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The eigenvalue complementarity problem

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Abstract In this paper an eigenvalue complementarity problem (EiCP) is studied, which finds its origins in the solution of a contact problem in mechanics. The EiCP is shown to be equivalent to a Nonlinear Complementarity Problem, a Mathematical Programming Problem with Complementarity Constraints and a Global Optimization Problem. A finite Reformulation–Linearization Technique (RLT)-based tree search algorithm is introduced for processing the EiCP via the lattermost of these formulations. Computational experience is included to highlight the efficacy of the above formulations and corresponding techniques for the solution of the EiCP.

Keywords Global optimization · Complementarity · Eigenvalue problems

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1 Introduction

Given the matrix $A \in \mathbb{R}^{n\times n}$ and the positive definite matrix $B \in \mathbb{R}^{n\times n}$, the Eigenvalue Complementarity Problem (EiCP) consists of finding a scalar $\lambda > 0$ and a vector $x \in \mathbb{R}^n \setminus \{0\}$ such that

$$
\begin{aligned}
&w = (\lambda B - A)x,\\
&w \ge 0, \qquad x \ge 0,\\
&x^T w = 0.
\end{aligned}
$$

This problem is a special case of the Generalized Eigenvalue Complementarity Problem (GEiCP_J), where $J \subseteq \{1,\dots,n\}$, which originally appeared in the study [4] of the states of static equilibrium of a finite dimensional mechanical system with unilateral friction contact. The problem GEiCP_J consists of finding a scalar $\lambda > 0$ and a vector $x \in \mathbb{R}^n \setminus \{0\}$ such that

$$
\begin{aligned}
&w = (\lambda B - A)x,\\
&w_J \ge 0, \qquad x_J \ge 0, \qquad x_J^T w_J = 0\\
&w_{\bar{J}} = 0,
\end{aligned}
\tag{1}
$$

where $x_J \equiv (x_j, j \in J)$, $w_J \equiv (w_j, j \in J)$, and $\bar{J} = \{1,\dots,n\} \setminus J$. Note that the EiCP is obtained from (1) when $J = \{1,\dots,n\}$. This problem arises in several important practical applications in engineering and physics, as for example in studying the resonance frequency of structures and the stability of dynamic systems [4].

Note that the GEiCP_J with $J = \emptyset$ ($w = 0$) leads to the so-called Generalized Eigenvalue Problem [8]. For any solution (λ, x) of GEiCP_J, the value of λ is called a *Complementary Eigenvalue* of (A, B) and x is a corresponding *Complementary Eigenvector*. Since the set of complementary eigenvectors associated with a given complementary eigenvalue is a cone, there is no loss of generality to consider only the solutions satisfying $\|x\|_2 = p$, with $p > 0$. These constraints ensure that $x \neq 0$. In the case of EiCP, the constraint $\|x\|_2 = p$ can be replaced by the linear constraint $\|x\|_1 = e^T x = p$, since $x \ge 0$, where e is a vector of ones.

It is easy to see that any solution of EiCP with $w = 0$ is a positive eigenvalue of (A, B) with a corresponding eigenvector satisfying a sign constraint. In general, we can prove [16] that for any solution (λ, x) to GEiCP_J, there is a set I satisfying $\bar{J} \subseteq I \subseteq \{1, 2, \dots, n\}$, such that λ is a positive eigenvalue of (A_{II}, B_{II}) and x_I is a corresponding eigenvector satisfying $x_{J\cap I} \ge 0$, where A_{II} and B_{II} are respective submatrices of A and B with rows and columns indexed by I. For the EiCP, this result means that given a solution (λ, x), λ is an eigenvalue of (A_{II}, B_{II}) and x_I is a corresponding nonnegative eigenvector. As a corollary of this result, the number of solutions of the EiCP and of the GEiCP is finite. Moreover, in [16], it is proved that GEiCP_J has at most $(n - |J| + 2)2^{|J|} - |J| - 2$ solutions. As EiCP is a particular case of GEiCP_J with $J = \{1,\dots,n\}$, we see that the number of solutions for EiCP is at most $2^{n+1} - n - 2$.

The set of solutions of GEiCP_J can be obtained through a complete enumeration. Although this method is not practical, it provides a necessary and sufficient condition for the feasibility of the GEiCP$_J$. However, it has been shown in [16] that determining the feasibility of GEiCP$_J$ is an NP-complete decision problem, since any solution (λ, x) of GEiCP$_J$ must verify $x_J \geq 0$ and $x^T Ax > 0$, which is an NP-complete problem [3]. Therefore, the GEiCP$_J$ is in general an NP-hard problem. However, the class of the matrix A plays a very important role in the solution of GEiCP$_J$. Indeed, for some classes of matrices, the feasibility of GEiCP$_J$ can be easily established [16].

The case where A and B are symmetric matrices was studied in [16], where it was shown that EiCP can be reduced to the problem of finding a stationary point of the Rayleigh function on the simplex.

In this paper, the more general case where at least one matrix is asymmetric is considered. A Nonlinear Complementarity Problem (NCP) and a Mathematical Programming with Complementarity Constraints (MPEC) are established as alternative formulations of EiCP. The solution of EiCP by exploiting these formulations is also studied. It is shown that a Complementarity Active-Set algorithm [12] for finding a Stationary Point of an MPEC can in many cases process the EiCP. On the other hand, a robust technique for NCP such as PATH, is not in general able to solve the EiCP by exploiting the NCP formulation. The EiCP can additionally be shown to be equivalent to a global optimization problem. A branch-and-bound method for the solution of this optimization problem is introduced, that is based on the Reformulation–Linearization Technique (RLT) of Sherali and Tuncbilek [18]. Numerical results are included showing that this algorithm is able to solve the EiCP when the order of the matrices A and B is small.

The remainder of this paper is organized as follows. In Sect. 2 the symmetric problem is briefly discussed. The formulations for the asymmetric case are presented in Sect. 3. The Reformulation–Linearization Technique based branch-and-bound method for the EiCP is introduced in Sect. 4. The importance of scaling the EiCP is treated in Sect. 5. Finally, some computational experience and some conclusions are presented in the last section.

2 The symmetric eigenvalue complementarity problem

In this section, the symmetric EiCP is considered, where the matrices A and B are both symmetric. In this case, the EiCP is closely related to the classical Eigenvalue Problem. The complementarity condition $x^T w = 0$ can be rewritten as $x^T(\lambda Bx - Ax) = 0$. Because $x \neq 0$, and B is positive definite, then

$$\lambda(x) = \frac{x^T Ax}{x^T Bx}$$

where $\lambda(x)$ is the generalized Rayleigh quotient function [8]. Since the gradient of this function is

$$\nabla\lambda(x) = \frac{2}{x^T Bx}[A - \lambda(x)B]x,$$

and $\nabla\lambda(x) = 0$ if and only if $[A - \lambda(x)B]x = 0$, then it is to be expected that any stationary point $(x, \lambda(x))$ of the generalized Rayleigh quotient in the nonnegative orthant with $\lambda(x) > 0$ provides a solution of EiCP. Indeed, if we consider the optimization problem

$$\text{OEiCP:} \quad \begin{cases} \text{Maximize} & \lambda(x) \\ \text{subject to} & x \geq 0, \\ & e^T x = p \end{cases}$$

where $e = (1, \ldots, 1)^T$ and $p > 0$, then the following result holds [16].

Theorem 1

- *x is a stationary point of OEiCP with $\lambda(x) > 0$ if and only if $(\lambda = \lambda(x), x)$ is a solution of EiCP.*
- *The EiCP has a solution if and only if there exists $0 \neq x \geq 0$ such that $x^T Ax > 0$.*
- *The symmetric EiCP is NP-hard.*
- *The EiCP has at most $2^n - 1$ eigenvalues.*

Taking into account this theorem, the following process for solving the symmetric EiCP can be designed:

1. Find a vector $\bar{x}$ such that $0 \neq \bar{x} \geq 0$, $\bar{x}^T A\bar{x} > 0$, and $e^T \bar{x} = p$.
2. Find a stationary point of

$$\begin{array}{ll} \text{Maximize} & \lambda(x) \\ \text{subject to} & x \geq 0, \\ & e^T x = p \end{array}$$

   using $\bar{x}$ as an initial solution by any suitable nonlinear optimization procedure.

It is important to point out that the first step of this process is not usually easy, since the underlying problem is in general NP-complete [3]. However, Table 1 presents some classes of matrices for which an initial solution for the OEiCP can be easily obtained [16] (e^i represents the ith canonical basis vector). Hence, the symmetric EiCP has a solution when the matrix A belongs to any of these classes. In particular, the symmetric EiCP has a solution if A is a symmetric PD matrix, a nonzero symmetric PSD matrix or a symmetric strictly copositive matrix [5].

Table 1 Initial solution for OEiCP

Class of A	Initial solution
$\exists i$: $a_{ii} > 0$	$x = e^i$
$A \geq 0$, $A \neq 0$	$x = e$
$\exists i, j$: $a_{ii} = 0$, $a_{jj} \leq 0$ and $a_{ij} > 0$	$x_l = \begin{cases} 1, & l = j, \\ \frac{1-a_{jj}}{2a_{ij}}, & l = i, \\ 0, & l \neq i, j \end{cases}$
$\exists x \geq 0$: $Ax > 0$	x is solution of LP: max y s.t. $Ax - ye \geq 0$, $e^T x = p$, $x \geq 0$, $y \in \mathbb{R}$.

3 The asymmetric eigenvalue complementarity problem

The expression of the gradient of the generalized Rayleigh quotient presented in the previous section is only valid when A is symmetric. If $A \neq A^T$ and B is symmetric PD, the expression of this gradient is:

$$\nabla\lambda(x) = \frac{1}{x^T Bx}\left[A + A^T - 2\lambda(x)B\right]x,$$

and the relation between the stationary points of OEiCP and the solutions of the EiCP is no longer valid. Thus, other methodologies for the solution of the asymmetric EiCP need to be investigated. The best approach is to reformulate the EiCP as a known optimization problem. Three of these suitable reformulations of EiCP are discussed below.

3.1 Nonlinear complementarity problem

The EiCP can be re-stated in the following way (henceforth referred to as EiCP itself):

$$\text{EiCP:}\quad \begin{array}{l} w = Bx - x_{n+1}Ax, \\ e^T x = p, \\ x_{n+1} > 0, \\ x \geq 0,\ w \geq 0, \quad x^T w = 0 \end{array} \tag{2}$$

where $x_{n+1} = \frac{1}{\lambda}$. If a new nonnegative variable w_{n+1} is introduced such that

$$w_{n+1} = -p + e^T x \quad \text{and} \quad w_{n+1}x_{n+1} = 0,$$

then the following Nonlinear Complementarity Problem is obtained

$$\text{NCP:}\quad \begin{array}{l} w = Bx - x_{n+1}Ax, \\ w_{n+1} = -p + e^T x, \\ x \geq 0, \quad w \geq 0, \quad x_{n+1} \geq 0, \quad w_{n+1} \geq 0, \\ x^T w = x_{n+1}w_{n+1} = 0. \end{array} \tag{3}$$

To check that the EiCP and the NCP are equivalent, if $x_{n+1} = 0$ in a given solution of NCP, then

$$\begin{array}{ll} w = Bx, & x \geq 0,\ w \geq 0, \\ x^T w = 0, & e^T x \geq p. \end{array}$$

This is impossible for $B \in$ PD, since $x^T w = x^T Bx = 0$ implies $x = 0$, which contradicts that $e^T x \geq p > 0$. On the other hand, if $x_{n+1} > 0$ and $w_{n+1} = 0$ in any solution of NCP, then

$$e^T x = p \quad \Rightarrow \quad x \neq 0$$

and

$$\begin{array}{l} w = Bx - x_{n+1}Ax, \quad w \geq 0,\ x \geq 0, \\ x^T w = 0. \end{array}$$

So, $(\lambda = \frac{1}{x_{n+1}}, x)$ is a solution of EiCP. The converse implication is obvious. As the NCP (3) is not monotone, complementarity algorithms [7] may have a great difficulty to process it. The algorithm PATH [2], which is included in the GAMS collection, is considered to be the most widely used in practice. In the last section of this paper, some computational experience is reported showing that PATH is, in many cases, unable to solve the NCP and thus to process the EiCP.

3.2 Mathematical programming problem with complementarity constraints

In order to reformulate the EiCP as a Mathematical Programming Problem with Complementarity Constraints (MPEC), a new nonnegative auxiliary vector y is introduced in the reformulation (2) of EiCP such that $y = x_{n+1}x$. Considering this equality and multiplying the condition $e^T x = p$ by x_{n+1}, it is obvious that any solution of the EiCP must verify

$$\begin{aligned} &w = Bx - Ay, \\ &e^T x = p, \\ &e^T y - p x_{n+1} = 0, \\ &x \geq 0, \quad w \geq 0, \quad y \geq 0, \quad x_{n+1} \geq 0, \\ &x^T w = 0, \\ &\|y - x_{n+1}x\|_2 = 0. \end{aligned}$$

We note that this new constraint $e^T y - p x_{n+1} = 0$ is an RLT restriction [18] that has been introduced for the purpose of ensuring the convergence of our algorithm proposed below (see Theorem 4).

Accordingly, consider the following optimization problem with complementarity constraints.

$$\begin{array}{lll} \text{MPEC:} & \text{Minimize} & (y - x_{n+1}x)^T (y - x_{n+1}x) \\ & \text{subject to} & \\ & \text{GLCP} & \left\{ \begin{array}{l} w = Bx - Ay, \\ e^T x = p, \\ e^T y - p x_{n+1} = 0, \\ x \geq 0, \quad w \geq 0, \quad y \geq 0, \quad x_{n+1} \geq 0, \\ x^T w = 0. \end{array} \right. \end{array} \tag{4}$$

Hence, (λ, x) is a solution of EiCP if and only if $(x_{n+1} = \frac{1}{\lambda}, x, y = x_{n+1}x)$ is a solution of MPEC with zero objective value. The following process for finding a solution for the EiCP can therefore be designed:

1. Solve the GLCP (defined by (4)).
2. Find a stationary point or a global minimum for the MPEC.

In order to find a solution for the GLCP the following result is useful.

Theorem 2 *If the EiCP has a solution, then there exists* $0 \neq \bar{y} \geq 0$ *such that* $(A\bar{y})_i > 0$, *for some* i.

Proof If $A \leq 0$, then the EiCP

$$\begin{aligned}
&w = Bx - A(x_{n+1}x), \quad x \geq 0,\ w \geq 0,\\
&x^T w = 0,\\
&x_{n+1} > 0, \qquad x \neq 0
\end{aligned}$$

has no solution. To see this, if $0 \neq x \geq 0$ and $x_{n+1} > 0$ then

$$A(x_{n+1}x) = x_{n+1}Ax \leq 0.$$

Therefore,

$$w \geq Bx \quad \Rightarrow \quad w^T x \geq x^T Bx > 0$$

and so the EiCP has no solution. Hence, A has at least one element $a_{rs} > 0$, and letting $\bar{y} = e^s$, the sth unit vector, we have

$$(A\bar{y})_r = \sum_{j=1}^{n} a_{rj} e_j^s = a_{rs} > 0.$$

□

Accordingly, let $\bar{y}$ be such that $0 \neq \bar{y} \geq 0$, $(A\bar{y})_i > 0$ for some i and let us consider LCP$(-A\bar{y}, B)$:

$$\text{LCP:} \quad \begin{cases} w = -A\bar{y} + Bx, \\ x \geq 0, \qquad w \geq 0, \\ x^T w = 0. \end{cases}$$

As $B \in$ PD, then the LCP has an unique solution $(\bar{w}, \bar{x})$ [5]. Moreover $0 \neq \bar{x} \geq 0$, since if $\bar{x} = 0$, we have

$$\bar{w}_i = (-A\bar{y})_i < 0.$$

If we now consider the vectors $\tilde{w}$, $\tilde{x}$, $\tilde{y}$, and $\tilde{x}_{n+1}$ such that

$$\tilde{w} = \frac{p\bar{w}}{e^T\bar{x}}, \qquad \tilde{x} = \frac{p\bar{x}}{e^T\bar{x}}, \qquad \tilde{y} = \frac{p\bar{y}}{e^T\bar{x}}, \qquad \tilde{x}_{n+1} = \frac{e^T\tilde{y}}{p}$$

then $(\tilde{x}, \tilde{y}, \tilde{w}, \tilde{x}_{n+1})$ is solution of the GLCP.

It is important to add that if B is the identity matrix, then the LCP can be easily solved as

$$\begin{cases} (A\bar{y})_i > 0 \quad \Rightarrow \quad \bar{x}_i = (A\bar{y})_i \\ (A\bar{y})_i \leq 0 \quad \Rightarrow \quad \bar{x}_i = 0 \end{cases} \quad \text{for all } i = 1, \ldots, n.$$

On the other hand, if $B \neq I$, the LCP can be efficiently solved by one of the algorithms discussed in [11], and the solution to GLCP can then be recovered as above.

After a solution of the GLCP is at hand, the Complementarity Active Set Algorithm (CASET) [12] can be used for finding a stationary point $(\bar{x}, \bar{y}, \bar{x}_{n+1})$ of the

MPEC associated with the EiCP. This procedure is based on the active-set strategy [15], which begins with the solution of GLCP and tries to reduce the objective value while maintaining complementarity during the entire procedure. It is possible to show that under reasonable hypotheses the algorithm is able to find a stationary point for the MPEC. If this stationary point satisfies $\|\bar{y} - \bar{x}_{n+1}\bar{x}\|_2 = 0$, then $(\bar{\lambda} = \frac{1}{\bar{x}_{n+1}}, \bar{x})$ is a solution of the EiCP. Note that $\bar{x}_{n+1} > 0$ in such a solution. In fact, if $\bar{x}_{n+1} = 0$ then $\bar{y} = 0$, and $\bar{w} = B\bar{x}$, $\bar{x} \neq 0$ implies that $\bar{w}^T\bar{x} = \bar{x}^T B\bar{x} > 0$, which is a contradiction.

If such a stationary point $(\bar{x}, \bar{y}, \bar{x}_{n+1})$ does not satisfy $\|\bar{y} - \bar{x}_{n+1}\bar{x}\|_2 = 0$, then a global optimization algorithm is necessary to find a global minimum for the MPEC. There are some algorithms to perform this task when the objective function is convex [1, 9, 10, 13]. However, the function of the MPEC (4) is not convex, which precludes the use of such techniques. Therefore, we have only tested the performance of the Complementarity Active-Set method for processing the EiCP by finding a stationary point of the associated MPEC. The design of a global optimization algorithm for dealing with this MPEC is proposed as a topic of future research. Instead, in this paper, we have used another methodology to treat the EiCP as a global optimization problem. This topic is discussed next.

3.3 Global optimization problem

Let us consider the following formulation of the EiCP.

$$
\begin{aligned}
\text{GOP:} \quad \text{Minimize} \quad & \sum_{i=1}^{n} w_i x_i \\
\text{subject to} \quad & w = -x_{n+1}Ax + Bx, && (5)\\
& e^T x = p, && (6)\\
& x_{n+1} \geq 0, \qquad x \geq 0, \qquad w \geq 0. && (7)
\end{aligned}
$$

As described in the previous section, $(\bar{\lambda} = \frac{1}{\bar{x}_{n+1}}, \bar{x})$ is a solution of EiCP if and only if $(\bar{x}, \bar{w}, \bar{x}_{n+1})$ solves GOP with

$$\min_{(x,w,x_{n+1})\in S} w^T x = 0$$

where S is the set satisfying the constraints of GOP. To facilitate the design of a global optimization algorithm, it is useful to restrict x_{n+1} to belong to a closed interval while searching for an optimal solution to GOP. Hence, the following problem can be considered

$$
\begin{aligned}
\text{GOP:} \quad \text{Minimize} \quad & \sum_{i=1}^{n} w_i x_i \\
\text{subject to} \quad & w = -x_{n+1}Ax + Bx, && (8)\\
& e^T x = p, && (9)\\
& \epsilon \leq x_{n+1} \leq \Delta, && (10)\\
& x \geq 0, \qquad w \geq 0 && (11)
\end{aligned}
$$

where $\Delta > 0$ is sufficiently large and $\epsilon > 0$ is very small. A branch-and-bound algorithm for solving this global optimization problem is described in the next section.

4 An RLT-based branch-and-bound method (BBRLT)

In this section, the solution of the EiCP using the Reformulation–Linearization Technique (RLT) [18] is studied. This technique involves two steps, namely the reformulation and the linearization phases. In the reformulation phase, a set of suitable nonnegative variable factors are defined and products with these factors and the original constraints are constructed to generate some nonlinear constraints. In the linearization phase, an appropriate technique of replacing nonlinear product terms by variables is used to linearize the reformulated nonlinear problem.

The RLT process yields a lower bounding problem that can be embedded in a branch-and-bound algorithm to solve the GOP problem presented in the previous section. As a feasible solution of GOP is a solution of EiCP if and only if the value of its objective function is zero, then a node is fathomed whenever the corresponding lower bound is positive. Moreover, whenever a feasible solution of the GOP with an objective value of zero is detected, the algorithm terminates with a solution for the EiCP at hand. For each node k of the binary branch-and-bound tree, two types of branching mechanisms can be considered, as depicted in Fig. 1.

Thus, for any given node subproblem, the added restrictions imposed on the branches on the chain from this node to the root node must be considered. Let us assume that these restrictions are given by

$$\begin{aligned} &l \le x_{n+1} \le u, \\ &w_i = 0, \quad \forall i \in I_0, \\ &x_i = 0, \quad \forall i \in J_0 \end{aligned}$$

where $\epsilon \le l < u \le \Delta$ and I_0 and J_0 are the sets of indices of the fixed variables at the current node ($I_0 \cap J_0 = \emptyset$). Let us consider the set

$$K_0 = I_0 \cup J_0$$

and let

$$\bar{I}_0 = \{1, \dots, n\} \setminus I_0, \qquad \bar{J}_0 = \{1, \dots, n\} \setminus J_0, \qquad \bar{K}_0 = \{1, \dots, n\} \setminus K_0.$$

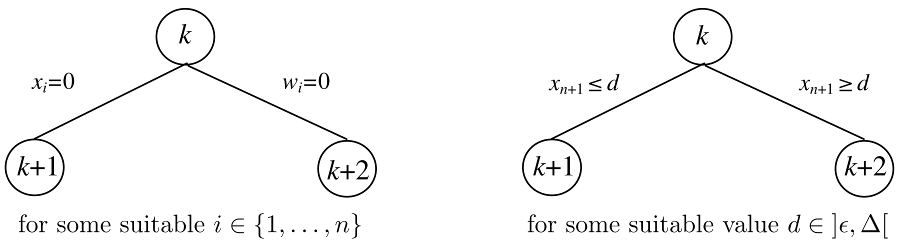

Fig. 1 Branching scheme of the algorithm

The subproblem at node k is then given by

$$\begin{aligned}
\text{GOP}(k):\quad & \text{Minimize} \quad \sum_{i\in\bar{K}_0} w_i x_i \\
& \text{subject to} \quad w = -x_{n+1}Ax + Bx, && (12)\\
& \qquad e^T x = p, && (13)\\
& \qquad l \le x_{n+1} \le u, && (14)\\
& \qquad (x, w) \ge 0, && (15)\\
& \qquad w_i = 0, \quad \forall i \in I_0, \qquad x_i = 0, \quad \forall i \in J_0. && (16)
\end{aligned}$$

The RLT technique is now used for finding a lower bound for this node subproblem. To do this, some transformations on the problem are performed in such a way that a lower bounding problem is obtained in a higher dimensional space. These transformations are presented below:

- In the Reformulation Phase, the following products are constructed, where $(12)_i$ is the ith row of (12):
  $(12)_i$ is multiplied by $x_i, \forall i \in \bar{J}_0$.
  (13) is multiplied by: $\begin{cases} x_{n+1} & \\ x_i, & \forall i \in \bar{J}_0, \\ x_{n+1}x_i, & \forall i \in \bar{J}_0. \end{cases}$
  (14) is multiplied by $x_i, \forall i \in \bar{J}_0$.
- In the Linearization Phase, the following variables are substituted in place of the stated polynomial terms:
  $z_i = w_i x_i, \forall i \in \bar{K}_0$ $(z_i = 0, \forall i \in K_0)$.
  $y_i = x_{n+1}x_i, \forall i \in \bar{J}_0$ $(y_i = 0, \forall i \in J_0)$.
  $v_{ij} = x_{n+1}x_i x_j, \forall i, j \in \bar{J}_0, i \le j$ ($v_{ij} = 0$ for all (i, j) with $i \in J_0$ or $j \in J_0$).
  $s_{ij} = x_i x_j, \forall i, j \in \bar{J}_0, i \le j$ ($s_{ij} = 0$ for all (i, j) with $i \in J_0$ or $j \in J_0$).

The application of the RLT technique yields the following linear program, where $v_{(ij)} \equiv v_{ij}$ if $i \le j$ and $v_{(ij)} \equiv v_{ji}$ if $j < i$ (similarly for $s_{(ij)}$).

$$\text{LP}(k):\quad \text{Minimize} \quad \sum_{i\in\bar{K}_0} z_i$$

subject to

$$w = -Ay + Bx, \tag{17}$$

$$z_i = -\sum_{j=1}^{n} a_{ij} v_{(ij)} + \sum_{j=1}^{n} b_{ij} s_{(ij)}, \quad \forall i \in \bar{J}_0, \tag{18}$$

$$e^T x = p, \tag{19}$$

$$e^T y = p x_{n+1}, \tag{20}$$

$$\sum_{j=1}^{n} s_{(ij)} = px_i, \quad \forall i \in \bar{J}_0 \tag{21}$$

$$\sum_{j=1}^{n} v_{(ij)} = py_i, \quad \forall i \in \bar{J}_0 \tag{22}$$

$$lx_i \leq y_i \leq ux_i, \quad \forall i \in \bar{J}_0 \tag{23}$$

$$\begin{array}{llllll} z_i = 0, & \forall i \in K_0, & x_i = 0, & \forall i \in J_0, & w_i = 0, & \forall i \in I_0, \\ y_i = 0, & \forall i \in J_0, & v_{(ij)} = 0 & \text{if } i \in J_0 \text{ or } j \in J_0, & & \\ s_{(ij)} = 0 & \text{if } i \in J_0 \text{ or } j \in J_0, & & & & \end{array} \tag{24}$$

$$\gamma = (z, w, x, x_{n+1}, y, v, s) \geq 0 \tag{25}$$

where $z = (z_i) \in \mathbb{R}^n$, $y = (y_i) \in \mathbb{R}^n$, $v = (v_{ij}) \in \mathbb{R}^{n(n+1)/2}$, and $s = (s_{ij}) \in \mathbb{R}^{n(n+1)/2}$.

Let $\bar{\gamma} = (\bar{z}, \bar{w}, \bar{x}, \bar{x}_{n+1}, \bar{y}, \bar{v}, \bar{s})$ be a solution of LP(k) with $\bar{\tau}$ being the objective value. If $\bar{\tau} > 0$ then this node k can be fathomed. On the other hand, let

$$\theta_1 = \max_{i \in \bar{K}_0} \{\bar{w}_i \bar{x}_i\}, \qquad \theta_2 = \max_{i \in \bar{J}_0} |\bar{y}_i - \bar{x}_{n+1}\bar{x}_i|.$$

The next result is an immediate consequence of the definition of the GOP and provides a stopping criterion for the algorithm.

Theorem 3 *If* $\theta_1 = \theta_2 = 0$, *then* $(\lambda = \frac{1}{\bar{x}_{n+1}}, \bar{x})$ *is a solution of EiCP.*

On the other hand, if $\theta = \max\{\theta_1, \theta_2\} \neq 0$ then branching is conducted in the following way:

- If $\theta_1 > \theta_2$, then branch on the dichotomy $\{w_i = 0\} \vee \{x_i = 0\}$ for i such that $\theta_1 = \bar{w}_i \bar{x}_i$.
- Otherwise, $\theta_2 \geq \theta_1$ with $\theta_2 > 0$, and we branch in the following way:

$$\{l \leq x_{n+1} \leq \tilde{x}_{n+1}\} \vee \{\tilde{x}_{n+1} \leq x_{n+1} \leq u\} \tag{26}$$

where

$$\tilde{x}_{n+1} = \begin{cases} \bar{x}_{n+1}, & \text{if } \min\{(\bar{x}_{n+1} - l),\ (u - \bar{x}_{n+1})\} \geq \eta(u - l), \\ \dfrac{(u + l)}{2}, & \text{otherwise,} \end{cases}$$

with $0 < \eta < 0.5$ (we recommend $\eta = 0.1$).

The convergence of this algorithm to a solution of the EiCP is proved in the next theorem.

Theorem 4 *Consider the above branch-and-bound algorithm with the stated partitioning rule. Then either the procedure terminates finitely with a solution to EiCP, including possibly an indication that no solution exists, or else, an infinite branch-and-bound tree is generated such that along any infinite branch of this tree, any accumulation point of the LP(*k*)-solutions generated solves EiCP.*

Proof The case of finite convergence is obvious. Else, suppose that an infinite tree is generated and consider any infinite branch. Let an accumulation point of LP(k)-solutions along this branch for an index set K yield limits

$$\{\gamma^k\}_K \to \gamma^*, \qquad \{[l_k, u_k]\}_K \to [l^*, u^*]$$

where $l_k \le x_{n+1}^k \le u_k$, $\forall k$, represents the node k bounds on x_{n+1}. Note that we can have $\theta_1 > \theta_2$ only finitely often for $k \in K$, because this results in fixing either some $w_i = 0$ or $x_i = 0$. Hence, there exists $k_1 \in \mathbb{N}$ such that for $k \ge k_1$, $k \in K$, we have $\theta_2 \ge \theta_1$ and we branch on x_{n+1} according to (26).

Owing to the branching process, following the argument in [18], we get

$$\{x_{n+1}^* = l^*\} \vee \{x_{n+1}^* = u^*\} \tag{27}$$

where we could have $l^* = u^*$.

Consider the case $x_{n+1}^* = l^*$ in (27) (the case of $x_{n+1}^* = u^*$ is similar).

From (23) we have in the limit that

$$y_i^* \ge l^* x_i^*, \quad \forall i. \tag{28}$$

Furthermore, (20) yields

$$\sum_{i=1}^{n} y_i^* = pl^*. \tag{29}$$

Note that if any inequality in (28) is strict, then using (28) and (19), we would have

$$\sum_{i=1}^{n} y_i^* > l^* \sum_{i=1}^{n} x_i^* = pl^*,$$

which would contradict (29). Hence,

$$y_i^* = l^* x_i^* = x_{n+1}^* x_i^*, \quad \forall i.$$

Consequently, from (17), we have,

$$w^* = -Ay^* + Bx^* = -x_{n+1}^* Ax^* + Bx^*. \tag{30}$$

Furthermore, since $\theta_2 \ge \theta_1 \ge 0$ for $k \ge k_1$, and $\theta_2 \to 0$, we have that $\theta_1 \to 0$, i.e.,

$$w_i^* x_i^* = 0, \quad \forall i = 1, \dots, n. \tag{31}$$

Moreover, since all active nodes have a lower bound of zero, we have $z^k = 0$, $\forall k \in K$, and the LP(k) objective value τ^k is also zero for all $k \in K$. Thus

$$\{\tau^k\} \to \tau^* = 0 \quad \text{and} \quad z_i^* = 0, \quad \forall i = 1, \dots, n.$$

In addition, $x_{n+1}^* = l^*$ along with the branching process implies that

$$0 < \epsilon \le x_{n+1}^* \le \Delta. \tag{32}$$

Thus by (19), (25), (30), and (32), (w^*, x^*, x^*_{n+1}) is feasible for the GOP with $x^*_{n+1} > 0$. But by (31), the objective value associated with this feasible solution is zero, and therefore, $(\lambda^* = \frac{1}{x^*_{n+1}}, x^*)$ is a solution of EiCP. □

This proof shows that the variables v_{ij} and s_{ij} and the restrictions (18), (21), and (22) can be omitted in the LP(k) problem without impairing the theoretical convergence of the branch-and-bound algorithm. The advantage of using these new variables and constraints has to do with the possibility of obtaining stronger lower bounds that may reduce the search process. On the other hand, this has the obvious disadvantage of drastically increasing the number of constraints and variables of the LP(k) programs that are required to be solved at each node k.

5 Scaling in the eigenvalue complementarity problem

In this section, we show that the EiCP can be scaled so that all elements of the matrices A and B have absolute values less than or equal to one. Let

$$\text{EiCP:}\quad \begin{array}{l} w = Bx - x_{n+1}Ax, \quad x \geq 0,\ w \geq 0, \\ x^T w = 0, \\ e^T x = p, \\ x_{n+1} \geq 0 \end{array}$$

with $p > 0$, $B \in \text{PD}$ (symmetric or asymmetric) of order $n \times n$, $A \in \mathbb{R}^{n\times n}$, $x, w \in \mathbb{R}^n$, and where $e \in \mathbb{R}^n$ is a vector of ones.

Let

$$\begin{cases} \beta = \max\limits_{(i,j)} |b_{i,j}| > 0, \\ \alpha = \max\limits_{(i,j)} |a_{i,j}| > 0. \end{cases}$$

Then,

$$w = Bx - x_{n+1}Ax \quad \Leftrightarrow \quad w = \left(\frac{1}{\beta}B\right)(\beta x) - \left(\frac{x_{n+1}\alpha}{\beta}\right)\left(\frac{1}{\alpha}A\right)(\beta x).$$

Moreover,

$$\begin{aligned} x_{n+1} > 0 \quad &\Leftrightarrow \quad \frac{x_{n+1}\alpha}{\beta} > 0 \quad \Leftrightarrow \quad \frac{\beta x_{n+1}}{\alpha} > 0, \\ \beta x \geq 0 \quad &\Leftrightarrow \quad x \geq 0, \\ w^T x = 0 \quad &\Leftrightarrow \quad (\beta x)^T w = \beta(x^T w) = 0, \\ e^T x = p \quad &\Leftrightarrow \quad e^T(\beta x) = \beta(e^T x) = \beta p, \end{aligned}$$

and

$$\left|\frac{1}{\beta}b_{ij}\right| = \frac{1}{\beta}|b_{ij}| \leq 1, \qquad \left|\frac{1}{\alpha}a_{ij}\right| = \frac{1}{\alpha}|a_{ij}| \leq 1, \quad \forall (i,j),$$

$$\frac{1}{\beta}B \in \text{PD (PSD)} \quad \Leftrightarrow \quad B \in \text{PD (PSD)}.$$

Note also that

$$A \in \text{PD (PSD)} \quad \Leftrightarrow \quad \frac{1}{\alpha} A \in \text{PD (PSD)}.$$

Therefore, the EiCP is equivalent to

$$\begin{aligned} \text{EiCP}_{\text{esc}}: \quad & w = \left(\frac{1}{\beta} B\right) x - x_{n+1} \left(\frac{1}{\alpha} A\right) x, \quad x \geq 0, \ w \geq 0, \\ & x^T w = 0, \qquad e^T x = \beta p \end{aligned}$$

and we have

$$\begin{cases} \bar{x} = (\bar{x}_i) \in \mathbb{R}^n \\ \bar{x}_{n+1} > 0 \end{cases} \quad \text{is a solution of EiCP}_{\text{esc}} \quad \Leftrightarrow \quad \begin{cases} \tilde{x} = \left(\frac{\bar{x}_i}{\beta}\right) \in \mathbb{R}^n, \\ \tilde{x}_{n+1} = \frac{\beta \bar{x}_{n+1}}{\alpha} > 0 \end{cases} \text{is a solution of EiCP.}$$

6 Computational experience

In this section, some computational experience is presented to illustrate the efficiency of the algorithms described in this paper for the solution of symmetric and asymmetric EiCPs. All computations have been performed on a Pentium IV 2.4 GHz machine having 256 MB of RAM.

6.1 Symmetric EiCPs

In this subsection, computational experience with the following algorithms is reported:

1. MINOS [14] for solving the Nonlinear Program OEiCP.
2. PATH [6] for solving the NCP defined by (3).
3. CASET [12] for finding a stationary point for MPEC defined by (4).

For our test problems, $B = I_n$, and $A \in \mathbb{R}^{n \times n}$ was randomly generated such that its elements are uniformly distributed in the interval [0, 1] or [−50, 50]. These two types of test problems are referred to by the sets S_1 and S_2, respectively. The parameter p in the restriction $e^T x = p$ was taken to be one, except for the problem s200, where $p = 10$ was used. To identify the test problems, the letters “s” and “a” are used according to whether the matrix A is symmetric or asymmetric. The number presented after these letters represents the order of the matrices A and B.

The behavior of the algorithms MINOS and CASET for solving symmetric EiCPs is presented in Table 2. In this table, as well as in the sequel, x_{n+1} is the inverse of the eigenvalue of the EiCP, NI is the total number of pivot steps, T is the total CPU time in seconds for solving the problem, OBJ is the obtained objective function value, and RES is the residual norm defined by

$$\|w - Bx + x_{n+1} Ax\|_2$$

where (x, w, x_{n+1}) is the solution found by CASET algorithm.

Table 2 Solution of the symmetric EiCP with algorithms MINOS and CASET

	Prob	MINOS			CASET				
		x_{n+1}	NI	T	x_{n+1}	NI	T	OBJ	RES
S_1	s200$^{(*)}$	0.0100	1537	20.92	0.0100	397	0.31	2.38E-15	8.31E-08
	s100	0.0197	286	1.01	0.0197	197	0.03	1.73E-13	1.58E-06
	s50	0.0396	154	0.15	0.0396	98	0.02	1.79E-16	3.64E-08
	s40	0.0495	125	0.09	0.0495	78	0.00	7.06E-15	2.10E-07
	s30	0.0665	97	0.12	0.0665	59	0.00	4.76E-19	1.21E-09
	s20	0.0987	69	0.04	0.0987	39	0.02	2.51E-16	1.96E-08
	s10	0.2065	32	0.03	0.2065	20	0.00	3.01E-18	1.79E-09
	s6	0.3447	19	0.03	0.3447	11	0.00	6.88E-14	1.58E-07
	s3	0.7878	8	0.03	0.7878	6	0.00	2.63E-14	5.12E-08
S_2	s200	0.0018	317	3.90	0.0019	2508	1.53	3.51E-08	2.35E-03
	s100	0.0024	163	0.05	0.0029	1023	0.19	4.57E-07	5.89E-03
	s50	0.0037	78	0.09	0.0037	195	0.02	8.34E-16	1.48E-07
	s40	0.0041	66	0.06	0.0045	219	0.03	2.03E-06	8.79E-03
	s30	0.0057	49	0.03	0.0068	117	0.02	5.07E-05	2.73E-02
	s20	0.0053	43	0.02	0.0053	57	0.02	3.73E-16	6.17E-08
	s10	0.0139	10	0.04	0.0157	27	0.03	9.05E-15	2.23E-07
	s6	0.0235	3	0.04	0.0235	2	0.03	6.65E-17	8.78E-09
	s3	0.0210	7	0.03	0.0210	6	0.02	9.99E-17	1.22E-08

The notation (*) is used whenever the algorithm MINOS has been unable to find a solution for the EiCP with $p = 1$.

In the computational experience performed with algorithm MINOS, the vector $x = e^i$ was used as an initial solution. This vector is the column i of the identity matrix associated with a positive diagonal element a_{ii} of the matrix A.

In the computational experience with the CASET algorithm, the test problems S_2 were scaled according to the described procedure in Sect. 5. Moreover, as $B = I$, the solution of the GLCP associated with MPEC is given by the procedure described after Theorem 2. In this process, $\bar{y}$ was chosen as the vector e^i corresponding to a positive diagonal element a_{ii} of A.

The results presented in Table 2 show that the CASET algorithm has been able to process all the symmetric EiCP problems of the set S_1 with a significantly lower computational effort than the commercial program MINOS. However, for the problems of the set S_2, in some cases, the CASET algorithm finds a stationary point for the MPEC that is not a solution for the EiCP. Nevertheless, it is interesting to note that the resulting residual is always very small in these cases.

The efficiency and robustness of the PATH method was tested by solving Problem EiCP via the Nonlinear Complementary Problem given by (3). The numerical results are not displayed, since the algorithm behaves quite badly for these examples. In fact the PATH method was unable to process about 63% of the test problems, and when it found a solution for the EiCP, the required computational effort was substantially greater than that for the CASET algorithm [17].

Despite the fact that the CASET algorithm does not guarantee a solution of the EiCP, its behavior appears to be, in general, efficient and robust when compared with methodologies based on other approaches.

6.2 Asymmetric EiCPs

In this subsection, the algorithms PATH and CASET for solving asymmetric EiCPs are compared. It is important to recall that PATH tries to solve the NCP (3) associated with the EiCP, while Algorithm CASET finds a stationary point of the MPEC (4) that is equivalent to the EiCP.

The matrix A was randomly generated in a similar manner to the case of the symmetric EiCP, resulting in two types of test problems designated by N_1 and N_2, respectively. Moreover, B was once again taken to be the identity matrix.

The results of the performance of these two algorithms are presented in Table 3. In this table, (****) is used whenever the PATH algorithm, with the default optional parameters, was unable to solve the EiCP. This occurred in 67% of the test problems in Table 3.

The results show that the CASET algorithm was able to solve all the test problems in the set N_1 in a more efficient and robust way than the PATH algorithm, where the latter was able to process only 75% of these problems.

This situation with PATH was drastically worsened for the N_2 problems, where it was unable to process all the problems. The behavior of the CASET algorithm turned

Table 3 Solution of the asymmetric EiCP with the algorithms PATH and CASET

	Prob	PATH				CASET				
		x_{n+1}	NI	T	Res	x_{n+1}	NI	T	Obj	Res
N_1	a200	****	521	4.63	1.00E+00	0.0010	397	0.33	1.70E-16	6.74E-08
	a100	0.0199	202	1.01	8.15E-07	0.0199	197	0.05	3.71E-14	6.46E-07
	a50	****	202	1.06	1.00E+00	0.0399	97	0.00	8.52E-13	2.07E-06
	a40	0.0493	133	0.75	1.62E-10	0.0493	78	0.00	1.75E-15	9.49E-08
	a30	0.0662	123	0.77	1.07E-08	0.0662	58	0.00	3.73E-17	9.18E-09
	a20	0.1006	114	0.78	5.18E-12	0.1006	40	0.00	2.91E-14	1.62E-07
	a10	0.1881	90	0.79	1.58E-07	0.1881	19	0.00	3.47E-14	1.86E-07
	a6	0.3268	90	0.79	3.04E-07	0.3268	11	0.00	1.13E-13	2.64E-07
	a3	****	122	1.12	1.00E+00	0.6182	7	0.00	1.38E-23	2.20E-12
N_2	a200	****	690	4.67	1.00E+00	0.0038	3173	2.05	7.84E-09	9.14E-04
	a100	****	234	1.39	1.00E+00	0.0047	974	0.19	9.34E-14	7.20E-06
	a50	****	130	1.12	1.00E+00	0.0077	303	0.03	2.56E-06	7.49E-03
	a40	****	383	1.08	1.00E+00	0.0064	231	0.00	1.32E-15	1.28E-07
	a30	****	141	1.09	1.00E+00	0.0125	178	0.02	9.04E-06	1.08E-02
	a20	****	130	1.10	1.00E+00	0.0112	66	0.02	2.13E-13	1.47E-06
	a10	****	130	1.12	1.00E+00	0.0110	21	0.02	1.78E-02	2.23E-07
	a6	****	517	1.98	1.00E+00	0.2233	5	0.03	1.78E-19	5.97E-10
	a3	****	119	1.13	1.00E+00	0.0258	3	0.05	6.91E-05	7.06E-03

out to be similar to that observed for the S_2 test problems, being able to process 65% of the cases. It is also important to add that the computational effort required by Algorithm CASET to obtain a solution for EiCP is significantly smaller than that required by PATH, and that the residual values of the resulting solutions are always very small. The class of problems for which the CASET algorithm is guaranteed to process the EiCP is a subject of future research.

6.3 Solution of the symmetric and asymmetric EiCPs with the RLT technique

In this subsection, we report numerical results obtained by the branch-and-bound algorithm, for solving some of the test problems S_2 and N_2. The experience with the test problems S_1 and N_1 is not reported since the CASET algorithm had a sufficiently efficient and robust behavior in these cases. The two types of test problems S_2 and N_2 were scaled according to the procedure described in Sect. 5.

The results are reported in Table 4 where, besides the previously used parameters, $[\epsilon, \Delta]$ indicates the interval for x_{n+1} on which the search of the solution of EiCP was restricted (we have used $\epsilon = 0$ in practice, because the algorithm performed better with this value than with a small positive number), p is the value used in the constraints of the LP(k) problem (we used $p = 1$ for the runs in Tables 2 and 3), ND is the number of enumerated nodes, NR is the number of performed branches, and NSD is the number of performed interval splits for x_{n+1}.

The numerical results clearly indicate that BBRLT was able to efficiently process problems of moderate dimension. However, the computational effort of the algorithm tends to increase drastically when the dimension of the EiCP increases. These results are in a way expected, noting the characteristics of the methodology and the high number of constraints in each node subproblem for computing the lower bounds required by the algorithm. We believe that removing some of these constraints and combining this method with the CASET algorithm will lead to a more efficient process for processing the EiCP. This topic is proposed for future research.

Table 4 Solution of the symmetric and asymmetric EiCPs with the BBRLT algorithm

	Prob	$[\epsilon, \Delta]$	p	x_{n+1}	NI	ND	NR	NSD	T
S_2	$s30$	$[0, 3]$	3	0.0054	31217	387	175	42	5.48
	$s20$	$[0, 0.5]$	3	0.0053	5483	162	77	11	0.80
	$s10$	$[0, 1]$	1	0.0135	335	39	13	16	0.14
	$s6$	$[0, 2]$	1	0.0290	396	87	20	27	0.17
	$s3$	$[0, 1]$	3	0.0210	38	43	3	19	0.11
N_2	a30	$[0, 3]$	3	0.0461	58673	4791	2142	289	123.61
	a20	$[0, 1]$	3	0.0112	32502	767	443	78	5.53
	a10	$[0, 3]$	1	0.0423	6286	539	217	74	1.36
	a6	$[0, 3]$	1	0.2233	302	93	22	31	0.61
	a3	$[0, 3]$	3	0.0311	23	3	2	0	0.39

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