V} -category of \mathbb{T} -Alg consisting of the (\mathbb{T}, E) -algebras, and we regard (\mathbb{T}, E) -Alg as a \mathscr{V} -category over \mathscr{E} by means of the \mathscr{V} -functor (\mathbb{T}, E) -Alg $\to \mathscr{E}$ obtained as a restriction of $U^{\mathbb{T}}$.
- **4.4.** Let $G: \mathscr{A} \to \mathscr{C}$ be a strictly \mathscr{J} -monadic \mathscr{V} -functor, so that \mathscr{A} is a strictly \mathscr{J} -monadic \mathscr{V} -category over \mathscr{C} , and let \mathbb{T} be the induced \mathscr{J} -ary \mathscr{V} -monad on \mathscr{C} . In view of 3.3, systems of diagrammatic \mathscr{J} -ary equations E on \mathscr{A} are in bijective correspondence with systems of parametrized \mathscr{J} -ary equations E over \mathbb{T} , so that we may identify these concepts. With this identification, $\mathscr{A}_{\mathsf{E}} \cong (\mathbb{T}, \mathsf{E})$ -Alg in \mathscr{V} -CAT/ \mathscr{C} , because $\mathscr{A} \cong \mathbb{T}$ -Alg and it is immediate from 3.3 and 3.6 that an object A of \mathscr{A} satisfies a given diagrammatic equation $\omega \doteq \nu$ on \mathscr{A} iff the associated \mathbb{T} -algebra (GA, a_A) satisfies the corresponding parametrized \mathbb{T} -equation, with the notation of 3.3. Hence, in order to show that \mathscr{A}_{E} is strictly \mathscr{J} -monadic over \mathscr{C} , it suffices to show that (\mathbb{T}, E) -Alg is so, and for this we shall need the following material. \square
- **4.5.** Given a \mathscr{J} -ary \mathscr{V} -monad \mathbb{T} on \mathscr{C} and an object J of \mathscr{J} , the identity morphism 1_{TJ} may be regarded as a parametrized \mathscr{J} -ary \mathbb{T} -term with arity J and parameter TJ, so for each \mathbb{T} -algebra A=(A,a) we obtain a parametrized \mathscr{J} -ary operation $[\![1_{TJ}]\!]_A=\mathscr{C}(TJ,a)\circ T_{JA}:\mathscr{C}(J,A)\to\mathscr{C}(TJ,A)$, which may be described also as the transpose of the interpretation morphism $\mathfrak{i}_J^A:TJ\to[\mathscr{C}(J,A),A]$ of 2.9. Given any parametrized \mathscr{J} -ary \mathbb{T} -term $t:C\to TJ$, the interpretation $[\![t]\!]_A=\mathscr{C}(t,a)\circ T_{JA}:\mathscr{C}(J,A)\to\mathscr{C}(C,A)$ is therefore the transpose of the composite $\mathfrak{i}_J^A\circ t:C\to[\mathscr{C}(J,A),A]$. Hence, given any parametrized \mathscr{J} -ary \mathbb{T} -equation $t\doteq u:C\rightrightarrows TJ$, $A\vDash t\doteq u$ iff $\mathfrak{i}_J^A\circ t=\mathfrak{i}_J^A\circ u$. It follows that if $(f_\lambda:C_\lambda\to C)_{\lambda\in\Lambda}$ is a jointly epimorphic family of morphisms in \mathscr{C} , then $A\vDash t\doteq u$ iff $A\vDash t\circ f_\lambda\doteq u\circ f_\lambda$ for all $\lambda\in\Lambda$. \square
- **4.6.** Given a traditional system of \mathscr{J} -ary equations $E = (t, u : \Gamma \rightrightarrows \mathbb{T})$ over a \mathscr{J} -ary \mathscr{V} -monad \mathbb{T} (2.8), we may regard E as a system of parametrized \mathscr{J} -ary equations $\mathsf{E} = (t_J \doteq u_J : \Gamma J \rightrightarrows TJ)_{J \in \mathsf{ob}}$ over \mathbb{T} , and it follows immediately from 4.5 that (\mathbb{T}, E) -Alg (\mathbb{T}, E) -Alg as \mathscr{V} -categories over \mathscr{C} . In the other direction, every system of parametrized \mathscr{J} -ary equations determines a traditional system of \mathscr{J} -ary equations, as follows:
- **Proposition 4.7.** Let $\mathsf{E} = (t_\delta \doteq u_\delta : C_\delta \rightrightarrows TJ_\delta)_{\delta \in \mathcal{D}}$ be a system of parametrized \mathscr{J} -ary equations over a \mathscr{J} -ary \mathscr{V} -monad \mathbb{T} . Then there is a traditional system of \mathscr{J} -ary equations $E = (\tilde{t}, \tilde{u} : \Gamma \rightrightarrows \mathbb{T})$ over \mathbb{T} such that (\mathbb{T}, E) -Alg = (\mathbb{T}, E) -Alg as \mathscr{V} -categories over \mathscr{C} .

Proof. Define a traditional \mathscr{J} -signature Γ by declaring that $\Gamma J = \coprod_{\delta \in \mathcal{D}, J_{\delta} = J} C_{\delta}$ for each $J \in \mathsf{ob} \mathscr{J}$, and define $\tilde{t}_J, \tilde{u}_J : \Gamma J \Rightarrow TJ$ to be the unique morphisms such that $\tilde{t}_J \circ \iota_{\delta} = t_{\delta}$ and $\tilde{u}_J \circ \iota_{\delta} = u_{\delta}$ for all $\delta \in \mathcal{D}$ with $J_{\delta} = J$, where $\iota_{\delta} : C_{\delta} \to \Gamma J$ is the coproduct insertion. The result now follows readily in view of 4.5, since the ι_{δ} are jointly epimorphic. \square

Theorem 4.8. Let (\mathbb{T}, E) be a system of parametrized \mathscr{J} -ary equations. Then there is a \mathscr{J} -ary \mathscr{V} -monad \mathbb{T}/E such that (\mathbb{T}/E) -Alg $\cong (\mathbb{T}, \mathsf{E})$ -Alg in \mathscr{V} -CAT/ \mathscr{C} .

Proof. Taking E as in 4.7 and letting $\mathbb{T}/\mathsf{E} := \mathbb{T}/E$ with the notation of 2.9, this follows from 4.7 and 2.9.

Theorem 4.9. Let E be a system of diagrammatic \mathscr{J} -ary equations on a strictly \mathscr{J} -monadic \mathscr{V} -category \mathscr{A} over \mathscr{C} . Then \mathscr{A}_E is a strictly \mathscr{J} -monadic \mathscr{V} -category over \mathscr{C} .

Proof. This follows from 4.8, by the discussion in 4.4.

Remark 4.10. In the situation of 4.9, if we write \mathbb{T} for the \mathscr{J} -ary \mathscr{V} -monad induced by \mathscr{A} (i.e., by the associated \mathscr{V} -adjunction $F \dashv G$, 2.3), then the \mathscr{J} -ary \mathscr{V} -monad induced by \mathscr{A}_{E} is \mathbb{T}/E (in view of 2.3, 4.4, and the proof of 4.9), and there is an associated regular epimorphism $q: \mathbb{T} \to \mathbb{T}/\mathsf{E}$ in $\mathsf{Mnd}_{\mathscr{I}}(\mathscr{C})$ (by 2.9 and the proof of 4.8).

5 Free-form and diagrammatic *f*-presentations

We now study objects equipped with parametrized operations whose arities and parameters are specified by a *signature* of the following kind:

Definition 5.1. A free-form \mathscr{J} -signature is a small set \mathcal{S} equipped with an assignment to each $\sigma \in \mathcal{S}$ a pair of objects J_{σ}, C_{σ} with $J_{\sigma} \in \mathsf{ob} \mathscr{J}$ and $C_{\sigma} \in \mathsf{ob} \mathscr{C}$. We call each element $\sigma \in \mathcal{S}$ an **operation symbol** with **arity** J_{σ} and **parameter** C_{σ} . We denote such a free-form \mathscr{J} -signature simply by \mathcal{S} , and we then call \mathcal{S} a free-form \mathscr{J} -signature with arities J_{σ} and parameters C_{σ} ($\sigma \in \mathcal{S}$).

Definition 5.2. Given a free-form \mathscr{J} -signature \mathscr{S} , with arities J_{σ} and parameters C_{σ} ($\sigma \in \mathscr{S}$), an \mathscr{S} -algebra is an object A of \mathscr{C} equipped with parametrized operations σ^A : $\mathscr{C}(J_{\sigma},A) \to \mathscr{C}(C_{\sigma},A)$ for all the operation symbols $\sigma \in \mathscr{S}$. An \mathscr{S} -algebra is equivalently an object A of \mathscr{C} with a family of morphisms $\sigma^A : \mathscr{C}(J_{\sigma},A) \otimes C_{\sigma} \to A$ in \mathscr{C} ($\sigma \in \mathscr{S}$), or equivalently, a family of morphisms $\sigma^A : C_{\sigma} \to [\mathscr{C}(J_{\sigma},A),A]$ in \mathscr{C} . In the special case where $\mathscr{C} = \mathscr{V}$, we may also write $\sigma^A : C_{\sigma} \otimes \mathscr{V}(J_{\sigma},A) \to A$. Given \mathscr{S} -algebras A and B, an \mathscr{S} -homomorphism $f : A \to B$ is a morphism in \mathscr{C} that makes the following diagram commute for each $\sigma \in \mathscr{S}$:

$$\mathcal{C}(J_{\sigma}, A) \xrightarrow{\sigma^{A}} \mathcal{C}(C_{\sigma}, A)$$

$$\mathcal{C}(J_{\sigma}, f) \downarrow \qquad \qquad \downarrow \mathcal{C}(C_{\sigma}, f)$$

$$\mathcal{C}(J_{\sigma}, B) \xrightarrow{\sigma^{B}} \mathcal{C}(C_{\sigma}, B).$$

We thus have an ordinary category S-Alg₀ of S-algebras and S-homomorphisms, which underlies a \mathscr{V} -category S-Alg in which each hom-object S-Alg(A, B) is defined as the *pairwise equalizer* [33, 2.1] of the following S-indexed family of parallel pairs in \mathscr{V} ($\sigma \in S$):

$$\mathscr{C}(A,B) \xrightarrow{\mathscr{C}(J_{\sigma},-)_{AB}} \mathscr{V}(\mathscr{C}(J_{\sigma},A),\mathscr{C}(J_{\sigma},B)) \xrightarrow{\mathscr{V}(1,\sigma^B)} \mathscr{V}(\mathscr{C}(J_{\sigma},A),\mathscr{C}(C_{\sigma},B)),$$

$$\mathscr{C}(A,B) \xrightarrow{\mathscr{C}(C_{\sigma},-)_{AB}} \mathscr{V}(\mathscr{C}(C_{\sigma},A),\mathscr{C}(C_{\sigma},B)) \xrightarrow{\mathscr{V}(\sigma^{A},1)} \mathscr{V}(\mathscr{C}(J_{\sigma},A),\mathscr{C}(C_{\sigma},B)) .$$

Composition in S-Alg is defined in the unique way that enables the resulting subobjects $U_{AB}^{S}: S$ -Alg $(A, B) \hookrightarrow \mathscr{C}(A, B)$ $(A, B \in S$ -Alg) to serve as the structural morphisms of a faithful \mathscr{V} -functor $U^{S}: S$ -Alg $\to \mathscr{C}$ that sends each S-algebra A to its carrier object $A \in \mathsf{ob}\mathscr{C}$.

Definition 5.3. Let \mathcal{S} be a free-form \mathscr{J} -signature. Given a \mathscr{V} -category \mathscr{X} , an \mathscr{X} -parametrized \mathcal{S} -algebra is a \mathscr{V} -functor $A: \mathscr{X} \to \mathscr{C}$ equipped with parametrized operations $\sigma^A: \mathscr{C}(J_{\sigma}, A-) \to \mathscr{C}(C_{\sigma}, A-)$ (3.2) for all $\sigma \in \mathcal{S}$. Writing \mathcal{S} -Alg[\mathscr{X}] to denote the (large) set of all \mathscr{X} -parametrized \mathcal{S} -algebras, we obtain an evident functor \mathcal{S} -Alg[-]: \mathscr{V} -CAT^{op} \to SET given on objects by $\mathscr{X} \mapsto \mathcal{S}$ -Alg[\mathscr{X}].

5.4. Let S be a free-form \mathscr{J} -signature. It readily follows from the definition of U^S : S-Alg $\to \mathscr{C}$ (5.2) that for each operation symbol $\sigma \in S$, the parametrized operations $\sigma^A: \mathscr{C}(J_{\sigma},A) \to \mathscr{C}(C_{\sigma},A)$ for S-algebras A constitute a parametrized operation $\sigma^{U^S}: \mathscr{C}(J_{\sigma},U^S-) \to \mathscr{C}(C_{\sigma},U^S-)$ on the \mathscr{V} -functor U^S . Thus, the \mathscr{V} -functor $U^S: S$ -Alg $\to \mathscr{C}$ carries the structure of an S-Alg-parametrized S-algebra. It is now straightforward to verify the following result, which characterizes S-Alg uniquely up to isomorphism:

Proposition 5.5. Let S be a free-form \mathscr{J} -signature. Then S-Alg is a representing object for the functor S-Alg[-]: \mathscr{V} -CAT $^{op} \to SET$, and the counit of this representation (in the sense of $[21, \S1.10]$) is U^S : S-Alg $\to \mathscr{C}$. Thus, there is a bijective correspondence, natural in $\mathscr{X} \in \mathscr{V}$ -CAT, between \mathscr{V} -functors $\mathscr{X} \to S$ -Alg and \mathscr{X} -parametrized S-algebras. \square

5.6. Every traditional \mathscr{J} -signature Σ (2.8) determines a (rather special) free-form \mathscr{J} signature \mathscr{S}_{Σ} that is defined by declaring that the set of operation symbols underlying \mathscr{S}_{Σ} is ob \mathscr{J} and that each operation symbol $J \in \mathsf{ob} \mathscr{J}$ has arity J and parameter ΣJ . With
this notation, it is straightforward to verify that Σ -Alg $\cong \mathscr{S}_{\Sigma}$ -Alg in \mathscr{V} -CAT/ \mathscr{C} . In fact,
if we formulate Σ -algebras equivalently in terms of structure morphisms of the form α_J : $\mathscr{C}(J,A) \to \mathscr{C}(\Sigma J,A)$ as mentioned in 2.9, then in fact Σ -Alg $= \mathscr{S}_{\Sigma}$ -Alg as \mathscr{V} -categories over \mathscr{C} .

5.7. In the opposite direction from 5.6, given a free-form \mathscr{J} -signature \mathscr{S} with arities J_{σ} and parameters C_{σ} ($\sigma \in \mathscr{S}$), we can construct a traditional \mathscr{J} -signature $\Sigma_{\mathscr{S}}$: ob $\mathscr{J} \to \mathscr{C}$ as follows: for each $J \in \mathsf{ob} \mathscr{J}$, we set $\Sigma_{\mathscr{S}} J := \coprod_{\sigma \in \mathscr{S}, J_{\sigma} = J} C_{\sigma}$. It then follows straightforwardly that $\Sigma_{\mathscr{S}}$ -algebras (in the sense of 2.6) are in bijective correspondence with \mathscr{S} -algebras, and that we have an isomorphism of \mathscr{V} -categories \mathscr{S} -Alg $\cong \Sigma_{\mathscr{S}}$ -Alg in \mathscr{V} -CAT/ \mathscr{C} . To prove the latter claim, note that $\Sigma_{\mathscr{S}}$ -Alg $= \mathscr{S}_{\Sigma_{\mathscr{S}}}$ -Alg by 5.6, so that by 5.5 it suffices to show that we have a bijective correspondence, natural in $\mathscr{X} \in \mathscr{V}$ -CAT, between \mathscr{X} -parametrized \mathscr{S} -algebras and \mathscr{X} -parametrized $\mathscr{S}_{\Sigma_{\mathscr{S}}}$ -algebras can reasonably be called \mathscr{X} -parametrized $\Sigma_{\mathscr{S}}$ -algebras.) \square

Theorem 5.8. Given a free-form \mathcal{J} -signature \mathcal{S} , the \mathcal{V} -category of \mathcal{S} -algebras \mathcal{S} -Alg is strictly \mathcal{J} -monadic over \mathcal{C} , i.e., there is a \mathcal{J} -ary \mathcal{V} -monad $\mathbb{T}_{\mathcal{S}}$ on \mathcal{C} such that \mathcal{S} -Alg $\cong \mathbb{T}_{\mathcal{S}}$ -Alg in \mathcal{V} -CAT/ \mathcal{C} .

Proof. Letting $\mathbb{T}_{\mathcal{S}} := \mathbb{T}_{\Sigma_{\mathcal{S}}}$ be the free \mathscr{J} -ary \mathscr{V} -monad on the traditional \mathscr{J} -signature $\Sigma_{\mathcal{S}}$ (2.6, 5.7), we find that \mathcal{S} -Alg $\cong \Sigma_{\mathcal{S}}$ -Alg $\cong \mathbb{T}_{\mathcal{S}}$ -Alg in \mathscr{V} -CAT/ \mathscr{C} by 5.7 and 2.6.

Definition 5.9. We call the \mathscr{V} -monad $\mathbb{T}_{\mathcal{S}} = \mathbb{T}_{\Sigma_{\mathcal{S}}}$ in 5.8 the \mathscr{J} -ary \mathscr{V} -monad generated by \mathcal{S} .

Definition 5.10. Let S be a free-form \mathscr{J} -signature. A \mathscr{V} -natural S-operation is a parametrized \mathscr{J} -ary operation on the \mathscr{V} -functor $U^S: S$ -Alg $\to \mathscr{C}$. Hence, a \mathscr{V} -natural S-operation ω with arity $J \in \mathsf{ob} \mathscr{J}$ and parameter $C \in \mathsf{ob} \mathscr{C}$ is precisely a family of morphisms

$$\omega_A: \mathscr{C}(J,A) \longrightarrow \mathscr{C}(C,A) \text{ in } \mathscr{V} \qquad (A \in \mathcal{S}\text{-Alg})$$

that is \mathscr{V} -natural in $A \in \mathcal{S}$ -Alg, where we write simply A to denote $U^{\mathcal{S}}A$. A **diagrammatic** \mathcal{S} -equation, denoted by $\omega \doteq \nu$, is a diagrammatic \mathscr{J} -ary equation on $U^{\mathcal{S}}$, i.e. a parallel pair of \mathscr{V} -natural \mathcal{S} -operations

$$\omega_A, \nu_A : \mathscr{C}(J, A) \rightrightarrows \mathscr{C}(C, A) \qquad (A \in \mathcal{S}\text{-Alg})$$

with a specified arity $J \in \mathsf{ob} \mathscr{J}$ and a specified parameter $C \in \mathsf{ob} \mathscr{C}$. In view of 3.1, a diagrammatic \mathcal{S} -equation may be written equivalently in the form

$$\omega_A, \nu_A : \mathscr{C}(J, A) \otimes C \rightrightarrows A \qquad (A \in \mathcal{S}\text{-Alg}).$$

With the terminology of 3.2, an S-algebra A satisfies the diagrammatic S-equation $\omega \doteq \nu$ if $\omega_A = \nu_A$. A system of diagrammatic S-equations is a system of diagrammatic \mathscr{J} -ary equations on U^S , i.e., a small family $\mathsf{E} = (\omega_\delta \doteq \nu_\delta)_{\delta \in \mathcal{D}}$ of diagrammatic S-equations $\omega_\delta \doteq \nu_\delta : \mathscr{C}(J_\delta, U^S -) \rightrightarrows \mathscr{C}(C_\delta, U^S -)$, indexed by a small set \mathcal{D} . The index set \mathcal{D} then carries the structure of a free-form \mathscr{J} -signature with arities J_δ and parameters C_δ ($\delta \in \mathcal{D}$).

5.11. By 5.4, if S is a free-form \mathscr{J} -signature, then each operation symbol $\sigma \in S$ determines a \mathscr{V} -natural S-operation $\sigma^A : \mathscr{C}(J_{\sigma}, A) \to \mathscr{C}(C_{\sigma}, A)$ $(A \in S$ -Alg). We shall use this fact very often in the examples of §6.

Definition 5.12. A diagrammatic \mathscr{J} -presentation is a pair $\mathcal{P} = (\mathcal{S}, \mathsf{E})$ consisting of a free-form \mathscr{J} -signature \mathcal{S} and a system of diagrammatic \mathcal{S} -equations E . A \mathcal{P} -algebra is an \mathcal{S} -algebra \mathcal{A} that satisfies every equation in E .

Definition 5.13. Let S be a free-form \mathscr{J} -signature. A **parametrized** S-term is a parametrized \mathscr{J} -ary \mathbb{T}_S -term $t: C \to T_S J$, where \mathbb{T}_S is the \mathscr{J} -ary \mathscr{V} -monad generated by S. A **parametrized** S-equation is a parametrized \mathscr{J} -ary \mathbb{T}_S -equation $t \doteq u$, i.e. a parallel pair $t, u: C \rightrightarrows T_S J$ with $J \in \mathsf{ob} \mathscr{J}$ and $C \in \mathsf{ob} \mathscr{C}$.

5.14. Applying 3.5 to the \mathscr{V} -functor $U^{\mathcal{S}}: \mathcal{S}$ -Alg $\to \mathscr{C}$, we may consider the interpretation $\llbracket t \rrbracket_A : \mathscr{C}(J,A) \to \mathscr{C}(C,A)$ of a parametrized \mathcal{S} -term $t: C \to T_{\mathcal{S}}J$ in an \mathcal{S} -algebra A, and we may ask whether A satisfies a parametrized \mathcal{S} -equation $t \doteq u: C \rightrightarrows T_{\mathcal{S}}J$, in which case we write $A \models t \doteq u$.

Definition 5.15. A free-form \mathscr{J} -presentation is a pair $\mathscr{P} = (\mathcal{S}, \mathsf{E})$ consisting of a free-form \mathscr{J} -signature \mathscr{S} and a system of parametrized \mathscr{J} -ary equations E over the \mathscr{J} -ary \mathscr{V} -monad $\mathbb{T}_{\mathcal{S}}$ generated by \mathscr{S} (i.e., a small family of parametrized \mathscr{S} -equations). A \mathscr{P} -algebra is an \mathscr{S} -algebra \mathscr{A} that satisfies every equation in E .

Remark 5.16. By 4.4, there is a bijection between diagrammatic \mathcal{J} -presentations and free-form \mathcal{J} -presentations, so that we may identify these concepts.

Definition 5.17. Given a free-form or diagrammatic \mathscr{J} -presentation $\mathcal{P} = (\mathcal{S}, \mathsf{E})$, we write \mathcal{P} -Alg to denote the full sub- \mathscr{V} -category of \mathcal{S} -Alg consisting of the \mathcal{P} -algebras, and we regard \mathcal{P} -Alg as a \mathscr{V} -category over \mathscr{C} by means of the \mathscr{V} -functor $U^{\mathcal{P}}: \mathcal{P}$ -Alg $\to \mathscr{C}$ obtained as a restriction of $U^{\mathcal{S}}$.

Remark 5.18. The notions of free-form \mathscr{J} -signature \mathscr{S} and diagrammatic \mathscr{J} -presentation \mathscr{P} can be defined relative to an arbitrary full sub- \mathscr{V} -category $\mathscr{J} \hookrightarrow \mathscr{C}$, without the assumptions in 2.1 and 2.2, as can the notion of \mathscr{P} -algebra and the \mathscr{V} -category of \mathscr{P} -algebras, \mathscr{P} -Alg. We shall make use of this observation in §8.

Remark 5.19. Given a diagrammatic \mathscr{J} -presentation $\mathcal{P}=(\mathcal{S},\mathsf{E}),\,\mathcal{P}$ -algebras are precisely E-models for the system of diagrammatic \mathscr{J} -ary equations E on \mathcal{S} -Alg, and moreover \mathcal{P} -Alg = \mathcal{S} -Alg_E as \mathscr{V} -categories over \mathscr{C} , so Theorems 4.9 and 5.8 (together with 4.10) entail the following:

Theorem 5.20. Let $\mathcal{P} = (\mathcal{S}, \mathsf{E})$ be a free-form or diagrammatic \mathcal{J} -presentation. Then the \mathcal{V} -category of \mathcal{P} -algebras \mathcal{P} -Alg is strictly \mathcal{J} -monadic over \mathcal{C} , i.e., there is a \mathcal{J} -ary \mathcal{V} -monad $\mathbb{T}_{\mathcal{P}}$ such that $\mathbb{T}_{\mathcal{P}}$ -Alg $\cong \mathcal{P}$ -Alg in \mathcal{V} -CAT/ \mathcal{C} . Explicitly, $\mathbb{T}_{\mathcal{P}} = \mathbb{T}_{\mathcal{S}}/\mathsf{E}$.

Definition 5.21. Given a free-form or diagrammatic \mathscr{J} -presentation \mathscr{P} , we call the \mathscr{V} -monad $\mathbb{T}_{\mathscr{P}} = \mathbb{T}_{\mathscr{S}}/\mathsf{E}$ in 5.20 the \mathscr{J} -ary \mathscr{V} -monad presented by \mathscr{P} .

Definition 5.22. Let $\mathcal{P} = (\mathcal{S}, \mathsf{E})$ be a diagrammatic \mathscr{J} -presentation and \mathscr{X} a \mathscr{V} -category. An \mathscr{X} -parametrized \mathscr{P} -algebra is an \mathscr{X} -parametrized \mathscr{S} -algebra $A: \mathscr{X} \to \mathscr{C}$ whose corresponding \mathscr{V} -functor $\underline{A}: \mathscr{X} \to \mathscr{S}$ -Alg (5.5) factors through \mathscr{P} -Alg $\hookrightarrow \mathscr{S}$ -Alg (equivalently, has the property that the \mathscr{S} -algebra $\underline{A}X$ is a \mathscr{P} -algebra for each object X of \mathscr{X}). In particular, by 5.5, the \mathscr{V} -functor $U^{\mathscr{P}}: \mathscr{P}$ -Alg $\to \mathscr{C}$ (5.17) carries the structure of a \mathscr{P} -Alg-parametrized \mathscr{P} -algebra that has the following universal property, which characterizes \mathscr{P} -Alg uniquely up to isomorphism:

Proposition 5.23. Let \mathcal{P} be a diagrammatic \mathscr{J} -presentation. Then \mathcal{P} -Alg is a representing object for the functor \mathcal{P} -Alg $[-]: \mathscr{V}$ -CAT^{op} \to SET that sends each \mathscr{V} -category \mathscr{X} to the (large) set \mathcal{P} -Alg $[\mathscr{X}]$ of all \mathscr{X} -parametrized \mathcal{P} -algebras. The counit of this representation is $U^{\mathcal{P}}: \mathcal{P}$ -Alg $\to \mathscr{C}$. Thus, there is a bijective correspondence, natural in $\mathscr{X} \in \mathscr{V}$ -CAT, between \mathscr{V} -functors $\mathscr{X} \to \mathcal{P}$ -Alg and \mathscr{X} -parametrized \mathcal{P} -algebras.

Proposition 5.24. Let \mathscr{A} be a strictly \mathscr{J} -monadic \mathscr{V} -category over \mathscr{C} . Then $\mathscr{A} \cong \mathcal{P}$ -Alg in \mathscr{V} -CAT/ \mathscr{C} for some diagrammatic (or, equivalently, free-form) \mathscr{J} -presentation \mathscr{P} .

Proof. By 2.8, there is a traditional \mathscr{J} -presentation $P = (\Sigma, E)$ with $\mathscr{A} \cong P$ -Alg in \mathscr{V} -CAT/ \mathscr{C} . Here, E is a traditional system of \mathscr{J} -ary equations over the \mathscr{V} -monad \mathbb{T}_{Σ} associated to the strictly \mathscr{J} -monadic \mathscr{V} -category Σ -Alg over \mathscr{C} (2.6, 2.8). By 4.6, E may be regarded as a system of parametrized \mathscr{J} -ary equations E over \mathbb{T}_{Σ} and so, equivalently, as a system of diagrammatic \mathscr{J} -ary equations on Σ -Alg (4.4). But Σ -Alg = \mathscr{S}_{Σ} -Alg as \mathscr{V} -categories over \mathscr{C} , by 5.6, and the result follows by taking $\mathscr{P} = (\mathscr{S}_{\Sigma}, \mathsf{E})$.

Definition 5.25. A \mathcal{V} -category over \mathscr{C} is a \mathscr{J} -ary variety if it is of the form \mathscr{P} -Alg for some diagrammatic (or free-form) \mathscr{J} -presentation \mathscr{P} . The (ordinary) category of \mathscr{J} -ary varieties $\mathsf{Var}_{\mathscr{J}}(\mathscr{C})$ is the full subcategory of \mathscr{V} -CAT/ \mathscr{C} whose objects are the \mathscr{J} -ary varieties. \square

Theorem 5.26. The category $Var_{\mathscr{J}}(\mathscr{C})$ of \mathscr{J} -ary varieties is dually equivalent to the category of \mathscr{J} -ary \mathscr{V} -monads on \mathscr{C} :

$$\mathsf{Var}_{\mathscr{I}}(\mathscr{C}) \simeq \mathsf{Mnd}_{\mathscr{I}}(\mathscr{C})^{\mathsf{op}}.$$

Proof. By 5.20 and 5.24, the repletion of $\mathsf{Var}_{\mathscr{J}}(\mathscr{C})$ in $\mathscr{V}\text{-}\mathsf{CAT}/\mathscr{C}$ is the full subcategory $\mathscr{J}\text{-}\mathsf{Monadic}^!$ of $\mathscr{V}\text{-}\mathsf{CAT}/\mathscr{C}$ consisting of the strictly $\mathscr{J}\text{-}\mathsf{monadic}\,\mathscr{V}\text{-}\mathsf{categories}$ over \mathscr{C} (2.3). Hence $\mathsf{Var}_{\mathscr{J}}(\mathscr{C}) \simeq \mathscr{J}\text{-}\mathsf{Monadic}^!$, but $\mathscr{J}\text{-}\mathsf{Monadic}^! \simeq \mathsf{Mnd}_{\mathscr{J}}(\mathscr{C})^\mathsf{op}$ by 2.3.

Remark 5.27. Theorem 5.26 generalizes certain results in the literature. If we take $\mathscr{V} = \mathscr{C} = \mathsf{Set}$ and $\mathscr{J} = \mathsf{SF}(\mathsf{Set})$ (i.e. the finite cardinals), then 5.26 specializes to the classical result that ordinary varieties (in the sense of universal algebra) are dually equivalent to finitary monads on Set . If we take $\mathscr{V} = \mathscr{C} = \mathsf{Pos}$ (the cartesian closed category of posets and monotone maps) and $\mathscr{J} = \mathsf{SF}(\mathsf{Pos})$, then 5.26 specializes to the result [1, Theorem 4.6] that $(\mathsf{SF}(\mathsf{Pos})\text{-ary})$ varieties of ordered algebras are dually equivalent to strongly finitary (i.e. $\mathsf{SF}(\mathsf{Pos})\text{-ary}$) enriched monads on Pos . If we take $\mathscr{V} = \mathsf{Set}$ and $\mathscr{C} = \mathsf{Pos}$, and we let $\mathscr{J} = \mathsf{Pos}_f$ be a skeleton of the full subcategory of finite (equivalently, finitely presentable) posets, then 5.26 specializes to the result [2, Corollary 4.5] that finitary (i.e. Pos_f -ary) varieties of ordered algebras are dually equivalent to finitary (unenriched) monads on Pos . Finally, if we take $\mathscr{V} = \mathscr{C} = \mathsf{Pos}$ and $\mathscr{J} = \mathsf{Pos}_f$, then 5.26 specializes to the result [2, Corollary 4.7] that finitary (i.e. Pos_f -ary) varieties of coherent ordered algebras are dually equivalent to finitary enriched monads on Pos .

6 Examples of diagrammatic *I*-presentations and *I*-ary varieties

In the present section, we develop several examples of diagrammatic \mathscr{J} -presentations and \mathscr{J} -ary varieties. We very often (implicitly) use the fact (5.11) that every operation symbol of a free-form \mathscr{J} -signature \mathscr{S} canonically induces a \mathscr{V} -natural \mathscr{S} -operation.

In our first two examples, we employ the system of arities $SF(\mathcal{V}) \hookrightarrow \mathcal{V}$ consisting of the finite copowers of the terminal object in a complete and cocomplete cartesian closed category

²The varieties of [1, 2] are technically defined differently than ours, in that the former varieties are defined using syntactic inequations; but, in view of 5.6 and 5.7, it follows from [1, p. 2] and [2, Remark 2.7, Example 3.19(9)] (cf. also the discussion in [40, §5]) that the two definitions are equivalent.

 \mathscr{V} (2.4, 2.5). For each finite cardinal $n \in \mathbb{N}$, we write simply n to denote the nth copower $n \cdot 1$ of the terminal object 1 of \mathscr{V} , so that for each object A of \mathscr{V} , the internal hom $\mathscr{V}(n, A)$ may be identified with the (conical) nth power A^n . Hence, a parametrized operation on A with arity n and parameter $C \in \mathsf{ob} \mathscr{V}$ is a morphism $\omega : C \times A^n \to A$ in \mathscr{V} , and in the case where C = 1 we recover the notion of n-ary operation $\omega : A^n \to A$ on A in the usual sense.

Example 6.1. Let $(R, +^R, \cdot^R, 0^R, 1^R)$ be an internal rig (or internal unital semiring) in a complete and cocomplete cartesian closed category \mathscr{V} (see e.g. [32, 2.7]). We now provide a diagrammatic $\mathsf{SF}(\mathscr{V})$ -presentation whose algebras will be the (left) R-modules in \mathscr{V} . Recall (see [32, 2.7]) that an R-module in \mathscr{V} is an object M of \mathscr{V} equipped with morphisms $+^M: M \times M \to M, \ 0^M: 1 \to M$ and $\bullet^M: R \times M \to M$ such that $(M, +^M, 0^M)$ is a commutative monoid in \mathscr{V} , \bullet^M is an associative, unital action of the monoid $(R, \cdot^R, 1^R)$ on M, and \bullet^M is a bimorphism of commutative monoids from $(R, +^R, 0^R), (M, +^M, 0^M)$ to $(M, +^M, 0^M)$. Explicitly, we require the satisfaction of the following equations, where we use the convenient notation employed in [32, 2.6] for expressing equations involving algebraic operations in categories with finite products; in particular, given objects $X_1, ..., X_n$ of \mathscr{V} , we write expressions of the form $(x_1, x_2, ..., x_n): X_1 \times X_2 \times ... \times X_n$ to mean that x_i denotes the ith projection morphism $x_i = \pi_i: X_1 \times X_2 \times ... \times X_n \to X_i$ for each $i \in \{1, ..., n\}$.

1.
$$m_1 +^M (m_2 +^M m_3) = (m_1 +^M m_2) +^M m_3 : M \times M \times M \to M$$
, where $(m_1, m_2, m_3) : M \times M \times M$;

2.
$$m +^M n = n +^M m : M \times M \to M$$
, where $(m, n) : M \times M$;

3.
$$m + {}^{M} 0^{M} = m : M \to M$$
 and $0^{M} + {}^{M} m = m : M \to M$, where $m : M$;

4.
$$1^R \bullet^M m = m : M \to M$$
, where $m : M$;

5.
$$r \bullet^M (s \bullet^M m) = (r \cdot^R s) \bullet^M m : R \times R \times M \to M$$
, where $(r, s, m) : R \times R \times M$;

6.
$$0^R \bullet^M m = 0^M : M \to M$$
 and $r \bullet^M 0^M = 0^M : R \to M$, where $m : M$ and $r : R$;

7.
$$(r + R s) \bullet^M m = (r \bullet^M m) + M (s \bullet^M m) : R \times R \times M \to M$$
, where $(r, s, m) : R \times R \times M$; and

8.
$$r \bullet^M (m + M^m n) = (r \bullet^M m) + M(r \bullet^M n) : R \times M \times M \to M$$
, where $(r, m, n) : R \times M \times M$.

For example, equation (8) expresses the commutativity of the following diagram

where Δ_R is the diagonal and the isomorphism \sim exchanges the middle two factors.

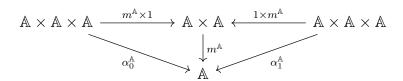
Our free-form $\mathsf{SF}(\mathscr{V})$ -signature \mathscr{S} for R-modules will thus have one operation symbol + of arity 2 and parameter 1, one operation symbol 0 of arity 0 and parameter 1, and one operation symbol \bullet of arity 1 and parameter R. As in the example of equation (8) above, each of the listed equations describes a diagrammatic \mathscr{S} -equation $C \times M^n \rightrightarrows M$ ($M \in \mathscr{S}$ -Alg), for some arity $n \in \mathbb{N}$ and parameter $C \in \mathsf{ob}\,\mathscr{V}$. We thus obtain a diagrammatic $\mathsf{SF}(\mathscr{V})$ -presentation $\mathscr{P} = (\mathscr{S}, \mathsf{E})$ such that \mathscr{P} -algebras are precisely R-modules in \mathscr{V} , and we may therefore regard R-Mod := \mathscr{P} -Alg as the \mathscr{V} -category of (left) R-modules in \mathscr{V} . By 5.20, this diagrammatic $\mathsf{SF}(\mathscr{V})$ -presentation \mathscr{P} presents a strongly finitary \mathscr{V} -monad $\mathbb{T}_{\mathscr{P}}$ on \mathscr{V} with $\mathbb{T}_{\mathscr{P}}$ -Alg $\cong R$ -Mod in \mathscr{V} -CAT/ \mathscr{V} .

Example 6.2. Letting \mathscr{V} be the complete and cocomplete cartesian closed category Cat of small categories, we now discuss a presentation of small monoidal categories in terms of the concepts of §4 and §5, relative to the subcategory of arities $\mathsf{SF}(\mathsf{Cat}) \hookrightarrow \mathsf{Cat}$. A small monoidal category is precisely a pseudomonoid in the monoidal 2-category Cat (see [37, 2.4]), i.e., a small category \mathbb{A} equipped with the following data, where we write $\tilde{\mathbf{2}}$ to denote the category with exactly two objects and an isomorphism between these objects:

- Functors $m: \mathbb{A}^2 \to \mathbb{A}$ and $i: 1 \to \mathbb{A}$ (where 1 denotes the terminal category);
- A natural isomorphism $\mathbb{A}^3 \xrightarrow{\alpha_0 \atop \alpha_1} \mathbb{A}$ (the associator) between the functors $\alpha_0 = m \circ (m \times 1)$ and $\alpha_1 = m \circ (1 \times m)$;
- A natural isomorphism $\mathbb{A} \xrightarrow{\lambda_0} \mathbb{A}$ (the *left unitor*) between functors $\lambda_0, \lambda_1 : \mathbb{A} \Rightarrow \mathbb{A}$, where λ_0 is the composite $\mathbb{A} \xrightarrow{\sim} 1 \times \mathbb{A} \xrightarrow{i \times 1} \mathbb{A} \times \mathbb{A} \xrightarrow{m} \mathbb{A}$ and $\lambda_1 = 1_{\mathbb{A}} : \mathbb{A} \to \mathbb{A}$;
- A natural isomorphism $\mathbb{A} \xrightarrow{\rho_0} \mathbb{A}$ (the *right unitor*) between functors $\rho_0, \rho_1 : \mathbb{A} \Rightarrow \mathbb{A}$, where $\rho_0 = 1_{\mathbb{A}} : \mathbb{A} \to \mathbb{A}$ and ρ_1 is the composite $\mathbb{A} \xrightarrow{\sim} \mathbb{A} \times 1 \xrightarrow{1 \times i} \mathbb{A} \times \mathbb{A} \xrightarrow{m} \mathbb{A}$; such that the following monoidal coherence laws hold:
- The pentagon identity requires that the composite natural isomorphism $m(m\times 1)(m\times 1\times 1) \xrightarrow{\alpha\circ 1} m(1\times m)(m\times 1\times 1) = m(m\times 1)(1\times 1\times m) \xrightarrow{\alpha\circ 1} m(1\times m)(1\times 1\times m)$ between functors $\mathbb{A}^4 \rightrightarrows \mathbb{A}$ be equal to the composite natural isomorphism $m(m\times 1)(m\times 1\times 1) \xrightarrow{1\circ (\alpha\times 1)} m(m\times 1)(1\times m\times 1) \xrightarrow{\alpha\circ 1} m(1\times m)(1\times m\times 1) \xrightarrow{1\circ (1\times \alpha)} m(1\times m)(1\times 1\times m).$
- The following automorphism of $m: \mathbb{A}^2 \to \mathbb{A}$ is required to be equal to the identity on $m: m = m \circ 1 \xrightarrow{1 \circ (\rho \times 1)} m(m \times 1)(1 \times i \times 1) \xrightarrow{\alpha \circ 1} m(1 \times m)(1 \times i \times 1) \xrightarrow{1 \circ (1 \times \lambda)} m \circ 1 = m$.

The associator α clearly corresponds to a functor $\tilde{\mathbf{2}} \to \mathbb{A}^{\mathbb{A}^3}$, and hence to a parametrized operation $\mathbb{A}^3 \times \tilde{\mathbf{2}} \to \mathbb{A}$ with arity 3 and parameter $\tilde{\mathbf{2}}$; and the left and right unitors λ, ρ clearly correspond to functors $\tilde{\mathbf{2}} \to \mathbb{A}^{\mathbb{A}}$, and hence to parametrized operations $\mathbb{A} \times \tilde{\mathbf{2}} \to \mathbb{A}$ with arity 1 and parameter $\tilde{\mathbf{2}}$.

We thus define a free-form $\mathsf{SF}(\mathsf{Cat})$ -signature $\mathcal S$ with one operation symbol m of arity 2 and parameter 1, one operation symbol i of arity 0 and parameter 1, one operation symbol i of arity 3 and parameter i and two operation symbols i and parameter i and intermediate step, we next provide a diagrammatic $\mathsf{SF}(\mathsf{Cat})$ -presentation $\mathcal P = (\mathcal S,\mathsf E)$ for which a i-algebra will be a small category i and equipped with functors i and i

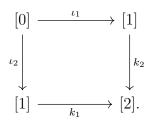


In view of 5.11, the two triangles in this diagram constitute diagrammatic \mathcal{S} -equations $m^{\mathbb{A}} \circ \left(m^{\mathbb{A}} \times 1\right) \doteq \alpha_0^{\mathbb{A}}, \ m^{\mathbb{A}} \circ \left(1 \times m^{\mathbb{A}}\right) \doteq \alpha_1^{\mathbb{A}} : \mathbb{A}^3 \rightrightarrows \mathbb{A}^1 \ (\mathbb{A} \in \mathcal{S}\text{-Alg}).$ In an exactly analogous way, we obtain diagrammatic \mathcal{S} -equations to ensure that $\lambda_0^{\mathbb{A}}, \lambda_1^{\mathbb{A}}, \rho_0^{\mathbb{A}}, \rho_1^{\mathbb{A}}$ all have their intended interpretations. We thus obtain a diagrammatic $\mathsf{SF}(\mathsf{Cat})$ -presentation $\mathcal{P} = (\mathcal{S}, \mathsf{E})$ for which a \mathcal{P} -algebra is a small category \mathbb{A} equipped with functors $m^{\mathbb{A}} : \mathbb{A}^2 \to \mathbb{A}, \ i^{\mathbb{A}} : \mathbb{A}^0 \to \mathbb{A}$ and natural isomorphisms $\alpha^{\mathbb{A}} : m^{\mathbb{A}} \circ \left(m^{\mathbb{A}} \times 1\right) \xrightarrow{\sim} m^{\mathbb{A}} \circ \left(1 \times m^{\mathbb{A}}\right), \ \lambda^{\mathbb{A}} : m^{\mathbb{A}} \circ \left(i^{\mathbb{A}} \times 1\right) \xrightarrow{\sim} 1$, and $\rho^{\mathbb{A}} : 1 \xrightarrow{\sim} m^{\mathbb{A}} \circ \left(1 \times i^{\mathbb{A}}\right)$.

We now define a further system of diagrammatic SF(Cat)-ary equations F on \mathcal{P} -Alg whose models (4.2) will be monoidal categories. Given a \mathcal{P} -algebra \mathbb{A} , we want \mathbb{A} to be an F-model iff \mathbb{A} is a monoidal category, i.e. iff \mathbb{A} satisfies the monoidal coherence laws. The pentagon identity requires the equality of two natural isomorphisms between functors $\mathbb{A}^4 \rightrightarrows \mathbb{A}$, and it can be expressed by a diagrammatic SF(Cat)-ary equation $\mathbb{A}^4 \rightrightarrows \mathbb{A}^2$ ($\mathbb{A} \in \mathcal{P}$ -Alg) on \mathcal{P} -Alg. The second monoidal coherence law requires the equality of two natural isomorphisms between functors $\mathbb{A}^2 \rightrightarrows \mathbb{A}$, and it can be expressed by a diagrammatic SF(Cat)-ary equation $\mathbb{A}^2 \rightrightarrows \mathbb{A}^2$ ($\mathbb{A} \in \mathcal{P}$ -Alg) on \mathcal{P} -Alg. We thus obtain a system of diagrammatic SF(Cat)-ary equations F on \mathcal{P} -Alg for which \mathcal{P} -Alg_F = MonCat_{strict}, the 2-category of small monoidal categories and strict monoidal functors. Hence, MonCat_{strict} is a strictly SF(Cat)-monadic 2-category over Cat, by 4.9 and 5.20, so MonCat_{strict} is isomorphic to the 2-category of algebras for a strongly finitary 2-monad on Cat, as noted in [22].

Example 6.3. We now employ diagrammatic \mathscr{J} -presentations to prove an extension of the well-known result that (small) categories are finitary monadic over graphs [11]. Indeed, whereas Wolff [43] extended the latter result by proving the monadicity of small \mathscr{V} -categories over small \mathscr{V} -graphs under only the assumption that \mathscr{V} is cocomplete, we now prove instead

that internal categories in $\mathscr V$ form a $\mathscr V$ -category that is α -ary monadic over internal graphs in \mathscr{V} as soon as \mathscr{V} is locally α -presentable as a closed category, which we assume for the remainder of Example 6.3. To this end, let $\mathscr{B}_{\Rightarrow}$ be the free \mathscr{V} -category on the ordinary category consisting of a single parallel pair $s,t:1 \Rightarrow 0$, and consider the presheaf \mathcal{V} category $\mathscr{C} = \mathsf{Gph}(\mathscr{V}) = [\mathscr{B}_{\rightrightarrows}, \mathscr{V}]$, the \mathscr{V} -category of internal graphs in \mathscr{V} , which is a locally α -presentable \mathscr{V} -category by [20, 3.1, 7.4]. An object G of \mathscr{C} can be regarded as a parallel pair $s_G, t_G : G_1 \rightrightarrows G_0$ in \mathscr{V} , i.e. as an internal graph in \mathscr{V} . Let $\mathscr{J} = \mathscr{C}_{\alpha}$ be a skeleton of the full sub- \mathscr{V} -category of \mathscr{C} spanned by the enriched α -presentable objects, so that \mathscr{J} is a bounded and eleutheric subcategory of arities by 2.4. In particular, J contains the representables $[i] := \mathscr{B}_{\rightrightarrows}(i,-) : \mathscr{B}_{\rightrightarrows} \to \mathscr{V} \ (i \in \{0,1\})$ and is closed under conical finite colimits. In the $\mathcal{V} = \mathsf{Set}$ case, the (ordinary) graph [0] consists of a single vertex and no edges, while the graph $[1] = (\cdot \to \cdot)$ consists of a single edge between two distinct vertices. We have the two morphisms $\iota_1 := \mathscr{B}_{\rightrightarrows}(s,-) : [0] \to [1]$ and $\iota_2 := \mathscr{B}_{\rightrightarrows}(t,-) : [0] \to [1]$; in the $\mathscr{V} = \mathsf{Set}$ case, ι_1 sends the single vertex of [0] to the source of the unique edge of [1], while ι_2 sends the single vertex of [0] to the target of this edge. For every internal graph $G: \mathscr{B}_{\rightrightarrows} \to \mathscr{V}$, the enriched Yoneda lemma enables us to identify $\mathscr{C}([i], G)$ with G_i $(i \in \{0,1\})$, so that we may identify $\mathscr{C}(\iota_1,G)$ with s_G and $\mathscr{C}(\iota_2,G)$ with t_G . We shall also need the arity $[2] \in \mathsf{ob}\,\mathscr{C}_{\alpha}$ defined via the following pushout in $\mathscr{C} = \mathsf{Gph}(\mathscr{V})$, which in the $\mathcal{V} = \mathsf{Set} \ \mathrm{case} \ \mathrm{is} \ \mathrm{the} \ \mathrm{graph} \ [2] = (\cdot \to \cdot \to \cdot).$



For every internal graph G in $\mathscr V$ we have a pullback square in $\mathscr V$ as on the left below,

$$\mathcal{C}([2], G) \xrightarrow{\mathcal{C}(k_2, G)} \mathcal{C}([1], G) \qquad G_2 := G_1 \times_{G_0} G_1 \xrightarrow{\pi_2} G_1$$

$$\mathcal{C}([1], G) \xrightarrow{\mathcal{C}(\iota_2, G)} \mathcal{C}([0], G) \qquad G_1 \xrightarrow{t_G} G_0$$

which we may identify with the pullback square on the right.

We now provide a presentation of internal categories in \mathscr{V} in terms of the concepts of §4 and §5; for a standard reference on internal categories, see e.g. [7, §8.1]. An internal category in \mathscr{V} is an internal graph $A = (s_A, t_A : A_1 \rightrightarrows A_0)$ in \mathscr{V} equipped with morphisms $e^A : A_0 \to A_1$ and $c^A : A_2 \to A_1$ (identity and composition) satisfying certain axioms. The identity and composition operations e^A and e^A may be written equivalently as $e^A : \mathscr{C}([0], A) \to \mathscr{C}([1], A)$ and $e^A : \mathscr{C}([2], A) \to \mathscr{C}([1], A)$ and so are parametrized \mathscr{J} -ary operations on the object e^A of $e^A = \mathsf{Gph}(\mathscr{V})$, where $e^A = \mathsf{Ob}\mathscr{C}_{\alpha}$. We thus require a free-form e^A -signature e^A with one

operation symbol e of arity [0] and parameter [1], and one operation symbol c of arity [2] and parameter [1]. We now construct a diagrammatic \mathscr{J} -presentation $\mathcal{P} = (\mathcal{S}, \mathsf{E})$ that will play a preliminary role, in which E consists of four diagrammatic \mathcal{S} -equations, expressing the interaction of the source and target morphisms with e^A, c^A in an \mathcal{S} -algebra A. In particular, we shall have a diagrammatic \mathcal{S} -equation $\mathscr{C}([2], A) \rightrightarrows \mathscr{C}([0], A)$ ($A \in \mathcal{S}$ -Alg) given by the diagram on the left below, which may be written as the more familiar diagram on the right

$$\mathcal{C}([2], A) \xrightarrow{c^A} \mathcal{C}([1], A) \qquad A_2 \xrightarrow{c^A} A_1$$

$$\downarrow^{s_A} \qquad \downarrow^{s_A} \qquad \downarrow^{s_A}$$

$$\mathcal{C}([1], A) \xrightarrow{s_A} \mathcal{C}([0], A) \qquad A_1 \xrightarrow{s_A} A_0,$$

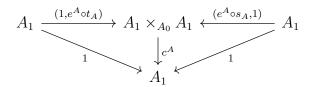
and we shall also have a diagrammatic S-equation $t_A \circ c^A \doteq t_A \circ \pi_2 : \mathscr{C}([2], A) \rightrightarrows \mathscr{C}([0], A)$ $(A \in S\text{-Alg})$. We shall also have two diagrammatic S-equations $s_A \circ e^A \doteq 1_{A_0}$, $t_A \circ e^A \doteq 1_{A_0} : \mathscr{C}([0], A) \rightrightarrows \mathscr{C}([0], A)$ $(A \in S\text{-Alg})$. We thus obtain a diagrammatic \mathscr{J} -presentation $\mathscr{P} = (S, \mathsf{E})$ for which \mathscr{P} -algebras will be internal graphs $A = (s_A, t_A : A_1 \rightrightarrows A_0)$ in \mathscr{V} equipped with identity and composition operations e^A, c^A satisfying (just) these four diagrammatic laws.

We now define a system of diagrammatic \mathscr{J} -ary equations F on \mathscr{P} -Alg whose models (4.2) will be internal categories in \mathscr{V} . First, given an internal graph G, we need to define an object of paths of length three in G. This object, which we write as G_3 , can be defined as the (vertex of the) pullback of $s_G \circ \pi_1 : G_2 \to G_0$ along $t_G : G_1 \to G_0$, or as the pullback of $s_G : G_1 \to G_0$ along $t_G \circ \pi_2 : G_2 \to G_0$, since it is well known that these pullbacks $G_1 \times_{G_0} (G_1 \times_{G_0} G_1)$ and $(G_1 \times_{G_0} G_1) \times_{G_0} G_1$ are canonically isomorphic. By a dual process, we can define a further arity $[3] \in \mathsf{ob}\,\mathscr{C}_\alpha$ by taking the pushout of $t_1 \circ t_1 : [0] \to [2]$ along $t_2 : [0] \to [1]$, or equivalently the pushout of $t_1 : [0] \to [1]$ along $t_2 \circ t_2 : [0] \to [2]$. In the $\mathscr{V} = \mathsf{Set}$ case, $[3] = (\cdot \to \cdot \to \cdot \to \cdot)$. For any internal graph G in \mathscr{V} , we may identify $\mathscr{C}([3], G)$ with $G_3 \cong G_1 \times_{G_0} (G_1 \times_{G_0} G_1) \cong (G_1 \times_{G_0} G_1) \times_{G_0} G_1$.

Given a \mathcal{P} -algebra A, we want A to be an F-model iff A is an internal category in \mathcal{V} , i.e. iff composition in A is associative and unital. The associativity of composition is expressed by the following diagram

which is a diagrammatic \mathscr{J} -ary equation $\mathscr{C}\left([3],A\right)\rightrightarrows\mathscr{C}\left([1],A\right)$ $(A\in\mathcal{P}\text{-}\mathsf{Alg})$ on $\mathcal{P}\text{-}\mathsf{Alg}$. The

identity axioms for an internal category are expressed by the following diagram



which may be regarded as a pair of diagrammatic \mathscr{J} -ary equations $\mathscr{C}([1], A) \rightrightarrows \mathscr{C}([1], A)$ ($A \in \mathcal{P}$ -Alg) on \mathcal{P} -Alg. We thus obtain a system of diagrammatic \mathscr{J} -ary equations F on \mathcal{P} -Alg for which F -models are internal categories in \mathscr{V} , so that we may regard $\mathsf{Cat}(\mathscr{V}) := \mathcal{P}$ -Alg_{F} as the \mathscr{V} -category of internal categories in \mathscr{V} . The following theorem now follows from 4.9 and 5.20:

Theorem 6.4. Let \mathscr{V} be locally α -presentable as a closed category. There is an α -accessible \mathscr{V} -monad \mathbb{T} on $\mathsf{Gph}(\mathscr{V}) = [\mathscr{B}_{\rightrightarrows}, \mathscr{V}]$ such that \mathbb{T} -Alg $\cong \mathsf{Cat}(\mathscr{V})$ in \mathscr{V} -CAT/ $\mathsf{Gph}(\mathscr{V})$, so that the \mathscr{V} -category $\mathsf{Cat}(\mathscr{V})$ of internal categories in \mathscr{V} is strictly α -ary monadic over $\mathsf{Gph}(\mathscr{V})$. \square

Example 6.5. We now provide a diagrammatic \mathscr{J} -presentation for Plotkin and Power's global state algebras [39], which they employed in treating the global state monad within their program of algebraic computational effects. Suppose that \mathscr{V} is cartesian closed, and let $j: \mathscr{J} \hookrightarrow \mathscr{V}$ be any bounded and eleutheric system of arities (so that \mathscr{J} contains the terminal object 1 and is closed under finite products, 2.5). For example, we may take $\mathscr{J} = \mathsf{SF}(\mathscr{V})$, or if \mathscr{V} is locally α -presentable as a cartesian closed category, then we may take $\mathscr{J} = \mathscr{V}_{\alpha}$, a skeleton of the full sub- \mathscr{V} -category of \mathscr{V} consisting of the α -presentable objects. Let $V \in \mathsf{ob} \mathscr{J}$ be a fixed object (the object of values), let L be a finite set (the set of locations), and let $L_2 := \{(\ell, \ell') \in L \times L \mid \ell \neq \ell'\}$. By an abuse of notation, we write $L := L \cdot 1 \in \mathsf{ob} \mathscr{V}$ (the coproduct of L copies of 1) and $L_2 := L_2 \cdot 1 \in \mathsf{ob} \mathscr{V}$. We define a free-form \mathscr{J} -signature \mathscr{S} with an operation symbol ℓ ("lookup") of arity ℓ and parameter ℓ , and an operation symbol ℓ ("update") of arity 1 and parameter ℓ and an update operation ℓ of ℓ equipped with a lookup operation ℓ : ℓ and an update operation ℓ and ℓ and an update operation ℓ and ℓ and an update operation ℓ and ℓ and ℓ and an update operation ℓ and ℓ and ℓ and an update operation ℓ and ℓ and ℓ and ℓ and an update operation ℓ and ℓ and

Each of the seven commutative diagrams of [39, Definition 1] for global state may be regarded as a diagrammatic S-equation $A^J \rightrightarrows A^C$ ($A \in S$ -Alg) for some arity $J \in \mathsf{ob} \mathscr{J}$ and parameter $C \in \mathsf{ob} \mathscr{V}$. Specifically, the first commutative diagram can be expressed as a diagrammatic S-equation $A \rightrightarrows A^L$ ($A \in S$ -Alg); the second as a diagrammatic S-equation $A^{V \times V} \rightrightarrows A^L$ ($A \in S$ -Alg); the third as a diagrammatic S-equation $A \rightrightarrows A^{L \times V \times V}$ ($A \in S$ -Alg); the fourth as a diagrammatic S-equation $A^V \rightrightarrows A^{L \times V}$ ($A \in S$ -Alg); the fifth as a diagrammatic S-equation $A^{V \times V} \rightrightarrows A^{L_2}$ ($A \in S$ -Alg); the sixth as a diagrammatic S-equation $A \rightrightarrows A^{L_2 \times V \times V}$ ($A \in S$ -Alg); and the last as a diagrammatic S-equation $A^V \rightrightarrows A^{L_2 \times V}$ ($A \in S$ -Alg). We thus obtain a diagrammatic S-presentation S-equation S-algebras are (the S-based analogue of) the global state algebras of [39, Definition 1], and by 5.20 we find that S-presents a S-ary S-monad S-ary S-monad S-ary S-Alg in S-Alg in S-Alg.

Example 6.6. We now provide diagrammatic \mathscr{J} -presentations corresponding to some of Staton's parametrized algebraic theories [42], namely the parametrized theory of reading many bits [42, §IV] and the parametrized theory of restriction [42, §V.B], and we thereby obtain \mathscr{J} -ary \mathscr{V} -monads determined by these parametrized algebraic theories. Suppose that \mathscr{V} is cartesian closed, and let us first define a diagrammatic $SF(\mathscr{V})$ -presentation for the theory of reading many bits, parametrized by an object $\alpha \in \mathsf{ob} \mathscr{V}$. The free-form $SF(\mathscr{V})$ -signature \mathscr{S} will have one operation symbol ? of arity 2 and parameter α , so that an \mathscr{S} -algebra is an object A of \mathscr{V} equipped with a morphism $?^A : A^2 \to A^\alpha$, or equivalently $?^A : A \times A \times \alpha \to A$. The desired diagrammatic $SF(\mathscr{V})$ -presentation $\mathscr{P} = (\mathscr{S}, \mathsf{E})$ will have diagrammatic \mathscr{S} -equations corresponding to the equations of [42] annotated as (idem- $?_a$), (dup- $?_a$), and ($?_a/?_b$) (for $a, b : \alpha$). The first equation (idem- $?_a$) is written there as $x \equiv x ?_a x$ for x : A and $a : \alpha$, and can be expressed as the commutativity of the diagram

$$A \times \alpha \xrightarrow{\Delta_A \times 1} A \times A \times \alpha \xrightarrow{?^A} A,$$

which is a diagrammatic S-equation $A \times \alpha \rightrightarrows A$ ($A \in S$ -Alg). The second equation (dup- $?_a$) is written as $(u ?_a v) ?_a (x ?_a y) \equiv u ?_a y$ for u, v, x, y : A and $a : \alpha$. Writing the product projections as $u, v, x, y : A^4 \times \alpha \to A$ and $a : A^4 \times \alpha \to \alpha$, this equation can be expressed as the commutativity of the diagram

$$A^{4} \times \alpha \xrightarrow{\left\langle ?^{A} \circ \langle u, v, a \rangle, ?^{A} \circ \langle x, y, a \rangle, a \right\rangle} A \times A \times \alpha \xrightarrow{?^{A}} A,$$

$$?^{A} \circ \langle u, y, a \rangle$$

which is a diagrammatic \mathcal{S} -equation $A^4 \times \alpha \rightrightarrows A$ ($A \in \mathcal{S}$ -Alg). The third equation $(?_a/?_b)$ is written as $(u ?_b v) ?_a (x ?_b y) \equiv (u ?_a x) ?_b (v ?_a y)$ for u, v, x, y : A and $a, b : \alpha$. Writing the product projections as $u, v, x, y : A^4 \times \alpha^2 \to A$ and $a, b : A^4 \times \alpha^2 \to \alpha$, this equation can be expressed as the commutativity of the diagram

which is a diagrammatic \mathcal{S} -equation $A^4 \times \alpha^2 \rightrightarrows A$ ($A \in \mathcal{S}$ -Alg). We thus obtain a diagrammatic $\mathsf{SF}(\mathscr{V})$ -presentation $\mathcal{P} = (\mathcal{S}, \mathsf{E})$ for this α -parametrized theory ($\alpha \in \mathsf{ob}\,\mathscr{V}$) of reading many bits, and 5.20 entails that \mathcal{P} presents a strongly finitary \mathscr{V} -monad $\mathbb{T}_{\mathcal{P}}$ on \mathscr{V} with $\mathbb{T}_{\mathcal{P}}$ -Alg $\cong \mathcal{P}$ -Alg in \mathscr{V} -CAT/ \mathscr{V} .

Continuing to suppose that \mathscr{V} is cartesian closed, now let $\mathscr{J} \hookrightarrow \mathscr{V}$ be any bounded and eleutheric *system* of arities; for example, as in Example 6.5, we may take $\mathscr{J} = \mathsf{SF}(\mathscr{V})$, or if \mathscr{V} is locally α -presentable as a cartesian closed category then we may take $\mathscr{J} = \mathscr{V}_{\alpha}$. Let $J \in \mathsf{ob} \mathscr{J}$ be a fixed arity. We define a diagrammatic \mathscr{J} -presentation for the J-parametrized

theory of restriction [42, §V.B]. The free-form \mathscr{J} -signature \mathscr{S} will have a single operation symbol ν of arity J and parameter 1, so that an \mathscr{S} -algebra is an object A of \mathscr{V} equipped with a morphism $\nu^A:A^J\to A$. The desired diagrammatic \mathscr{J} -presentation $\mathscr{P}=(\mathscr{S},\mathsf{E})$ will have diagrammatic \mathscr{S} -equations corresponding to the equations of [42] annotated as (idem- ν) and (ν/ν) . The first equation (idem- ν) is written there as $\nu(\lambda j.x) \equiv x$ for x:A, and can be expressed as the commutativity of the diagram

$$A \xrightarrow{\Delta_A} A^J \xrightarrow{\nu^A} A$$

(where $\Delta_A : A \to A^J$ is the exponential transpose of the first projection $\pi_1 : A \times J \to A$), which is a diagrammatic \mathcal{S} -equation $A \rightrightarrows A$ ($A \in \mathcal{S}$ -Alg). The second equation (ν/ν) is written as $\nu (\lambda j.\nu (\lambda k.f(j,k))) \equiv \nu (\lambda j.\nu (\lambda k.f(k,j)))$ for $f : A^{J\times J}$, and it can be expressed as a diagrammatic \mathcal{S} -equation of the form $A^{J\times J} \rightrightarrows A$ ($A \in \mathcal{S}$ -Alg) via the diagram

$$A^{J \times J} \xrightarrow{\kappa_{JJ}} (A^{J})^{J} \xrightarrow{(\nu^{A})^{J}} A^{J} \xrightarrow{\nu^{A}} A$$

$$A^{s} \downarrow \qquad \qquad \qquad \parallel$$

$$A^{J \times J} \xrightarrow{\kappa_{JJ}} (A^{J})^{J} \xrightarrow{(\nu^{A})^{J}} A^{J} \xrightarrow{\nu^{A}} A$$

where $\kappa_{XY}: A^{X\times Y} \xrightarrow{\sim} (A^Y)^X$ denotes the canonical isomorphism $(X,Y\in\mathcal{V})$, and $s: J\times J\xrightarrow{\sim} J\times J$ is the 'twist' isomorphism. We thus obtain a diagrammatic \mathscr{J} -presentation $\mathcal{P}=(\mathcal{S},\mathsf{E})$ for this J-parametrized theory of restriction, and 5.20 entails that \mathcal{P} presents a \mathscr{J} -ary \mathscr{V} -monad $\mathbb{T}_{\mathcal{P}}$ on \mathscr{V} such that $\mathbb{T}_{\mathcal{P}}$ -Alg $\cong \mathcal{P}$ -Alg in \mathscr{V} -CAT/ \mathscr{V} . By combining the J-parametrized theory of restriction with the α -parametrized theory of reading many bits, for $\alpha=J$, and adding a single additional equation that requires that ν commutes with ? in the sense of §7 below, we obtain a diagrammatic \mathscr{J} -presentation for Staton's parametrized theory of instantiating and reading bits [42, §V.B], assuming that \mathscr{J} contains $\mathsf{SF}(\mathscr{V})$. \square

7 Constructions on diagrammatic *f*-presentations: Sum and tensor

We can define the *sum* and *tensor* of two diagrammatic (or free-form) \mathscr{J} -presentations \mathcal{P}_1 and \mathcal{P}_2 as follows. For each $i \in \{1,2\}$, let us write $\mathcal{P}_i = (\mathcal{S}_i, \mathsf{E}_i)$, where \mathcal{S}_i is a free-form \mathscr{J} -signature with arities $J_{i\sigma}$ and parameters $C_{i\sigma}$ ($\sigma \in \mathcal{S}_i$).

To define the sum $\mathcal{P}_1 + \mathcal{P}_2$, we first define a free-form \mathscr{J} -signature $\mathcal{S}_1 + \mathcal{S}_2$ by equipping the disjoint union of the (small) sets of operation symbols \mathcal{S}_1 and \mathcal{S}_2 with the associated arities $J_{i\sigma}$ and parameters $C_{i\sigma}$ ($i \in \{1, 2\}, \sigma \in \mathcal{S}_i$). An $(\mathcal{S}_1 + \mathcal{S}_2)$ -algebra is then an object A of \mathscr{C} carrying the structure of both an \mathcal{S}_1 -algebra and an \mathcal{S}_2 -algebra, and we readily deduce by 5.5 that moreover $(\mathcal{S}_1 + \mathcal{S}_2)$ -Alg $\cong \mathcal{S}_1$ -Alg $\times_{\mathscr{C}} \mathcal{S}_2$ -Alg in \mathscr{V} -CAT/ \mathscr{C} . For each $i \in \{1, 2\}$, E_i is a family of diagrammatic \mathcal{S}_i -equations $\omega^{i\delta} \doteq \nu^{i\delta}$ ($\delta \in \mathcal{D}_i$), each of which consists of morphisms $\omega_A^{i\delta}, \nu_A^{i\delta} : \mathscr{C}(J_{i\delta}, A) \rightrightarrows \mathscr{C}(C_{i\delta}, A)$ that are \mathscr{V} -natural in $A \in \mathcal{S}_i$ -Alg (where we assume without loss of generality that the sets \mathcal{S}_i and \mathcal{D}_i are disjoint). Given $i \in \{1, 2\}$,

if we denote the S_i -algebra underlying each $(S_1 + S_2)$ -algebra A also by A, then for each $\delta \in \mathcal{D}_i$ we obtain a parallel pair of \mathscr{V} -natural families $\omega_A^{i\delta}, \nu_A^{i\delta} : \mathscr{C}(J_{i\delta}, A) \rightrightarrows \mathscr{C}(C_{i\delta}, A)$ $(A \in (S_1 + S_2)$ -Alg), so that we obtain a diagrammatic $(S_1 + S_2)$ -equation that we may write also as $\omega^{i\delta} \doteq \nu^{i\delta}$. Collectively, these equations $\omega^{i\delta} \doteq \nu^{i\delta}$ $(i \in \{1, 2\}, \delta \in \mathcal{D}_i)$ constitute a system of diagrammatic $(S_1 + S_2)$ -equations that we may write as $E_1 + E_2$.

Thus we obtain a diagrammatic \mathscr{J} -presentation $\mathcal{P}_1 + \mathcal{P}_2 := (\mathcal{S}_1 + \mathcal{S}_2, \mathsf{E}_1 + \mathsf{E}_2)$, for which a $(\mathcal{P}_1 + \mathcal{P}_2)$ -algebra is an object A of \mathscr{C} equipped with the structure of both a \mathcal{P}_1 -algebra and a \mathcal{P}_2 -algebra, and moreover $(\mathcal{P}_1 + \mathcal{P}_2)$ -Alg $\cong \mathcal{P}_1$ -Alg $\times_{\mathscr{C}} \mathcal{P}_2$ -Alg in \mathscr{V} -CAT/ \mathscr{C} as a consequence of the above isomorphism $(\mathcal{S}_1 + \mathcal{S}_2)$ -Alg $\cong \mathcal{S}_1$ -Alg $\times_{\mathscr{C}} \mathcal{S}_2$ -Alg.

Now supposing that $\mathscr{C} = \mathscr{V}$ and that $\mathscr{J} \hookrightarrow \mathscr{V}$ is a *system* of arities (see 2.5), we next add further diagrammatic $(S_1 + S_2)$ -equations to the sum $\mathcal{P}_1 + \mathcal{P}_2$ to obtain the *tensor product* $\mathcal{P}_1 \otimes \mathcal{P}_2$ of the diagrammatic \mathscr{J} -presentations $\mathcal{P}_1, \mathcal{P}_2$. To do this, we employ the following terminology: given any two parametrized operations $\omega_i : [J_i, A] \to [C_i, A]$ ($i \in \{1, 2\}$) on an object A of \mathscr{V} , where $J_i, C_i \in \mathsf{ob} \mathscr{V}$ and we write the internal hom in \mathscr{V} as [-, -], we obtain a pair of parametrized operations

$$\omega_1 * \omega_2, \ \omega_1 \tilde{*} \omega_2 : [J_1 \otimes J_2, A] \rightrightarrows [C_1 \otimes C_2, A]$$

that are called the first and second Kronecker products of ω_1 and ω_2 (cf. [33]) and are defined as the following composites, respectively:

$$[J_1 \otimes J_2, A] \cong [J_2, [J_1, A]] \xrightarrow{[1,\omega_1]} [J_2, [C_1, A]] \cong [C_1, [J_2, A]] \xrightarrow{[1,\omega_2]} [C_1, [C_2, A]] \cong [C_1 \otimes C_2, A],$$

$$[J_1 \otimes J_2, A] \cong [J_1, [J_2, A]] \xrightarrow{[1,\omega_2]} [J_1, [C_2, A]] \cong [C_2, [J_1, A]] \xrightarrow{[1,\omega_1]} [C_2, [C_1, A]] \cong [C_1 \otimes C_2, A].$$
We say that ω_1 commutes with ω_2 if $\omega_1 * \omega_2 = \omega_1 * \omega_2$.

Given operation symbols $\sigma_1 \in \mathcal{S}_1$ and $\sigma_2 \in \mathcal{S}_2$, we can consider the parametrized operations $\sigma_i^A : [J_{i\sigma_i}, A] \to [C_{i\sigma_i}, A]$ ($i \in \{1, 2\}$) carried by each $(\mathcal{S}_1 + \mathcal{S}_2)$ -algebra A. The first and second Kronecker products of σ_1^A and σ_2^A are therefore morphisms

$$\sigma_1^A * \sigma_2^A, \quad \sigma_1^A \tilde{*} \sigma_2^A : [J_{1\sigma_1} \otimes J_{2\sigma_2}, A] \rightrightarrows [C_{1\sigma_1} \otimes C_{2\sigma_2}, A]$$
 (7.0.i)

that are \mathscr{V} -natural in $A \in (\mathcal{S}_1 + \mathcal{S}_2)$ -Alg, in view of their definitions above, since σ_1^A and σ_2^A are \mathscr{V} -natural in $A \in (\mathcal{S}_1 + \mathcal{S}_2)$ -Alg by 5.11. Hence, the morphisms in (7.0.i) constitute \mathscr{V} -natural $(\mathcal{S}_1 + \mathcal{S}_2)$ -operations $\sigma_1 * \sigma_2$ and $\sigma_1 * \sigma_2$, so we obtain a diagrammatic $(\mathcal{S}_1 + \mathcal{S}_2)$ -equation $\sigma_1 * \sigma_2 \doteq \sigma_1 * \sigma_2$. Adjoining these equations $\sigma_1 * \sigma_2 \doteq \sigma_1 * \sigma_2$ ($\sigma_1 \in \mathcal{S}_1, \sigma_2 \in \mathcal{S}_2$) to $\mathsf{E}_1 + \mathsf{E}_2$, we obtain a system of diagrammatic $(\mathcal{S}_1 + \mathcal{S}_2)$ -equations $\mathsf{E}_1 \otimes \mathsf{E}_2$.

Thus we obtain a diagrammatic \mathscr{J} -presentation $\mathcal{P}_1 \otimes \mathcal{P}_2 = (\mathcal{S}_1 + \mathcal{S}_2, \mathsf{E}_1 \otimes \mathsf{E}_2)$, for which a $(\mathcal{P}_1 \otimes \mathcal{P}_2)$ -algebra is an object A of \mathscr{V} equipped with both the structure of a \mathcal{P}_1 -algebra and a \mathcal{P}_2 -algebra such that σ_1^A commutes with σ_2^A for each pair of operation symbols $\sigma_1 \in \mathcal{S}_1$ and $\sigma_2 \in \mathcal{S}_2$, and $(\mathcal{P}_1 \otimes \mathcal{P}_2)$ -Alg is isomorphic (in \mathscr{V} -CAT/ \mathscr{V}) to the full sub- \mathscr{V} -category of \mathcal{P}_1 -Alg $\times_{\mathscr{V}} \mathcal{P}_2$ -Alg consisting of these objects. For example, by bringing together the diagrammatic \mathscr{J} -presentations \mathcal{P}_1 and \mathcal{P}_2 for reading many bits and restriction (6.6) and forming the tensor product $\mathcal{P}_1 \otimes \mathcal{P}_2$, we arrive at a diagrammatic \mathscr{J} -presentation for Staton's parametrized theory of instantiating and reading bits; the diagrammatic equation expressing that \mathscr{V} commutes with ? corresponds to Staton's syntactic equation (\mathscr{V} /?) [42, §V.B].

8 \mathcal{J} -ary varieties of algebras for \mathcal{K} -relative monads

The notion of abstract clone of Philip Hall provides an equivalent way of encoding the data for a Lawvere theory in terms of a family of sets H(n) indexed by the finite cardinals n, with some further structure; see [12]. Variants and generalizations of this concept include Kleisli triples [36], enriched clones [41, 17], and relative monads [3]. By transporting the notion of relative monad to the setting of a subcategory of arities $\mathscr{K} \hookrightarrow \mathscr{C}$ enriched in \mathscr{V} , we obtain a notion of \mathcal{K} -relative \mathcal{V} -monad (8.1). In the special case where \mathcal{K} is eleutheric, \mathcal{K} -relative \mathscr{V} -monads provide an equivalent way of describing \mathscr{K} -ary \mathscr{V} -monads in terms of families of objects $H = (HK)_{K \in ob \mathscr{K}}$ of \mathscr{C} equipped with further structure (8.3). But \mathscr{K} -relative \mathscr{V} -monads are of interest also in cases where \mathscr{K} is not eleutheric, and in the present section we do not impose such an assumption on \mathcal{K} . Instead, we consider the case where \mathcal{K} is merely contained in a bounded and eleutheric subcategory of arities $\mathscr{J} \hookrightarrow \mathscr{C}$ (under the assumptions of 2.1 and 2.2), and in this case we show that for each \mathcal{K} -relative \mathcal{V} -monad H, the \mathscr{V} -category of H-algebras H-Alg is precisely the \mathscr{V} -category of \mathcal{P}_H -algebras for a diagrammatic \mathcal{J} -presentation \mathcal{P}_H , so that H-Alg is a \mathcal{J} -ary variety and hence is \mathcal{J} -ary monadic over \mathscr{C} . For example, this result is applicable to any small subcategory of arities \mathscr{K} in a locally presentable \mathscr{V} -category \mathscr{C} over a locally presentable closed category \mathscr{V} , since in this case \mathscr{K} is contained in the bounded and eleutheric subcategory of arities $\mathscr{C}_{\alpha} \hookrightarrow \mathscr{C}$ for a suitable α (2.4, 8.11). But the presentations \mathcal{P}_H obtained in this way are important even in the case where $\mathcal{H} = \mathcal{J}$, because by applying this construction to the \mathcal{J} -relative \mathscr{V} -monad corresponding to a \mathscr{J} -ary \mathscr{V} -monad \mathbb{T} we obtain a diagrammatic \mathscr{J} -presentation of \mathbb{T} that is more economical than the canonical \mathscr{J} -presentation of \mathbb{T} obtained by way of the monadicity of \mathcal{J} -ary \mathcal{V} -monads (2.6, 2.8); examples are discussed in 8.15.

Definition 8.1. Given a full sub- \mathcal{V} -category $\mathcal{K} \hookrightarrow \mathcal{C}$, a \mathcal{K} -relative \mathcal{V} -monad (on \mathcal{C}) is a triple (H, e, m) consisting of a \mathcal{V} -functor $H : \mathcal{K} \to \mathcal{C}$ and \mathcal{V} -natural families of morphisms $e_J : J \to HJ$ $(J \in \mathcal{K})$ and $m_{JK} : \mathcal{C}(J, HK) \to \mathcal{C}(HJ, HK)$ $(J, K \in \mathcal{K})$ making the following diagrams commute for all $J, K, L \in \mathsf{ob}\,\mathcal{K}$:

$$I \xrightarrow{e_{J}} \mathscr{C}(J, HJ) \xrightarrow{m_{JJ}} \mathscr{C}(HJ, HJ)$$

$$\mathscr{C}(J, HK) \xrightarrow{m_{JK}} \mathscr{C}(HJ, HK) \xrightarrow{\mathscr{C}(e_{J}, 1)} \mathscr{C}(J, HK)$$

$$\downarrow 1$$

$$\mathscr{C}(J, HK) \otimes \mathscr{C}(K, HL) \xrightarrow{m_{JK} \otimes m_{KL}} \mathscr{C}(HJ, HK) \otimes \mathscr{C}(HK, HL)$$

$$\downarrow 1 \otimes m_{KL} \qquad \qquad \downarrow c$$

$$\mathscr{C}(J, HK) \otimes \mathscr{C}(HK, HL) \xrightarrow{c} \mathscr{C}(J, HL) \xrightarrow{m_{JL}} \mathscr{C}(HJ, HL),$$

where each c is a composition morphism of \mathscr{C} .

Remark 8.2. In Definition 8.1, it actually suffices to just assume that we have a family of objects $(HJ)_{J\in ob\,\mathscr{K}}$ of \mathscr{C} and families of morphisms $e_J:J\to HJ$ $(J\in ob\,\mathscr{K})$ and

 $m_{JK}: \mathscr{C}(J, HK) \to \mathscr{C}(HJ, HK) \ (J, K \in \mathsf{ob}\,\mathscr{K})$ making the diagrams of 8.1 commute, because then (as in the ordinary context [3]) it follows straightforwardly that the object assignment $J \mapsto HJ \ (J \in \mathsf{ob}\,\mathscr{K})$ extends to a \mathscr{V} -functor $H: \mathscr{K} \to \mathscr{C}$, with respect to which the families of morphisms e_J and m_{JK} are \mathscr{V} -natural.

8.3. In the special case where \mathscr{K} is a subcategory of arities $\mathscr{K} = \mathscr{J} \hookrightarrow \mathscr{C}$ satisfying the assumptions of 2.1 and 2.2 (including eleuthericity³), there is an equivalence between \mathcal{J} -relative \mathcal{V} -monads and \mathcal{J} -ary \mathcal{V} -monads. To see this, first note that \mathcal{V} -CAT $(\mathcal{J},\mathcal{E})$ is a monoidal category, whose monoidal structure is defined by transporting the monoidal structure of $\operatorname{End}_{\mathscr{I}}(\mathscr{C})$ along the equivalence $\mathscr{V}\operatorname{-CAT}(\mathscr{J},\mathscr{C})\simeq\operatorname{End}_{\mathscr{I}}(\mathscr{C})$ discussed in 2.3. Thus, the unit of the monoidal category $\mathscr{V}\text{-CAT}(\mathscr{J},\mathscr{C})$ is the \mathscr{V} -functor $j:\mathscr{J}\hookrightarrow\mathscr{C}$, while the monoidal product of $H, H': \mathscr{J} \to \mathscr{C}$ is defined by $H * H' := (\mathsf{Lan}_j H) \circ H'$. A monoid in the monoidal category $\mathscr{V}\text{-CAT}(\mathscr{J},\mathscr{C})$ is thus a triple (H,e,m) consisting of a \mathscr{V} -functor $H: \mathscr{J} \to \mathscr{C}$ and \mathscr{V} -natural transformations $e: j \to H$ and $m: H*H \to H$ satisfying the associativity and unit equations. Unpacking m, we have for each $K \in \mathsf{ob} \mathscr{J}$ a morphism $m_K: (H*H)K \to HK$, where $(H*H)K = (\mathsf{Lan}_j H)HK = \int^{J \in \mathscr{J}} \mathscr{C}(J, HK) \otimes HJ$ by the coend formula for left Kan extensions [21, 4.25]. Hence m is equivalently given by a family of morphisms $m_{JK}: \mathscr{C}(J, HK) \otimes HJ \to HK$ or equivalently $m_{JK}: \mathscr{C}(J, HK) \to \mathscr{C}(HJ, HK)$, \mathcal{V} -natural in $J, K \in \mathcal{J}$. Expressing m in this way, the unit and associativity axioms for a monoid (H, e, m) in $\mathscr{V}\text{-CAT}(\mathscr{J}, \mathscr{C})$ translate into the axioms for a \mathscr{J} -relative \mathscr{V} -monad in 8.1. Thus, \mathcal{J} -relative \mathcal{V} -monads may be identified with monoids in \mathcal{V} -CAT $(\mathcal{J},\mathcal{C})$ and so are the objects of a category that is equivalent to the category of \mathscr{J} -ary \mathscr{V} -monads on \mathscr{C} . In particular, if (H, e, m) is a \mathscr{J} -relative \mathscr{V} -monad on \mathscr{C} , then the induced \mathscr{V} -endofunctor $\mathsf{Lan}_j H : \mathscr{C} \to \mathscr{C}$ underlies a \mathscr{J} -ary \mathscr{V} -monad on \mathscr{C} , and if \mathbb{T} is a \mathscr{J} -ary \mathscr{V} -monad on \mathscr{C} , then the \mathscr{V} -functor $Tj: \mathscr{J} \to \mathscr{C}$ underlies a \mathscr{J} -relative \mathscr{V} -monad.

The following is an enrichment (and a slight variation, 8.5) of a definition in [3]:

Definition 8.4. Let $\mathcal{K} \hookrightarrow \mathcal{C}$ be a full sub- \mathcal{V} -category, and let H = (H, e, m) be a \mathcal{K} -relative \mathcal{V} -monad on \mathcal{C} . An H-algebra is an object A of \mathcal{C} equipped with a \mathcal{V} -natural family of morphisms $\alpha_J : \mathcal{C}(J,A) \to \mathcal{C}(HJ,A) \ (J \in \mathcal{K})$ making the following diagrams commute for all $J, K \in \mathsf{ob} \mathcal{K}$:

$$\mathscr{C}(J,A) \xrightarrow{\alpha_J} \mathscr{C}(HJ,A) \xrightarrow{\mathscr{C}(e_J,1)} \mathscr{C}(J,A)$$
(8.4.i)

$$\mathscr{C}(J, HK) \otimes \mathscr{C}(K, A) \xrightarrow{m_{JK} \otimes \alpha_K} \mathscr{C}(HJ, HK) \otimes \mathscr{C}(HK, A)$$

$$\downarrow^{c} \qquad \qquad \downarrow^{c} \qquad (8.4.ii)$$

$$\mathscr{C}(J, HK) \otimes \mathscr{C}(HK, A) \xrightarrow{c} \mathscr{C}(J, A) \xrightarrow{\alpha_J} \mathscr{C}(HJ, A),$$

where each c is a composition morphism of \mathscr{C} .

³In fact, the remarks in this paragraph apply when \mathscr{J} is an arbitrary (possibly large) eleutheric subcategory of arities in an arbitrary \mathscr{V} -category \mathscr{C} over an arbitrary closed category \mathscr{V} .

Remark 8.5. In Definition 8.4, we may omit the requirement that the morphisms α_J are \mathscr{V} -natural in $J \in \mathscr{K}$, as this follows automatically from the commutativity of the diagrams in 8.4 (cf. [3] for the same fact in the ordinary context). Note also that the structural morphisms α_J of an H-algebra may equivalently be written as $\alpha_J : \mathscr{C}(J, A) \otimes HJ \to A$. \square

8.6. In the special case of 8.3 where $\mathscr{K} = \mathscr{J} \hookrightarrow \mathscr{C}$, the equivalence of monoidal categories $\mathscr{V}\text{-CAT}(\mathscr{J},\mathscr{C}) \simeq \operatorname{End}_{\mathscr{J}}(\mathscr{C})$ induces an *action* of the monoidal category $\mathscr{V}\text{-CAT}(\mathscr{J},\mathscr{C})$ on the $\mathscr{V}\text{-category}\,\mathscr{C}$, in the form of a strong monoidal functor $\Phi: \mathscr{V}\text{-CAT}(\mathscr{J},\mathscr{C}) \to \mathscr{V}\text{-CAT}(\mathscr{C},\mathscr{C})$ given by left Kan extension. If H = (H,e,m) is a \mathscr{J} -relative \mathscr{V} -monad, regarded as a monoid in $\mathscr{V}\text{-CAT}(\mathscr{J},\mathscr{C})$ (8.3), then the corresponding \mathscr{J} -ary \mathscr{V} -monad is the monoid $\Phi(H)$ in $\mathscr{V}\text{-CAT}(\mathscr{C},\mathscr{C})$ obtained canonically by applying Φ . An H-algebra is equivalently given by an algebra for the \mathscr{V} -monad $\Phi(H)$, which from the point of view of actions of monoidal categories we may call an H-module in \mathscr{C} for the monoid H in $\mathscr{V}\text{-CAT}(\mathscr{J},\mathscr{C})$. \square

The algebras for a \mathcal{K} -relative \mathcal{V} -monad can be described in terms of diagrammatic \mathcal{K} -presentations in the sense of 5.18:

Proposition 8.7. Let H = (H, e, m) be a \mathcal{K} -relative \mathcal{V} -monad for a small full sub- \mathcal{V} -category $\mathcal{K} \hookrightarrow \mathcal{C}$. Then there is a diagrammatic \mathcal{K} -presentation \mathcal{P}_H such that \mathcal{P}_H -algebras are precisely H-algebras.

Proof. The free-form \mathcal{K} -signature \mathcal{S} for H-algebras will have an operation symbol α_J of arity $J \in \mathsf{ob}\,\mathcal{K}$ and parameter $HJ \in \mathsf{ob}\,\mathcal{C}$ for each $J \in \mathsf{ob}\,\mathcal{K}$, so that an \mathcal{S} -algebra is an object A of \mathcal{C} equipped with structural morphisms $\alpha_J^A: \mathcal{C}(J,A) \to \mathcal{C}(HJ,A)$ for all $J \in \mathsf{ob}\,\mathcal{K}$. For each object J of \mathcal{K} , the diagram (8.4.i) describes a diagrammatic \mathcal{S} -equation $\mathcal{C}(J,A) \rightrightarrows \mathcal{C}(J,A)$ ($A \in \mathcal{S}$ -Alg). For each pair J,K of objects of \mathcal{K} , the diagram (8.4.ii) describes a diagrammatic \mathcal{S} -equation $\mathcal{C}(K,A) \rightrightarrows \mathcal{C}(\mathcal{C}(J,HK) \otimes HJ,A)$ ($A \in \mathcal{S}$ -Alg). We thus obtain a diagrammatic \mathcal{K} -presentation $\mathcal{P}_H = (\mathcal{S},\mathsf{E})$, and \mathcal{P}_H -algebras are precisely H-algebras, in view of 8.5.

Definition 8.8. Given a \mathscr{K} -relative \mathscr{V} -monad H for a small full sub- \mathscr{V} -category $\mathscr{K} \hookrightarrow \mathscr{C}$, the \mathscr{V} -category of H-algebras is by definition the \mathscr{V} -category H-Alg of \mathcal{P}_H -algebras (5.18, 8.7).

Remark 8.9. In the special case of 8.3 and 8.6, where $\mathscr{K} = \mathscr{J}$, the \mathscr{V} -category H-Alg is isomorphic to the \mathscr{V} -category $\Phi(H)$ -Alg of algebras for the associated \mathscr{J} -ary \mathscr{V} -monad $\Phi(H)$, which we may equally call the \mathscr{V} -category of H-modules in \mathscr{C} for the monoid H in \mathscr{V} -CAT(\mathscr{J},\mathscr{C}).

Theorem 8.10. Let \mathscr{C} and \mathscr{V} be as in 2.1, and let $\mathscr{K} \hookrightarrow \mathscr{C}$ be a full sub- \mathscr{V} -category that is contained in some bounded and eleutheric subcategory of arities $\mathscr{J} \hookrightarrow \mathscr{C}$. For each \mathscr{K} -relative \mathscr{V} -monad H, the \mathscr{V} -category H-Alg of H-algebras is strictly \mathscr{J} -monadic over \mathscr{C} (as it is a \mathscr{J} -ary variety).

Proof. Under the given hypotheses, we may regard the diagrammatic \mathcal{K} -presentation \mathcal{P}_H also as a diagrammatic \mathcal{J} -presentation, and the result follows from Theorem 5.20.

Example 8.11. Let \mathscr{V} be a locally α -presentable closed category and \mathscr{C} a locally α -presentable \mathscr{V} -category. Then any small full sub- \mathscr{V} -category $\mathscr{K} \hookrightarrow \mathscr{C}$ is contained in a bounded and eleutheric subcategory of arities $\mathscr{J} \hookrightarrow \mathscr{C}$. Indeed, by [20, 7.4] we may choose some regular cardinal $\beta \geq \alpha$ such that every object of \mathscr{K} is β -presentable in the enriched sense, and hence \mathscr{K} is contained in the bounded and eleutheric subcategory of arities $\mathscr{J} = \mathscr{C}_{\beta} \hookrightarrow \mathscr{C}$ discussed in 2.4(1), where we noted that these data satisfy the assumptions of 2.1 and 2.2. Hence, Theorem 8.10 entails the following:

Corollary 8.12. Let $\mathscr V$ be a locally α -presentable closed category, let $\mathscr C$ be a locally α -presentable $\mathscr V$ -category, and let $\mathscr K \hookrightarrow \mathscr C$ be a small full sub- $\mathscr V$ -category. Then there is a regular cardinal $\beta \geq \alpha$ such that for each $\mathscr K$ -relative $\mathscr V$ -monad H on $\mathscr C$, the $\mathscr V$ -category of H-algebras H-Alg is strictly β -ary monadic over $\mathscr C$.

Definition 8.13. Returning to the setting of a given subcategory of arities $\mathscr{J} \hookrightarrow \mathscr{C}$ satisfying the assumptions in 2.1 and 2.2, let \mathbb{T} be a \mathscr{J} -ary \mathscr{V} -monad on \mathscr{C} , recalling that $Tj: \mathscr{J} \to \mathscr{C}$ then carries the structure of a \mathscr{J} -relative \mathscr{V} -monad that we denote simply by Tj (8.3). By 8.8 and 8.9, we have a diagrammatic \mathscr{J} -presentation \mathcal{P}_{Tj} with \mathcal{P}_{Tj} -Alg = Tj-Alg in \mathscr{V} -CAT/ \mathscr{C} . We call \mathcal{P}_{Tj} the **standardized** \mathscr{J} -**presentation of** \mathbb{T} . Note that a \mathcal{P}_{Tj} -algebra is an object A of \mathscr{C} equipped with morphisms $\mathscr{C}(J,A) \to \mathscr{C}(TJ,A)$ or $\mathscr{C}(J,A) \otimes TJ \to A$ ($J \in \mathsf{ob} \mathscr{J}$) satisfying associativity and unit axioms (8.4, 8.5).

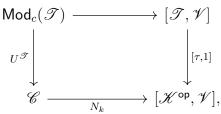
8.14. Let $\mathscr{C} = \mathscr{V}$, and suppose that $\mathscr{J} \hookrightarrow \mathscr{V}$ is a *system* of arities. By 2.5, $\mathsf{Mnd}_{\mathscr{J}}(\mathscr{V})$ is equivalent to the category $\mathsf{Th}_{\mathscr{J}}$ of \mathscr{J} -theories. Let \mathscr{T} be a \mathscr{J} -theory with associated identity-on-objects \mathscr{J} -cotensor-preserving \mathscr{V} -functor $\tau: \mathscr{J}^{\mathsf{op}} \to \mathscr{T}$. Writing \mathbb{T} for the \mathscr{J} -ary \mathscr{V} -monad on \mathscr{V} corresponding to \mathscr{T} , we may consider also the \mathscr{J} -relative \mathscr{V} -monad on \mathscr{V} corresponding to \mathbb{T} , whose underlying \mathscr{V} -functor is $\mathscr{T}(\tau-,I): \mathscr{J} \to \mathscr{V}$ (see [33, 10.3]). By 8.13, this \mathscr{J} -relative \mathscr{V} -monad determines a diagrammatic \mathscr{J} -presentation \mathscr{P} for which \mathscr{P} -Alg is isomorphic to the \mathscr{V} -category of \mathbb{T} -algebras and hence to the \mathscr{V} -category of normal \mathscr{T} -algebras, which is in turn equivalent to the \mathscr{V} -category of (all) \mathscr{T} -algebras (see 2.5). A \mathscr{P} -algebra is then given by an object A of \mathscr{V} equipped with structural morphisms $\alpha_J^A: \mathscr{T}(J,I) \to \mathscr{V}(\mathscr{V}(J,A),A)$ or $\alpha_J^A: \mathscr{T}(J,I) \otimes \mathscr{V}(J,A) \to A$ $(J \in \mathsf{ob} \mathscr{J})$ satisfying associativity and unit axioms.

Example 8.15. As in 6.1, let \mathscr{V} be a complete and cocomplete cartesian closed category, and let R be an internal rig in \mathscr{V} . Employing the system of arities $\mathsf{SF}(\mathscr{V}) \hookrightarrow \mathscr{V}$ (2.5), in 6.1 we exhibited a diagrammatic $\mathsf{SF}(\mathscr{V})$ -presentation whose algebras are the (left) R-modules in \mathscr{V} . We may also regard R-modules equivalently as the normal algebras of an $\mathsf{SF}(\mathscr{V})$ -theory, namely the \mathscr{V} -category of R-matrices Mat_R , which has hom-objects $\mathsf{Mat}_R(m,n) = R^{n\times m}$ ($m,n\in\mathbb{N}$), with composition given by internal matrix multiplication [32, 6.4.6]. Invoking 8.14, we thus obtain another diagrammatic $\mathsf{SF}(\mathscr{V})$ -presentation for R-modules. As in 8.14, an algebra for this diagrammatic $\mathsf{SF}(\mathscr{V})$ -presentation is an object A of \mathscr{V} equipped with structural morphisms $\alpha_n^A: \mathsf{Mat}_R(n,1) \times A^n \to A$, i.e. $\alpha_n^A: R^n \times A^n \to A$, for all $n \in \mathbb{N}$, satisfying associativity and unit axioms, since $\mathsf{Mat}_R(n,1) = R^{1\times n} = R^n$. The structural morphism $\alpha_n^A: R^n \times A^n \to A$ can be regarded as the parametrized operation of taking n-ary R-linear combinations in A.

As in [32, 8.5], there is a (non-full) sub- \mathscr{V} -category $\mathsf{Mat}_R^{\mathsf{aff}}$ of Mat_R whose objects are again the natural numbers and whose hom-objects are subobjects $\mathsf{Mat}_R^{\mathsf{aff}}(m,n) \hookrightarrow R^{n\times m}$ that describe those matrices in which each row has sum 1. By [32, 8.5], $\mathsf{Mat}_R^{\mathsf{aff}}$ is an $\mathsf{SF}(\mathscr{V})$ -theory whose normal algebras in \mathscr{V} are called (internal left) R-affine spaces; also see [35, 11.2.7]. Hence, by 8.14, we obtain a diagrammatic $\mathsf{SF}(\mathscr{V})$ -presentation \mathscr{P} for which the associated notion of \mathscr{P} -algebra provides an equivalent way of defining the notion of internal R-affine space, as an object A of \mathscr{V} equipped with structural morphisms $\alpha_n^A: R^{n,\mathsf{aff}} \times A^n \to A$, satisfying associativity and unit axioms, where $R^{n,\mathsf{aff}}:=\mathsf{Mat}_R^{\mathsf{aff}}(n,1) \hookrightarrow R^n$ is the subobject of n-tuples of R whose sum is 1. The structural morphism α_n^A can thus be regarded as the parametrized operation of taking n-ary R-affine combinations in A. We call R-Aff := \mathscr{P} -Alg the \mathscr{V} -category of (left) R-affine spaces.

9 A more general monad-pretheory adjunction

In the recent paper [10], Bourke and Garner fix a small subcategory of arities $k: \mathcal{K} \to \mathcal{C}$ in a locally presentable \mathcal{V} -category \mathcal{C} over a locally presentable closed category \mathcal{V} , and they consider the notion of a \mathcal{K} -pretheory, which is by definition a \mathcal{V} -category \mathcal{T} equipped with a \mathcal{V} -functor $\tau: \mathcal{K}^{op} \to \mathcal{T}$ that is identity-on-objects⁴, so that \mathcal{K} -pretheories are the objects of a full subcategory $\mathsf{Preth}_{\mathcal{K}}(\mathcal{C})$ of the coslice $\mathcal{K}^{op}/\mathcal{V}$ -Cat. Letting $\mathsf{Mnd}(\mathcal{C})$ be the category of \mathcal{V} -monads on \mathcal{C} , they define a functor $\Phi: \mathsf{Mnd}(\mathcal{C}) \to \mathsf{Preth}_{\mathcal{K}}(\mathcal{C})$ that sends a \mathcal{V} -monad \mathbb{T} on \mathcal{C} to the Kleisli \mathcal{K} -pretheory $\mathcal{K}^{op}_{\mathbb{T}}$, i.e. the opposite of the full sub- \mathcal{V} -category $\mathcal{K}^{op}_{\mathbb{T}}$ of the Kleisli \mathcal{V} -category $\mathcal{C}_{\mathbb{T}}$ on the objects of \mathcal{K} , equipped with the \mathcal{V} -functor $\mathcal{K}^{op} \to \mathcal{K}^{op}_{\mathbb{T}}$ obtained by restricting the Kleisli left adjoint $F_{\mathbb{T}}: \mathcal{C} \to \mathcal{C}_{\mathbb{T}}$. By [10, Theorem 6], Φ has a left adjoint $\Psi: \mathsf{Preth}_{\mathcal{K}}(\mathcal{C}) \to \mathsf{Mnd}(\mathcal{C})$ that sends each \mathcal{K} -pretheory \mathcal{T} to a \mathcal{V} -monad $\Psi(\mathcal{T})$ whose \mathcal{V} -category of algebras $\Psi(\mathcal{T})$ -Alg is isomorphic (in \mathcal{V} -CAT/ \mathcal{C}) to the \mathcal{V} -category $\mathsf{Mod}_c(\mathcal{T})$ of concrete \mathcal{T} -models. This \mathcal{V} -category is defined as the following pullback in \mathcal{V} -CAT



where $N_k : \mathscr{C} \to [\mathscr{K}^{op}, \mathscr{V}]$ is the k-nerve \mathscr{V} -functor defined by $N_k C = \mathscr{C}(k-,C)$ ($C \in \mathscr{C}$). A concrete \mathscr{T} -model is therefore an object A of \mathscr{C} equipped with a \mathscr{V} -functor $M : \mathscr{T} \to \mathscr{V}$ that extends the k-nerve $\mathscr{C}(k-,A) : \mathscr{K}^{op} \to \mathscr{V}$ along $\tau : \mathscr{K}^{op} \to \mathscr{T}$, i.e. $M \circ \tau = \mathscr{C}(k-,A)$. Since τ is identity-on-objects, M is then given on objects by $MK = \mathscr{C}(K,A)$ ($K \in \mathsf{ob} \mathscr{T} = \mathsf{ob} \mathscr{K}$).

Our objective is now to employ diagrammatic \mathcal{J} -presentations to generalize this monadpretheory adjunction from the locally presentable context of [10] to the more general context

⁴In [10], the dual notion is used, while in the present paper we employ a notion of pretheory that accords with the concept previously studied (in the unenriched context) by Linton [29, p. 20] under the name *clone*, and later by Diers [13] under the name *theory*.

⁵This definition (in the unenriched context) goes back to Linton [29] and Diers [13].

of a small subcategory of arities $\mathscr{K} \hookrightarrow \mathscr{C}$ that is contained in some bounded and eleutheric subcategory of arities $\mathscr{J} \hookrightarrow \mathscr{C}$ (2.2), where \mathscr{C} and \mathscr{V} need only satisfy the weaker hypotheses of the present paper (2.1). We shall then recover the monad-pretheory adjunction of [10] as a special case, since in the locally presentable setting of [10], every small subcategory of arities $\mathscr{K} \hookrightarrow \mathscr{C}$ is contained in a bounded and eleutheric subcategory of arities (8.11). Towards these ends, we first prove the following result, now transporting the above notions of pretheory and concrete model to the general setting of the present paper, under just the assumptions of 2.1:

Theorem 9.1. Let $\mathscr C$ and $\mathscr V$ be as in 2.1, and let $k:\mathscr K\hookrightarrow\mathscr C$ be a subcategory of arities that is contained in some bounded and eleutheric subcategory of arities $j:\mathscr J\hookrightarrow\mathscr C$. For any $\mathscr K$ -pretheory $\tau:\mathscr K^{\mathsf{op}}\to\mathscr T$, there is a diagrammatic $\mathscr J$ -presentation $\mathscr P_\mathscr T$ with $\mathscr P_\mathscr T$ -Alg $\cong \mathsf{Mod}_c(\mathscr T)$ in $\mathscr V$ -CAT/ $\mathscr C$, so $\mathsf{Mod}_c(\mathscr T)$ is a strictly $\mathscr J$ -monadic $\mathscr V$ -category over $\mathscr C$.

Proof. A concrete \mathscr{T} -model A consists of an object A of \mathscr{C} equipped with a \mathscr{V} -functor $M^A: \mathscr{T} \to \mathscr{V}$ satisfying certain conditions, including the requirement that $M^AK = \mathscr{C}(K,A)$ for all $K \in \mathsf{ob}\,\mathscr{T} = \mathsf{ob}\,\mathscr{K}$. In particular, M^A has structural morphisms $M^A_{KK'}: \mathscr{T}(K,K') \to \mathscr{V}(\mathscr{C}(K,A),\mathscr{C}(K',A))$ $(K,K' \in \mathsf{ob}\,\mathscr{K})$, which are equivalently given by morphisms $M^A_{KK'}: \mathscr{C}(K,A) \otimes \mathscr{T}(K,K') \otimes K' \to A$, since \mathscr{C} is tensored. We now define a free-form \mathscr{J} -signature \mathscr{S} consisting of an operation symbol $M_{KK'}$ of arity $K \in \mathsf{ob}\,\mathscr{K} \subseteq \mathsf{ob}\,\mathscr{J}$ and parameter $\mathscr{T}(K,K') \otimes K' \in \mathsf{ob}\,\mathscr{C}$ for each pair $(K,K') \in \mathsf{ob}\,\mathscr{K} \times \mathsf{ob}\,\mathscr{K}$. An \mathscr{S} -algebra is thus an object A of \mathscr{C} equipped with a \mathscr{V} -graph morphism $M^A: \mathscr{T} \to \mathscr{V}$ given on objects by $M^AK = \mathscr{C}(K,A)$ $(K \in \mathsf{ob}\,\mathscr{T} = \mathsf{ob}\,\mathscr{K})$, where we write simply \mathscr{T} and \mathscr{V} for the \mathscr{V} -graphs underlying \mathscr{T} and \mathscr{V} .

We now define a diagrammatic \mathscr{J} -presentation $\mathscr{P}_{\mathscr{T}} = (\mathcal{S}, \mathsf{E})$ for which $\mathscr{P}_{\mathscr{T}}$ -algebras will be precisely the concrete \mathscr{T} -models. We need diagrammatic \mathscr{S} -equations expressing preservation of composition and identities by a concrete \mathscr{T} -model. For an \mathscr{S} -algebra A to be a concrete \mathscr{T} -model, we need the following diagram to commute for each triple K, K', K'' of objects of \mathscr{K} :

$$\mathscr{C}(K,A)\otimes\mathscr{T}(K,K')\otimes\mathscr{T}(K',K'')\otimes K'' \xrightarrow{1\otimes c_{KK'K''}\otimes 1} \mathscr{C}(K,A)\otimes\mathscr{T}(K,K'')\otimes K'' \\ \downarrow^{M_{KK''}^A} \\ A \\ \uparrow^{M_{KK''}^A} \\ \mathscr{C}(K',\mathscr{C}(K,A)\otimes\mathscr{T}(K,K')\otimes K')\otimes\mathscr{T}(K',K'')\otimes K'' \xrightarrow{\mathscr{C}\left(1,M_{KK'}^A\right)\otimes 1} \mathscr{C}(K',A)\otimes\mathscr{T}(K',K'')\otimes K'',$$

where $c_{KK'K''}$ is a composition morphism of \mathscr{T} and **coev** is the unit morphism at $\mathscr{C}(K,A) \otimes \mathscr{T}(K,K') \in \mathsf{ob}\,\mathscr{V}$ of the tensor-hom adjunction $(-) \otimes K' \dashv \mathscr{C}(K',-) : \mathscr{C} \to \mathscr{V}$. This diagram describes a diagrammatic \mathscr{S} -equation $\mathscr{C}(K,A) \otimes \mathscr{T}(K,K') \otimes \mathscr{T}(K',K'') \otimes K'' \rightrightarrows A$ $(A \in \mathscr{S}\text{-Alg})$ with arity K and parameter $\mathscr{T}(K,K') \otimes \mathscr{T}(K',K'') \otimes K''$. For each object K of \mathscr{K} , we also require the commutativity of the following diagram, which describes a

diagrammatic S-equation $\mathscr{C}(K,A) \otimes K \rightrightarrows A \ (A \in S\text{-Alg})$:

$$\mathscr{C}(K,A)\otimes K \xrightarrow{\sim} \mathscr{C}(K,A)\otimes I\otimes K \xrightarrow{1\otimes u_K\otimes 1} \mathscr{C}(K,A)\otimes \mathscr{T}(K,K)\otimes K \xrightarrow{\text{ev}} A,$$

where $u_K: I \to \mathcal{F}(K, K)$ is an identity morphism of \mathcal{F} and ev is the counit morphism at $A \in \operatorname{ob} \mathcal{C}$ of the tensor-hom adjunction $(-) \otimes K \dashv \mathcal{C}(K, -) : \mathcal{C} \to \mathcal{V}$.

Lastly, we need diagrammatic S-equations expressing that, for a concrete \mathscr{T} -model A, the \mathscr{V} -functor $M^A: \mathscr{T} \to \mathscr{V}$ extends $\mathscr{C}(k-,A): \mathscr{K}^{\mathsf{op}} \to \mathscr{V}$ along $\tau: \mathscr{K}^{\mathsf{op}} \to \mathscr{T}$. So for an S-algebra A to be a concrete \mathscr{T} -model, we need the following diagram to commute for all objects K, K' of \mathscr{K} :

$$\begin{split} \mathscr{C}(K,A) \otimes \mathscr{K}(K',K) \otimes K' & \xrightarrow{1 \otimes \tau_{KK'} \otimes 1} & \mathscr{C}(K,A) \otimes \mathscr{T}(K,K') \otimes K' \\ \downarrow & & \downarrow^{M_{KK'}^A} \\ \mathscr{C}(K',K) \otimes \mathscr{C}(K,A) \otimes K' & \xrightarrow{c_{K'KA} \otimes 1} & \mathscr{C}(K',A) \otimes K' & \xrightarrow{\text{ev}} & A, \end{split}$$

where $c_{K'KA}$ is a composition morphism of \mathscr{C} . This diagram describes a diagrammatic \mathscr{S} -equation $\mathscr{C}(K,A)\otimes\mathscr{K}(K',K)\otimes K' \rightrightarrows A \ (A\in \mathscr{S}\text{-Alg})$. We thus obtain a diagrammatic \mathscr{J} -presentation $\mathscr{P}_{\mathscr{T}}$ for which $\mathscr{P}_{\mathscr{T}}$ -algebras are in bijective correspondence with concrete \mathscr{T} -models, and it is straightforward to verify (using 5.23 and the definition of $\mathsf{Mod}_c(\mathscr{T})$ as a pullback in $\mathscr{V}\text{-CAT}$) that this bijection on objects extends to an isomorphism $\mathscr{P}_{\mathscr{T}}\text{-Alg} \cong \mathsf{Mod}_c(\mathscr{T})$ in $\mathscr{V}\text{-CAT}/\mathscr{C}$. Hence $\mathsf{Mod}_c(\mathscr{T})$ is strictly \mathscr{J} -monadic over \mathscr{C} by 5.20.

In the current general context, where \mathscr{C} and \mathscr{V} satisfy the background hypotheses of 2.1 and $\mathscr{K} \hookrightarrow \mathscr{C}$ is a small subcategory of arities, we can again define a functor $\Phi : \mathsf{Mnd}(\mathscr{C}) \to \mathsf{Preth}_{\mathscr{K}}(\mathscr{C})$ exactly as described at the beginning of this section. Moreover, under the hypotheses of 9.1, we can prove the following generalization of [10, Theorem 6]:

Theorem 9.2. Let \mathscr{C} and \mathscr{V} be as in 2.1, and let $k: \mathscr{K} \hookrightarrow \mathscr{C}$ be a subcategory of arities that is contained in some bounded and eleutheric subcategory of arities $j: \mathscr{J} \hookrightarrow \mathscr{C}$. Then $\Phi: \mathsf{Mnd}(\mathscr{C}) \to \mathsf{Preth}_{\mathscr{K}}(\mathscr{C})$ has a left adjoint $\Psi: \mathsf{Preth}_{\mathscr{K}}(\mathscr{C}) \to \mathsf{Mnd}(\mathscr{C})$ that sends each \mathscr{K} -pretheory \mathscr{T} to a \mathscr{V} -monad $\Psi(\mathscr{T})$ such that $\Psi(\mathscr{T})$ -Alg $\cong \mathsf{Mod}_c(\mathscr{T})$ in \mathscr{V} -CAT/ \mathscr{C} . Furthermore, $\Psi(\mathscr{T})$ is a \mathscr{J} -ary \mathscr{V} -monad for each \mathscr{K} -pretheory \mathscr{T} .

Proof. By [10, Theorem 2], which (as Bourke and Garner emphasize) does not require the local presentability of \mathscr{C} or \mathscr{V} , and (hence) still holds under the more general hypotheses of 2.1, if the forgetful \mathscr{V} -functor $U^{\mathscr{T}}: \mathsf{Mod}_c(\mathscr{T}) \to \mathscr{C}$ has a left adjoint for each \mathscr{K} -pretheory \mathscr{T} , then Φ has a left adjoint Ψ that sends \mathscr{T} to the \mathscr{V} -monad $\Psi(\mathscr{T})$ induced by this adjunction. But, under the present hypotheses, $U^{\mathscr{T}}$ is strictly \mathscr{J} -monadic by Theorem 9.1.

Theorem 9.2 significantly generalizes [10, Theorem 6] by removing the assumptions that \mathscr{C} and \mathscr{V} are locally presentable. Thus, Theorem 9.2 is applicable to all of the examples of 2.4, in most of which \mathscr{V} need not be locally presentable. In view of the discussion preceding 9.1, the hypotheses of Theorem 9.2 are automatically satisfied in the locally presentable context by taking $\mathscr{J} = \mathscr{C}_{\beta}$ for a suitable regular cardinal β , so that as a special case of 9.2 we recover [10, Theorem 6]: If $\mathscr{K} \hookrightarrow \mathscr{C}$ is a small subcategory of arities in a locally presentable \mathscr{V} -category \mathscr{C} over a locally presentable closed category \mathscr{V} , then $\Phi : \mathsf{Mnd}(\mathscr{C}) \to \mathsf{Preth}_{\mathscr{K}}(\mathscr{C})$ has a left adjoint Ψ such that $\Psi(\mathscr{T})$ -Alg $\cong \mathsf{Mod}_c(\mathscr{T})$ in \mathscr{V} -CAT/ \mathscr{C} for each \mathscr{K} -pretheory \mathscr{T} .

Definition 9.3. Let $\mathscr C$ and $\mathscr V$ be as in 2.1, and let $\mathscr K \hookrightarrow \mathscr C$ be a small subcategory of arities. A $\mathscr V$ -monad $\mathbb T$ on $\mathscr C$ is $\mathscr K$ -nervous if $\mathbb T \cong \Psi(\mathscr T)$ for some $\mathscr K$ -pretheory $\mathscr T$.

This is not the original definition of \mathcal{K} -nervous \mathcal{V} -monad given in [10, Definition 17], but it is equivalent to their definition (in their locally presentable context) by [10, Corollary 21].

Corollary 9.4. Let \mathscr{C} and \mathscr{V} be as in 2.1, and let $\mathscr{K} \hookrightarrow \mathscr{C}$ be a subcategory of arities that is contained in some bounded and eleutheric subcategory of arities $\mathscr{J} \hookrightarrow \mathscr{C}$. Then every \mathscr{K} -nervous \mathscr{V} -monad \mathbb{T} is \mathscr{J} -ary.

Proof. There is some \mathscr{K} -pretheory \mathscr{T} with $\mathbb{T} \cong \Psi(\mathscr{T})$, and $\Psi(\mathscr{T})$ is \mathscr{J} -ary by 9.2.

Corollary 9.5. Let $\mathcal{K} \hookrightarrow \mathcal{C}$ be a small subcategory of arities in a locally α -presentable \mathcal{V} -category \mathcal{C} over a locally α -presentable closed category \mathcal{V} . Then there is a regular cardinal $\beta \geq \alpha$ such that every \mathcal{K} -nervous \mathcal{V} -monad on \mathcal{C} is a β -ary \mathcal{V} -monad (2.4).

Proof. In view of 8.11, this follows from 9.4. \Box

10 A Birkhoffian Galois connection for \mathscr{J} -ary varieties enriched in \mathscr{V}

Throughout this final section, we fix a free-form \mathcal{J} -signature \mathcal{S} .⁶ If \mathcal{P} is a free-form (or, equivalently, diagrammatic) \mathcal{J} -presentation of the form $\mathcal{P} = (\mathcal{S}, \mathsf{E})$, then we say that \mathcal{P} is a **free-form** \mathcal{J} -presentation over \mathcal{S} , and we call \mathcal{P} -Alg a \mathcal{J} -ary variety over \mathcal{S} (cf. 5.25). Hence, a \mathcal{J} -ary variety over \mathcal{S} is equivalently given by a (possibly large) set of \mathcal{S} -algebras of the form $\mathsf{ob}(\mathcal{P}\text{-Alg})$ for a free-form \mathcal{J} -presentation \mathcal{P} over \mathcal{S} , and we refer to such sets also as \mathcal{J} -ary varieties over \mathcal{S} .

For the remainder of the section, we also fix a generating set \mathscr{G} for \mathscr{C}_0 , i.e. a set of objects $\mathscr{G} \subseteq \mathsf{ob}\,\mathscr{C}$ such that the functors $\mathscr{C}_0(G,-):\mathscr{C}_0 \to \mathsf{Set}\ (G \in \mathscr{G})$ are jointly faithful; we do not assume that \mathscr{G} is small (e.g. we allow $\mathscr{G} = \mathsf{ob}\,\mathscr{C}$). Finally, we assume for this section that (in addition to the assumptions of 2.1) the category \mathscr{C}_0 is complete, so that the \mathscr{V} -category \mathscr{C} is complete (being tensored and cotensored).

In this section, we establish and study a Galois connection between sets of S-algebras and sets of G-parametrized S-equations (10.1) whose fixed points on one side are precisely

⁶Although we have chosen to focus on a free-form \mathscr{J} -signature \mathscr{S} and the \mathscr{J} -ary \mathscr{V} -monad $\mathbb{T}_{\mathscr{S}}$ generated by \mathscr{S} (5.8), the results of this section also hold when replacing $\mathbb{T}_{\mathscr{S}}$ by a general \mathscr{J} -ary \mathscr{V} -monad \mathbb{T} .

the \mathscr{J} -ary varieties over \mathscr{S} , thus generalizing the classical Galois connection of Birkhoff [6]. We shall conclude the section by investigating conditions under which a \mathscr{J} -ary variety over \mathscr{S} is generated by a given \mathscr{S} -algebra or a given set of \mathscr{S} -algebras.

Definition 10.1. A \mathscr{G} -parametrized \mathscr{S} -equation is a parametrized \mathscr{S} -equation $t \doteq u : G \rightrightarrows T_{\mathscr{S}}J$ (5.13) whose parameter G is an object of \mathscr{G} . We then have the (possibly large) set $\mathsf{Eqn}_{\mathscr{G}}(\mathscr{S})$ of \mathscr{G} -parametrized \mathscr{S} -equations, which may also be described as the disjoint union

$$\mathsf{Eqn}_{\mathscr{G}}(\mathcal{S}) \ = \coprod_{J \in \mathsf{ob} \ \mathscr{J}, \ G \in \mathscr{G}} \mathscr{C}_0 \left(G, T_{\mathcal{S}} J \right)^2.$$

We let $\mathscr{P}(\mathsf{Eqn}_{\mathscr{G}}(\mathcal{S}))$ be the (possibly large) set of all subsets of $\mathsf{Eqn}_{\mathscr{G}}(\mathcal{S})$ and $\mathscr{P}(\mathcal{S}\text{-Alg})$ the (possibly large) set of all subsets of $\mathsf{ob}(\mathcal{S}\text{-Alg})$.

10.2. Note that if $\mathscr{G} = \mathsf{ob}\,\mathscr{C}$, then an $\mathsf{ob}\,\mathscr{C}$ -parametrized \mathcal{S} -equation is just a parametrized \mathcal{S} -equation (5.13). We write

$$\mathsf{Eqn}(\mathcal{S}) := \mathsf{Eqn}_{\mathsf{ob}\,\mathscr{C}}(\mathcal{S})$$

to denote the set of all parametrized S-equations. Note also that if $\mathcal{E} \subseteq \mathsf{Eqn}(S)$ is small, then \mathcal{E} may be regarded as a system of parametrized \mathscr{J} -ary equations over $\mathbb{T}_{\mathcal{S}}$ (4.3), so that $\mathcal{P} = (\mathcal{S}, \mathcal{E})$ is a free-form \mathscr{J} -presentation (5.15).

Definition 10.3. We now introduce some notation related to the satisfaction relation \models for \mathcal{S} -algebras and parametrized \mathcal{S} -equations (5.14), recalling that \models is a binary relation from ob (\mathcal{S} -Alg) to Eqn(\mathcal{S}). If $\mathcal{A} \in \mathscr{P}(\mathcal{S}$ -Alg) and $\mathcal{E} \in \mathscr{P}(\mathsf{Eqn}(\mathcal{S}))$, then we write $\mathcal{A} \models \mathcal{E}$ to mean that $A \models t \doteq u$ (5.14) for all $A \in \mathcal{A}$ and all $(t, u) \in \mathcal{E}$. If $\mathcal{A} = \{A\}$, then we write $\mathcal{A} \models \mathcal{E}$ rather than $\{A\} \models \mathcal{E}$, and if $\mathcal{E} = \{(t, u)\}$, then we write $\mathcal{A} \models t \doteq u$ rather than $\mathcal{A} \models \{(t, u)\}$. Given a (possibly large) set \mathcal{E} of parametrized \mathcal{S} -equations, we write

$$\mathsf{Sat}^*(\mathcal{E}) := \{ A \in \mathsf{ob}(\mathcal{S}\text{-}\mathsf{Alg}) \mid A \models \mathcal{E} \}. \tag{10.3.i}$$

Given a (possibly large) set of S-algebras A, we write

$$\operatorname{Sat}_*(\mathcal{A}) := \{(t, u) \in \operatorname{Eqn}(\mathcal{S}) \mid \mathcal{A} \models t \doteq u\},$$

$$\operatorname{Sat}_*^{\mathscr{G}}(\mathcal{A}) := \{(t, u) \in \operatorname{Eqn}_{\mathscr{G}}(\mathcal{S}) \mid \mathcal{A} \models t \doteq u\}.$$

$$(10.3.ii)$$

Remark 10.4. In view of 10.2, a set of S-algebras A is a \mathscr{J} -ary variety over S if and only if $A = \mathsf{Sat}^*(\mathcal{F})$ for some *small* set \mathcal{F} of parametrized S-equations.

As far as the satisfaction relation \models is concerned, sets of parametrized S-equations can be represented faithfully as sets of \mathscr{G} -parametrized S-equations, by way of the following construction:

Definition 10.5. Given a parametrized S-equation $t \doteq u : C \rightrightarrows T_{\mathcal{S}}J$, we write $[t \doteq u]_{\mathscr{G}}$ to denote the set of all \mathscr{G} -parametrized S-equations of the form $t \circ x \doteq u \circ x : G \rightrightarrows T_{\mathcal{S}}J$ where G is an object in \mathscr{G} and $x : G \to C$ is a morphism in \mathscr{C} . Given instead a set \mathscr{E} of parametrized S-equations, we define $[\mathscr{E}]_{\mathscr{G}}$ to be the union of the sets $[t \doteq u]_{\mathscr{G}}$ associated to the parametrized S-equations $t \doteq u$ in \mathscr{E} .

Proposition 10.6. Let \mathcal{A} be a set of \mathcal{S} -algebras, and let \mathcal{E} be a set of parametrized \mathcal{S} -equations. Then $\mathcal{A} \models \mathcal{E}$ if and only if $\mathcal{A} \models [\mathcal{E}]_{\mathscr{G}}$.

Proof. Since \mathscr{G} is a generating set for \mathscr{C}_0 , this follows readily from 4.5.

Corollary 10.7. Let \mathcal{E} be a set of parametrized \mathcal{S} -equations. Then $\mathsf{Sat}^*(\mathcal{E}) = \mathsf{Sat}^*([\mathcal{E}]_{\mathscr{G}})$. \square

Remark 10.8. Given a set \mathcal{A} of \mathcal{S} -algebras, each of the sets $\mathsf{Sat}_*(\mathcal{A})$ and $\mathsf{Sat}_*^{\mathscr{G}}(\mathcal{A})$ can be expressed in terms of the other, as follows: firstly, $\mathsf{Sat}_*^{\mathscr{G}}(\mathcal{A})$ is clearly the intersection of $\mathsf{Sat}_*(\mathcal{A})$ with $\mathsf{Eqn}_{\mathscr{G}}(\mathcal{S})$, and secondly, it follows from 10.6 that the $\mathsf{Sat}_*(\mathcal{A})$ is the set of all parametrized \mathcal{S} -equations $t \doteq u$ such that $\mathcal{A} \models [t \doteq u]_{\mathscr{G}}$, i.e. such that $[t \doteq u]_{\mathscr{G}} \subseteq \mathsf{Sat}_*^{\mathscr{G}}(\mathcal{A})$.

Regarding $\mathscr{P}(\mathsf{Eqn}_\mathscr{G}(\mathcal{S}))$ and $\mathscr{P}(\mathcal{S}\text{-Alg})$ as (possibly large) partially ordered sets under inclusion, the following result now follows immediately from the definitions and is an instance of the well-known fact that any relation between sets induces a Galois connection between their power sets:

Proposition 10.9. There is a Galois connection

$$\mathscr{P}(\mathsf{Eqn}_\mathscr{G}(\mathcal{S})) \xrightarrow[\mathsf{Sat}^\mathscr{G}]{\mathsf{Sat}^\mathscr{G}} \mathscr{P}(\mathcal{S}\text{-Alg})^\mathsf{op}$$

in which Sat^* and $\mathsf{Sat}^{\mathscr{G}}_*$ are given by (10.3.i) and (10.3.ii), respectively. That is, Sat^* and $\mathsf{Sat}^{\mathscr{G}}_*$ are order-reversing maps, and for all $\mathcal{A} \in \mathscr{P}(\mathcal{S}\text{-Alg})$ and $\mathcal{E} \in \mathscr{P}(\mathsf{Eqn}_{\mathscr{G}}(\mathcal{S}))$ we have

$$\mathcal{A}\subseteq \mathsf{Sat}^*(\mathcal{E}) \iff \mathcal{E}\subseteq \mathsf{Sat}_*^\mathscr{G}(\mathcal{A}). \endalign{\medskip}\Box$$

In the case where $\mathscr{G} = \mathsf{ob}\mathscr{C}$, 10.9 specializes to yield a Galois connection $\mathsf{Sat}^* \dashv \mathsf{Sat}_* : \mathscr{P}(\mathcal{S}\mathsf{-Alg})^\mathsf{op} \to \mathscr{P}(\mathsf{Eqn}(\mathcal{S})).$

We recover the classical Galois connection of Birkhoff [6] from 10.9 by taking $\mathscr{C} = \mathscr{V} = \mathsf{Set}$, $\mathscr{J} = \mathsf{SF}(\mathsf{Set})$ (the finite cardinals), and $\mathscr{G} = \{1\}$. In the classical setting, varieties over a given signature \mathcal{S} are the fixed points of the idempotent monad on $\mathscr{P}(\mathcal{S}\text{-Alg})$ induced by this Galois connection. The same is true in our current general setting if the generator \mathscr{G} is small (10.30). But in general we do not assume that \mathscr{G} is small, and while the fixed points of the idempotent monad on $\mathscr{P}(\mathcal{S}\text{-Alg})$ induced by 10.9 are those of the form $\mathsf{Sat}^*(\mathcal{E})$ for a set of \mathscr{G} -parametrized \mathcal{S} -equations \mathcal{E} , the set \mathcal{E} need not be small, so in view of 10.4 it is not clear that $\mathsf{Sat}^*(\mathcal{E})$ would be a \mathscr{J} -ary variety. Indeed, we intend to accommodate, for example, the case where $\mathscr{G} = \mathsf{ob}\,\mathscr{C}$.

For this reason, we now define certain mild size conditions on sets of parametrized Sequations and S-algebras, and we proceed to show that the Galois connection of 10.9 restricts
to sets satisfying these conditions:

Definition 10.10. A set \mathcal{E} of parametrized \mathcal{S} -equations is **tame** if there is a *small* set \mathcal{F} of parametrized \mathcal{S} -equations such that $\mathsf{Sat}^*(\mathcal{E}) = \mathsf{Sat}^*(\mathcal{F})$. A set \mathcal{A} of \mathcal{S} -algebras is **tame** if there is a *small* set \mathcal{B} of \mathcal{S} -algebras such that $\mathsf{Sat}_*(\mathcal{A}) = \mathsf{Sat}_*(\mathcal{B})$.

Remark 10.11. In view of 10.4, a set \mathcal{E} of parametrized \mathcal{S} -equations is tame iff $\mathsf{Sat}^*(\mathcal{E})$ is a \mathscr{J} -ary variety over \mathcal{S} .

Note that Definition 10.10 does not involve \mathscr{G} at all. But 10.8 immediately entails the following characterization of tame sets of \mathcal{S} -algebras in terms of \mathscr{G} -parametrized \mathcal{S} -equations:

Proposition 10.12. A set \mathcal{A} of \mathcal{S} -algebras is tame if and only if there is a small set \mathcal{B} of \mathcal{S} -algebras with $\mathsf{Sat}_*^{\mathscr{G}}(\mathcal{A}) = \mathsf{Sat}_*^{\mathscr{G}}(\mathcal{B})$.

Lemma 10.13. If \mathcal{F} is a small set of parametrized \mathcal{S} -equations, then $[\mathcal{F}]_{\mathscr{G}}$ is a tame set of \mathscr{G} -parametrized \mathcal{S} -equations.

Proof. This follows immediately from 10.7.

Lemma 10.14. Let \mathcal{A} be a small set of \mathcal{S} -algebras. Then $\mathsf{Sat}_*^{\mathscr{G}}(\mathcal{A})$ is a tame set of parametrized \mathcal{S} -equations.

Proof. For each $J \in \text{ob} \mathscr{J}$, let $\bar{\mathfrak{i}}_J : T_{\mathcal{S}}J \to \prod_{A \in \mathcal{A}} \langle A, A \rangle J$ be the morphism in \mathscr{C} induced by the interpretation morphisms $\mathfrak{i}_J^A : T_{\mathcal{S}}J \to \langle A, A \rangle J$ $(A \in \mathcal{A})$ of 2.9, where we regard each $A \in \mathcal{A}$ equivalently as a $\mathbb{T}_{\mathcal{S}}$ -algebra (5.8), and let $p_J^1, p_J^2 : C_J \rightrightarrows T_{\mathcal{S}}J$ be the kernel pair of $\bar{\mathfrak{i}}_J$. Now let \mathcal{F} be the set consisting of all the parametrized \mathcal{S} -equations $p_J^1 \doteq p_J^2$ with $J \in \text{ob} \mathscr{J}$. Then \mathcal{F} is small, so by 10.13 it suffices to show that $\mathsf{Sat}_*^{\mathscr{G}}(\mathcal{A}) = [\mathcal{F}]_{\mathscr{G}}$. For each \mathscr{G} -parametrized \mathcal{S} -equation $t, u : G \rightrightarrows T_{\mathcal{S}}J$ $(J \in \text{ob} \mathscr{J}, G \in \mathscr{G})$, we have that $(t, u) \in [\mathcal{F}]_{\mathscr{G}}$ iff t, u factors through the kernel pair of $\bar{\mathfrak{i}}_J$, iff $\bar{\mathfrak{i}}_J \circ t = \bar{\mathfrak{i}}_J \circ u$, iff $\bar{\mathfrak{i}}_J^A \circ t = \bar{\mathfrak{i}}_J^A \circ u$ for all $A \in \mathcal{A}$, iff $A \models t \doteq u$ for all $A \in \mathcal{A}$ (by 4.5), iff $(t, u) \in \mathsf{Sat}_*^{\mathscr{G}}(\mathcal{A})$, as desired.

Proposition 10.15. Let \mathcal{A} be a tame set of \mathcal{S} -algebras. Then $\mathsf{Sat}_*^{\mathscr{G}}(\mathcal{A})$ is a tame set of parametrized \mathcal{S} -equations.

Proof. By 10.12, there is a small set \mathcal{B} of \mathcal{S} -algebras with $\mathsf{Sat}_*^{\mathscr{G}}(\mathcal{A}) = \mathsf{Sat}_*^{\mathscr{G}}(\mathcal{B})$, and the latter is tame by 10.14.

10.16. Let $\mathcal{P} = (\mathcal{S}, \mathsf{E})$ be a free-form \mathscr{J} -presentation over \mathcal{S} , let $\mathbb{T}_{\mathcal{P}}$ be the \mathscr{J} -ary \mathscr{V} -monad presented by \mathcal{P} (5.20), and let $q: \mathbb{T}_{\mathcal{S}} \to \mathbb{T}_{\mathcal{P}}$ be the regular epimorphism in $\mathsf{Mnd}_{\mathscr{J}}(\mathscr{C})$ that presents $\mathbb{T}_{\mathcal{P}} = \mathbb{T}_{\mathcal{S}}/\mathsf{E}$ as a quotient (2.9, 4.10, 5.20). Then each $\mathbb{T}_{\mathcal{P}}$ -algebra A = (A, a) corresponds via 5.20 to a \mathcal{P} -algebra that we denote also by A, but then A is in particular an \mathcal{S} -algebra and corresponds via 5.8 to the $\mathbb{T}_{\mathcal{S}}$ -algebra $(A, a \circ q_A)$, by 2.9. Hence, 3.10 immediately entails the following:

Proposition 10.17. In the situation of 10.16, let $t, u : C \Rightarrow T_{\mathcal{S}}J$ be a parametrized \mathcal{S} -equation (where $J \in \mathsf{ob} \mathscr{J}$ and $C \in \mathsf{ob}\mathscr{C}$). Then the following are equivalent: (1) every \mathcal{P} -algebra satisfies $t \doteq u$; (2) every free \mathcal{P} -algebra on an object of \mathscr{J} satisfies $t \doteq u$; (3) the free \mathcal{P} -algebra $T_{\mathcal{P}}J$ on J satisfies $t \doteq u$; (4) $q_J \circ t = q_J \circ u$.

Corollary 10.18. Let \mathcal{P} be a free-form \mathscr{J} -presentation over \mathcal{S} . Then $\mathsf{Sat}^{\mathscr{G}}_*(\mathsf{ob}(\mathcal{P}\mathsf{-Alg})) = \mathsf{Sat}^{\mathscr{G}}_*(\{T_{\mathcal{P}}J \mid J \in \mathsf{ob} \mathscr{J}\})$.

Since \mathcal{J} is small, 10.18 entails the following (in view of 10.12):

Corollary 10.19. Every \mathscr{J} -ary variety over \mathcal{S} is a tame set of \mathcal{S} -algebras.

Corollary 10.20. Let \mathcal{E} be a tame set of parametrized \mathcal{S} -equations. Then $\mathsf{Sat}^*(\mathcal{E})$ is a tame set of \mathcal{S} -algebras (as it is a \mathcal{J} -ary variety over \mathcal{S} , by 10.11).

Definition 10.21. If \mathcal{E} is a tame set of parametrized \mathcal{S} -equations, then we call

$$\mathbb{V}(\mathcal{E}) := \mathsf{Sat}^*(\mathcal{E})$$

the variety described by \mathcal{E} . If \mathcal{A} is a tame set of \mathcal{S} -algebras, then we call

$$\Theta_{\mathscr{G}}(\mathcal{A}) := \mathsf{Sat}_*^{\mathscr{G}}(\mathcal{A})$$

the equational theory of \mathcal{A} (over \mathcal{S} with parameters in \mathcal{G}). An equational theory over \mathcal{S} (with parameters in \mathcal{G}) is a set \mathcal{E} of \mathcal{G} -parametrized \mathcal{S} -equations such that $\mathcal{E} = \Theta_{\mathcal{G}}(\mathcal{A})$ for some tame set \mathcal{A} of \mathcal{S} -algebras (noting that \mathcal{E} itself is then tame by 10.15).

From 10.15 and 10.20 we now immediately obtain the following result.

Theorem 10.22. The Galois connection of 10.9 restricts to a Galois connection between the poset $\mathscr{P}(\mathsf{Eqn}_{\mathscr{G}}(\mathcal{S}))_{\mathsf{tame}}$ of all tame sets of \mathscr{G} -parametrized \mathscr{S} -equations and the poset $\mathscr{P}(\mathcal{S}\mathsf{-Alg})_{\mathsf{tame}}$ of all tame sets of \mathscr{S} -algebras:

$$\mathscr{P}(\mathsf{Eqn}_\mathscr{G}(\mathcal{S}))_{\mathsf{tame}} \xrightarrow{\mathbb{V}} \mathscr{P}(\mathcal{S}\text{-Alg})^{\mathsf{op}}_{\mathsf{tame}} \qquad \square$$

We now show that the \mathscr{J} -ary varieties over \mathcal{S} are precisely the fixed points on one side of the Galois connection (10.22):

Theorem 10.23. Let \mathcal{A} be a (possibly large) set of \mathcal{S} -algebras. Then the following are equivalent: (1) \mathcal{A} is a \mathscr{J} -ary variety over \mathcal{S} ; (2) \mathcal{A} is tame and $\mathcal{A} = \mathbb{V}(\Theta_{\mathscr{G}}(\mathcal{A}))$; (3) there is a tame set \mathcal{E} of \mathscr{G} -parametrized \mathcal{S} -equations such that $\mathcal{A} = \mathbb{V}(\mathcal{E})$.

Proof. (2) and (3) are equivalent by 10.22, and the standard theory of Galois connections, and (3) implies (1) by 10.11. If (1) holds, then there is a small set \mathcal{F} of parametrized \mathcal{S} -equations with $\mathcal{A} = \mathsf{Sat}^*(\mathcal{F}) = \mathsf{Sat}^*([\mathcal{F}]_{\mathscr{G}})$, by 10.4 and 10.7, but $[\mathcal{F}]_{\mathscr{G}}$ is a tame set of \mathscr{G} -parametrized \mathcal{S} -equations since \mathcal{F} is small (10.13).

Invoking 10.22 and 10.23 in the case where $\mathscr{G} = \mathsf{ob}\,\mathscr{C}$, we obtain the following:

Corollary 10.24. Writing $\Theta := \Theta_{\mathsf{ob}\mathscr{C}}$, there is a Galois connection

$$\mathbb{V}\dashv\Theta:\mathscr{P}(\mathcal{S}\text{-Alg})^{\mathsf{op}}_{\mathsf{tame}}\to\mathscr{P}(\mathsf{Eqn}(\mathcal{S}))_{\mathsf{tame}}\;,$$

and the following are equivalent for a set of S-algebras A: (1) A is a \mathscr{J} -ary variety over S; (2) A is tame and $A = \mathbb{V}(\Theta(A))$; (3) there is a tame set \mathcal{E} of parametrized S-equations such that $A = \mathbb{V}(\mathcal{E})$.

10.25. The Galois connection of 10.22 induces a closure operator (i.e. an idempotent monad) $\mathbb{V}\Theta_{\mathscr{G}}$ on $\mathscr{P}(\mathcal{S}\text{-Alg})_{\mathsf{tame}}$ whose fixed points are precisely the \mathscr{J} -ary varieties over \mathcal{S} , by 10.23. Hence, given a tame set \mathcal{A} of \mathcal{S} -algebras, $\mathbb{V}\Theta_{\mathscr{G}}(\mathcal{A})$ is the smallest \mathscr{J} -ary variety over \mathcal{S} that contains \mathcal{A} . As a special case, similar conclusions are obtained with regard to the idempotent monad $\mathbb{V}\Theta$ on $\mathscr{P}(\mathcal{S}\text{-Alg})_{\mathsf{tame}}$ obtained via 10.24, and we thus deduce that the monads $\mathbb{V}\Theta$ and $\mathbb{V}\Theta_{\mathscr{G}}$ are identical. For each tame set \mathcal{A} of \mathcal{S} -algebras, we call $\mathbb{V}\Theta_{\mathscr{G}}(\mathcal{A}) = \mathbb{V}\Theta(\mathcal{A})$ the \mathscr{J} -ary variety over \mathcal{S} generated by \mathcal{A} . We may also consider the idempotent monad $\Theta_{\mathscr{G}}\mathbb{V}$ on $\mathscr{P}(\mathsf{Eqn}_{\mathscr{G}}(\mathcal{S}))_{\mathsf{tame}}$, whose fixed points are the equational theories of 10.21. Given a tame set \mathcal{E} of \mathscr{G} -parametrized \mathcal{S} -equations, we call $\Theta_{\mathscr{G}}\mathbb{V}(\mathcal{E})$ the equational theory over \mathcal{S} generated by \mathcal{E} (with parameters in \mathscr{G}), as it is the smallest equational theory over \mathcal{S} that contains \mathcal{E} .

If the generating set \mathscr{G} of \mathscr{C}_0 is small, then we can show that *every* set of parametrized \mathcal{S} -equations is tame, and that *every* set of \mathcal{S} -algebras is tame. We begin with the following observation:

Lemma 10.26. Suppose that \mathcal{G} is small. Then every set of \mathcal{G} -parametrized \mathcal{S} -equations is small.

Proof. $\mathsf{Eqn}_{\mathscr{G}}(\mathcal{S})$ is small since \mathscr{G} is small, \mathscr{J} is small, and \mathscr{C}_0 is locally small, and the result follows.

Proposition 10.27. Suppose that \mathcal{G} is small. Then every set \mathcal{E} of parametrized \mathcal{S} -equations is tame.

Proof. By 10.7, $[\mathcal{E}]_{\mathscr{G}}$ is a set of \mathscr{G} -parametrized \mathscr{S} -equations with $\mathsf{Sat}^*(\mathcal{E}) = \mathsf{Sat}^*([\mathcal{E}]_{\mathscr{G}})$, and $[\mathcal{E}]_{\mathscr{G}}$ is small by 10.26.

Lemma 10.28. Let \mathcal{A} be a set of \mathcal{S} -algebras, and suppose that $\mathsf{Sat}_*^{\mathscr{G}}(\mathcal{A})$ is a tame set of parametrized \mathcal{S} -equations. Then \mathcal{A} is tame.

Proof. By 10.20, $\mathsf{Sat}^*\mathsf{Sat}^{\mathscr{G}}_*(\mathcal{A})$ is a tame set of \mathcal{S} -algebras, so by 10.12 there is a small set \mathcal{B} of \mathcal{S} -algebras such that $\mathsf{Sat}^{\mathscr{G}}_*\mathsf{Sat}^*\mathsf{Sat}^*\mathsf{Sat}^{\mathscr{G}}_*(\mathcal{A}) = \mathsf{Sat}^{\mathscr{G}}_*(\mathcal{B})$. But $\mathsf{Sat}^{\mathscr{G}}_*\mathsf{Sat}^*\mathsf{Sat}^{\mathscr{G}}_*(\mathcal{A}) = \mathsf{Sat}^{\mathscr{G}}_*(\mathcal{A})$ by 10.22 and a general property of Galois connections, and the result follows, by 10.12.

Proposition 10.29. Suppose that \mathcal{G} is small. Then every set \mathcal{A} of \mathcal{S} -algebras is tame.

Proof. By 10.27, $\mathsf{Sat}^{\mathscr{G}}_*(\mathcal{A})$ is a tame set of parametrized \mathcal{S} -equations, so \mathcal{A} is tame by 10.28. **Theorem 10.30.** Suppose that \mathcal{C}_0 has a small generating set. Then, for any (possibly large) generating set \mathscr{G} for \mathscr{C}_0 , the Galois connections of 10.9 and of 10.22 are identical and so may be written as $\mathbb{V} \dashv \Theta_{\mathscr{G}} : \mathscr{P}(\mathcal{S}\text{-Alg})^{\mathsf{op}} \to \mathscr{P}(\mathsf{Eqn}_{\mathscr{G}}(\mathcal{S}))$. With this notation, the following are equivalent, for a (possibly large) set A of S-algebras: (1) A is a \mathscr{J} -ary variety over S; (2) $\mathcal{A} = \mathbb{V}(\Theta_{\mathscr{G}}(\mathcal{A}))$; (3) there is a (possibly large) set \mathcal{E} of \mathscr{G} -parametrized \mathcal{S} -equations such that $\mathcal{A} = \mathbb{V}(\mathcal{E})$. *Proof.* By hypothesis, \mathscr{C}_0 has some small generating set \mathscr{H} , which need not coincide with \mathscr{G} . Nevertheless, we may apply 10.27 and 10.29 with respect to \mathcal{H} in order to deduce that every set of parametrized S-equations is tame and that every set of S-algebras is tame. Hence, the result now follows from 10.23. Corollary 10.31. Suppose that \mathcal{C}_0 has a small generating set. Then there is a Galois connection $\mathbb{V} \dashv \Theta : \mathscr{P}(S\text{-Alg})^{\mathsf{op}} \to \mathscr{P}(\mathsf{Eqn}(S))$ for which the fixed points of the associated idempotent monad $V\Theta$ on $\mathscr{P}(S\text{-Alg})$ are precisely the \mathscr{J} -ary varieties over S. *Proof.* This follows from 10.30 in the case where $\mathscr{G} = \mathsf{ob}\,\mathscr{C}$. **Definition 10.32.** Let \mathcal{A} be a tame set of \mathcal{S} -algebras. We say that a \mathscr{J} -ary variety \mathcal{B} over \mathcal{S} is generated by \mathcal{A} if $\mathcal{B} = \mathbb{V}\Theta(\mathcal{A})$ (equivalently, if $\mathcal{B} = \mathbb{V}\Theta_{\mathscr{G}}(\mathcal{A})$, 10.25). As a special case, given an S-algebra A, we say that the \mathcal{J} -ary variety \mathcal{B} over \mathcal{S} is **generated by** A if it is generated by $\{A\}$. With this terminology, 10.18 entails the following: Corollary 10.33. Every \mathcal{J} -ary variety $\mathcal{B} = \mathsf{ob}(\mathcal{P}\text{-Alg})$ over \mathcal{S} is generated by the set $\{T_{\mathcal{P}}J \mid J \in \mathsf{ob} \, \mathscr{J}\}\$ consisting of the free \mathcal{P} -algebras $T_{\mathcal{P}}J$ on objects J of \mathscr{J} . We shall conclude this section with some examples of \mathcal{J} -ary varieties generated by a single \mathcal{S} -algebra, for which we shall need the following results: **Lemma 10.34.** Let \mathcal{P} be a free-form \mathcal{J} -presentation over \mathcal{S} . Then (1) \mathcal{P} -Alg is closed under (weighted) limits in S-Alg, and (2) \mathcal{P} -Alg is closed under subobjects in S-Alg, in the sense that if $m:A\to B$ is a monomorphism in S-Alg and B is a P-algebra, then A is a \mathcal{P} -algebra.

Proof. (1) follows from the fact that both $U^{\mathcal{S}}: \mathcal{S}\text{-Alg} \to \mathscr{C}$ and $U^{\mathcal{P}}: \mathcal{P}\text{-Alg} \to \mathscr{C}$ are strictly monadic \mathscr{V} -functors (by 5.8 and 5.20) and so preserve and create all limits. If $m: A \to B$ is a monomorphism in $\mathcal{S}\text{-Alg}$ and B satisfies a given parametrized $\mathcal{S}\text{-equation }t \doteq u$, then it is straightforward to verify that A satisfies $t \doteq u$, using the naturality of $[\![t]\!]$ and $[\![u]\!]$. (2) now follows.

Proposition 10.35. Let \mathcal{P} be a free-form \mathcal{J} -presentation over \mathcal{S} , and let \mathcal{A} be a tame set of \mathcal{P} -algebras. Suppose that for each $J \in \mathsf{ob} \mathcal{J}$, the free \mathcal{P} -algebra $T_{\mathcal{P}}J$ is a subobject of some (weighted) limit, in \mathcal{S} -Alg, of objects in \mathcal{A} . Then \mathcal{A} generates the \mathcal{J} -ary variety $\mathsf{ob}(\mathcal{P}\text{-Alg})$ over \mathcal{S} .

Proof. Let \mathcal{B} be the \mathscr{J} -ary variety over \mathcal{S} generated by \mathcal{A} . Then $\mathcal{B} \subseteq \mathsf{ob}(\mathcal{P}\text{-Alg})$, since \mathcal{A} is a subset of the \mathscr{J} -ary variety $\mathsf{ob}(\mathcal{P}\text{-Alg})$ over \mathcal{S} . By 10.34, \mathcal{B} is closed in $\mathcal{S}\text{-Alg}$ under limits and subobjects. Hence, our hypothesis entails that $\{T_{\mathcal{P}}J \mid J \in \mathsf{ob}\mathscr{J}\} \subseteq \mathcal{B}$, so by 10.33 we deduce that $\mathsf{ob}(\mathcal{P}\text{-Alg}) \subseteq \mathcal{B}$.

Example 10.36. Let R be an internal rig in a complete and cocomplete cartesian closed category \mathscr{V} , and let $\mathcal{P} = (\mathcal{S}, \mathsf{E})$ be the diagrammatic (or, equivalently, free-form) $\mathsf{SF}(\mathscr{V})$ -presentation of internal (left) R-modules discussed in 6.1, so that $\mathcal{P}\text{-Alg} = R\text{-Mod}$. Given a finite cardinal n, let us write also n to denote the nth copower of 1 in \mathscr{V} and note that the free internal R-module on n is the (conical) power R^n in R-Mod (e.g. by [32, 6.4.5]), where R itself is regarded as an R-module by multiplication on the left. Hence, the single R-module R generates the $\mathsf{SF}(\mathscr{V})$ -ary variety $\mathsf{ob}(R\text{-Mod})$ over \mathscr{S} , by 10.35.

Example 10.37. Again let R be an internal rig in a complete and cocomplete cartesian closed category \mathscr{V} , but this time let $\mathcal{P} = (\mathcal{S}, \mathsf{E})$ be the diagrammatic $\mathsf{SF}(\mathscr{V})$ -presentation of (internal left) R-affine spaces discussed in 8.15, so that \mathcal{P} -Alg = R-Aff. Let us regard R itself as a left R-affine space (taking R-affine combinations in R in the usual way, [32, 8.6]). Given a finite cardinal n, the free R-affine space on n (or rather, on the nth copower of 1 in \mathscr{V}) is the subobject $R^{n,\text{aff}} \hookrightarrow R^n$ in R-Aff of n-tuples with sum 1, by [32, §8] (or by 8.15). Hence, the single R-affine space R generates the $\mathsf{SF}(\mathscr{V})$ -ary variety $\mathsf{ob}(R$ -Aff) over \mathscr{S} , by 10.35. \square

Noting that 10.34 involves closure properties for \mathscr{J} -ary varieties over \mathscr{S} that are related to the Birkhoff variety theorem [5], it is natural to ask whether a generalization of the latter theorem can be proved in the present general setting, perhaps under additional hypotheses; we leave this question for future work.

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